ON SMOOTHING PROPERTIES AND TAO'S GAUGE TRANSFORM OF THE BENJAMIN-ONO EQUATION ON THE TORUS

(RÉGULARISATION ET TRANSFORMATION DE JAUGE DE TAO POUR L'ÉQUATION DE BENJAMIN–ONO SUR LE TORE)

by

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Abstract. — We prove smoothing properties of the solutions of the Benjamin-Ono equation in the Sobolev space $H^s(\mathbb{T},\mathbb{R})$ for any $s \geq 0$. To this end we show that Tao's gauge transform is a high frequency approximation of the nonlinear Fourier transform for the Benjamin-Ono equation, constructed in our previous work. The results of this paper are manifestations of the quasi-linear character of the Benjamin-Ono equation.

Résumé. — Nous établissons des propriétés de régularisation pour les solutions de l'équation de Benjamin-Ono dans l'espace de Sobolev $H^s(\mathbb{T};\mathbb{R})$ pour tout $s \geq 0$. À cette fin nous montrons que la transformation de jauge de Tao est une approximation à haute fréquence de la transformation de Fourier non linéaire pour l'équation de Benjamin-Ono, construite dans notre précédent travail. Les résultats de cet article sont des manifestations du caractère quasi-linéaire de l'équation de Benjamin-Ono.

Contents

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1. Introduction

In this paper we consider the Benjamin-Ono (BO) equation on the torus,

$$\partial_t v = \mathrm{H}[\partial_x^2 v] - \partial_x v^2 \,, \qquad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}\,,\ t \in \mathbb{R}, \tag{1}$$

where $v \equiv v(t,x)$ is real valued and H denotes the Hilbert transform, defined for $f = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{inx}$, $\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$, by

$$\mathrm{H}[f](x) := \sum_{n\in\mathbb{Z}} -i\ \mathrm{sign}(n)\widehat{f}(n)\ e^{inx}$$

with $\mathrm{sign}(\pm n) := \pm 1$ for any $n \geq 1$, whereas $\mathrm{sign}(0) := 0$.

Equation (1) has been introduced by Benjamin [2] and Davis & Acrivos [3] to model long, uni-directional internal waves in a two-layer fluid. It has been extensively studied, both on the real line $\mathbb{R}$ and on the torus $\mathbb{T}$. Let us briefly summarize some of the by now classical results on the well-posedness problem of (1), relevant for this paper – we refer to [27] for an excellent survey as well as a derivation of (1). Based on work of Saut [26], Abdelouhab & Bona & Felland & Saut proved in [1] that for $s > 3/2$, equation (1) is globally in time well-posed on the Sobolev space $H^s_r \equiv H^s(\mathbb{T},\mathbb{R})$ (endowed with the standard norm $\|\cdot\|_s$, defined by (31) below), meaning the following:

- (S1) *Existence and uniqueness of classical solutions:* For any initial data $v_0 \in H^s_r$, there exists a unique curve $v : \mathbb{R} \to H^s_r$ in $C(\mathbb{R}, H^s_r) \cap C^1(\mathbb{R}, H^{s-2}_r)$ so that $v(0) = v_0$ and for any $t \in \mathbb{R}$, equation (1) is satisfied in H^{s-2}_r. (Since H^s_r is an algebra, one has $\partial_x(v(t)^2) \in H^{s-1}_r$ for any time $t \in \mathbb{R}$.)
- (S2) *Continuity of solution map:* The solution map $\mathcal{S} : H^s_r \to C(\mathbb{R}, H^s_r)$ is continuous, meaning that for any $v_0 \in H^s_r$, $T > 0$, and $\varepsilon > 0$ there exists $\delta > 0$, so that for any $w_0 \in H^s_r$ with $\|w_0 - v_0\|_s < \delta$, the solutions $w(t) = \mathcal{S}(t, w_0)$ and $v(t) = \mathcal{S}(t, v_0)$ of (1) with initial data $w(0) = w_0$ and, respectively, $v(0) = v_0$ satisfy $\sup_{|t|\leq T} \|w(t) - v(t)\|_s \leq \varepsilon$.

In the sequel, further progress has been made on the well-posedness of (1) on Sobolev spaces of low regularity. The best results so far in this direction were obtained by Molinet, using as a key ingredient the gauge transform, introduced by Tao [28] for the Benjamin-Ono equation on $\mathbb{R}$. Molinet's results in [22] (cf. also [23]) imply that the solution map $\mathcal{S}$, introduced in $(S2)$ above, continuously extends to any Sobolev space H^s_r with $0 \leq s \leq 3/2$. More precisely, for any such s, $\mathcal{S} : H^s_r \to C(\mathbb{R}, H^s_r)$ is continuous and for any $v_0 \in H^s_r$, $\mathcal{S}(t, v_0)$ satisfies equation (1) in H^{s-2}_r. Finally, in the recent paper [9] we proved that (1) is wellposed in the Sobolev space H^s_r for any $s > -1/2$, but illposed for $s \leq -1/2$.

In a straightforward way one verifies that for any solution $v(t) \equiv \mathcal{S}(t, v_0)$ of (1) in H^s_r with $s > -1/2$, the mean $\langle v(t)|1\rangle$ is conserved. Here $\langle\cdot\,|\,\cdot\rangle$ denotes the extension of the L^2-inner product,

$$\langle f|g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f\overline{g}dx\,, \qquad \forall\, f, g \in L^2_c \equiv L^2(\mathbb{T},\mathbb{C}) \tag{2}$$

to the dual pairing $H^s_r \times H^{-s}_r \to \mathbb{C}$. As a consequence, for any $s > -1/2$, the subspace

$$H^s_{r,0} := \{v \in H^s_r \,:\, \langle v|1\rangle = 0\}\,, \tag{3}$$

of H^s_r is invariant by the flow of (1). (For $s = 0$, we usually write $L^2_{r,0}$ for $H^0_{r,0}$.) Since for any $a \in \mathbb{R}$ and any solution $v(t) = \mathcal{S}(t, v_0)$ of (1) in H^s_r with $s > -1/2$, $v_a(t, x) := a + v(t, x - 2at)$ is again a solution of (1) in H^s_r, for our purposes, it suffices to consider solutions in $H^s_{r,0}$.

The main goal of this paper is to prove smoothing properties of solutions of (1). A first key ingredient in their proof is Tao's gauge transform, which we denote by $\mathcal{G}$. To define it, we first need to introduce some more notation. For any $f \in H^s_c \equiv H^s(\mathbb{T}, \mathbb{C})$, $s \in \mathbb{R}$, the Szegő projection Πf of $f = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{inx}$ is defined as $\sum_{n\geq 0} \widehat{f}(n)e^{inx}$. Clearly, Π defines a bounded linear operator $H^s_c \to H^s_+$ where

$$H^s_+ := \{f \in H^s_c \;:\; \widehat{f}(n) = 0 \;\; \forall\, n < 0\}\,.$$

Furthermore, we denote by ∂_x^{-1} the operator

$$\partial_x^{-1} : H^s_c \to H^{s+1}_{c,0}, f \mapsto \sum_{n\neq 0} \frac{1}{in}\widehat{f}(n)e^{inx}\,,$$

where for any $s \in \mathbb{R}$, $H^s_{c,0} := \{v \in H^s_c \,:\, \langle v|1\rangle = 0\}$. For notational convenience, the restriction of ∂_x^{-1} to $H^s_{c,0}$ is also denoted by ∂_x^{-1}.

For our purposes, it suffices to consider solutions of (1) in the Sobolev spaces $H^s_{r,0}$ with $s \geq 0$. For any $u \in H^s_{r,0}$ with $s \geq 0$, we denote by $\mathcal{G}(u)$ (the following version of) Tao's gauge transform of u [28, 23],

$$\mathcal{G}(u) := \partial_x \Pi e^{-i\partial_x^{-1} u}\,. \tag{4}$$

It was pointed out in [28] that $\mathcal{G}$ can be viewed as a complex version of the Cole-Hopf transform, which was introduced independently by Cole and Hopf in the early fifties to convert Burgers' equation $\partial_t u = \partial_x(\partial_x u - u^2)$ into the heat equation. See e.g. [7, Section 4.4].

Note that

$$\partial_x \Pi[e^{-i\partial_x^{-1} u}] = \Pi[\partial_x e^{-i\partial_x^{-1} u}] = -i\Pi[u e^{-i\partial_x^{-1} u}]$$

and that for any $s \geq 0$,

$$\mathcal{G} : H^s_{r,0} \to H^s_{+,0}\,,\; u \mapsto \partial_x \Pi e^{-i\partial_x^{-1} u}\,,$$

is a real analytic map, where

$$H^s_{+,0} := \{f \in H^s_+ \,:\, \langle f|1\rangle = 0\}\,, \qquad H_{+,0} \equiv H^0_{+,0}\,.$$

It turns out that for any $s \geq 0$, $\mathcal{G}$ is a diffeomorphism onto an open proper subset of $H^s_{+,0}$. See Appendix B for a proof.

Given any initial data $u_0 \in H^s_{r,0}$ with $s \geq 0$, let $u(t) = \mathcal{S}(t, u_0)$ and denote by $w(t) = \mathcal{G}(u(t))$ the gauge transform of $u(t)$, i.e.,

$$w(t) = \partial_x \Pi[e^{-i\partial_x^{-1} u(t)}]\,.$$

For notational convenience, we will often not explicitly indicate the dependence of u, v, and w on t in the sequel. Let us derive the equation, satisfied by $w(t)$. Since $\partial_x^{-1}\partial_x(u^2) = u^2 - \langle u^2|1\rangle$ one sees that $v(t) := \partial_x^{-1}u(t)$ satisfies

$$\partial_t v = \mathrm{H}[\partial_x^2 v] - (\partial_x v)^2 + \langle(\partial_x v)^2|1\rangle\,, \qquad v(0) = \partial_x^{-1}u_0\,. \tag{5}$$

Furthermore, using that $\partial_t w = \partial_x \Pi[-i\partial_t v \cdot e^{-iv}]$ and

$$\partial_x^2 w = \partial_x \Pi[-i\partial_x^2 v \cdot e^{-iv} - (\partial_x v)^2 e^{-iv}]\,,$$

one computes

$$\partial_t w + i\partial_x^2 w = \partial_x \Pi[-i\partial_t v \cdot e^{-iv} + \partial_x^2 v \cdot e^{-iv} - i(\partial_x v)^2 e^{-iv}]\,.$$

Since for any $f \in H_c^s$, the Hilbert transform $\mathrm{H}[f]$ of f satisfies $\mathrm{H}[f] = -if + 2i(\mathrm{Id} - \Pi)[f]$ one infers that

$$\partial_x^2 v = i\mathrm{H}[\partial_x^2 v] + 2(\mathrm{Id} - \Pi)[\partial_x^2 v]\,.$$

Combining the latter identity with (5) then yields

$$\partial_t w + i\partial_x^2 w = \partial_x \Pi\big[-i\langle(\partial_x v)^2|1\rangle e^{-iv} + 2(e^{-iv} \cdot (\mathrm{Id} - \Pi)(\partial_x^2 v)\big]\,.$$

Finally, writing

$$e^{-iv} = \Pi e^{-iv} + (\mathrm{Id} - \Pi)[e^{-iv}]\,, \quad \Pi e^{-iv} = \partial_x^{-1} w + \langle e^{-iv}|1\rangle\,,$$

and using that

$$\Pi\big[\langle e^{-iv}|1\rangle \cdot (\mathrm{Id} - \Pi)(\partial_x^2 v)\big] = 0\,, \quad \Pi\big[(\mathrm{Id} - \Pi)e^{-iv} \cdot (\mathrm{Id} - \Pi)(\partial_x^2 v)\big] = 0\,,$$

one arrives at

$$\partial_t w + i\partial_x^2 w = -i\langle(\partial_x v)^2|1\rangle w + 2\partial_x \Pi[\partial_x^{-1} w \cdot (\mathrm{Id} - \Pi)(\partial_x^2 v)]\,,$$

or, expressing the latter equation in terms of $u = \partial_x v$ instead of v,

$$\partial_t w + i\partial_x^2 w + i\langle u^2|1\rangle w = 2\partial_x \Pi[\partial_x^{-1} w \cdot (\mathrm{Id} - \Pi)(\partial_x u)]\,. \tag{6}$$

One verifies in a straightforward way that $\langle u^2|1\rangle$ is conserved along the flow of (1) so that the left hand side of (6) can be viewed as a linear expression in w with constant coefficients. We are now ready to state the smoothing properties of $u(t)$.

1.1. Approximation of $u(t)$.

— For any $u_0 \in H_{r,0}^s$ with $s \geq 0$, let $w_0 := \mathcal{G}(u_0)$. Furthermore, denote by $w_L(t)$ the solution of the linear initial value problem

$$\partial_t w + i\partial_x^2 w + i\langle u_0^2|1\rangle\, w = 0\,, \qquad w(0) = w_0\,.$$

Then $w_L(t)$ is given by

$$w_L(t) = \sum_{n\geq 1} e^{it(n^2 - \langle u_0^2|1\rangle)} \widehat{w}_0(n) e^{inx}\,. \tag{7}$$

Finally, we define

$$\sigma(s) := \begin{cases} 1 & \text{if } s > 1/2 \\ 1- & \text{if } s = 1/2 \\ 2s & \text{if } 0 \leq s < 1/2 \end{cases} \tag{8}$$

where $a-$ means $a - \varepsilon$ for any $\varepsilon > 0$.

Theorem 1.1. — *For any $u_0 \in H^s_{r,0}$ with $s \geq 0$ there exists $M_s > 0$ so that for any $t \in \mathbb{R}$,*

$$\|w(t) - w_L(t)\|_{s+\sigma(s)} \leq M_s \langle t \rangle \,, \qquad \langle t \rangle := 1 + |t| \,, \tag{9}$$

$$u(t) = 2\mathrm{Re}\big(e^{i\partial_x^{-1} u(t)} i w_L(t)\big) + r(t) \,, \qquad \|r(t)\|_{s+\sigma(s)} \leq M_s \langle t \rangle \,. \tag{10}$$

The constant $M_s > 0$ can be chosen uniformly for bounded subsets of initial data u_0 in $H^s_{r,0}$. Furthermore, for any $0 \leq s < 1/2$, there exists $u_0 \in H^s_{r,0}$ so that for any $t \neq 0$ and any $\varepsilon > 0$, $w(t) - w_L(t)$ does not belong to $H_+^{s+\sigma(s)+\varepsilon}$ and $r(t)$ not to $H_r^{s+\sigma(s)+\varepsilon}$.

Remark 1.2. — *The estimate (9) in Theorem 1.1 improves on* [15, Theorem 1.2] *in the following ways: (i) the estimate holds for $H^s_{r,0}$ with $s > 0$ arbitrary instead of $1/6 < s \leq 1$; (ii) the estimate holds for any $t \in \mathbb{R}$ with an explicit growth rate in t instead for compact time intervals $[0, T]$; (iii) for $1/2 < s \leq 1$, the order of smoothing is 1 instead of $(1/3)-$, and for $1/6 < s < 1/2$, it is $2s$ instead of $(s - 1/6)-$; (iv) for any $0 \leq s < 1/2$, the estimate is sharp.*

Remark 1.3. — *The estimate (10) for $s > 1/2$ answers a question, raised by Tzvetkov, and improves the estimate he conjectured.*

1.2. Enhanced approximation of $u(t)$. — It turns out that an enhanced version $w_{L,*}(t)$ of $w_L(t)$ is obtained by replacing for any $n \geq 1$ the frequency $n^2 - \langle u_0^2 | 1 \rangle$ in the nth summand in (7) by the nth BO frequency $\omega_n \equiv \omega_n(u_0)$. To define ω_n, let us recall the definition of the Lax operator of (1),

$$L_u := \frac{1}{i}\partial_x - T_u \,, \tag{11}$$

acting on the Hardy space $H_+ \equiv H^0_+$ with domain H^1_+. Here T_u denotes the Toeplitz operator with symbol u, given by

$$T_u[f] := \Pi[uf] \,, \qquad \forall f \in H^1_+ \,.$$

The operator L_u is self-adjoint and bounded from below. Since it has a compact resolvent, its spectrum is discrete. We list the eigenvalues of L_u in increasing order and with their multiplicites, $\lambda_0(u) \leq \lambda_1(u) \leq \lambda_2(u) \leq \cdots$. By [8, Proposition 2.1]

$$\gamma_n := \lambda_n - \lambda_{n-1} - 1 \geq 0 \,, \quad \forall n \geq 1 \,, \tag{12}$$

and by [8, Proposition 3.1], the following trace formulas hold,

$$\lambda_n = n - \sum_{k \geq n+1} \gamma_k \,, \quad \forall n \geq 0 \,, \qquad \|u\|_0^2 = 2 \sum_{n \geq 1} n \gamma_n \,, \tag{13}$$

where for notational convenience, we (often) do not indicate the dependence of λ_n ($n \geq 0$) and γ_n ($n \geq 1$) on u. In particular, all eigenvalues of L_u are simple. The nth BO frequency is then given by (cf. [8, formula (8.4)])

$$\omega_n = n^2 - \langle u_0^2 \,|\, 1 \rangle + 2 \sum_{k > n} (k - n) \gamma_k \,. \tag{14}$$

By [9, Proposition 5], it follows that $\omega_n(u_0) - (n^2 - \langle u_0^2|1\rangle) = O(n^{-2s})$ as $n \to \infty$, uniformly on bounded subsets of initial data u_0 in $H^s_{r,0}$. Now we can define the enhanced approximation of w,

$$w_{L,*}(t) := \sum_{n\geq 1} e^{it\omega_n} \widehat{w}_0(n) e^{inx} \,. \tag{15}$$

To state our enhanced approximation result, we introduce

$$\tau(s) := \begin{cases} 1 & \text{if } \ s > 1/2 \\ 1- & \text{if } \ s = 1/2 \\ s + \frac{1}{2} & \text{if } \ 0 \leq s < 1/2 \end{cases} \tag{16}$$

Theorem 1.4. — *For any $u_0 \in H^s_{r,0}$ with $s \geq 0$ there exists $M_s > 0$ so that for any $t \in \mathbb{R}$,*

$$\|w(t) - w_{L,*}(t)\|_{s+\tau(s)} \leq M_s \,, \tag{17}$$

$$u(t) = 2\mathrm{Re}\big(e^{i\partial_x^{-1} u(t)} i w_{L,*}(t)\big) + r_*(t) \,, \qquad \|r_*(t)\|_{s+\tau(s)} \leq M_s \,. \tag{18}$$

The constant $M_s > 0$ can be chosen uniformly for bounded subsets of initial data u_0 in $H^s_{r,0}$.

