Prescribing the behaviour of geodesics in negative curvature

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Given a family of (almost) disjoint strictly convex subsets of a complete negatively curved Riemannian manifold M, such as balls, horoballs, tubular neighbourhoods of totally geodesic submanifolds, etc, the aim of this paper is to construct geodesic rays or lines in M which have exactly once an exactly prescribed (big enough) penetration in one of them, and otherwise avoid (or do not enter too much in) them. Several applications are given, including a definite improvement of the unclouding problem of [PP1], the prescription of heights of geodesic lines in a finite volume such M, or of spiraling times around a closed geodesic in a closed such M. We also prove that the Hall ray phenomenon described by Hall in special arithmetic situations and by Schmidt-Sheingorn for hyperbolic surfaces is in fact only a negative curvature property.

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1 Introduction

The problem of constructing obstacle-avoiding geodesic rays or lines in negatively curved Riemannian manifolds has been studied in various different contexts. For example, Dani [Dan] and others [Str, AL, KW] have constructed (many) geodesic rays that are bounded (i.e. avoid a neighbourhood of infinity) in noncompact Riemannian manifolds. This work has deep connections with Diophantine approximation problems, see for instance the papers by Sullivan [Sul], Kleinbock-Margulis [KM] and Hersonsky-Paulin [HP5]. Hill and Velani [HV] and others (see for instance [HP3]) have studied the shrinking target problem for the geodesic flow. Schroeder [Schr] and others [BSW] have worked on the construction of geodesic lines avoiding given subsets, see also the previous work [PP1] of the authors on the construction of geodesic rays and lines avoiding a uniformly shrunk family of horoballs.

In this paper, we are interested in constructing geodesic rays or lines in negatively curved Riemannian manifolds which, given some family of obstacles, have exactly once an exactly prescribed (big enough) penetration in one of them, and otherwise avoid (or do not enter too much in) them. We also study an asymptotic version of this problem. This introduction contains a sample of our results (see also [PP2]).

Let H be either a horoball H of center ξ or a ball of center x and radius r in a $\mathrm{CAT}(-1)$ metric space (such as a complete simply connected Riemannian manifold of sectional curvature at most -1). For every $t \geq 0$, let $H[t]$ be the concentric horoball or ball contained in H, whose boundary is at distance t from the boundary of H (with $H[t]$ empty if H is a ball of radius r and $t > r$). The following result (see Section 4.1) greatly improves the main results, Theorem 1.1 and Theorem 4.5, of [PP1]. The fact that the constant μ_0 is universal (and not very big, though not optimal) is indeed remarkable.

Theorem 1.1 *Let X be a proper geodesic CAT(-1) metric space with arcwise connected boundary $\partial_\infty X$ and extendible geodesics, let $(H_\alpha)_{\alpha\in\mathscr{A}}$ be a family of balls or horoballs with pairwise disjoint interiors in X, and let $\mu_0 = 1.534$. For every x in $X - \bigcup_{\alpha\in\mathscr{A}} H_\alpha$, there exists a geodesic ray starting from x and avoiding $H_\alpha[\mu_0]$ for every α.*

From now on, we denote by M a complete connected Riemannian manifold with sectional curvature at most -1.

If M has finite volume and e is an end of M, let V_e be the maximal Margulis neighbourhood of e (see for instance [BK, Bow, HP5] and Section 5.1). If ρ_e is a minimizing geodesic ray in M starting from a point in the boundary of V_e and converging to e, let $\mathrm{ht}_e : M \to \mathbb{R}$ be the height map defined by $\mathrm{ht}_e(x) = \lim_{t\to\infty} (t - d(\rho_e(t), x))$. The *maximum height spectrum* $\mathrm{MaxSp}(M, e)$ of the pair (M, e) is the subset of $]-\infty, +\infty]$ consisting of elements of the form $\sup_{t\in\mathbb{R}} \mathrm{ht}_e(\gamma(t))$ where γ is a locally geodesic line in M.

As a consequence of Theorem 1.1 (see Corollary 4.4), we prove that if M is noncompact and has finite volume, then there exist universally low closed geodesics in M.

From now on, we assume that the dimension of M is at least 3. The following statements are true or expected to be true in the constant curvature 2-dimensional case, but are expected to be false in variable curvature and dimension 2. We first have the following result on the upper part of the maximum height spectrum.

Theorem 1.2 *If M has finite volume and e is an end of M, then* $\mathrm{MaxSp}(M, e)$ *contains the interval* $[4.2, +\infty]$.

For more precise analogous statements when M is geometrically finite, and for finite subsets of cusps of M, see Section 5.1. Schmidt and Sheingorn [SS] proved the two-dimensional analog of Theorem 1.2 in constant curvature -1. They showed that the maximum height spectrum of a finite area hyperbolic surface with respect to any cusp contains the interval $[4.61, +\infty]$.

The previous result is obtained by studying the penetration properties of geodesic lines in a family of horoballs. Our next theorem concerns families of balls (see Section 5.1 for generalizations). See for instance [HP6] for the almost everywhere properties of the geodesic lines passing at very small distance from a given point.

Theorem 1.3 *Let x be point in M with $r = \operatorname{inj}_M x \geq 56$. Then, for every $d \in [2, r-54]$, there exists a locally geodesic line γ passing at distance exactly d from x at time 0 and remaining at distance greater than d from x at any nonzero time.*

Given a closed geodesic L in M, the behaviour of a locally geodesic ray γ in M with respect to L is typically that γ spirals around L for some time, then wanders away from L, then spirals again for some time around L, then wanders away, etc. Our next aim is to construct such a γ which has exactly one (big enough) exactly prescribed spiraling length, and all of whose other spiraling lengths are bounded above by some uniform constant. Let us make this precise.

Let L be an embedded compact totally geodesic submanifold in M with $1 \leq \dim L \leq \dim M - 1$, and $\epsilon > 0$ small enough so that the (closed) ϵ-neighbourhood $\mathscr{N}_\epsilon L$ of L is a tubular neighbourhood. For every locally geodesic line γ in M, the set of $t \in \mathbb{R}$ such that $\gamma(t)$ belongs to $\mathscr{N}_\epsilon L$ is the disjoint union of maximal closed intervals $[s_n, t_n]$, with $s_n \leq t_n < s_{n+1}$. Let $\widetilde{\gamma}$ be any lift of γ to a Riemannian universal cover of M. Let $\widetilde{C}_n$ be the lift of C at distance at most ϵ from $\widetilde{\gamma}(s_n)$. Let $p_{\widetilde{\gamma}_-}$ and $p_{\widetilde{\gamma}_+}$ be the orthogonal projections on $\widetilde{C}_n$ of the points at infinity of $\widetilde{\gamma}$. The distance between $p_{\widetilde{\gamma}_-}$ and $p_{\widetilde{\gamma}_+}$ will be called a *fellow-traveling time* of γ along L (see Section 5.2).

Theorem 1.4 *Let L be as above. There exist constants $c, c' > 0$, depending only on ϵ, such that for every $h \geq c$, there exists a locally geodesic line in M, having one fellow-traveling time exactly h, all others being at most c'.*

See Section 5.2 for an extension of Theorem 1.4 when L is not necessarily embedded, and to finitely many disjoint such neighbourhoods $\mathscr{N}_\epsilon L$. If M has finite volume, we also construct bounded locally geodesic lines with the above property (with a control of the heights uniform in ϵ). In constant curvature, we can also prescribe one of the penetration lengths $|t_n - s_n|$ at least c, while keeping all the other ones at most c'. Schmidt and Sheingorn [SS] sketch a proof of a result for hyperbolic surfaces which is analogous to Theorem 1.4 with a different way of measuring the affinity of locally geodesic lines. Other results about the spiraling properties of geodesic lines around closed geodesics are given in [HP6, PP3].

For our next result, we specialize to the case where M is a hyperbolic 3-manifold. See Section 5.3 for a more general statement, and for instance [MT] for references on 3-manifolds and Kleinian groups.

Theorem 1.5 *Let N be a compact, connected, orientable, irreducible, acylindrical, atoroidal, boundary incompressible 3-manifold with boundary, with ∂N having exactly one torus component e. For every compact subset K in the space $\mathcal{GF}(N,e)$ of (isotopy classes of) complete geometrically finite hyperbolic metrics in the interior of N with one cusp, there exists a constant $c \geq 0$ such that for every $h \geq c$ and every $\sigma \in K$, there exists a locally geodesic line γ contained in the convex core of σ such that the maximum height of γ is exactly h.*

If M has finite volume and e is an end of M, define the *asymptotic height spectrum* $\mathrm{LimsupSp}(M,e)$ of the pair (M,e) to be the subset of $]-\infty,+\infty]$ consisting of elements of the form $\limsup_{t\in\mathbb{R}} \mathrm{ht}_e(\gamma(t))$ where γ is a locally geodesic line in M.

Theorem 1.6 (The ubiquity of Hall rays) *If M has finite volume and e is an end of M, then* $\mathrm{LimsupSp}(M,e)$ *contains* $[6.8,\infty]$.

The interval given by Theorem 1.6 is called a *Hall ray*. Note that the value 6.8 is uniform on all couples (M,e), but we do not know the optimal value. If M is the one-ended hyperbolic 2-orbifold $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$ where $\mathbb{H}^2_{\mathbb{R}}$ is the real hyperbolic plane with sectional curvature -1, then the existence of a Hall ray follows from the work of Hall [Hal1, Hal2] on continued fractions. Freiman [Fre] (see also [Slo]) has determined the maximal Hall ray of $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$, which is approximately $[0.8,+\infty]$. The generality of Theorem 1.6 proves in particular that the Hall ray phenomenon is neither an arithmetic nor a constant curvature property. See Section 5.4 for a more precise version of Theorem 1.6, which is valid also in the geometrically finite case.

The results of Hall and Freiman cited above were originally formulated in terms of Diophantine approximation of real numbers by rationals. The projective action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on the upper halfplane provides a way to obtain the geometric interpretation. We conclude this sample of our results by giving applications of our methods to Diophantine approximation problems (see Section 6 for generalizations in the framework of Diophantine approximation on negatively curved manifolds, developped in [HP3, HP4, HP5]). These results were announced in [PP2].

Theorem 1.7 *Let m be a squarefree positive integer, and let $\mathscr{I}$ be a non-zero ideal in an order $\mathscr{O}$ in the ring of integers $\mathscr{O}_{-m}$ of the imaginary quadratic number field $\mathbb{Q}(i\sqrt{m})$. For every $x \in \mathbb{C} - \mathbb{Q}(i\sqrt{m})$, let*

$$c(x) = \liminf_{(p,q)\in\mathscr{O}\times\mathscr{I},\ \langle p,q\rangle=\mathscr{O},\ |q|\to\infty} \quad |q|^2\Big|x - \frac{p}{q}\Big|$$

be the approximation constant of the complex number x by elements of $\mathscr{O}\mathscr{I}^{-1}$, and Sp_{Lag} the Lagrange spectrum consisting of the real numbers of the form $c(x)$ for some $x \in \mathbb{C} - \mathbb{Q}(i\sqrt{m})$. Then Sp_{Lag} contains the interval $[0\,,\,0.0005]$.

Theorem 1.7 follows from Hall's result and from the work of Poitou [Poi] in the particular case $\mathscr{I} = \mathscr{O} = \mathscr{O}_{-m}$. Other arithmetic applications of our geometric methods can be obtained by varying the (nonuniform) arithmetic lattice in the isometry group of a negatively curved symmetric space. We only state the following result in this introduction (with the notation of Section 6.1), see Section 6.4 and [PP2] for other ones.

Theorem 1.8 *Let $\mathscr{Q}(\mathbb{R})$ be the real quadric $\{(z,w) \in \mathbb{C}^2 \;:\; 2\,\mathrm{Re}\ z - |w|^2 = 0\}$ endowed with the Lie group law $(z,w)\cdot(z',w') = (z+z'+w'\overline{w}, w+w')$ and $\mathscr{Q}(\mathbb{Q}) = \mathscr{Q}(\mathbb{R})\cap\mathbb{Q}(i)^2$ be its rational points. If $r = (p/q, p'/q) \in \mathscr{Q}(\mathbb{Q})$ with $p,p',q \in \mathbb{Z}[i]$ relatively prime, let $h(r) = |q|$. Let d'_{Cyg} be the left-invariant distance on $\mathscr{Q}(\mathbb{R})$ such that $d'_{\mathrm{Cyg}}((z,w),(0,0)) = \sqrt{2|z| + |w|^2}$. For every $x \in \mathscr{Q}(\mathbb{R}) - \mathscr{Q}(\mathbb{Q})$, let*

$$c(x) = \liminf_{r\in\mathscr{Q}(\mathbb{Q}),\ h(r)\to\infty} h(r)\ d'_{\mathrm{Cyg}}(x,r)$$

be the approximation constant of x by rational points, and Sp_{Lag} the Lagrange spectrum consisting of the real numbers of the form $c(x)$ for some $x \in \mathscr{Q}(\mathbb{R}) - \mathscr{Q}(\mathbb{Q})$. Then Sp_{Lag} contains the interval $[0\,,\,0.001]$.

The paper is organized as follows. In Section 2, we define a class of uniformly strictly convex subsets of metric spaces, that we call *ϵ-convex subsets*. We study the interaction of geodesic rays and lines with ϵ-convex sets in CAT(-1)-spaces. In particular, we give various estimates on the distance between the entering and exiting points in an ϵ-convex set of two geodesic rays starting from a fixed point in the space and of two geodesic lines starting from a fixed point in the boundary at infinity. Section 3 is devoted to defining and studying several penetration maps which are used to measure the penetration of geodesic rays and lines in an ϵ-convex set. We emphasize the case of penetration maps in horoballs, balls and tubular neighbourhoods of totally geodesic submanifolds. We show that in a number of geometrically interesting cases, it is possible to adjust the penetration of a geodesic line or ray in one ϵ-convex set while keeping the penetration in another set fixed. Section 4 contains the inductive construction that gives geodesic rays and lines with prescribed maximal penetration with respect to a given collection of ϵ-convex sets. As a warm-up for the construction, we prove Theorem 1.1 in Subsection 4.1. The other theorems in the introduction besides the last two and a number of others are proved in Section 5 where the results of Section 4 are applied in the cases studied in Section 3. Finally, we give our arithmetic applications in Section 6.

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2 On strict convexity in CAT(-1) spaces

2.1 Notations and background

In this section, we introduce some of the objects which are central in this paper. We refer to [BH, GH] for the definitions and basic properties of CAT(-1) spaces. Our reference for hyperbolic geometry is [Bea].

Let (X,d) be a proper geodesic CAT(-1) metric space, and $X\cup\partial_\infty X$ be its compactification by the asymptotic classes of geodesic rays. By a *geodesic line* (resp. *ray* or *segment*) in X, we mean an isometric map $\gamma:\mathbb{R}\to X$ (resp. $\gamma:[\iota_\gamma,+\infty[\;\to X$ with $\iota_\gamma\in\mathbb{R}$ or $\gamma:[a,b]\;\to X$, with $a\leq b$). We sometimes also denote by γ the image of this map. For x,y in X, we denote by $[x,y]$ the (unique) closed geodesic segment between x,y, with the obvious extension to open and half-open geodesic segments, rays and lines (with one or two endpoints in $\partial_\infty X$). We say that X *has extendible geodesics* if every geodesic segment can be extended to a geodesic line.

We denote by T^1X the space of geodesic lines in X, endowed with the compact-open topology. When X is a Riemannian manifold, the space T^1X coincides with the usual definition of the unit tangent bundle, upon identifying a geodesic line γ and its (unit) tangent vector $\dot\gamma(0)$ at time $t=0$. For every geodesic ray or line γ, we denote by $\gamma(+\infty)$ the point of $\partial_\infty X$ to which $\gamma(t)$ converges as $t\rightarrow+\infty$, and we define $\gamma(-\infty)$ similarly when γ is a geodesic line. We say that a geodesic line (resp. ray) γ *starts* from a point $\xi\in\partial_\infty X$ (resp. $\xi\in X$) if $\xi=\gamma(-\infty)$ (resp. $\gamma(\iota_\gamma)=\xi$). For every ξ in $X\cup\partial_\infty X$, we denote by $T^1_\xi X$ the space of geodesic lines (if $\xi\in\partial_\infty X$) or rays (if $\xi\in X$) starting from ξ, endowed with the compact-open topology.

If Y is a subset of X and ξ a point in $X\cup\partial_\infty X$, the *shadow of Y seen from ξ* is the set $\mathscr{O}_\xi Y$ of points $\gamma(+\infty)$ where γ is a geodesic ray or line starting from ξ and meeting Y.

The *Busemann function* $\beta_\xi : X\times X\rightarrow\mathbb{R}$ at a point ξ in $\partial_\infty X$ is defined by

$$\beta_\xi(x,y)=\lim_{t\rightarrow+\infty}\big(d(x,\rho(t))-d(y,\rho(t))\big),$$

where ρ is any geodesic ray ending at ξ. The function $y\mapsto\beta_\xi(x,y)$ can be thought of as a normalized signed distance to $\xi\in\partial_\infty X$, or as the *height* of the point y with respect to ξ (relative to y). Accordingly, if $\beta_\xi(x,y)=\beta_\xi(x,y')$, then the points y and y' are said to be *equidistant* to ξ. If $\xi\in X$, we define

$$\beta_\xi(x,y)=d(x,\xi)-d(y,\xi)\ .$$

This is convenient in Section 4.2 and in the proof of Corollary 5.5. For every x,y,z in X and $\xi\in X\cup\partial_\infty X$, we have

$$\beta_\xi(x,y)+\beta_\xi(y,z)=\beta_\xi(x,z),$$

$\beta_\xi(x,x)=0$, and $|\beta_\xi(x,y)|\leq d(x,y)$.

A *horoball* in X centered at $\xi\in\partial_\infty X$ is the preimage of $[s,+\infty[$ for some s in $\mathbb{R}$ by the map $y\mapsto\beta_\xi(x,y)$ for some x in X. If

$$H=\{y\in X\ :\ \beta_\xi(x,y)\geq s\}$$

is a horoball, we define its *boundary horosphere* by

$$\partial H=\{y\in X\ :\ \beta_\xi(x,y)=s\},$$

and for every $t\geq0$, its *t-shrunk horoball* by

$$H[t]=\{y\in X\ :\ \beta_\xi(x,y)\geq s+t\}.$$

(In [PP1], we denoted $H[t]$ by $H(t)$.) Similarly, if B is a ball of center x and radius r, for every $t\leq r$, we denote by $B[t]$ the ball of center x and radius $r-t$. By convention, if $t>r$, define $B[t]=\emptyset$. Note that for every ball or horoball H, we have $H[t']\subset H[t]$ if $t'\geq t$. The point at infinity of a horoball H is denoted by $H[\infty]$. Note that, in this paper, all balls and horoballs in X are assumed to be closed.

Recall that a subset C in a $\mathrm{CAT}(-1)$ metric space is *convex* if C contains the geodesic segment between any two points in C. Let C be a convex subset in X. We denote by $\partial_\infty C$ its set of points at infinity, and by ∂C its boundary in X. If C is nonempty and closed, for every ξ in $\partial_\infty X$, we define *the closest point to ξ on the convex set C* to be the following point p in $C \cup \partial_\infty C$: if $\xi \notin \partial_\infty C$, then p belongs to C and maximizes the map $y \mapsto \beta_\xi(x_0, y)$ for some (hence any) given point x_0 in X; if $\xi \in \partial_\infty C$, then we define $p = \xi$. This point p exists, is unique, and depends continuously on ξ, by the properties of $\mathrm{CAT}(-1)$-spaces.

If $x, y, z \in X \cup \partial_\infty X$, we denote by (x, y, z) the triangle formed by the three geodesic segments, rays or lines with endpoints in $\{x, y, z\}$. Recall that if $\alpha : t \mapsto \alpha_t$ and $\beta : t \mapsto \beta_t$ are two (germs of) geodesic segments starting from a point x_0 in X at time $t = 0$, if $(\overline{x}_0, \overline{\alpha}_t, \overline{\beta}_t)$ for $t > 0$ small enough is a comparison triangle for (x_0, α_t, β_t) in the real hyperbolic plane $\mathbb{H}^2_\mathbb{R}$, then the *comparison angle* between α and β at x_0 is the limit, which exists, of the angle $\angle_{\overline{x_0}}(\overline{\alpha_t}, \overline{\beta_t})$ as t tends to 0.

If $x, y \in X$ and $\xi \in \partial_\infty X$, then a triple $(\overline{x}, \overline{y}, \overline{\xi})$ with $\overline{x}, \overline{y} \in \mathbb{H}^2_\mathbb{R}$, $\overline{\xi} \in \partial_\infty \mathbb{H}^2_\mathbb{R}$, $d(\overline{x}, \overline{y}) = d(x, y)$ and $\beta_{\overline{\xi}}(\overline{x}, \overline{y}) = \beta_\xi(x, y)$ is called a *comparison triangle* for (x, y, ξ). Clearly, this comparison triangle exists, and is unique up to isometry. The natural map from $]\overline{\xi}, \overline{x}] \cup [\overline{x}, \overline{y}] \cup [\overline{y}, \overline{\xi}[$ to $]\xi, x] \cup [x, y] \cup [y, \xi[$ is 1-Lipschitz, and for every $z \in [x, y]$, if $\overline{z}$ is its corresponding point on $[\overline{x}, \overline{y}]$, then $\beta_\xi(z, x) \leq \beta_{\overline{\xi}}(\overline{z}, \overline{x})$.

We end this section with the following (well known) exercises in hyperbolic geometry.

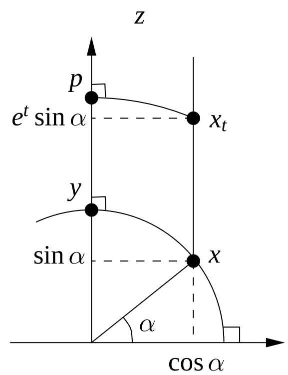
Lemma 2.1 *For all points x, y in X and z in $X \cup \partial_\infty X$, and every t in $[0, d(x, z)]$ (finite if $z \in \partial_\infty X$), if x_t is the point on $[x, z]$ at distance t from x, then*

$$d(x_t, [y, z]) \leq e^{-t} \sinh d(x, y) \leq \frac{1}{2}\, e^{-t + d(x,y)} \,.$$

Proof. By comparison, we may assume that $X = \mathbb{H}^2_\mathbb{R}$. As it does not decrease $d(x_t, [y, z])$ to replace z by the point at infinity of the geodesic ray starting from x and passing through z, we may assume that z is the point at infinity in the upper half-space model of $\mathbb{H}^2_\mathbb{R}$. Let p be the orthogonal projection of x_t on the geodesic line γ through y and z. Assume first that p belongs to $[y, z[$.

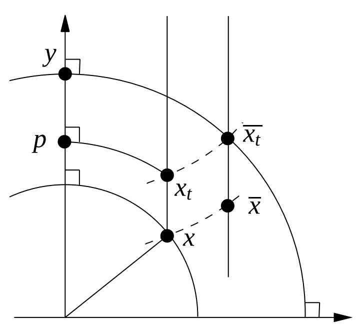
If we replace y by the orthogonal projection of x on γ, then we decrease $d(x, y)$, and do not change t and $d(x_t, [y, z])$. Hence we may assume that $y = i$ and x is on the (Euclidean) circle of center 0 and radius 1. If α is the (Euclidean) angle at 0 between the horizontal axis and the (Euclidean) line from 0 passing through x, then an easy computation in hyperbolic geometry (see also [Bea], page 145) gives $\sinh d(x,y) = \cos\alpha/\sin\alpha$. Similarly, $\sinh d(x_t,[y,z]) = \cos\alpha/(e^t\sin\alpha)$. So that

$$d(x_t,[y,z]) \leq \sinh d(x_t,[y,z]) = e^{-t}\sinh d(x,y)\ .$$



Assume now that p does not belong to $[y,z[$. In particular, $y \neq x_t$. Let $\overline{x_t}$ be the point at same distance from y as x_t (and on the same side) such that y is the orthogonal projection of $\overline{x_t}$ on γ, so that

$$d(\overline{x_t},[y,z]) = d(\overline{x_t},y) = d(x_t,y) = d(x_t,[y,z]).$$



Let $\overline{x}$ be the intersection of the geodesic line from z through $\overline{x_t}$ with the (hyperbolic) circle of center y and radius $d(x,y)$, so that $d(\overline{x},y) = d(x,y)$. Then, with $\bar{t} = d(\overline{x_t},\overline{x})$, we have $\bar{t} \geq t$, as the angle at $\overline{x_t}$ of $[\overline{x_t},\overline{x}]$ with the outgoing unit vector of the geodesic ray from y through $\overline{x_t}$ is bigger than the corresponding one for x_t and x. Hence we may assume that $x_t = \overline{x_t}$ and $x = \overline{x}$. As then the orthogonal projection of x_t on the geodesic line through y and z is y, this reduces the situation to the first case treated above. □

Lemma 2.2 *For every $\epsilon > 0$, if*

(- 1 -) $$c_0(\epsilon) = 2\log\left(\frac{2(1+e^{\epsilon/2})\sinh\epsilon}{\epsilon}\right)\ ,$$

then for all points a, b, a', b' in X such that

$$d(a, a') \leq \epsilon\ , \quad d(b, b') \leq \epsilon\ , \quad d(a, b) \geq c_0(\epsilon)\ ,$$

if m is the midpoint of the geodesic segment $[a, b]$, then $d(m, [a', b']) \leq \frac{\epsilon}{2}$.

Proof. Let p be the point in $[a, b']$ the closest to m, and q the point of $[a', b']$ the closest to p. Let $t = d(a, m) = d(b, m) = d(a, b)/2$. By Lemma 2.1, we have

$$d(m, p) \leq e^{-d(b,m)} \sinh d(b, b') \leq e^{-t} \sinh \epsilon$$

and, as $d(m, p) \leq \epsilon/2$ by convexity,

$$d(p, q) \leq e^{-d(a,p)} \sinh d(a, a') \leq e^{-d(a,m)+d(m,p)} \sinh d(a, a') \leq e^{-t+\epsilon/2} \sinh \epsilon\ .$$

Hence $d(m, q) \leq d(m, p) + d(p, q) \leq e^{-t}(1 + e^{\epsilon/2}) \sinh \epsilon$, and the result follows by the assumption on $d(a, b)$. □

Remark. If we want a simpler expression, we can also take $c_0(\epsilon) = 3\epsilon + 4 \log 2$.

2.2 Entering and exiting ϵ-convex subsets

For every subset A in X and $\epsilon > 0$, we denote by $\mathscr{N}_\epsilon A$ the closed ϵ-neighbourhood of A in X. For every $\epsilon > 0$, a subset C of X will be called *ϵ-convex* if there exists a convex subset C' in X such that $C = \mathscr{N}_\epsilon C'$. As the metric space X is $\mathrm{CAT}(-1)$, it is easy to see that an ϵ-convex subset C is closed, convex, equal to the closure of its interior, and *strictly convex* in the sense that for every geodesic line γ meeting C in at least two points, the segment $\gamma \cap C$ is the closure of $\gamma \cap \overset{\circ}{C}$. If X is a smooth Riemannian manifold, then an ϵ-convex subset has a $\mathrm{C}^{1,1}$-smooth boundary, see [Wal].

Examples. (1) For every $\epsilon > 0$, any ball of radius at least ϵ is ϵ-convex, and any horoball is ϵ-convex. Conversely, as proved below, if a subset $C \subset X$ is ϵ-convex for every $\epsilon > 0$, then C is X, $\emptyset$ or a horoball. Accordingly, we will sometimes refer to horoballs as ∞-convex subsets.

To prove the above statement, assume that $C \neq X, \emptyset$ and that for all $\epsilon > 0$, there exists a convex subset $C_{-\epsilon}$ in X such that $C = \mathscr{N}_\epsilon C_{-\epsilon}$. For every x in ∂C (note that ∂C is non empty as $C \neq X, \emptyset$) and every $t \geq 0$, let x_t be the point of the closed convex subset $\overline{C_{-t}}$ which is the closest to x. Then $t \mapsto x_t$ is a geodesic ray, which converges to a point called x_∞. We claim that $x_\infty = y_\infty$ for every x, y in ∂C. Otherwise, the geodesic segment between x_t and y_t, contained in $\overline{C_{-t}}$ by convexity, converges to the geodesic line between $x_\infty = y_\infty$. Hence, the point x_t would not be the closest one to

x, for t big enough. Therefore ∂C is a horosphere whose point at infinity is x_∞, and by convexity, C is a horoball.

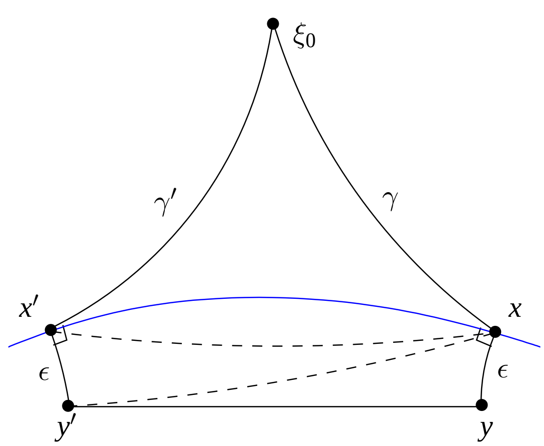
(2) When X is a Riemannian manifold and C is a closed convex subset C with nonempty interior and $\mathrm{C}^{1,1}$-smooth boundary, the property of C being ϵ-convex is related with extrinsic curvature properties of its boundary, see for instance [PP4] and references therein. In particular, if X has constant curvature $-a^2$, then C is ϵ-convex if and only if the eigenvalues of the second fundamental form of ∂C (for the inner pointing normal unit vector field along it) belong to $[a\tanh(a\epsilon), a\coth(a\epsilon)]$ almost everywhere (see loc. cit.).

The rest of this section is devoted to several lemmas concerning the relative distances between entering points and exiting points, in and out of an ϵ-convex subset of X, of two geodesic rays or lines starting from the same point. The asymptotic behaviour of the various constants appearing in this section is described in Remark 2.7.

Lemma 2.3 *Let C be a convex subset in X, let $\epsilon > 0$ and let $\xi_0 \in (X \cup \partial_\infty X) - (\mathscr{N}_\epsilon C \cup \partial_\infty C)$. If two geodesic segments, rays or lines γ, γ' which start from ξ_0 intersect $\mathscr{N}_\epsilon C$, then the first intersection points x, x' of γ, γ' respectively with $\mathscr{N}_\epsilon C$ are at a distance at most*

$$c_1'(\epsilon) = 2\,\mathrm{arsinh}(\coth \epsilon).$$

Proof. Let y and y' be the closest points in C to x and x' respectively. As $x, x' \in \partial\mathscr{N}_\epsilon[y, y']$, it is sufficient to prove the result when $C = [y, y']$.



We may assume that $x \neq x'$, and, by a continuity argument, that $y \neq y'$. Let us construct a pentagon in $\mathbb{H}^2_\mathbb{R}$ with vertices $\overline{\xi_0}, \overline{x}, \overline{y}, \overline{x'}, \overline{y'}$ by gluing together the comparison triangles of (ξ_0, x, x'), (x, x', y') and (x, y', y). By comparison (see for instance [BH, Prop. 1.7.(4)]), the comparison angles at $\overline{x}, \overline{y}, \overline{x'}, \overline{y'}$ are at least $\pi/2$. Hence, the segments or rays $]\overline{\xi_0}, \overline{x}[$ and $]\overline{\xi_0}, \overline{x'}[$ do not meet $\mathscr{N}_\epsilon[\overline{y}, \overline{y'}]$, and the point $\overline{y}$ is the closest point on $[\overline{y}, \overline{y'}]$ to $\overline{x}$.

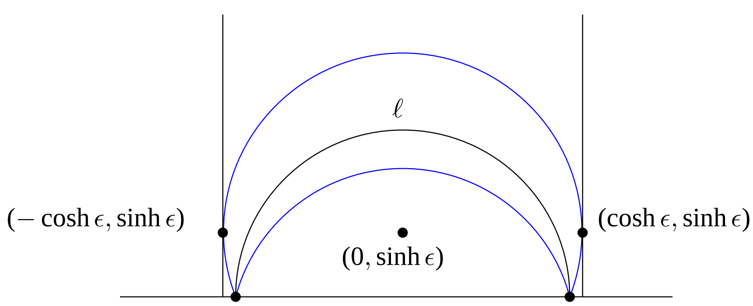
Furthermore, $\overline{y'}$ is the closest point on $[\overline{y}, \overline{y'}]$ to $\overline{x'}$. Indeed, the angle at $\overline{y'}$ of the pentagon is at most $3\pi/2$ since $\angle_{\overline{y'}}(\overline{y}, \overline{x}) \leq \pi/2$ and $\angle_{\overline{y'}}(\overline{x}, \overline{x'}) \leq \pi$. Therefore, if by absurd $\overline{z} \in [\overline{y}, \overline{y'}[$ is closest to $\overline{x'}$, the geodesic segment $[\overline{x'}, \overline{z}]$ intersects $[\overline{y'}, \overline{x}]$ at a point $\overline{u}$. If $z \in [y', y]$ and $u \in [y', x]$ are such that $d(y', z) = d(\overline{y'}, \overline{z})$ and $d(y', u) = d(\overline{y'}, \overline{u})$, then by comparison

$$d(x', z) \leq d(x', u) + d(u, z) \leq d(\overline{x'}, \overline{u}) + d(\overline{u}, \overline{z}) = d(\overline{x'}, \overline{z}) < d(\overline{x'}, \overline{y'}) = d(x', y') ,$$

a contradiction.

As $d(x, x') = d(\overline{x}, \overline{x'})$, we only have to prove that $d(\overline{x}, \overline{x'}) \leq c_1'(\epsilon)$, i.e. we may assume that $X = \mathbb{H}^2_{\mathbb{R}}$. Up to replacing ξ_0 by the point at infinity of the geodesic ray starting at x and passing through ξ_0, we may assume that ξ_0 is at infinity. By homogeneity, we may assume that ξ_0 is the point at infinity ∞ in the upper halfplane model of $\mathbb{H}^2_{\mathbb{R}}$. As a geodesic line starting from ∞ and meeting the ϵ-neighbourhood of a vertical geodesic segment enters it in the sphere of radius ϵ centered at its highest point, we may assume that C is a segment (possibly a point) of the geodesic line ℓ between the points -1 and 1 of the real line.

Claim. *There are points $x_\sharp$ and $x'_\sharp$ that are first meeting points with $\mathscr{N}_\epsilon \ell$ of geodesic lines starting from ∞, such that $d(x, x') \leq d(x_\sharp, x'_\sharp)$.*



Proof. Note that $\partial\mathscr{N}_\epsilon\ell$ is the union of two arcs of Euclidean circles, meeting at -1 and 1, and let $\partial^+\mathscr{N}_\epsilon\ell$ be the upper one, which is the intersection with $\mathbb{H}^2_{\mathbb{R}}$ of the Euclidean circle through $(\pm1, 0)$ and $(0, e^\epsilon)$. The horizontal line with equation $y = \sinh\epsilon$ goes through the Euclidean center $(0, \sinh\epsilon)$ of this circle, and the points of $\partial^+\mathscr{N}_\epsilon\ell$ above or on this line are exactly the first hitting points with $\mathscr{N}_\epsilon\ell$ of the geodesic lines starting from ∞.

We may assume, up to permuting x and x' that the horizontal coordinate of x is strictly less than the one of x'.

If both x and x' are in $\partial\mathscr{N}_\epsilon\ell$ or have vertical coordinate at least $\sinh\epsilon$, then replace them by the points $x_\sharp$ and $x'_\sharp$, respectively to their left and right, on $\partial^+\mathscr{N}_\epsilon\ell$ at the same vertical coordinate. These points satisfy the claim.

Otherwise, assume for instance that x does not lie on $\partial\mathscr{N}_\epsilon\ell$ and has vertical coordinate strictly less than $\sinh\epsilon$. In particular, x belongs to the hyperbolic circle S of radius ϵ centered at one endpoint of the segment C. Moreover, the horizontal Euclidean diameter of S is at vertical height strictly less than $\sinh\epsilon$.

Since the Euclidean normal line to S and $\partial^+\mathscr{N}_\epsilon\ell$ at their common tangency point goes through both their Euclidean centers, this tangency point is below the horizontal Euclidean diameter of S. Hence both x and x' lie on S, since the other points of $\partial\mathscr{N}_\epsilon C$ are not first hitting points with $\mathscr{N}_\epsilon C$ of the geodesic lines starting from ∞. By horizontal translations and homotheties, which are hyperbolic isometries preserving the geodesic lines starting from ∞, we may assume that C is reduced to the closest point on ℓ to ∞. Hence we are again in a situation when both x and x' have vertical coordinates at least $\sinh\epsilon$, which has already been considered. $\square$

The above claim allows us to assume that $C=\ell$. The distance $d(x,x')$ is then maximized when the geodesic lines are tangent to $\mathscr{N}_\epsilon\ell$ on both sides (see the figure above). Thus, we may assume that the points x,x' are $(\pm\cosh\epsilon,\sinh\epsilon)$. The computation of $d(x,x')$ yields the result. $\square$

The following technical result will be used in Lemma 2.5. Define, for every $\epsilon>0$,

$$c''(\epsilon)=\frac{2}{\epsilon}\operatorname{arcosh}(2\cosh(\epsilon/2))\ .$$

For future use, it is easy to check that, for every $\epsilon>0$,

$$c_0(\epsilon)\geq\epsilon\, c''(\epsilon)\ . \tag{-2-}$$

Lemma 2.4 *For every $\epsilon>0$, for every convex subset C in X, for every a,b in $\mathscr{N}_\epsilon C$ and for every a_0 in $[a,b]$, if $d(a,b)\geq c_0(\epsilon)$ and*

$$\eta=\frac{1}{c''(\epsilon)}\min\{d(a_0,a),d(a_0,b)\}\leq\frac{\epsilon}{2}\ ,$$

then $d(a_0,C)\leq\epsilon-\eta$.

Proof. Let $\epsilon > 0$. Let C, a, b, a_0, η be as in the statement, and let us prove that $d(a_0, C) \leq \epsilon - \eta$. By an easy computation, we have $c''(\epsilon)\epsilon \leq c_0(\epsilon)$. By symmetry, we may assume that $d(a, a_0) \leq d(b, a_0)$, so that our assumptions give the following inequalities:

$$d(a, a_0) = c''(\epsilon)\eta \leq c''(\epsilon)\epsilon/2 \leq c_0(\epsilon)/2 \leq d(a, b)/2 \ . \tag{-3-}$$

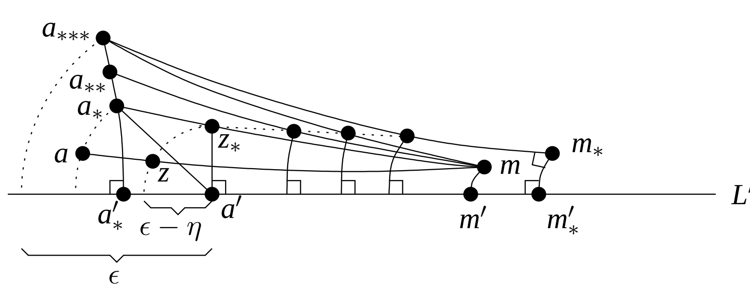
Let a', b' be the points in C the closest to a, b respectively. As $[a', b']$ is contained in C, we may assume that $C = [a', b']$. Let m be the midpoint of $[a, b]$, and m' its closest point on $[a', b']$. By Lemma 2.2, we have $d(m, m') \leq \frac{\epsilon}{2}$.

As $\eta \leq \epsilon/2$, if $d(a, a') \leq \epsilon - \eta$, then by convexity every point in $[a, m]$ is at distance at most $\epsilon - \eta$ from C. In particular, this is true for a_0, since $d(a, a_0) \leq d(a, m)$. Hence, we may assume that $d(a, a') > \epsilon - \eta$.

Consider the quadruple (a, a', m, m') of points of X, which satisfies

- $\epsilon - \eta < d(a, a') \leq \epsilon$,
- $d(m, m') \leq \frac{\epsilon}{2}$,
- a' is the point in $[a', m']$ the closest to a, and
- m' is the point in $[a', m']$ the closest to m.

Define $t = t(a, a', m, m')$ as the distance between a and the point $z = z(a, a', m, m')$ in $[a, m]$ at distance $\epsilon - \eta$ from $[a', m']$ (which exists and is unique by convexity), see the figure below.



We claim that $t \leq c''(\epsilon)\eta = d(a, a_0)$. Before proving this claim, we note that it implies by Equation (-3-) that $t \leq d(a, a_0) \leq d(a, b)/2$, hence, by convexity, $d(a_0, [a', m']) \leq \epsilon - \eta$, and Lemma 2.4 will follow.

We will make several reductions, in order to reach a situation where easy computations will be possible.

