AN EXPLICIT FORMULA FOR THE BENJAMIN–ONO EQUATION

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Abstract. We establish an explicit formula for the general solution of the Benjamin–Ono equation on the torus and on the line.

Contents

1. Introduction

1.1. **The Benjamin–Ono equation.** The Benjamin–Ono equation was introduced by Benjamin [1] (see also Davis–Acrivos [3], Ono [15]) to model long, one-way internal gravity waves in a two-layer fluid. It reads

$$\partial_t u = \partial_x(|D|u - u^2)\ . \tag{1}$$

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Here $u = u(t, x)$ denotes a real valued function and $|D|$ denotes the Fourier multiplier associated to the symbol $|\xi|$. There is a vast literature about this equation, and we refer to the book by Klein and Saut [13] for a recent survey. We consider both the case of periodic boundary conditions $u(t, x + 2\pi) = u(t, x)$, which we refer as $x \in \mathbb{T}$, and the case where $u(t, x)$ cancels as x tends to $\pm\infty$, which we refer as $x \in \mathbb{R}$. In both cases, we will restrict ourselves to sufficiently smooth solutions, which can be proved to exist globally by a combination of standard quasilinear scheme and appropriate conservation laws.
More precisely, for every $s \in \mathbb{R}$, let us denote by H^s the Sobolev space of tempered distributions with s derivatives in L^2, and by H^s_r the real subspace of H^s made of real valued distributions. Then one can prove the following result.

Theorem 1 (Saut, 1979 [16]). *For every $u_0 \in H^2_r$, there exists a unique solution $u \in C(\mathbb{R}, H^2_r)$ of (1) with $u(0) = u_0$.*

Our goal in this paper is to provide an explicit formula of the solution $u(t)$ in terms of the initial datum u_0.

For this, we need to introduce the Lax pair structure for (1).

1.2. **The Lax pair.** On $\mathbb{T}$ or $\mathbb{R}$, we denote by L^2_+ the Hardy space corresponding to L^2 functions having a Fourier transform supported in the domain $\xi \geq 0$. Recall that both Hardy spaces identify to some spaces of holomorphic functions. The space $L^2_+(\mathbb{T})$ identifies to holomorphic functions f on the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ such that

$$\sup_{r<1} \int_0^{2\pi} \left|f(re^{ix})\right|^2 \, dx < +\infty,$$

while $L^2_+(\mathbb{R})$ identifies to holomorphic functions on the upper half plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ such that

$$\sup_{y>0} \int_{\mathbb{R}} |f(x + iy)|^2 \, dy < +\infty \ .$$

We denote by Π the orthogonal projector from L^2 onto L^2_+. Remarkable operators on L^2_+ are Toeplitz operators, associated to functions $b \in L^\infty$ by the formula

$$\forall f \in L^2_+ \ , \ T_b f = \Pi(bf) \ .$$

For every $u \in H^2_r$, we denote by L_u the semi–bounded selfadjoint operator defined on L^2_+ by

$$\mathrm{Dom}(L_u) = H^1_+ := H^1(\mathbb{T}) \cap L^2_+ \ , \ L_u f = Df - T_u f \ ,$$

where

$$D := \frac{1}{i} \frac{d}{dx} \ .$$

We also consider, for $u \in H^2_r$, the bounded antiselfadjoint operator defined by

$$B_u = i(T_{|D|u} - T_u^2) \ .$$

Then one can check the following identity (see e.g. [4], [18], [6], [8]).