Remark 1.5. — *Smoothing properties can also be proved for solutions in some Sobolev spaces $H^s_{r,0}$ with s negative. In order to limit the size of the paper, we decided to focus on solutions in $H^s_{r,0}$ with $s \geq 0$.*

Note that, in addition to providing a better gain of regularity for s in the interval $0 \leq s < 1/2$, the estimates of Theorem 1.4 are *uniform in time*. These improvements are obtained by taking into account that the BO equation is *integrable* and as a consequence that the BO dynamics are determined by the BO frequencies.

1.3. High frequency approximation of the nonlinear Fourier transform. — A second key ingredient in the proof of the smoothing properties of solutions of (1) is the high frequency approximation of the nonlinear Fourier transform Φ of the Benjamin-Ono equation, which was constructed in [8], [9]. Let us review the definition of Φ and the properties of Φ needed to state our smoothing results for solutions of (1). To this end, we first need to review further properties of the Lax operator L_u, introduced in the previous subsection. It is shown in [8] that L_u admits an orthonormal basis of eigenfunctions $f_n \equiv f_n(\cdot, u) \in H^1_+$, $n \geq 0$, uniquely determined by the normalisation conditions

$$\langle f_0|1\rangle > 0 \,, \qquad \langle f_n|e^{ix} f_{n-1}\rangle > 0 \,, \quad \forall\, n \geq 1 \,. \tag{19}$$

For any $s \in \mathbb{R}$, denote by $\mathfrak{h}^s \equiv \mathfrak{h}^s(\mathbb{N}, \mathbb{C})$ the weighted ℓ^2-sequence space

$$\mathfrak{h}^s := \{z = (z_n)_{n\geq 1} \subset \mathbb{C} \,:\, \|z\|_s < \infty\} \,, \qquad \|z\|_s := \Big(\sum_{n\geq 1} n^{2s}|z_n|^2\Big)^{1/2} \,.$$

In [8], we introduced the map

$$\Phi : H^0_{r,0} \to \mathfrak{h}^{1/2},\ u \mapsto (\zeta_n(u))_{n\ge 1}\,, \qquad \zeta_n(u) := \frac{\langle 1|f_n(\cdot,u)\rangle}{\sqrt{\kappa_n(u)}}\,, \tag{20}$$

and proved that Φ is a homeomorphism and that

$$|\zeta_n(u)|^2 = \gamma_n(u)\,, \qquad \forall\, n\ge 1\,,\ \forall\, u \in H^0_{r,0}\,. \tag{21}$$

Here $\kappa_n \equiv \kappa_n(u) > 0$, $n \ge 1$, are defined as absolutely convergent infinite products,

$$\kappa_n = \frac{1}{\lambda_n - \lambda_0} \prod_{1\le p\ne n} \Big(1 - \frac{\gamma_p}{\lambda_p - \lambda_n}\Big)\,. \tag{22}$$

It is shown in [9] that for any $s \ge 0$, the restriction of Φ to $H^s_{r,0}$ takes values in $\mathfrak{h}^{s+1/2}$ and in [11]-[12] that

$$\Phi : H^s_{r,0} \to \mathfrak{h}^{s+1/2} \tag{23}$$

is a real analytic diffeomorphism.

One of the principal features of Φ is that it can be used to solve the initial value problem of (1). Indeed, it is shown in [9] that for any initial data $u_0 \in H^s_{r,0}$ with $s \ge 0$, the solution $t \mapsto u(t) \in H^s_{r,0}$ of (1) with initial data $u(0) = u_0$ satisfies

$$\Phi(u(t)) = (e^{it\omega_n}\zeta_n(u_0))_{n\ge 1}\,, \tag{24}$$

where $\omega_n \equiv \omega_n(u_0)$, $n \ge 1$, denote the BO frequencies of u_0, introduced in (14) above. The high frequency approximation of $\Phi : H^0_{r,0} \to \mathfrak{h}^{1/2}$ is then defined as the map $\Phi_0 : H^0_{r,0} \to \mathfrak{h}^{1/2}$, given by

$$\Phi_0(u) := \big(\sqrt{n}\langle 1|g_\infty e^{inx}\rangle\big)_{n\ge 1}\,, \qquad g_\infty \equiv g_\infty(\cdot,u) := e^{i\partial_x^{-1}u}\,. \tag{25}$$

Note that Φ_0 is a quasi-linear perturbation of the Fourier transform. Indeed, since

$$n\langle 1\,|\,g_\infty e^{inx}\rangle = \langle \overline{g_\infty}\,|\,ne^{inx}\rangle = \langle \overline{g_\infty}\,|\,\frac{1}{i}\partial_x e^{inx}\rangle\,,$$

integration by parts yields

$$n\langle 1\,|\,g_\infty e^{inx}\rangle = \langle \frac{1}{i}\partial_x\overline{g_\infty}\,|\,e^{inx}\rangle = -\langle u\overline{g_\infty}\,|\,e^{inx}\rangle\,. \tag{26}$$

This shows that

$$\Phi_0(u) = (-\frac{1}{\sqrt{n}}\langle u\,|\,g_\infty e^{inx}\rangle)_{n\ge 1}\,. \tag{27}$$

Since g_∞ is a function of $\partial_x^{-1}u$, the map Φ_0 can be viewed, up to scaling, as a quasi-linear perturbation of the Fourier transform.

The following smoothing properties of $\Phi - \Phi_0$, which are of independent interest, are key ingredients in the proofs of Theorem 1.1 and Theorem 1.4.

Theorem 1.6. — *For any $s \ge 0$, the map $\Phi - \Phi_0$ is smoothing of order $\tau(s)$ with $\tau(s)$ as defined in (16), i.e., $\Phi - \Phi_0$ is a continuous map from $H^s_{r,0}$ with values in $\mathfrak{h}^{s+1/2+\tau(s)}$. Furthermore, there exists $u \in H^{1/2}_{r,0}$ with the property that $\Phi(u) - \Phi_0(u) \notin$*

$\mathfrak{h}^2$ *and similarly, for any* $0 < s < 1/2$*, there exists* $u \in H^s_{r,0}$ *so that for any* $\varepsilon > 0$*,* $\Phi(u) - \Phi_0(u) \notin \mathfrak{h}^{s+1/2+\tau(s)+\varepsilon}$*.*

Remark 1.7. — *(i) Theorem 1.6 says that* Φ_0 *can be viewed as a* quasi-linear *high frequency approximation of* Φ*.*
(ii) Note that for any $u \in H^0_{r,0}$*, one has*

$$\mathcal{G}(u) = \partial_x \Pi(e^{-i\partial_x^{-1} u}) = \partial_x \Pi(\overline{g_\infty}) = \sum_{n\geq 1} in\langle \overline{g_\infty} | e^{inx}\rangle e^{inx}$$

and hence

$$\Phi_0(u) = \big(- \frac{i}{\sqrt{n}} \langle \mathcal{G}(u) \,|\, e^{inx}\rangle\big)_{n\geq 1} . \tag{28}$$

It then follows from Theorem B.1 in Appendix B that for any $s \geq 0$*,* $\Phi_0 : H^s_{r,0} \to \mathfrak{h}^{s+1/2}$ *is a diffeomorphism onto an open proper subset of* $\mathfrak{h}^{s+1/2}$*.*
(iii) In [28], Tao asks whether the gauge transform $\mathcal{G}$ *is related to the integrability of the Benjamin–Ono equation. In view of the formula (28) for* Φ_0*, Theorem 1.6 answers Tao's question for the BO equation on* $\mathbb{T}$ *by proving that (up to scaling) the Fourier transform of* $\mathcal{G}$ *is a high frequency approximation of the Birkhoff map* Φ*.*

Remark 1.8. — *In Appendix C, we provide high frequency approximations of the differentials of* Φ *and of* Φ^{-1}*. Such approximations are useful when studying the pullback of vector fields by* Φ *or* Φ^{-1}*.*

As a corollary of Theorem 1.6, we obtain smoothing properties of solutions of (1), expressed in the coordinates provided by Φ. To this end, we introduce the following evolution maps: given any $\alpha \geq \frac{1}{2}$ and $t \in \mathbb{R}$, define for any initial data $\zeta \in \mathfrak{h}^\alpha$,

$$\begin{aligned} \mathcal{S}_L(t,\zeta) &:= \big(e^{it(n^2 - 2\|\zeta\|^2_{1/2})}\zeta_n\big)_{n\geq 1} \in \mathfrak{h}^\alpha , \\ \mathcal{S}_{L,*}(t,\zeta) &:= \big(e^{it(n^2 - 2\|\zeta\|^2_{1/2} + \delta_n(\zeta))}\zeta_n\big)_{n\geq 1} \in \mathfrak{h}^\alpha , \end{aligned}$$

where

$$\delta_n(\zeta) := 2\sum_{k>n}(k-n)|\zeta_k|^2 .$$

Corollary 1.9. — *For any* $u_0 \in H^s_{r,0}$ *with* $s \geq 0$*, there exists* $M_s > 0$ *so that for any* $t \in \mathbb{R}$*,*

$$\|\Phi(\mathcal{S}(t,u_0)) - \mathcal{S}_L(t, \Phi_0(u_0))\|_{s+\frac{1}{2}+\sigma(s)} \leq M_s \langle t\rangle , \tag{29}$$

$$\|\Phi(\mathcal{S}(t,u_0)) - \mathcal{S}_{L,*}(t, \Phi_0(u_0))\|_{s+\frac{1}{2}+\tau(s)} \leq M_s . \tag{30}$$

The constant $M_s > 0$ *can be chosen uniformly on bounded subsets of initial data in* $H^s_{r,0}$*.*

1.4. Applications. — In Section 4, we apply Theorem 1.1 to study the action of the Benjamin–Ono flow $\mathcal{S}(t)$ on the Hölder spaces $C^\alpha(\mathbb{T},\mathbb{R})$. In particular, we prove that there exists a subset $N \subset \mathbb{R}$ of Lebesgue measure 0 so that for any $t \notin N$ and any $1/2 < \alpha < 1$, $\mathcal{S}(t)$ does not map $\bigcap_{\varepsilon>0} C^{\alpha-\varepsilon}(\mathbb{T},\mathbb{R})$ into $\bigcup_{\varepsilon>0} C^{\alpha-1/2+\varepsilon}(\mathbb{T},\mathbb{R})$. On the other hand it is easy to check that for any $t \in \mathbb{R}$, $\mathcal{S}(t)$ maps $\bigcap_{\varepsilon>0} C^{\alpha-\varepsilon}(\mathbb{T},\mathbb{R})$ into $\bigcap_{\varepsilon>0} C^{\alpha-1/2-\varepsilon}(\mathbb{T},\mathbb{R})$. We refer to Section 4 for additional results.

1.5. Comments. — (i) Birkhoff maps have been constructed for integrable PDEs such as the KdV equation [16], the mKdV equation, and the defocusing NLS equation [14]. Each of these maps admits a high frequency approximation, similar to the one of the Birkhoff map of the Benjamin-Ono equation, stated in Theorem 1.6. But in contrast to the Benjamin-Ono equation, it is given (up to scaling) by the Fourier transform. See [17] (cf. also [21]), [18, 19]. Hence for these equations, the Birkhoff map can be viewed as a *semilinear* perturbation of the Fourier transform.
(ii) Smoothing properties, similar to the ones stated in Theorem 1.1 and Theorem 1.4 for the Benjamin-Ono equation, have been established previously for solutions of integrable PDEs such as the KdV equation [4, 6, 17] and the defocusing NLS equation [5, 6, 20]. In contrast to (10), (1.4), these smoothing properties are obtained by approximating solutions of these equations by solutions of the Airy equation (in the case of the KdV equation) and by solutions of the linear Schrödinger equation (in the case of the defocusing NLS equation) or by enhanced versions of solutions of these linear equations, involving the KdV and NLS frequencies.
(iii) Reference [13], posted by the first author after the first version of this paper was submitted, provides an explicit formula for the solution of the Benjamin–Ono in terms of the Lax operator L_{u_0} associated to the initial datum u_0 which is useful for studying low regularity solutions. However, in order to study the high frequency phenomena which are the core of the present paper, it seems impossible to avoid a thorough spectral analysis of L_{u_0}, which precisely led us to the Birkhoff map introduced in [8] and to Theorem 1.6.

1.6. Organization of the paper. Notation. — The paper is organized as follows. In Section 2 we prove Theorem 1.6, which then is used in Section 3 to derive Theorems 1.1 and 1.4 and Corollary 1.9. In Section 4, we apply Theorem 1.1 to study the action of the Benjamin–Ono flow on Hölder spaces. In Appendix A, we record smoothing properties of Hankel operators, which are used throughout the main body of the paper. In Appendix B, we prove diffeomorphism properties of Tao's gauge transform. Finally, in Appendix C, we derive high frequency approximations of the differential of Φ and the one of Φ^{-1}.

By and large, we will use the notation established in [8]. In particular, the H^s-norm of an element v in the Sobolev space $H^s_c \equiv H^s(\mathbb{T},\mathbb{C})$, $s \in \mathbb{R}$, will be denoted by $\|v\|_s$. It is defined by

$$\|v\|_s = \big(\sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} |\widehat{v}(n)|^2\big)^{1/2}, \qquad \langle n\rangle = \max\{1,|n|\}\,. \tag{31}$$

For $s = 0$, we usually write $\|v\|$ for $\|v\|_0$. By $\langle \cdot \,|\, \cdot \rangle$, we will also denote the extension of the L^2-inner product, introduced in (2), to $H_c^{-s} \times H_c^s$, $s \in \mathbb{R}$, by duality. By $H_+ \equiv H_+^0$ we denote the Hardy space, consisting of elements $f \in L^2(\mathbb{T}, \mathbb{C}) \equiv H_c^0$ with the property that $\widehat{f}(n) = 0$ for any $n < 0$. More generally, for any $s \in \mathbb{R}$, H_+^s denotes the subspace of H_c^s, consisting of elements $f \in H_c^s$ with the property that $\widehat{f}(n) = 0$ for any $n < 0$. By $\mathfrak{h}^s \equiv \mathfrak{h}^s(\mathbb{N}, \mathbb{C})$ we denote the weighted ℓ^2-sequence space

$$\mathfrak{h}^s := \{ z = (z_n)_{n \geq 1} \subset \mathbb{C} \, : \, \|z\|_s < \infty \} \,, \qquad \|z\|_s := (\sum_{n \geq 1} n^{2s} |z_n|^2)^{1/2} \,.$$

For notational convenience, we often write $z_n = \mathfrak{h}_n^s$ for a sequence $(z_n)_{n \geq 1}$ in $\mathfrak{h}^s$. The same notation is also used for other sequence spaces such as $\ell^1 \equiv \ell^1(\mathbb{N}, \mathbb{R})$. Finally, for any a, b in $\mathbb{R}$, the expression $a \lesssim b$ means that there exists $C > 0$ so that $a \leq Cb$.

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2. Proof of Theorem 1.6

In this section we prove Theorem 1.6. Throughout this section we assume that $s \geq 0$. Recall that Φ denotes the Birkhoff map,

$$\Phi : H_{r,0}^s \to \mathfrak{h}^{s+1/2}, \, u \mapsto (\zeta_n(u))_{n \geq 1} \,, \qquad \zeta_n(u) = \frac{\langle 1 | f_n(\cdot, u) \rangle}{\sqrt{\kappa_n(u)}} \,,$$

where $f_n \equiv f_n(\cdot, u)$, $n \geq 0$, is the orthonormal basis of eigenfunctions of the Lax operator L_u (cf. (11)), uniquely determined by the normalization conditions (19), and $\kappa_n \equiv \kappa_n(u) > 0$ are scaling factors given by (22). To prove that $\Phi_0(u) = \big(\sqrt{n} \langle 1 | g_\infty e^{inx} \rangle\big)_{n \geq 1}$, defined in (25), approximates Φ, we introduce the auxiliary map

$$\Phi_1 : H_{r,0}^0 \to \mathfrak{h}^{1/2}, \, u \mapsto \big(\sqrt{n} \langle 1 | f_n \rangle\big)_{n \geq 1} \,. \tag{32}$$

Since $\lambda_n \langle 1 | f_n \rangle = \langle 1 | L_u f_n \rangle = -\langle u | f_n \rangle$, the map Φ_1 can be viewed (up to scaling) as a version of the Fourier transform where the orthonormal basis e^{inx}, $n \geq 0$, of the Hardy space H_+ is replaced by the basis f_n, $n \geq 0$, of eigenfunctions of L_u.

In a first step we study the difference $\Phi(u) - \Phi_1(u)$. Its nth component is given by

$$\zeta_n(u) - \sqrt{n} \langle 1 | f_n \rangle = \sqrt{n} \left(\frac{1}{\sqrt{n \kappa_n}} - 1 \right) \langle 1 | f_n \rangle \,. \tag{33}$$

We begin by deriving an estimate for $n\kappa_n$.