First we may assume, by comparison, that $X = \mathbb{H}^2_{\mathbb{R}}$. If the segment $[a,m]$ cuts the segment $[a',b']$ in a point u, then replacing m and m' by the intersection point u gives a new quadruple with the same t. By an approximation argument, we may assume that $[a,m]\cap[a',b']$ is empty and that $a' \neq m' \neq m$. The assumptions on the quadruple (a,a',m,m') then imply that the angles $\angle_{a'}(a,m')$ and $\angle_{m'}(m,a')$ are at least $\frac{\pi}{2}$. Let L' be the geodesic line through a' and m'.

If $[a,m]$ does not enter $\mathcal{N}_{\epsilon-\eta}C$ in the sphere $\partial B(a',\epsilon-\eta)$ (in which case a and m are on the same side of L'), then define $a_* = a$. Otherwise, replace a by the point a_* at distance equal to $d(a,a')$ from a', such that the geodesic segment between a_* and m goes through the point $z_* \in \partial B(a',\epsilon-\eta)\cap\partial\mathcal{N}_{\epsilon-\eta}L'$ (on the same side of L' as m). This gives a new quadruple (a_*,a',m,m') satisfying the same properties, whose t has not decreased, by convexity.

Replace a' by a'_* and a_* by a_{**} such that $\angle_{a'_*}(a_{**},m') = \frac{\pi}{2}$, $d(a_{**},a'_*) = d(a,a')$, and $a_* \in [a_{**},a'_*]$. Clearly, this does not decrease t. Now replace a_{**} by the point a_{***} such that $d(a_{***},a'_*) = \epsilon$ and $[a_{**},a'_*] \subset [a_{***},a'_*]$. Let m_* be the point on $\mathcal{N}_{\epsilon/2}C$ such that there is a geodesic line through a_{***} and m_* which is tangent to $\mathcal{N}_{\epsilon/2}C$ at m_*. Let m'_* be its closest point in L'. Again, the value of t for the quadruple (a_{***},a'_*,m_*,m'_*) has not decreased.

Hence, after these reductions, we may assume that $X = \mathbb{H}^2_{\mathbb{R}}$, that the quadrilateral (a,a',m,m') has right angles at a',m',m, and that $d(a,a') = 2d(m,m') = \epsilon$.

Now, let $\ell = d(m,a) - t$ be the distance between m and the point on $[a,m]$ at distance $\epsilon-\eta$ from $[a',m']$. An easy computation (see [Bea, page 157]) shows that

$$\cosh(t+\ell) = \frac{\sinh(\epsilon)}{\sinh(\epsilon/2)} \quad \text{and} \quad \cosh\ell = \frac{\sinh(\epsilon-\eta)}{\sinh(\epsilon/2)}\ .$$

Consider the map $f_\epsilon : s \mapsto \operatorname{arcosh}\frac{\sinh(\epsilon+s)}{\sinh(\epsilon/2)}$. This function is increasing and concave on $[-\epsilon/2,0]$, with $f_\epsilon(-\epsilon/2) = 0$. By concavity, the graph of f_ϵ on $[-\epsilon/2,0]$ is above the line passing through its endpoints $(-\epsilon/2,0)$ and $(0,f_\epsilon(0))$. Hence, for every s in $[0,\epsilon/2]$, by the definition of $c''(\epsilon)$, we have $f_\epsilon(0) - f_\epsilon(-s) \leq c''(\epsilon)s$. Therefore $t = f_\epsilon(0) - f_\epsilon(-\eta) \leq c''(\epsilon)\eta$ as $\eta \leq \epsilon/2$. This proves our claim, and ends the proof of Lemma 2.4. $\square$

Here is a finer version of Lemma 2.3 which shows that the entering point of a geodesic which enters an ϵ-convex set for a long enough time and the entering point of any nearby geodesic are close. For every $\epsilon > 0$, we define

$$c'_2(\epsilon) = \max\left\{\ c''(\epsilon)+1\ ,\ \frac{2c'_1(\epsilon)}{\epsilon}\ ,\ \sqrt{\frac{\cosh\epsilon}{\cosh\epsilon - 1}}\ \frac{\sinh c'_1(\epsilon)}{c'_1(\epsilon)}\ \right\}. \tag{-4-}$$

Lemma 2.5 *For every $\epsilon > 0$, every ξ_0 in $X \cup \partial_\infty X$, every convex subset C in X, and all geodesic rays or lines γ, γ' in X which start at ξ_0 and enter $\mathscr{N}_\epsilon C$ at the points x, x' in X respectively, if the length of $\gamma' \cap \mathscr{N}_\epsilon C$ is at least $c_0(\epsilon)$, then we have*

$$d(x, x') \leq c_2'(\epsilon) d(x, \gamma') \ .$$

Remarks. (1) Without assuming that the geodesic ray or line γ' has a sufficiently big penetration distance inside $\mathscr{N}_\epsilon C$, the result is false, as can be seen by taking C a point, γ' a geodesic line tangent to $\partial \mathscr{N}_\epsilon C$ and x very close to x' on $\partial \mathscr{N}_\epsilon C$.

(2) The curvature assumption is necessary, as can be seen by considering geodesics which enter a half-plane in $\mathbb{R}^2$ almost parallel to the boundary.

Proof. Let $\epsilon > 0$ and assume that $\xi_0, C, \gamma, \gamma', x, x'$ are as in the statement. We may assume that $x \neq x'$. In particular $\xi_0 \notin C$. Let p' be the point of γ' the closest to x. Let $[x', y']$ be the intersection of γ' with $\mathscr{N}_\epsilon C$ (or $[x', y'[$ with $y' \in \partial_\infty X$ if $\gamma' \cap \mathscr{N}_\epsilon C$ is unbounded). By assumption, $d(x', y') \geq c_0(\epsilon)$.

Case 1: Assume first that p' does not belong to $[\xi_0, x']$. If $d(x', p') \leq \frac{\epsilon}{2} c''(\epsilon)$, then let a_0 be the point p'. Otherwise let a_0 be the point in $[x', y'[$ at distance $\frac{\epsilon}{2} c''(\epsilon)$ from x'. This point exists and is at distance at least $\frac{\epsilon}{2} c''(\epsilon) \geq d(a_0, x')$ from y', as $d(x', y') \geq c_0(\epsilon) \geq \epsilon\, c''(\epsilon)$ by Equation (-2-). By Lemma 2.4, we have $d(a_0, C) \leq \epsilon - \frac{1}{c''(\epsilon)} d(a_0, x')$.

Hence, if $a_0 = p'$, then

$$\frac{1}{c''(\epsilon)} d(p', x') = \frac{1}{c''(\epsilon)} d(a_0, x') \leq \epsilon - d(a_0, C) = d(x, C) - d(p', C) \leq d(x, p') \ .$$

Thus,

$$d(x, x') \leq d(x, p') + d(p', x') \leq (1 + c''(\epsilon))\, d(x, p') \ ,$$

which proves the result, by the definition of $c_2'(\epsilon)$.

If $a_0 \neq p'$, then $p' \notin [a_0, \xi_0[$. Let us prove that $d(x, p') \geq \frac{\epsilon}{2}$. This implies, by Lemma 2.3, that

$$d(x, x') \leq c_1'(\epsilon) \leq \frac{2 c_1'(\epsilon)}{\epsilon}\, d(p', x) \ ,$$

which proves the result, by the definition of $c_2'(\epsilon)$. Let b_0 be the point in $[x', y']$ at distance $\frac{\epsilon}{2} c''(\epsilon)$ from y' (or $b_0 = y'$ if y' is at infinity). By Lemma 2.4, we have

$$\max\{d(a_0, C), d(b_0, C)\} \leq \epsilon - \frac{1}{c''(\epsilon)} \min\{d(a_0, x'), d(b_0, y')\} = \frac{\epsilon}{2} \ .$$

Assume by absurd that $d(x,p') < \frac{\epsilon}{2}$. If $p' \in [a_0,b_0]$, then by convexity $d(p',C) \le \frac{\epsilon}{2}$, therefore $d(x,C) \le d(x,p') + d(p',C) < \epsilon$, a contradiction. If otherwise $p' \notin [a_0,b_0]$, then $b_0 \in [p',\xi_0[$. Therefore there exists a point z in $[x,\xi_0[$ whose closest point to $[p',\xi_0[$ is b_0. By convexity, $d(z,b_0) < \frac{\epsilon}{2}$. Hence $d(z,C) \le d(z,b_0) + d(b_0,C) < \epsilon$, which contradicts the fact that γ enters $\mathscr{N}_\epsilon C$ at x.

Case 2: Assume now that p' belongs to $[\xi_0,x']$. Let a' and b' be the points of C the closest to x' and y' respectively. They are at distance $\epsilon > 0$ from x' and y' respectively (except that $b' = y'$ if y' is at infinity). Let ϕ be the comparison angle at x' between the geodesic segments $[x',a']$ and $[x',y'[$. We claim that $\sin\phi \le \frac{1}{\sqrt{\cosh\epsilon}}$.

To prove this claim, if $y' \in X$, we construct a comparison quadrilateral with vertices $\overline{x'},\overline{a'},\overline{b'},\overline{y'} \in \mathbb{H}^2_\mathbb{R}$ by gluing together the comparison triangles $(\overline{x'},\overline{a'},\overline{y'})$ of (x',a',y') and $(\overline{a'},\overline{b'},\overline{y'})$ of (a',b',y') along their isometric edges $[\overline{a'},\overline{y'}]$. If $y' \notin X$, then $b' = y'$, and the above quadrilateral is replaced by the comparison triangle with vertices $\overline{x'},\overline{a'},\overline{y'} \in \mathbb{H}^2_\mathbb{R} \cup \{\infty\}$. By comparison, all angles in the quadrilateral in $\mathbb{H}^2_\mathbb{R}$ with vertices $\overline{x'},\overline{a'},\overline{b'},\overline{y'}$ are greater than or equal to those in the quadrilateral in X with vertices x',a',b',y'. In particular, if the angle at $\overline{x'}$ is $\overline{\phi}$, we have $\phi \le \overline{\phi}$. If the quadrilateral with vertices $\overline{x'},\overline{a'},\overline{b'},\overline{y'}$ is replaced by the one with vertices $\overline{x'},\overline{a'}_*,\overline{b'}_*,\overline{y'}$ with $d(\overline{x'},\overline{a'}_*) = \epsilon = d(\overline{y'},\overline{b'}_*)$ and right angles at $\overline{a'}_*$ and $\overline{b'}_*$, the angle $\overline{\phi}_*$ at $\overline{x'}$ of this quadrilateral is at least $\overline{\phi}$. Furthermore, this quadrilateral is symmetric: the angle at $\overline{y'}$ is also $\overline{\phi}_*$. Thus, we get an upper bound for ϕ by estimating $\overline{\phi}_*$.

Let $[m,m']$ be the common perpendicular segment between $[\overline{x'},\overline{y'}]$ and $[\overline{a'}_*,\overline{b'}_*]$, with $m \in [\overline{x'},\overline{y'}]$. We have (see for instance [Bea, page 157]),

$$\sin\overline{\phi}_* = \frac{\cosh d(m,m')}{\cosh\epsilon} \quad \text{and} \quad \cosh d(\overline{x'},m) = \frac{\sinh\epsilon}{\sinh d(m,m')} .$$

Hence, as $d(x',y') \ge c_0(\epsilon)$,

$$\sin\overline{\phi}_* = \frac{\sqrt{1+(\sinh^2\epsilon)/(\cosh^2 d(\overline{x'},m))}}{\cosh\epsilon} \le \frac{\sqrt{1+(\sinh^2\epsilon)/(\cosh^2(c_0(\epsilon)/2))}}{\cosh\epsilon} \le \frac{1}{\sqrt{\cosh\epsilon}} ,$$

as, by Equation (-2-), $c_0(\epsilon) \ge \epsilon c''(\epsilon) \ge 2\ \mathrm{arcosh}(\sqrt{2}\cosh(\epsilon/2))$. This proves the claim.

By convexity, the comparison angle at x' between the geodesic segments $[x',x]$ and $[x',a']$ is at most $\frac{\pi}{2}$. Hence the comparison angle θ at x' between $[x',x]$ and $[x',\xi_0[$ (which lies in $[0,\frac{\pi}{2}[$ since $p' \in [\xi_0,x'[$) is at least $\pi - \frac{\pi}{2} - \phi = \frac{\pi}{2} - \phi$. In particular,

$$\frac{1}{\sin\theta} \le \frac{1}{\sin(\frac{\pi}{2}-\phi)} = \frac{1}{\sqrt{1-\sin^2\phi}} \le \sqrt{\frac{\cosh\epsilon}{\cosh\epsilon - 1}} .$$

If $p' = x'$, then the result holds, since $c'_2(\epsilon) \geq 1$. Otherwise, since γ' enters $\mathscr{N}_\epsilon C$ at x', the points x, x', p' are pairwise distinct. Let $(\overline{x}, \overline{x'}, \overline{p'})$ be a comparison triangle in $\mathbb{H}^2_\mathbb{R}$ of the geodesic triangle (x, x', p'). By comparison, $\overline{\theta} = \angle_{\overline{x'}}(\overline{x}, \overline{p'}) \geq \theta$ and $\overline{p'}$ is the closest point to $\overline{x}$ on $[\overline{p'}, \overline{x'}]$. In particular, the angle $\angle_{\overline{p'}}(\overline{x}, \overline{x'})$ is at least $\frac{\pi}{2}$. By the formulae in right-angled hyperbolic triangles, we have

$$\frac{\sinh d(x,p')}{\sinh d(x,x')} = \frac{\sinh d(\overline{x}, \overline{p'})}{\sinh d(\overline{x}, \overline{x'})} \geq \sin \overline{\theta} \geq \sin \theta \ .$$

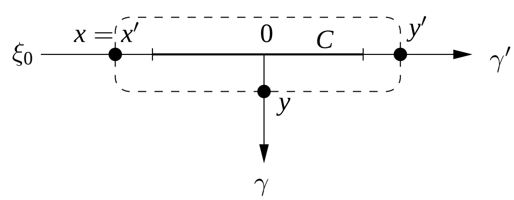
Since p' is the closest point to x on γ', we have $d(x,p') \leq d(x,x') \leq c'_1(\epsilon)$. In particular, by the convexity of the map $t \mapsto \sinh t$ on $[0, +\infty[$, we have

$$\sinh d(x,p') \leq \frac{\sinh c'_1(\epsilon)}{c'_1(\epsilon)} \ d(x,p') \ .$$

Hence, by the definition of $c'_2(\epsilon)$,

$$d(x,x') \leq \sinh d(x,x') \leq \frac{\sinh d(x,p')}{\sin \theta} \leq c'_2(\epsilon) \ d(x,p') \ . \quad \Box$$

In general, there is no estimate analogous to Lemma 2.5 for the distance between the points y, y' where two geodesic rays or lines γ, γ' starting from a point ξ_0 exit an ϵ-convex subset $\mathscr{N}_\epsilon C$. For instance, the geodesic line γ could be tangent to $\mathscr{N}_\epsilon C$, and γ' could enter for a long time in $\mathscr{N}_\epsilon C$, so that y and y' would not be close. But the result is not true even if we assume that both γ and γ' meet $\mathscr{N}_\epsilon C$ in a long segment. Here is a counterexample when X is a tree (but this phenomenon is not specific to trees).



Let γ, γ' be two geodesic lines in a tree X, coinciding on their negative subrays, starting at $\xi_0 \in \partial_\infty X$, and with disjoint positive subrays. Let $\epsilon = \eta = 1$, and $C = \gamma'([-\ell, +\ell])$. Then the entering points of γ, γ' in $\mathscr{N}_\epsilon C$ are $x = x' = \gamma'(-\ell - 1)$. Besides, $y = \gamma(1)$, $y' = \gamma'(\ell + 1)$ and $d(y, \gamma') \leq 1$. But we have $d(y, y') = \ell + 2$, which goes to $+\infty$ as $\ell \to +\infty$.

This explains the dichotomy in the following result on the exiting points from an ϵ-convex sets of two geodesic lines which start from the same point at infinity. For every $\epsilon, \eta > 0$, we define

(-5-) $$h'(\epsilon, \eta) = \max \{ \ 2\eta + \max\{0, -2 \log \frac{\epsilon}{2}\} \ , \ \eta + c'_1(\epsilon) + c_0(\epsilon) \ \}$$

and

$$c_3'(\epsilon) = 3 + \frac{2c_1'(\epsilon)}{\epsilon}\,. \tag{- 6 -}$$

Lemma 2.6 *Let $\epsilon, \eta > 0$. Let C be a convex subset in X, $\xi_0 \in X \cup \partial_\infty X$, and γ, γ' geodesic rays or lines starting from ξ_0. If γ enters $\mathscr{N}_\epsilon C$ at a point $x \in X$ and exits $\mathscr{N}_\epsilon C$ at a point $y \in X$ such that $d(x, y) \geq h'(\epsilon, \eta)$ and $d(y, \gamma') \leq \eta$, then γ' meets $\mathscr{N}_\epsilon C$, entering it at a point $x' \in X$, exiting it at a point $y' \in X \cup \partial_\infty X$ such that*

$$d(y, y') \leq c_3'(\epsilon) d(y, \gamma') \;\text{ or }\; d(x', y') > d(x, y)\,.$$

Proof. Let p' be the closest point on γ' to y. Let q be the closest point on γ to p'. The point q belongs to $[y, \xi_0]$ and satisfies $d(y, q) \leq d(y, p') \leq \eta$, as closest point maps do not increase the distances. By the properties of geodesic triangles in $\mathrm{CAT}(-1)$ spaces, we have

$$d(p', q) \leq \operatorname{arsinh} 1 = \log(1 + \sqrt{2}).$$

Let us first prove that γ' meets $\mathscr{N}_\epsilon C$. Let m be the midpoint of $[x, y]$. As

$$d(y, m) = d(x, y)/2 \geq h'(\epsilon, \eta)/2 \geq \eta \geq d(y, q)\,,$$

the point q belongs to $[m, y]$. Furthermore,

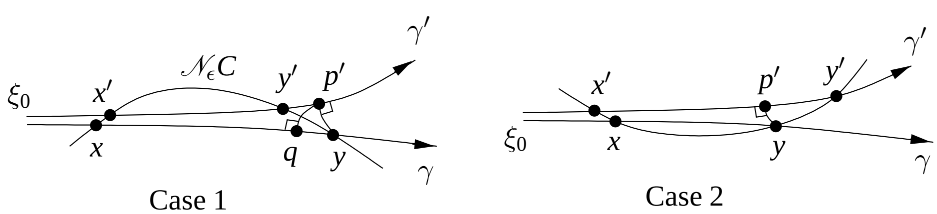
$$d(q, m) = d(y, m) - d(y, q) \geq h'(\epsilon, \eta)/2 - \eta \geq -\log\frac{\epsilon}{2}\,,$$

by the definition of $h'(\epsilon, \eta)$. By Lemma 2.1, we have

$$d(m, \gamma') \leq e^{-d(q,m)} \sinh d(q, p') \leq \frac{\epsilon}{2}\,.$$

By Lemma 2.2, as $d(x, y) \geq h'(\epsilon, \eta) \geq c_0(\epsilon)$ by the definition of $h'(\epsilon, \eta)$, we have $d(m, C) \leq \frac{\epsilon}{2}$. Hence the point m' of γ' the closest to m belongs to $\mathscr{N}_\epsilon C$, which is what we wanted.

Let x' and y' be the entering point in $\mathscr{N}_\epsilon C$ and exiting point out of $\mathscr{N}_\epsilon C$ of γ' respectively. The point y' could for the moment be at infinity, in which case the second possibility below would hold.



Case 1 : Assume that $p' \notin [y', \xi_0]$. Let $\eta_\epsilon = \epsilon\, c''(\epsilon)/2$. There are two subcases. First assume that $d(y, p') \geq \epsilon/2$. Let

$$t_\epsilon = \max\,\{\eta_\epsilon, -\log\frac{\epsilon}{2}\}.$$

Note that $h'(\epsilon, \eta) \geq \eta_\epsilon + t_\epsilon + \eta$ by the definition of $h'(\epsilon, \eta)$, as by Equation (-2-) we have $c_0(\epsilon) \geq \epsilon\, c''(\epsilon) = 2\eta_\epsilon$, and by a discussion on the value of t_ϵ. Hence we have

$$d(y, x) - d(y, q) - t_\epsilon \geq h'(\epsilon, \eta) - \eta - t_\epsilon \geq \eta_\epsilon \geq 0\ .$$

Therefore, the point y_0 in $[x, q]$ at distance t_ϵ of q exists and satisfies $d(y_0, x) \geq \eta_\epsilon$. Furthermore, $d(y_0, q) = t_\epsilon \geq \eta_\epsilon$ and

$$d(x, q) = d(x, y) - d(y, q) \geq h'(\epsilon, \eta) - \eta \geq c_0(\epsilon) \geq 2\eta_\epsilon\ ,$$

by the definition of $h'(\epsilon, \eta)$. Let a_0 and b_0 be the points in $[x, q]$ at distance η_ϵ from x and q respectively, which are at distance at least η_ϵ from q and x respectively. By Lemma 2.4, we have $d(a_0, C) \leq \epsilon - \eta_\epsilon/c''(\epsilon) = \epsilon/2$, and similarly $d(b_0, C) \leq \epsilon/2$. Note that y_0 belongs to $[a_0, b_0]$. Hence by convexity, we have $d(y_0, C) \leq \epsilon/2$. By Lemma 2.1, we have

$$d(y_0, \gamma') \leq e^{-t_\epsilon} \sinh d(q, p') \leq \frac{\epsilon}{2}\ .$$

Therefore the point q' on γ' the closest to y_0 belongs to $\mathscr{N}_\epsilon C$. As y' is the exiting point of γ' from $\mathscr{N}_\epsilon C$, it belongs to $[q', p']$. As closest point maps do not increase the distances, we have $d(p', q') \leq d(y, y_0)$. Hence

$$\begin{aligned} d(y, y') &\leq d(y, p') + d(p', y') \leq d(y, p') + d(p', q') \leq d(y, p') + d(y, y_0) \\ &\leq d(y, p') + d(y, q) + d(q, y_0) \leq 2d(y, p') + t_\epsilon \\ &\leq (2 + 2t_\epsilon/\epsilon) d(y, p') \leq c_3'(\epsilon)\, d(y, p')\ , \end{aligned}$$

as it can be checked that $2c_1'(\epsilon)/\epsilon + 1 \geq 2t_\epsilon/\epsilon$.

Assume now that $d(y, p') \leq \epsilon/2$. Since

$$d(x, y) \geq h'(\epsilon, \eta) \geq c_0(\epsilon) \geq 2\eta_\epsilon \geq 2c''(\epsilon) d(y, p')\ ,$$

the point y_0 in $[x, y]$ at distance $c''(\epsilon) d(y, p')$ from y exists and $d(y_0, x) \geq c''(\epsilon) d(y, p')$. Hence by Lemma 2.4, we have $d(y_0, C) \leq \epsilon - d(y, p')$. Let q' be the point on γ' the closest to y_0. By convexity, q' is at distance at most $d(y, p')$ from y_0, hence belongs to $\mathscr{N}_\epsilon C$. As y' is the exiting point of γ' from $\mathscr{N}_\epsilon C$, it belongs to $[q', p']$. As closest point maps do not increase distances, we have $d(q', p') \leq d(y_0, y)$. Hence, as above, and by the definition of y_0,

$$d(y, y') \leq d(y, p') + d(y_0, y) \leq \big(1 + c''(\epsilon)\big)\, d(y, p')\ ,$$

which proves the result, by the definition of $c'_3(\epsilon)$, as $2c'_1(\epsilon)/\epsilon + 1 \geq c''(\epsilon)$.

Case 2 : Assume that $p' \in \,]y', \xi_0]$. Lemma 2.3 implies that $d(x, x') \leq c'_1(\epsilon)$. Note that $p' \notin [x', \xi_0]$. Otherwise, with q and s the closest points to p' and x' on γ respectively, we would have $s \notin \,]q, \xi_0]$ by convexity. As $q \in [y, \xi_0]$, we would then have

$$d(x, y) \leq d(x, s) + d(q, y) \leq d(x, x') + d(p', y) \leq c'_1(\epsilon) + \eta < h'(\epsilon, \eta) \;,$$

by the definition of $h'(\epsilon, \eta)$, a contradiction.

Assume first that $d(y, p') < \epsilon/2$. We start by observing that $d(p', y') \leq c''(\epsilon) d(y, p')$. Indeed, suppose by absurd that $d(p', y') > c''(\epsilon) d(y, p')$. By continuity of the closest point maps, let y_0 be a point on γ that does not belong to $\mathscr{N}_\epsilon C$, but is close enough to y, so that the closest point q' to y_0 on γ' belongs to $[p', y']$ and satisfies $d(y_0, q') \leq \epsilon/2$ and $d(q', y') \geq c''(\epsilon) d(q', y_0)$. Hence, using the definition of $h'(\epsilon, \eta)$ and Equation (- 2 -), we have

$$\begin{aligned} d(y', x') &\geq d(q', x') \geq d(p', x') \geq d(x, y) - d(p', y) - d(x, x') \\ &\geq h'(\epsilon, \eta) - \eta - c'_1(\epsilon) \geq c_0(\epsilon) \geq \epsilon\, c''(\epsilon) \geq 2c''(\epsilon) d(q', y_0) \;. \end{aligned} \tag{- 7 -}$$

Let a_0 and b_0 be the points in $[x', y']$ at distance $c''(\epsilon) d(q', y_0) \leq \epsilon\, c''(\epsilon)/2$ from x' and y' respectively. The estimate (- 7 -) implies that a_0 and b_0 are at distance at least $c''(\epsilon) d(q', y_0)$ from y' and x' respectively. By Lemma 2.4, we have $d(a_0, C) \leq \epsilon - d(q', y_0)$ and $d(b_0, C) \leq \epsilon - d(q', y_0)$. Hence, the point q', which belongs to $[a_0, b_0]$ by Formula (- 7 -) and the construction of q', is by convexity at distance at most $\epsilon - d(q', y_0)$ from C. Therefore by the triangular inequality, $d(y_0, C) \leq \epsilon$, which is a contradiction. Hence $d(p', y') \leq c''(\epsilon) d(y, p')$, and

$$d(y, y') \leq d(y, p') + d(p', y') \leq (1 + c''(\epsilon)) d(y, p') \;,$$

which proves the result, as in the end of Case 1.

Assume now that $d(y, p') \geq \epsilon/2$. Suppose first that $d(p', y') > d(y, p') + c'_1(\epsilon)$. Then, as $p' \in [x', y']$,

$$d(x', y') = d(x', p') + d(p', y') \geq d(p', y') + d(x, y) - d(y, p') - d(x, x') > d(x, y) \;,$$

which is one of the two possible conclusions. Otherwise,

$$d(y, y') \leq d(y, p') + d(p', y') \leq 2d(y, p') + c'_1(\epsilon) \leq \big(2 + \frac{2c'_1(\epsilon)}{\epsilon}\big) d(y, p') \leq c'_3(\epsilon) d(y, p') \;,$$

by the definition of $c'_3(\epsilon)$. This is the other possible conclusion. □

Remark 2.7 The asymptotic behaviour of the constants when ϵ is very big or very small is as follows.

- $c_0(\epsilon) \sim 3\epsilon$ as $\epsilon \to +\infty$ and $\lim_{\epsilon\to 0} c_0(\epsilon) = 4\log 2 \approx 2.77$.
- $\lim_{\epsilon\to+\infty} c_1'(\epsilon) = c_1'(\infty) = 2\log(1+\sqrt{2}) \approx 1.76$, and $c_1'(\epsilon) \sim -2\log\epsilon$ as $\epsilon \to 0$. Note that $\epsilon \mapsto c_1'(\epsilon)$ is decreasing.
- $\lim_{\epsilon\to+\infty} c''(\epsilon) = 1$, and $c''(\epsilon) \sim \frac{2}{\epsilon}\log(2+\sqrt{3})$ as $\epsilon \to 0$.
- For ϵ big, $c_2'(\epsilon) = c''(\epsilon) + 1$, hence $\lim_{\epsilon\to+\infty} c_2'(\epsilon) = 2$. For $\epsilon > 0$ small,
$$c_2'(\epsilon) = \sqrt{\frac{\cosh\epsilon}{\cosh\epsilon - 1}}\ \frac{\sinh c_1'(\epsilon)}{c_1'(\epsilon)} \sim \frac{\sqrt{2}}{4\epsilon^3\log(1/\epsilon)}\ .$$
- $\lim_{\epsilon\to+\infty} c_3'(\epsilon) = 3$, and $c_3'(\epsilon) \sim -\frac{4}{\epsilon}\log\epsilon$ as $\epsilon \to 0$.
- $h'(\epsilon,\eta) \sim 3\epsilon$ as $\epsilon \to +\infty$, and $h'(\epsilon,\eta) \sim -2\log\epsilon$ as $\epsilon \to 0$, uniformly on compact subsets of η's.

When ϵ goes to $+\infty$, $c_1'(\epsilon)$ and $c_3'(\epsilon)$ have finite limits, and the limiting values apply for the horoball case, see Lemmas 2.9 and 2.12 below. On the other hand, the constants $c_0(\epsilon)$ and $h'(\epsilon,\eta)$ behave badly as $\epsilon \to \infty$, and we will improve them in Section 2.3.

When X is a tree, the constants $c_3'(\epsilon)$ and $h'(\epsilon,\eta)$ can be simplified, we can take $c_3'(\epsilon) = 2$ and any $h'(\epsilon,\eta) > 2\eta$, as the following more precise result shows, improving Lemma 2.6 for trees. Note that the versions of Lemmas 2.3 and 2.5 for trees simply say that we can take $c_0(\epsilon) = \epsilon$, and $c_1'(\epsilon) = c_2'(\epsilon) = 0$, since for every point or end ξ_0 of a (real) tree, for every convex subset C, for all geodesic rays or lines γ, γ'' starting from ξ_0 and entering C in x, x' respectively, we have $x = x'$.

Remark 2.8 *Let X be an $\mathbb{R}$-tree and $\epsilon > 0$. Let C be a convex subset in X, $\xi_0 \in X \cup \partial_\infty X$, and γ, γ' geodesic rays or lines starting from ξ_0. If γ enters $\mathscr{N}_\epsilon C$ at a point $x \in X$ and exits $\mathscr{N}_\epsilon C$ at a point $y \in X$ such that $d(x,y) > 2d(y,\gamma')$, then γ' meets $\mathscr{N}_\epsilon C$, entering it at $x' = x$, exiting it at a point y' (possibly at infinity) such that*
$$d(y,y') \le 2\,d(y,\gamma') \;\text{ or }\; d(x',y') > d(x,y).$$

Proof. Let p' be the closest point to y on γ'. Note that p' belongs to $]\xi_0, y]$, as X is a tree and γ' also starts from ξ_0. If $p' \in\,]\xi_0, x[$, then $d(y,\gamma') > d(x,y)$, a contradiction. Hence $p' \in [x,y] \subset \mathscr{N}_\epsilon C$, and γ' enters $\mathscr{N}_\epsilon C$ at $x' = x$.

Suppose first that $d(x,y) < 2\epsilon$. Then the closest point z to y in C does not belong to $[x,y]$. Let q be the midpoint of $[x,y]$, which is also the closest point to z on $[x,y]$. As

$d(x,y) > 2d(y,\gamma')$, the point p' belongs to $]q,y]$, hence $d(y,y') = 2d(y,\gamma')$, which is fine.

Assume now that $d(x,y) \geq 2\epsilon$. If x_ϵ and y_ϵ are the points in $[x,y]$ at distance ϵ from x and y respectively, then $[x,y] \cap C = [x_\epsilon, y_\epsilon]$. If p' belongs to $]y_\epsilon, y]$, then y_ϵ is also the closest point to y' in C, and $d(p',y) = d(p',y')$, so that $d(y,y') = 2d(y,\gamma')$, which is fine. Otherwise, we have $d(y,\gamma') \geq \epsilon$. If $d(x',y') \leq d(x,y)$, then $d(p',y') \leq d(p',y)$. Hence

$$d(y,y') = d(y,p') + d(p',y') \leq 2d(y,p') = 2d(y,\gamma') \ . \quad \square$$

2.3 Hitting horoballs

As shown in Remark 2.7, the constants $c_0(\epsilon)$ and $h'(\epsilon,\eta)$, used to describe the penetration of geodesic lines inside ϵ-convex subsets, do not have a finite limit as ϵ goes to $+\infty$. Horoballs are ϵ-convex subsets for every ϵ, and we could use for instance $\epsilon = 1$ in these constants to get numerical values. But in order to get better values, we will prove analogs for horoballs of the lemmas 2.3, 2.4, 2.5 and 2.6. The proofs of the lemmas below follow the same lines as the ones for the general case of ϵ-convex subsets given in Section 2.2, with many simplifications.

As $c'_1(\epsilon)$ tends to

$$c'_1(\infty) = 2\log(1+\sqrt{2}),$$

the next lemma follows by passing to the limit in Lemma 2.3. It is not hard to see (for instance by considering the real hyperbolic plane) that the constant $c'_1(\infty)$ is optimal.

Lemma 2.9 *For every horoball H in X, for every ξ_0 in $(X \cup \partial_\infty X) - (H \cup H[\infty])$, for all geodesic rays or lines γ and γ' starting from ξ_0 and entering H in x and x' respectively, we have*

$$d(x,x') \leq c'_1(\infty) = 2\log(1+\sqrt{2}) \ . \quad \square$$

The following result, Lemma 2.10, improves Lemma 2.4 for horoballs, and says that when the ϵ-convex subset under consideration is a horoball, we can replace $c_0(\epsilon)$ by

$$c_0(\infty) = 4.056 \ , \qquad \text{(- 8 -)}$$

and $c''(\epsilon)$ by $c''(\infty) = \frac{3}{2}$. Lemma 2.11 below is the analog of Lemma 2.5 for horoballs, and says that when the ϵ-convex subset under consideration is a horoball, we can replace $c_0(\epsilon)$ by $c_0(\infty) = 4.056$ and $c'_2(\epsilon)$ by

$$c'_2(\infty) = \frac{5}{2} \ . \qquad \text{(- 9 -)}$$

Note that $c''(\infty), c_0(\infty)$ and $c'_2(\infty)$ are not limits as ϵ goes to ∞ of $c''(\epsilon), c_0(\epsilon)$ and $c'_2(\epsilon)$, but this mnemonic notation will be useful in Section 4, where it will be used in a similar way whether ϵ is finite or not.

Lemma 2.10 *For every horoball H, for every a and b in ∂H with $d(a,b) \geq c_0(\infty)$, for every a_0 in $[a,b]$, we have*

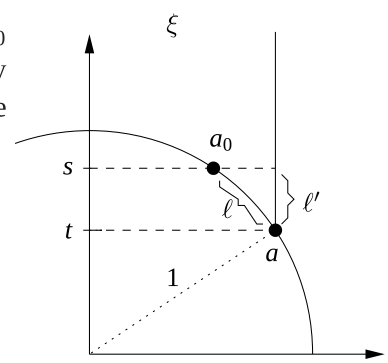
$$a_0 \in H[\,\tfrac{2}{3}\min\{d(a_0,a), d(a_0,b)\}\,]\ .$$

Proof. Let $\xi = H[\infty]$ be the point at infinity of H. Up to exchanging a and b, we may assume that $\ell = d(a_0,a) = \min\{d(a_0,a), d(a_0,b)\}$.

Let $(\overline{a}, \overline{b}, \overline{\xi} = \infty)$ be a comparison triangle of (a,b,ξ) in $\mathbb{H}^2_{\mathbb{R}}$. By comparison (see the paragraph before Lemma 2.1), the (non negative) difference ℓ' of the heights of a and a_0 with respect to ξ is at least the corresponding quantity $\overline{\ell'}$ for the comparison points $\overline{a}$ and $\overline{a_0}$. Thus, in order to show that $\ell' \geq \frac{2}{3}\ell$, it is sufficient to show that $\overline{\ell'} \geq \frac{2}{3}\ell$, and the question reduces to the case $X = \mathbb{H}^2_{\mathbb{R}}$. We assume that $[b,a]$ lies on the unit circle, with a (and hence a_0, as a and b have the same (Euclidean) vertical coordinate) in the closed positive quadrant.

Let s be the (Euclidean) vertical coordinate of a_0 and t the one of a, with $0 < t \leq s \leq 1$. An easy computation in hyperbolic geometry (see also the proof of Lemma 2.1) gives $\ell' = \log \frac{s}{t}$ and

$$\ell = \operatorname{arsinh} \frac{\sqrt{1-t^2}}{t} - \operatorname{arsinh} \frac{\sqrt{1-s^2}}{s}$$
$$= \log \frac{s}{t} + \log \frac{1+\sqrt{1-t^2}}{1+\sqrt{1-s^2}}\ .$$



Hence, to prove that $\ell \leq \frac{3}{2}\ell'$, we only have to show that $\log \frac{1+\sqrt{1-t^2}}{1+\sqrt{1-s^2}} \leq \frac{1}{2}\log\frac{s}{t}$, which is equivalent to $\sqrt{t}(1+\sqrt{1-t^2}) \leq \sqrt{s}(1+\sqrt{1-s^2})$. The map $f : x \mapsto \sqrt{x}(1+\sqrt{1-x^2})$ on $[0,1]$ is increasing from $f(0) = 0$ to $f(\frac{\sqrt{5}}{3})$, and then decreasing to $f(1) = 1$. Let $t' = 0.25873$. As $f(t') < 1$ and $s \geq t$, to prove that $f(t) \leq f(s)$, it is sufficient to show that $t \leq t'$. Let a' and b' be the two points of the unit circle at (Euclidean) height t'. As a and b are at the same (Euclidean) height t on the unit circle, to prove that $t \leq t'$, we only have to show that $d(a',b') \leq d(a,b)$. By the definition of $c_0(\infty)$, we have

$$d(a',b') = 2\operatorname{arsinh}\frac{\sqrt{1-t'^2}}{t'} \leq c_0(\infty) \leq d(a,b)\ .$$

Hence the result follows. $\square$

Lemma 2.11 *For every horoball H in X, for every ξ_0 in $X \cup \partial_\infty X$, for all geodesic rays or lines γ and γ' starting from ξ_0 and entering H in $x \in X$ and $x' \in X$ respectively, if the length of $\gamma' \cap H$ is at least $c_0(\infty)$, then*

$$d(x, x') \leq \frac{5}{2}\, d(x, \gamma')\ .$$

Proof. Let p' be the point of γ' the closest to x. Let ξ be the point at infinity of H. Define y' by $[x', y'] = \gamma' \cap H$ if this intersection is bounded, and $y' = \xi$ otherwise. We may assume that $x \neq x'$. In particular, $\xi_0 \notin H \cup \{\xi\}$.

Assume first that p' does not belong to $[\xi_0, x'[$. As closest point projections do not increase distances and by Lemma 2.9, we have $d(x', p') \leq d(x, x') \leq c_1'(\infty)$. Since

$$d(x', y') \geq c_0(\infty) \geq 2c_1'(\infty), \tag{- 10 -}$$

the point p' belongs to H, and $d(p', y') \geq d(p', x')$. Let z be the point of intersection of $]\xi, x']$ with the horosphere centered at ξ passing through p', so that in particular $d(x, p') \geq d(x', z)$. By Lemma 2.10, we have $d(x', z) \geq \frac{2}{3}\, d(x', p')$. Hence

$$d(x, x') \leq d(x, p') + d(p', x') \leq d(x, p') + \frac{3}{2}\, d(x', z) \leq \frac{5}{2}\, d(x, p')\ .$$

Assume now that p' belongs to $[\xi_0, x'[$. Let β be the comparison angle at x' between the (nontrivial) geodesic segments or rays $[x', y'[$ and $[x', \xi[$. By comparison, β is at most the angle $\overline{\beta}$ between $[\overline{x'}, \overline{y'}[$ and $[\overline{x'}, \overline{\xi}[$, where $\overline{x'}$ and $\overline{y'}$ are two points, at distance $d(x', y')$, on a horosphere in $\mathbb{H}^2_{\mathbb{R}}$ centered at $\overline{\xi}$. An easy computation in the upper half space model and the inequality (- 10 -) show that

$$\tan \overline{\beta} = \left(\sinh \frac{1}{2} d(\overline{x'}, \overline{y'})\right)^{-1} \leq \left(\sinh\left(2 \log(1 + \sqrt{2})\right)\right)^{-1} \leq \frac{1}{\sqrt{3}}\ .$$

As $0 \leq \overline{\beta} \leq \frac{\pi}{2}$, this implies that $\beta \leq \overline{\beta} \leq \frac{\pi}{6}$.