Theorem 2. *Under the conditions of Theorem 1, we have*

$$\forall t \in \mathbb{R} \ , \ \frac{d}{dt} L_{u(t)} = [B_{u(t)}, L_{u(t)}] \ .$$

Corollary 1. *Under the conditions of Theorem 1, denote by $U = U(t)$ the operator–valued solution of the linear ODE*

$$U'(t) = B_{u(t)} U(t) \ , \ U(0) = \mathrm{Id} \ .$$

Then, for every $t \in \mathbb{R}$, $U(t)$ is unitary on L^2_+, and

$$L_{u(t)} = U(t) L_{u(0)} U(t)^* \ .$$

1.3. **The explicit formula on the torus.** Let us mention some more properties of the Hardy space on the torus. The Hardy space $L^2_+(\mathbb{T})$ is equipped with the shift operator $S := T_{\mathrm{e}^{ix}}$ and with its adjoint $S^* = T_{\mathrm{e}^{-ix}}$. With this notation, our main result on the torus reads as follows.

Theorem 3. *Let $u \in C(\mathbb{R}, H^2_r(\mathbb{T}))$ be the solution of the Benjamin–Ono equation on the torus $\mathbb{T}$ with $u(0) = u_0$.*
Then $u(t) = \Pi u(t) + \overline{\Pi u}(t) - \langle u_0 | 1 \rangle$, with

$$\forall z \in \mathbb{D} \ , \ \Pi u(t,z) = \left\langle (\mathrm{Id} - z\mathrm{e}^{it}\mathrm{e}^{2itL_{u_0}} S^*)^{-1} \Pi u_0 | 1 \right\rangle \ .$$

Remark 1. *The above formula can be equivalently stated as a characterization of Fourier coefficients of the solution u,*

$$\begin{aligned} \forall k \geq 0 \ , \ \hat{u}(t,k) &= \left\langle (\mathrm{e}^{it}\mathrm{e}^{2itL_{u_0}} S^*)^k \Pi u_0 | 1 \right\rangle \\ &= \left\langle \Pi u_0 | (S\mathrm{e}^{-it}\mathrm{e}^{-2itL_{u_0}})^k \, 1 \right\rangle \ . \end{aligned}$$

Under this form, it extends to much more singular data, for which the flow of the Benjamin–Ono has been proved to extend continuously. According to [7], this is the case if u_0 belongs to $H^s_r(\mathbb{T})$ for some $s > -1/2$. Indeed, in this case, L_{u_0} is selfadjoint, semibounded, and the domain of the square root of $L_{u_0} + K\mathrm{Id}$, for K big enough, is the space $H^{1/2}_+ := H^{1/2}(\mathbb{T}) \cap L^2_+$. Consequently, the operator $S\mathrm{e}^{-it}\mathrm{e}^{-2itL_{u_0}}$ acts on $H^{1/2}_+(\mathbb{T})$, so that the inner product in the second line above is well defined.

1.4. **The explicit formula on the line.** On $L^2_+(\mathbb{R})$, the shift operator S must be replaced by the Lax–Beurling semi–group $(S(\eta))_{\eta \geq 0}$ of isometries defined as

$$\forall f \in L^2_+(\mathbb{R}) \ , \ S(\eta) f(x) = \mathrm{e}^{i\eta x} f(x) \ .$$

We denote by G the adjoint of the operator of multiplication by x on $L^2_+(\mathbb{R})$. Notice that $-iG$ is the infinitesimal generator of the adjoint semi–group of contractions $(S(\eta)^*)_{\eta\geq 0}$, so that

$$\forall \eta \geq 0\ ,\ S(\eta)^* = \mathrm{e}^{-i\eta G}\ .$$

It is easy to check that the domain of G consists of those functions $f \in L^2_+(\mathbb{R})$ such that the restriction of $\hat f$ to the half–line $(0,+\infty)$ belongs to the Sobolev space $H^1(0,+\infty)$, and that

$$\widehat{Gf}(\xi) = i\frac{d}{d\xi}[\hat f(\xi)]\, \mathbf{1}_{\xi>0}\ .$$

As a consequence, for every $f \in \mathrm{Dom}(G)$, one may define

$$I_+(f) := \hat f(0^+)\ .$$

This definition can be extended to any $f \in L^2_+$ such that the restriction of $\hat f$ to some interval $(0,\delta)$ belongs to the Sobolev space $H^1(0,\delta)$ for some $\delta > 0$, and we shall use it as well.
With this notation, our main result on the line reads as follows.