Lemma 2.1. — *For any $u \in H_{r,0}^s$ with $s \geq 0$,*

$$n \kappa_n(u) = 1 + O\left(\frac{1}{n}\right) \,. \tag{34}$$

As a consequence

$$\frac{1}{\sqrt{n\kappa_n(u)}} = 1 + O\left(\frac{1}{n}\right) \ . \tag{35}$$

For any given $s \geq 0$, the estimates for $n\kappa_n(u)$ and $\frac{1}{\sqrt{n\kappa_n(u)}}$ hold uniformly on bounded subsets of potentials u in $H^s_{r,0}$.

Proof. — In view of (22) we write $n\kappa_n - 1 = I_n + II_n$ where

$$I_n = \left(\frac{n}{\lambda_n - \lambda_0} - 1\right) \prod_{p \neq n} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n}\right) , \qquad II_n = \prod_{p \neq n} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n}\right) - 1 \, .$$

Let us first estimate I_n. Since for any $m \geq 0$, $\lambda_m \equiv \lambda_m(u)$ satisfies $\lambda_m = m - \sum_{k \geq m+1} \gamma_k$ (cf. (13)) one has

$$\lambda_n - \lambda_0 = n + \sum_{k=1}^{n} \gamma_k \, , \qquad \forall\, n \geq 1, \tag{36}$$

and in turn

$$\frac{n}{\lambda_n - \lambda_0} - 1 = -\frac{1}{n} \frac{\sum_{k=1}^{n} \gamma_k}{1 + \frac{1}{n}\sum_{k=1}^{n} \gamma_k} \, .$$

The product $\prod_{p \neq n}(1 - \frac{\gamma_p}{\lambda_p - \lambda_n})$ can be estimated as follows. Taking into account that

$$\left| \prod_{p \neq n} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n}\right) \right| \leq \prod_{p \neq n} \left(1 + \frac{\gamma_p}{|\lambda_p - \lambda_n|}\right) = \exp\left(\sum_{p \neq n} \log\left(1 + \frac{\gamma_p}{|\lambda_p - \lambda_n|}\right)\right)$$

and that $0 \leq \log(1 + a) \leq a$ for any $a \geq 0$ one sees that

$$\left| \prod_{p \neq n} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n}\right) \right| \leq \exp\left(\sum_{p \neq n} \frac{\gamma_p}{|\lambda_p - \lambda_n|}\right) \leq \exp\left(\sum_{p \neq n} \gamma_p\right) ,$$

where for the latter inequality we used that $|\lambda_p - \lambda_n| \geq 1$ for any $p \neq n$. By (13) it then follows that

$$I_n = O\left(\frac{1}{n}\right) \, . \tag{37}$$

Next let us consider II_n. We start from the following identity,

$$1 - \prod_{p=1}^{N} (1 - a_p) = \sum_{p=1}^{N} a_p \prod_{1 \leq q < p} (1 - a_q) \ , \tag{38}$$

which can be easily checked by induction on N for any given sequence of real (or complex) numbers a_n. Note that (38) continues to hold for $N = \infty$ if the series of a_p is absolutely convergent. Since $|1 - a_p| \leq 1 + |a_p| \leq \exp(|a_p|)$ for any $p \geq 1$, one is led to the estimate

$$\left| 1 - \prod_{p=1}^{\infty} (1 - a_p) \right| \leq \left(\sum_{p=1}^{\infty} |a_p|\right) \exp\left(\sum_{p=1}^{\infty} |a_p|\right) \, . \tag{39}$$

Using again that $|\lambda_p - \lambda_n| \geq 1$, it then follows that

$$|II_n| \leq \left(\sum_{1 \leq p \neq n} \frac{\gamma_p}{|\lambda_p - \lambda_n|} \right) \exp\left(\sum_{q=1}^{\infty} \gamma_q \right) .$$

Since by (13) for any $p > n$,

$$\lambda_p - \lambda_n = p - n + \sum_{n<k\leq p} \gamma_k \geq p - n \,,$$

one infers that for any p, $|\lambda_p - \lambda_n| \geq |p-n|$. Hence

$$\sum_{|p-n|\geq \frac{n}{2}} \frac{\gamma_p}{|\lambda_p - \lambda_n|} \leq \frac{2}{n} \sum_p \gamma_p$$

and

$$\sum_{0<|p-n|<\frac{n}{2}} \frac{\gamma_p}{|\lambda_p - \lambda_n|} \leq \sum_{0<|p-n|<\frac{n}{2}} \gamma_p \leq \frac{2^{1+2s}}{n^{1+2s}} \sum_{p\geq 1} p^{1+2s} \gamma_p \,.$$

By (21) and [9, Proposition 5] it then follows that

$$II_n = O\left(\frac{1}{n}\right) , \tag{40}$$

which together with estimate (37) yields (34).

By (21) and [9, Proposition 5], the estimate for $n\kappa_n(u)$ hold uniformly on bounded subsets of potentials in $H^s_{r,0}$ with $s \geq 0$.

To see that the estimate for $\frac{1}{\sqrt{n\kappa_n(u)}}$ also holds uniformly on bounded subsets of potentials in $H^s_{r,0}$, it remains to find a uniform positive lower bound for $n\kappa_n$, $n \geq 1$, on such subsets. To this end note that by (22) and (36)

$$n\kappa_n = \frac{1}{1 + \frac{1}{n}\sum_{k=1}^n \gamma_k} \prod_{1\leq p<n} \left(1 + \frac{\gamma_p}{\lambda_n - \lambda_p}\right) \cdot \prod_{p>n} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n}\right) ,$$

yielding, when combined with (13),

$$n\kappa_n \geq \frac{1}{1 + |\lambda_0|} \exp\left(- \sum_{p>n} - \log\left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n}\right)\right) .$$

Applying the estimate

$$-\log(1-a) = \int_0^a \frac{1}{1-x} dx \leq \frac{a}{1-a} \,, \qquad \forall\, 0 < a < 1 \,,$$

to $a = \frac{\gamma_p}{\lambda_p - \lambda_n}$ and using that for any $p > n$,

$$\lambda_p - \lambda_n - \gamma_p = p - n + \sum_{n<k<p} \gamma_k \geq 1 \,,$$

one concludes that for any $p > n$

$$\frac{a}{1-a} = \frac{\gamma_p}{\lambda_p - \lambda_n - \gamma_p} \leq \gamma_p$$

and hence we obtain the following positive lower bound for $n\kappa_n$, $n \geq 1$,

$$n\kappa_n \geq \frac{1}{1+|\lambda_0|} \exp\left(-\sum_{p>n} \gamma_p\right) \geq \frac{1}{1+|\lambda_0|} e^{-|\lambda_0|}, \qquad \forall n \geq 1 .$$

By (13) the latter lower bound is uniformly bounded away from 0 on bounded subsets of potentials u in $H^0_{r,0}$. □

Combining (33) and (35) then leads to the following

Corollary 2.2. — *For any $s \geq 0$, the difference $\Phi - \Phi_1$ is one-smoothing, meaning that it can be viewed as a continuous map from $H^s_{r,0}$ with values in $\mathfrak{h}^{s+3/2}$.*

Proof. — Going through the arguments of the proof of Lemma 2.1 one sees that $\Phi - \Phi_1 : H^s_{r,0} \to \mathfrak{h}^{s+3/2}$ is continuous for any $s \geq 0$.

□

Next we investigate $\Phi_1 - \Phi_0$. To this end we first derive asymptotic estimates for the scaling factors μ_n, introduce in [8, Section 4]. Recall that for any $n \geq 1$, $0 < \mu_n \leq 1$ is given by

$$\mu_n = \langle f_n | e^{ix} f_{n-1} \rangle^2 \tag{41}$$

and admits the following infinite product representation (cf. [8, (4.9)])

$$\mu_n = \left(1 - \frac{\gamma_n}{\lambda_n - \lambda_0}\right) \prod_{p \neq n} (1 - b_{np}), \qquad b_{np} = \gamma_n \frac{\gamma_p}{(\lambda_p - \lambda_n)(\lambda_{p-1} - \lambda_{n-1})} .$$

Lemma 2.3. — *For any $u \in H^s_{r,0}$ with $s \geq 0$,*

$$0 \leq 1 - \sqrt{\mu_n} \leq 1 - \mu_n = \ell^{1,2+2s}_n ,$$

meaning that $(1-\mu_n)_{n\geq 1} \in \ell^{1,2+2s}(\mathbb{N}, \mathbb{R})$ (cf. "Notation" in Section 1). As a consequence

$$0 \leq \sqrt{1 - \sqrt{\mu_n}} \leq \sqrt{1 - \mu_n} = \mathfrak{h}^{1+s}_n .$$

For any given $s \geq 0$, these estimates hold uniformly on bounded subsets of potentials u in $H^s_{r,0}$, $s \geq 0$.

Proof. — Using (39) and $|\lambda_p - \lambda_n| \geq |p - n|$, we have

$$0 \leq 1 - \mu_n \leq S_n \exp(S_n) , \qquad S_n := \frac{\gamma_n}{n} + \gamma_n \sum_{p \neq n} \frac{\gamma_p}{(p-n)^2} .$$

Since by (21) and [9, Proposition 5], $(\gamma_n)_{n\geq 1} \in \ell^{1,1+2s}(\mathbb{N}, \mathbb{R})$ and hence $(\frac{\gamma_n}{n})_{n\geq 1} \in \ell^{1,2+2s}(\mathbb{N}, \mathbb{R})$ it follows that

$$\gamma_n \sum_{|p-n|>n/2} \frac{\gamma_p}{(p-n)^2} \leq \frac{4\gamma_n}{n^2} \sum_{p\geq 1} \gamma_p = \ell^{1,3+2s}_n$$

and

$$\gamma_n \sum_{0<|p-n|\le n/2} \frac{\gamma_p}{(p-n)^2} \le \gamma_n \sum_{0<|p-n|\le n/2} \gamma_p$$
$$\le \frac{2^{1+2s}\gamma_n}{n^{1+2s}} \sum_{p\ge 1} p^{1+2s}\gamma_p = \ell_n^{1,2+4s}\,.$$

By (21) and [9, Proposition 5], the stated estimates hold uniformly on bounded subsets of potentials u in $H^s_{r,0}$ with $s \ge 0$. □

To estimate $\Phi_1 - \Phi_0$, introduce $\Xi : H^s_{r,0} \to \mathfrak{h}^s$, $u \mapsto (\Xi_n(u))_{n\ge 1}$ where

$$\Xi_n(u) := \sqrt{n}(\Phi_1(u) - \Phi_0(u))_n = n\langle 1|f_n\rangle - n\langle 1|g_\infty e^{inx}\rangle\,, \quad \forall\, n \ge 1\,. \tag{42}$$

We recall that the exponent $\tau(s)$, $s > 0$, of the gain of regularity has been introduced in (16).

Lemma 2.4. — *Let $s \ge 0$. Then for any $u \in H^s_{r,0}$,*

$$(\Xi_n(u))_{n\ge 1} \in \mathfrak{h}^{s+\tau(s)}$$

and $(\Xi_n(u))_{n\ge1}$ is uniformly bounded on bounded subsets of potentials u in $H^s_{r,0}$. Furthermore, there exists $u \in H^{1/2}_{r,0}$ with the property that $(\Xi_n(u))_{n\ge1} \notin \mathfrak{h}^{1/2+1}$ and for $0 < s < 1/2$, there exists $u \in H^s_{r,0}$ so that for any $\varepsilon > 0$, $(\Xi_n(u))_{n\ge1} \notin \mathfrak{h}^{s+\tau(s)+\varepsilon}$.

Proof. — Assume that $u \in H^s_{r,0}$ with $s \ge 0$. Then for any $n \ge 1$,

$$n\langle 1|f_n\rangle = (n-\lambda_n)\langle 1|f_n\rangle + \langle 1|L_u f_n\rangle = (n-\lambda_n)\langle 1|f_n\rangle - \langle u|f_n\rangle \tag{43}$$

and by (26)

$$n\langle 1|g_\infty e^{inx}\rangle = -\langle u|g_\infty e^{inx}\rangle\,. \tag{44}$$

Substituting (43) and (44) into the definition of $\Xi_n \equiv \Xi_n(u)$, one gets

$$\Xi_n = T_{1,n} + \langle u|g_\infty e^{inx} - f_n\rangle\,, \qquad T_{1,n} := (n-\lambda_n)\langle 1|f_n\rangle\,.$$

We then write $\langle u|\, g_\infty e^{inx} - f_n\rangle = T_{2,n} + T_{3,n}$ where

$$T_{2,n} := \langle (\mathrm{Id} - \Pi)u\,|\, g_\infty e^{inx}\rangle\,, \qquad T_{3,n} := \langle \Pi u\,|\, g_\infty e^{inx} - f_n\rangle\,.$$

We thus have

$$\Xi_n(u) = \sum_{j=1}^{3} T_{j,n}(u)\,, \qquad T_{j,n} \equiv T_{j,n}(u)\,, \quad 1 \le j \le 3\,.$$

We begin by estimating $T_{1,n}$. By (13),

$$0 \le n - \lambda_n = \sum_{k\ge n+1} \gamma_k \le \frac{1}{n^{1+2s}} \sum_{k\ge n+1} k^{1+2s}\gamma_k\,. \tag{45}$$

Since by (20), Lemma 2.1 , and [9, Proposition 5], one has

$$\langle 1|f_n\rangle = \mathfrak{h}_n^{1+s}\,, \tag{46}$$

the inequality (45) implies that

$$T_{1,n} = (n-\lambda_n)\langle 1|f_n\rangle = \mathfrak{h}_n^{s+(2+2s)}\,. \tag{47}$$

To estimate $T_{2,n}$ we note that

$$T_{2,n} = \langle u| \, (\mathrm{Id} - \Pi)[g_\infty \mathrm{e}^{inx}] \rangle = \langle \overline{g_\infty} (\mathrm{Id} - \Pi) u | \, \mathrm{e}^{inx} \rangle \, . \tag{48}$$

Hence for any $\rho \in \mathbb{R}$, $(T_{2,n})_{n\geq 1} \in \mathfrak{h}^\rho$ if and only if $\Pi(\overline{g}_\infty (\mathrm{Id} - \Pi) u) \in H^\rho_+$. If $u \in H^s_{r,0}$ with $s \geq 0$, then $g_\infty \in H^{s+1}_c$ and by the smoothing properties of Hankel operators recorded in Lemma A.1(i),(ii),(iii) with $\alpha = 1$, we infer that

$$(T_{2,n}(u))_{n\geq 1} \in \mathfrak{h}^{s+\tau(s)} \, , \qquad \forall \, u \in H^s_{r,0} \, . \tag{49}$$

To estimate $T_{3,n}$, we write

$$T_{3,n} = \langle \Pi u | \, e^{inx} (g_\infty - g_n) \rangle \, , \qquad g_n := f_n \, e^{-inx} \, . \tag{50}$$

By writing $g_\infty - g_n$ as a telescoping sum,

$$g_\infty - g_n = \sum_{k\geq n} (g_{k+1} - g_k) \, , \tag{51}$$

we proved in [10] (cf. [10, Proposition 9]) that for any $u \in H^s_{r,0}$ with $s \geq 0$, g_n converges in H^{1+s}_c to g_∞ as $n \to \infty$. To improve the estimates for $g_\infty - g_n$, obtained in [10], we write

$$g_\infty - g_n = \langle g_\infty - g_n | \, g_\infty \rangle g_\infty + r_n \, .$$

Whereas $\langle g_\infty - g_n | \, g_\infty \rangle$ will be estimated using (51), we need to analyze the remainder term $r_n \equiv r_n(u)$ further. To this end we introduce for any $u \in H^0_{r,0}$ and $n \geq 1$ the operators

$$K_n \equiv K_n(u) : \, H_c^{\frac{1}{2}+\varepsilon} \to H^1_c, \; f \mapsto g_\infty D^{-1} [\overline{g}_\infty \Pi_{<-n}(uf)] \, ,$$
$$K'_n \equiv K'_n(u) : \, H^0_c \to H^1_c, \; f \mapsto (n - \lambda_n) g_\infty D^{-1} [\overline{g_\infty} f] \, ,$$

where $D^{-1} := i\partial_x^{-1}$ and $\varepsilon > 0$.

Lemma 2.5. — *For any $u \in H^0_{r,0}$ and $n \geq 1$,*

$$(Id + K_n + K'_n)[g_\infty - g_n] = \langle g_\infty - g_n | \, g_\infty \rangle g_\infty + K_n g_\infty \, . \tag{52}$$

Furthermore, for any $u \in H^s_{r,0}$ with $0 \leq s \leq 1/2$, $0 < \varepsilon < 1/2 + s$, and $n \geq 1$, there exists a constant $C_{s,\varepsilon} > 0$ so that

$$\|K_n f\|_{1/2+\varepsilon} + \|K'_n f\|_{1/2+\varepsilon} \leq \frac{C_{s,\varepsilon}}{n^{1/2+s-\varepsilon}} \|f\|_{1/2+\varepsilon} \; , \tag{53}$$

$$\|K_n f\| + \|K'_n f\| \leq \frac{C_{s,\varepsilon}}{n^{1+s}} \|f\|_{1/2+\varepsilon} \; . \tag{54}$$

Similarly, for any $u \in H^s_{r,0}$ with $s > 1/2$, there exists a constant $C_s > 0$ so that

$$\|K_n f\|_s + \|K'_n f\|_s \leq \frac{C_s}{n} \|f\|_s \, , \tag{55}$$

$$\|K_n f\| + \|K'_n f\| \leq \frac{C_s}{n^{1+s}} \|f\|_s \, . \tag{56}$$

The constants $C_{s,\varepsilon}$ and C_s can be chosen uniformly for $n \geq 1$ and for bounded subsets of potentials $u \in H^s_{r,0}$.