Let α be the comparison angle at x' between the (nontrivial) geodesic segments $[x', p']$ and $[x', x]$, which is at most $\frac{\pi}{2}$, as p' is the closest point to x on $]x', \xi_0]$. As the geodesic segment $[x, x']$ lies in H, we have $\alpha \geq \pi - \frac{\pi}{2} - \beta \geq \frac{\pi}{3}$. By Lemma 2.9, we have $d(x, p') \leq d(x, x') \leq c_1'(\infty)$. Using the formulae for right-angled hyperbolic triangles (see [Bea]) and the comparison triangle in $\mathbb{H}^2_{\mathbb{R}}$ to the triangle (x, x', p') in X, we have, by convexity of $t \mapsto \sinh t$,

$$d(x, x') \leq \sinh d(x, x') \leq \frac{1}{\sin \alpha} \sinh d(x, p') \leq \frac{2}{\sqrt{3}} \frac{\sinh c_1'(\infty)}{c_1'(\infty)} d(x, p') \leq \frac{5}{2}\, d(x, p')\ .$$

This proves the result. $\square$

The following Lemma is the analog of Lemma 2.6 for horoballs. It says that when the ϵ-convex subset under consideration is a horoball, we can replace $c_3'(\epsilon)$ and $h'(\epsilon,\eta)$ by

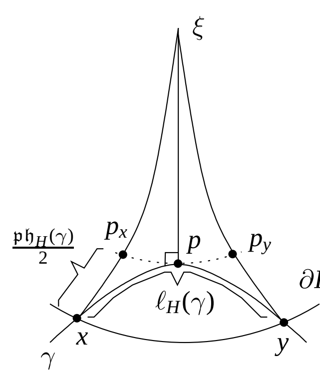
$$c_3'(\infty) = \frac{5}{2} \quad \text{and} \quad h'(\infty,\eta) = 3\eta + c_0(\infty) + c_1'(\infty) \approx 3\eta + 5.8188\ , \tag{- 11 -}$$

and that the first of the two possible conclusions of Lemma 2.6 always holds. Note that $c_3'(\infty)$ is not the limit as ϵ goes to $+\infty$ of $c_3'(\epsilon)$, and that $h'(\epsilon,\eta)$ diverges as $\epsilon \to \infty$. However, in both cases, this mnemonic notation will be useful in Section 4, where it will be used in a similar way whether ϵ is finite or not.

Lemma 2.12 *For every horoball H in X, for every ξ_0 in $(X \cup \partial_\infty X) - (H \cup H[\infty])$, for all geodesic rays or lines γ, γ' starting from ξ_0, if γ enters H at a point $x \in X$ and exits H at a point $y \in X$, and if $d(x,y) \geq h'(\infty, d(y,\gamma'))$, then γ' meets H, exiting it at a point $y' \in X$ such that*

$$d(y,y') \leq \frac{5}{2}\, d(y,\gamma')\ .$$

Proof. Let ξ be the point at infinity of H, let p be the closest point on $[x,y]$ to ξ, and let p_x and p_y be the points of intersection of the horosphere ∂H_p centered at ξ passing through p with the geodesic rays $[x,\xi[$ and $[y,\xi[$ respectively. By comparison, we have $d(p_x,p_y) \leq 2\log(1+\sqrt{2}) = c_1'(\infty)$. Thus, the triangle inequality, along with the fact that p_y is the closest point to y on ∂H_p and the assumption on $d(x,y)$, gives



$$2\min\{d(y,p), d(x,p)\} \geq d(y,p_y) + d(x,p_x) \geq$$
$$d(x,y) - 2\log(1+\sqrt{2}) \geq c_0(\infty) \geq 3\ .$$

In particular, as $d(x,y) \geq c_0(\infty)$, Lemma 2.10 implies that p belongs to $H[1]$. By Lemma 2.1 and the assumption on $d(x,y)$, we have

$$d(p,\gamma') \leq \frac{1}{2}\, e^{-d(y,p)+d(y,\gamma')} \leq \frac{1}{2}\, e^{-\frac{1}{2}d(x,y)+\log(1+\sqrt{2})+d(y,\gamma')} \leq \frac{1}{2}\ .$$

This implies that γ' meets H, because $\mathscr{N}_{\frac{1}{2}}(H[1]) = H[\frac{1}{2}]$ is contained in H.

Let x' and y' be the entering point in H and the exiting point out of H of γ', respectively. Let p' be the point on γ' the closest to y.

Case 1 : Assume that $p' \notin [y', \xi_0]$. Note that

$$d(x,y) - \frac{3}{2}\,d(y,p') \geq \frac{3}{2}\,d(y,p') \geq 0\ ,$$

as $d(x,y) \geq 3\,d(y,\gamma')$, by the definition of $h'(\infty,\eta)$. Hence, there is a point y_0 in $[x,y]$ at distance $\frac{3}{2}\,d(y,p')$ of y which satisfies $d(x,y_0) \geq \frac{3}{2}\,d(y,p')$. By Lemma 2.10, we have $y_0 \in H[d(y,p')]$. Let q' be the point of γ' the closest to y_0. By convexity, we have $d(y_0,q') \leq d(y,p')$. Hence q' belongs to H. By the intermediate value theorem, the point y' belongs to $[q',p']$. As closest point maps do not increase the distances, we have $d(p',q') \leq d(y,y_0) = \frac{3}{2}\,d(y,p')$. Therefore,

$$d(y,y') \leq d(y,p') + d(p',y') \leq d(y,p') + d(p',q') \leq \frac{5}{2}\,d(y,p')\ ,$$

which proves the result.

Case 2 : Assume that $p' \in\,]y', \xi_0]$. By the same argument as in Case 2 of the proof of Lemma 2.6, we have $p' \notin [x', \xi_0]$. If $d(p',y') \leq \frac{3}{2}\,d(y,p')$, then $d(y,y') \leq \frac{5}{2}\,d(y,p')$, and the result is proved. Therefore, assume by absurd that $d(p',y') > \frac{3}{2}\,d(y,p')$. By the continuity of the closest point maps, there exists a point y_0 in γ that does not belong to H, whose closest point q' on γ', which lies in $\gamma' -\,]p',\xi_0]$, satisfies $d(q',y') \geq \frac{3}{2}\,d(y_0,q')$ and $d(y_0,q') \leq d(y,p') + \frac{1}{2}\,c_0(\infty)$. Lemma 2.9 implies that $d(x,x') \leq c_1'(\infty)$. Thus, by the assumption on $d(x,y)$,

$$\begin{aligned} d(x',y') \geq d(x',q') \geq d(p',x') &\geq d(x,y) - d(x,x') - d(y,p') \\ &\geq 3\,d(y,p') + c_0(\infty) + c_1'(\infty) - c_1'(\infty) - d(y,p') \\ &\geq 2d(y,p') + c_0(\infty) \geq \max\{2d(y_0,q'), c_0(\infty)\}\ . \end{aligned}$$

In particular, $d(y_0,q') \leq \frac{1}{2}d(x',q') \leq \frac{2}{3}d(q',x')$ and we already had $d(y_0,q') \leq \frac{2}{3}d(q',y')$. Hence, by Lemma 2.10, we have $q' \in H[d(y_0,q')]$. This implies that y_0 belongs to H, a contradiction. □

3 Properties of penetration in ϵ-convex sets

3.1 Penetration maps

Let X be a proper geodesic CAT(-1) space, and $\xi_0 \in X \cup \partial_\infty X$. We are interested in controlling the penetration of geodesic rays or lines starting from ξ_0 in ϵ-convex subsets of X. One way to measure this penetration is the intersection length. If C is a closed convex subset in X such that $\xi_0 \notin C \cup \partial_\infty C$, we define a map $\ell_C : T^1_{\xi_0}X \to$

$[0,+\infty]$, called the *penetration length map*, which associates to every γ in $T^1_{\xi_0}X$ the length of the intersection $\gamma\cap C$ (which is connected by convexity).

When we study specific geometric situations, such as collections of horoballs and ϵ-neighbourhoods of geodesics, there are further natural ways of measuring the penetration. These will be used in many applications in Section 5 and in [PP3]. If C is an ϵ-convex subset of X such that $\xi_0\notin C\cup\partial_\infty C$, we will require our penetration maps $f: T^1_{\xi_0}X\rightarrow[0,+\infty]$ in C to have one or two of the following properties, the first one depending on a constant $\kappa\geq 0$. The sup–norm of a real valued function f on $T^1_{\xi_0}X$ is denoted by $\|f\|_\infty$.

(i) **(Penetration property)** For any γ in $T^1_{\xi_0}X$, $f(\gamma)=+\infty$ if and only if $\ell_C(\gamma)=+\infty$, and the restrictions of f and ℓ_C to the set where both functions are finite satisfy $\|f-\ell_C\|_\infty\leq\kappa$.

(ii) **(Lipschitz property)** For every γ,γ' in $T^1_{\xi_0}X$ which intersect C, if $\gamma\cap C=[a,b]$ and $\gamma'\cap C=[a',b']$ with a,b,a',b' in X, then

$$|f(\gamma)-f(\gamma')|\leq 2\max\{d(a,a'),d(b,b')\}\ .$$

If C is an ϵ-convex subset of X such that $\xi_0\notin C\cup\partial_\infty C$, and $f: T^1_{\xi_0}X\rightarrow[0,+\infty[$ is a map which satisfies (i) for some $\kappa\geq 0$, we say that f is a κ*-penetration map in (the* ϵ*-convex set)* C. We also say that (C,f) is an (ϵ,κ)*-penetration pair*. In the condition (ii), we could have replaced 2 by some $\lambda\geq 2$, but if f also satisfies the property (i), then only $\lambda=2$ is really relevant in the large scale.

Note that if (C,f) is an (ϵ',κ')-penetration pair, if $\epsilon'\geq\epsilon$ and $\kappa'\leq\kappa$, then (C,f) is an (ϵ,κ)-penetration pair. If C is ∞-convex and (C,f) is an (ϵ,κ)-penetration pair in every $\epsilon>0$ then f will be called a κ*-penetration map in (the* ∞*-convex set)* C.

Penetration maps in general ϵ-convex subsets. If C is a closed convex subset of X, the map ℓ_C is in general not continuous on $T^1_{\xi_0}X$, as can be seen by taking C to be a geodesic segment of positive length. The following result shows that the situation is nicer for ϵ-convex subsets. Note that the statement of Lemma 3.1 is not true in $\mathbb{R}^n$ (which is not a CAT(-1) space).

Lemma 3.1 *Let* $\epsilon>0$ *and let* C *be an* ϵ*-convex subset of* X *such that* $\xi_0\notin C\cup\partial_\infty C$. *The map* $\ell_C: T^1_{\xi_0}X\rightarrow[0,+\infty]$ *is a continuous* 0*-penetration map in* C *satisfying the Lipschitz property* (ii).

Proof. The Lipschitz property (ii) of the penetration length map ℓ_C follows from the triangular inequality. It remains to show the continuity of the map.

Choose a convex subset C' such that $C = \mathscr{N}_\epsilon(C')$, and note that by the definition of the topology of $X \cup \partial_\infty X$, the subsets C and C' have the same points at infinity. Let $\gamma_0 \in T^1_{\xi_0}X$, and let us prove that ℓ_C is continuous at γ_0.

Assume first that $\gamma_0(+\infty)$ is a point at infinity of C. Then there exists a geodesic ray contained in C' ending at this point at infinity. As geodesic rays converging to the same point at infinity become exponentially close, this implies that $\ell_C(\gamma_0) = \infty$. Let $A > 0$. As $\gamma_0 \cap C$ is the closure of $\gamma_0 \cap \overset{\circ}{C}$, let $[x, y]$ be a geodesic segment of length $A + 2$ contained in $\gamma_0 \cap \overset{\circ}{C}$. Let $\eta \in \,]0, 1]$ be such that the balls B and B' of radius η and of center x and y respectively are contained in $\overset{\circ}{C}$. If $\gamma \in T^1_{\xi_0}X$ is close enough to γ_0, then γ meets B and B', and by convexity, $\ell_C(\gamma) \geq A$, which proves the result.

Assume now that $\gamma_0(+\infty)$ is not a point at infinity of C, but that γ_0 does meet C. Then $\gamma_0 \cap C$ is a nonempty compact segment $[a, b]$. For every $\eta > 0$, let a_+, b_+ be points in $\gamma_0 - [a, b]$, at distance at most $\eta/4$ from a, b respectively, and, if $d(a, b) > 0$, let a_-, b_- be points in $]a, b[$ at distance at most $\eta/4$ from a, b respectively. As C is closed and $\gamma_0 \cap C$ is the closure of $\gamma_0 \cap \overset{\circ}{C}$ if $a \neq b$, there exists $\eta' \in \,]0, \eta/4]$ such that the balls $B(a_+), B(b_+)$ of radius η' and centers a_+, b_+ respectively are contained in $X - C$ and, if $d(a, b) > 0$, the balls $B(a_-), B(b_-)$ of radius η' and centers a_-, b_- respectively are contained in the interior of C. If $\gamma \in T^1_{\xi_0}X$ is close enough to γ_0, then γ meets $B(a_+), B(b_+)$ (and hence $B(a_-), B(b_-)$ by convexity, if $d(a, b) > 0$). It is easy to see then that $|\ell_C(\gamma) - \ell_C(\gamma_0)| \leq \eta$.

Assume now that γ_0 does not meet C. Let U, V be neighbourhoods of the endpoints of γ_0 in $X \cup \partial_\infty X$ that are disjoint from $C \cup \partial_\infty C$. Let $\eta > 0$ be such that the η-neighbourhood of γ_0 is disjoint from C, which exists, as $\inf_{x \in \gamma_0} d(x, C) > 0$. If $\gamma \in T^1_{\xi_0}X$ is close enough to γ_0, then (the image of) γ lies in $U \cup V \cup \mathscr{N}_\eta \gamma_0$, hence does not meet C. So that $\ell_C(\gamma) = \ell_C(\gamma_0) = 0$. $\square$

In particular, if H is a horoball such that $\xi_0 \notin H \cup \partial_\infty H$, then ℓ_H is a continuous 0-penetration map for H satisfying the Lipschitz property (ii).

Let C be a convex subset of X such that $\xi_0 \notin C \cup \partial_\infty C$. For every γ in $T^1_{\xi_0}X$, let $\gamma_- = \xi_0$ and $\gamma_+ = \gamma(+\infty)$, and let $q_{\gamma_\pm}$ be the closest point on C to $\gamma_\pm$. Define the *boundary-projection penetration map* $\mathfrak{bp}_C : T^1_{\xi_0}X \to [0, +\infty]$ by

$$\mathfrak{bp}_C(\gamma) = d(q_{\gamma_-}, q_{\gamma_+}) \ ,$$

with the obvious convention that $\mathfrak{bp}_C(\gamma) = +\infty$ if q_{γ_+} is at infinity.

Lemma 3.2 *Let C be an ϵ-convex subset of X such that $\xi_0 \notin C \cup \partial_\infty C$. The map $\mathfrak{bp}_C$ is a continuous $2c_1'(\epsilon)$-penetration map in C.*

Proof. The continuity of $\mathfrak{bp}_C$ follows from the continuity of the projection maps and the endpoint maps. Let us prove that $\mathfrak{bp}_C$ has the Penetration property (i) with $\kappa = 2c_1'(\epsilon)$. Let $\gamma \in T^1_{\xi_0}X$. If γ_+ is a point at infinity of C, then $\mathfrak{bp}_C(\gamma) = \ell_C(\gamma) = +\infty$, and the claim is true for these geodesics. Otherwise, if γ meets C, then γ enters C at x and exits C at y, with x, y in X. By Lemma 2.3, we hence have

$$|d(x,y) - d(q_{\gamma_-}, q_{\gamma_+})| \leq d(x, q_{\gamma_-}) + d(q_{\gamma_+}, y) \leq 2c_1'(\epsilon) ,$$

and the result follows.

If γ does not meet C, let $[p,q]$ be the shortest connecting segment between a point p in γ and a point q in C. By angle comparison, the geodesic segment or ray between q and $\gamma_\pm$ meets C exactly in q. Hence, by Lemma 2.3,

$$d(q_{\gamma_-}, q_{\gamma_+}) \leq d(q_{\gamma_-}, q) + d(q, q_{\gamma_+}) \leq 2c_1'(\epsilon) .$$

As $\ell_C(\gamma) = 0$, the result follows. □

Penetration maps in horoballs. If H is a horoball in X, with ξ its point at infinity, such that $\xi_0 \notin H \cup \{\xi\}$, and if x_0 is any point in the boundary of H in X, define a 1-Lipschitz map $\beta_H : X \to [0, +\infty[$, called the *height map* of H by

$$\beta_H : x \mapsto \max\{\beta_\xi(x_0, x), 0\},$$

whose values are positive in the interior of H, and 0 outside H. By convention, define $\beta_H(\xi) = +\infty$. Note that β_H is independent of the choice of the point x_0. For every γ in $T^1_{\xi_0}X$, let p_γ be the closest point to $\gamma(+\infty)$ on the geodesic line between ξ_0 and ξ, with $p_\gamma = \xi$ if $\gamma(+\infty) = \xi$.

We will study two penetration maps associated with the height map. The map $\mathfrak{ph}_H : T^1_{\xi_0}X \to [0, +\infty]$ defined by

$$\mathfrak{ph}_H(\gamma) = 2 \sup_{t \in \mathbb{R}} \beta_H(\gamma(t))$$

will be called the *penetration height map* inside H. The map $\mathfrak{ipp}_H : T^1_{\xi_0}X \to [0, +\infty]$ defined by

$$\mathfrak{ipp}_H(\gamma) = 2\, \beta_H(p_\gamma)$$

will be called the *inner-projection penetration map* inside H. Note that for every $t \geq 0$ and $\gamma \in T^1_{\xi_0}X$, we have $\mathfrak{ph}_{H[t]}(\gamma) = \max\{0, \mathfrak{ph}_H(\gamma) - 2t\}$ and $\mathfrak{ipp}_{H[t]}(\gamma) = \max\{0, \mathfrak{ipp}_H(\gamma) - 2t\}$.

Lemma 3.3 *Let H be a horoball in X, such that $\xi_0 \notin H \cup \partial_\infty H$. The maps $\mathfrak{ph}_H, \mathfrak{ipp}_H : T^1_{\xi_0} X \to [0, +\infty]$ are continuous $2\log(1+\sqrt{2})$-penetration maps for H, and $\mathfrak{ph}_H$ has the Lipschitz property (ii). Furthermore,*

$$\|\mathfrak{ph}_H - \mathfrak{ipp}_H\|_\infty \le 2\log(1+\sqrt{2}).$$

Remark. In $\mathbb{H}^n_{\mathbb{R}}$, the equality $\mathfrak{ipp}_H(\gamma) = \mathfrak{ph}_H(\gamma) + \log 2$ holds for any horoball H with $\xi_0 \notin H \cup H[\infty]$ and $\gamma \in T^1_{\xi_0} X$ meeting H. Thus, in $\mathbb{H}^n_{\mathbb{R}}$ the map $\mathfrak{ipp}_H$ satisfies the Lipschitz property (ii). We do not know whether $(H, \mathfrak{ipp}_H)$ satisfies the Lipschitz property (ii) in general.

Proof. Let us prove that $(H, \mathfrak{ph}_H)$ satisfies the Lipschitz property (ii). Let γ, γ' be elements in $T^1_{\xi_0} X$ such that $\gamma \cap H = [a, b]$ and $\gamma' \cap H = [a', b']$. Then, for every x in $[a, b]$, if x' is the point on $[a', b']$ the closest to x, we have, with ξ the point at infinity of H,

$$|\beta_\xi(a, x) - \beta_\xi(a, x')| = |\beta_\xi(x, x')| \le d(x, x') \le \max\{d(a, a'), d(b, b')\}$$

by convexity. Taking x the highest point in $[a, b]$, we get

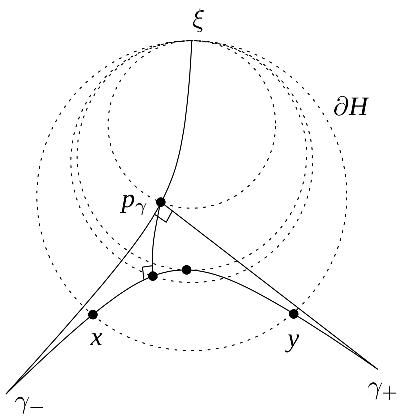
$$\begin{aligned} \mathfrak{ph}_H(\gamma) = 2\beta_\xi(a, x) &\le 2\max\{d(a, a'), d(b, b')\} + 2\beta_\xi(a, x') \\ &\le 2\max\{d(a, a'), d(b, b')\} + \mathfrak{ph}_H(\gamma') \,. \end{aligned}$$

Using a symmetry argument, the result follows.

Let us prove that $(H, \mathfrak{ph}_H)$ satisfies the Penetration property (i) with $\kappa = c'_1(\infty) = 2\log(1+\sqrt{2})$. Let $\gamma \in T^1_{\xi_0} X$. Note that γ enters the interior of H if and only if $\ell_H(\gamma) > 0$, and if and only if $\mathfrak{ph}_H(\gamma) > 0$. Hence we may assume that γ meets H in a segment $[x, y]$. By the first paragraph of the proof of Lemma 2.12, we have

$$\mathfrak{ph}_H(\gamma) \le \ell_H(\gamma) \le \mathfrak{ph}_H(\gamma) + 2\log(1+\sqrt{2}) \,.$$

Let us prove that $(H, \mathfrak{ipp}_H)$ satisfies the Penetration property (i) with $\kappa = 2\log(1+\sqrt{2})$. Let $\gamma \in T^1_{\xi_0} X$. If $p_\gamma = \xi$, then $\mathfrak{ipp}_H(\gamma) = \ell_H(\gamma) = +\infty$, and the result holds, hence we may assume that p_γ belongs to X. If p_γ does not belong to H, as the closest point projection of γ on the geodesic line $]\gamma_-, \xi[$ is $]\gamma_-, p_\gamma[$, then γ does not enter H, and hence $\mathfrak{ipp}_H(\gamma) = \ell_H(\gamma) = 0$, and the result is proven.



Assume that p_γ belongs to H, and note that by comparison and an easy hyperbolic estimate, we have $d(p_\gamma,\gamma) \leq \log(1+\sqrt{2})$. In particular, if γ does not enter H, then $0 \leq \beta_H(p_\gamma) \leq d(p_\gamma,\gamma) \leq \log(1+\sqrt{2})$, and $|\mathfrak{ipp}_H(\gamma)-\ell_H(\gamma)| \leq 2\log(1+\sqrt{2})$, hence the result holds. Therefore we may assume that γ enters H at the point x and exits H at the point y. We then have $\mathfrak{ph}_H(\gamma) \leq \mathfrak{ipp}_H(\gamma) \leq \mathfrak{ph}_H(\gamma)+2d(p_\gamma,\gamma)$. Hence,

$$\ell_H(\gamma) - 2\log(1+\sqrt{2}) \leq \mathfrak{ipp}_H(\gamma) \leq \ell_H(\gamma) + 2\log(1+\sqrt{2})\ ,$$

and the result is proven. The continuity of $\mathfrak{ipp}_H$ follows from the continuity of the endpoint maps, of the closest point projection maps and of $\beta_H : X \cup \{\xi\} \to [0,+\infty]$. To prove the continuity of $\mathfrak{ph}_H$ at a point γ_0 of $T^1_{\xi_0}X$, note that if $\gamma_0(+\infty) = \xi$, then $\mathfrak{ph}_H(\gamma_0) = +\infty$, and the continuity follows from the Penetration property (i) of $(H,\mathfrak{ph}_H)$ and the continuity of ℓ_H. Otherwise, $\gamma_0 \cap H$ is a compact segment. If it is nonempty, then if γ is close enough to γ', the argument in the proof of Lemma 3.1 shows that the Hausdorff distance between $\gamma \cap H$ and $\gamma_0 \cap H$ is as small as wanted. The result follows then since β_H is 1-Lipschitz (and vanishes outside H). If γ_0 does not meet H, then if γ is close enough to γ', the argument in the proof of Lemma 3.1 shows that γ also avoids H, hence $\mathfrak{ph}_H(\gamma) = \ell_H(\gamma) = 0$. □

Penetration maps in balls. If B is a ball of center x_0 and radius r_0 in X with $\xi_0 \notin B$, define a 1-Lipschitz map $\beta_B : X \to [0,+\infty[$, called the *height map*, by

$$\beta_B : x \mapsto \max\{r_0 - d(x_0,x), 0\}\ ,$$

whose values are positive in the interior of B, and 0 outside B. For every γ in $T^1_{\xi_0}X$, let p_γ be the closest point to $\gamma(+\infty)$ on the geodesic segment or ray between ξ_0 and x_0.

The map $\mathfrak{ph}_B : T^1_{\xi_0}X \to [0,2r_0]$ defined by

$$\mathfrak{ph}_B(\gamma) = 2\ \sup_{t\in\mathbb{R}} \beta_B(\gamma(t))$$

will be called the *penetration height map* inside B. The map $\mathfrak{ipp}_B : T^1_{\xi_0}X \to [0,2\,r_0]$ defined by

$$\mathfrak{ipp}_B(\gamma) = 2\ \beta_B(p_\gamma)$$

will be called the *inner-projection penetration map* inside B.

We claim that the maps $\mathfrak{ph}_B, \mathfrak{ipp}_B$ are continuous $2\log(1+\sqrt{2})$-penetration maps, and that $\mathfrak{ph}_B$ has the Lipschitz property *(ii)*; furthermore,

$$\|\mathfrak{ph}_B - \mathfrak{ipp}_B\|_\infty \leq 2\log(1+\sqrt{2}).$$

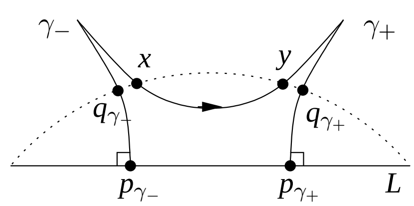
This is proved as in Lemma 3.3, except that in the proof of the Penetration property of $\mathfrak{ipp}_B$, the discussion is on whether p_γ is equal to x_0 or not, and if $p_\gamma = x_0$, then, by comparison, $d(\gamma, x_0) \leq \log(1+\sqrt{2})$, and the claim follows in this case as $\mathfrak{ipp}_B(\gamma) = 2r_0$.

If a sequence of balls $(B_i)_{i\in\mathbb{N}}$ converges to a horoball H (for the Hausdorff distance on compact subsets of X), then the maps $\mathfrak{ph}_{B_i}, \mathfrak{ipp}_{B_i}$ converge, uniformly on compact subsets of $T^1_{\xi_0}X$, to $\mathfrak{ph}_H, \mathfrak{ipp}_H$ respectively.

Penetration maps in tubular neighbourhoods of totally geodesic subspaces.

We define two functions on $T^1_{\xi_0}X$ which describe the closeness of a geodesic line to a totally geodesic subspace L. If ξ_0 is in the boundary at infinity of X, then these functions are defined without reference to an ϵ-neighbourhood of L. However, we show that they are penetration maps in the ϵ-neighbourhood of L, with explicit constants which depend only on ϵ.

Let $\epsilon > 0$, and let L be a complete totally geodesic subspace of X, with set of points at infinity $\partial_\infty L$, such that $\xi_0 \notin \mathscr{N}_\epsilon L \cup \partial_\infty L$. For every γ in $T^1_{\xi_0}X$, let $\gamma_- = \xi_0$ and $\gamma_+ = \gamma(+\infty)$, and let $p_{\gamma_\pm}$ be the point on L the closest to $\gamma_\pm$.



We define the *fellow-traveller penetration map* $\mathfrak{ftp}_L : T^1_{\xi_0}X \to [0,+\infty]$ by

$$\mathfrak{ftp}_L(\gamma) = d(p_{\gamma_-}, p_{\gamma_+}) \ ,$$

with the convention that this distance is $+\infty$ if p_{γ_+} is in $\partial_\infty L$.

Lemma 3.4 *Let $\epsilon > 0$, and let L be a complete totally geodesic subspace of X such that $\xi_0 \notin \mathscr{N}_\epsilon L \cup \partial_\infty L$. The map $\mathfrak{ftp}_L$ is a continuous $(2c_1'(\epsilon)+2\epsilon)$-penetration map in $\mathscr{N}_\epsilon L$ and $\|\mathfrak{ftp}_L - \mathfrak{bp}_{\mathscr{N}_\epsilon L}\|_\infty \leq 2\epsilon$.*

Proof. The continuity of $\mathfrak{ftp}_L$ follows from the continuity of the projection maps and of the endpoint maps. Note that, for every γ in $T^1_{\xi_0}X$, the geodesic segment or ray from $p_{\gamma_\pm}$ to $\gamma_\pm$ exits $\mathscr{N}_\epsilon L$ at the closest point $q_{\gamma_\pm}$ on $\mathscr{N}_\epsilon L$ to $\gamma_\pm$. Hence, by the triangular inequality, and as closest point maps do not increase distances, we have

$$0 \leq \mathfrak{bp}_{\mathscr{N}_\epsilon L}(\gamma) - \mathfrak{ftp}_L(\gamma) \leq 2\epsilon\ .$$

Therefore the fact that $\mathfrak{ftp}_L(\gamma)$ satisfies the Penetration property (i) with $\kappa = 2c'_1(\epsilon) + 2\epsilon$ follows from Lemma 3.2. □

If L is one-dimensional and $\xi_0 \in \partial_\infty X - \partial_\infty L$, a natural penetration map is defined using the crossratios of the endpoints of L and γ. Let $\partial_4 X$ be the set of quadruples (a,b,c,d) in $(\partial_\infty X)^4$ such that $a \neq b$ and $c \neq d$. The *crossratio* $[a,b,c,d] \in [-\infty,+\infty]$ of a quadruple (a,b,c,d) in $\partial_4 X$ is defined as follows (see for instance [Ota, Bou, Pau]). If a_t, b_t, c_t, d_t are any geodesic rays converging to a,b,c,d respectively, then

$$[a,b,c,d] = \frac{1}{2}\lim_{t\to+\infty} d(a_t,c_t) - d(c_t,b_t) + d(b_t,d_t) - d(d_t,a_t).$$

Note that the order conventions differ in the references, we are using the ones of [Bou, HP2]), and that our crossratio is the logarithm of the crossratio used in [Bou]. As suggested by the referee, $[a,b,c,d]$ should rather be called "crossdifference", but we will stick to the "crossratio" terminology, which is more widely known, because of formula (- 12 -), and because taking a log from the very beginning makes our subsequent formulas shorter.

Let us give other formulae for the crossratio. The *visual distance* of two points a and b in $\partial_\infty X$ with respect to a given point x_0 in X is

$$d_{x_0}(a,b) = \lim_{t\to\infty} e^{-\frac{1}{2}\left(d(x_0,a_t)+d(x_0,b_t)-d(a_t,b_t)\right)}.$$

If $\xi \in \partial_\infty X$, if H is a horosphere centered at ξ, and a,b are points in $\partial_\infty X - \{\xi\}$, and $t \mapsto x_t$ is a geodesic ray with $x_0 \in H$ which converges to ξ, the *Hamenstädt distance* (defined in [Ham], [HP2, Appendix]) of a and b in $\partial_\infty X - \{\xi\}$ normalized with respect to H is

$$d_H(a,b) = \lim_{t\to\infty} e^t d_{x_t}(a,b).$$

Note that if H' is another horosphere centered at ξ, then there exists a constant $c > 0$ such that $d_{H'} = c\, d_H$. In particular, for every $\xi' \in \partial_\infty X - \{\xi\}$ and $r > 0$, the sphere of center ξ' and radius r for d_H coincides with the sphere of center ξ' and radius cr for $d_{H'}$.

It is easy to see that for any $x_0 \in X$ and any horoball H, we have for every $(a,b,c,d) \in \partial_4 X$

$$[a,b,c,d] = \log \frac{d_{x_0}(a,c)}{d_{x_0}(c,b)} \frac{d_{x_0}(b,d)}{d_{x_0}(d,a)} = \log \frac{d_H(a,c)}{d_H(c,b)} \frac{d_H(b,d)}{d_H(d,a)} , \tag{- 12 -}$$

if, in the second equation, a,b,c,d are in $\partial_\infty X - H[\infty]$. Note that each expression in the above two equalities is $-\infty$ if $a = c$ or $b = d$, and $+\infty$ if $c = b$ or $a = d$. If the points ξ and a coincide, the expression of the crossratio simplifies to

$$[\xi,b,c,d] = \log \frac{d_H(b,d)}{d_H(c,b)} .$$

The crossratio is continuous on $\partial_4 X$, it is invariant under the diagonal action of the isometry group of Γ, and it has the following symmetries

$$[c,d,a,b] = [a,b,c,d] \text{ and } [a,b,d,c] = [b,a,c,d] = -[a,b,c,d].$$

If $X = \mathbb{H}^n_\mathbb{R}$ and ξ is the point at infinity ∞ in the upper halfspace model of $\mathbb{H}^n_\mathbb{R}$, then the Hamenstädt distance coincides with a constant multiple of the Euclidean distance of $\partial_\infty \mathbb{H}^n_\mathbb{R} - \{\infty\} = \mathbb{R}^{n-1}$ (see for instance [HP3]). In particular, if $n = 2$ or $n = 3$, then our crossratio is the logarithm of the modulus of the classical crossratio of four points in $\mathbb{C} \cup \{\infty\}$.

If $\xi_0 \in \partial_\infty X - \partial_\infty L$, we define the *crossratio penetration map* $\mathfrak{crp}_L : T^1_{\xi_0} X \to [0,+\infty]$ as follows. Let γ be a geodesic line starting at $\gamma_- = \xi_0$, and ending at $\gamma_+ \in \partial_\infty X$. Let L_1, L_2 be the endpoints of L. Set

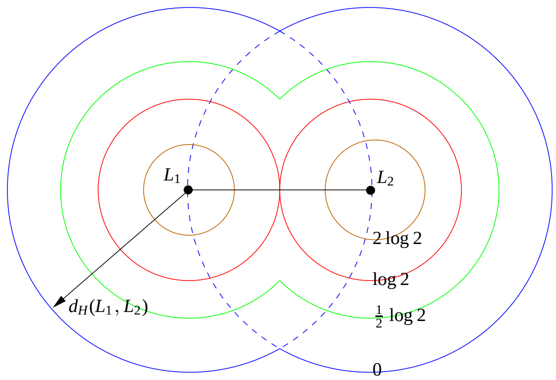
$$\mathfrak{crp}_L(\gamma) = \max \{0, [\gamma_-, L_1, \gamma_+, L_2], [\gamma_-, L_2, \gamma_+, L_1]\} .$$

Note that $\mathfrak{crp}_L(\gamma) = +\infty$ if γ_+ is equal to L_1 or L_2. The map $\mathfrak{crp}_L$ is clearly continuous, and is independent of the ordering L_1, L_2 of the endpoints of L.

If H is a horosphere centered at ξ_0, then

$$[\xi_0, L_1, \gamma_+, L_2] = \log \frac{d_H(L_1, L_2)}{d_H(\gamma_+, L_1)},$$

and the level sets for $\mathfrak{crp}_L$ have a simple form: $[\xi_0, L_1, \gamma_+, L_2] = c$ if and only if γ_+ is on the sphere of radius $e^{-c} d_H(L_1, L_2)$ centered at L_1 with respect to the Hamenstädt metric. Thus, in particular, the boundary of the zero set of $\mathfrak{crp}_L$ is the boundary of the union of the two balls of radius $d_H(L_1, L_2)$ centered at L_1 and L_2. Furthermore, if $c > \log 2$, then the level set ${\mathfrak{crp}_L}^{-1}(c)$ is the union of two spheres for the Hamenstädt distance d_H of centers L_1 and L_2 and radius $e^{-c} d_H(L_1, L_2)$. These two spheres are disjoint by the triangle inequality. Each of them separates ξ_0 from exactly one of the endpoints of L. We will use this in the proof of Lemma 3.9.



Note that if X is a negatively curved symmetric space, then the spheres and balls of the Hamenstädt distance are topological spheres and balls in the topological sphere $\partial_\infty X$ (see [HP3] if $X = \mathbb{H}^n_{\mathbb{R}}$ and [HP4] if $X = \mathbb{H}^n_{\mathbb{C}}$). We do not know (and in fact we doubt it) whether this always holds in the general variable curvature case.

Lemma 3.5 *Let $(a,b,c,d) \in \partial_4 X$. If $b = d$, we define by convention $p = q = b$ and $d(p,q) = 0$. Otherwise, let p and q be the closest points on $[b,d]$ of a and c respectively.*

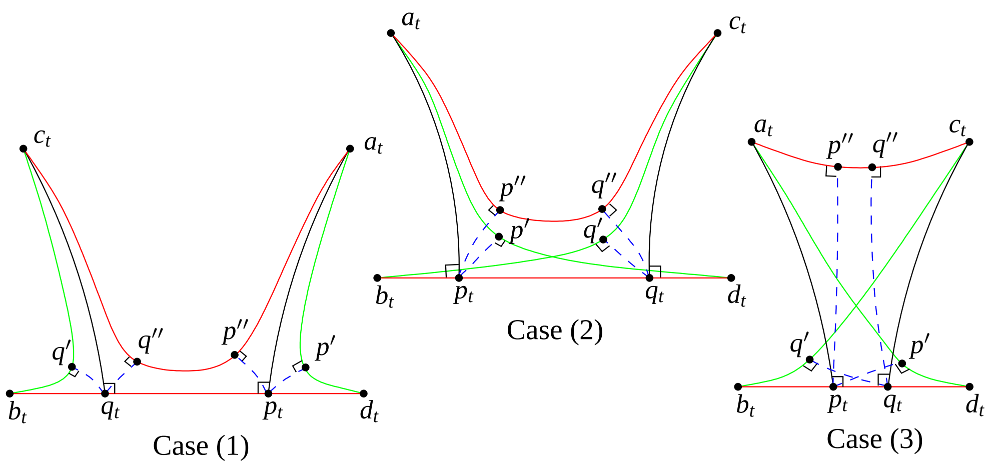
(1) *If b,q,p,d are in this order on $[b,d]$ and $d(p,q) \geq c_1'(\infty)$, then $|[a,b,c,d] - d(p,q)| \leq 2c_1'(\infty)$.*

(2) *If b,p,q,d are in this order on $[b,d]$ and $d(p,q) \geq c_1'(\infty)$, then $[a,b,c,d] \leq c_1'(\infty)$.*

(3) *If $d(p,q) \leq c_1'(\infty)$, then $[a,b,c,d] \leq 2\, c_1'(\infty)$.*

Proof. If $a = d$ or $c = b$, then $p = d$ or $q = b$, hence we are in case (1) with $b \neq d$, and $[a,b,c,d] = d(p,q) = +\infty$, which proves the result. If $a = c$ or $b = d$, then $p = q$, we are in case (3) and $[a,b,c,d] = -\infty$, which proves the result. Hence we may assume that a,b,c,d are pairwise disjoint.

Let $t \mapsto a_t, b_t, c_t, d_t$ be geodesic rays converging to respectively a,b,c,d as $t \to \infty$, and let p_t and q_t be the closest points to a_t and c_t respectively on $[b_t, d_t]$. Let $p' \in [a_t, d_t]$ and $p'' \in [a_t, c_t]$ be the closest points to p_t on $[a_t, d_t]$ and $[a_t, c_t]$, and let $q' \in [b_t, c_t]$ and $q'' \in [a_t, c_t]$ be the closest points to q_t on $[b_t, c_t]$ and $[a_t, c_t]$.



Recall that by an easy comparison argument, for pairwise distinct points u, v, w in $X \cup \partial_\infty X$, if r is the closest point to w on $]u, v[$, then r is at distance less than $\delta = \log(1+\sqrt{2})$ from a point on $]u, w[$. We will apply this remark to $r = p_t$ and $r = q_t$. Recall also that $c_1'(\infty) = 2\,\delta$.

Case (1). Under the assumptions of Assertion (1), if t is big enough, then the points b_t, q_t, p_t, d_t are in this order on $[b_t, d_t]$. Using the triangle inequality on $d(a_t, c_t)$ and $d(b_t, d_t)$, and inserting the points p' and q' in $[d_t, a_t]$ and $[c_t, b_t]$, we have

$$\begin{aligned}
&d(a_t, c_t) - d(c_t, b_t) + d(b_t, d_t) - d(d_t, a_t)\\
&\le d(a_t, p_t) + d(p_t, q_t) + d(q_t, c_t) - d(c_t, q_t) - d(q_t, b_t) + 2\,d(q', q_t)\\
&\quad + d(b_t, q_t) + d(q_t, p_t) + d(p_t, d_t) - d(d_t, p_t) - d(p_t, a_t) + 2\,d(p', p_t)\\
&\le 2\,d(p_t, q_t) + 4\,\delta\ .
\end{aligned}$$

By comparison and a standard argument on hyperbolic quadrilaterals with three right angles (see [Bea, page 157]), for every $\epsilon > 0$, if t is big enough, we have that $d(q_t, q'') \le 2\,\delta + \epsilon/4$, as $d(p_t, q_t) \to d(p, q) \ge c_1'(\infty)$. If we insert the points p'' and q'' in $[a_t, c_t]$, we get, as above,

$$\begin{aligned}
&d(a_t, c_t) - d(c_t, b_t) + d(b_t, d_t) - d(d_t, a_t)\\
&\ge d(a_t, p_t) + d(p_t, q_t) + d(q_t, c_t) - 2\,d(p_t, p'') - 2\,d(q_t, q'') - d(c_t, q_t) - d(q_t, b_t)\\
&\quad + d(b_t, q_t) + d(q_t, p_t) + d(p_t, d_t) - d(d_t, p_t) - d(p_t, a_t)\\
&\ge 2\,d(p_t, q_t) - 8\,\delta + \epsilon\ .
\end{aligned}$$

This proves Assertion (1) in Lemma 3.5.