Theorem 4. *Let $u \in C(\mathbb{R}, H^2_r(\mathbb{R}))$ be the solution of the Benjamin–Ono equation on the line $\mathbb{R}$ with $u(0) = u_0$.*
Then $u(t) = \Pi u(t) + \overline{\Pi u}(t)$, with

$$\forall z \in \mathbb{C}_+\ ,\ \Pi u(t,z) = \frac{1}{2i\pi} I_+[(G - 2tL_{u_0} - z\mathrm{Id})^{-1}\Pi u_0]\ .$$

Notice that, in the above formula, the function

$$f_{z,t} := (G - 2tL_{u_0} - z\mathrm{Id})^{-1}\Pi u_0$$

belongs to the domain of $G - 2tL_{u_0}$ — see the end of section 3 for more detail —, and therefore its Fourier transform satisfies $\hat f \in H^1(0,\delta)$ for every finite $\delta > 0$, hence one can define $I_+(f_{z,t})$.

Remark 2. *At this time, the wellposedness theory for (1) on the line is slightly less advanced than on the torus, see [14] for a detailed account of this, with extension of the flow map to $L^2_r(\mathbb{R})$. However, one can easily prove – see section 3 below — that the above formula makes sense for u_0 in the space $L^\infty(\mathbb{R}) \cap L^2_r(\mathbb{R})$.*

1.5. **Organization of the paper.** Sections 2 and 3 are respectively devoted to the proofs of Theorems 3 and 4. The main idea is to take advantage of commutation identities between the operators of the shift structure of the Hardy space and the operators L_u and B_u of the Lax pair, in the spirit of what was done in [5] for the cubic Szegő equation on the torus. At the end of Section 3, we also provide a short discussion of the meaning of the formula, leading to an extension to real valued initial data in $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. Section 4 briefly draws possible applications and extensions to other equations.

2. Proof of the explicit formula on the torus

The proof is based on the following lemma.

Lemma 1. *For every* $u \in H^2_r(\mathbb{T})$,

$$[S^*, B_u] = i((L_u + \mathrm{Id})^2 S^* - S^* L_u^2) \ .$$

Let us postpone the proof of Lemma 1 and complete the proof of Theorem 3. Since $u(t)$ is real valued, we have the identity

$$u(t) = \Pi u(t) + \overline{\Pi u}(t) - \langle u(t)|1\rangle,$$

and $\langle u(t)|1\rangle = \langle u_0|1\rangle$ because of the equation. It remains to identify $\Pi u(t)$ as a holomorphic function on the disc. For this, we proceed as in [5], where a similar formula was established for the cubic Szegő equation. We have, for every $z \in \mathbb{D}$,

$$\Pi u(t,z) = \sum_{n=0}^{\infty} z^n \langle \Pi u(t), \mathrm{e}^{inx}\rangle = \langle (\mathrm{Id} - zS^*)^{-1}\Pi u(t)|1\rangle \ .$$

We denote by $U = U(t)$ the solution of the linear initial value problem in $\mathscr{L}(L^2_+(\mathbb{T}))$, (see Corollary 1),

$$U'(t) = B_{u(t)}U(t) \ , \ U(0) = \mathrm{Id} \ .$$

Since $B_{u(t)}$ is anti–selfadjoint, $U(t)$ is unitary, and we can write

$$\Pi u(t,z) = \langle (\mathrm{Id} - zU(t)^* S^* U(t))^{-1} U(t)^* \Pi u(t) | U(t)^* 1\rangle \ . \tag{2}$$