Proof of Lemma 2.5. — Since for any $u \in H^0_{r,0}$ and $n \geq 0$, $f_n \in H^1_+$ is an eigenfunction of the Lax operator L_u with eigenvalue λ_n, $Df_n - \Pi[uf_n] = \lambda_n f_n$, the function $g_n = e^{-inx} f_n \in H^1_c$ satisfies

$$(D-u)g_n = (\lambda_n - n)g_n - \Pi_{<-n}[ug_n]\,, \tag{57}$$

where $\Pi_{<-n}$ is the L^2-orthogonal projector, defined by

$$f = \sum_{k\in\mathbb{Z}} \widehat{f}_k e^{ikx} \mapsto \Pi_{<-n}[f] := \sum_{k<-n} \widehat{f}(k)e^{ikx}\,.$$

Substracting (57) from $(D-u)g_\infty = 0$ we obtain

$$(D-u)[g_\infty - g_n] = (n-\lambda_n)g_n + \Pi_{<-n}[ug_n]$$

or, after multiplying left and right hand side of the latter identity by $\overline{g_\infty}$, and using that $D[\overline{g_\infty}\,(g_\infty - g_n)] = -u\overline{g_\infty}\,(g_\infty - g_n) + \overline{g_\infty}\,D(g_\infty - g_n)$,

$$D[\overline{g_\infty}\,(g_\infty - g_n)] = (n-\lambda_n)\overline{g_\infty}\,g_n + \overline{g_\infty}\,\Pi_{<-n}[ug_n]\,.$$

Applying $g_\infty D^{-1}$ to the left and right hand side of the latter identity, we obtain

$$g_\infty - g_n - \langle \overline{g_\infty}(g_\infty - g_n)|\,1\rangle g_\infty = (K_n' + K_n)[g_n]\,.$$

Since $K_n'(g_\infty) = 0$, identity (52) follows.

Next we prove the estimates (53) - (54) for the operators K_n, $n \geq 1$. First assume that $u \in H^s_{r,0}$ with $0 \leq s \leq 1/2$ and $0 < \varepsilon < 1/2 + s$. Since $g_\infty \in H^{1+s}_c \subset H^{1/2+\varepsilon}_c$, it follows that for any $f \in H^{1/2+\varepsilon}_c$,

$$\begin{aligned} \|K_n f\|_{1/2+\varepsilon} &= \|g_\infty D^{-1}[\overline{g_\infty}\,\Pi_{<-n}(uf)]\|_{1/2+\varepsilon} \\ &\lesssim \|\Pi_{<-n}[uf]\|_{-1/2+\varepsilon} \\ &\lesssim \frac{1}{n^{1/2+s-\varepsilon}}\|uf\|_s \lesssim \frac{1}{n^{1/2+s-\varepsilon}}\|f\|_{1/2+\varepsilon}\,. \end{aligned}$$

Similarly,

$$\begin{aligned} \|K_n f\| &= \|g_\infty D^{-1}[\overline{g_\infty}\,\Pi_{<-n}(uf)]\| \lesssim \|\Pi_{<-n}[uf]\|_{-1} \\ &\lesssim \frac{1}{n^{1+s}}\|uf\|_s \lesssim \frac{1}{n^{1+s}}\|f\|_{1/2+\varepsilon}\,. \end{aligned}$$

Now assume $u \in H^s_{r,0}$ with $s > 1/2$. Note that for such s, H^s_c is an algebra, and H^{1+s}_c acts on H^{s-1}_c by multiplication. Hence for any $f \in H^s_c$, one gets

$$\begin{aligned} \|K_n f\|_s &= \|g_\infty D^{-1}[\overline{g_\infty}\,\Pi_{<-n}(uf)]\|_s \\ &\lesssim \|\Pi_{<-n}[uf]\|_{s-1} \lesssim \frac{1}{n}\|uf\|_s \lesssim \frac{1}{n}\|f\|_s\,. \end{aligned}$$

Similarly,

$$\begin{aligned} \|K_n f\| &= \|g_\infty D^{-1}[\overline{g_\infty}\,\Pi_{<-n}(uf)]\| \\ &\lesssim \|\Pi_{<-n}[uf]\|_{-1} \lesssim \frac{1}{n^{1+s}}\|uf\|_s \lesssim \frac{1}{n^{1+s}}\|f\|_s\,. \end{aligned}$$

Since for any $u \in H^s_{r,0}$ one has (cf. trace formula (13), properties (21), (23) of Φ)

$$0 \le n - \lambda_n = \sum_{k>n} \gamma_k \le n^{-1-2s} \sum_{k>n} k^{1+2s} \gamma_k \lesssim \frac{1}{n^{1+2s}},$$

the proofs of the claimed estimates for the operators K'_n, $n \ge 1$, are easier and hence we omit them. Going through the arguments of the proof one verifies that the constants $C_{s,\varepsilon}$ and C_s can be chosen uniformly for bounded subsets of potentials $u \in H^s_{r,0}$. □

Let us now continue with the proof of Lemma 2.4. The identity (52) and the estimates of the operators K_n and K'_n of Lemma 2.5 allow to write $g_\infty - g_n$ for $u \in H^s_{r,0}$ with $s \ge 0$ and n sufficiently large as a Neumann series, $g_\infty - g_n = \varepsilon_n g_\infty + r_n$, where $\varepsilon_n := \langle g_\infty - g_n | g_\infty \rangle$ and

$$r_n := (1 - \varepsilon_n) \sum_{j=1}^{\infty} (-1)^{j+1} (K_n + K'_n)^j g_\infty \, . \tag{58}$$

Substituting $\varepsilon_n g_\infty + r_n$ for $g_\infty - g_n$ in the formula for $T_{3,n}$ one obtains $T_{3,n} = \varepsilon_n \langle \Pi u | \, e^{inx} g_\infty \rangle + (1 - \varepsilon_n) \langle \Pi u | \, e^{inx} r_n \rangle$, which we decompose further as

$$T_{3,n} = U_n + V_n + W_n \, , \tag{59}$$

where

$$U_n \equiv U_n(u) := \varepsilon_n \langle \overline{g_\infty} \, \Pi u | \, e^{inx} \rangle \, , \tag{60}$$

$$V_n \equiv V_n(u) := (1 - \varepsilon_n) \sum_{r=2}^{\infty} (-1)^{r+1} \langle \Pi u | \, e^{inx} (K_n + K'_n)^r [g_\infty] \rangle \, , \tag{61}$$

and (using that $K'_n[g_\infty] = 0$)

$$W_n \equiv W_n(u) := (1 - \varepsilon_n) \langle \Pi u | \, e^{inx} K_n [g_\infty] \rangle \, . \tag{62}$$

We first estimate ε_n. Representing $g_\infty - g_n$ by the telescoping sum (51), we get

$$\varepsilon_n = \langle g_\infty - g_n | \, g_\infty \rangle = \sum_{k \ge n} \langle g_{k+1} - g_k | \, g_\infty - g_k \rangle + \sum_{k \ge n} (\langle g_{k+1} | \, g_k \rangle - 1) \, .$$

Note that by the definition (41) of μ_{k+1} ,

$$\langle g_{k+1} | \, g_k \rangle = \sqrt{\mu_{k+1}} \, ,$$

and therefore, in view of Lemma 2.3,

$$\Big| \sum_{k \ge n} (\langle g_{k+1} | \, g_k \rangle - 1) \Big| \lesssim \frac{1}{n^{2+2s}} \, . \tag{63}$$

Moreover, $\| g_{k+1} - g_k \|^2 = \| f_{k+1} - e^{ix} f_k \|^2$ can be computed as

$$\| g_{k+1} - g_k \|^2 = 2 - \langle g_{k+1} | \, g_k \rangle - \langle g_k | \, g_{k+1} \rangle = 2 - 2\sqrt{\mu_{k+1}}$$

and hence $\|g_{k+1} - g_k\| = \sqrt{2}(1 - \sqrt{\mu_{k+1}})^{1/2}$. By Lemma 2.3, the Cauchy–Schwarz inequality, and the assumption $s \geq 0$, one then infers that

$$\begin{aligned} \|g_\infty - g_n\| &\leq \sum_{k\geq n} \|g_{k+1} - g_k\| \leq \sum_{k\geq n} \sqrt{2}(1 - \sqrt{\mu_{k+1}})^{1/2} \\ &\lesssim \sum_{k\geq n} \left(k^{1+s}(1 - \sqrt{\mu_{k+1}})^{1/2}\right) \cdot \frac{1}{k^{1+s}} \lesssim \frac{1}{n^{(1+2s)/2}} \,, \end{aligned} \tag{64}$$

and hence by the Cauchy-Schwarz inequality,

$$\begin{aligned} &\Big|\sum_{k\geq n} \langle g_{k+1} - g_k | \, g_\infty - g_k \rangle \Big| \leq \sum_{k\geq n} \sqrt{2}(1 - \sqrt{\mu_{k+1}})^{1/2} \|g_\infty - g_k\| \\ &\lesssim \sum_{k\geq n} \left(k^{1+s}(1 - \sqrt{\mu_{k+1}})^{1/2}\right) \cdot \frac{1}{k^{1+s+(1+2s)/2}} \lesssim \frac{1}{n^{1+2s}} \,. \end{aligned} \tag{65}$$

Combining (63) and (65) we obtain

$$|\varepsilon_n| \lesssim \frac{1}{n^{1+2s}} \,. \tag{66}$$

Using that $\overline{g_\infty}\,\Pi u \in H^s_c$ and taking into account the estimates (66) it then follows that U_n, defined by (60), satisfies

$$U_n = \mathfrak{h}_n^{s+1+2s} \,.$$

Next we estimate V_n, defined by (61). First we consider the case where $0 \leq s \leq 1/2$. From (53) and (54), we have

$$\begin{aligned} |V_n| &\lesssim \sum_{r=2}^{\infty} \|(K_n + K_n')^r [g_\infty]\| \\ &\lesssim \sum_{r=2}^{\infty} \frac{1}{n^{1+s+(r-1)(1/2+s-\varepsilon)}} \|g_\infty\|_{1/2+\varepsilon} \lesssim \frac{1}{n^{2s+3/2-\varepsilon}} \,. \end{aligned}$$

Choosing $0 < \varepsilon < 1/4$, we get

$$V_n = \mathfrak{h}_n^{2s+3/4} \,.$$

In the case where $s > 1/2$, we use (55) and (56) to conclude that

$$\begin{aligned} |V_n| \;&\lesssim\; \sum_{r=2}^{\infty} \|(K_n + K_n')^r [g_\infty]\| \\ &\lesssim\; \sum_{r=2}^{\infty} \frac{1}{n^{1+s+r-1}} \lesssim \frac{1}{n^s} \sum_{r=2}^{\infty} \frac{1}{n^r} \lesssim \frac{1}{n^{s+2}} \,. \end{aligned}$$

Hence for any $\varepsilon > 0$,

$$V_n = \mathfrak{h}_n^{s+3/2-\varepsilon} \,.$$

It remains to estimate W_n, defined by (62), for $u \in H^s_{r,0}$ with $s \geq 0$. First we note that

$$\begin{aligned} K_n g_\infty &= g_\infty D^{-1}[\overline{g_\infty}\, \Pi_{<-n}(Dg_\infty)] \\ &= g_\infty D^{-1} D[\overline{g_\infty} \cdot \Pi_{<-n} g_\infty] - g_\infty D^{-1}[(D\overline{g_\infty}) \cdot \Pi_{<-n} g_\infty] \\ &= \Pi_{<-n} g_\infty - \|\Pi_{<-n} g_\infty\|^2 g_\infty + g_\infty D^{-1}[u\overline{g_\infty} \cdot \Pi_{<-n} g_\infty] \,. \end{aligned}$$

Since $\langle \Pi u |\, e^{inx} \Pi_{<-n} g_\infty \rangle = 0$, it then follows that

$$W_n = -\|\Pi_{<-n} g_\infty\|^2 \langle \Pi u |\, e^{inx} g_\infty \rangle + \langle \overline{g_\infty}\, \Pi u |\, e^{inx} D^{-1}[u\overline{g_\infty}\, \Pi_{<-n} g_\infty] \rangle$$

or

$$W_n = \mathfrak{h}_n^{2+2s+s} + \langle \overline{g_\infty}\, \Pi u |\, e^{inx} D^{-1}[u\overline{g_\infty}\, \Pi_{<-n} g_\infty] \rangle \,. \tag{67}$$

To estimate the latter term, let

$$f_1 := \overline{g_\infty}\, \Pi u \in H^s_c \,, \qquad f_2 := \overline{g_\infty}\, u \in H^s_c \,.$$

Then

$$\begin{aligned} &|\langle \overline{g_\infty}\, \Pi u |\, e^{inx} D^{-1}[u\overline{g_\infty}\, \Pi_{<-n} g_\infty] \rangle|^2 \\ &= \Big| \sum_{k \in \mathbb{Z}\setminus\{n\}} \overline{\widehat{f}_1(k)} \frac{1}{k-n} \sum_{p=1}^{\infty} \widehat{f}_2(k+p) \widehat{g}_\infty(-n-p) \Big|^2 \\ &\leq \|f_1\|_s^2 \sum_{k \neq n} \frac{1}{(k-n)^2 \langle k \rangle^{2s}} \Big| \sum_{p=1}^{\infty} \widehat{f}_2(k+p) \widehat{g}_\infty(-n-p) \Big|^2 \end{aligned}$$

Since by the Cauchy-Schwarz inequality,

$$\begin{aligned} &\Big| \sum_{p=1}^{\infty} \widehat{f}_2(k+p) \widehat{g}_\infty(-n-p) \Big|^2 \\ &\leq \Big(\sum_{p=1}^{\infty} |\widehat{f}_2(k+p)|^2 \langle k+p \rangle^{2s} \Big) \cdot \Big(\sum_{p=1}^{\infty} \frac{1}{\langle k+p \rangle^{2s}} |\widehat{g}_\infty(-n-p)|^2 \Big) \\ &\leq \|f_2\|_s^2 \Big(\sum_{p=1}^{\infty} \frac{1}{\langle k+p \rangle^{2s}} |\widehat{g}_\infty(-n-p)|^2 \Big) \,, \end{aligned}$$

we get

$$\begin{aligned} &|\langle \overline{g_\infty}\, \Pi u |\, e^{inx} D^{-1}[u\overline{g_\infty}\, \Pi_{<-n} g_\infty] \rangle|^2 \\ &\leq \|f_1\|_s^2 \, \|f_2\|_s^2 \sum_{p=1}^{\infty} B_{n,p} |\widehat{g}_\infty(-n-p)|^2 \,, \end{aligned} \tag{68}$$

where

$$B_{n,p} := \sum_{k \neq n} \frac{1}{(k-n)^2 \langle k \rangle^{2s} \langle k+p \rangle^{2s}} \,.$$

Splitting the latter sum into two sums with domains $0 < |k-n| < n/2$ and $|k-n| \geq n/2$ respectively, one concludes that for any $\varepsilon > 0$,

$$\sum_{0<|k-n|<n/2} \frac{1}{(k-n)^2\langle k\rangle^{2s}\langle k+p\rangle^{2s}} \lesssim \frac{1}{\langle n\rangle^{2s}(n+p)^{2s}} \sum_{0<|k-n|<n/2} \frac{1}{(k-n)^2},$$

$$\sum_{|k-n|\geq n/2} \frac{1}{(k-n)^2\langle k\rangle^{2s}\langle k+p\rangle^{2s}} \lesssim \frac{1}{n^2} \sum_{|k|<3n/2} \frac{1}{\langle k\rangle^{2s}} + \frac{1}{n^{1-\varepsilon}} \sum_{|k|\geq 3n/2} \frac{1}{\langle k\rangle^{1+\varepsilon+4s}},$$

and hence

$$B_{n,p} \lesssim \frac{1}{n^{2s}(n+p)^{2s}} + \frac{1}{n^{1+2s-\varepsilon}}. \tag{69}$$

Given any $\gamma > 0$, it then follows from (67)–(69) that $\sum_{n=1}^{\infty} n^{2(s+\gamma)}|W_n|^2$ can be bounded by

$$\begin{aligned} &\lesssim \sum_{n=1}^{\infty} n^{2\gamma-4-4s}\,\ell_n^1 + \sum_{n=1}^{\infty} n^{2\gamma} \sum_{q>n} |\widehat{g}_\infty(-q)|^2(q^{-2s} + n^{-1+\varepsilon}) \\ &\lesssim \sum_{n=1}^{\infty} n^{2\gamma-4-4s}\,\ell_n^1 + \sum_{q=1}^{\infty}(q^{2\gamma+1-2s} + q^{2\gamma+\varepsilon})|\widehat{g}_\infty(-q)|^2 . \end{aligned}$$

Using that $g_\infty \in H_c^{1+s}$ and choosing γ as

$$\tau_1(s) = \begin{cases} (1+s)- & \text{if } s > \frac{1}{2} \\ \frac{3}{2}- & \text{if } s = \frac{1}{2} \\ \frac{1}{2} + 2s & \text{if } 0 \leq s < \frac{1}{2} \end{cases}$$

we conclude that for any $u \in H^s_{r,0}$ the latter two sums are finite.

In summary, we have proved that for any $u \in H^s_{r,0}$ with $s \geq 0$, $T_{3,n} = \mathfrak{h}_n^{s+\tau_1(s)}$. Note that by the definition (16) of $\tau(s)$ and the one of $\tau_1(s)$, one has

$$\tau(0) = \frac{1}{2} = \tau_1(0), \qquad \tau(s) < \tau_1(s), \quad \forall\, s > 0.$$

Combining the estimate (47) of $T_{1,n}$ and the estimate (49) of $T_{2,n}$ with the estimate of $T_{3,n}$, we conclude that for any $u \in H^s_{r,0}$,

$$\Xi_n(u) = \sum_{1\leq j\leq 3} T_{j,n}(u) = \mathfrak{h}_n^{s+\tau(s)}$$

and that for any $u \in H^s_{r,0}$ with $s > 0$,

$$(\Xi_n(u) - T_{2,n}(u))_{n\geq 1} \in \mathfrak{h}^{s+\tau_1(s)}, \qquad \tau_1(s) > \tau(s). \tag{70}$$

Furthermore, going through the arguments of the proof one verifies that for any $s \geq 0$, the stated estimates hold uniformly on bounded subsets of potentials u in $H^s_{r,0}$.