Case (2). The proof of Assertion (2) is almost identical to the one of the upper bound in the first inequality in Case (1). The different order of the points p_t and q_t now causes cancellations:

$$\begin{aligned}
&d(a_t,c_t)-d(c_t,b_t)+d(b_t,d_t)-d(d_t,a_t)\\
&\leq d(a_t,p_t)+d(p_t,q_t)+d(q_t,c_t)-d(c_t,q_t)-d(q_t,p_t)-d(p_t,b_t)+2\,d(q',q_t)\\
&\quad +d(b_t,p_t)+d(p_t,q_t)+d(q_t,d_t)-d(d_t,q_t)-d(q_t,p_t)-d(p_t,a_t)+2\,d(p',p_t)\\
&\leq 4\,\delta\;.
\end{aligned}$$

Case (3). Assume that $d(p,q)\leq c_1'(\infty)$ and let $\epsilon>0$. By taking t big enough, we can assume that $d(p_t,q_t)\leq\; 2\,\delta+\epsilon$. Inserting the points p'' and q'' in $[a_t,c_t]$ and using the fact that closest point maps do not increase distances, we have $d(p'',q'')\leq 2\,\delta+\epsilon$, $d(c_t,q_t)\geq d(c_t,q'')$ and $d(a_t,p_t)\geq d(a_t,p'')$. Thus, as in the cases above,

$$\begin{aligned}
&d(a_t,c_t)-d(c_t,b_t)+d(b_t,d_t)-d(d_t,a_t)\\
&\leq d(a_t,p'')+d(p'',q'')+d(q'',c_t)-d(c_t,q_t)-d(q_t,b_t)+2\,d(q',q_t)\\
&\quad +d(b_t,q_t)+d(q_t,p_t)+d(p_t,d_t)-d(d_t,p_t)-d(p_t,a_t)+2\,d(p',p_t)\\
&\leq 8\,\delta+2\epsilon\;.
\end{aligned}$$

As this holds for any $\epsilon>0$, the result follows. □

Lemma 3.6 *Let $\epsilon>0$, let L be a geodesic line in X, and assume that $\xi_0\in\partial_\infty X-\partial_\infty L$. The map $\mathfrak{crp}_L$ is a continuous $(2c_1'(\epsilon)+2c_1'(\infty)+2\epsilon)$-penetration map in the ϵ-convex set $\mathscr{N}_\epsilon L$ and $\|\mathfrak{crp}_L-\mathfrak{ftp}_L\|_\infty\leq 2c_1'(\infty)$.*

Proof. Let $\gamma\in T^1_{\xi_0}X$, let $\gamma_-=\xi_0$ and γ_+ be the endpoints of γ, and L_1 and L_2 be the endpoints of L. Let p and q be the closest points to γ_- and γ_+ on L respectively, so that $\mathfrak{ftp}_L(\gamma)=d(p,q)$.

If $d(p,q)\leq c_1'(\infty)$, then Lemma 3.5 (3) implies that $0\leq\mathfrak{crp}_L(\gamma)\leq 2\;c_1'(\infty)$, and thus $|\mathfrak{crp}_L(\gamma)-\mathfrak{ftp}_L(\gamma)|\leq 2\;c_1'(\infty)$.

If $d(p,q)>c_1'(\infty)$, then up to renaming the endpoints of L, we have by Lemma 3.5 (2) and (1) that $[\gamma_-,L_2,\gamma_+,L_1]\leq c_1'(\infty)$ and $-2\,c_1'(\infty)+\mathfrak{ftp}_L(\gamma)\leq[\gamma_-,L_1,\gamma_+,L_2]\leq 2\,c_1'(\infty)+\mathfrak{ftp}_L(\gamma)$, which implies the result, using Lemma 3.4. □

Remark. The penetration maps can be defined for any fixed starting point which is outside the ϵ-convex set C, except for $\mathfrak{crp}_L$, and its boundary at infinity. Thus, the penetration maps $\ell_C,\mathfrak{bp}_C,\mathfrak{ph}_H,\mathfrak{ipp}_H,\mathfrak{ph}_B,\mathfrak{ipp}_B,\mathfrak{ftp}_L$ considered in this section are all restrictions to $T^1_{\xi_0}X$ of maps defined, and continuous (as an inspection of the above proof shows) on $\bigcup_{\xi\notin C'\cup\partial_\infty C'} T^1_\xi X \subset T^1X$ with C' respectively $C, C, H, H, B, B, \mathscr{N}_\epsilon L$. The penetration map $\mathfrak{crp}_L$ is defined and continuous on $\bigcup_{\xi\in\partial_\infty X-\partial_\infty C'} T^1_\xi X$. This point of view is used in cases (3) and (4) of Proposition 3.7 below, and will be useful to apply Corollary 4.11.

3.2 Prescribing the penetration

In Section 4, we will use the following operation repeatedly: a geodesic ray or line γ starting from a given point ξ_0 is given that penetrates two ϵ-convex sets C and C' with penetration maps f and f', first entering C with $f(\gamma) = h$, and then C' with $f'(\gamma) \geq h'$. We will need to pick a new geodesic ray or line γ' starting from ξ_0 which intersects C before C', for which we still have $f(\gamma') = h$, and for which we now have the equality $f'(\gamma') = h'$. In the following result, we show that this operation is possible in a number of geometric cases. These cases will be used in Section 5 for various applications.

Proposition 3.7 *Let X be a complete, simply connected Riemannian manifold with sectional curvature at most -1 and dimension at least 3. Let $\epsilon > 0$ and $\delta, h, h' \geq 0$. Let C and C' be ϵ-convex subsets of X, and $\xi_0 \in (X\cup\partial_\infty X) - (C\cup\partial_\infty C)$. Let f and f' be maps $T^1_{\xi_0}X \to [0, +\infty]$, with f' continuous and $\kappa' = \|f' - \ell_{C'}\|_\infty < +\infty$. Consider the following cases:*

(1) *C is a horoball with $\operatorname{diam}(C\cap C') \leq \delta$; f is either the penetration height map $\mathfrak{ph}_C$ or the inner-projection penetration map $\mathfrak{ipp}_C$;*
$$h \geq h^{\min} = 2c_1'(\epsilon) + 2\delta + \|f - \mathfrak{ph}_C\|_\infty$$
and $h' \geq h_0^{\min} = \kappa' + 2\delta$; if C' is also a horoball we may take $\epsilon = +\infty$ in the definition of $h^{\min}$.

(2) *C is a ball of radius R $(\geq \epsilon)$ with $\operatorname{diam}(C\cap C') \leq \delta$; f is either the penetration height map $\mathfrak{ph}_C$ or the inner-projection penetration map $\mathfrak{ipp}_C$;*
$$h > h^{\min} = 2c_1'(\epsilon) + 2\delta + \|f - \mathfrak{ph}_C\|_\infty \;\leq\; h \;\leq\; 2R - 2c_1'(\epsilon) - \|f - \mathfrak{ph}_C\|_\infty = h^{\max}$$
and $h' \geq h_0^{\min} = \kappa' + 2\delta$;

(3) *C is the ϵ-neighbourhood of a complete totally geodesic subspace L of dimension at least 2, with $\operatorname{diam}(C\cap C') \leq \delta$; either $f = \ell_C$ and X has constant curvature, or f is the fellow-traveller penetration map $\mathfrak{ftp}_L$;*
$$h \geq h^{\min} = 4c_1'(\epsilon) + 2\epsilon + \delta + 2\|f - \mathfrak{ftp}_L\|_\infty$$
and $h' > h_0^{\min} = \kappa' + \delta$;

(4)
- *C is the ϵ-neighbourhood of a geodesic line L;*
- *$h \geq h^{\min} = 4c'_1(\epsilon) + 2\epsilon + \delta + \|f - \mathfrak{ftp}_L\|_\infty$;*
- *either $f = \ell_C$ and X has constant curvature, or f is the fellow-traveller penetration map $\mathfrak{ftp}_L$, or $\xi_0 \in \partial_\infty X$, $f = \mathfrak{crp}_L$, and the metric spheres of the Hamenstädt distance on $\partial_\infty X - \{\xi_0\}$ are topological spheres;*
- *either C' is any ϵ-convex subset that does not meet C (in which case $\delta = 0$) and $h' > h_0^{\min} = \kappa'$, or C' is the ϵ-neighbourhood of a totally geodesic subspace with codimension at least two such that* $\operatorname{diam}(C \cap C') \leq \delta$ *and*

$$h' \geq h_0^{\min} = 3c'_1(\epsilon) + 3\epsilon + \delta + \|f' - \mathfrak{ftp}_{L'}\|_\infty .$$

Assume that one of the above cases holds. If there exists a geodesic ray or line γ starting from ξ_0 which meets first C and then C' with $f(\gamma) = h$ and $f'(\gamma) \geq h'$, then there exists a geodesic ray or line $\overline{\gamma}$ starting from ξ_0 which meets first C and then C' with $f(\overline{\gamma}) = h$ and $f'(\overline{\gamma}) = h'$.

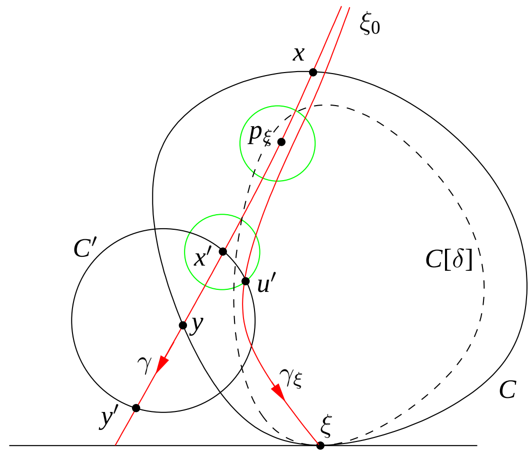
Proof. Let γ be as in the statement, and x (resp. y) be the point where γ enters (resp. exits) C (with y in X since $f(\gamma) = h < +\infty$). Let x' (resp. y') be the point where γ enters (resp. exits) C' (with $x' \in X$ but possibly with y' at infinity). By convexity, $\xi_0 \notin C' \cup \partial_\infty C'$. For every $h \geq 0$, we define A as the set of points $\alpha(+\infty)$ where $\alpha \in T^1_{\xi_0}X$ satisfies $f(\alpha) = h$. Let A_0 be the arcwise connected component of A containing $\gamma(+\infty)$. By considering the various cases, we will prove below the following two claims :

a) every geodesic ray or line, starting from ξ_0 and meeting C', first meets C and then C';

b) there exists a geodesic ray or line $\overline{\gamma}_0$ starting from ξ_0 with $\overline{\gamma}_0(+\infty)$ belonging to A_0, and $f'(\overline{\gamma}_0) \leq h_0^{\min}$.

As f' is continuous and A_0 is arcwise connected, the intermediate value theorem implies the existence of a geodesic $\overline{\gamma}$ with the desired properties, and Proposition 3.7 is proven.

Case (1). Let $\kappa = \|f - \mathfrak{ph}_C\|_\infty$. Let ξ be the point at infinity of C, which is different from ξ_0, and let p_ξ be the closest point to ξ on γ. As $f(\gamma) = h > 0$, the point p_ξ belongs to the interior of the horoball C. Let γ_ξ be the geodesic ray or line starting from ξ_0 with $\gamma_\xi(+\infty) = \xi$.



We start by proving the (stronger) first claim that every geodesic ray or line starting from ξ_0 and meeting C' meets $C[\delta]$ first (hence it meets C before C' and Assertion a) holds). Note that

$$d(y,p_\xi) \geq d(p_\xi,\partial C) = \frac{\mathfrak{ph}_C(\gamma)}{2} \geq \frac{f(\gamma)-\kappa}{2} = \frac{h-\kappa}{2} \geq \frac{h^{\min}-\kappa}{2} = c_1'(\epsilon)+\delta > \delta\ .$$

As $f'(\gamma) \geq h' \geq h_0^{\min} = \kappa' + 2\delta$, we have $\ell_{C'}(\gamma) \geq f'(\gamma) - \kappa' > \delta$, unless $\ell_{C'}(\gamma) = \delta = 0$. Note that $\gamma \cap C'$ is not contained in the geodesic segment $[x,y]$. Otherwise, this would contradict the assumption that $\operatorname{diam}(C \cap C') \leq \delta$ when $\ell_{C'}(\gamma) > \delta$. When $\ell_{C'}(\gamma) = \delta = 0$, as γ meets C', the segment $\gamma \cap C'$ would be reduced to a point by the convexity of C'. This point would be $\{x\}$ or $\{y\}$ (as C' is not a singleton). But then the tangent vector of γ at x or its opposite at y would both enter strictly C and be tangent to C', which contradicts the fact that $\delta = 0$.

As γ meets C before C', this implies, in particular, that the geodesic ray $[y,\gamma(+\infty)[$ meets C', and that the point p_ξ belongs to $]x',\xi_0]$: otherwise $C \cap C'$ would contain a segment of length at least $d(p_\xi,y) > \delta$, which is impossible. Hence, by convexity, any geodesic ray or line starting from ξ_0 and meeting $B(x',c_1'(\epsilon))$ first meets $B(p_\xi,c_1'(\epsilon))$. By Lemma 2.3, every geodesic ray or line, starting from ξ_0 and meeting C', meets the ball $B(x',c_1'(\epsilon))$ before entering C' (and we may take $\epsilon = +\infty$ if C' is also a horoball, by Lemma 2.9). This proves the first claim, as the ball $B(p_\xi,c_1'(\epsilon))$ is contained in $C[\delta]$, since $d(p_\xi,\partial C) \geq c_1'(\epsilon)+\delta$, as seen above.

Let us prove now the (stronger) second claim that there exists a geodesic ray or line $\overline{\gamma}_0$ starting from ξ_0 with $\overline{\gamma}_0(+\infty)$ belonging to A_0, and avoiding the interior of C', which implies assertion b), as then $f'(\overline{\gamma}_0) \leq \ell_{C'}(\overline{\gamma}_0) + \kappa' = \kappa' \leq h_0^{\min}$.

The subspace A of $\partial_\infty X$ is a codimension 1 topological submanifold of the topological sphere $\partial_\infty X$, which is homeomorphic to the sphere $\mathbb{S}^{n-2}$, hence it is arcwise connected. Indeed, if $f = \mathfrak{ph}_C$, then A is the subset of endpoints of the geodesic rays or lines starting from ξ_0 that are tangent to $\partial\big(C[h/2]\big)$. If $f = \mathfrak{ipp}_C$, the subset A is the preimage of a point in $]\xi_0, \xi[$ by the closest point map from $\partial_\infty X$ to $[\xi_0, \xi]$, which is, over $]\xi_0, \xi[$, a trivial topological bundle with fibers homeomorphic to $\mathbb{S}^{n-2}$.

Note that f is continuous, $f(\gamma_\xi) = \infty > h$, and $f(\alpha) = 0$ if α is a geodesic ray or line starting from ξ_0 with $\alpha(+\infty)$ close enough to $\gamma(-\infty)$. Therefore A separates $\gamma(-\infty)$ and $\gamma_\xi(+\infty)$, as the connected components of $\partial_\infty X - A$ are arcwise connected.

If the (stronger) second claim is not true, then the topological sphere $A_0 = A$ of dimension $n-2$ is contained in the interior of the shadow $\mathscr{O}_{\xi_0}C'$. As $\xi_0 \notin C' \cup \partial_\infty C'$, this shadow is homeomorphic to a ball of dimension $n-1$. Thus, by Jordan's theorem, one of the two connected components of $\partial_\infty X - A$ is contained in the interior of $\mathscr{O}_{\xi_0}C'$. As $\gamma(-\infty)$ does not belong to $\mathscr{O}_{\xi_0}C'$ and A separates $\gamma(-\infty)$ and $\gamma_\xi(+\infty)$, this implies that $\gamma_\xi(+\infty)$ belongs to the interior of $\mathscr{O}_{\xi_0}C'$. Hence γ_ξ meets the interior of C'.

Therefore, by the first claim, the geodesic ray or line γ_ξ meets $C[\delta]$ before meeting C'. Let u' be the entering point of γ_ξ in C'. As ξ is the point at infinity of $C[\delta]$, the points ξ_0, u', ξ are in this order on γ_ξ. Hence by convexity, this implies that u' belongs to $C[\delta]$. As $f'(\gamma) \geq h' \geq h_0^{\min} = \kappa' + 2\delta$, we have

$$d(x', y') = \ell_{C'}(\gamma) \geq f'(\gamma) - \kappa' \geq 2\delta\ .$$

Hence by the triangular inequality, one of the two distances $d(u', x'), d(u', y')$ is at least δ, and by the strict convexity of the distance, it is strictly bigger than δ (as u' does not belong to γ (as $\gamma \neq \gamma_\xi$). Hence, if u'' is a point close enough to u' in $]u', \xi[$, then u'' belongs to the interior of C' and to the interior of $C[\delta]$, and is at distance strictly greater than δ from either $z' = x'$ or $z' = y'$. Therefore, the geodesic segment $[u'', z'] \cap C$ has length strictly bigger than δ, and is contained in the intersection $C \cap C'$. This contradicts the assumption that $\operatorname{diam}(C \cap C') \leq \delta$.

Case (2). The proof is completely similar to Case (1). Let now z be the center of the ball C, let p_z be the point of γ the closest to z, let γ_z be the geodesic ray or line starting from ξ_0 and passing through z, and let $\kappa = \|f - \mathfrak{ph}_C\|_\infty$. Note that $R > h^{\max}/2 \geq h^{\min}/2 \geq \delta$, so that $C[\delta]$ is non empty. We only have to replace ξ by z, p_ξ by p_z and γ_ξ by γ_z, and to replace two arguments in the above proof, the one in order to show that A separates $\gamma(-\infty)$ from $\gamma_z(+\infty)$, and the one in order to show that ξ_0, u', z are in this order on γ_z, where u' is the entering point of γ_z in C'.

To prove that A separates $\gamma(-\infty)$ from $\gamma_z(+\infty)$, we simply use now that $f(\gamma_z) = 2R > h^{\max} \geq h$ instead of $f(\gamma_\xi) = \infty > h$. Let us prove that ξ_0, u', z are in this order on γ_z. We have

$$d(z, p_z) = R - \frac{\mathfrak{ph}_C(\gamma)}{2} \geq R - \frac{f(\gamma) + \kappa}{2} \geq R - \frac{h^{\max} + \kappa}{2} = c'_1(\epsilon) \ .$$

By Lemma 2.3, we have $d(x', u') \leq c'_1(\epsilon)$. As γ_z meets the interior of C', by the same argument as in Case 1, we even have $d(u', \gamma) < c'_1(\epsilon)$. Hence by strict convexity, we do have $u' \in \,]z, \xi_0[$. The rest of the argument in the proof of Case (1) is unchanged.

Before studying the last two cases, we start by proving two lemmas. The first one implies the first of the two claims we need to prove in Cases (3), (4), and the second one gives the topological information on A that we will need in these last two cases.

Lemma 3.8 *Let L be a complete totally geodesic subspace with dimension at least 1, $\epsilon > 0$, $C = \mathscr{N}_\epsilon L$, $\xi_0 \in (X \cup \partial_\infty X) - (C \cup \partial_\infty L)$, and C' be an ϵ-convex subset of X such that $\operatorname{diam}(C \cap C') \leq \delta$. Let $f, f' : T^1_{\xi_0} X \to [0, +\infty]$ be maps such that $\kappa = \|f - \mathfrak{ftp}_L\|_\infty < +\infty$, $\kappa' = \|f' - \ell_{C'}\|_\infty < +\infty$. Let γ be a geodesic ray or line starting from ξ_0, entering C before entering C', such that $4c'_1(\epsilon) + 2\epsilon + \delta + \kappa \leq f(\gamma) < +\infty$ and $f'(\gamma) > \delta + \kappa'$. If $\widetilde{\gamma}$ is a geodesic ray or line starting from ξ_0 which meets C', then $\widetilde{\gamma}$ meets the interior of C before meeting C'.*

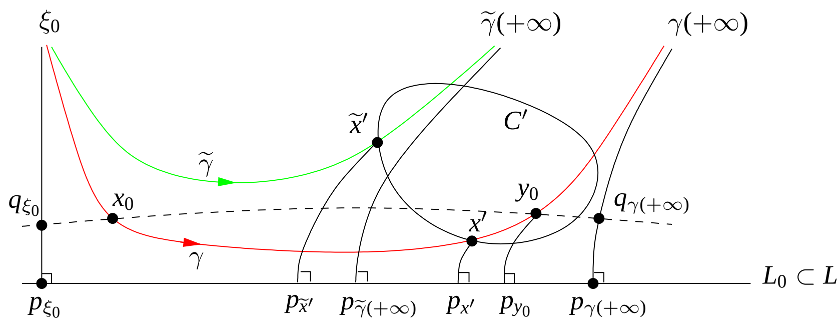
Proof. Note that $\xi_0 \notin C' \cup \partial_\infty C'$, by convexity and the assumptions on γ, as $\xi_0 \notin C \cup \partial_\infty C$. Let L_0 be the geodesic line passing through the closest points $p_{\xi_0}, p_{\gamma(+\infty)}$ on L of $\xi_0, \gamma(+\infty)$, respectively. Note that

$$d(p_{\xi_0}, p_{\gamma(+\infty)}) = \mathfrak{ftp}_L(\gamma) \geq f(\gamma) - \kappa \geq 4c'_1(\epsilon) + 2\epsilon + \delta > 0 \ . \tag{- 13 -}$$

Hence, by Lemma 3.4, and as $\mathfrak{ftp}_{L_0}(\gamma) = \mathfrak{ftp}_L(\gamma)$, we have

$$\ell_{\mathscr{N}_\epsilon L_0}(\gamma) \geq \mathfrak{ftp}_{L_0}(\gamma) - 2c'_1(\epsilon) - 2\epsilon > 0.$$

In particular, γ enters $\mathscr{N}_\epsilon L_0$ at a point x_0 and exits it at a point y_0 in X (as $\gamma(+\infty) \notin \partial_\infty L$ since $f(\gamma) < +\infty$). Let $u \mapsto p_u$ be the closest point map from $X \cup \partial_\infty X$ onto $L_0 \cup \partial_\infty L_0$. Recall that this map does not increase the distances (and even decreases them, unless the two points under consideration are on L_0), and that it *preserves betweenness*, that is, if $u'' \in [u, u']$, then $p_{u''} \in [p_u, p_{u'}]$. Let x' (resp. $\widetilde{x}'$) be the point where γ (resp. $\widetilde{\gamma}$) enters C', and q_{ξ_0} and $q_{\gamma(+\infty)}$ be the closest point to ξ_0 and $\gamma(+\infty)$ respectively on $\mathscr{N}_\epsilon L_0$.



Recall that by Lemma 2.3, the distances $d(\widetilde{x}',x'), d(x_0,q_{\xi_0}), d(y_0,q_{\gamma(+\infty)})$ are at most $c_1'(\epsilon)$. Note that $\widetilde{x}'\in[\xi_0,\widetilde{\gamma}(+\infty)]$. Hence, as betweenness is preserved,

$$\begin{aligned}\mathfrak{ftp}_{L_0}(\widetilde{\gamma}) = d(p_{\xi_0},p_{\widetilde{\gamma}(+\infty)}) \geq d(p_{\xi_0},p_{\widetilde{x}'}) \geq d(p_{\xi_0},p_{x'}) - d(p_{x'},p_{\widetilde{x}'})\\ \geq d(p_{\xi_0},p_{x'}) - d(x',\widetilde{x}') \geq d(p_{\xi_0},p_{x'}) - c_1'(\epsilon)\ .\end{aligned}$$

Note that $d(p_{\xi_0},p_{x'}) \geq d(p_{\xi_0},p_{y_0})$ when ξ_0,y_0,x' are in this order on γ. When ξ_0,y_0,x' are not in this order on γ, as γ enters in C before C', as $\ell_{C'}(\gamma)\geq f'(\gamma)-\kappa'>\delta$ and as $\operatorname{diam}(C\cap C')\leq\delta$, we have $d(x',y_0)\leq\delta$; hence

$$d(p_{\xi_0},p_{x'}) \geq d(p_{\xi_0},p_{y_0}) - d(p_{y_0},p_{x'}) \geq d(p_{\xi_0},p_{y_0}) - d(y_0,x') \geq d(p_{\xi_0},p_{y_0}) - \delta\ .$$

Therefore, in both cases, as $y_0\in[\xi_0,\gamma(+\infty)]$ and $u\mapsto p_u$ preserves the betweenness, and since $p_{\gamma(+\infty)} = p_{q_{\gamma(+\infty)}}$, we have, using (- 13 -),

$$\begin{aligned}\mathfrak{ftp}_{L_0}(\widetilde{\gamma}) &\geq d(p_{\xi_0},p_{\widetilde{x}'}) \geq d(p_{\xi_0},p_{x'}) - c_1'(\epsilon) \geq d(p_{\xi_0},p_{y_0}) - \delta - c_1'(\epsilon)\\ &\geq d(p_{\xi_0},p_{\gamma(+\infty)}) - d(p_{q_{\gamma(+\infty)}},p_{y_0}) - c_1'(\epsilon) - \delta\\ &> \mathfrak{ftp}_L(\gamma) - d(q_{\gamma(+\infty)},y_0) - c_1'(\epsilon) - \delta \geq\\ &\mathfrak{ftp}_L(\gamma) - 2c_1'(\epsilon) - \delta \geq 2c_1'(\epsilon) + 2\epsilon\ .\end{aligned}$$

By Lemma 3.4, we hence have

$$\ell_{\mathscr{N}_\epsilon L}(\widetilde{\gamma}) \geq \ell_{\mathscr{N}_\epsilon L_0}(\widetilde{\gamma}) \geq \mathfrak{ftp}_{L_0}(\widetilde{\gamma}) - 2c_1'(\epsilon) - 2\epsilon > 0.$$

In particular, $\widetilde{\gamma}$ does enter the interior of C, at a point $\widetilde{x}$. Note that the geodesic from ξ_0 through p_{ξ_0} enters C at q_{ξ_0}. Now by absurd, if $\widetilde{\gamma}$ enters the interior of C after it enters C', then $\widetilde{x}'\in[\xi_0,\widetilde{x}]$, so that

$$c_1'(\epsilon) \geq d(q_{\xi_0},\widetilde{x}) \geq d(p_{\xi_0},p_{\widetilde{x}}) \geq d(p_{\xi_0},p_{\widetilde{x}'}) > 2c_1'(\epsilon) + 2\epsilon\ ,$$

as seen above, a contradiction. □

Lemma 3.9 *Let X be a complete, simply connected Riemannian manifold with sectional curvature at most -1 and dimension at least 3. Let $\epsilon,h>0$. Let L be a complete totally geodesic submanifold with dimension at least 1 and $\xi_0\in(X\cup\partial_\infty X)-(\mathscr{N}_\epsilon L\cup\partial_\infty L)$. Assume either that*

(1) *$f = \ell_{\mathscr{N}_\epsilon L}$, X has constant curvature and $h \in [4c'_1(\epsilon) + 2\epsilon, +\infty[$, or*

(2) *$\xi_0 \in \partial_\infty X$, $\dim L = 1$, $f = \mathfrak{crp}_L$, $h \in \,]\log 2, +\infty[$, and the metric spheres of the Hamenstädt distance on $\partial_\infty X - \{\xi_0\}$ are topological spheres, or*

(3) *$f = \mathfrak{ftp}_L$.*

Then

$$A = \{\alpha(+\infty) \;:\; \alpha \in T^1_{\xi_0}X,\ f(\alpha) = h\}$$

is a codimension 1 topological submanifold of the topological sphere $\partial_\infty X$, which is homeomorphic to the torus $\mathbb{S}^{\dim L - 1} \times \mathbb{S}^{\operatorname{codim} L - 1}$. Furthermore,

(a) *if $\dim L = 1$, then A has two arcwise connected components, homeomorphic to a sphere of dimension $n-2$. If $f = \mathfrak{crp}_L$ or if $h > c'_1(\epsilon)$, then each of them separates $\gamma(-\infty)$ and exactly one of the two points at infinity of L, for every geodesic ray or line γ starting from ξ_0 if $f = \mathfrak{crp}_L$, and for those meeting $\mathscr{N}_\epsilon L$ if $f \neq \mathfrak{crp}_L$.*

(b) *if $\operatorname{codim} L = 1$, then A has two arcwise connected components, homeomorphic to a sphere of dimension $n-2$, separated by $\partial_\infty L$.*

(c) *if $\dim L \geq 2$ and $\operatorname{codim} L \geq 2$, then A is arcwise connected.*

In cases (b) and (c), for every component A_0 of A, for every geodesic ray ρ in L with $\rho(0)$ the closest point to ξ_0 on L, there exists $\eta \in A_0$ such that $\rho(h)$ is at distance at most $\|f - \mathfrak{ftp}_L\|_\infty$ from the closest point to η on ρ.

Proof. Let $\pi_L : X \cup \partial_\infty X \to L \cup \partial_\infty L$ be the closest point map, and $p_0 = \pi_L(\xi_0)$. Note that L has codimension at least one, by the existence of ξ_0.

Assume first that $f = \mathfrak{ftp}_L$. As $h > 0$, the subspace A of $\partial_\infty X$ is the preimage of the sphere (of dimension $\dim L - 1$) of center p_0 and radius h in L, by π_L. As $\pi_L : \partial_\infty X \setminus \partial_\infty L \to L$ is a trivial topological bundle whose fibers are spheres of dimension $\operatorname{codim} L - 1$, the topological structure (including the assertions (b) and (c)) of A is immediate. The final statement on (b) and (c) is trivial as, by definition, $\rho(h)$ is the closest point to some point in A_0.

If $\dim L = 1$ and $h > c'_1(\epsilon)$, if $\gamma \in T^1_{\xi_0}X$ meets $\mathscr{N}_\epsilon L$, then by Lemma 2.3 and by convexity, $d(\pi_L(\gamma(-\infty)), p_0) \leq c'_1(\epsilon) < h$. Hence, the separation statement in (a) follows.

Assume then that $f = \mathfrak{crp}_L$, and that the hypotheses of (2) are satisfied. The result in this case (only assertion (a) needs to be checked) follows from the discussion before Lemma 3.5.

Assume now that $f=\ell_{\mathscr{N}_\epsilon L}$, X has constant curvature, and $h\in[4c_1'(\epsilon)+2\epsilon,+\infty[$. Let $S_0=\pi_L^{-1}(p_0)\cap\partial_\infty X$. Using normal coordinates along L, the topological sphere $\partial_\infty X$ is homeomorphic to the topological join of the spheres $\partial_\infty L$ of dimension $\dim L-1$ and S_0 of dimension $\operatorname{codim} L-1$

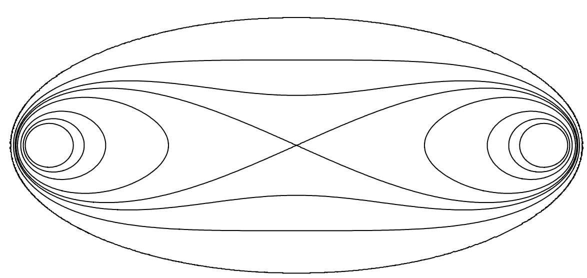
$$S_0\vee\partial_\infty L=\big(S_0\times[0,+\infty]\times\partial_\infty L\big)/\sim\,,$$

where $\sim$ is the equivalence relation generated by $(a,0,b)\sim(a,0,b')$ as well as $(a,+\infty,b)\sim(a',+\infty,b)$, for every a,a' in S_0 and b,b' in $\partial_\infty L$. We denote by $[a,t,b]$ the equivalence class of (a,t,b). We choose the parametrization of $\partial_\infty X$ by $S_0\vee\partial_\infty L$ such that $[a,0,b]=a$, $[a,+\infty,b]=b$, $d(\pi_L([a,t,b]),p_0)=t$, and the geodesic rays $\big[\pi_L([a,t,b]),[a,t,b]\big[$ are parallel transports of $[p_0,a[$ along the geodesic ray $[p_0,b[$, for $0<t<+\infty$.

For every t in $]0,+\infty[$ and every (a,b) in $S_0\times\partial_\infty L$, let $\gamma_{[a,t,b]}$ be the geodesic ray or line starting from ξ_0 and ending at $[a,t,b]$. By the proof of Lemma 3.4 and by Lemma 3.2, we have

$$-2c_1'(\epsilon)\le\ell_{\mathscr{N}_\epsilon L}(\alpha)-\mathfrak{ftp}_L(\alpha)\le 2c_1'(\epsilon)+2\epsilon,$$

for every $\alpha\in T^1_{\xi_0}X$. In particular, if $t=\mathfrak{ftp}_L(\gamma_{[a,t,b]})>2c_1'(\epsilon)$, then $\ell_{\mathscr{N}_\epsilon L}(\gamma_{[a,t,b]})>0$, that is $\gamma_{[a,t,b]}$ meets the interior of $\mathscr{N}_\epsilon L$. The points p_0,a,b,ξ_0 are contained in an isometrically embedded copy of $\mathbb{H}^3_{\mathbb{R}}$ in X, and therefore we can restrict to the case when X is the upper halfspace model of $\mathbb{H}^3_{\mathbb{R}}$ and L has dimension 1. If we normalize so that $\xi_0=\infty$ and the endpoints of L are $b=1$ and $-b$, then the level sets of $\ell_{\mathscr{N}_\epsilon L}$ for positive values are drawn in the following picture. The level set $\ell^{-1}_{\mathscr{N}_\epsilon L}(2\epsilon)$ is the unique figure-8 curve. Each level set $\ell^{-1}_{\mathscr{N}_\epsilon L}(t)$ for $t>2\epsilon$ has exacty two components, one in each bounded component of the complement of $\ell^{-1}_{\mathscr{N}_\epsilon L}(2\epsilon)$.



The curve $t\mapsto[a,t,b]$, $t\ge 0$, is a segment of a circle through b and $-b$ that connects the point a on the imaginary axis to b. Thus, it is easy to see that the map from $[2c'_1(\epsilon), +\infty[$ to $[0, +\infty[$ defined by $t \mapsto \ell_{\mathscr{N}_\epsilon L}(\gamma_{[a,t,b]})$ is continuous and strictly increasing, for every fixed (a, b) in $S_0 \times \partial_\infty L$.

Hence, as Lemma 3.4 gives

$$\ell_{\mathscr{N}_\epsilon L}(\gamma_{[a,2c'_1(\epsilon),b]}) \leq 4c'_1(\epsilon) + 2\epsilon \leq h < +\infty,$$

there exists a unique $t_{a,b} \in [2c'_1(\epsilon), +\infty[$, depending continuously on (a, b), such that $\ell_{\mathscr{N}_\epsilon L}(\gamma_{[a,t_{a,b},b]}) = h$. In particular, the subset of points of $\partial_\infty X$ of the form $[a, t_{a,b}, b]$ for some (a, b) in $S_0 \times \partial_\infty L$ is indeed a codimension 1 topological submanifold of $\partial_\infty X$, which is homeomorphic to the torus $\mathbb{S}^{\dim L-1} \times \mathbb{S}^{\operatorname{codim} L-1}$. The statements (b) and (c) follow.

If L has dimension 1, and if $\gamma \in T^1_{\xi_0}X$ meets $\mathscr{N}_\epsilon L$, then by Lemma 2.3 and by convexity, $d(\pi_L(\gamma(-\infty)), p_0) \leq c'_1(\epsilon)$. For every ξ in a component A_0 of A, if as above $\xi = [a, t_{a,b}, b]$, then we have $d(\pi_L(\xi), p_0) = t_{a,b} \geq 2c'_1(\epsilon)$, hence A_0 separates $\gamma(-\infty)$ and b. This proves (a).

Let us prove the last assertion of the lemma. Let $\kappa = \|f - \mathfrak{ftp}_L\|_\infty$, and let A_0 be a connected component of A. For every u in L such that $d(u, p_0) = h$, let $\eta_0 = [a, h, b]$, on the same side of $\partial_\infty L$ as A_0 if $\operatorname{codim} L = 1$, be such that $\pi_L(\eta_0) = u$. Let $\eta_t = [a, h + t, b]$, which is on the same side of $\partial_\infty L$ as A_0 if $\operatorname{codim} L = 1$. Note that

$$f(\gamma_{[a,h+\kappa,b]}) \geq \mathfrak{ftp}_L(\gamma_{[a,h+\kappa,b]}) - \kappa = h,$$

and similarly, $f(\gamma_{[a,h-\kappa,b]}) \leq h$. By the intermediate value theorem, there exists $t \in [-\kappa, +\kappa]$ such that $\eta_t \in A_0$. Hence $d(u, \pi_L(\eta_t)) = |t| \leq \kappa$. □

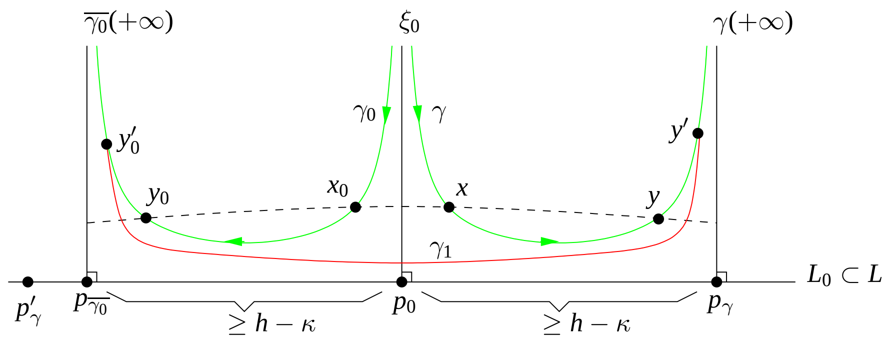
Now we proceed with the proof of the remaining parts of Proposition 3.7.

Case (3). By Lemma 3.8, we only have to prove the second claim, that there exists a geodesic ray or line $\overline{\gamma_0}$ starting from ξ_0 with $\overline{\gamma_0}(+\infty)$ belonging to A_0, such that $f'(\overline{\gamma_0}) \leq h_0^{\min}$.

Let $\kappa = \|f - \mathfrak{ftp}_L\|_\infty$. Let p_0 (respectively p_γ) be the point of L the closest to ξ_0 (respectively $\gamma(+\infty)$), so that, in particular,

$$d(p_0, p_\gamma) = \mathfrak{ftp}_L(\gamma) \geq f(\gamma) - \kappa = h - \kappa > 0.$$

Let p'_γ be the point on the geodesic line L_0 (contained in L) passing through p_0 and p_γ on the opposite side of p_γ with respect to p_0, and at distance h from p_0. By Lemma 3.9, there exists a geodesic line $\overline{\gamma_0}$ starting from ξ_0 and ending at a point in A_0 whose closest point $p_{\overline{\gamma_0}}$ on L is at distance at most κ from p'_γ.



Assume by absurd that $f'(\overline{\gamma_0}) > h_0^{\min}$. We have

$$\ell_{C'}(\gamma) \geq f'(\gamma) - \kappa' \geq h' - \kappa' \geq h_0^{min} - \kappa' > \delta.$$

Similarly, $\ell_{C'}(\overline{\gamma_0}) > \delta$, and, in particular, $\overline{\gamma_0}$ enters C'. Let y' (resp. y'_0) be the point, possibly at infinity, where γ (resp. $\overline{\gamma_0}$) exits C'. By Lemma 3.8, $\overline{\gamma_0}$ meets C before C'. Let x_0 (resp. y_0) be the point where $\overline{\gamma_0}$ enters in (resp. exits) C. As $\operatorname{diam}(C \cap C') \leq \delta$, we have $y'_0 \in \,]y_0, \gamma_0(+\infty)[$ and $y' \in \,]y, \gamma(+\infty)[$, so that in particular $d(y'_0, L) > \epsilon$ and $d(y', L) > \epsilon$.