Let us calculate

$$\begin{aligned}
\frac{d}{dt}U(t)^*1 &= -U(t)^*B_{u(t)}1 = -iU(t)^*[(T_{|D|u(t)} - T^2_{u(t)})(1)] \\
&= -iU(t)^*[D\Pi u(t) - T_{u(t)}\Pi u(t)] = -iU(t)^* L_{u(t)}\Pi u(t) \\
&= iU(t)^* L^2_{u(t)}(1) = iL^2_{u_0}U(t)^*1 \ ,
\end{aligned}$$

where we have used Corollary 1, from which we conclude

$$U(t)^*1 = \mathrm{e}^{itL^2_{u_0}}(1) \ .$$

Consequently, using again Corollary 1,

$$U(t)^*\Pi u(t) = -U(t)^*L_{u(t)}(1) = -L_{u_0}U(t)^*(1) = -L_{u_0}\mathrm{e}^{itL^2_{u_0}}(1) = \mathrm{e}^{itL^2_{u_0}}\Pi u_0 \ .$$

Finally, using Lemma 1,

$$\begin{aligned}
\frac{d}{dt}U(t)^*S^*U(t) &= U(t)^*[S^*, B_{u(t)}]U(t) = U(t)^*[i((L_{u(t)} + \mathrm{Id})^2 S^* - S^* L^2_{u(t)}]U(t) \\
&= i(L_{u_0} + \mathrm{Id})^2 U(t)^*S^*U(t) - iU(t)^*S^*U(t)L^2_{u_0} \ ,
\end{aligned}$$

from which we infer

$$U(t)^*S^*U(t) = \mathrm{e}^{it(L_{u_0} + \mathrm{Id})^2} S^* \mathrm{e}^{-itL^2_{u_0}} \ .$$

Plugging the previous identites into (1), we conclude

$$\begin{aligned}\Pi u(t,z) &= \langle (\mathrm{Id} - z\mathrm{e}^{it(L_{u_0}+\mathrm{Id})^2}S^*\mathrm{e}^{-itL_{u_0}^2})^{-1}\mathrm{e}^{itL_{u_0}^2}\Pi u_0 | \mathrm{e}^{itL_{u_0}^2}(1)\rangle \\ &= \langle (\mathrm{Id} - z\mathrm{e}^{-itL_{u_0}^2}\mathrm{e}^{it(L_{u_0}+\mathrm{Id})^2}S^*)^{-1}\Pi u_0 | 1\rangle\end{aligned}$$

which yields the claimed formula. □

Finally, let us prove Lemma 1. First of all, it easy to check the following commutation identity with the Toeplitz operators,

$$\forall b \in L^\infty(\mathbb{T}), [S^*, T_b] = \langle\ .\ |1\rangle S^*\Pi b\ \ .$$

On the other hand, from the adjoint Leibniz formula, we have

$$S^*D = DS^* + S^*\ .$$

Combining the two above identities, we infer

$$S^*L_u = (L_u + \mathrm{Id})S^* - \langle\ .\ |1\rangle S^*\Pi u$$

and finally

$$\begin{aligned}[S^*, B_u] &= i([S^*, T_{|D|u}] - T_u[S^*, T_u] - [S^*, T_u]T_u) \\ &= i(\langle\ .\ |1\rangle S^*D\Pi u - T_u\langle\ .\ |1\rangle S^*\Pi u - \langle\ .\ (|1\rangle S^*\Pi u)T_u) \\ &= i(\langle\ .\ |1\rangle (DS^*\Pi u - T_uS^*\Pi u + S^*\Pi u) - \langle\ .\ |T_u1\rangle S^*\Pi u) \\ &= i(\langle\ .\ |1\rangle (L_uS^*\Pi u + S^*\Pi u) + \langle\ .\ |L_u1\rangle S^*\Pi u) \\ &= i((L_u + \mathrm{Id})\langle\ .\ |1\rangle S^*\Pi u + (\langle\ .\ |1\rangle S^*\Pi u)L_u) \\ &= i((L_u + \mathrm{Id})((L_u + \mathrm{Id})S^* - S^*L_u) + ((L_u + \mathrm{Id})S^* - S^*L_u)L_u) \\ &= i((L_u + \mathrm{Id})^2S^* - S^*L_u^2)\ .\end{aligned}$$