It remains to study the optimality of these estimates for $0 < s \leq 1/2$. First consider the case $0 < s < 1/2$. Assume that $u \in H^s_{r,0}$ satisfies $(\Xi_n(u))_{n\geq 1} \in \mathfrak{h}^{s+\tau(s)+\varepsilon}$ for some

$\varepsilon > 0$. In view of the estimate (70) of $(\Xi_n(u) - T_{2,n})_{n\geq 1}$, it then follows that for $\varepsilon > 0$ small enough, $(T_{2,n}(u))_{n\geq 1} \in \mathfrak{h}^{2s+1/2+\varepsilon}$ or, by the formula (48) for $T_{2,n}$,

$$\Pi(\overline{g}_\infty(\mathrm{Id} - \Pi)u) \in H^{2s+1/2+\varepsilon}\,.$$

Set $v := -\Pi u$. Since u is real–valued and has vanishing mean, one has $(\mathrm{Id}-\Pi)u = -\overline{v}$ and hence

$$\Pi[\overline{g}_\infty(\mathrm{Id} - \Pi)u)] = -\Pi[e^{-i\partial_x^{-1}\overline{v}}e^{-i\partial_x^{-1}v}\overline{v}]\,.$$

Since $e^{-i\partial_x^{-1}v}$ is in the Hardy space H_+^{s+1} and $e^{-i\partial_x^{-1}\overline{v}}$ in H_-^{s+1}, it follows that

$$\Pi[e^{-i\partial_x^{-1}\overline{v}}(\mathrm{Id} - \Pi)(e^{-i\partial_x^{-1}v}\overline{v})] = 0$$

and hence

$$\begin{aligned}\Pi[\overline{g}_\infty(\mathrm{Id} - \Pi)u)] &= -\Pi[e^{-i\partial_x^{-1}\overline{v}}\Pi(e^{-i\partial_x^{-1}v}\overline{v})] \\ &= -T_{e^{-i\partial_x^{-1}\overline{v}}}[\Pi(e^{-i\partial_x^{-1}v}\overline{v})]\,.\end{aligned}$$

Note that the Toeplitz operator $T_{e^{-i\partial_x^{-1}\overline{v}}}$ with symbol $e^{-i\partial_x^{-1}\overline{v}}$ is a linear isomorphism on H_+^ρ for any $0 \leq \rho \leq 1 + s$ and that its inverse is given by $T_{e^{i\partial_x^{-1}\overline{v}}}$ (cf. e.g. [24], [12, Section 6]). Since $1 + s \geq 1/2 + 2s + \varepsilon$ for ε small enough, we then conclude that

$$\Pi[e^{-i\partial_x^{-1}v}\overline{v}] \in H^{2s+1/2+\varepsilon}\,. \tag{71}$$

Let us choose $u = -v - \overline{v}$ with

$$v(x) = \sum_{k=1}^{\infty} \frac{e^{ikx}}{k^{1/2+s}\log(1 + k)}\,. \tag{72}$$

Clearly, $v \in H_+^s$ and $u \in H_{r,0}^s$. We claim that (71) fails for every $\varepsilon > 0$. Indeed, observe that the Fourier coefficients of v and of $i\partial_x^{-1}v$ are positive, and so are the Fourier coefficients of $(i\partial_x^{-1}v)^p\overline{v}$ for every integer $p \geq 1$. Expanding

$$e^{i\partial_x^{-1}v} = \sum_{p=0}^{\infty} \frac{(i\partial_x^{-1}v)^p}{p!}\,,$$

we infer that the kth Fourier coefficient of $\Pi(e^{i\partial_x^{-1}v}\overline{v})$ is larger than the kth Fourier coefficient of $\Pi((i\partial_x^{-1}v)\overline{v})$, and hence (71) implies

$$\Pi[(i\partial_x^{-1}v)\overline{v}] \in H^{2s+1/2+\varepsilon}\,. \tag{73}$$

For any $k \geq 1$, the kth Fourier coefficient of $\Pi[(i\partial_x^{-1}v)\overline{v}]$ can be computed as

$$a_k = \sum_{j\in\mathbb{Z}} \frac{\widehat{v}(k-j)\widehat{\overline{v}(j)}}{k-j} = \sum_{j\leq -1} \frac{\widehat{v}(k-j)\widehat{v(-j)}}{k-j} = \sum_{\ell=1}^{\infty} \frac{\widehat{v}(k+\ell)\widehat{v}(\ell)}{k+\ell}$$

and thus by the definition (72) of v, one has

$$\begin{aligned} a_k &= \sum_{\ell=1}^{\infty} \frac{1}{(k+\ell)^{3/2+s}\log(1+k+\ell)\ell^{1/2+s}\log(1+\ell)} \\ &\geq \sum_{\ell=1}^{k} \frac{1}{(k+\ell)^{3/2+s}\log(1+k+\ell)\ell^{1/2+s}\log(1+\ell)} \\ &\geq \frac{2^{-3/2-s}}{k^{1+2s}\log(1+2k)\log(1+k)}\,, \end{aligned}$$

so that

$$k^{2s+1/2+\varepsilon}a_k \geq b_k := \frac{2^{-3/2-s}}{k^{1/2-\varepsilon}[\log(1+2k)]^2}\,.$$

Clearly, for any $\varepsilon > 0$, $(b_k)_{k\geq 1}$ is not an ℓ^2-sequence. This contradicts (73) and shows that for $u = -v - \overline{v}$ with v given by (72), $(\Xi_n(u))_{n\geq 1} \notin \mathfrak{h}^{2s+1/2+\varepsilon}$ for any $\varepsilon > 0$.

In the case $s = 1/2$, we argue similarly as in the case $0 < s < 1/2$. Choose $u = -v - \overline{v}$ with

$$v(x) = \sum_{k=1}^{\infty} \frac{e^{ikx}}{k[\log(1+k)]^{\alpha}}\,,$$

where $1/2 < \alpha < 3/4$. Since $1/2 < \alpha$ it follows that $u \in H^{1/2}_{r,0}$. In this case, the kth Fourier coefficient of $\Pi[i\partial_x^{-1}v\overline{v}]$ is

$$\begin{aligned} a_k &= \sum_{\ell=1}^{\infty} \frac{\hat{v}(k+\ell)\hat{v}(\ell)}{k+\ell} = \sum_{\ell=1}^{\infty} \frac{1}{(k+\ell)^2[\log(1+k+\ell)]^{\alpha}\ell[\log(1+\ell)]^{\alpha}} \\ &\geq \sum_{\ell=1}^{k} \frac{1}{(k+\ell)^2[\log(1+k+\ell)]^{\alpha}\ell[\log(1+\ell)]^{\alpha}} \\ &\geq \frac{4^{-1}}{k^2[\log(1+2k)]^{\alpha}[\log(1+k)]^{\alpha}} \sum_{\ell=1}^{k}\frac{1}{\ell} \\ &\gtrsim \frac{1}{k^2[\log(k)]^{2\alpha-1}} \end{aligned}$$

so that

$$k^{3/2}a_k \gtrsim b_k := \frac{1}{k^{1/2}[\log(k)]^{2\alpha-1}}\,.$$

Since $\alpha < 3/4$, $(b_k)_{k\geq 1}$ is not an ℓ^2-sequence, hence $\Pi[(i\partial_x^{-1}v)\overline{v}]$ does not belong to $H^{3/2}$, and consequently $(\Xi_n(u))_{n\geq 1} \notin \mathfrak{h}^{3/2}$.

This finishes the proof of Lemma 2.4. □

In view of the definition (42) of Ξ, Lemma 2.4 yields the following

Corollary 2.6. — *For any $s \geq 0$, the difference $\Phi_1 - \Phi_0$ is $\tau(s)$-smoothing, meaning that for any $s \geq 0$, $\Phi_1 - \Phi_0$ is a continuous map from $H^s_{r,0}$ with values in $\mathfrak{h}^{s+1/2+\tau(s)}$.*

Furthermore, there exists $u \in H^{1/2}_{r,0}$ *so that* $\Phi_1(u) - \Phi_0(u) \notin \mathfrak{h}^2$ *and for* $0 < s < 1/2$, *there exists* $u \in H^s_{r,0}$ *so that for any* $\varepsilon > 0$, $\Phi_1(u) - \Phi_0(u) \notin \mathfrak{h}^{2s+1+\varepsilon}$.

Proof of Theorem 1.6. — The claimed results directly follow from Corollary 2.2 and Corollary 2.6. □

3. Approximations of the Benjamin–Ono flow

In this section we apply Theorem 1.6 to prove smoothing properties of the flow map of the Benjamin-Ono equation, stated in Theorem 1.1, Theorem 1.4 and Corollary 1.9.

Recall from Section 1 that for any $u_0 \in H^s_{r,0}$ with $s \geq 0$, we denote by $u(t) = \mathcal{S}(t, u_0)$ the solution of the Benjamin-Ono equation constructed in [9] and by $w(t)$ the gauge transformation of $u(t)$ (cf. (4)),

$$w(t) = \mathcal{G}(u(t)) = \partial_x \Pi(e^{-i\partial_x^{-1} u(t)}) \,, \quad w_0 := w(0) = \partial_x \Pi(e^{-i\partial_x^{-1} u_0}) \,.$$

Furthermore, we introduced (cf. (7), (15))

$$w_L(t) = \sum_{n \geq 1} e^{it(n^2 - \langle u_0^2 | 1 \rangle)} \widehat{w}_0(n) e^{inx} \,, \tag{74}$$

$$w_{L,*}(t) = \sum_{n \geq 1} e^{it\omega_n} \widehat{w}_0(n) e^{inx} \,. \tag{75}$$

Proof of Theorem 1.4. — First we prove (17), saying that for any bounded subset $\mathcal{B}$ of $H^s_{r,0}$ with $s \geq 0$, there exists $M_s > 0$ so that

$$\sup_{t \in \mathbb{R}} \| w(t) - w_{L,*}(t) \|_{s + \tau(s)} \leq M_s \,, \qquad \forall \, u_0 \in \mathcal{B} \,. \tag{76}$$

Recall that by [9, Corollary 8, Appendix A], there exists a bounded subset $\tilde{\mathcal{B}}$ of $H^s_{r,0}$ so that $\mathcal{S}(t, u_0) \in \tilde{\mathcal{B}}$ for any $t \in \mathbb{R}$ and $u_0 \in \mathcal{B}$. By Theorem 1.6, applied to $u = u(t)$ and (28) it follows that

$$\widehat{w(t)}(n) = i\sqrt{n} \zeta_n(u(t)) + \rho_n(t) \,, \qquad \forall \, n \geq 1 \,, \tag{77}$$

where $(\rho_n(t))_{n \geq 1}$ is uniformly bounded in $\mathfrak{h}^{s+\tau(s)}$ with respect to $t \in \mathbb{R}$. Since by (24),

$$i\sqrt{n} \zeta_n(u(t)) = i\sqrt{n} e^{it\omega_n} \zeta_n(u_0) \,,$$

and by (77) for $t = 0$,

$$\widehat{w(0)}(n) = i\sqrt{n} \zeta_n(u_0) + \rho_n(0) \,,$$

we conclude that for any $t \in \mathbb{R}$,

$$\widehat{w(t)}(n) - e^{it\omega_n} \widehat{w(0)}(n) = \rho_n(t) - e^{it\omega_n} \rho_n(0) \,, \qquad \forall n \geq 1 \,, \tag{78}$$

which proves (76). It remains to prove estimate (18). Following a suggestion by N. Tzvetkov, we write $u(t) = e^{i\partial_x^{-1} u(t)} e^{-i\partial_x^{-1} u(t)} u(t)$ as

$$u(t) = e^{i\partial_x^{-1} u(t)} \Pi[e^{-i\partial_x^{-1} u(t)} u(t)] + e^{i\partial_x^{-1} u(t)} (\mathrm{Id} - \Pi)[e^{-i\partial_x^{-1} u(t)} u(t)] \,.$$

Since $\Pi[e^{-i\partial_x^{-1}u(t)}u(t)] = iw(t)$, one has

$$e^{i\partial_x^{-1}u(t)}\Pi[e^{-i\partial_x^{-1}u(t)}u(t)] = e^{i\partial_x^{-1}u(t)}iw_{L,*}(t) + e^{i\partial_x^{-1}u(t)}i(w(t) - w_{L,*}(t))\,.$$

Recall that $u(t)$ is real valued and satisfies $\langle u(t)|1\rangle = 0$. Hence $u(t) = 2\mathrm{Re}(\Pi u(t))$. Splitting the term $2\mathrm{Re}\big(\Pi[e^{i\partial_x^{-1}u(t)}iw_{L,*}(t)]\big)$ as

$$2\mathrm{Re}(e^{i\partial_x^{-1}u(t)}iw_{L,*}(t)) - 2\mathrm{Re}\big((\mathrm{Id} - \Pi)[e^{i\partial_x^{-1}u(t)}iw_{L,*}(t)]\big)$$

one then concludes that

$$u(t) = 2\mathrm{Re}\big(e^{i\partial_x^{-1}u(t)}iw_{L,*}(t)\big) + r_*(t)$$

where $r_*(t) := I(t) + II(t) + III(t)$ and

$$I(t) := -2\mathrm{Re}\big((\mathrm{Id} - \Pi)[e^{i\partial_x^{-1}u(t)}iw_{L,*}(t)]\big)\,,$$

$$II(t) := 2\mathrm{Re}\big(\Pi[e^{i\partial_x^{-1}u(t)}i(w(t) - w_{L,*}(t))]\big)\,,$$

$$III(t) := 2\mathrm{Re}\big(\Pi[e^{i\partial_x^{-1}u(t)}(\mathrm{Id} - \Pi)(e^{-i\partial_x^{-1}u(t)}u(t))]\big)\,.$$

Since $\partial_x^{-1}u(t) \in H^{s+1}_{r,0}$, the claimed estimate of $r_*(t)$ is obtained by estimating the term $II(t)$ with the help of estimate (17) and the terms $III(t)$, $I(t)$ by Lemma A.1 in Appendix A. □

Proof of Theorem 1.1. — First we prove estimate (9), saying that for any bounded subset $\mathcal{B}$ of $H^s_{r,0}$ with $s \geq 0$, there exists $M_s > 0$ so that

$$\sup_{t\in\mathbb{R}} \|w(t) - w_L(t)\|_{s+\sigma(s)} \leq M_s\langle t\rangle\,, \qquad \forall\, u_0 \in \mathcal{B}\,. \tag{79}$$

To this end note that by (14),

$$|e^{it\omega_n} - e^{it(n^2 - \langle u_0^2|1\rangle)}| \leq 2|t|\sum_{k>n}(k-n)\gamma_k(u_0) \leq \frac{C|t|}{n^{2s}}\,,$$

where the constant $C > 0$ can be chosen uniformly for $u_0 \in \mathcal{B}$. Combined with (78), estimate (79) follows. Estimate (10) can be proved in a similar way as the estimate (18) in the proof of Theorem 1.4 and hence we omit the details.

It remains to prove the optimality statement of Theorem 1.1. Let $0 < s < 1/2$, and consider $u_0 \in H^s_{r,0}$ with the property that

$$\gamma_n(u_0) = \frac{1}{n^{2+2s}[\log(1+n)]^2}\,, \qquad \forall\, n \geq 1\,. \tag{80}$$

(Such u_0 exist since by [9, Proposition 5 in Appendix A], $\Phi : H^s_{r,0} \to \mathfrak{h}^{s+1/2}$ is onto.) By the definition (8), one has $\sigma(s) = 2s$. Assume that there exist $t \neq 0$ and $\varepsilon > 0$ so that

$$w(t) - w_L(t) \in H^{3s+\varepsilon}_+\,. \tag{81}$$

Since $3s < 2s + 1/2 = s + \tau(s)$, estimate (17) of Theorem 1.4 then implies that for ε sufficiently small, (81) is equivalent to

$$w_{L,*}(t) - w_L(t) \in H^{3s+\varepsilon}\,. \tag{82}$$

By the above formulas (74), (75), this is equivalent to

$$\left|e^{it\omega_n} - e^{it(n^2-\langle u_0^2|1\rangle)}\right| \cdot |\widehat{w}_0(n)| = \mathfrak{h}_n^{3s+\varepsilon}$$

(recall that $w_0 = \partial_x \Pi(e^{-i\partial_x^{-1}u_0})$), or

$$\Big(\sum_{k>n}(k-n)\gamma_k(u_0)\Big)|\widehat{w}_0(n)| = \mathfrak{h}_n^{3s+\varepsilon}\,, \tag{83}$$

where we used that for n sufficiently large (cf. (14)),

$$\begin{aligned}\left|e^{it\omega_n} - e^{it(n^2-\langle u_0^2|1\rangle)}\right| &= 2\sum_{k>n}(k-n)\gamma_k(u_0)\left|\int_0^t e^{is2\sum_{k>n}(k-n)\gamma_k(u_0)}ds\right| \\ &\sim |t|\sum_{k>n}(k-n)\gamma_k(u_0)\,.\end{aligned}$$

Since by (80),

$$\begin{aligned}\sum_{k>n}(k-n)\gamma_k(u_0) &\geq \frac{1}{2}\sum_{k>2n}k\gamma_k(u_0) \\ &\gtrsim \frac{1}{n^{2s}[\log(n)]^2}\,,\end{aligned}$$

it then follows from (83) that

$$n^{s+\varepsilon}(\log(n))^{-2}|\hat{w}_0(n)| = \ell_n^2$$

and hence $w_0 \in H_+^{s+\delta}$ for any $\delta < \varepsilon$. On the other hand, by the definition of w_0,

$$\begin{aligned} i\Pi[e^{i\partial_x^{-1}u_0}w_0] &= \Pi[e^{i\partial_x^{-1}u_0}\Pi(u_0e^{-i\partial_x^{-1}u_0})] \\ &= \Pi u_0 - \Pi[e^{i\partial_x^{-1}u_0}(\mathrm{Id}-\Pi)(u_0e^{-i\partial_x^{-1}u_0})]\,.\end{aligned}$$

By Lemma A.1(iii) (with $\alpha = 1$, $\beta = s+1/2$) it then follows that $i\Pi[e^{i\partial_x^{-1}u_0}w_0] = \Pi u_0 + H_+^{2s+1/2}$. Since $e^{i\partial_x^{-1}u_0} \in H_c^{1+s}$, $w_0 \in H_+^{s+\delta}$, and hence $i\Pi[e^{i\partial_x^{-1}u_0}w_0] \in H_+^{s+\delta}$ one concludes that $u_0 \in H_{r,0}^{s+\delta}$. Hence for any $\delta < \varepsilon$,

$$\sum_{n=1}^{\infty} n^{1+2(s+\delta)}\gamma_n(u_0) < \infty\,,$$

which is in contradiction to (80). Therefore (82) is false and hence so is (81).