Let γ_1 be the geodesic line through y' and y'_0. The points at infinity of γ_1 do not belong to $\partial_\infty L_0$, so that $\mathfrak{ftp}_{L_0}(\gamma_1)$ and $\ell_C(\gamma_1)$ are finite. Note that by strict convexity and by Lemma 2.3, we have

$$d(y', [p_\gamma, \gamma(+\infty)[) < d(y, [p_\gamma, \gamma(+\infty)[) \leq c'_1(\epsilon),$$

and, similarly, $d(y'_0, [p_{\overline{\gamma_0}}, \overline{\gamma_0}(+\infty)[) < c'_1(\epsilon)$. Hence, with π_{L_0} the closest point map to L_0, which preserves betweenness and does not increase distances,

$$\begin{aligned}\mathfrak{ftp}_{L_0}(\gamma_1) &\geq d(\pi_{L_0}(y'_0), \pi_{L_0}(y')) \\ &> d(p_{\overline{\gamma_0}}, p_\gamma) - 2c'_1(\epsilon) = d(p_{\overline{\gamma_0}}, p_0) + d(p_0, p_\gamma) - 2c'_1(\epsilon) \\ &\geq h - \kappa + h - \kappa - 2c'_1(\epsilon) = 2h - 2\kappa - 2c'_1(\epsilon)\ .\end{aligned}$$

In particular, by Lemma 3.4,

$$\ell_C(\gamma_1) \geq \ell_{\mathscr{N}_\epsilon L_0}(\gamma_1) \geq \mathfrak{ftp}_{L_0}(\gamma_1) - 2c'_1(\epsilon) - 2\epsilon > 2h - 2\kappa - 4c'_1(\epsilon) - 2\epsilon \geq \delta\ ,$$

by the definition of $h^{\min}$. Hence γ_1 meets C in a segment I of length $> \delta$. But as y'_0 and y' are at a distance strictly bigger than ϵ of L, the segment I is contained in $[y', y'_0]$, which is contained in C', by convexity. This contradicts the assumption that $\operatorname{diam}(C \cap C') \leq \delta$.

Case (4)**.** Let $\kappa'' = \|f' - \mathfrak{ftp}_{L'}\|_\infty$. Note that $f'(\gamma) \geq h' \geq h_0^{\min} > \delta + \kappa'$. This is true under both assumptions on the value of $h_0^{\min}$, as when $h_0^{\min} = 3c'_1(\epsilon) + 3\epsilon + \delta + \kappa''$, we have, by Lemma 3.4,

$$\delta + \kappa' \leq \delta + \kappa'' + 2c'_1(\epsilon) + 2\epsilon < h_0^{\min} \ .$$

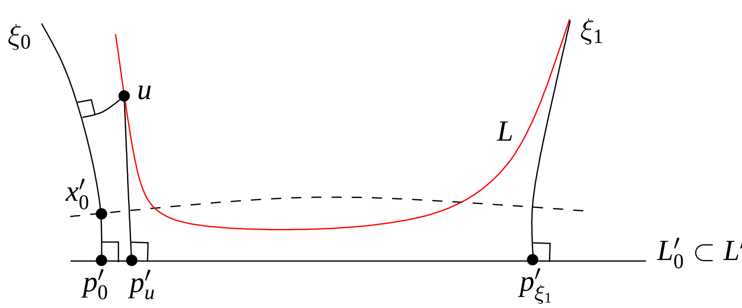
By Lemma 3.8, we only have to prove the second claim that there exists a geodesic line $\overline{\gamma}_0$ starting from ξ_0 with $\overline{\gamma}_0(+\infty)$ belonging to A_0, such that $f'(\overline{\gamma}_0) \leq h_0^{\min}$.

We first consider the case $C' = \mathscr{N}_\epsilon L'$ where L' is a totally geodesic subspace of codimension at least 2, with $\operatorname{diam}(C \cap C') \leq \delta$, and $h' \geq h_0^{\min} = 3c'_1(\epsilon) + 3\epsilon + \delta + \kappa''$. Assume by absurd that every geodesic ray or line α starting from ξ_0 with $\alpha(\infty) \in A_0$ satisfies $f'(\alpha) > h_0^{\min}$. In particular, every such α satisfies $\ell_{C'}(\alpha) \geq f'(\alpha) - \kappa' \geq h_0^{\min} - \kappa' > \delta \geq 0$, hence α meets the interior of C'. Let

$$B' = \{\beta(\infty) : \beta \in T^1_{\xi_0} X,\ \mathfrak{ftp}_{L'}(\beta) > h_0^{\min} - \kappa''\} \ .$$

By the absurdity hypothesis and the definition of κ'', we have $A_0 \subset B'$. Let p'_0 be the closest point to ξ_0 on L'. Note that B' is a (topological) open tubular neighbourhood of $\partial_\infty L'$, whose fiber over a point ξ in $\partial_\infty L'$ is the preimage of $\rho_\xi(]h_0^{\min} - \kappa'', +\infty])$ by the closest point map from $\partial_\infty X$ to $L' \cup \partial_\infty L'$, where ρ_ξ is the geodesic ray with $\rho(0) = p'_0$ and $\rho(+\infty) = \xi$.

By Lemma 3.9 (a), let ξ_1 be the point at infinity of L separated from $\gamma(-\infty)$ by A_0. Let $p'_{\gamma(-\infty)}$ be the closest point to $\gamma(-\infty)$ on L'. Recall that γ enters C' at x', and $\xi_0 \notin C'$, so that $\xi_0 \in [\gamma(-\infty), x'[$. Hence, by Lemma 2.3 and the fact that closest point maps preserve betweenness and do not increase the distances, we have $d(p'_{\gamma(-\infty)}, p'_0) \leq c'_1(\epsilon) < h_0^{\min} - \kappa''$ by the definition of $h_0^{\min}$. Hence the complement of B' in $\partial_\infty X$, which is connected as $\operatorname{codim} L' \geq 2$, contains $\gamma(-\infty)$. As A_0 separates $\gamma(-\infty)$ from ξ_1 and is contained in B', it follows that B' contains ξ_1.



Let x'_0 be the intersection point of $]\xi_0, p'_0]$ with $\partial C'$. Lemma 2.3 then implies that $d(x', x'_0) \leq c'_1(\epsilon)$. Hence, by convexity and as γ first meets C and then C', we have

$d\big(x,]\xi_0, p'_0]\big) \leq c'_1(\epsilon)$. If $u \in L$ is the closest point to x on L, we therefore have $d(u,]\xi_0, p'_0]) \leq c'_1(\epsilon) + \epsilon$. Let p'_{ξ_1} be the closest point to ξ_1 on L', and L'_0 the geodesic line (contained in L') through p'_0 and p'_{ξ_1}. As the closest point map does not increase distances, the closest point p'_u to u on L'_0 satisfies $d(p'_0, p'_u) \leq c'_1(\epsilon) + \epsilon$. Then, since the closest point map to L'_0 preserves betweenness, by the triangle inequality, and as ξ_1 belongs to B', we have

$$\mathfrak{ftp}_{L'_0}(L) \geq d(p'_u, p'_{\xi_1}) \geq d(p'_{\xi_1}, p'_0) - d(p'_0, p'_u) > h_0^{\min} - \kappa'' - c'_1(\epsilon) - \epsilon\ .$$

Therefore, using Lemma 3.4,

$$\begin{aligned}\operatorname{diam}(C \cap C') \geq \operatorname{diam}(C \cap \mathscr{N}_\epsilon L'_0) \geq \ell_{\mathscr{N}_\epsilon L'_0}(L) \geq \mathfrak{ftp}_{L'_0}(L) - 2c'_1(\epsilon) - 2\epsilon \\ > h_0^{\min} - \kappa'' - 3c'_1(\epsilon) - 3\epsilon = \delta\ ,\end{aligned}$$

a contradiction.

Assume now that C' is any ϵ-convex subset such that $C \cap C' = \emptyset$, and that $h' > h_0^{\min} = \kappa''$. Let us prove that there exists a geodesic ray or line $\overline{\gamma}_0$ starting from ξ_0 with $\overline{\gamma}_0(+\infty)$ in A_0, and avoiding the interior of C'. This implies the result as in Case (1).

By absurd, suppose that for every ξ in A_0, the geodesic ray or line γ_ξ starting from ξ_0 and ending at ξ meets the interior of C'. By Lemma 3.8 (applied with $\delta = 0$), γ_ξ meets the interior of C before meeting C'. Let x'_ξ be the entering point of γ_ξ in C' and y_ξ be its exiting point out of C. As C and C' are disjoint, note that $\xi_0, y_\xi, x'_\xi, \xi$ are in this order along γ_ξ. The maps $\xi \mapsto y_\xi$ and $\xi \mapsto x'_\xi$ are injective and continuous on A_0 (by the strict convexity of C and C', as γ_ξ meets the interior of C and C'). We know that A_0 is a topological sphere, by Lemma 3.9 (a), separating the endpoints of L. Hence the subsets A_0 and $S' = \{x'_\xi\ : \xi \in A_0\}$ are spheres, that are homotopic (by the homotopy along the geodesic ray or line γ_ξ that does not meet L between x'_ξ and ξ) in the complement of L in $X \cup \partial_\infty X$. By a homology argument, every disc with boundary S' in $X \cup \partial_\infty X$ has to meet L. But by convexity of C', there exists a disc contained in C' with boundary S' (fix a point of S' and take the union of the geodesic arcs from this point to the other points of S'). This contradicts the fact that $C \cap C' = \emptyset$. $\square$

Remarks. (1) In Case (2), we have $h^{\max} \geq h^{\min}$ if R is big enough, as $c'_1(\epsilon)$ has a finite limit as $\epsilon \to \infty$.

(2) In Case (3), if the codimension of L is 1, then we may assume that $\overline{\gamma}$ meets L if γ meets L. Indeed, as we have seen in Lemma 3.9(b), $L \cup \partial_\infty L$ separates $X \cup \partial_\infty X$ into

two connected components, and A (defined in the beginning of the proof) has exactly two components separated by $L\cup\partial_\infty L$. If A_0^+ is the component of A on the same side of $L\cup\partial_\infty L$ as ξ_0, and A_0^- the component of A on the other side, then a geodesic ray or line starting from ξ_0 and ending in A_0^+ does not meet L (as L is totally geodesic), and any geodesic line starting from ξ_0 and ending in A_0^- meets L, by separation. This observation on the crossing property will be used in the proof of Corollary 5.12 to make sure that the locally geodesic ray or line constructed in the course of the proof stays in the convex core.

(3) Case (4) is not true if C' is assumed to be any ϵ-convex subset, as shown by taking X the real hyperbolic 3-space, and C' the ϵ-neighbourhood of the (totally geodesic) hyperbolic plane perpendicular to L at a point at distance h from the closest point to ξ_0 on L: any geodesic ray or line α starting from ξ_0, with $\mathfrak{ftp}_L(\alpha)=h$ and meeting C' satisfies $f'(\alpha)=+\infty$ for every f' which is a κ'-penetration map in C'.

4 The main construction

4.1 Unclouding the sky

The aim of this section is to prove the following result, improving on our result in [PP1]. The first claim of Theorem 4.1 was stated as Theorem 1.1 in the introduction.

Theorem 4.1 *Let X be a proper geodesic CAT(-1) metric space (having at least two points), with arcwise connected boundary $\partial_\infty X$ and extendible geodesics. Let $(H_\alpha)_{\alpha\in\mathscr{A}}$ be any family of balls or horoballs with pairwise disjoint interiors. Let $\mu_0=1.534$.*

(1) *For every x in $X-\bigcup_{\alpha\in\mathscr{A}}H_\alpha$, there exists a geodesic ray starting from x and avoiding $H_\alpha[\mu_0]$ for every α.*

(2) *For every α_0 in $\mathscr{A}$ such that H_{α_0} is an horoball, there exists a geodesic line starting from the point at infinity of H_{α_0} and avoiding $H_\alpha[\mu_0]$ for every $\alpha\neq\alpha_0$.*

Remarks. (1) Note that by its generality, Theorem 4.1 greatly improves the main results, Theorem 1.1 and Theorem 4.5, of [PP1], where (except for trees) X was always assumed to be a manifold, strict assumptions were made on the boundary of X, and no definite value of μ_0 was given except in special cases. But besides this, an important point is that its proof is a much simplified version of the upcoming main construction of Section 4, and hence could be welcome as a guide for reading Section 4.2.

(2) Note that the constant μ_0 is not optimal, but not by much. For simplicial trees all of whose vertices have degree at least 3, the result is true, with any $\mu_0 > 1$ and this is optimal (though they do not satisfy the hypotheses of the above result, the proof is easy for them, see for instance [PP1, Theo. 7.2 (3)]). We proved in [PP1] that the optimal value for the second assertion of the theorem, when $X = \mathbb{H}^n_{\mathbb{R}}$ and all H_α's are horoballs, is $\mu_0 = -\log(4\sqrt{2}-5) \approx 0.42$. Hence Theorem 4.1 (2) is not far from optimal, despite its generality. Furthermore, when $X = \mathbb{H}^n_{\mathbb{R}}$ and all H_α's are horoballs, a possible value of μ_0 for the first assertion of the theorem that was given in [PP1, Theo. 7.1] was $\log(2+\sqrt{5}) - \log(4\sqrt{2}-5) \approx 1.864$. Hence Theorem 4.1 (1) is even better than the corresponding result in [PP1] when $X = \mathbb{H}^n_{\mathbb{R}}$, despite its generality.

Proof. We start with the following geometric lemma. For every $\mu \geq 0$, define

$$\nu(\mu) = \frac{2\, e^{-\mu}}{1+\sqrt{1-e^{-2\mu}}}\ , \qquad \text{(- 14 -)}$$

which is positive and decreasing from 2 to 0 as μ goes from 0 to $+\infty$.

Lemma 4.2 *Let X be a proper geodesic CAT(-1) space. Let H be a ball or a horoball in X and $\xi_0 \in (X \cup \partial_\infty X) - (H \cup H[\infty])$. Let $\mu \geq \log 2$ be at most the radius of H, and let γ and γ' be geodesic rays or lines starting at ξ_0, meeting $H[\mu]$, parametrized such that $\gamma'(s), \gamma(s)$ are equidistant to ξ_0 for some (hence every) s, and that γ enters H at time 0.*

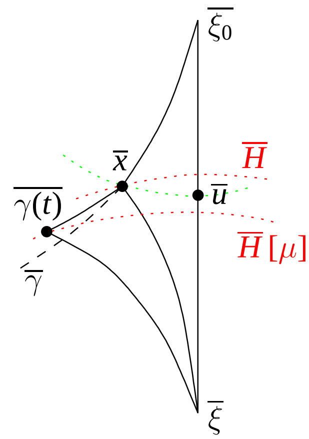
(1) *If $x = \gamma(0)$ and x' are the points of entry in H of γ and γ' respectively, then $d(x, x') \leq \nu(\mu)$.*

(2) *For every $s \geq 0$, we have*

$$d(\gamma(-s), \gamma'(-s)) \leq \nu(\mu)\, e^{-s}\ .$$

Proof. Let ξ be the center or point at infinity of H, and let t, t' be the entrance times of γ, γ' respectively in $H[\mu]$. Note that $t \geq 0$ as $\mu \geq 0$. Let us prove first that $t' \geq 0$ too. We refer to Section 2.1 for the definition and properties of the map β_{ξ_0}, especially when $\xi_0 \in X$. Let u be the point on the geodesic $]\xi_0, \xi[$ such that $\beta_{\xi_0}(x, u) = 0$. By the convexity of the balls and horoballs, the point u is the closest point to ξ on the sphere or horosphere centered at ξ_0 passing through x. Since $\beta_{\xi_0}(x, \gamma'(0)) = 0$, we hence have $\beta_\xi(x, \gamma'(0)) \leq \beta_\xi(x, u)$. Let us prove that $\beta_\xi(x, u) \leq \mu$, which will hence imply that γ' enters $H[\mu]$ at a non-negative time (which is t').

Glue the two comparison triangles $(\overline{\xi_0}, \overline{x}, \overline{\xi}\,)$ and $(\overline{\xi}, \overline{x}, \overline{\gamma(t)}\,)$ in $\mathbb{H}^2_\mathbb{R}$ for the geodesic triangles (ξ_0, x, ξ) and $(\xi, x, \gamma(t))$ along their sides $[\,\overline{x}, \overline{\xi}\,]$. Let $\overline{H}$ be the ball or horoball centered at $\overline{\xi}$ such that $\overline{x} \in \partial\overline{H}$. By comparison, we have $\angle_{\overline{x}}(\,\overline{\xi_0}, \overline{\xi}\,) \leq \pi \leq \angle_{\overline{x}}(\,\overline{\xi_0}, \overline{\xi}\,) + \angle_{\overline{x}}(\,\overline{\xi}, \overline{\gamma(t)}\,)$. Hence the geodesic ray or line $\overline{\gamma}$ starting from $\overline{\xi_0}$ and passing through $\overline{x}$ meets $[\,\overline{\gamma(t)}, \overline{\xi}\,]$, therefore it enters $\overline{H}[\mu]$. Let $\overline{u}$ be the point on the geodesic $[\,\overline{\xi_0}, \overline{\xi}\,]$ such that $\beta_{\overline{\xi_0}}(\overline{x}, \overline{u}) = 0$. As $\beta_\xi(x, u) = \beta_{\overline{\xi}}(\overline{x}, \overline{u})$, we only have to prove the result if X is the upper halfspace model of the hyperbolic plane $\mathbb{H}^2_\mathbb{R}$.



We may then assume that ξ_0 is the point at infinity ∞, and that H is the horoball with point at infinity 0 and Euclidean diameter 1 (see the figure below). But then, the vertical coordinate of $\gamma(0)$ is at least $\frac{1}{2}$, since γ enters H at $\gamma(0)$. As $e^{-\mu} \leq \frac{1}{2}$, the result follows: any geodesic line, starting from ξ_0 and meeting $H[\mu]$, meets the horizontal horosphere containing $\gamma(0)$ before $H[\mu]$.

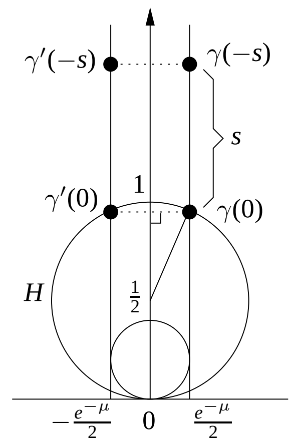
Now, in order to prove both assertions of Lemma 4.2, let us show that we may assume that $X = \mathbb{H}^2_\mathbb{R}$.

For the first one, glue the two comparison triangles $(\,\overline{\xi_0}, \overline{x}, \overline{\xi}\,)$ and $(\,\overline{\xi_0}, \overline{x'}, \overline{\xi}\,)$ for the geodesic triangles (ξ_0, x, ξ) and (ξ_0, x', ξ) along their sides $[\,\overline{\xi_0}, \overline{\xi}\,]$. As seen above, the geodesic lines $\overline{\gamma}$ (resp. $\overline{\gamma'}$) starting from $\overline{\xi_0}$ and passing through $\overline{x}$ (resp. $\overline{x'}$) enter $\overline{H}[\mu]$. And by comparison, we have $d(x, x') \leq d(\overline{x}, \overline{x'})$.

For the second assertion, we glue the two comparison triangles $(\,\overline{\xi_0}, \overline{\gamma(t)}, \overline{\gamma'(t')}\,)$ and $(\,\overline{\xi}, \overline{\gamma(t)}, \overline{\gamma'(t')}\,)$ for the geodesic triangles $(\xi_0, \gamma(t), \gamma'(t'))$ and $(\xi, \gamma(t), \gamma'(t'))$ along their isometric segments $[\,\overline{\gamma(t)}, \overline{\gamma'(t')}\,]$. As in the beginning of the proof of Lemma 2.3, the geodesic segment or ray $]\,\overline{\xi_0}, \overline{\gamma(t)}\,[$ does not meet the ball or horoball $\overline{H[\mu]}$ centered at $\overline{\xi}$ whose boundary goes through $\overline{\gamma(t)}$ and $\overline{\gamma'(t')}$. By comparison, if H' is the ball or horoball centered at $\overline{\xi}$ whose boundary passes through the point $\overline{\gamma(0)}$ on $]\,\overline{\xi_0}, \overline{\gamma(t)}\,[$ at distance t from $\overline{\gamma(t)}$, then $H'[\mu]$ contains $\overline{H[\mu]}$, so that $]\,\overline{\xi_0}, \overline{\gamma(t)}\,]$ and $]\,\overline{\xi_0}, \overline{\gamma'(t')}\,]$ meet $H'[\mu]$. For every $s \geq 0$, as $t, t' \geq 0$, if $\overline{\gamma(-s)}, \overline{\gamma'(-s)}$ are the corresponding points to $\gamma(-s), \gamma'(-s)$ on $]\,\overline{\xi_0}, \overline{\gamma(t)}\,[$, $]\,\overline{\xi_0}, \overline{\gamma'(t')}\,[$ respectively, then by comparison $d(\gamma(-s), \gamma'(-s)) \leq d(\,\overline{\gamma(-s)}, \overline{\gamma'(-s)}\,)$.

Hence we may assume that X is the upper halfspace model of the real hyperbolic plane $\mathbb{H}^2_\mathbb{R}$. Up to replacing ξ_0 by the point at infinity ξ'_0 of a geodesic ray starting perpendicularly from the boundary of H and passing through ξ_0, and γ, γ' by the geodesic lines starting from ξ'_0 and passing through x, x', we may assume that ξ_0 is at infinity.

By homogeneity and monotonicity, it is sufficient to prove the result for ξ_0 the point at infinity ∞, for H the horoball with point at infinity 0 and Euclidean diameter 1, and with γ and γ' different and both tangent to $H[\mu]$. Then, by an easy computation, the Euclidean height of the point $\gamma(0)$ is $\nu'(\mu) = \frac{1}{2}(1+\sqrt{1-e^{-2\mu}})$, so that the Euclidean height of the point $\gamma(-s)$ is $\nu'(\mu)\ e^{s}$. The hyperbolic distance between $\gamma(-s)$ and $\gamma'(-s)$ is hence at most $\frac{e^{-\mu}}{\nu'(\mu)e^{s}} = \nu(\mu)\ e^{-s}$. With the case $s=0$, this proves both assertions. $\square$



Proof of Theorem 4.1. Let X and $(H_\alpha)_{\alpha\in\mathscr{A}}$ be as in the statement. Let ξ_0 be either a point in $X-\bigcup_{\alpha\in\mathscr{A}} H_\alpha$ or the point at infinity of H_{α_0} for some α_0 in $\mathscr{A}$ such that H_{α_0} is a horoball. For every $\mu_1 \geq \log 2$, define the following constants, with ν the map introduced before Lemma 4.2,

$$\mu_2 = \nu(\mu_1) > 0\ , \quad \mu_3 = \mu_1 + \mu_2 > 0\ , \quad \mu_4 = 2\mu_1 - 2\mu_2\ .$$

As $\mu_1 \geq \log 2$, ν is decreasing and $\nu(\log 2) < \log 2$, we have $\mu_4 > 0$. We define by induction an initial segment $\mathscr{N}$ in $\mathbb{N}$ and the following finite or infinite sequences

- $(\gamma_k)_{k\in\mathscr{N}}$ of geodesic rays or lines starting from ξ_0,
- $(\alpha_k)_{k\in\mathscr{N}-\{0\}}$ of elements in $\mathscr{A}$,
- $(t_k)_{k\in\mathscr{N}}$ of non-negative real numbers,
- $(u_k)_{k\in\mathscr{N}}$ of maps $u_k : [0,+\infty[\ \to\ ]0,+\infty[$,

such that for every k in $\mathscr{N}$, the following assertions hold:

(1) If $\xi_0 \in X$, then $\gamma_k(0) = \xi_0$. Otherwise, γ_k meets ∂H_{α_0} at time 0.

(2) If $k \geq 1$, then γ_k enters H_{α_k} at the point $\gamma_k(t_k)$ and meets $H_{\alpha_k}[\mu_1]$ in one and only one point.

(3) If $k \geq 1$, then $u_k(t) = u_{k-1}(t) + \mu_2\, e^{t-t_k}$ if $t \leq t_{k-1}$, and $u_k(t) = \mu_3$ if $t > t_{k-1}$.

(4) If $k \geq 1$, then $t_k \geq \mu_4 + t_{k-1}$.

(5) If $t \in [0, t_k[$, then the point $\gamma_k(t)$ does not belong to $\bigcup_{\alpha\in\mathscr{A}} H_\alpha[u_k(t)]$.

If $\xi_0 \in X$, let γ_0 be a geodesic ray starting from ξ_0 at time 0. Otherwise, let γ_0 be a geodesic line starting from ξ_0 and exiting H_{α_0} at time 0. Such a γ_0 exists by the assumptions on X. Define u_0 as the constant map $t \mapsto \mu_3$. Let $t_0 = 0$. The assertions (1)–(5) are satisfied for $k = 0$. Assume that $\gamma_k, t_k, \alpha_k, u_k$ are constructed for $0 \le k \le n$ verifying the assertions (1)–(5).

If the geodesic ray $\gamma_n(]t_n, +\infty[)$ does not enter in the interior of any element of the family $(H_\alpha[\mu_1])_{\alpha\in\mathscr{A}}$, then define $\mathscr{N} = [0, n] \cap \mathbb{N}$, and the construction terminates. Otherwise, let $H_{\alpha_{n+1}}[\mu_1]$ be the first element of the family $(H_\alpha[\mu_1])_{\alpha\in\mathscr{A}}$ such that the geodesic ray $\gamma_n(]t_n, +\infty[)$ enters in its interior. Such an element exists as the H_α's have disjoint interiors. Note that $\alpha_{n+1} \neq \alpha_n$, as γ_n does not meet the interior of $H_{\alpha_n}[\mu_1]$ by (2).

If $\xi_0 \in X$, let γ_{n+1} be a geodesic ray starting from ξ_0 at time 0 and meeting $H_{\alpha_{n+1}}[\mu_1]$ in one and only one point. This is possible as there exists a geodesic ray starting from ξ_0 and avoiding $H_{\alpha_{n+1}}$ by the properties of X (consider for instance the extension to $]-\infty, 0]$ of γ_n) and since $\partial_\infty X$ is arcwise connected. If $\xi_0 \notin X$, let γ_{n+1} be a geodesic line starting from ξ_0, and meeting $H_{\alpha_{n+1}}[\mu_1]$ in one and only one point. Again, this is possible as $\partial_\infty X$ is arcwise connected. Parametrize γ_{n+1} such that γ_{n+1} exits H_{α_0} at time 0. In particular, in both cases, the assertion (1) for $k = n+1$ is satisfied.

Define $t_{n+1} \ge 0$ such that γ_{n+1} enters $H_{\alpha_{n+1}}$ at the point $\gamma_{n+1}(t_{n+1})$, so that the assertion (2) for $k = n+1$ is satisfied. As γ_n and γ_{n+1} both meet $H_{\alpha_{n+1}}[\mu_1]$ and as $\mu_1 \ge \log 2$, it follows from Lemma 4.2 (2) that, for every $t \le t_{n+1}$,

$$d(\gamma_{n+1}(t), \gamma_n(t)) \le \mu_2\, e^{t - t_{n+1}} \ . \tag{- 15 -}$$

Define $\tau_n \ge t_n$ as the entrance time of γ_n in $H_{\alpha_{n+1}}$. By Lemma 4.2 (1), as both γ_n and γ_{n+1} meet $H_{\alpha_{n+1}}[\mu_1]$ and $\mu_1 \ge \log 2$, we have

$$d(\gamma_{n+1}(t_{n+1}), \gamma_n(\tau_n)) \le \mu_2 \ .$$

As $H_{\alpha_{n+1}}$ and H_{α_n} have disjoint interiors, and since H_{α_n} and $H_{\alpha_n}[\mu_1]$ are at distance μ_1, we have $d(\gamma_n(t_n), \gamma_n(\tau_n)) \ge 2\mu_1$. Hence

$$\begin{aligned} d(\gamma_n(t_n), \gamma_n(t_{n+1})) &\ge d(\gamma_n(t_n), \gamma_n(\tau_n)) - d(\gamma_n(\tau_n), \gamma_{n+1}(t_{n+1})) - d(\gamma_{n+1}(t_{n+1}), \gamma_n(t_{n+1})) \\ &\ge 2\mu_1 - 2\mu_2 = \mu_4 > 0 \ . \end{aligned}$$

In particular $t_{n+1} - t_n$ is positive (otherwise $\gamma_n(t_n)$ belongs to $[\gamma_n(t_{n+1}), \gamma_n(\tau_n)]$, hence

$$\begin{aligned} 2\mu_1 \le d(\gamma_n(t_n), \gamma_n(\tau_n)) &\le d(\gamma_n(t_{n+1}), \gamma_n(\tau_n)) \\ &\le d(\gamma_n(\tau_n), \gamma_{n+1}(t_{n+1})) + d(\gamma_{n+1}(t_{n+1}), \gamma_n(t_{n+1})) \le 2\mu_2 \ , \end{aligned}$$

a contradiction). Therefore $t_{n+1}-t_n$ is at least μ_4, which proves the assertion (4) for $k=n+1$.

Define $t\mapsto u_{n+1}(t)$ by the induction formula in Assertion (3). The only remaining assertion to verify is (5). By absurd, assume that there exist some t in $[0,t_{n+1}[$ and some $\alpha\in\mathscr{A}$ such that $\gamma_{n+1}(t)$ belongs to $H_\alpha[u_{n+1}(t)]$. As $u_{n+1}(t)>0$, the element α is different from α_0 if $\xi_0\in\partial_\infty X$, and it is also different from α_{n+1} by construction. By Equation (- 15 -), the point $\gamma_n(t)$ belongs to $H_\alpha[u_{n+1}(t)-\mu_2\, e^{t-t_{n+1}}]$.

Assume first that $t>t_n$, so that $u_{n+1}(t)=\mu_3$. As $\mu_3-\mu_2\, e^{t-t_{n+1}}>\mu_1$ (since $t<t_{n+1}$ and by the definition of μ_3), this implies that $\gamma_n(t)$ belongs to the interior of $H_\alpha[\mu_1]$. This contradicts the fact that $H_{\alpha_{n+1}}[\mu_1]$ is the first element of the family $(H_\alpha[\mu_1])_{\alpha\in\mathscr{A}}$ encountered by $\gamma_n(]t_n,+\infty[)$ in its interior.

Assume now that $t\leq t_n$. Then $\gamma_n(t)$ belongs to $H_\alpha[u_n(t)]$. This contradicts the assertion (5) at step n. Thus, the assertions (1)–(5) hold for all $k\in\mathscr{N}$.

Let us prove that the maps u_n are uniformly bounded from above by

$$\mu_5=\mu_3+\frac{\mu_2}{e^{\mu_4}-1}\;.$$

As $\mu_4>0$, the sequence $(t_k)_{k\in\mathbb{N}}$ increases to $+\infty$. Fix $t\geq 0$. Let $k=k(t)$ be the unique non-negative integer such that t belongs to $]t_{k-1},t_k]$ (by convention, $t_{-1}=-\infty$). Let us prove, by induction on n, that

$$u_n(t)\leq\mu_3+\mu_2\sum_{j=1}^{n-k}e^{-\mu_4 j}\;.$$

(Recall that an empty sum is 0). This implies that $u_n(t)\leq\mu_5$.

This is true if $n=0$, as $u_0(t)=\mu_3$. Assume that the result is true for n. If $t>t_n$, then $u_{n+1}(t)=\mu_3$, and the result is true. Otherwise, by the property (3), we have $u_{n+1}(t)=u_n(t)+\mu_2 e^{t-t_{n+1}}$. Note that $t_k-t_{n+1}\leq-\mu_4(n+1-k)$ by the property (4), and that $t\leq t_k$. Hence, by induction,

$$u_{n+1}(t)\leq\mu_3+\mu_2\sum_{j=1}^{n-k}e^{-\mu_4 j}+\mu_2 e^{-\mu_4(n+1-k)}=\mu_3+\mu_2\sum_{j=1}^{n+1-k}e^{-\mu_4 j}\;.$$

This proves the induction.

Summarizing the above construction, there exist a sequence of geodesic rays or lines $(\gamma_n)_{n\in\mathbb{N}}$ starting from ξ_0, and a sequence of times $(t_n)_{n\in\mathbb{N}}$ converging to $+\infty$, such that for every t in $[0,t_n]$, the point $\gamma_n(t)$ does not belong to $\bigcup_{\alpha\in\mathscr{A}}H_\alpha[\mu_5]$. (Take an eventually constant sequence $(\gamma_n)_{n\in\mathbb{N}}$ if the construction stops at a finite stage, which is possible as $\mu_5 > \mu_1$.) As $(t_n)_{n\in\mathbb{N}}$ grows at least linearly, the formula (-15-) implies that $(\gamma_n(t))_{n\in\mathbb{N}}$ is a Cauchy sequence, uniformly on every compact subset of non-negative t's. Hence, the geodesic rays or lines γ_n converge to a geodesic ray or line avoiding $\bigcup_{\alpha\in\mathscr{A}} H_\alpha[\mu_5-\epsilon]$, for every $\epsilon>0$. Taking $\mu_1 = 1.042 \geq \log 2$, we can check that $\mu_5 < 1.5332$, hence the result follows. □

Corollary 4.3 *Let X and $(H_\alpha)_{\alpha\in\mathscr{A}}$ be as in Theorem 4.1. For every $x\in X$, there exist $t>0$ and a geodesic ray γ starting at x such that $\gamma([t,\infty[)$ is contained in the complement of $\bigcup_{\alpha\in\mathscr{A}} H_\alpha[\mu_0]$.*

Proof. We may assume that $x\in H_{\alpha_0}$ for some $\alpha_0\in\mathscr{A}$, otherwise, Theorem 4.1 (1) applies (with $t=0$). Let $H'_\alpha = H_\alpha$ if $\alpha\neq\alpha_0$, and $H'_{\alpha_0} = H_{\alpha_0}[d(x,\partial H_{\alpha_0})+1]$. Then $x\notin X-\bigcup_{\alpha\in\mathscr{A}} H'_\alpha$. By Theorem 4.1 (1), let γ be a geodesic ray starting from x and avoiding the $H'_\alpha[\mu_0]$'s. Let $t = d(x,\partial H_{\alpha_0})+2+2\mu_0+2\log(1+\sqrt{2})$. Recall that $\|\ell_{H_{\alpha_0}} - \mathfrak{ph}_{H_{\alpha_0}}\|_\infty \leq 2\log(1+\sqrt{2})$ by Subsection 3.1. With $\overline{\gamma}$ any extension of γ to a full geodesic line, the length of the geodesic segment $\gamma\cap H_{\alpha_0}$ is at most

$$\ell_{H_{\alpha_0}}(\overline{\gamma}) - d(x,\partial H_{\alpha_0}) \leq \mathfrak{ph}_{H_{\alpha_0}} + 2\log(1+\sqrt{2}) - d(x,\partial H_{\alpha_0}) < t\ ,$$

since γ avoids $H_{\alpha_0}[d(x,\partial H_{\alpha_0})+1+\mu_0]$. Hence the geodesic ray $\gamma([t,\infty[)$ does not meet H_{α_0}. The result follows. □

Let e be an end of a finite volume complete negatively curved Riemannian manifold V. Let ht_e be the Busemann function of e normalized to be zero on the boundary of the maximal Margulis neighbourhood of e (see for instance [BK, HP3, PP1], as well as the paragraph above Corollary 5.4). Our next result improves Theorem 7.4 (hence Corollary 1.2) in [PP1], with the same proof as in [loc. cit.], by removing the technical assumptions on the manifold, and giving a universal upper bound on $h_e(V)$.

Corollary 4.4 *Let V be a finite volume complete Riemannian manifold with dimension at least 2 and sectional curvature $K\leq -1$. Then there exists a closed geodesic in V whose maximum height (with respect to ht_e) is at most 1.534.* □

4.2 The inductive construction

Fix arbitrary constants $\epsilon_0\in\mathbb{R}_+^*\cup\{\infty\}$ and $\delta_0,\kappa_0\geq 0$, and fix an arbitrary point ξ_0 in $X\cup\partial_\infty X$. Let $(C_n)_{n\in\mathbb{N}}$ be a family of ϵ_0-convex subsets of X such that $\xi_0\notin C_0\cup\partial_\infty C_0$, and let f_0 be a κ_0-penetration map for C_0.

The aim of this section is to construct by induction a sequence of geodesic rays or lines in X, starting from ξ_0 and having a suitable penetration behaviour in the C_n's.

Prescription of constants. The following constants will appear in the statement, or in the proof, of the inductive construction:

- $c_1 = c_1'(\epsilon_0) > 0$ given by Lemma 2.3 if $\epsilon_0 \neq \infty$ and by Lemma 2.9 if $\epsilon_0 = \infty$ and $(f_0, \delta_0) \neq (\mathfrak{ph}_{C_0}, 0)$; otherwise $c_1 = \frac{1}{19}$;
- $c_2 = c_2'(\epsilon_0) > 0$ given by Equation (-4-) if $\epsilon_0 \neq \infty$ and by Equation (-9-) otherwise;
- $c_3 = 2 \sinh c_1 + c_2\, e^{2c_1} \sinh c_1$, which is positive, and depends on ϵ_0;
- $c_4 = c_3'(\epsilon_0) \sinh(c_1 + \delta_0) + c_2\; e^{-3c_3'(\epsilon_0) \sinh(c_1+\delta_0) - \log 2} \sinh c_1$, where $c_3'(\cdot)$ is given by Equation (-6-) if $\epsilon_0 \neq \infty$ and by Equation (-11-) otherwise. Note that c_4 is positive, and depends on ϵ_0, δ_0;
- $c_5 = c_5(\epsilon_0, \delta_0) = 2 \max\{c_2, c_3'(\epsilon_0)\} \sinh(c_1 + \delta_0)$, which is positive, and depends on ϵ_0, δ_0;
- $c_6 = 3c_4 + \log 2$, which is positive, and depends on ϵ_0, δ_0;
- $h_0 = h_0(\epsilon_0, \delta_0, \kappa_0) = \max\{\ \delta_0 + \kappa_0\ ,\ c_0(\epsilon_0) + \kappa_0\ ,\ h'(\epsilon_0, \sinh(\delta_0 + c_1))\ ,\ \delta_0 + 2c_1 + c_6\ ,\ c_6 + \kappa_0 + \delta_0 - 2c_5\ \}$, where $c_0(\cdot)$ is given by Equation (-1-) if $\epsilon_0 \neq \infty$ and by Equation (-8-) otherwise, and $h'(\cdot,\cdot)$ is given by Equation (-5-) if $\epsilon_0 \neq \infty$ and by Equation (-11-) otherwise;
- For every $h_0' \geq 0$, let $h_1' = h_1'(\epsilon_0, \delta_0, h_0') = h_0' + 2c_5$.

Fix $h_0' \geq h_0$ and $h \geq h_1'$.

Assumptions on the family $(C_n)_{n\in\mathbb{N}}$. Assume that there exists at least one geodesic ray or line γ_0 starting from ξ_0 with $f_0(\gamma_0) = h$ (this implies that γ_0 meets C_0, since f_0 is a κ_0-penetration map in C_0 and $h \geq h_1' \geq h_0' \geq h_0 > \kappa_0$, hence $\ell_{C_0}(\gamma_0) > 0$), and that the following conditions are satisfied.

(*iii*) **(Almost disjointness property)** For every m, n in $\mathbb{N}$ with $m \neq n$, the diameter of $C_n \cap C_m$ is at most δ_0.

(*iv*) **(Local prescription property)** For every n in $\mathbb{N} - \{0\}$ such that $\xi_0 \notin C_n \cup \partial_\infty C_n$, if there exists a geodesic ray or line α starting from ξ_0 which meets first C_0 and then C_n with $f_0(\alpha) = h$ and $\ell_{C_n}(\alpha) \geq h_0'$, then there exists a geodesic ray or line α', starting from ξ_0 which meets first C_0 and then C_n with $f_0(\alpha') = h$ and $\ell_{C_n}(\alpha') = h_0'$.

Note that (*iii*) is satisfied with $\delta_0 = 0$ if the C_n's have disjoint interior. In Section 5, we will use Proposition 3.7 to check (*iv*) for various applications, with $h'_0 = \max\{h_0, h_0^{\min}\}$ and $h \geq \max\{h'_1, h^{\min}\}$, for the various values of $h_0^{\min}, h^{\min}$ defined in Proposition 3.7.

For every n in $\mathbb{N}$ such that $\xi_0 \notin C_n \cup \partial_\infty C_n$, define $f_n = \ell_{C_n} : T^1_{\xi_0} X \to [0, +\infty]$, and for every geodesic ray or line γ starting from ξ_0 and meeting C_n, let $t_n^-(\gamma), t_n^+(\gamma) \in \,]-\infty, +\infty]$ be the entrance time and exit time of γ in and out of the convex subset C_n respectively. The following remark will be used later on.