The proof of Theorem 3 is complete. □

3. Proof of the explicit formula on the line

We start with the inverse Fourier transform for every $f \in L^2_+(\mathbb{R})$, which we can write in the upper–half plane, as an absolutely convergent integral,

$$\forall z \in \mathbb{C}_+\ ,\ f(z) = \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{iz\xi}\hat{f}(\xi)\, d\xi\ ,$$

while, in view of the Plancherel theorem, we have, in $L^2(0, +\infty)$,

$$\hat{f}(\xi) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \mathrm{e}^{-ix\xi}\frac{f(x)}{1 + i\varepsilon x}\, dx = \lim_{\varepsilon\to 0}\langle S(\xi)^*f|\chi_\varepsilon\rangle\ ,$$

where χ_ε denotes the following function in $L^2_+(\mathbb{R})$,

$$\chi_\varepsilon(x) := \frac{1}{1 - i\varepsilon x}\ .$$

Plugging the second formula into the first one, we infer

$$\begin{aligned} f(z) &= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_0^\infty \mathrm{e}^{iz\xi}\langle S(\xi)^* f|\chi_\varepsilon\rangle d\xi \\ &= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_0^\infty \mathrm{e}^{iz\xi}\langle \mathrm{e}^{-i\xi G} f|\chi_\varepsilon\rangle d\xi \\ &= \lim_{\varepsilon\to 0}\frac{1}{2i\pi}\langle (G-z\mathrm{Id})^{-1}f|\chi_\varepsilon\rangle \\ &= \frac{1}{2i\pi}I_+[(G-z\mathrm{Id})^{-1}f]\ . \end{aligned}$$

Since $u(t)$ is real valued, we can write $u(t)=\Pi u(t)+\overline{\Pi u(t)}$, and it remains to characterize $\Pi u(t,z)$ for $z\in\mathbb{C}_+$. We are going to proceed as in the previous paragraph, using the family $U(t)$ of unitary operators in $\mathscr{L}(L^2_+(\mathbb{R}))$ defined by Corollary 1,

$$U'(t)=B_{u(t)}U(t)\ ,\ U(0)=\mathrm{Id}\ .$$

For every $z\in\mathbb{C}_+$, we have

$$\begin{aligned} \Pi u(t,z) &= \lim_{\varepsilon\to 0}\frac{1}{2i\pi}\langle U(t)^*(G-z\mathrm{Id})^{-1}\Pi u(t)|U(t)^*\chi_\varepsilon\rangle \\ &= \lim_{\varepsilon\to 0}\frac{1}{2i\pi}\langle (U(t)^*GU(t)-z\mathrm{Id})^{-1}U(t)^*\Pi u(t)|U(t)^*\chi_\varepsilon\rangle\ . \end{aligned}$$

We have the following lemma.

Lemma 2. *For every $u\in H^2_r(\mathbb{T})$,*

$$[G,B_u]=-2L_u+i[L_u^2,G]\ .$$

Let us postpone the proof of Lemma 2 and complete the proof of Theorem 4. We calculate

$$\begin{aligned} \frac{d}{dt}U(t)^*GU(t) &= U(t)^*[G,B_{u(t)}]U(t) \\ &= U(t)^*(-2L_{u(t)}+i[L^2_{u(t)},G])U(t) \\ &= -2L_{u_0}+i[L^2_{u_0},U(t)^*GU(t)]\ . \end{aligned}$$

Integrating this ODE, we get

$$U(t)^*GU(t)=-2tL_{u_0}+\mathrm{e}^{itL^2_{u_0}}G\mathrm{e}^{-itL^2_{u_0}}\ .$$

Let us determine the other terms in the inner product. First of all, we recall an identity coming directly from (1) (see also [17]),

$$\begin{aligned} \partial_t\Pi u &= \partial_x D\Pi u-\partial_x\Pi(u^2)=iD^2\Pi u-\partial_x((\Pi u)^2)-2\Pi\partial_x(|\Pi u|^2) \\ &= iL_u^2(\Pi u)+T_u\partial_x\Pi u+\partial_x(T_u\Pi u)-iT_u^2\Pi u-\partial_x((\Pi u)^2)-2\Pi\partial_x(|\Pi u|^2) \\ &= iL_u^2(\Pi u)+\Pi u\partial_x\Pi u+\Pi(\overline{\Pi u}\partial_x\Pi u)-\Pi\partial_x(|\Pi u|^2)-iT_u^2\Pi u \\ &= iL_u^2(\Pi u)+B_u(\Pi u)\ . \end{aligned}$$