To finish the proof of Theorem 1.1, assume that $r(t)$ is $H_r^{3s+\varepsilon}$. Since by (10) and (18),

$$r(t) - r_*(t) = 2\mathrm{Re}[e^{i\partial_x^{-1}u(t)}(w_L(t) - w_{L,*}(t))]\,,$$

estimate (18) implies that for $\varepsilon > 0$ sufficiently small

$$2\mathrm{Re}[e^{i\partial_x^{-1}u(t)}(w_L(t) - w_{L,*}(t))] \in H^{3s+\varepsilon}\,.$$

Note that $\Pi\big(2\mathrm{Re}[e^{i\partial_x^{-1}u(t)}(w_L(t) - w_{L,*}(t))]\big) \in H^{3s+\varepsilon}$ equals

$$\Pi[e^{i\partial_x^{-1}u(t)}\big(w_L(t) - w_{L,*}(t)\big)] + \Pi[e^{-i\partial_x^{-1}u(t)}\big(\overline{w_L(t)} - \overline{w_{L,*}(t)}\big)]\,.$$

By Lemma A.1(iii) (with $\alpha = 1$ and $\beta = s + 1/2$), one then concludes that for ε sufficiently small

$$T_{e^{i\partial_x^{-1}u(t)}}[w_L(t) - w_{L,*}(t)] = \Pi[e^{i\partial_x^{-1}u(t)}(w_L(t) - w_{L,*}(t))] \in H_+^{3s+\varepsilon}\,.$$

Since the Toeplitz operator $T_{e^{i\partial_x^{-1}u(t)}} : H_+^{3s+\varepsilon} \to H_+^{3s+\varepsilon}$ is a linear isomorphism, one obtains

$$w_L(t) - w_{L,*}(t) \in H^{3s+\varepsilon}\,,$$

which contradicts the above conclusion that (82) is false.

This finishes the proof of Theorem 1.1. □

Proof of Corollary 1.9. — The claimed results can be proved in a similar way as Theorem 1.1, using again Theorem 1.6, and hence we leave the details of the proof to the reader. □

4. Benjamin–Ono flow and Hölder spaces

As an illustration of possible applications of Theorem 1.1, we show in this section how this theorem can be used to study the action of the Benjamin–Ono flow $\mathcal{S}(t)$ on Hölder spaces $C^\alpha(\mathbb{T},\mathbb{R})$, $1/2 < \alpha < 1$. The main result of this section is Proposition 4.8 below, which states that for almost any time $t \in \mathbb{R}$ and any $1/2 < \alpha < 1$, $\mathcal{S}(t)$ does not map $C^{\alpha-}(\mathbb{T},\mathbb{R})$ into $\bigcup_{\varepsilon>0} C^{\alpha-1/2+\varepsilon}(\mathbb{T},\mathbb{R})$, whereas by the Sobolev embedding theorem, $\mathcal{S}(t)$ maps $C^{\alpha-}(\mathbb{T},\mathbb{R})$ into $C^{(\alpha-1/2)-}(\mathbb{T},\mathbb{R})$. Here for any $0 < \beta < 1$,

$$C^{\beta-}(\mathbb{T},\mathbb{C}) := \bigcap_{\gamma<\beta} C^\gamma(\mathbb{T},\mathbb{C})\,.$$

First we need make some preliminary considerations. We begin by reviewing a result on the flow map of the linear Schrödinger equation on $\mathbb{T}$,

$$-i\partial_t\psi = \partial_x^2\psi\,,$$

related to the Talbot effect. See [6, Section 2.3 and references therein]. First we need to introduce some additional notation. For any $0 < \alpha < 1$, we denote by $C^\alpha(\mathbb{T},\mathbb{C})$ the Banach space of α-Hölder continuous functions $\psi : \mathbb{T} \to \mathbb{C}$, endowed with the standard norm,

$$\|\psi\|_{C^\alpha} := \sup_{x\in\mathbb{T}} |\psi(x)| + \sup_{x\neq y} \frac{|\psi(y) - \psi(x)|}{(d(x,y))^\alpha}\,,$$

where $d(x,y)$ denotes the distance between x and y in $\mathbb{T}$.

Theorem 4.1. — [6, Theorem 2.16] *There exists a subset $N \subset \mathbb{R}$ of Lebesgue measure 0 so that for any $t \notin N$ and any function $\psi : \mathbb{T} \to \mathbb{C}$ of bounded variation, $e^{it\partial_x^2}\psi \in C^{1/2-}(\mathbb{T},\mathbb{C})$.*

Remark 4.2. — *It follows from the proof of Theorem 2.16 in* [6, p.37-39] *that the set N of Theorem 4.1 can be chosen independently of ψ.*

Theorem 4.1 can be used to analyze the action of the flow map $\mathrm{e}^{it\partial_x^2}$ on $C^{1/2-}(\mathbb{T},\mathbb{C})$. To state our result we make the following preliminary considerations. The $\mathbb{C}$-vector space $C^{1/2-}(\mathbb{T},\mathbb{C})$ is endowed with the countable family of the norms of $C^{1/2-1/p}(\mathbb{T},\mathbb{C})$, $p \in \mathbb{Z}_{\geq 3}$. In this way, $C^{1/2-}(\mathbb{T},\mathbb{C})$ becomes a Fréchet space with the property that $C^{\infty}(\mathbb{T},\mathbb{C})$ is dense in $C^{1/2-}(\mathbb{T},\mathbb{C})$, since $C^{\infty}(\mathbb{T},\mathbb{C})$ is dense in $C^{\alpha}(\mathbb{T},\mathbb{C})$, when considered with the norm of $C^{\alpha-\varepsilon}(\mathbb{T},\mathbb{C})$, $\varepsilon > 0$.

Corollary 4.3. — *Let N be any set satisfying the conclusions of Theorem 4.1. Then for any $t \notin N' := -N$, $\mathrm{e}^{it\partial_x^2}$ does not map $C^{1/2-}(\mathbb{T},\mathbb{C})$ into $L^{\infty}(\mathbb{T},\mathbb{C})$.*

Proof. — Let $t \notin N'$ and suppose that for any $\psi \in C^{1/2-}(\mathbb{T},\mathbb{C})$, $\mathrm{e}^{it\partial_x^2}\psi \in L^{\infty}(\mathbb{T},\mathbb{C})$. Since for any $n \in \mathbb{Z}$, $\widehat{\mathrm{e}^{it\partial_x^2}\psi}(n) = \mathrm{e}^{-itn^2}\widehat{\psi}(n)$, it then easily follows from the closed graph theorem that the linear map

$$\mathrm{e}^{it\partial_x^2} : C^{1/2-}(\mathbb{T},\mathbb{C}) \to L^{\infty}(\mathbb{T},\mathbb{C})$$

is continuous. In particular, since $C^{\infty}(\mathbb{T},\mathbb{C})$ is dense in $C^{1/2-}(\mathbb{T},\mathbb{C})$ by the considerations above, one concludes that the image of $C^{1/2-}(\mathbb{T},\mathbb{C})$ by $\mathrm{e}^{it\partial_x^2}$ is contained in the closure of $C^{\infty}(\mathbb{T},\mathbb{C})$ in $L^{\infty}(\mathbb{T},\mathbb{C})$. As a consequence, the image of $C^{1/2-}(\mathbb{T},\mathbb{C})$ by $\mathrm{e}^{it\partial_x^2}$ consists of functions which are almost everywhere equal to a continuous function.

Consider a function ψ of bounded variation, which is not continuous, e.g., a step function. Then ψ is not almost everywhere equal to a continuous function. On the other hand, since $-t \notin N$, it follows from Theorem 4.1 that

$$\phi := \mathrm{e}^{-it\partial_x^2}\psi \in C^{1/2-}(\mathbb{T},\mathbb{C}),$$

and hence in contradiction to our choice of ψ, $\mathrm{e}^{it\partial_x^2}\phi = \psi$ would have to be almost everywhere equal to a continuous function. □

Let us now turn to the flow map $\mathcal{S}(t)$ of the Benjamin-Ono equation. In view of Corollary 4.3, we begin by studying the action of $\mathcal{S}(t)$ on $C^{1/2-}(\mathbb{T},\mathbb{C})$. To this end we need to establish the following auxiliary result on the action of $\mathcal{S}(t)$ on the Besov space $B^1_{1,1}(\mathbb{T},\mathbb{R})$. Recall that for any $s \geq 0$ and $p \geq 1$, $B^s_{p,1}(\mathbb{T},\mathbb{K})$, $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, is a Banach space, endowed with the norm

$$\|f\|_{B^s_{p,1}} := \sum_{j\geq 0} 2^{sj}\|P_j f\|_{L^p}, \tag{84}$$

where P_j, $j \geq 0$, are Littlewood-Paley projections (cf. e.g. [25, Apppendix 2.6]). Note that elements in $B^1_{1,1}(\mathbb{T},\mathbb{R})$ are absolutely continuous and hence of bounded variation. Furthermore, $B^1_{1,1}(\mathbb{T},\mathbb{R})$ is a Banach algebra.

Lemma 4.4. — *Let N be any set satisfying the conclusions of Theorem 4.1. Then for any $t \notin N$ and any $u_0 \in B^1_{1,1}(\mathbb{T},\mathbb{R})$ with $\langle u_0 \,|\, 1\rangle = 0$, $\mathcal{S}(t,u_0) \in C^{1/2-}(\mathbb{T},\mathbb{R})$.*

Proof. — For any given $u_0 \in B^1_{1,1}(\mathbb{T},\mathbb{R})$ with $\langle u_0 \,|\, 1\rangle = 0$, let

$$w_0 := -i\Pi(u_0\mathrm{e}^{-i\partial_x^{-1}u_0}).$$

Note that the Szegő projection Π maps $B^1_{1,1}(\mathbb{T},\mathbb{C})$ into itself (cf. e.g. [25, Apppendix 2.6]) and hence $w_0 \in B^1_{1,1}(\mathbb{T},\mathbb{C})$. Since any function in $B^1_{1,1}(\mathbb{T},\mathbb{C})$ is of bounded variation, it then follows by Theorem 4.1 that for any $t \notin N$,

$$w_L(t) := \mathrm{e}^{it(\partial_x^2 + \langle u_0^2 | 1\rangle)} w_0 \in C^{1/2-}(\mathbb{T},\mathbb{C}) .$$

Furthermore, $B^1_{1,1}(\mathbb{T},\mathbb{R}) \subset B^{1/2}_{2,1}(\mathbb{T},\mathbb{R}) \subset H^{1/2}(\mathbb{T},\mathbb{R})$, and therefore

$$u(t) := \mathcal{S}(t,u_0) \in H^{1/2}_r , \qquad \mathrm{e}^{i\partial_x^{-1} u(t)} \in H^{3/2}_r \subset \bigcap_{\alpha<1} C^{\alpha}(\mathbb{T},\mathbb{R}) .$$

As a consequence, by Theorem 1.1,

$$u(t) = 2\mathrm{Re}\left(\mathrm{e}^{i\partial_x^{-1} u(t)} i w_L(t)\right) + r(t) ,$$

where

$$r(t) \in H^{3/2-}_r \subset \bigcap_{\alpha<1} C^{\alpha}(\mathbb{T},\mathbb{R}) .$$

Altogether we thus proved that $u(t) \in C^{1/2-}(\mathbb{T},\mathbb{R})$. □

As already advertised above, the following proposition states a result on the action of the BO flow map on $C^{1/2-}(\mathbb{T},\mathbb{R})$. It should be compared with the result of Corollary 4.3 on the action of the flow map $\mathrm{e}^{it\partial_x^2}$ of the linear Schrödinger equation on $C^{1/2-}(\mathbb{T},\mathbb{C})$.

Proposition 4.5. — *Let N be any set satisfying the conclusions of Theorem 4.1. Then for any $t \notin N' := -N$, $\mathcal{S}(t)$ does not map $C^{1/2-}(\mathbb{T},\mathbb{R})$ into $\cup_{\varepsilon>0} C^{\varepsilon}(\mathbb{T},\mathbb{R})$.*

Proof. — Given any $u_0 \in B^1_{1,1}(\mathbb{T},\mathbb{R})$ with $\langle u_0 \,|\, 1\rangle = 0$, it follows from Lemma 4.4 that for any $t \notin N'$ (and hence $-t \notin N$), $v := \mathcal{S}(-t,u_0) \in C^{1/2-}(\mathbb{T},\mathbb{R})$. Since $\mathcal{S}(t,v) = u_0$, the proposition is proved by choosing

$$u_0 \in B^1_{1,1}(\mathbb{T},\mathbb{R}) \setminus \bigcup_{\varepsilon>0} C^{\varepsilon}(\mathbb{T},\mathbb{R}) .$$

A possible choice is

$$u_0(x) = \mathrm{Re}\Big(\sum_{j=0}^{\infty} \frac{1}{j^2} \mathrm{e}^{i2^j x} \chi_j(x)\Big) , \qquad \chi_j(x) := \sum_{k\in\mathbb{Z}} \chi(2^j(x - 2k\pi)) ,$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is C^∞-smooth with $\chi(0) = 1$ and support contained in the open interval $(-1/4, 1/4)$. □

Let us now turn to the main result of this section, which concerns the action of the BO flow map on $C^{\alpha-}(\mathbb{T},\mathbb{R})$ with $1/2 < \alpha < 1$. To prove this, we first extend Theorem 4.1 and Lemma 4.4. The extension of Theorem 4.1 is obtained from the proof of the latter in a straightforward way and reads as follows.

Corollary 4.6. — *Let N be any set satisfying the conclusions of Theorem 4.1. Then for any $t \notin N$ and any $1/2 < \alpha < 1$,*

$$\mathrm{e}^{it\partial_x^2} : B^{\alpha+1/2}_{1,1}(\mathbb{T},\mathbb{C}) \to C^{\alpha-}(\mathbb{T},\mathbb{C}) .$$

Proof. — Let $\beta := \alpha - 1/2$. Given $\psi \in B^{\alpha+1/2}_{1,1}(\mathbb{T},\mathbb{C})$, we write ψ as

$$\psi = \langle \psi \,|\, 1\rangle + |D|^{-\beta}\phi\,,$$

where $\phi := |D|^{\beta}\psi \in B^{1}_{1,1}(\mathbb{T},\mathbb{C})$. Then

$$\mathrm{e}^{it\partial_x^2}\psi = \langle \psi \,|\, 1\rangle + |D|^{-\beta}\mathrm{e}^{it\partial_x^2}\phi \in C^{\alpha-}(\mathbb{T},\mathbb{C})$$

This completes the proof. □

By the arguments used in its proof, Lemma 4.4 can be extended as follows.

Lemma 4.7. — *Let N be the set of Lebesgue measure zero of Theorem 4.1 and $1/2 < \alpha < 1$. Then for any $t \notin N$ and any $u_0 \in B^{\alpha+1/2}_{1,1}(\mathbb{T},\mathbb{R})$ with $\langle u_0 \,|\, 1\rangle = 0$, $S(t,u_0) \in C^{\alpha-}(\mathbb{T},\mathbb{R})$.*

Now we are ready to state our result on the action of $\mathcal{S}(t)$ on $C^{\alpha-}(\mathbb{T},\mathbb{R})$.

Proposition 4.8. — *For any $1/2 < \alpha < 1$ the following holds.*
(i) For any $t \in \mathbb{R}$, $\mathcal{S}(t)$ maps $C^{\alpha-}(\mathbb{T},\mathbb{R})$ continuously into $C^{(\alpha-1/2)-}(\mathbb{T},\mathbb{R})$.
(ii) Let N be any set satisfying the conclusions of Theorem 4.1. Then for any $t \notin N' := -N$, $\mathcal{S}(t)$ does not map $C^{\alpha-}(\mathbb{T},\mathbb{R})$ into $\bigcup_{\varepsilon>0} C^{\alpha-1/2+\varepsilon}(\mathbb{T},\mathbb{R})$.