Lemma 4.5 *For every $n > 0$, for every geodesic ray or line γ starting from ξ_0 and entering C_0 at time $t = 0$, such that $f_0(\gamma) = h$ and $\gamma(]\delta_0, +\infty[)$ meets C_n, we have $\xi_0 \notin C_n \cup \partial_\infty C_n$ and $t_n^-(\gamma) > 0$.*

Proof. Otherwise, as $\gamma(]\delta_0, +\infty[)$ meets C_n and by convexity, there exists $\epsilon > 0$ such that the geodesic segment $\gamma([0, \delta_0 + \epsilon])$ is contained in C_n. By the Penetration property (*i*) of f_0, the length of $\gamma \cap C_0$ is at least $h - \kappa_0$, which is bigger than δ_0 as $h \geq h'_1 > h'_0 \geq h_0 \geq \delta_0 + \kappa_0$ by the definitions of h'_1 and h_0. As γ enters C_0 at time $t = 0$, up to taking $\epsilon > 0$ smaller, this implies that the geodesic segment $\gamma([0, \delta_0 + \epsilon])$ is also contained in C_0. This contradicts the Almost disjointness property (*iii*) as $n \neq 0$. □

Statement of the inductive construction. We will define by induction an initial segment $\mathscr{N}$ in $\mathbb{N}$, and finite or infinite sequences

- $(\gamma_k)_{k \in \mathscr{N}}$ of geodesic rays or lines starting from ξ_0,
- $(n_k)_{k \in \mathscr{N}}$ of integers such that $\xi_0 \notin C_{n_k} \cup \partial_\infty C_{n_k}$,
- $(u_k)_{k \in \mathscr{N}}$ of maps $u_k : [0, +\infty[\to [h'_0, h'_1]$,

such that the following assertions hold, for every k in $\mathscr{N}$, where we use $t_k^\pm = t_{n_k}^\pm(\gamma_k)$ to simplify notations.

(1) The geodesic ray or line γ_k enters C_0 at time $t = 0$ and $f_0(\gamma_k) = h$.

(2) If $k \geq 1$, then γ_k meets C_{n_k} with $t_k^- \geq 0$ and $f_{n_k}(\gamma_k) = h'_0$.

(3) If $k \geq 1$, then $d(\gamma_k(t), \gamma_{k-1}(t)) \leq c_3\, e^{t - t_k^-}$ for every t in $[0, t_k^-]$.

(4) If $k \geq 1$, then

$$u_k(t) = \sup_{s \in [0, +\infty[\ :\ |s-t| \leq c_4\, e^{t - t_k^-}} u_{k-1}(s) \;+\; c_5\, e^{t - t_k^-}$$

for $t \in [0, t_k^-]$ and $u_k(t) = h'_0$ if $t > t_k^-$.

(5) If $k \geq 1$, then $t_k^- \geq t_{k-1}^- + c_6$.

(6) If $k \geq 1$, for every n in $\mathbb{N} - \{0\}$ such that $\gamma_k(]\delta_0, +\infty[)$ meets C_n with $t_n^-(\gamma_k) \leq t_k^-$, we have $f_n(\gamma_k) \leq u_k(t_n^+(\gamma_k) - \delta_0)$.

Note that by Lemma 4.5 and by (1), if $\gamma_k(]\delta_0, +\infty[)$ meets C_n for some $n \geq 1$, then $\xi_0 \notin C_n \cup \partial_\infty C_n$, so that, in particular, $t_n^\pm(\gamma_k)$ are well defined, and (6) does make sense.

Proof of the inductive construction. By the assumptions, let γ_0 be a geodesic ray or line starting from ξ_0 and entering C_0 at time $t_0^- = 0$, such that $f_0(\gamma_0) = h$. Let $n_0 = 0$. Let $u_0 : [0, +\infty[\to [h'_0, h'_1]$ be the constant map with value h'_0. As the conditions (2)–(6) are empty if $k = 0$, the construction is done at step 0.

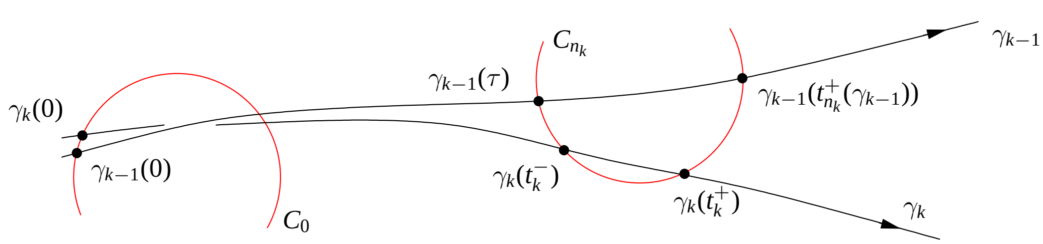
Let $k \geq 1$, and assume that $\gamma_0, n_0, u_0, \ldots, \gamma_{k-1}, n_{k-1}, u_{k-1}$ are constructed. Note that $u_{k-1} \geq h'_0$ by induction. If for every n in $\mathbb{N} - \{0\}$ such that $\gamma_{k-1}(]\delta_0, +\infty[)$ meets C_n, we have $f_n(\gamma_{k-1}) \leq u_{k-1}(t_n^+(\gamma_{k-1}) - \delta_0)$, then we stop and we define $\mathscr{N} = \{0, 1 \ldots, k-1\}$.

Otherwise, let τ be the greatest lower bound of the $t_n^-(\gamma_{k-1})$'s taken over all n in $\mathbb{N} - \{0\}$ such that $\gamma_{k-1}(]\delta_0, +\infty[)$ meets C_n with $f_n(\gamma_{k-1}) > u_{k-1}(t_n^+(\gamma_{k-1}) - \delta_0)$.

Let us prove that this lower bound is in fact a minimum, attained for only one such n. Let $\epsilon > 0$ such that $h'_0 > \delta_0 + \epsilon$, which is possible by the definition of h_0, as $h'_0 \geq h_0$. If $t_n^-(\gamma_{k-1})$ and $t_m^-(\gamma_{k-1})$ belong to $[\tau, \tau + \epsilon]$ with $f_n(\gamma_{k-1}) > u_{k-1}(t_n^+(\gamma_{k-1}) - \delta_0)$ and $f_m(\gamma_{k-1}) > u_{k-1}(t_m^+(\gamma_{k-1}) - \delta_0)$, assume for instance that $t_n^-(\gamma_{k-1}) \leq t_m^-(\gamma_{k-1})$. As $f_n = \ell_{C_n}$, $f_m = \ell_{C_m}$, $u_{k-1} \geq h'_0$ and $t_m^-(\gamma_{k-1}) - t_n^-(\gamma_{k-1}) \leq \epsilon$, the subsets C_n and C_m meet along a segment of length at least $h'_0 - \epsilon > \delta_0$. By the Almost disjointness property (iii), this implies that $n = m$. In particular, we have $\tau = t_n^-(\gamma_{k-1})$ for a unique $n \in \mathbb{N} - \{0\}$, and we denote this n by $n_k \in \mathbb{N} - \{0\}$, so that $\gamma_{k-1}(]\delta_0, +\infty[)$ meets C_{n_k} with

$$f_{n_k}(\gamma_{k-1}) > u_{k-1}(t_{n_k}^+(\gamma_{k-1}) - \delta_0) \geq h'_0 \; . \tag{-16-}$$

In particular, $\xi_0 \notin C_{n_k} \cup \partial_\infty C_{n_k}$ by Lemma 4.5 and by Assertion (1) at rank $k-1$. Note that $n_k \neq n_{k-1}$, as $f_{n_{k-1}}(\gamma_{k-1}) = h'_0$ by the assertion (2) at rank $k-1$, which would contradict Equation (-16-) if $n_k = n_{k-1}$.



By Lemma 4.5, the geodesic ray or line γ_{k-1} first enters C_0 and then C_{n_k}. Furthermore, γ_{k-1} satisfies (1) and $f_{n_k}(\gamma_{k-1}) \geq h'_0$. Hence, by the Local prescription property (iv), there exists a geodesic ray or line γ_k starting from ξ_0 that first enters C_0 and then C_{n_k}, with $f_0(\gamma_k) = h$ and $f_{n_k}(\gamma_k) = h'_0$. Choose the parametrization in such a way that γ_k enters C_0 at time 0. In particular, (1) and (2) hold for γ_k, and $t_k^- = t_{n_k}^-(\gamma_k) > 0$. Define $u_k : [0, +\infty[\to [0, +\infty[$ by using the induction formula given in the assertion (4). Before checking (3)–(6) for γ_k, n_k, u_k, let us make two preliminary remarks.

Lemma 4.6 *We have $d(\gamma_{k-1}(\tau), \gamma_k(t_k^-)) \leq c_1$ and $d(\gamma_{k-1}(0), \gamma_k(0)) \leq c_1$.*

Proof. By Lemma 2.3 if $\epsilon_0 \neq \infty$ and Lemma 2.9 otherwise, we have the inequalities $d(\gamma_{k-1}(\tau), \gamma_k(t_k^-)) \leq c'_1(\epsilon_0)$ and $d(\gamma_{k-1}(0), \gamma_k(0)) \leq c'_1(\epsilon_0)$. By the definition of c_1, we hence only have to prove Lemma 4.6 when $\epsilon_0 = \infty$, $\delta_0 = 0$ and $f_0 = \mathfrak{ph}_{C_0}$. In this case, as $c_1 = 1/19$, $c_2 = 5/2$, $c_0(\infty) = 4.056$, $\kappa_0 = 2\log(1+\sqrt{2}) = c'_1(\infty)$, $c'_3(\infty) = 5/2$, easy computations show that

$$h_0 = h'(\infty, \sinh c_1) = 3\sinh c_1 + c_0(\infty) + c'_1(\infty) \approx 5.9767$$

and, for future use,

$$h'_1(\infty, 0, h_0(\infty, 0, c'_1(\infty))) \approx 6.5032. \tag{- 17 -}$$

As $\mathfrak{ph}_{C_0}(\gamma_k)$ and $\mathfrak{ph}_{C_0}(\gamma_{k-1})$ are equal to $h \geq h'_1 \geq h'_0 \geq h_0$, and since $h_0/2 \geq \log 2$, it follows from the definition of the map $\mathfrak{ph}_{C_0}$ and from Lemma 4.2 (1) and (2) that $d(\gamma_{k-1}(0), \gamma_k(0))$ and similarly $d(\gamma_{k-1}(\tau), \gamma_k(t_k^-))$ are at most $\nu(h_0/2)$, where $\nu(.)$ is defined by Equation (- 14 -). An easy computation shows that $\nu(h_0/2) \leq c_1 = 1/19$, which proves the result. $\square$

Lemma 4.7 *We have $|\tau - t_k^-| \leq 2c_1$.*

Proof. Lemma 4.5, applied to $n = n_k$ and $\gamma = \gamma_{k-1}$, implies that $\tau > 0$. We have seen that $t_k^- > 0$. By the triangular inequality and the above lemma, we have $|\tau - t_k^-| \leq 2c_1$. $\square$

Verification of (5). If $k = 1$, then since $n_1 \neq 0$, $t_1^- > 0$, and $f_{n_1}(\gamma_1) = h'_0 \geq h_0 > \delta_0$, we have $t_1^- \geq \ell_{C_0}(\gamma_1) - \delta_0$ by the Almost disjointness property (iii). Therefore

$$\begin{aligned} t_1^- - t_0^- = t_1^- &\geq \ell_{C_0}(\gamma_1) - \delta_0 \geq f_0(\gamma_1) - \delta_0 - \kappa_0 = h - \kappa_0 - \delta_0 \\ &\geq h'_1 - \kappa_0 - \delta_0 = h'_0 + 2c_5 - \kappa_0 - \delta_0 \geq h_0 + 2c_5 - \kappa_0 - \delta_0 \geq c_6 \,, \end{aligned}$$

by the definition of h_0.

Assume now that $k \geq 2$. Note that $\tau = t^-_{n_k}(\gamma_{k-1}) > t^-_{k-1}$. Otherwise, as

$$t^+_{n_k}(\gamma_{k-1}) = \tau + f_{n_k}(\gamma_{k-1}) \geq \tau + h'_0 \geq \tau + h_0 > \delta_0$$

by Equation (- 16 -) and by the definition of h_0, we have, by the assertion (6) at step $k-1$, the inequality $f_{n_k}(\gamma_{k-1}) \leq u_{k-1}(t^+_{n_k}(\gamma_{k-1}) - \delta_0)$, which contradicts the definition of n_k, see Equation (- 16 -).

Let us first prove that $\tau \geq t^-_{k-1} + h_0 - \delta_0$. Assume first that $\tau \geq t^+_{k-1}$. Since $k \geq 2$, we have

$$\tau - t^-_{k-1} \geq t^+_{k-1} - t^-_{k-1} = f_{n_{k-1}}(\gamma_{k-1}) = h'_0 \geq h_0. \tag{- 18 -}$$

Hence the result holds. Otherwise, $t^-_{k-1} < \tau < t^+_{k-1}$. By convexity, $\gamma_{k-1}(\tau)$ belongs to $C_{n_{k-1}}$. Note that $\gamma_{k-1}([\tau, \tau + h_0])$ is contained in C_{n_k}, since τ is the entrance time of γ_{k-1} in C_{n_k}, and $f_{n_k}(\gamma_{k-1}) \geq h'_0 \geq h_0$. If $\tau + \delta_0 < t^+_{k-1}$, as $\ell_{C_{n_k}}(\gamma_{k-1}) \geq h_0 > \delta_0$ by the definition of h_0, then $C_{n_k} \cap C_{n_{k-1}}$ contains a geodesic segment of length bigger than δ_0. This contradicts the Almost disjointness property (*iii*) since $n_k \neq n_{k-1}$. Hence

$$\tau \geq t^+_{k-1} - \delta_0 \geq t^-_{k-1} + f_{n_{k-1}}(\gamma_{k-1}) - \delta_0 \geq t^-_{k-1} + h_0 - \delta_0 \ ,$$

and the result holds.

Now, by Lemma 4.7,

$$t^-_k - t^-_{k-1} \geq \tau - 2c_1 - t^-_{k-1} \geq h_0 - \delta_0 - 2c_1 \geq c_6$$

by the definition of h_0. Therefore, the assertion (5) holds at rank k.

Verification of (4). We only have to check that u_k has values in $[h'_0, h'_1]$. This will follow from the following easy but tedious general lemma.

Lemma 4.8 *Let $c, c', c'', h_* \geq 0$, let $\mathscr{M}$ be an initial segment in $\mathbb{N}$, let $(t_n)_{n\in\mathscr{M}}$ be a sequence of non-negative real numbers, and let $(u_n : [0, +\infty[\;\to [0, +\infty[)_{n\in\mathscr{M}}$ be a sequence of maps. Assume that u_0 has constant value h_*, and that for every n in $\mathscr{M} - \{0\}$, we have $t_n - t_{n-1} \geq c''$, $u_n(t) = h_*$ if $t > t_n$ and if $t \leq t_n$, then*

$$u_n(t) = c\, e^{t-t_n} + \sup_{s\,\in\,[0,+\infty[\ :\ |s-t|\,\leq\, c'\, e^{t-t_n}} u_{n-1}(s) \ .$$

If $c'' \geq 3c' + \log 2$, then for every $t \in [0, +\infty[$, for every n in $\mathscr{M}$, we have

$$h_* \leq u_n(t) \leq h_* + 2c \ .$$

To prove that u_k has values in $[h'_0, h'_1]$, we apply Lemma 4.8 with $c = c_5$, $c' = c_4$, $c'' = c_6$, $h_* = h'_0$, $\mathscr{M} = \{0, 1, \dots, k\}$ and $(t_i)_{i \in \mathscr{M}} = (t_i^-)_{1 \le i \le k}$. Its hypotheses are satisfied by the definition of the constant c_6, by the assertion (5) at rank less than or equal to k, that we just proved, and by the definition of u_k and the assertion (4) for u_i with $0 \le 1 \le k-1$. Hence the map u_k does have values in $[h'_0, h'_1]$, by the definition of h'_1.

Proof of Lemma 4.8. First note that by an easy induction, whatever the value of c'' is, for every $t \in [0, +\infty[$ and $n \in \mathscr{M}$, we have $u_n(t) \ge h_*$.

Let $c'' \ge 3c' + \log 2$, $t \in [0, +\infty[$ and $n \in \mathscr{M}$. Let us prove that $u_n(t) \le h_* + 2c$. We may assume that $t \le t_n$ and that $n \ge 1$. Define $t_{-1} = -2c' - 1$. Let m be the unique element in $\mathscr{M}$ such that $t_{m-1} + 2c' < t \le t_m + 2c'$. Set $N = n - m$, which is non-negative (otherwise $n \le m-1$ and $t_n \le t_{m-1} \le t_{m-1} + 2c' < t_n$, a contradiction). Note that for every integer k with $0 \le k \le N$, we have $t_{n-k} - t_m \ge (n-m-k)c''$ hence

$$t - t_{n-k} \le 2c' - (N-k)c'' \ . \tag{- 19 -}$$

Consider the finite sequence $(x_k)_{0 \le k \le N}$ defined by $x_0 = 0$ and

$$x_{k+1} = x_k + e^{c' x_k - (N-k)c'' + 2c'}$$

for $0 \le k \le N-1$. Let us prove by induction on k that $x_k \le e^{-(N-k)c''}$, which in particular implies that

$$x_N \le 1 \ . \tag{- 20 -}$$

Indeed, the result is true for $k = 0$. Assume it to be true for some $k \le N-1$. Then

$$\begin{aligned} x_{k+1} &\le e^{-(N-k)c''} + e^{c' e^{-(N-k)c''} - (N-k)c'' + 2c'} \\ &\le e^{-(N-k-1)c''} \left(e^{-c''} + e^{-c'' + 3c'} \right) \le e^{-(N-k-1)c''} \end{aligned}$$

as $c'' \ge 3c' + \log 2 \ge \log(1 + e^{3c'})$.

Let us now prove by induction on k that, for $0 \le k \le N$, we have

$$u_n(t) \le \sup_{|s-t| \le c' x_k} u_{n-k}(s) \ + c\, x_k \ . \tag{- 21 -}$$

This is true if $k = 0$, assume it is true for some $k \le N-1$. In particular, $n - k \ge 1$. For every $s \in [0, +\infty[$ such that $|s-t| \le c' x_k$, we have

$$u_{n-k}(s) \le \sup_{|s'-s| \le c' e^{s - t_{n-k}}} u_{n-k-1}(s') \ + c\, e^{s - t_{n-k}}$$

(this is true by definition if $s\le t_{n-k}$, and also true otherwise as then $u_{n-k}(s)=h_*$ and $u_{n-k-1}(s')\ge h_*$ for every s'). Hence by the triangular inequality and Equation (- 19 -),

$$\begin{aligned} u_n(t) &\le \sup_{|s'-t|\,\le c'\,x_k+c'e^{t+c'x_k-t_{n-k}}} u_{n-k-1}(s')\ + c\,x_k + ce^{t+c'x_k-t_{n-k}} \\ &\le \sup_{|s'-t|\,\le c'\,x_k+c'e^{c'x_k+2c'-(N-k)c''}} u_{n-k-1}(s')\ + c\,x_k + ce^{c'x_k+2c'-(N-k)c''} \\ &= \sup_{|s'-t|\,\le c'\,x_{k+1}} u_{n-k-1}(s')\ + c\,x_{k+1}\ , \end{aligned}$$

which proves the inductive formula (- 21 -).

Finally, let us prove that $u_n(t)\le h_*+2c$, which finishes the proof of the lemma. Take $k=N$ in the inductive formula (- 21 -), and note that $n-N=m$. For every $\epsilon>0$, let $s\in[0,+\infty[$ with $|s-t|\ \le\ c'\,x_N$ such that $\sup_{|s'-t|\,\le c'\,x_N} u_m(s')\le u_m(s)+\epsilon$. If $s>t_m$ or $m=0$, then $u_m(s)=h_*$, hence by the inequality (- 20 -),

$$u_n(t)\le \sup_{|s'-t|\,\le c'\,x_N} u_m(s')\ + c\,x_N\le h_*+\epsilon+c\ ,$$

and the result holds. Otherwise, $s\le t_m$ and $m\ge 1$. For every $s'\in[0,+\infty[$ such that $|s'-s|\ \le\ c'\,e^{s-t_m}$, by Equation (- 20 -), we have $s'\ge s-c'\ge t-c'x_N-c'\ge t-2c'>t_{m-1}$. Again, the definition of s and the inequality (- 20 -) gives

$$\begin{aligned} u_n(t) &\le u_m(s)+\epsilon+c\,x_N \\ &= \sup_{|s'-s|\,\le c'\,e^{s-t_m}} u_{m-1}(s')\ + c\,e^{s-t_m}+\epsilon+c\,x_N\le \epsilon+h_*+2c\ , \end{aligned}$$

and the result also holds. □

Verification of (3). Let t be in $[0,t_k^-]$. Recall that $d(\gamma_{k-1}(\tau),\gamma_k(t_k^-))\ \le\ c_1$ by Lemma 4.6. By Lemma 2.1 applied with $x=\gamma_{k-1}(\tau), y=\gamma_k(t_k^-), z=\xi_0$, we have $d(\gamma_{k-1}(0),\gamma_k)\le e^{-\tau}\sinh c_1$. By the Penetration property (i) of f_0 and the definition of h_0, we have

$$\ell_{C_0}(\gamma_k)\ge f_0(\gamma_k)-\kappa_0=h-\kappa_0\ge h_0-\kappa_0\ge c_0(\epsilon_0)\ .$$

Thus, by Lemma 2.5 if $\epsilon_0\ne\infty$ and by Lemma 2.11 if $\epsilon_0=\infty$, and by the definition of c_2, we have

$$d(\gamma_{k-1}(0),\gamma_k(0))\le c_2\,e^{-\tau}\sinh c_1\ . \tag{- 22 -}$$

We refer to Section 2.1 for the definition and properties of the map β_{ξ_0}. It follows from the inequality (- 22 -) that

(- 23 -)
$$|\beta_{\xi_0}(\gamma_{k-1}(t),\gamma_k(t))|=|\beta_{\xi_0}(\gamma_{k-1}(0),\gamma_k(0))|\le d(\gamma_{k-1}(0),\gamma_k(0))\le c_2\,e^{-\tau}\sinh c_1\ .$$

For every s in $\mathbb{R}$, let $\gamma_{k-1}(s')$ be the point on the geodesic line γ_{k-1} such that the equality $\beta_{\xi_0}(\gamma_{k-1}(s'),\gamma_k(s)) = 0$ holds. For every point $p \in \gamma_{k-1}$, we have

$$d(p,\gamma_{k-1}(t')) = |\beta_{\xi_0}(p,\gamma_{k-1}(t'))| = |\beta_{\xi_0}(p,\gamma_k(t))| \leq d(p,\gamma_k(t)) \ , \tag{- 24 -}$$

Using the triangle inequality with the point p the closest to $\gamma_k(t)$ on γ_{k-1}, Lemma 2.1 and Lemma 4.6, we hence have the following inequalities

$$\begin{aligned} d(\gamma_k(t),\gamma_{k-1}(t')) &\leq 2\,d(\gamma_k(t),\gamma_{k-1}) \leq 2\,e^{t-t_k^-}\ \sinh d(\gamma_k(t_k^-),\gamma_{k-1}(\tau)) \\ &\leq 2\,e^{t-t_k^-}\ \sinh c_1 \ . \end{aligned} \tag{- 25 -}$$

Note that, using Equation (- 24 -) with $p = \gamma_{k-1}(t)$ and the inequalities (- 23 -),

$$d(\gamma_{k-1}(t),\gamma_{k-1}(t')) = |\beta_{\xi_0}(\gamma_{k-1}(t),\gamma_k(t))| \leq c_2\,e^{-\tau}\ \sinh c_1 \ .$$

Hence, by the inequality (- 25 -), we have

$$\begin{aligned} d(\gamma_k(t),\gamma_{k-1}(t)) \leq&\ d(\gamma_k(t),\gamma_{k-1}(t')) + d(\gamma_{k-1}(t'),\gamma_{k-1}(t)) \\ \leq&\ 2\,e^{t-t_k^-}\ \sinh c_1 + c_2\ e^{-\tau}\ \sinh c_1 \ . \end{aligned}$$

As $\tau \geq t_k^- - 2c_1$ by Lemma 4.7, and by the definition of c_3, we get

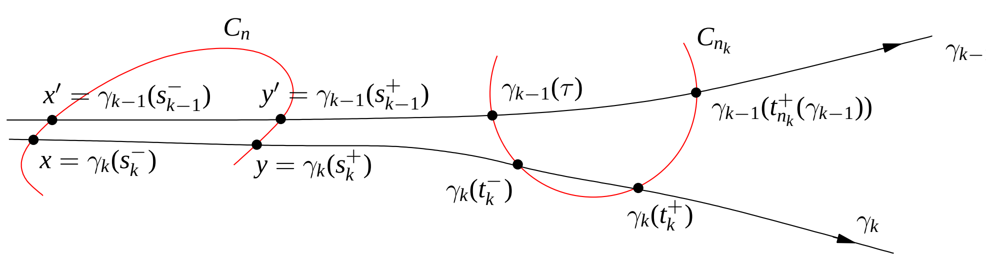
$$d(\gamma_k(t),\gamma_{k-1}(t)) \leq c_3\ e^{t-t_k^-} \ ,$$

which proves the assertion (3) at rank k.

Verification of (6). By absurd, assume that there exists $n \in \mathbb{N}-\{0\}$ such that $\gamma_k(]\delta_0,+\infty[)$ meets C_n (so that in particular $\xi_0 \notin C_n \cup \partial_\infty C_n$ by Lemma 4.5), with $t_n^-(\gamma_k) \leq t_k^-$ and

$$f_n(\gamma_k) > u_k(t_n^+(\gamma_k) - \delta_0) \ . \tag{- 26 -}$$

To simplify notation, let $s_k^\pm = t_n^\pm(\gamma_k)$, $x = \gamma_k(s_k^-), y = \gamma_k(s_k^+)$, and, as we will prove later on that γ_{k-1} also meets C_n, let $s_{k-1}^\pm = t_n^\pm(\gamma_{k-1})$, $x' = \gamma_{k-1}(s_{k-1}^-), y' = \gamma_{k-1}(s_{k-1}^+)$.



Note that $s_k^+ \leq t_k^- + \delta_0$. Otherwise, as $s_k^- \leq t_k^-$ and by convexity, there exists $\epsilon > 0$ such that $\gamma_k([t_k^-, t_k^- + \delta_0 + \epsilon])$ is contained in C_n. As $t_k^+ - t_k^- = h'_0 \geq h_0 > \delta_0$, up to making ϵ smaller, the geodesic segment $\gamma_k([t_k^-, t_k^- + \delta_0 + \epsilon])$ is also contained in C_{n_k}. Hence n is equal to n_k by the Almost disjointness property (iii). But $f_{n_k}(\gamma_k) = h'_0$ and, by Equation (- 26 -), we have $f_n(\gamma_k) > u_k(s_k^+ - \delta_0) \geq h'_0$, so that n cannot be equal to n_k.

By Lemma 2.1 applied to the geodesic triangle with vertices $\gamma_k(t_k^- + \delta_0), \gamma_{k-1}(\tau), \xi_0$, and as $d(\gamma_k(t_k^-), \gamma_{k-1}(\tau)) \leq c_1$ by Lemma 4.6, we have

$$\begin{aligned} d(y, \gamma_{k-1}) &\leq e^{-d(\gamma_k(t_k^- + \delta_0), y)} \sinh d(\gamma_k(t_k^- + \delta_0), \gamma_{k-1}(\tau)) \\ &\leq e^{s_k^+ - t_k^- - \delta_0} \sinh(\delta_0 + c_1) \qquad (-27-) \end{aligned}$$

which is, in particular, at most $\sinh(\delta_0 + c_1)$.

Note that

$$d(x,y) = f_n(\gamma_k) > h'_0 \geq h_0 \geq h'(\epsilon_0, \sinh(\delta_0 + c_1)) \geq \sinh(\delta_0 + c_1) + c'_1(\epsilon_0)\ , \qquad (-28-)$$

by the definition of h_0 and of $h'(\cdot,\cdot)$ in Equation (- 5 -) if $\epsilon_0 \neq \infty$ and in Equation (- 11 -) otherwise. It follows from Lemma 2.6 if $\epsilon_0 \neq \infty$ and by Lemma 2.12 otherwise, that the geodesic line γ_{k-1} meets C_n (thus x' and y' indeed exist) and that one of the following two assertions hold :

$$d(y, y') \leq c'_3(\epsilon_0)\, d(x', \gamma_k) \qquad (-29-)$$

or

$$d(x', y') \geq d(x, y)\ . \qquad (-30-)$$

Note that $d(x, x') \leq c'_1(\epsilon_0)$ by Lemma 2.3 if $\epsilon_0 \neq \infty$ and Lemma 2.9 otherwise. Let q' be the closest point to y on γ_{k-1}, so that

$$d(y, q') = d(y, \gamma_{k-1}) \leq \sinh(\delta_0 + c_1)\ .$$

By the cocycle property of β_{ξ_0}, we have

$$\beta_{\xi_0}(q', x') = \beta_{\xi_0}(q', y) + \beta_{\xi_0}(y, x) + \beta_{\xi_0}(x, x') \geq d(y, x) - d(y, q') - d(x, x')\ ,$$

which is non-negative by the two previous assertions and Equation (- 28 -). Hence ξ_0, x', q' are in this order on γ_k. Therefore, by convexity,

$$d(x', \gamma_k) \leq d(q', \gamma_k) \leq d(q', y) = d(y, \gamma_{k-1})\ . \qquad (-31-)$$

By Equation (- 28 -) and the definition of h_0, we have $d(x,y) \geq h_0 \geq c_0(\epsilon_0)$. Hence, by Lemma 2.5 if $\epsilon_0 \neq \infty$ and Lemma 2.11 otherwise, and by the inequalities (- 31 -) and (- 27 -), we have

$$d(x,x') \leq c_2\, d(x',\gamma_k) \leq c_2\, e^{s_k^+ - t_k^- - \delta_0}\, \sinh(\delta_0 + c_1)\ . \tag{- 32 -}$$

Before obtaining a contradiction from both assertions (- 29 -) and (- 30 -), we prove a technical result.

Lemma 4.9 *We have $\delta_0 < s_{k-1}^- < \tau$, so that the geodesic ray $\gamma_{k-1}(]\delta_0,+\infty[)$ meets C_n with $t_n^-(\gamma_{k-1}) < \tau$.*

Proof. Assume first by absurd that $s_{k-1}^- \leq \delta_0$. If $s_{k-1}^- \in\ ]0,\delta_0]$, we have by the triangular inequality, Lemma 4.6 and the inequality (- 32 -),

$$\begin{aligned} s_k^- &= d(\gamma_k(0),\gamma_k(s_k^-)) \\ &\leq d(\gamma_k(0),\gamma_{k-1}(0)) + d(\gamma_{k-1}(0),\gamma_{k-1}(s_{k-1}^-)) + d(\gamma_{k-1}(s_{k-1}^-),\gamma_k(s_k^-)) \\ &\leq c_1 + \delta_0 + c_2\, \sinh(\delta_0 + c_1)\ . \end{aligned}$$

Let z_0 and $z_{s_{k-1}^-}$ be the closest points on γ_k to $\gamma_{k-1}(0)$ and $x' = \gamma_{k-1}(s_{k-1}^-)$, respectively. If $s_{k-1}^- \leq 0$, then as the closest point map does not increase distances, we have

$$\begin{aligned} s_k^- &= d(\gamma_k(0),\gamma_k(s_k^-)) \leq d(\gamma_k(0),z_0) + d(z_0,\gamma_k(s_k^-)) \\ &\leq d(\gamma_k(0),z_0) + d(z_{s_{k-1}^-},\gamma_k(s_k^-)) \\ &\leq d(\gamma_k(0),\gamma_{k-1}(0)) + d(\gamma_{k-1}(s_{k-1}^-),\gamma_k(s_k^-)) \\ &\leq c_1 + c_2\, \sinh(\delta_0 + c_1)\ . \end{aligned}$$

Hence, by the definition of c_5 and as $c_3'(\epsilon_0) \geq 1$ (see the equation (- 6 -) if $\epsilon_0 \neq \infty$ or (- 11 -) otherwise), we have

$$c_5 \geq c_2\, \sinh(\delta_0 + c_1) + c_3'(\epsilon_0) \sinh(\delta_0 + c_1) \geq c_2\, \sinh(\delta_0 + c_1) + \delta_0 + c_1 \geq s_k^-\ .$$

Now $\ell_{C_n}(\gamma_k) \geq h_0 > \delta_0$ by the definition of h_0, and

$$\ell_{C_0}(\gamma_k) \geq f_0(\gamma_k) - \kappa_0 = h - \kappa_0 \geq h_1' - \kappa_0 \geq h_0 + 2c_5 - \kappa_0 > \delta_0 + c_5 \geq \delta_0 + s_k^-\ .$$

As the entrance time s_k^- of γ_k in C_n is positive by Lemma 4.5, this implies that $\operatorname{diam}(C_0 \cap C_n) > \delta_0$. As $n \neq 0$, this contradicts the Almost disjointness property (iii). Hence $\delta_0 < s_{k-1}^-$.

Assume now by absurd that $s_{k-1}^- \geq \tau$. Then as in the case $s_{k-1}^- \leq 0$, we get

$$t_k^- - s_k^- \leq d(\gamma_{k-1}(\tau),\gamma_k(t_k^-)) + d(\gamma_{k-1}(s_{k-1}^-),\gamma_k(s_k^-)) \leq c_1 + c_1'(\epsilon_0)\ , \tag{- 33 -}$$

by Lemma 4.6, and by Lemma 2.3 if $\epsilon_0 \neq \infty$ and Lemma 2.9 otherwise. We have seen in the inequalities (-28-) that

$$h_0 \geq \sinh(\delta_0 + c_1) + c'_1(\epsilon_0) > \delta_0 + c_1 + c'_1(\epsilon_0).$$

Hence

$$t_k^- \geq s_k^+ - \delta_0 \geq s_k^- + h_0 - \delta_0 > s_k^- + c_1 + c'_1(\epsilon_0)\ ,$$

a contradiction to Equation (-33-). Hence $s_{k-1}^- < \tau$. □

Assume first that the inequality (-29-) holds. As $s_k^- \geq 0$ by Lemma 4.5 and by the definition of h_0, we have

$$s_k^+ > h'_0 + s_k^- \geq h_0 \geq \delta_0 + 2c_1 + c_6\ .$$

Hence, as $t_k^- \leq \tau + 2c_1$ by Lemma 4.7, we have $e^{-c_6}\ e^{s_k^+ - \delta_0 - t_k^-} \geq e^{-\tau}$. By the definition of c_6 and of c_4, we have

$$c_6 = 3c_4 + \log 2 \geq 3\ c'_3(\epsilon_0) \sinh(c_1 + \delta_0) + \log 2\ .$$

By the triangular inequality since $s_{k-1}^+ \geq 0$ by Lemma 4.9, by the equations (-29-), (-31-), (-27-), and (-22-), and by the definition of c_4, we hence have

$$\begin{aligned} |s_k^+ - s_{k-1}^+| &\leq d(y, y') + d(\gamma_k(0), \gamma_{k-1}(0)) \\ &\leq c'_3(\epsilon_0)\ e^{s_k^+ - t_k^- - \delta_0}\ \sinh(\delta_0 + c_1) + c_2\, e^{-\tau} \sinh c_1 \\ &\leq c_4\ e^{s_k^+ - \delta_0 - t_k^-}\ . \end{aligned} \tag{-34-}$$

By the Lipschitz property (*ii*) of $f_n = \ell_{C_n}$ (as $n \neq 0$), by the inequalities (-32-), (-29-), (-31-), and (-27-), and by the definition of c_5, we have

$$\begin{aligned} &|f_n(\gamma_{k-1}) - f_n(\gamma_k)| \leq 2 \max\{d(x, x'), d(y, y')\} \\ &\leq 2 \max\{c_2\ e^{s_k^+ - \delta_0 - t_k^-}\ \sinh(\delta_0 + c_1), c'_3(\epsilon_0)\ e^{s_k^+ - \delta_0 - t_k^-}\ \sinh(\delta_0 + c_1)\} \\ &\leq c_5\ e^{s_k^+ - \delta_0 - t_k^-}\ . \end{aligned} \tag{-35-}$$

By the inequalities (-26-) and (-35-), by the definition of τ and by Lemma 4.9, we have

$$u_k(s_k^+ - \delta_0) < f_n(\gamma_k) \leq f_n(\gamma_{k-1}) + c_5\ e^{s_k^+ - \delta_0 - t_k^-} \leq u_{k-1}(s_{k-1}^+ - \delta_0) + c_5\ e^{s_k^+ - \delta_0 - t_k^-}\ .$$

Assume now that the inequality (-30-) holds instead of the inequality (-29-). Then $f_n(\gamma_k) \leq f_n(\gamma_{k-1})$, so we again have that

$$u_k(s_k^+ - \delta_0) < u_{k-1}(s_{k-1}^+ - \delta_0) + c_5\ e^{s_k^+ - \delta_0 - t_k^-}.$$

As $|(s^+_{k-1}-\delta_0)-(s^+_k-\delta_0)|\le c_4\, e^{s^+_k-\delta_0-t^-_k}$ by the inequality (-34-), this contradicts the assertion (4) on the map u_k. Hence the assertion (6) at rank k is verified.

The main corollary of the construction. The above inductive construction will only be used in this paper through the following summarising statement.

Proposition 4.10 *Let X be a proper geodesic $\mathrm{CAT}(-1)$ metric space. Let ϵ_0 in $\mathbb{R}_+^*\cup\{\infty\}$, $\delta_0,\kappa_0\ge 0$ and $\xi_0\in X\cup\partial_\infty X$. Let $h_0'\ge h_0(\epsilon_0,\delta_0,\kappa_0)$ and $h\ge h_1'=h_1'(\epsilon_0,\delta_0,h_0')$. Let $(C_n)_{n\in\mathbb{N}}$ be a collection of ϵ_0-convex subsets of X which satisfies the assertions (iii) and (iv), and with $\xi_0\notin C_0\cup\partial_\infty C_0$. Let $f_0:T^1_{\xi_0}X\to[0,+\infty]$ be a continuous κ_0-penetration map in C_0. Assume that there exists a geodesic ray or line γ_0 starting from ξ_0 with $f_0(\gamma_0)=h$. Then there exists a geodesic ray or line γ_∞ starting from ξ_0, entering C_0 at time $t=0$ with $f_0(\gamma_\infty)=h$, such that $\ell_{C_n}(\gamma_\infty)\le h_1'$ for every n in $\mathbb{N}-\{0\}$ such that $\gamma_\infty(]\delta_0,+\infty[)$ meets C_n.*

Proof. Apply the main construction of the previous subsections with initial input a geodesic ray or line γ_0 entering C_0 at time $t=0$ with $f_0(\gamma_0)=h$, to get finite or infinite sequences $(\gamma_k)_{k\in\mathscr{N}}$, $(n_k)_{k\in\mathscr{N}}$, $(u_k)_{k\in\mathscr{N}}$ satisfying the assertions (1)–(6).

If $\mathscr{N}$ is finite, with maximum N, define $\gamma_k=\gamma_N$ for $k>N$. Then the sequence $(\gamma_k)_{k\in\mathbb{N}}$ converges to a geodesic ray or line $\gamma_\infty=\gamma_N$ in $T^1_{\xi_0}X$. If $\mathscr{N}$ is infinite, as X is complete, it follows from the assertions (3) and (5), by an easy geometric series argument, that the sequence $(\gamma_k)_{k\in\mathscr{N}}$ converges in $T^1_{\xi_0}X$ to a geodesic ray or line γ_∞ starting from ξ_0. Since C_0 is closed, the point $\gamma_\infty(0)$ belongs to C_0. If γ_∞ does not enter in C_0 at time $t=0$, then there exists $\eta>0$ such that $\gamma_\infty(-2\eta)\in C_0$. By the strict convexity of C_0, the geodesic segment $]\gamma_\infty(-2\eta),\gamma_\infty(0)[$ is contained in the interior of C_0. Hence if n is big enough, we have $\gamma_n(-\eta)\in C_0$, which is impossible. Hence γ_∞ does enter in C_0 at time $t=0$. By the continuity of f_0 and the assertion (1), we have $f_0(\gamma_\infty)=h$.

Suppose by absurd that there exists n in $\mathbb{N}-\{0\}$ such that $\gamma_\infty(]\delta_0,+\infty[)$ meets C_n and $\ell_{C_n}(\gamma_\infty)>h_1'>0$. In particular, $\gamma_\infty(]\delta_0,+\infty[)$ meets the interior of C_n and $\xi_0\notin C_n\cup\partial_\infty C_n$ by Lemma 4.5. Furthermore, it follows from the definition of the stopping time, and the fact that $u_k\le h_1'$ for every k, that $\mathscr{N}$ is infinite. Hence, as the γ_k's converge to γ_∞, and by the continuity of ℓ_{C_n}, if k is big enough, then $\gamma_k(]\delta_0,+\infty[)$ meets C_n and $\ell_{C_n}(\gamma_k)>h_1'$.