We infer, using Corollary 1,

$$\frac{d}{dt}U(t)^*\Pi u(t) = U(t)^*(\partial_t \Pi u(t) - B_{u(t)}\Pi u(t)) = iU(t)^* L_{u(t)}^2 \Pi u(t) = iL_{u_0}^2 U(t)^*\Pi u(t) \ ,$$

from which we conclude

$$U(t)^*\Pi u(t) = \mathrm{e}^{itL_{u_0}^2}\Pi u_0 \ .$$

Finally, we have

$$\frac{d}{dt}U(t)^*\chi_\varepsilon = -U(t)^* B_{u(t)}\chi_\varepsilon = -iU(t)^*(T_{|D|u(t)}\chi_\varepsilon - T_{u(t)}^2\chi_\varepsilon)$$

and the right hand side converges in L^2_+ to

$$\begin{aligned}
&-iU(t)^*(D\Pi u(t) - T_{u(t)}\Pi u(t)) = -iU(t)^* L_{u(t)}\Pi u(t)\\
&= -iL_{u_0}U(t)^*\Pi u(t) = -iL_{u_0}\mathrm{e}^{itL_{u_0}^2}\Pi u_0\\
&= \lim_{\varepsilon\to 0} iL_{u_0}^2 \mathrm{e}^{itL_{u_0}^2}\chi_\varepsilon \ .
\end{aligned}$$

By integrating in time, we infer

$$U(t)^*\chi_\varepsilon - \mathrm{e}^{itL_{u_0}^2}\chi_\varepsilon \to 0$$

in L^2_+. Plugging these informations into the formula which gives $\Pi u(t,z)$, we infer

$$\begin{aligned}
\Pi u(t,z) &= \lim_{\varepsilon\to 0}\frac{1}{2i\pi}\langle\left(\mathrm{e}^{itL_{u_0}^2}G\mathrm{e}^{-itL_{u_0}^2} - 2tL_{u_0} - z\mathrm{Id}\right)^{-1}\mathrm{e}^{itL_{u_0}^2}\Pi u_0 | \mathrm{e}^{itL_{u_0}^2}\chi_\varepsilon\rangle\\
&= \lim_{\varepsilon\to 0}\frac{1}{2i\pi}\langle (G - 2tL_{u_0} - z\mathrm{Id})^{-1}\Pi u_0|\chi_\varepsilon\rangle\\
&= \frac{1}{2i\pi} I_+[(G - 2tL_{u_0} - z\mathrm{Id})^{-1}\Pi u_0] \ .
\end{aligned}$$

It remains to prove Lemma 2. We shall appeal to the following elementary identity, whose proof can be found in [17], [10].

Lemma 3. *For every $f \in \mathrm{Dom}(G)$, $b \in H^1(\mathbb{R})$, $T_b f \in \mathrm{Dom}(G)$ and*

$$[G, T_b]f = \frac{i}{2\pi} I_+(f)\Pi b \ .$$

Using Lemma 3 and the simple observation that $[G, D] = i\mathrm{Id}$, we obtain (see also [17]),

$$\forall f \in \mathrm{Dom}(G)\cap \mathrm{Dom}(L_u) \ , \ [G, L_u]f = if - \frac{i}{2\pi} I_+(f)\Pi u \ .$$