Proof. — (i) Note that for any $0 < \beta < 1$, $C^{\beta}(\mathbb{T},\mathbb{R}) \subset H^{\beta-}_r := \cap_{\varepsilon>0} H^{\beta-\varepsilon}_r$. Furthermore, if $\beta > 1/2$, then $H^{\beta}_r \subset C^{\beta-1/2}(\mathbb{T},\mathbb{R})$ by the Sobolev embedding theorem. Now let $u_0 \in C^{\alpha-}(\mathbb{T},\mathbb{R})$ with $1/2 < \alpha < 1$. Then $u_0 \in H^{\alpha-}_r$ and hence $\mathcal{S}(t,u_0) \in H^{\alpha-}_r \subset C^{(\alpha-1/2)-}(\mathbb{T},\mathbb{R})$ for any $t \in \mathbb{R}$.
(ii) We argue as in the proof of Proposition 4.5. Given any $u_0 \in B^{\alpha+1/2}_{1,1}(\mathbb{T},\mathbb{R})$ with $\langle u_0 \,|\, 1\rangle = 0$, it follows from Lemma 4.7 that for any $t \notin N'$ (and hence $-t \notin N$), $v := \mathcal{S}(-t,u_0) \in C^{\alpha-}(\mathbb{T},\mathbb{R})$. Since $\mathcal{S}(t,v) = u_0$, the proposition is proved by choosing

$$u_0 \in B^{\alpha+1/2}_{1,1}(\mathbb{T},\mathbb{R}) \setminus \bigcup_{\varepsilon>0} C^{\beta+\varepsilon}(\mathbb{T},\mathbb{R})\,, \qquad \beta := \alpha - 1/2\,.$$

A possible choice is

$$u_0(x) = \mathrm{Re}\Big(\sum_{j=0}^{\infty} \frac{2^{-j\beta}}{j^2}\mathrm{e}^{i2^j x}\chi_j(x)\Big)\,, \qquad \chi_j(x) := \sum_{k\in\mathbb{Z}} \chi(2^j(x-2k\pi))\,,$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is C^{∞}-smooth with $\chi(0) = 1$ and support contained in the open interval $(-1/4, 1/4)$. □

Appendix A
Smoothing properties of Hankel operators

In this appendix, we record results on smoothing properties of Hankel operators, which are used throughout the paper. First we need to introduce some more notation. For any $s \in \mathbb{R}$, H^s_- denotes the Hardy space with Sobolev exponent s, i.e.,

$$H^s_- := \{f \in H^s_c \ : \ \widehat{f}(n) = 0 \ \ \forall n > 0\}$$

and Π^- the corresponding projection,

$$\Pi^- : H^s_c \to H^s_- \, , \ f = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{inx} \mapsto f = \sum_{n\leq 0} \widehat{f}(n) e^{inx} \, .$$

For any $u \in H^1_c$, denote by $H_u : H^0_- \to H^0_+$ and $H^-_u : H^0_+ \to H^0_-$ the Hankel operators with symbol u, defined as

$$H_u f := \Pi[uf], \quad \forall\, f \in H^0_- \, , \qquad H^-_u f := \Pi^-[uf], \quad \forall\, f \in H^0_+ \, .$$

Actually, H_u and H^-_u extend as bounded linear operators,

$$H_u : H^s_- \to H^s_+ \, , \qquad H^-_u : H^s_+ \to H^s_- \, ,$$

for any $-1/2 < s < 0$ (see e.g. [10, Lemma 1]). The following lemma shows that the operators H_u and H^-_u can be defined for symbols u in $H^{1/2}_c$ on appropriate Hardy spaces and that they have smoothing properties, which depend on s. For notational convenience, we itemize them according to the size of the gain of regularity. For $s < 1/2$ and $\alpha \geq 0$, let

$$\beta \equiv \beta(s, \alpha) := \alpha + s - \frac{1}{2} < \alpha \, . \tag{85}$$

Lemma A.1. — *(**Smoothing properties of Hankel operators**) For any $u \in H^{s+\alpha}_c$ and $f \in H^s_-$ with $s \in \mathbb{R}$, $\alpha \geq 0$, the following holds:*

(i) $\|H_u[f]\|_{s+\alpha} \lesssim_{s,\alpha} \|u\|_{s+\alpha} \, \|f\|_s$, *if* $s > \frac{1}{2}$, $\alpha \geq 0$.

(ii) $\|H_u[f]\|_{\frac{1}{2}+\alpha-\varepsilon} \lesssim_{\alpha,\varepsilon} \|u\|_{\frac{1}{2}+\alpha} \, \|f\|_{\frac{1}{2}}$, *if* $s = \frac{1}{2}$, $\alpha \geq 0$, $\varepsilon > 0$.

(iii) $\|H_u[f]\|_{s+\beta} \lesssim_{s,\alpha} \|u\|_{s+\alpha} \, \|f\|_s$, *if* $0 \leq s < \frac{1}{2}$, $\alpha \geq \frac{1}{2} - s$.

(iv) $\|H_u[f]\|_{s+\beta} \lesssim_{s,\alpha} \|u\|_{s+\alpha} \, \|f\|_s$, *if* $s < 0$, $\alpha \geq \frac{1}{2} - s$, $\alpha > -2s$,

where $\beta \equiv \beta(s, \alpha)$ is given by (85). Corresponding results hold for the operator H^-_u.

Proof.— Let $u = \sum_{k\in\mathbb{Z}} \widehat{u}(k) e^{ikx} \in H^{s+\alpha}_c$ and $f = \sum_{p\geq 0} \widehat{f}(-p) e^{-ipx} \in H^s_-$. Then with $n := -(k + p)$,

$$g(x) := \Pi \left(\sum_{k\in\mathbb{Z}, p\geq 0} \widehat{u}(-k) \widehat{f}(-p) e^{-i(k+p)x} \right) = \sum_{n\geq 0} \widehat{g}(n) e^{inx}$$

where

$$\widehat{g}(n) := \sum_{p\geq 0} \widehat{u}(n + p) \widehat{f}(-p) \, , \qquad \forall\, n \geq 0 \, .$$

By the Cauchy Schwarz inequality, one obtains

$$|\widehat{g}(n)|^2 \leq \|f\|_s^2 \sum_{p\geq 0} \frac{1}{\langle p\rangle^{2s}} |\widehat{u}(n+p)|^2 , \qquad \forall\, n \geq 0 ,$$

and thus for any $\gamma \in \mathbb{R}$*,*

$$\|g\|_{s+\gamma}^2 \leq \|f\|_s^2 \sum_{\ell\geq 0} |\widehat{u}(\ell)|^2 \sum_{p,n\geq 0, p+n=\ell} \frac{\langle n\rangle^{2(s+\gamma)}}{\langle p\rangle^{2s}} . \tag{86}$$

(i) In the case $s > 1/2$*,* $\alpha \geq 0$ *one has* $2s > 1$ *and hence*

$$\sum_{p,n\geq 0, p+n=\ell} \frac{\langle n\rangle^{2(s+\alpha)}}{\langle p\rangle^{2s}} \leq \langle \ell\rangle^{2(s+\alpha)} \sum_{0\leq p\leq \ell} \frac{1}{\langle p\rangle^{2s}} \lesssim_s \langle \ell\rangle^{2(s+\alpha)},$$

so that by (86)*,*

$$\|g\|_{s+\alpha} \lesssim_s \|u\|_{s+\alpha}\|f\|_s.$$

(ii) Recall that $s = 1/2$*,* $\alpha \geq 0$ *and note that without loss of generality, we can assume that* $0 < \varepsilon < 1/2$*. Then*

$$\sum_{p,n\geq 0, p+n=\ell} \frac{\langle n\rangle^{2(s+\alpha-\varepsilon)}}{\langle p\rangle^{2s}} \leq \langle \ell\rangle^{2(\frac{1}{2}+\alpha-\varepsilon)} \sum_{0\leq p\leq \ell} \frac{1}{\langle p\rangle} \lesssim \langle \ell\rangle^{2(\frac{1}{2}+\alpha-\varepsilon)} \log\langle \ell\rangle.$$

Hence (86) *implies that*

$$\|g\|_{\frac{1}{2}+\alpha-\varepsilon} \lesssim_{s,\varepsilon} \|u\|_{\frac{1}{2}+\alpha}\|f\|_{\frac{1}{2}} .$$

(iii) In the case $0 \leq s < 1/2$*,* $\alpha \geq 1/2 - s$*, one has* $1 - 2s > 0$ *and* $s + \beta = s + \alpha - (1/2 - s) \geq 0$*, implying that*

$$\sum_{p,n\geq 0, p+n=\ell} \frac{\langle n\rangle^{2(s+\beta)}}{\langle p\rangle^{2s}} \leq \langle \ell\rangle^{2(s+\beta)} \sum_{0\leq p\leq \ell} \frac{1}{\langle p\rangle^{2s}} \lesssim_s \langle \ell\rangle^{2(s+\beta)}\langle \ell\rangle^{1-2s}.$$

Since $s + \beta + 1/2 - s = s + \alpha$*, it then follows by* (86) *that*

$$\|g\|_{s+\beta} \lesssim_s \|u\|_{s+\alpha}\|f\|_s .$$

(iv) In the case $s < 0$ *and* $\alpha \geq \frac{1}{2} + |s|$*,* $\alpha > 2|s|$*, one has*

$$s + \beta = -\frac{1}{2} + \alpha - 2|s| > -1/2$$

and hence

$$\sum_{p,n\geq 0, p+n=\ell} \frac{\langle n\rangle^{2(s+\beta)}}{\langle p\rangle^{2s}} \leq \langle \ell\rangle^{-2s} \sum_{0\leq n\leq \ell} \langle n\rangle^{2(s+\beta)} \lesssim_s \langle \ell\rangle^{-2s}\langle \ell\rangle^{1+2s+2\beta} .$$

Since $1/2 + \beta = s + \alpha$*, it then follows by* (86) *that*

$$\|g\|_{s+\beta} \lesssim_s \|f\|_s \, \|u\|_{s+\alpha} . \qquad \square$$

Appendix B
Diffeomorphism properties of Tao's gauge transform

The aim of this appendix is to prove diffeomorphism properties of Tao's gauge transform. Without further reference, we will use the notation introduced in the main body of the paper.

Theorem B.1. — *For any $s \geq 0$, Tao's gauge transform*

$$\mathcal{G} : H^s_{r,0} \to H^s_{+,0},\ u \mapsto \partial_x \Pi(e^{-i\partial_x^{-1} u})\,,$$

is a real analytic diffeomorphism onto an open proper subset of $H^s_{+,0}$. Furthermore, the functions ine^{inx}, $n \geq 1$, do not belong to the range of $\mathcal{G}$ and the differential of $\mathcal{G}$ at $u = 0$ reads $d_0\mathcal{G} = -i\Pi$.

Before proving Theorem B.1, we make some preliminary considerations. For any given any $u \in L^2_{r,0}$, let $w = \mathcal{G}(u)$ and introduce

$$v := e^{-i\partial_x^{-1}\Pi u} \in H^1_+\,. \tag{87}$$

Since $u = \Pi u + \overline{\Pi u}$, one has

$$e^{-i\partial_x^{-1} u} = e^{-i\partial_x^{-1}\Pi u} \cdot e^{-i\partial_x^{-1}\overline{\Pi u}} = \frac{v}{\bar v} \tag{88}$$

and consequently,

$$\partial_x^{-1} w = \partial_x^{-1}\partial_x \Pi[e^{-i\partial_x^{-1} u}] = \Pi\left[\frac{v}{\bar v}\right] + a\,, \quad a := -\langle e^{-i\partial_x^{-1} u} | 1\rangle\,. \tag{89}$$

Given $g \in H^1_c$, we denote by $\check H_g$ the anti-linear Hankel operator of symbol g,

$$\check H_g : H_+ \to H_+,\ h \mapsto \Pi[g\bar h]\,.$$

Lemma B.2. — *For any $w \in \mathcal{G}(L^2_{r,0})$, the nullspace $\ker(\mathrm{Id} - \check H_{\partial_x^{-1} w})$ of the linear operator $\mathrm{Id} - \check H_{\partial_x^{-1} w} : H_+ \to H_+$ satisfies*

$$\ker(\mathrm{Id} - \check H_{\partial_x^{-1} w}) \cap H_{+,0} = \{0\}\,.$$

Proof of Lemma B.2. — Let h be an element in $\ker(\mathrm{Id} - \check H_{\partial_x^{-1} w}) \cap H_{+,0}$. Then $h = \Pi[(\partial_x^{-1} w)\bar h]$ and by (89)

$$\Pi\left[\frac{v}{\bar v}\,\bar h\right] = \Pi\left[\left(\Pi\left[\frac{v}{\bar v}\right]\right)\bar h\right] = \Pi[(\partial_x^{-1} w)\,\bar h] - a\,\Pi[\,\bar h\,]\,.$$

Since $h \in H_{+,0}$, one has $\Pi[\,\bar h\,] = \langle 1 | h\rangle = 0$. Using that $h = \Pi[(\partial_x^{-1} w)\bar h]$ it then follows that

$$\Pi\left[\frac{v}{\bar v}\,\bar h\right] = h\,(= \Pi h)\,.$$

Hence there exists $f \in H_{+,0}$ so that

$$\frac{v}{\bar v}\,\bar h = h + \bar f\,. \tag{90}$$

This implies that

$$\bar v\,\bar f = v\bar h - \bar v h = \overline{(\bar v h - v\,\bar h)} = -vf \in H_+\,, \tag{91}$$

where we used that by (87), v is in H^1_+. Consequently [1], vf is a constant function. Furthermore, since v and f both belong to H_+ and $\langle f|1\rangle = 0$,

$$\langle vf|1\rangle = \langle v|1\rangle \cdot \langle f\,|\,1\rangle = 0\,.$$

Therefore $vf = 0$ and in turn, since v does never vanish, $f = 0$. Coming back to (90), we conclude that

$$\frac{\bar h}{\bar v} = \frac{h}{v} \in H_+\,.$$

We thus again conclude that the function h/v is constant. Since

$$\left\langle \frac{h}{v}\Big|\, 1\right\rangle = \langle h|\,1\rangle \cdot \left\langle \frac{1}{v}\Big|\, 1\right\rangle = 0$$

we infer that $h = 0$. □

Proof of Theorem B.1. — In a first step we consider the case where $s = 0$. It is straightforward to verify that $\mathcal{G} : L^2_{r,0} \to H_{+,0}$ is real analytic and $d_0\mathcal{G} = -i\Pi$. Next we prove that for any integer $n \geq 1$, the function $f_n(x) := ine^{inx}$ is not an element in the image $\mathcal{G}(L^2_{r,0})$ of $\mathcal{G}$. In the case where $n \geq 2$ we argue as follows. Note that for $n \geq 2$,

$$h_n(x) := e^{ix} + e^{i(n-1)x} \in H_{+,0}$$

and since $\partial_x^{-1} f_n = e^{inx}$, one has $h_n - \check H_{\partial_x^{-1} f_n}[h_n] = 0$ and hence

$$h_n \in \ker(\mathrm{Id} - \check H_{\partial_x^{-1} f_n}) \cap H_{+,0}\,.$$

By Lemma B.2 one then concludes that $f_n \notin \mathcal{G}(L^2_{r,0})$. In the case $n = 1$ we have to argue differently since $h_1 := e^{ix} + 1$ is not an element in $H_{+,0}$. We note that $\mathcal{G}$ possesses the following scaling invariance : for any $u \in L^2_{r,0}$ and any integer $n \geq 1$,

$$\mathcal{G}(u_n)(x) = n\mathcal{G}(u)(nx)\,, \qquad u_n(x) := nu(nx)\,.$$

Consequently, if $f_1(x) = e^{ix}$ were to belong to the range of $\mathcal{G}$, then so would $f_n(x) = ne^{inx} = nf_1(nx)$ for any $n \geq 2$, in contradiction to what we just have proved.

Next we establish that $\mathcal{G} : L^2_{r,0} \to H_{+,0}$ is injective. Assume that $u_1,\, u_2 \in L^2_{r,0}$ satisfy $\mathcal{G}(u_1) = \mathcal{G}(u_2)$. Set

$$v_1 := e^{-i\partial_x^{-1}\Pi u_1} \in H^1_+\,, \qquad v_2 := e^{-i\partial_x^{-1}\Pi u_2} \in H^1_+\,.$$

By (88), the assumption $\mathcal{G}(u_1) = \mathcal{G}(u_2)$ can then be written as

$$\partial_x \Pi\left[\frac{v_1}{\bar v_1} - \frac{v_2}{\bar v_2}\right] = 0\,. \tag{92}$$

It means that there exists $f \in H_+$ so that

$$\frac{v_1}{\bar v_1} - \frac{v_2}{\bar v_2} = \bar f\,. \tag{93}$$

Arguing as in (91) it then follows that

$$\bar v_1\, \bar v_2\;\; \bar f = v_1\, \bar v_2 - v_2\, \bar v_1 = -\overline{(v_1\, \bar v_2 - v_2\, \bar v_1)} = -v_1 v_2\, f\,,$$

1. Throughout this appendix, we make frequent use of the elementary observation that any function $f \in H_+$ with $\overline{f} \in H_+$, is constant.

implying that $v_1 v_2\, f$ is a constant function, $v_1 v_2\, f = a \in \mathbb{C}$. Coming back to (93), we obtain

$$\frac{v_1}{v_2} - \frac{\bar v_1}{\bar v_2} = \frac{\bar a}{v_2\, \bar v_2} \,. \tag{94}$$

Note that

$$v_1 = 1 + r\,, \qquad r := \sum_{k=1}^{\infty} \frac{(-i\partial_x^{-1}\Pi u_1)^k}{k!} \in H_{+,0}\,. \tag{95}$$

Hence $\langle v_1 \,|\, 1\rangle = 1$ and in turn $\langle \bar v_1 \,|\, 1\rangle = 1$. Substituting $-u_2$ for u_1, (95) yields $\langle \frac{1}{v_2} \,|\, 1\rangle = 1$ and $\langle \frac{1}{\bar v_2} \,|\, 1\rangle = 1$. Finally, since v_1 and $1/v_2$ both belong to H_+, one has

$$\left\langle \frac{v_1}{v_2} \,\middle|\, 1 \right\rangle = \langle v_1 \,|\, 1\rangle \cdot \left\langle \frac{1}{v_2} \,\middle|\, 1 \right\rangle = 1\,, \tag{96}$$

and in turn $\langle \frac{\bar v_1}{\bar v_2} \,|\, 1\rangle = 1$. We then conclude from (94) that

$$0 = \left\langle \frac{\bar a}{v_2 \bar v_2} \,\middle|\, 1 \right\rangle = \bar a \left\| \frac{1}{v_2} \right\|^2 ,$$

implying that $a = 0$. By (94) it then follows that $\frac{v_1}{v_2} = \frac{\bar v_1}{\bar v_2}$ is a constant, which by (96) equals 1. We thus have proved that $v_1 = v_2$ and therefore

$$\Pi[u_1] = \frac{1}{v_1} i\partial_x v_1 = \frac{1}{v_2} i\partial_x v_2 = \Pi[u_2]\,,$$

yielding $u_1 = u_2$. This proves the injectivity of $\mathcal{G}$.