In particular, $t^+_n(\gamma_k)>\delta_0$. Since the point $\gamma_k(t^-_n(\gamma_k))$ is at distance at most $c_1'(\epsilon_0)$ from the point $\gamma_\infty(t^-_n(\gamma_\infty))$ by Lemma 2.3 if $\epsilon_0\ne\infty$ and by Lemma 2.9 otherwise, the sequence of times $\big(t^-_n(\gamma_k)\big)_{k\in\mathbb{N}}$ is bounded. Hence if k is big enough, then $t^-_n(\gamma_k)$

is less than t_k^-, since t_k^- converges to $+\infty$ when $k \to +\infty$ by the assertion (5). This contradicts the assertion (6), as $u_k \le h_1'$. □

Remark. If $X, \epsilon_0, \delta_0, \kappa_0, \xi_0, h_0', h, (C_n)_{n\in\mathbb{N}}, f_0$ satisfy the hypotheses in the statement of Proposition 4.10, and if for every n such that $\xi_0 \notin C_n \cup \partial_\infty C_n$, we have a κ-penetration map $g_n : T^1_{\xi_0}X \to [0, +\infty]$ for some constant $\kappa \ge 0$, then Proposition 4.10 implies that there exists a geodesic ray or line γ_∞ starting from ξ_0, entering C_0 at time $t = 0$ with $f_0(\gamma_\infty) = h$, such that $g_n(\gamma_\infty) \le h_1' + \kappa$ for every n in $\mathbb{N} - \{0\}$ such that $\gamma_\infty(]\delta_0, +\infty[)$ meets C_n. We will apply this observation to more general penetration maps than the ℓ_{C_n}'s, in Section 5.

The next corollary yields geodesic lines with a prescribed penetration in C_0, and that essentially avoid the C_n's not only for positive times, but also for negative ones. The penetration in the sets C_n for $n \ne 0$ cannot be made quite as small as in Proposition 4.10.

Corollary 4.11 *Let X be a proper geodesic $\mathrm{CAT}(-1)$ metric space. Let ϵ_0 in $\mathbb{R}_+^* \cup \{\infty\}$, $\delta_0, \kappa_0 \ge 0$. Let C_0 be a proper ϵ_0-convex subset of X, and let*

$$f_0 : \bigcup_{\xi \in \partial_\infty X - \partial_\infty C_0} T^1_\xi X \to [0, +\infty]$$

be a continuous map such that $f_{0|T^1_{\xi_0}X}$ is a κ_0-penetration map in C_0 for every $\xi_0 \in \partial_\infty X - \partial_\infty C_0$. Let $h_0' \ge h_0 = h_0(\epsilon_0, \delta_0, \kappa_0)$, $h \ge h_1' = h_1'(\epsilon_0, \delta_0, h_0')$, and

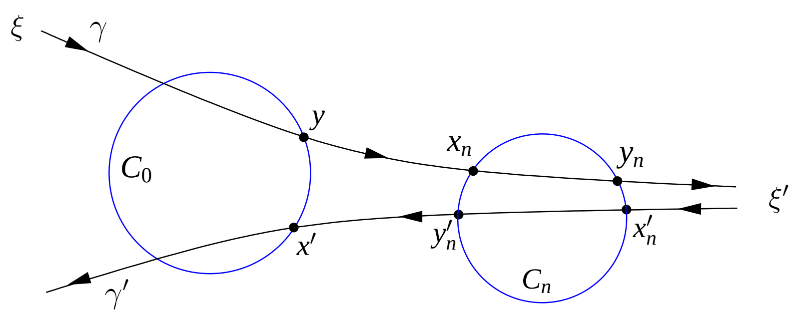
$$h_1'' = h_1'(\epsilon_0, \delta_0, h_0') + c_3'(\epsilon_0)(\delta_0 + c_1) + c_1'(\epsilon_0)\ .$$

Assume that there exists a geodesic line γ_0 in X with $f_0(\gamma_0) = h$. For every n in $\mathbb{N} - \{0\}$, let C_n be an ϵ_0-convex subset of X, such that $(C_n)_{n\in\mathbb{N}}$ satisfies the assertions (iii) and (iv) with respect to every $\xi_0 \in \partial_\infty X - \partial_\infty C_0$. Then there exists a geodesic line γ' in X entering C_0 at time $t = 0$ with $f_0(\gamma') = h$, such that $\ell_{C_n}(\gamma') \le h_1''$ for every n in $\mathbb{N} - \{0\}$.

Proof. Let γ_0 be a geodesic line in X with $f_0(\gamma_0) = h$, and let ξ be the starting point at infinity of γ_0, which does not belong to $\partial_\infty C_0$ as $h < +\infty$. Applying Proposition 4.10 with $\xi_0 = \xi$, as $h \ge h_1'$, there exists a geodesic line γ starting from ξ and entering C_0 at time 0, such that $f_0(\gamma) = h$ and $\ell_{C_n}(\gamma) \le h_1'$ for every $n \in \mathbb{N} - \{0\}$ such that $\gamma(]\delta_0, +\infty[)$ meets C_n.

Let ξ' be the other endpoint at infinity of γ, which does not belong to $\partial_\infty C_0$ as $h < +\infty$. Applying Proposition 4.10 again with now $\xi_0 = \xi'$, we get that there exists a geodesic line γ' starting from ξ' and entering C_0 at time 0, such that $f_0(\gamma') = h$ and $\ell_{C_n}(\gamma') \leq h'_1$ for every $n \in \mathbb{N} - \{0\}$ such that $\gamma'(]\delta_0, +\infty[)$ meets C_n.

Assume by absurd that there exists $n \in \mathbb{N} - \{0\}$ such that $\ell_{C_n}(\gamma') > h''_1 > 0$. Then γ' enters C_n at a point x'_n, exiting it at a point y'_n at time at most δ_0, as $h''_1 > h'_1$ by the definition of h''_1. In particular, if $x' = \gamma'(0)$ is the entering point of γ' in C_0, then $d(y'_n, x') \leq \delta_0$ if x', y'_n, x'_n, ξ' are not in this order on γ'.



Let y be the exiting point of γ out of C_0. Note that

$$h \geq h'_1 \geq h'_0 \geq h_0 \geq h'(\epsilon_0, \sinh(\delta_0 + c_1)) \geq h'(\epsilon_0, \delta_0 + c_1) \tag{- 36 -}$$

by the definitions of h'_1, h_0, h'. By Lemma 2.3 if $\epsilon_0 \neq \infty$ and by Lemma 2.9 if $\epsilon_0 = \infty$ and $(f_0, \delta_0) \neq (\mathfrak{ph}_{C_0}, 0)$, and as in the proof of Lemma 4.6 if $(\epsilon_0, f_0, \delta_0) = (\infty, \mathfrak{ph}_{C_0}, 0)$ since $h \geq h_0$, we have $d(x', y) \leq c_1$. Hence by convexity,

$$d(y'_n, \gamma) \leq d(x', \gamma) + \delta_0 \leq d(x', y) + \delta_0 \leq \delta_0 + c_1 \ . \tag{- 37 -}$$

Note that

$$d(x'_n, y'_n) = \ell_{C_n}(\gamma') > h''_1 \geq h'_1 \geq h'(\epsilon_0, \delta_0 + c_1)$$

by the definition of h''_1 and by the inequalities (- 36 -). Hence, by Lemma 2.6 if $\epsilon_0 \neq \infty$ and by Lemma 2.12 otherwise, the geodesic line γ enters C_n at a point x_n and exits it at a point y_n such that

$$d(y'_n, x_n) \leq c'_3(\epsilon_0) d(y'_n, \gamma) \quad \text{or} \quad d(x_n, y_n) \geq d(x'_n, y'_n) \ . \tag{- 38 -}$$

Let us prove by absurd that $\gamma(]\delta_0, +\infty[)$ meets C_n. Otherwise, since $\gamma^{-1}(y) \geq 0$, by convexity, and by Lemma 2.3 if $\epsilon_0 \neq \infty$ or Lemma 2.9 if $\epsilon_0 = \infty$, we have

$$d(x', x'_n) \leq d(x', y) + \delta_0 + d(y_n, x'_n) \leq c_1 + \delta_0 + c'_1(\epsilon_0) \ . \tag{- 39 -}$$

By the definition of h''_1 and since $c'_3(\epsilon_0) \geq 2$ by Equation (- 6 -) if $\epsilon_0 \neq \infty$, and by Equation (- 11 -) otherwise, we have

$$d(x', x'_n) \geq d(x'_n, y'_n) - \delta_0 > h''_1 - \delta_0 \geq h'_1 + 2(\delta_0 + c_1) + c'_1(\epsilon_0) - \delta_0 \geq c_1 + \delta_0 + c'_1(\epsilon_0) \ .$$

This contradicts the inequalities (- 39 -).

Assume first that the second of the inequalities (- 38 -) holds. As $d(x'_n, y'_n) > h'_1$, this contradicts the construction of γ. Hence the first of the inequalities (- 38 -) is satisfied, and by Equation (- 37 -), we have

$$d(y'_n, x_n) \leq c'_3(\epsilon_0) d(y'_n, \gamma) \leq c'_3(\epsilon_0)(\delta_0 + c_1) \ .$$

But then, by the triangular inequality and by the definition of h''_1,

$$d(x_n, y_n) \geq d(x'_n, y'_n) - d(x_n, y'_n) - d(y_n, x'_n) > h''_1 - c'_3(\epsilon_0)(\delta_0 + c_1) - c'_1(\epsilon_0) = h'_1 \ ,$$

which contradicts the construction of γ. □

5 Prescribing the penetration of geodesic lines

In this section, we apply Proposition 4.10 to prove a number of results on the geodesic flow of negatively curved Riemannian manifolds.

The following constants appear in the theorems, depending on $\epsilon \in \mathbb{R}_+^* \cup \{\infty\}, \delta, \kappa \geq 0$.

- $c''_1(\epsilon, \delta, \kappa) = \max\{2c'_1(\epsilon) + 2\delta + \kappa, h'_1(\epsilon, \delta, h_0(\epsilon, \delta, c'_1(\infty)))\}$.
- $c''_2(\epsilon) = c''_1(\epsilon, 0, 0) + c'_1(\infty) + 2c_1$, where $c_1 = c'_1(\epsilon)$ if $\epsilon \neq \infty$, and $c_1 = 1/19$ otherwise. Note that $c''_2(\infty) = h'_1(\infty, 0, h_0(\infty, 0, c'_1(\infty))) + c'_1(\infty) + 2c_1 \approx 8.3712$ by the definition of $c''_1(\epsilon, \delta, \kappa)$ and the approximation (- 17 -).

Recall that the constants $c'_1(\epsilon)$ are given by Lemmas 2.3 and 2.9, and that $h_0(\cdot, \cdot, \cdot)$ and $h'_1(\cdot, \cdot, \cdot)$ are given in the list of constants in the beginning of Subsection 4.2.

5.1 Climbing in balls and horoballs

In this subsection, we construct geodesic rays or lines having prescribed penetration properties in a ball or a horoball, while essentially avoiding a family of almost disjoint convex subsets. Let us consider the penetration height and inner projection penetration maps first in horoballs and then in balls. Note that in these cases, if $f_0 = \mathfrak{ph}_{C_0}$, then $\|f_0 - \mathfrak{ph}_{C_0}\|_\infty = 0$ and if $f_0 = \mathfrak{ipp}_{C_0}$, then $\|f_0 - \mathfrak{ph}_{C_0}\|_\infty \leq c'_1(\infty)$ by Section 3.1.

Theorem 5.1 *Let $\epsilon \in \mathbb{R}_+^* \cup \{\infty\}$, $\delta, \kappa \geq 0$; let X be a complete simply connected Riemannian manifold with sectional curvature at most -1 and dimension at least 3; let $\xi_0 \in X \cup \partial_\infty X$; let C_0 be a horoball such that $\xi_0 \notin C_0 \cup \partial_\infty C_0$; let $f_0 = \mathfrak{ph}_{C_0}$*

or $f_0 = \mathfrak{ipp}_{C_0}$*; let* $(C_n)_{n\in\mathbb{N}-\{0\}}$ *be a family of* ϵ*-convex subsets of* X*; for every* $n \in \mathbb{N} - \{0\}$ *such that* $\xi_0 \notin C_n \cup \partial_\infty C_n$*, let* $f_n : T^1_{\xi_0}X \to [0, +\infty]$ *be a* κ*-penetration map in* C_n*. If* $\operatorname{diam}(C_n \cap C_m) \leq \delta$ *for all* n, m *in* $\mathbb{N}$ *with* $n \neq m$*, then, for every* $h \geq c''_1(\epsilon, \delta, \|f_0 - \mathfrak{ph}_{C_0}\|_\infty)$*, there exists a geodesic ray or line* γ *starting from* ξ_0 *and entering* C_0 *at time* 0*, such that* $f_0(\gamma) = h$ *and* $f_n(\gamma) \leq c''_1(\epsilon, \delta, \|f_0 - \mathfrak{ph}_{C_0}\|_\infty) + \kappa$ *for every* $n \geq 1$ *such that* $\gamma\big(]\delta, +\infty[\big)$ *meets* C_n*.*

Proof. Let $h \geq c''_1 = c''_1(\epsilon, \delta, \|f_0 - \mathfrak{ph}_{C_0}\|_\infty)$. In order to apply Proposition 4.10, define $\epsilon_0 = \epsilon, \delta_0 = \delta, \kappa_0 = c'_1(\infty) = 2\log(1+\sqrt{2})$ and $h'_0 = h_0(\epsilon_0, \delta_0, \kappa_0)$. Recall that $\mathfrak{ph}_{C_0}$ and $\mathfrak{ipp}_{C_0}$ are κ_0-penetration maps for C_0 by Lemma 3.3. For every $n \in \mathbb{N} - \{0\}$ such that $\xi_0 \notin C_n \cup \partial_\infty C_n$, let us apply Proposition 3.7 Case (1) to $C = C_0$, $C' = C_n$, $f = f_0$, $f' = \ell_{C_n}$, $h' = h'_0$, so that $h^{\min} = 2\,c'_1(\epsilon) + 2\delta + \|f_0 - \mathfrak{ph}_{C_0}\|_\infty$ and $h_0^{\min} = 2\delta$. Note that $h_0^{\min} \leq h'_0$, as

$$h'_0 \geq h'(\epsilon, \sinh(\delta + c_1)) \geq 2\sinh(\delta + c_1) \geq 2\delta\ ,$$

by the definitions of h_0 and of $h'(\cdot,\cdot)$. As $h \geq c''_1 \geq h^{\min}$ by the definition of c''_1, Proposition 3.7 Case (1) hence implies that $(C_n)_{n\in\mathbb{N}}$ satisfies the Local prescription property (iv). Thus by Proposition 4.10, there exists a geodesic ray or line γ starting at ξ_0 such that $f_0(\gamma) = h$ and $\ell_{C_n}(\gamma) \leq h'_1(\epsilon_0, \delta_0, h_0(\epsilon_0, \delta_0, \kappa_0))$, which implies that $f_n(\gamma) \leq c''_1 + \kappa$, for every $n \geq 1$ such that $\gamma(]\delta, +\infty[)$ meets C_n. $\square$

The proof of the corresponding result when C_0 is a ball of radius $R \geq \epsilon$ is the same, using Case (2) of Proposition 3.7 instead of Case (1). This requires $h \leq h^{\max} = 2R - 2c'_1(\epsilon) - \|f_0 - \mathfrak{ph}_{C_0}\|_\infty$. To be nonempty, the following result requires

$$R \geq c''_1(\epsilon, \delta, \|f_0 - \mathfrak{ph}_{C_0}\|_\infty)/2 + \ c'_1(\epsilon) + \|f_0 - \mathfrak{ph}_{C_0}\|_\infty/2\ .$$

Theorem 5.2 *Let* $\epsilon > 0$*,* $\delta, \kappa \geq 0$*; let* X *be a complete simply connected Riemannian manifold with sectional curvature at most* -1 *and dimension at least* 3*; let* C_0 *be a ball of radius* $R \geq \epsilon$*; let* $\xi_0 \in (X \cup \partial_\infty X) - C_0$*; let* $f_0 = \mathfrak{ph}_{C_0}$ *or* $f_0 = \mathfrak{ipp}_{C_0}$*; let* $(C_n)_{n\in\mathbb{N}-\{0\}}$ *be a family of* ϵ*-convex subsets of* X*; for every* $n \in \mathbb{N} - \{0\}$ *such that* $\xi_0 \notin C_n \cup \partial_\infty C_n$*, let* $f_n : T^1_{\xi_0}X \to [0, +\infty]$ *be a* κ*-penetration map in* C_n*. If* $\operatorname{diam}(C_n \cap C_m) \leq \delta$ *for all* n, m *in* $\mathbb{N}$ *with* $n \neq m$*, then, for every*

$$h \in \Big[c''_1\big(\epsilon, \delta, \|f_0 - \mathfrak{ph}_{C_0}\|_\infty\big),\ 2R - 2c'_1(\epsilon) - \|f_0 - \mathfrak{ph}_{C_0}\|_\infty\ \Big]\ ,$$

there exists a geodesic ray or line γ *starting from* ξ_0 *and entering* C_0 *at time* 0*, such that* $f_0(\gamma) = h$ *and* $f_n(\gamma) \leq c''_1\big(\epsilon, \delta, \|f_0 - \mathfrak{ph}_{C_0}\|_\infty\big) + \kappa$ *for every* $n \geq 1$ *such that* $\gamma(]\delta, +\infty[)$ *meets* C_n*.* $\square$

Varying the family $(C_n)_{n\in\mathbb{N}-\{0\}}$ of ϵ-convex subsets appearing in Theorems 5.1 and 5.2, among balls of radius at least ϵ, horoballs, ϵ-neighbourhoods of totally geodesic subspaces, etc, we get several corollaries. We will only state two of them, Corollaries 5.3 and 5.5, which have applications to equivariant families. The proofs of these results are simplified versions of the proof of Corollary 4.11, giving better (though very probably not optimal) constants.

Corollary 5.3 *Let X be a complete simply connected Riemannian manifold with sectional curvature at most -1 and dimension at least 3, and let $\big(H_n\big)_{n\in\mathbb{N}}$ be a family of horoballs in X with pairwise disjoint interiors. Then, for every $h \ge c''_1(\infty,0,0) \approx 6.5032$, there exists a geodesic line γ' such that $\mathfrak{ph}_{H_0}(\gamma') = h$ and $\mathfrak{ph}_{H_n}(\gamma') \le c''_2(\infty) \approx 8.3712$ for every $n \ge 1$.*

Proof. Let $c''_1 = c''_1(\infty,0,0)$ and $c''_2 = c''_2(\infty)$. Let $C_0 = H_0$ and let ξ be a point in $\partial_\infty X - \partial_\infty C_0$. We apply Theorem 5.1 with $\epsilon = \infty$, $\delta = 0$, $\kappa = 0$, $\xi_0 = \xi$, $C_n = H_n$ for every n in $\mathbb{N}$, $f_0 = \mathfrak{ph}_{C_0}$, and $f_n = \ell_{C_n}$ for every $n \neq 0$ such that $\xi_0 \notin C_n \cup \partial_\infty C_n$. Note that for every $n \in \mathbb{N} - \{0\}$, the map f_n is a κ-penetration map in C_n. As $h \ge c''_1$, there exists a geodesic line γ starting from ξ and entering C_0 at time 0, such that $\mathfrak{ph}_{C_0}(\gamma) = h$ and $\ell_{C_n}(\gamma) \le c''_1$ for every $n \in \mathbb{N} - \{0\}$ such that γ meets C_n at a positive time.

Let ξ' be the other endpoint of γ. This point is not in $\partial_\infty C_0$. Applying Theorem 5.1 again, as above except that now $\xi_0 = \xi'$, we get that there exists a geodesic line γ' starting from ξ' and entering C_0 at time 0, such that $\mathfrak{ph}_{C_0}(\gamma') = h$ and $\ell_{C_n}(\gamma') \le c''_1$ for every $n \in \mathbb{N} - \{0\}$ such that γ' meets C_n at a positive time.

Assume by absurd that there exists $n \in \mathbb{N} - \{0\}$ such that $\mathfrak{ph}_{C_n}(\gamma') > c''_2 > 0$. Then γ' enters C_n at the point x'_n, exiting it at the point y'_n at a nonpositive time, as $c''_2 > c''_1$. In particular, if $x' = \gamma'(0)$ is the entering point of γ' in C_0, then x', y'_n, x'_n, ξ' are in this order on γ' (see the picture in the proof of Corollary 4.11).

Let y be the exiting point of γ out of H_0. With $c_1 = 1/19$, as in the proof of Lemma 4.6, since $\mathfrak{ph}_{C_0}(\gamma)$ and $\mathfrak{ph}_{C_0}(\gamma')$ are equal to

$$h \ge c''_1 \ge h'_1(\infty, 0, h_0(\infty, 0, c'_1(\infty))) \ge h_0(\infty, 0, c'_1(\infty)) = h'(\infty, \sinh c_1)$$

by the definition of c''_1, h'_1, h_0, we have $d(x', y) \le c_1$.

Let ξ_n be the point at infinity of H_n. Let p' be the point in $[x'_n, y'_n]$ the closest to ξ_n, so that

$$d(p', y'_n) \ge \beta_{\xi_n}(y'_n, p') = \mathfrak{ph}_{C_n}(\gamma')/2 > c''_2/2.$$

Let p be the point of γ the closest to p'. By convexity and the definition of c''_2, we have

$$d(p',p)=d(p',\gamma)\leq d(x',\gamma)\leq d(x',y)\leq c_1<c''_2/2\;.$$

Hence p belongs to the interior of C_n. If $p\in\,]\xi,y]$, then the closest point to x' on γ lies in $]\xi,p[$ and by convexity,

$$c''_2/2<d(p',y'_n)\leq d(p',x')\leq d(p',p)+d(p,x')\leq d(p',p)+d(y,x')\leq 2c_1\;,$$

a contradiction, as by the definition of c''_2, of c''_1 and of $h'(\epsilon,\eta)$ (see the equations (- 5 -) and (- 11 -)), we have

$$c''_2\geq c''_1+2\,c_1\geq h'(\epsilon,\sinh c_1)+2\,c_1\geq 2\sinh c_1+2c_1>4\,c_1\;.$$

Hence $p\in\,]y,\xi'[\,\subset\gamma(]0,+\infty[)$, so that γ meets C_n at a positive time. But, by Lemma 3.3 and the definition of c''_2,

$$\begin{aligned}\ell_{C_n}(\gamma)&\geq \mathfrak{ph}_{C_n}(\gamma)-c'_1(\infty)\geq 2\beta_{\xi_n}(y'_n,p)-c'_1(\infty)\\&\geq 2(\beta_{\xi_n}(y'_n,p')-d(p,p'))-c'_1(\infty)>2(c''_2/2-c_1)-c'_1(\infty)=c''_1\;.\end{aligned}$$

This contradicts the construction of γ. □

Let M be a complete nonelementary geometrically finite Riemannian manifold with sectional curvature at most -1 (see for instance [Bow] for a general reference). Recall that a *cusp* of M is an asymptotic class of minimizing geodesic rays in M along which the injectivity radius converges to 0. If M has finite volume, then the set of cusps of M is in bijection with the (finite) set of ends of M, by the map which associates to a representative of a cusp the end of M towards which it converges. Let $\pi:\widetilde{M}\rightarrow M$ be a universal Riemannian covering of M, with covering group Γ. If e is a cusp of M, and ρ_e a minimizing geodesic ray in the class e, as M is geometrically finite and nonelementary, there exists a horoball H_e in $\widetilde{M}$ centered at the point at infinity ξ_e of a fixed lift $\widetilde{\rho}_e$ of ρ_e in $\widetilde{M}$, such that γH_e and H_e have disjoint interiors if $\gamma\in\Gamma$ does not fix ξ_e (see for instance [BK, Bow, HP5]). This horoball is unique if maximal (for the inclusion). The image V_e of H_e in M is called a *Margulis neighbourhood of* e, and the *maximal Margulis neighbourhood of* e if H_e is maximal. In this Section 5, we assume that H_e is maximal, and that $\widetilde{\rho}_e$ starts from the boundary of H_e. Let $\mathrm{ht}_e:M\rightarrow\mathbb{R}$ be the map defined by

$$\mathrm{ht}_e(x)=\lim_{t\rightarrow\infty}\,(t-d(\rho_e(t),x))\;,$$

called the *height function with respect to* e. Let $\mathrm{maxht}_e:T^1M\rightarrow\mathbb{R}$ be defined by

$$\mathrm{maxht}_e(\gamma)=\sup_{t\in\mathbb{R}}\mathrm{ht}_e(\gamma(t))\;.$$

The *maximum height spectrum* of the pair (M,e) is the subset of $]-\infty,+\infty]$ defined by
$$\mathrm{MaxSp}(M,e)=\mathrm{maxht}_e(T^1M)\ .$$

Corollary 5.4 *Let M be a complete, nonelementary geometrically finite Riemannian manifold with sectional curvature at most -1 and dimension at least 3, and let e be a cusp of M. Then $\mathrm{MaxSp}(M,e)$ contains $[c_2''(\infty)/2,+\infty]$.*

Note that $c_2''(\infty)/2\approx 4.1856$, hence Theorem 1.2 of the introduction follows.

Proof. With the above notations, let $(H_n)_{n\in\mathbb{N}}$ be the Γ-equivariant family of horoballs in $\widetilde{M}$ with pairwise disjoint interiors such that $H_0=H_e$. Apply Corollary 5.3 to this family to get, for every $h\geq c_2''(\infty)\geq c_1''(\infty,0,0)$, a geodesic line $\widetilde{\gamma}$ in $\widetilde{M}$ with $\mathfrak{ph}_{H_0}(\widetilde{\gamma})=h$ and $\mathfrak{ph}_{H_n}(\widetilde{\gamma})\leq c_2''(\infty)$ for every $n\geq 1$. Let γ be the locally geodesic line in M image by π of $\widetilde{\gamma}$. Observe that $\mathrm{ht}_e\circ\pi=\beta_{H_n}$ in H_n and that $\mathfrak{ph}_{H_n}(\widetilde{\gamma})=2\sup_{t\in\mathbb{R}}\beta_{H_n}(\widetilde{\gamma}(t))$ (see Section 3.1). Hence $\sup_{t\in\mathbb{R}}\mathrm{ht}_e(\gamma(t))=h/2$ and the result follows. $\square$

Schmidt and Sheingorn [SS] treated the case of two-dimensional manifolds of constant curvature -1 (hyperbolic surfaces) with a cusp. They showed that in that case $\mathrm{MaxSp}(M,e)$ contains the interval $[\log 100,+\infty]\approx[4.61,+\infty]$. This paper [SS] was a starting point of our investigations, although the method we use is quite different from theirs.

Let $\mathscr{P}$ be a (necessarily finite) nonempty set of cusps of M. For every e in $\mathscr{P}$, choose a maximal horoball H_e, with point at infinity ξ_e as above Corollary 5.4. The horoballs of the family $(gH_e)_{g\in\Gamma/\Gamma_{\xi_e}\,,\,e\in\mathscr{P}}$ may have intersecting interiors. But as M is geometrically finite and nonelementary, there exists (see [BK, Bow]) $t\geq 0$ such that two distinct elements in $(gH_e[t])_{g\in\Gamma/\Gamma_{\xi_e}\,,\,e\in\mathscr{P}}$ have disjoint interiors. Let $t_{\mathscr{P}}$ be the lower bound of all such t's. For every $\gamma\in T^1M$, define
$$\mathrm{maxht}_{\mathscr{P}}(\gamma)=\max_{e\in\mathscr{P}}\mathrm{maxht}_e(\gamma)\ \text{ and }\ \mathrm{MaxSp}(M,\mathscr{P})=\mathrm{maxht}_{\mathscr{P}}(T^1M)\ .$$

Remark. Let $\mathscr{C}$ be the set of all cusps of M. Under the same hypotheses as in Corollary 5.4, the following two assertions hold, by applying Corollary 5.3 to the family of horoballs $(gH'_{e'})_{g\in\Gamma/\Gamma_{\xi_{e'}}\,,\,e'\in\mathscr{C}}$ with $H'_{e'}=H_e$ if $e'=e$, and $H'_{e'}=H_{e'}[t]$ for some t big enough otherwise, for the first assertion, and to the family $(gH_e[t_{\mathscr{P}}])_{g\in\Gamma/\Gamma_{\xi_e}\,,\,e\in\mathscr{P}}$ for the second one.

(1) For every cusp e of M, there exists a constant $t \geq 0$ such that for every $h \geq t$, there exists a locally geodesic line γ in M such that $\mathrm{maxht}_e(\gamma) = h$ and $\mathrm{maxht}_{e'}(\gamma) \leq t$ for every cusp $e' \neq e$ in M.

(2) Let $\mathscr{P}$ be a nonempty set of cusps of M. Then $\mathrm{MaxSp}(M, \mathscr{P})$ contains the halfline $[c''_2(\infty)/2 + t_{\mathscr{P}}, +\infty]$.

Now, we prove the analogs of Corollaries 5.3 and 5.4 for families of balls with disjoint interiors. Let

$$R_0^{\min} = 7 \sinh c'_1(\infty) + \frac{3}{2} c'_1(\infty) \approx 22.4431.$$

Corollary 5.5 *Let X be a complete simply connected Riemannian manifold with sectional curvature at most -1 and dimension at least 3, and let $\big(B_n\big)_{n\in\mathbb{N}}$ be a family of balls in X with disjoint interiors such that the radius R_0 of B_0 is at least $R_0^{\min}$. For every $h \in \big[c''_1(R_0^{\min}, 0, 0), 2R_0 - 2\, c'_1(R_0^{\min})\big]$, there exists a geodesic line γ in X with $\mathfrak{ph}_{B_0}(\gamma) = h$ and $\mathfrak{ph}_{B_n}(\gamma) \leq c''_2(R_0^{\min})$ for all $n \geq 1$.*

Proof. We start by some computations. Let $\epsilon > 0$. With $c_1 = c'_1(\epsilon)$ and $c_5 = c_5(\epsilon, 0)$ as in Subsection 4.2, we have $c_5 \geq 6 \sinh c_1$ since $c'_3(\epsilon) \geq 3$ by the definition of $c'_3(\epsilon)$ in Equation (-6-). By the definition of h_0 in Subsection 4.2 and of h' in Equation (-5-), we have $h_0(\epsilon, 0, c'_1(\infty)) \geq h'(\epsilon, \sinh c_1) \geq 2 \sinh c_1$. Hence, by the definitions of $c''_2(\epsilon), c''_1(\epsilon, 0, 0), h'_1$ and as $\epsilon \mapsto c'_1(\epsilon)$ is decreasing,

$$\begin{aligned} c''_2(\epsilon) &= c''_1(\epsilon, 0, 0) + c'_1(\infty) + 2\, c_1 \\ &= \max\{2\, c_1, h_0(\epsilon, 0, c'_1(\infty)) + 2\, c_5\} + c'_1(\infty) + 2\, c_1 \\ &\geq 2 \sinh c_1 + 12 \sinh c_1 + c'_1(\infty) + 2\, c_1 \geq 14 \sinh c'_1(\infty) + 3\, c'_1(\infty)\ . \end{aligned}$$

Define now $\epsilon = R_0^{\min}$, so that $2\epsilon \leq c''_2(\epsilon)$ and $R_0 \geq \epsilon$. For every $n \neq 0$, let R_n be the radius of the ball B_n. If for some $n \neq 0$ we have $2R_n \leq c''_2(\epsilon)$, then $\mathfrak{ph}_{B_n}(\gamma) \leq c''_2(\epsilon)$ and the last assertion of Corollary 5.5 holds for this n. Hence up to removing balls, we may assume that $R_n \geq c''_2(\epsilon)/2 \geq \epsilon$ for every $n \neq 0$, so that the balls in $(B_n)_{n\in\mathbb{N}}$ are ϵ-convex.

The end of the proof is now exactly as the proof of Corollary 5.3, with the following modifications: $\epsilon = R_0^{\min}$; $c''_1 = c''_1(\epsilon, 0, 0)$; $c''_2 = c''_2(\epsilon)$; ξ is any point in $\partial_\infty X$; $C_n = B_n$ for every n in $\mathbb{N}$; we apply Theorem 5.2 instead of Theorem 5.1, which is possible by the range assumption on h; we take now $c_1 = c'_1(\epsilon)$, so that we still have $d(x', y) \leq c_1$ by Proposition 2.3; ξ_n is now the center of B_n, and $\beta_{\xi_n}(u, v) = d(u, \xi_n) - d(v, \xi_n)$ (see Section 2.1). Besides that, the proof is unchanged. □

A heavy computation shows that

$$c'_1(R_0^{\min}) \approx 1.7627, c''_1(R_0^{\min},0,0) \approx 101.4169 \quad\text{and}\quad c''_2(R_0^{\min}) \approx 106.7051\ .$$

Note that the above corollary is nonempty only if $R_0 \geq c''_1(R_0^{\min},0,0)/2 + c'_1(R_0^{\min}) \approx 52.4712$. The constants in the following corollary are not optimal. Theorem 1.3 in the introduction follows from it.

Corollary 5.6 *Let M be a complete Riemannian manifold with sectional curvature at most -1 and dimension at least 3, let $(x_i)_{i\in I}$ be a finite or countable family of points in M with $r_i = \operatorname{inj}_M x_i$, such that $d(x_i, x_j) \geq r_i + r_j$ if $i \neq j$ and such that $r_{i_0} \geq 56$ for some $i_0 \in I$. Then, for every $d \in [2, r_{i_0} - 54]$, there exists a locally geodesic line γ passing at distance exactly d from x_{i_0} at time 0, remaining at distance greater than d from x_{i_0} at any nonzero time, and at distance at least $r_i - 56$ from x_i for every $i \neq i_0$. In particular,*

$$\min_{t\in\mathbb{R}} d(\gamma(t), x_{i_0}) = d.$$

Proof. Let $\pi : \widetilde{M} \to M$ be a universal covering of M, with covering group Γ, and fix a lift $\widetilde{x}_i$ of x_i for every $i \in I$. Let B_i be the ball $B_{\widetilde{M}}(\widetilde{x}_i, r_i)$. Apply Corollary 5.5 to the family of balls $(g\,B_i)_{g\in\Gamma\,,\,i\in I}$ in $X = \widetilde{M}$, which have pairwise disjoint interiors by the definition of r_i and the assumption on $d(x_i, x_j)$. Note that $r_{i_0} \geq 56 \geq R_0^{\min}$ (see the definition of $R_0^{\min}$). Let $h = 2(r_{i_0} - d)$, which belongs to $[108, 2r_{i_0} - 4]$, which is contained in $[c''_1(R_0^{\min},0,0), 2r_{i_0} - 2\,c'_1(R_0^{\min})]$ by the previous computations. Then Corollary 5.5 implies that there exists a geodesic line $\widetilde{\gamma}$ in $\widetilde{M}$ such that $\mathfrak{ph}_{B_{i_0}}(\widetilde{\gamma}) = h$ and $\mathfrak{ph}_{gB_i}(\widetilde{\gamma}) \leq c''_2(R_0^{\min}) < 108$ for all $(g,i) \neq (1, i_0)$. Parametrize $\widetilde{\gamma}$ such that its closest point to $\widetilde{x}_{i_0}$ is at time $t = 0$. Let $\gamma = \pi \circ \widetilde{\gamma}$, then the result follows by the definition of $\mathfrak{ph}_C$ (see Subsection 3.1). □

5.2 Spiralling around totally geodesic subspaces

In this subsection, we apply Proposition 4.10 and Corollary 4.11 when C_0 is a tubular neighbourhood of a totally geodesic submanifold. We only give a few of the various possible applications, others can be obtained by varying the objects $(C_n)_{n\in\mathbb{N}-\{0\}}$, as well as the various subcases in Proposition 3.7 (3) and (4).

Theorem 5.7 *Let $\epsilon > 0$, $\delta \geq 0$. Let X be a complete simply connected Riemannian manifold with sectional curvature at most -1 and dimension at least 3. Let L be a complete totally geodesic submanifold of X with dimension at least 2, different from*

X, and $C_0 = \mathscr{N}_\epsilon L$. Let $(C_n)_{n\in\mathbb{N}-\{0\}}$ be a family of ϵ-convex subsets in X such that $\operatorname{diam}(C_n \cap C_m) \leq \delta$ *for all $n \neq m$ in $\mathbb{N}$. Let either $f_0 = \mathfrak{ftp}_L$ or $f_0 = \ell_{\mathscr{N}_\epsilon L}$, with X having constant curvature in this second case. Let*

$$h'_0 = h_0\big(\epsilon_0, \delta_0, \max\{\|f_0 - \ell_{\mathscr{N}_\epsilon L}\|_\infty, 2\|f_0 - \mathfrak{ftp}_L\|_\infty + 2\epsilon - 8\,c'_1(\epsilon)\}\big)$$

and $h \geq h'_1 = h'_1(\epsilon, \delta, h'_0)$.

- *For every $\xi \in (X \cup \partial_\infty X) - (C_0 \cup \partial_\infty C_0)$, there exists a geodesic ray or line γ starting from ξ and entering $\mathscr{N}_\epsilon L$ at time 0 with $f_0(\gamma) = h$, and with $\ell_{C_n}(\gamma) \leq h'_1$ for every $n \neq 0$ such that $\gamma(]\delta, +\infty[)$ meets C_n.*
- *There exists a geodesic line γ in X with $f_0(\gamma) = h$, and with*

  $$\ell_{C_n}(\gamma) \leq h'_1 + c'_3(\epsilon)\big(\delta + c'_1(\epsilon)\big) + c'_1(\epsilon)$$

  for all $n \neq 0$.

Note that if $\ell_{C_n}(\gamma) \leq c$, then $f(\gamma) \leq c + \kappa$ for any κ-penetration map f in C_n.

Proof. We apply Proposition 4.10 and Corollary 4.11 with $\xi_0 = \xi$, $\epsilon_0 = \epsilon$, $\delta_0 = \delta$,

$$\kappa_0 = \max\{\|f_0 - \ell_{\mathscr{N}_\epsilon L}\|_\infty, 2\|f_0 - \mathfrak{ftp}_L\|_\infty + 2\epsilon - 8\,c'_1(\epsilon)\}\ ,$$

and $h'_0 = h_0(\epsilon_0, \delta_0, \kappa_0)$. In particular, f_0 is a continuous κ_0-penetration map in C_0. As L is a complete totally geodesic submanifold of dimension and codimension at least 1, there does exist a geodesic line γ_0 in X starting from ξ_0 such that $f_0(\gamma_0) = h$. Let $h_0^{\min} = \delta_0$ and $h^{\min} = 4\,c'_1(\epsilon) + 2\epsilon + \delta + 2\|f_0 - \mathfrak{ftp}_L\|_\infty$. By the definitions of $h'_1(\cdot,\cdot,\cdot), h_0(\cdot,\cdot,\cdot), c_5(\cdot,\cdot)$ in Subsection 4.2, we have

$$h'_0 = h_0(\epsilon_0, \delta_0, \kappa_0) > \delta_0 = h_0^{\min}\ ,$$

and

$$\begin{aligned} h \geq h'_1 &= h_0(\epsilon_0, \delta_0, \kappa_0) + 2\,c_5(\epsilon_0, \delta_0) \geq \kappa_0 + 12\sinh(c'_1(\epsilon_0) + \delta_0) \\ &\geq \kappa_0 + 12\,c'_1(\epsilon) + \delta \geq h^{\min}\ . \end{aligned}$$

The family $(C_n)_{n\in\mathbb{N}}$ hence satisfies the Local prescription property (*iv*) by Proposition 3.7 (3). Therefore, the result follows from Proposition 4.10 and Corollary 4.11 □

Remark 5.8 If the C_n's are disjoint from $\mathscr{N}_\epsilon L$ (and $\delta = 0$), then the same result as Theorem 5.7 also holds when L has dimension 1, by replacing Proposition 3.7 (3) by Proposition 3.7 (4) in the above proof and $h_0^{\min} = \delta_0$ by $h_0^{\min} = 0$.