We infer, for $f \in \mathrm{Dom}(G) \cap \mathrm{Dom}(L_u^2)$,

$$\begin{aligned}[G, B_u]f &= i([G, T_{|D|u}]f - T_u[G, T_u]f - [G, T_u]T_u f) \\ &= \frac{i}{2\pi}(iI_+(f)(D\Pi u - T_u\Pi u) - iI_+(T_u f)\Pi u) \\ &= \frac{i}{2\pi}(iI_+(f)L_u\Pi u + iI_+(L_u f)\Pi u) \\ &= i(L_u(if - [G, L_u]f) + iL_u f - [G, L_u]L_u f) \\ &= -2L_u f + i[L_u^2, G]f \ .\end{aligned}$$

The proof of Theorem 4 is complete. □

Let us conclude this section by discussing the formula of Theorem 4 for more singular data u_0. First of all, let us observe that, for every $t \in \mathbb{R}$, the operator

$$A_t := -i(G - 2tL_0)$$

is maximally dissipative. Recall that we use the following definition : an unbounded operator A on a Hilbert space H is maximally dissipative if, for every $\lambda > 0$, $\lambda \mathrm{Id} - A : \mathrm{Dom}(A) \to H$ is onto and if, for every $f \in \mathrm{Dom}(A)$, $\mathrm{Re}\langle Af|f\rangle \le 0$. Indeed, the expression of A_t in the Fourier representation is given by

$$\widehat{A_t f}(\xi) = \frac{d}{d\xi}\hat{f}(\xi) + 2it\xi\hat{f}(\xi) \ ,$$

and therefore it is easy to check by explicit calculations that

$$\mathrm{Dom}(A_t) = \{f \in L^2_+(\mathbb{R}) : \mathrm{e}^{it\xi^2}\hat{f} \in H^1(0, \infty)\}$$

with

$$\forall f \in \mathrm{Dom}(A_t) \ , \ \mathrm{Re}\langle A_t f|f\rangle \le 0,$$

and that $A_t + iz\mathrm{Id} : \mathrm{Dom}(A_t) \to L^2_+(\mathbb{R})$ is bijective for every $z \in \mathbb{C}_+$. From standard perturbation theory, we infer that, for every bounded antiselfadjoint operator B on $L^2_+(\mathbb{R})$, $A_t + B$ is maximally dissipative. In particular, if $u_0 \in L^\infty(\mathbb{R}) \cap L^2_r(\mathbb{R})$,

$$-i(G - 2tL_{u_0}) = A_t - 2itT_{u_0}$$

is maximally dissipative, so that the formula of Theorem 4 still holds. Consequently, Theorem 4 provides a formula for the extension of the flow map of the Benjamin–Ono equation to $H^s_r(\mathbb{R})$ for every $s > 1/2$ [14].

4. Final remarks

In the case of finite gap potentials on the torus [6], or multisolitons on the line [17], formulae of Theorems 3 and 4 take place in finite dimensional vector spaces, and they reduce to calculations on finite dimensional matrices, as already observed in these references.

We expect Theorems 3 and 4 to be useful for the study of long time behaviour of solutions to the Benjamin–Ono equation. This is particularly important on the line, where soliton resolution for generic data is still an open problem (see however [12] for partial results in this direction).

Let us now briefly discuss applications of a similar approach to other integrable equations. First of all, it is clear that Theorems 3 and 4 easily extend to the spin Benjamin–Ono system [2], [8]. Furthermore, these formulae could probably be very useful in the study of the small dispersion limits of these equations, in particular the half–wave maps equation [9], [2]. We also expect similar formulae to hold for the recently introduced Calogero–Moser equation [10], since the Lax pair of operators for this equation enjoys similar commutation properties with the shift structure of the Hardy space. Finally, as we already observed, a similar formula is known to hold for the cubic Szegő equation on the torus [5], and it is possible to adapt the approach with the operator G developed in this paper in order to get an explicit formula for the cubic Szegő equation on the line [11]. On the other hand, we have no clue whether such explicit formulae could be extended to KdV, cubic NLS or DNLS equations.

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