It remains to show that $\mathcal{G}$ is a local diffemorphism. As already pointed out, $\mathcal{G} : L^2_{r,0} \to H_{+,0}$ is a real analytic map. Hence by the inverse function theorem, we just need to prove that for any $u \in L^2_{r,0}$, $d_u\mathcal{G} : L^2_{r,0} \to H_{+,0}$ is a linear isomorphism.

An easy computation yields that for any $h \in L^2_{r,0}$,

$$d_u\mathcal{G}[h] = -i\partial_x \Pi[(\partial_x^{-1} h) e^{-i\partial_x^{-1} u}] = -i\partial_x \Pi\left[(\partial_x^{-1} h) \frac{v}{\bar v}\right]\,, \tag{97}$$

where $v := e^{-i\partial_x^{-1}\Pi u} \in H^1_+$ (cf. (88)). First we prove that $d_u\mathcal{G}$ is one-to-one. Assume that h belongs to the kernel of $d_u\mathcal{G}$. Then $h \in H_{+,0}$ and there exists $f \in H_+$ so that

$$(\partial_x^{-1} h) \frac{v}{\bar v} = \bar f\,.$$

It follows that

$$\bar v^2\, \bar f = \partial_x^{-1} h\, v\, \bar v$$

is real valued and belongs to H_+. Hence $\bar v^2\, \bar f$ is a constant function, $\bar v^2\, \bar f = a \in \mathbb{C}$, implying that

$$\partial_x^{-1} h = \frac{a}{v\, \bar v}\,.$$

Taking the inner products of both sides of the latter identity with 1, we get $a = 0$. Since $h \in H_{+,0}$, we conclude that $h = 0$, proving that $d_u\mathcal{G}$ is one-to-one.

It remains to show that $d_u\mathcal{G} : L^2_{r,0} \to H_{+,0}$ is onto. Since $d_u\mathcal{G}$ is one-to-one, it suffices to prove that for any $u \in L^2_{r,0}$, $d_u\mathcal{G}$ is a compact perturbation of a linear isomorphism. For any $h \in L^2_{r,0}$, one has

$$h = h_1 + h_2\,, \qquad h_1 = \Pi[h] \in H_{+,0}\,, \quad h_2 = \bar{h}_1 = (\mathrm{Id} - \Pi)[h]\,,$$

and hence by (97),

$$d_u\mathcal{G}[h] = L_1[h_1] + L_2[h_2] + L_3[h_1] + L_4[h_2]\,, \qquad \forall h \in L^2_{r,0}\,,$$

where $L_1, L_3 : H_+ \to H_+$ are the bounded linear operators,

$$L_1[g] := -i\Pi\left[\frac{v}{\bar{v}}\,g\right]\,, \qquad L_3[g] := -i\Pi\left[\partial_x\left(\frac{v}{\bar{v}}\right)\cdot \partial_x^{-1}g\right]\,,$$

and $L_2, L_4 : H_- \to H_+$ the bounded linear operators,

$$L_2[g] := -i\Pi\left[\frac{v}{\bar{v}}\,g\right]\,, \qquad L_4[g] := -i\Pi\left[\partial_x\left(\frac{v}{\bar{v}}\right)\cdot \partial_x^{-1}g\right]\,.$$

By the Sobolev embedding theorem and Rellich's theorem, the bounded linear operator $\partial_x^{-1} : L^2_{r,0} \to H^1_{r,0}$ gives rise to a compact linear operator $L^2_{+,0} \to L^\infty_{r,0}$, which we again denote by ∂_x^{-1}. Hence L_3 and L_4 are compact operators. Furthermore, $\Pi[\frac{v}{\bar{v}}] \in H^1_+$ and L_2 is the Hankel operator $H_{-i\Pi[\frac{v}{\bar{v}}]}$ with symbol $-i\Pi[\frac{v}{\bar{v}}]$,

$$L_2[g] = -iH_{\Pi[\frac{v}{\bar{v}}]}[g]\,, \qquad \forall\, g \in H_-\,.$$

By the smoothing properties of Hankel operators (cf. Lemma A.1(iii) in Appendix A with $\alpha = 1, s = 0, \beta = 1/2$) it then follows that $L_2 : H_- \to H_+$ is compact. Finally, $L_1 : H_+ \to H_+$ is a Toeplitz operator with symbol $-i\frac{v}{\bar{v}}$,

$$L_1[g] = -iT_{\frac{v}{\bar{v}}}[g]\,, \qquad \forall\, g \in H_+\,,$$

which is invertible with inverse given by (cf. e.g. [12, Lemma 6.5])

$$L_1^{-1}[f] = i\frac{1}{v}\,\Pi\left[\frac{1}{\bar{v}}\,f\right]\,, \qquad \forall\, f \in H_+\,.$$

Denote by Π_1 the projection

$$\Pi_1 : H_+ \to H_{+,0}\,,\; g \mapsto g - \langle g|1\rangle\,.$$

Since $d_u\mathcal{G} : L^2_{r,0} \to H_{+,0}$ it follows that for any $h \in L^2_{+,0}$, $d_u\mathcal{G}[h]$ equals

$$\Pi_1 \circ L_1[\Pi h] + \Pi_1 \circ L_3[\Pi h] + \Pi_1 \circ L_2[(\mathrm{Id} - \Pi)h] + \Pi_1 \circ L_4[(\mathrm{Id} - \Pi)h]$$

Clearly, the linear operators $\Pi_1 \circ L_3 \circ \Pi : L^2_{r,0} \to H_{+,0}$ and

$$\Pi_1 \circ L_2 \circ (\mathrm{Id} - \Pi) : L^2_{r,0} \to H_{+,0}\,, \quad \Pi_1 \circ L_4 \circ (\mathrm{Id} - \Pi) : L^2_{r,0} \to H_{+,0}$$

are compact. Furthermore, one verifies in a straightforward way that $\Pi_1 \circ L_1 \circ \Pi : L^2_{+,0} \to H_{+,0}$ is a linear isomorphism (cf. [12, Lemma 6.5]). Altogether, we thus have proved that $d\mathcal{G}(u) : L^2_{r,0} \to H_{r,0}$ is a compact perturbation of a linear isomorphism and hence a Fredholm operator of index zero. This completes the proof of Theorem B.1 in the case $s = 0$.

By the same arguments as in the above proof one verifies that for any $s > 0$, $\mathcal{G} : H^s_{r,0}(\mathbb{T}) \to H^s_{+,0}(\mathbb{T})$ is a diffeomorphism onto an open proper subset of $H^s_{+,0}(\mathbb{T})$. $\square$

The following considerations add to the results on the image of Tao's gauge transform of Theorem B.1. Consider the family of one gap potentials $u_\alpha \in \bigcap_{s\geq 0} H^s_{r,0}$, *given by*

$$(\Pi u_\alpha)(x) = \frac{\alpha e^{ix}}{1-\alpha e^{ix}} = i\partial_x \log(1-\alpha e^{ix}) \tag{98}$$

where $\alpha \in \mathbb{C}$ *satisfies* $0 < |\alpha| < 1$. *Such potentials, studied in* [8, Appendix B], *give rise to traveling wave solutions of the BO equation. They are one gap potentials in the sense that*

$$\gamma_1(u_\alpha) = \frac{|\alpha|^2}{1-|\alpha|^2}\,, \qquad \gamma_n(u_\alpha) = 0\,, \quad \forall\, n \geq 2\,.$$

Using the identity (88), *the definition of* $\mathcal{G}(u)$, *and the second identity in* (98) *one sees that*

$$\begin{aligned}\mathcal{G}(u_\alpha) = \partial_x \Pi\left[\frac{1-\alpha e^{ix}}{1-\bar\alpha e^{-ix}}\right] &= \partial_x\Pi\left[(1-\alpha e^{ix})\sum_{k\geq 0}(\bar\alpha e^{-ix})^k\right] \\ &= \partial_x\left((1-\alpha e^{ix}) - \alpha e^{ix}\bar\alpha e^{-ix}\right) = -i\alpha e^{ix}\,.\end{aligned}$$

In particular, it follows that any $\beta \in \mathbb{C}$ *with* $0 \leq |\beta| < 1$, βe^{ix} *is in the image* $\mathcal{G}(L^2_{r,0})$ *of* $\mathcal{G}$. *It is then natural to ask whether* βe^{ix} *is in* $\mathcal{G}(L^2_{r,0})$ *for some* $\beta \in \mathbb{C}$ *with* $|\beta| \geq 1$.

Proposition B.3. — *For any* $\beta \in \mathbb{C}$ *with* $|\beta| \geq 1$ βe^{ix} *is* not *in* $\mathcal{G}(L^2_{r,0})$.

Proof. — Assume that $u \in L^2_{r,0}$ has the property that $\mathcal{G}(u) = \beta e^{ix}$ where $\beta \in \mathbb{C}$. Following (87), define $v := e^{-i\partial_x^{-1}\Pi u} \in H^1_+$. By (88) one has $e^{-i\partial_x^{-1}u} = \frac{v}{\bar v}$, implying that (cf. (89))

$$\mathcal{G}(u) = \partial_x \Pi\left[\frac{v}{\bar v}\right]\,.$$

Applying ∂_x^{-1} to both sides of the latter identity, one concludes that there exists a constant $a \in \mathbb{C}$ so that $-i\beta e^{ix} = \Pi[\frac{v}{\bar v}] + a$. It means that there exists $f \in H_+$ so that $-i\beta e^{ix} - \frac{v}{\bar v} = \bar f$ or, multiplying both sides of the latter equation by $\overline{v}$,

$$-\,i\beta e^{ix}\,\bar v - v = \bar f\,\bar v\,. \tag{99}$$

Since by (95), $\langle v\,|\,1\rangle = 1$ and $\beta e^{ix}(\bar v - 1) \in \bar H_+$ and hence

$$-i\beta e^{ix}\bar v \in -i\beta e^{ix} + \bar H_+,$$

it then follows from (99) that $-i\beta e^{ix} - v \in \bar H_+$ and hence is a constant. Using that $\langle v\,|\,1\rangle = 1$ we then conclude that

$$v = 1 - i\beta e^{ix} \tag{100}$$

and hence

$$\partial_x v = \beta e^{ix} \tag{101}$$

On the other hand, one has $v = e^{-i\partial_x^{-1}\Pi u}$ and hence by (100)

$$\partial_x v = -iv\Pi u = -i(1 - i\beta e^{ix})\Pi u. \tag{102}$$

Combining (101) and (102) it then follows that

$$\Pi u = \frac{i\beta e^{ix}}{1 - i\beta e^{ix}} = \sum_{n\geq 1} (i\beta e^{ix})^n = \sum_{n\geq 1} (i\beta)^n e^{inx}.$$

Since by assumption $u \in L^2_{r,0}$, $\|\Pi u\|^2 = \sum_{n\geq 1} |\beta|^{2n} < \infty$ and hence we conclude that $|\beta| < 1$. $\square$

Appendix C
Approximation of the differential of Φ

So far, no high frequency approximation has been found for Φ^{-1}. Our goal is to derive such an approximation at least for the differential of Φ^{-1}. At the same time we derive a high frequency approximation for the differential of Φ. Such approximations are useful for analyzing the pullback of vector fields by the maps Φ and Φ^{-1}.

Denote by $\mathcal{F}^+_{1/2}$ the (partial) weighted Fourier transform

$$\mathcal{F}^+_{1/2} : H^s_c \to \mathfrak{h}^{s+\frac{1}{2}},\ u \mapsto \left(\frac{1}{\sqrt{n}}\widehat{u}(n)\right)_{n\geq 1}$$

and by $\mathcal{G}$ Tao's gauge transform, defined for any given $s \geq 0$ by

$$\mathcal{G} : H^s_{r,0} \to H^s_{+,0},\ u \mapsto \partial_x \Pi[\bar{g}_\infty]\,, \tag{103}$$

where we recall that $g_\infty \equiv g_\infty(\cdot, u) = e^{i\partial_x^{-1}u}$. By (28), Φ_0 can be expressed in terms of $\mathcal{G}$ and $\mathcal{F}^+_{1/2}$ as

$$\Phi_0(u) = \frac{1}{i}\mathcal{F}^+_{1/2}[\mathcal{G}(u)]\,. \tag{104}$$

By Theorem 1.6, $\frac{1}{i}\mathcal{F}^+_{1/2} \circ \mathcal{G}$ is a high frequency approximation of Φ. In more detail, for any $s \geq 0$, $u \mapsto \Phi(u) - \frac{1}{i}\mathcal{F}^+_{1/2}[\mathcal{G}(u)]$ is a continuous map from $H^s_{r,0}$ with values in $\mathfrak{h}^{s+\frac{1}{2}+\tau(s)}$. By [11, 12], for any $s \geq 0$, the Birkhoff map $\Phi : H^s_{r,0} \to \mathfrak{h}^{s+\frac{1}{2}}$ is a real analytic diffeomorphism and by Theorem B.1 in Appendix B, so is Φ_0 from $H^s_{r,0}$ onto an open proper subset of $\mathfrak{h}^{s+\frac{1}{2}}$. To state our high frequency approximation of the differential of Φ and of Φ^{-1} we introduce

$$\tau_2(s) := \begin{cases} 1 & \textit{if } \ s > 3/2 \\ 1- & \textit{if } \ s = 3/2 \\ \frac{s}{2} + \frac{1}{4} & \textit{if } \ 1/2 \leq s < 3/2 \\ s & \textit{if } \ 0 < s < 1/2 \end{cases} .$$

Note that for any $s > 0$,

$$\min\{s,\ s - \tau_2(s) + \tau(s - \tau_2(s))\} = s\ . \tag{105}$$

We then obtain the following corollary of Theorem 1.6 and Theorem B.1.

Corollary C.1. — *(i) For any $s \geq 0$, $\Phi - \Phi_0 : H^s_{r,0} \to \mathfrak{h}^{s+\frac{1}{2}+\tau(s)}$ is real analytic. As a consequence, for any $u \in H^s_{r,0}$, $s \geq 0$, $d_u\Phi_0$ is a high frequency approximation of $d_u\Phi$, i.e., for any $u \in H^s_{r,0}$ with $s \geq 0$,*

$$d_u\Phi - d_u\Phi_0 : H^s_{r,0} \to \mathfrak{h}^{s+\frac{1}{2}+\tau(s)}$$

is a bounded linear operator.
(ii) For any $u \in H^s_{r,0}$ with $s > 0$, $(d_u\Phi_0)^{-1}$ is a high frequency approximation of $(d_u\Phi)^{-1}$ in the sense that $(d_u\Phi)^{-1} - (d_u\Phi_0)^{-1}$ maps $\mathfrak{h}^{s+\frac{1}{2}-\tau_2(s)}$ into $H^s_{r,0}$ and

$$(d_u\Phi)^{-1} - (d_u\Phi_0)^{-1} : \mathfrak{h}^{s+\frac{1}{2}-\tau_2(s)} \to H^s_{r,0}$$

is bounded.

Proof. — (i) By the above considerations, $\Phi - \Phi_0 : H^s_{r,0} \to \mathfrak{h}^{s+\frac{1}{2}}$ is real analytic for any $s \geq 0$. In particular, each component of $\Phi - \Phi_0$ is a real analytic map $H^s_{r,0} \to \mathbb{C}$. Since by Theorem 1.6, $\Phi - \Phi_0 : H^s_{r,0} \to \mathfrak{h}^{s+\frac{1}{2}+\tau(s)}$ is continuous for any $s \geq 0$, one infers from [16, Theorem A.5] that $\Phi - \Phi_0 : H^s_{r,0} \to \mathfrak{h}^{s+\frac{1}{2}+\tau(s)}$ is real analytic.
(ii) For any given $u \in H^s_{r,0}$ with $s > 0$, introduce the linear operators

$$A(u) := d_u\Phi - d_u\Phi_0 : L^2_{r,0} \to \mathfrak{h}^{\frac{1}{2}} ,$$

$$B(u) := d_u\Phi^{-1} - d_u\Phi_0^{-1} : \mathfrak{h}^{\frac{1}{2}} \to L^2_{r,0} .$$

Note that

$$\begin{aligned} \mathrm{Id} &= d_u\Phi \circ (d_u\Phi)^{-1} = d_u\Phi \circ ((d_u\Phi_0)^{-1} + B(u)) \\ &= (d_u\Phi_0 + A(u)) \circ (d_u\Phi_0)^{-1} + d_u\Phi \circ B(u) \\ &= \mathrm{Id} + A(u) \circ (d_u\Phi_0)^{-1} + d_u\Phi \circ B(u) . \end{aligned}$$

It then follows that

$$B(u) = -(d_u\Phi)^{-1} \circ A(u) \circ (d_u\Phi_0)^{-1} .$$

Furthermore, by item (i) and (105), $A(u)$ maps $H^{s-\tau_2(s)}_{r,0}$ into $\mathfrak{h}^{s+\frac{1}{2}}$ and

$$A(u) : H^{s-\tau_2(s)}_{r,0} \to \mathfrak{h}^{s+\frac{1}{2}} , \tag{106}$$

is bounded. Since $(d_u\Phi_0)^{-1} : \mathfrak{h}^{s+\frac{1}{2}-\tau_2(s)} \to H^{s-\tau_2(s)}_{r,0}$ and $(d_u\Phi)^{-1} : \mathfrak{h}^{s+\frac{1}{2}} \to H^s_{r,0}$ are bounded linear operators, we then conclude that $B(u)$ maps $\mathfrak{h}^{s+\frac{1}{2}-\tau_2(s)}$ into $H^s_{r,0}$ and that $B(u) : \mathfrak{h}^{s+\frac{1}{2}-\tau_2(s)} \to H^s_{r,0}$ is bounded. □

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