Theorem 5.9 *Let $\epsilon > 0$, $\delta \geq 0$. Let X be a complete simply connected Riemannian manifold with sectional curvature at most -1 and dimension at least 3. Let $(L_n)_{n\in\mathbb{N}}$ be a family of geodesic lines in X, such that $\operatorname{diam}(\mathscr{N}_\epsilon L_n \cap \mathscr{N}_\epsilon L_m) \leq \delta$ for all $n \neq m$ in $\mathbb{N}$. Let either $f_0 = \mathfrak{ftp}_{L_0}$, or $f_0 = \ell_{\mathscr{N}_\epsilon L_0}$ if X has constant curvature, or $f_0 = \mathfrak{crp}_{L_0}$ if the metric spheres for the Hamenstädt distances (on $\partial_\infty X - \{\xi\}$ for any $\xi \in \partial_\infty X$) are topological spheres. Let*

$$h'_0 = \max\{5\,c'_1(\epsilon)+5\epsilon+\delta, h_0(\epsilon, \delta, \max\{\|f_0 - \ell_{\mathscr{N}_\epsilon L_0}\|_\infty, \|f_0 - \mathfrak{ftp}_{L_0}\|_\infty + 2\epsilon - 8\,c'_1(\epsilon)\})\}$$

and $h \geq h'_1 = h'_1(\epsilon, \delta, h'_0)$.

- *For every $\xi \in (X \cup \partial_\infty X) - (\mathscr{N}_\epsilon L_0 \cup \partial_\infty L_0)$ (and $\xi \in \partial_\infty X - \partial_\infty L_0$ if $f_0 = \mathfrak{crp}_{L_0}$), there exists a geodesic ray or line γ starting from ξ and entering $\mathscr{N}_\epsilon L_0$ at time 0 with $f_0(\gamma) = h$, such that $\ell_{\mathscr{N}_\epsilon L_n}(\gamma) \leq h'_1$ for every $n \neq 0$ such that $\gamma(]\delta, +\infty[)$ meets $\mathscr{N}_\epsilon L_n$.*
- *There exists a geodesic line γ in X such that $f_0(\gamma) = h$, and, if $n \neq 0$, then $\ell_{\mathscr{N}_\epsilon L_n}(\gamma) \leq h'_1 + c'_3(\epsilon)\big(\delta + c'_1(\epsilon)\big) + c'_1(\epsilon)$.*

Note that if $\ell_{\mathscr{N}_\epsilon L_n}(\gamma) \leq c$, then $f(\gamma) \leq c + \kappa$ for any κ-penetration map f in $\mathscr{N}_\epsilon L_n$.

Proof. As in the previous proof, we apply Proposition 4.10 and Corollary 4.11 with $C_n = \mathscr{N}_\epsilon L_n$, $\epsilon_0 = \epsilon$, $\delta_0 = \delta$, $\xi_0 = \xi$,

$$\kappa_0 = \max\{\|f_0 - \ell_{\mathscr{N}_\epsilon L_0}\|_\infty, \|f_0 - \mathfrak{ftp}_{L_0}\|_\infty + 2\epsilon - 8\,c'_1(\epsilon)\}\ .$$

For every $n \neq 0$, let $h_0^{\min} = 3\,c'_1(\epsilon) + 3\epsilon + \delta + \|\ell_{\mathscr{N}_\epsilon L_n} - \mathfrak{ftp}_{L_n}\|_\infty$ and $h^{\min} = 4\,c'_1(\epsilon) + 2\epsilon + \delta + \|f_0 - \mathfrak{ftp}_L\|_\infty$. In particular, $h'_0 \geq 5\,c'_1(\epsilon) + 5\epsilon + \delta \geq h_0^{\min}$ by Lemma 3.4. As in the end of the previous proof, the family $(C_n)_{n\in\mathbb{N}}$ hence satisfies the property (*iv*) by Proposition 3.7 (4), and the result follows. □

Let M be a complete Riemannian manifold with sectional curvature at most -1 and dimension $n \geq 3$. Fix a universal cover $\widetilde{M} \to M$ of M. For $\epsilon > 0$, $\delta \geq 0$, a (possibly not connected, but such that any two components have equal dimension) immersed complete totally geodesic submanifold L (of dimension at least 1 and at most $n-1$) will be called (ϵ, δ)-*separated* if the diameter of the intersection of the ϵ-neighbourhoods of two lifts to $\widetilde{M}$ of two components of L is at most δ.

Examples. (1) If L is compact and embedded, then there exists $\epsilon > 0$ such that L is $(\epsilon, 0)$-separated. For instance, a finite family of disjoint simple closed geodesics is $(\epsilon, 0)$-separated for ϵ small enough.

(2) If L is compact, and if L is *self-transverse* (i.e. if the tangent spaces at every double point of L are transverse), then for every $\epsilon > 0$ small enough, L is $(\epsilon, 1)$-separated. In particular, a finite family of closed geodesics (possibly nonsimple) is $(\epsilon, 1)$-separated for ϵ small enough.

(3) The lift of a locally geodesic line $\gamma : \mathbb{R} \to M$ to the unit tangent bundle T^1M is the map $\widetilde{\gamma} : \mathbb{R} \to T^1M$ (or by abuse its image) given by $\widetilde{\gamma}(t) = (\gamma(t), \gamma'(t))$ for every $t \in \mathbb{R}$. For every $\rho > 0$, if the ρ-neighbourhood (for the standard Riemannian metric of T^1M) of the lift of γ to T^1M is a tubular neighbourhood, then there exists $\delta(\rho) \geq 0$ such that γ is $(\rho, \delta(\rho))$-separated. Indeed, if the intersection of the ρ-neighbourhoods of two different lifts to a universal cover of γ has diameter big enough (depending only on ρ), then by arguments similar to the ones in the proof of Lemma 2.2, two subsegments of the two lifts will follow themselves closely for some time, hence the tangent vectors at two points on these two lifts will be closer than ρ.

Let L be an (ϵ, δ)-separated immersed complete totally geodesic submanifold. Let $(\widetilde{L}_\alpha)_{\alpha \in \mathscr{A}}$ be the family of (connected) complete totally geodesic submanifolds of $\widetilde{M}$, that are the lifts to $\widetilde{M}$ of the components of L. Note that in particular, the family $(\mathscr{N}_\epsilon(\widetilde{L}_\alpha))_{\alpha \in \mathscr{A}}$ is locally finite.

Let f be one of the symbols $\ell, \mathfrak{bp}, \mathfrak{ftp}, \mathfrak{crp}$ and assume that L has dimension 1 if $f = \mathfrak{crp}$. Let κ_f be respectively 0, $2\,c_1'(\epsilon)$, $2\,c_1'(\epsilon) + 2\epsilon$, $2\,c_1'(\epsilon) + 2\,c_1'(\infty) + 2\epsilon$. For every $\alpha \in \mathscr{A}$, let $f_\alpha = \ell_{\mathscr{N}_\epsilon L_\alpha}, \mathfrak{bp}_{L_\alpha}, \mathfrak{ftp}_{L_\alpha}, \mathfrak{crp}_{L_\alpha}$ respectively, which is a κ_f-penetration map in $\mathscr{N}_\epsilon L_\alpha$ by Subsection 3.1. For every locally geodesic line γ in M, consider a lift $\widetilde{\gamma}$ of γ to $\widetilde{M}$.

The family $(f_\alpha(\widetilde{\gamma}))_{\alpha \in \mathscr{A}}$ will be called the family of *spiraling times* of γ along L with respect to f (and *length spiraling times*, *fellow-traveling times* or *crossratio spiraling times* if $f = \ell, \mathfrak{ftp}, \mathfrak{crp}$ respectively). Up to permutation of $\mathscr{A}$, it does not depend on the choice of the lift $\widetilde{\gamma}$ of γ. The entering times of $\widetilde{\gamma}$ in the sets $\mathscr{N}_\epsilon L_\alpha$ with $f_\alpha(\widetilde{\gamma}) > \delta + \kappa_f$, where α varies in $\mathscr{A}$, form a discrete subset (with multiplicity one) of $\mathbb{R}$, as $\mathscr{N}_\epsilon L_\alpha \cap \mathscr{N}_\epsilon L_\beta$ has diameter at most δ if $\alpha \neq \beta$. We will only be interested in the corresponding spiraling times. It is also then possible to order these spiraling times using the order given by the parametrisation on $\widetilde{\gamma}$, but we will not need this here.

Corollary 5.10 *Let M be a complete Riemannian manifold with sectional curvature at most -1 and dimension $n \geq 3$. Let $\epsilon > 0$, $\delta \geq 0$. Let L be an (ϵ, δ)-separated immersed complete totally geodesic submanifold (of dimension at least 1 and at most $n - 1$). Let f be one of the symbols $\ell, \mathfrak{ftp}, \mathfrak{crp}$, and $\kappa_f' = \max\{0, 6\epsilon - 4c_1'(\epsilon)\}$, $2c_1'(\epsilon) + 2\epsilon$, $2c_1'(\epsilon) + 2\epsilon + 2c_1'(\infty)$ respectively. If $f = \ell$, assume that M has constant* *curvature. If $f=\mathfrak{crp}$, assume that L has dimension 1 and that the metric spheres for the Hamenstädt distances (on the punctured boundary of a universal cover of M) are topological spheres.*

For every

$$h\geq h'_1=h'_1\big(\epsilon,\delta,\max\{5\,c'_1(\epsilon)+5\epsilon+\delta\;,\;h_0(\epsilon,\delta,\kappa'_f)\}\big),$$

there exists a locally geodesic line γ in M having one spiraling time with respect to f exactly h, and all others being at most $h'_1+c'_3(\epsilon)\big(\delta+c'_1(\epsilon)\big)+c'_1(\epsilon)$.

If furthermore M is nonelementary and geometrically finite, then for every cusp e of M, we may also assume that the locally geodesic line γ does not enter too much into the maximal Margulis neighbourhood of e, i.e. that γ satisfies

$$\operatorname{maxht}_e(\gamma)\leq\sup_{x\in L}\operatorname{ht}_e(x)+\epsilon+\frac{1}{2}\big(h'_1+c'_3(\epsilon)(\delta+c'_1(\epsilon))+c'_1(\epsilon)\big)\;.$$

Proof. Let $\pi:\widetilde{M}\to M$ be a universal cover of M, with covering group Γ. With κ_0 the constant in the proofs of Theorem 5.7 if the dimension of L is at least 2, and Theorem 5.9 otherwise, it is easy to check, using Section 3.1, that $\kappa'_f\geq\kappa_0$ for every case of f. Note that $h_0(\cdot,\cdot,\cdot)$ and $h'_1(\cdot,\cdot,\cdot)$ are non-decreasing in the third variable, by their definitions at the beginning of section 4.2 The first assertion follows from Theorem 5.9 applied to the family $(L_n)_n$ of the lifts of the components of L to $\widetilde{M}$, if the dimension of L is 1, and from Theorem 5.7 otherwise.

To prove the last assertion, with the notations of Section 5.1, let $t_e=\sup_{x\in L}\operatorname{ht}_e(x)+\epsilon$. We add to the family of convex subsets in Theorem 5.7 if $\dim L\geq 2$, and in the proof of Theorem 5.9 otherwise, the family of horoballs $\gamma H_e[t_e]$ for γ in Γ (modulo the stabilizer Γ_{ξ_e}). Note that these horoballs have pairwise disjoint interiors, and that their interiors are disjoint from the ϵ-neighbourhood of every lift of a component of L, so that Proposition 3.7 (4) does apply when the dimension of L is 1. □

Theorem 1.4 in the introduction follows from this one, by the above example (1).

Remark. (1) If we wanted to have the same locally geodesic line γ for every cusp e of M in the second assertion of Corollary 5.10, we should add the bigger family of horoballs $(\gamma H_e[t_e])_{\gamma\in\Gamma/\Gamma_{\xi_e}\,,\,e\in\mathscr{C}}$, and replace there t_e by $\max\{t_{\mathscr{C}},\max_{e\in\mathscr{C}}t_e\}$, where $\mathscr{C}$ is the set of cusps of M, and $t_{\mathscr{C}}$ is the greatest lower bound of $t\geq 0$ such that two distinct elements in $(\gamma H_e[t])_{\gamma\in\Gamma/\Gamma_{\xi_e}\,,\,e\in\mathscr{C}}$ have disjoint interiors (see the definition above Corollary 5.5), in order for the new horoballs to have disjoint interiors.

(2) With M and L as above, let f be one of the symbols $\ell, \mathfrak{bp}, \mathfrak{ftp}, \mathfrak{crp}$. Define, for every locally geodesic line γ in M,

$$\operatorname{maxspt}_{L,f}(\gamma) = \sup_{\alpha\in\mathscr{A}} f_\alpha(\widetilde{\gamma}) \;,$$

the least upper bound of the spiraling times of γ around L with respect to f. Let

$$\operatorname{MaxSp}_{L,f}(M) = \{\operatorname{maxspt}_{L,f}(\gamma) \;:\; \gamma \in T^1M\}$$

be the *maximum spiraling spectrum* $\operatorname{MaxSp}_{L,f}(M)$ around L with respect to f. Theorem 5.10 gives, in particular, sufficient conditions for the maximum spiraling spectrum to contain a ray $[c, +\infty]$.

5.3 Recurrent geodesics and related results

In this subsection, when M is geometrically finite, we construct locally geodesic lines that have a prescribed height in a cusp neighbourhood of M, and furthermore satisfy some recurrence properties. We will use the notation introduced in Section 5.1 concerning the cusps e, and the objects $\operatorname{ht}_e, V_e, H_e, \xi_e$.

Corollary 5.11 *Let M be a complete, nonelementary, geometrically finite Riemannian manifold with compact totally geodesic boundary, with sectional curvature at most -1 and dimension at least 3. Let e be a cusp of M. Then there exists a constant $c_3'' = c_3''(e,M)$ such that for every $h' \geq c_3''$, there exists a locally geodesic line γ in M with $\operatorname{maxht}_e(\gamma) = h'$, such that the spiraling length times of γ along the boundary ∂M are at most c_3''.*

Up to changing the constant c_3'', we may also assume that γ stays away from some fixed (small enough) cusp neighbourhood of every cusp different from e. Note that, up to changing the constant c_3'', the last assertion of the corollary does not depend on the choice of $f = \ell, \mathfrak{bp}, \mathfrak{ftp}, \mathfrak{crp}$, with respect to which the spiraling times are computed, and we will use $f = \ell$.

Proof. As ∂M is compact, there exists $\epsilon' \in\,]0,1[$ such that the ϵ'-neighbourhood of the geodesic boundary ∂M is a tubular neighbourhood of ∂M. By definition of manifolds with totally geodesic boundary, there exists a complete simply connected Riemannian manifold $\widetilde{M}$, a nonelementary, torsion-free, geometrically finite discrete subgroup Γ of isometries of $\widetilde{M}$, a Γ-equivariant collection $(L_k^+)_{k\in\mathbb{N}}$ of pairwise disjoint open halfspaces with totally geodesic boundary $(L_k)_{k\in\mathbb{N}}$, such that M is isometric with $\Gamma\backslash(\widetilde{M}-\bigcup_{k\in\mathbb{N}} L_k^+)$. We will identify M and $\Gamma\backslash(\widetilde{M}-\bigcup_{k\in\mathbb{N}} L_k^+)$ by such an isometry from now on. Note that $(\mathscr{N}_{\epsilon'}L_k^+)_{k\in\mathbb{N}}$ is a family of pairwise disjoint ϵ'-convex subsets in $\widetilde{M}$.

Let $t_{e,\partial M} = \max\{0, \max_{x\in\partial M} \mathrm{ht}_e(x)\} \geq 0$, which exists since ∂M is compact. Note that the family $(gH_e[t_{e,\partial M}+1])_{g\in\Gamma/\Gamma_{\xi_e}}$ is a Γ-equivariant family of pairwise disjoint horoballs in $\widetilde{M}$, which are disjoint from $\mathscr{N}_{\epsilon'}L_n^+$ for all $n\in\mathbb{N}$. Let us relabel this family of horoballs as $(H_k)_{k\in\mathbb{N}}$ such that $H_0 = H_e[t_{e,\partial M}+1]$. Note that the horoballs H_k, $k\in\mathbb{N}$, are ϵ'-convex.

Define

$$c_3'' = h_1'\big(\epsilon', 0, h_0(\epsilon', 0, c_1'(\infty))\big) + t_{e,\partial M} + 1 + c_1'(\epsilon')(c_3'(\epsilon')+1)\ .$$

and let $h' \geq c_3''$. We apply Corollary 4.11 with $X=\widetilde{M}$; $\epsilon_0=\epsilon'$; $\delta_0=0$; $\kappa_0 = c_1'(\infty)$; $C_0 = H_0$; $f_0 = \mathfrak{ph}_{C_0}$; $h_0' = h_0(\epsilon_0,\delta_0,\kappa_0)$; $C_{2k+1} = \mathscr{N}_{\epsilon'}L_k^+$; $C_{2k}=H_k$; $h = 2h' - 2(t_{e,\partial M}+1)$. Note that f_0 is a κ_0-penetration map in C_0 by Lemma 3.3, and that $h \geq h_1'(\epsilon_0,\delta_0,h_0')$, as $h'\geq c_3''$. As $\widetilde{M}$ is a manifold of dimension at least 2, there does exist a geodesic line γ_0 in X with $f_0(\gamma_0)=h$. The family $(C_n)_{n\in\mathbb{N}}$, whose elements have pairwise disjoint interiors, satisfies the assertion (iii). It also satisfies (iv), by Proposition 3.7 Case (1), as $h \geq 2c_3'' - 2(t_{e,\partial M}+1) \geq 2c_1'(\epsilon')$. Hence, by Corollary 4.11, there exists a geodesic line $\widetilde{\gamma}$ in X with $\mathfrak{ph}_{H_0}(\widetilde{\gamma}) = h$ and

$$\ell_{C_n}(\widetilde{\gamma}) \leq h_1'' = h_1'(\epsilon_0,\delta_0,h_0') + c_1'(\epsilon_0)(c_3'(\epsilon_0)+1) \leq c_3''$$

for all $n\neq 0$.

As $\ell_{C_{2n+1}}(\widetilde{\gamma})$ is finite, the geodesic $\widetilde{\gamma}$ does not cross the boundary of L_n^+, hence it stays in $\widetilde{M}-\bigcup_{k\in\mathbb{N}} L_k^+$. Let $\pi:\widetilde{M}-\bigcup_{k\in\mathbb{N}} L_k^+ \to M$ be the canonical projection, and let $\gamma = \pi\circ\widetilde{\gamma}$. Hence, the length spiraling times of γ are at most c_3''.

Note that

$$\mathfrak{ph}_{H_e}(\widetilde{\gamma}) = \mathfrak{ph}_{C_0}(\widetilde{\gamma}) + 2(t_{e,\partial M}+1) = h + 2(t_{e,\partial M}+1) = 2h',$$

by the paragraph above Lemma 3.3. Furthermore, if $g\in(\Gamma-\Gamma_{\xi_e})/\Gamma_{\xi_e}$, then there exists k in $\mathbb{N}-\{0\}$ such that

$$\begin{aligned}\mathfrak{ph}_{gH_e}(\widetilde{\gamma}) &= \mathfrak{ph}_{C_{2k}}(\widetilde{\gamma}) + 2(t_{e,\partial M}+1) \leq \ell_{C_{2k}}(\widetilde{\gamma}) + c_1'(\infty) + 2(t_{e,\partial M}+1)\\ &\leq h_1'' + c_1'(\infty) + 2(t_{e,\partial M}+1) \leq 2c_3'' \leq 2h'\ .\end{aligned}$$

Therefore $\max\mathrm{ht}_e(\gamma) = h'$ by the same proof as in the end of the proof of Corollary 5.4. □

Let M be a compact, connected, orientable, irreducible, acylindrical, atoroidal, boundary incompressible 3-manifold with nonempty boundary (see for instance [MT] for references on 3-manifolds and Kleinian groups). A *hyperbolic structure* on a manifold is a complete Riemannian metric with constant sectional curvature -1. A cusp e of a geometrically finite hyperbolic structure is *maximal* if the maximal Margulis neighbourhood of e is a neighbourhood of an end of the manifold. Let P be the union of the torus components of ∂M, and $\mathcal{GF}(M) = \mathcal{GF}(M,P)$ be the (nonempty) space of complete geometrically finite hyperbolic structures in the interior of M whose cusps are maximal, up to isometries isotopic to the identity. Recall that $\mathcal{GF}(M)$ is homeomorphic to the Teichmüller space of $\partial_0 M = \partial M - P$.

For every σ in $\mathcal{GF}(M)$, the cusps of σ are in one-to-one correspondence with the torus components of ∂M, as any minimizing geodesic ray representing a cusp converges to an end of the interior of M corresponding to a torus component of ∂M. If e is a torus component of ∂M, let $\operatorname{maxht}_{\sigma,e}(\gamma)$ denote the maximum height of a locally geodesic line γ in σ with respect to the cusp corresponding to e. The *convex core* of a structure σ in $\mathcal{GF}(M)$ is the smallest closed convex subset of the interior of M, whose injection in the interior of M induces an isomorphism on the fundamental groups.

The following result generalizes Theorem 1.5 in the introduction to the case of several cusps.

Corollary 5.12 *Let M be a compact, connected, orientable, irreducible, acylindrical, atoroidal, boundary incompressible 3-manifold with boundary, and let e be a torus component of ∂M. For every compact subset K in $\mathcal{GF}(M)$, there exists a constant $c''_4 = c''_4(K)$ such that for every $h \geq c''_4$ and every $\sigma \in K$, there exists a locally geodesic line γ contained in the convex core of σ such that $\operatorname{maxht}_{\sigma,e}(\gamma) = h$, and $\operatorname{maxht}_{\sigma,e'}(\gamma) \leq c''_4$ for every torus component $e' \neq e$ of ∂M.*

Proof. For a subset A of $\partial_\infty \mathbb{H}^3_{\mathbb{R}}$, we denote by $\operatorname{Conv} A$ the hyperbolic convex hull of A in $\mathbb{H}^3_{\mathbb{R}}$. A subgroup Γ of $\pi_1 M$ is called a *boundary subgroup* if there are an element $\gamma \in \pi_1 M$, a component C of $\partial_0 M$, and a point $x \in C$ such that $\Gamma = \gamma \operatorname{Im}\big(\pi_1(C,x) \to \pi_1(M,x)\big)\,\gamma^{-1}$. Let $(\Gamma_n)_{n\in\mathbb{N}}$ be the collection of boundary subgroups of $\pi_1 M$. Let $(\Gamma'_n)_{n\in\mathbb{N}}$ be the collection of maximal (rank 2) abelian subgroups of $\pi_1 M$, with Γ'_0 conjugate to $\pi_1 e$.

Let $\rho_\sigma : \pi_1 M \to \operatorname{Isom}(\mathbb{H}^3_{\mathbb{R}})$ be a holonomy representation corresponding to $\sigma \in K$, appropriately normalized to depend continuously on σ. By assumption, $\Gamma = \rho_\sigma(\pi_1 M)$ is a (particular) web group (see for instance [AM]). More precisely, for all $n \in \mathbb{N}$, $\rho_\sigma(\Gamma_n)$ is a quasifuchsian subgroup of Γ stabilizing a connected, simply connected component $\Omega_{n,\sigma}$ of the domain of discontinuity of $\rho_\sigma\pi_1 M$, such that $\Omega_{n,\sigma}$ and $\Omega_{m,\sigma}$ have disjoint closures if $n \neq m$, and that $\partial\Omega_{n,\sigma}$ contains no parabolic fixed points of Γ. Let $(H_{k,\sigma})_{k\in\mathbb{N}}$ be a maximal family of horoballs with pairwise disjoint interiors such that $H_{k,\sigma}$ is $\rho_\sigma(\Gamma'_k)$-invariant (such a family is unique if M has only one torus component). To make this family canonical over $\mathscr{G}\mathscr{F}(M)$, we may fix an ordering $e_1 = e, e_2, \dots, e_m$ of the torus components of ∂M, and take by induction $H_{k,\sigma}$, for the k's in $\mathbb{N}$ such that Γ'_k is conjugate to $\pi_1(e_i)$, to be equivariant and maximal with respect to having pairwise disjoint interiors as well as having their interior disjoint with the interior of $H_{k_*,\sigma}$, for the k_*'s in $\mathbb{N}$ such that Γ'_{k_*} is conjugate to $\pi_1(e_j)$ with $j < i$. Note that the $H_{k,\sigma}$'s, besides the ones such that Γ'_k is conjugate to $\pi_1 e$, are not the maximal horospheres that allow to define the height functions, but this changes their values only by a constant (uniform on K).

Hence, as K is compact, there exists $\delta > 0$ such that for every $\sigma \in K$, the 1-convex subsets $\mathscr{N}_1(\mathrm{Conv}\ \Omega_{n,\sigma})$ and $H_{k,\sigma}$ for $n, k \in \mathbb{N}$ meet pairwise with diameter at most δ.

The claim follows as in Corollary 5.11 by applying Corollary 4.11 to $X = \mathbb{H}^3_\mathbb{R}$, $\epsilon_0 = 1$, $\delta_0 = \delta$, $\kappa_0 = c'_1(\infty)$, $C_0 = H_{0,\sigma}$, $f_0 = \mathfrak{ph}_{C_0}$, $h'_0 = h_0(\epsilon_0, \delta_0, \kappa_0)$, $C_{2n+1} = \mathscr{N}_1(\mathrm{Conv}\ \Omega_{n,\sigma})$, $C_{2n} = H_{n,\sigma}$ to get a geodesic line $\widetilde{\gamma}$ in X with prescribed penetration in C_0, and penetration bounded by a constant in C_n for $n \neq 0$. The finiteness of the intersection lengths $\ell_{C_{2n+1}}(\widetilde{\gamma})$ for $n \in \mathbb{N}$ implies that $\widetilde{\gamma}$ stays in the convex hull of the limit set of Γ. □

Remark. The fact that a locally geodesic line stays in the convex core of the manifold and does not converge (either way) to a cusp is equivalent with the locally geodesic line being two-sided recurrent.

5.4 Prescribing the asymptotic penetration behaviour

Let X be a proper geodesic CAT(-1) metric space and let $\xi \in X \cup \partial_\infty X$. Let $\epsilon \in \mathbb{R}^*_+ \cup \{\infty\}$, $\delta, \kappa \geq 0$. Let $(C_\alpha)_{\alpha\in\mathscr{A}}$ be a family of ϵ-convex subsets of X which satisfies the Almost disjointness condition (*iii*) with parameter δ. For each $\alpha \in \mathscr{A}$, let f_α be a κ-penetration map in C_α. Let γ be a geodesic ray or line, with 0 in the domain of definition of γ (as we are only interested in the asymptotic behaviour, the choice of time 0 is unimportant). These assumptions guarantee that the set $\mathscr{E}_\gamma$ of times $t \geq 0$ such that γ enters in some C_α at time t with $f_\alpha(\gamma) > \delta + \kappa$ is discrete in $[0, +\infty[$, and that $\alpha = \alpha_t$ is then unique. The set $\mathscr{E}_\gamma$ may be finite, for instance if $f_\beta(\gamma) = +\infty$ for some β. Hence $\mathscr{E}_\gamma = (t_i)_{i\in\mathscr{N}}$ for some initial segment $\mathscr{N}$ in $\mathbb{N}$, with $t_i < t_{i+1}$ for $i, i+1$ in $\mathscr{N}$. With $a_i(\gamma) = f_{\alpha_{t_i}}(\gamma)$, the (finite or infinite) sequence $\big(a_i(\gamma)\big)_{n\in\mathscr{N}}$ will be called the (nonnegative) *penetration sequence* of γ with respect to $(C_\alpha, f_\alpha)_{\alpha\in\mathscr{A}}$. In this section, we study the asymptotic behaviour of these penetration sequences. We will only state some results when the C_α's are balls or horoballs, but similar ones are valid, for instance for ϵ-neighbourhoods of geodesic lines in X (see for instance [PP3, Section 5]). We may also prescribe the asymptotic penetration in one cusp, while keeping the heights in the other cusps (uniformly) bounded.

In the following results, we show how to prescribe the asymptotic behaviour of the penetration sequence of a geodesic ray or line with respect to horoballs and their penetration height functions. First, we prove a general result, and we give the more explicit result for Riemannian manifolds as Corollary 5.14.

Theorem 5.13 *Let X be a proper geodesic* CAT(-1) *metric space, with $\partial_\infty X$ infinite. Let $(H_\alpha)_{\alpha\in\mathscr{A}}$ be a family of horoballs with pairwise disjoint interiors. Assume that there exists $K \in [0,+\infty[$ and a dense subset Y in $\partial_\infty X$ such that, for every geodesic ray γ in X with $\gamma(+\infty) \in Y$, we have*

$$\liminf_{t\to+\infty} d(\gamma(t), \bigcup_{\alpha\in\mathscr{A}} H_\alpha) \leq K\ .$$

Let $\xi \in X \cup \partial_\infty X$ and $c, c' \geq 0$. Assume that for every $h \geq c$ and $\alpha \in \mathscr{A}$ such that $\xi \notin H_\alpha \cup H_\alpha[\infty]$, there exists a geodesic ray or line γ starting from ξ and entering H_α at time $t = 0$ with $\mathfrak{ph}_{H_\alpha}(\gamma) = h$, and with $\mathfrak{ph}_{H_\beta}(\gamma) \leq c'$ for every β in $\mathscr{A} - \{\alpha\}$ such that $\gamma(]0,+\infty[)$ meets H_β. Let $\big(a_i(\gamma')\big)_{n\in\mathscr{N}}$ be the penetration sequence of a geodesic ray or line γ' with respect to $(H_\alpha, \mathfrak{ph}_{H_\alpha})_{\alpha\in\mathscr{A}}$.

Then, for every

$$h \geq h_* = \max\{c\,,\ c' + 3\,c'_1(\infty) + 10^{-5}\},$$

there exists a geodesic ray or line γ starting from ξ such that

$$\limsup_{i\to+\infty}\ a_i(\gamma) = h\ .$$

Proof. To simplify notation, let $f_\alpha = \mathfrak{ph}_{H_\alpha}$, $c_* = c' + 3\,c'_1(\infty) + 10^{-5}$, so that $h_* = \max\{c_*, c\}$. If a geodesic ray or line γ starting from ξ meets H_α such that $\xi \notin H_\alpha \cup H_\alpha[\infty]$, let $t^-_\alpha(\gamma)$ and $t^+_\alpha(\gamma)$ be the entrance and exit times.

Let $h \geq h_*$, and let $\alpha_0 \in \mathscr{A}$ such that $\xi \notin H_{\alpha_0} \cup H_{\alpha_0}[\infty]$, which exists by the assumptions. As $h \geq h_* \geq c$, there exists a geodesic ray or line γ_0 starting from ξ, entering H_{α_0} at time 0, such that $f_{\alpha_0}(\gamma_0) = h$, and $f_\alpha(\gamma_0) \leq c'$ for every $\alpha \neq \alpha_0$ such that $\gamma_0(]0,+\infty[)$ meets H_α.

We construct, by induction, sequences $(\gamma_k)_{k\in\mathbb{N}}$ of geodesic rays or lines starting from ξ, $(\alpha_k)_{k\in\mathbb{N}}$ of elements of $\mathscr{A}$, and $(t_k)_{k\in\mathbb{N}-\{0\}}$ of elements in $[0,+\infty[$ converging to $+\infty$, such that for every $k\in\mathbb{N}$,

(1) γ_k enters the interior of H_{α_0} at time 0, with $d(\gamma_k(0),\gamma_{k-1}(0))\le\frac{1}{2^k}$ if $k\ge 1$;

(2) γ_k enters H_{α_k}, $\xi\notin H_{\alpha_k}\cup H_{\alpha_k}[\infty]$, and $f_{\alpha_k}(\gamma_k)=h$;

(3) if $0\le j\le k-1$, then $\gamma_k(]0,+\infty[)$ enters the interior of H_{α_j} before entering H_{α_k} with $t^-_{\alpha_j}(\gamma_k)<t_k=t^+_{\alpha_{k-1}}(\gamma_k)<t^+_{\alpha_k}(\gamma_k)$;

(4) if $k\ge 1$, then for every α such that $\gamma_k(]0,+\infty[)$ meets H_α, we have $|f_\alpha(\gamma_k)-f_\alpha(\gamma_{k-1})|<\frac{1}{2^k}$ if $t^-_\alpha(\gamma_k)<t_k$, and $f_\alpha(\gamma_k)\le c_*$ if $t_k\le t^-_\alpha(\gamma_k)<t^-_{\alpha_k}(\gamma_k)$, and $f_\alpha(\gamma_k)\le c'$ if $t^-_\alpha(\gamma_k)\ge t^+_{\alpha_k}(\gamma_k)$.

Let us first prove that the existence of such sequences implies Theorem 5.13. By the assertion (1), as $\gamma_k(0)$ stays at bounded distance from $\gamma_0(0)$, up to extracting a subsequence, the sequence $(\gamma_k)_{k\in\mathbb{N}}$ converges to a geodesic ray or line γ_∞ starting from ξ, entering in H_{α_0} at time $t=0$, by continuity of the entering point in an ϵ-convex subset. Let us prove that $\limsup_{i\to+\infty}\ a_i(\gamma_\infty)=h$.

The lower bound $\limsup_{i\to+\infty}\ a_i(\gamma_\infty)\ge h$ is immediate by a semicontinuity argument. Indeed, for every $k>i$ in $\mathbb{N}$, we have by the assertions (2), (3) and (4),

$$|f_{\alpha_i}(\gamma_k)-h|=|f_{\alpha_i}(\gamma_k)-f_{\alpha_i}(\gamma_i)|\le\sum_{j=i}^{k-1}|f_{\alpha_i}(\gamma_{j+1})-f_{\alpha_i}(\gamma_j)|\le\sum_{j=i}^{k-1}\frac{1}{2^{j+1}}\le\frac{1}{2^i}\ .$$

Hence by continuity of f_{α_i}, we have the inequality $f_{\alpha_i}(\gamma_\infty)\ge h-\frac{1}{2^i}$, whose right side converges to h as i tends to $+\infty$, which proves the lower bound, as $h>c_1'(\infty)$ and f_{α_i} is a $c_1'(\infty)$-penetration map in H_{α_i} (see Section 3.1).

To prove the upper bound, assume by absurd that there exists $\epsilon>0$ such that for every $\lambda>0$, there exists $\alpha=\alpha(\lambda)\in\mathscr{A}$ such that γ_∞ enters H_α with $f_\alpha(\gamma_\infty)\ge h+\epsilon$ and $t^-_\alpha(\gamma_\infty)>\lambda+2\,c_1'(\infty)$. Take $\lambda_0=\max\{t_{i+1}:\frac{1}{2^i}\ge\frac{\epsilon}{2}\}$, and $\alpha=\alpha(\lambda_0)$

By continuity of f_α, if k is big enough, we have $f_\alpha(\gamma_k)\ge h+\frac{\epsilon}{2}\ge h_*\ge c_*\ge c'$. Thus, γ_k meets H_α as $h_*>0$. The entry time is positive, as $d(\gamma_k(0),\gamma_\infty(0))\le c_1'(\infty)$ and the entrance points of γ_k and γ_∞ in H_α are at distance at most $c_1'(\infty)$, both by Lemma 2.3, and as the entrance time of γ_∞ in H_α is strictly bigger than $2c_1'(\infty)$. Hence, by the assertion (4), we have $t^-_\alpha(\gamma_k)<t_k$. Let $i\le k-1$ be the minimum element of $\mathbb{N}$ such that for $j=i,\dots,k-1$, the geodesic γ_{j+1} meets H_α at a positive time with $t^-_\alpha(\gamma_{j+1})<t_{j+1}$. By the triangular inequality, we have

$$\begin{aligned}|t^-_\alpha(\gamma_{i+1})-t^-_\alpha(\gamma_\infty)|&\le d\big(\gamma_{i+1}(t^-_\alpha(\gamma_{i+1})),\gamma_\infty(t^-_\alpha(\gamma_\infty))\big)+d\big(\gamma_{i+1}(0),\gamma_\infty(0)\big)\\&\le 2\,c_1'(\infty)\ .\end{aligned}$$

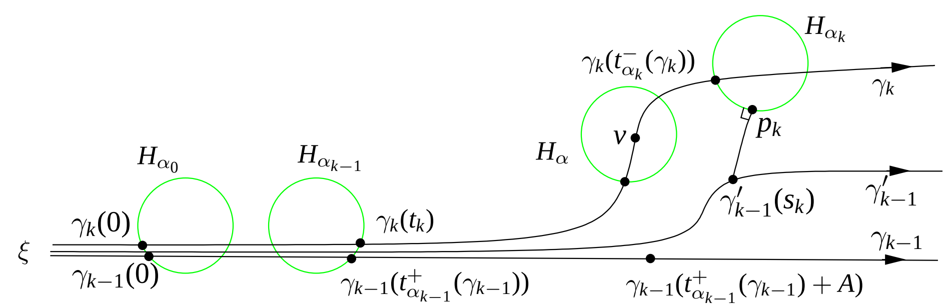
Hence

$$t_{i+1} > t_\alpha^-(\gamma_{i+1}) \geq t_\alpha^-(\gamma_\infty) - 2\,c_1'(\infty) > \lambda_0 + 2\,c_1'(\infty) - 2\,c_1'(\infty) = \lambda_0 .$$

By the definition of λ_0, we hence have $\frac{1}{2^i} < \frac{\epsilon}{2}$. By the definition of i and by the assertion (4), we have

$$\begin{aligned} f_\alpha(\gamma_i) &= f_\alpha(\gamma_k) + \sum_{j=i}^{k-1}\big(f_\alpha(\gamma_j) - f_\alpha(\gamma_{j+1})\big) \geq h + \frac{\epsilon}{2} - \sum_{j=i}^{k-1}\frac{1}{2^{j+1}} \\ &\geq h + \frac{\epsilon}{2} - \frac{1}{2^i} \geq h \geq h_* \ , \end{aligned}$$

and in particular by the same argument as for γ_k above, γ_i enters H_α at a positive time and $t_\alpha^-(\gamma_i) < t_i$. This contradicts the minimality of i. This completes the proof, assuming the existence of a sequence with properties (1)–(4).



Let us now construct the sequences $(\gamma_k)_{k\in\mathbb{N}}$, $(\alpha_k)_{k\in\mathbb{N}}$, $(t_k)_{k\in\mathbb{N}-\{0\}}$. We have defined γ_0, α_0, and they satisfy the properties (1)–(4). Let $k \geq 1$, and assume that γ_{k-1}, α_{k-1}, as well as t_{k-1} if $k \geq 2$, have been constructed.

As Y is dense in $\partial_\infty X$, for every $A > 0$, there exists a geodesic ray or line γ'_{k-1} starting from ξ with $\gamma'_{k-1}(+\infty) \in Y$, entering in H_{α_0} at time $t = 0$, which is very close to γ_{k-1} on $[0, t^+_{\alpha_{k-1}}(\gamma_{k-1}) + A]$. By the definition of K, let s_k be the first time $t \geq t^+_{\alpha_{k-1}}(\gamma_{k-1}) + A$ such that there exists α in $\mathscr{A}$ with $d(\gamma'_{k-1}(t), H_\alpha) \leq K + 1$, and let α_k be such an α with $d(\gamma'_{k-1}(s_k), H_\alpha)$ minimum. Let p_k be the closest point of H_{α_k} to $\gamma'_{k-1}(s_k)$. Note that $\xi \notin H_{\alpha_k} \cup H_{\alpha_k}[\infty]$, if A is big enough (in particular compared to K), as H_{α_0} and H_{α_k} have disjoint interiors.

By the hypothesis, let γ_k be a geodesic ray or line starting from ξ with $f_{\alpha_k}(\gamma_k) = h$ (which proves the assertion (2) at rank k as $h > 0$) and $f_\alpha(\gamma_k) \leq c'$ for every α such that $\gamma_k(]t^-_{\alpha_k}(\gamma_k), +\infty[)$ enters H_α. As a CAT(-1) metric space is $\log(1+\sqrt{2})$-hyperbolic, the geodesic $]\xi, p_k]$ is contained in the $\log(1+\sqrt{2})$-neighbourhood of the union $]\xi, \gamma'_{k-1}(s_k)] \cup [\gamma'_{k-1}(s_k), p_k]$. By Lemma 2.3, we have $d(\gamma_k(t^-_{\alpha_k}(\gamma_k)),]\xi, p_k]) \leq$

$c'_1(\infty)$, and therefore $]\xi,\gamma_k(t^-_{\alpha_k}(\gamma_k))]$ is contained in the $(c'_1(\infty)+\log(1+\sqrt{2}))$-neighbourhood of $]\xi,\gamma'_{k-1}(s_k)]\cup[\gamma'_{k-1}(s_k),p_k]$. Up to choosing A big enough, we may hence assume that γ_k is very close to γ_{k-1} between the times 0 and $t^+_{\alpha_{k-1}}(\gamma_{k-1})+1$. Using this and properties (1) and (3) at rank $k-1$, we have

- γ_k does enter the interior of H_{α_0}, at a time that we may assume to be 0, with $d(\gamma_k(0),\gamma_{k-1}(0))\leq\frac{1}{2^k}$ (this proves the assertion (1) at rank k);
- for $0\leq j\leq k-1$, as γ_{k-1} passes in the interior of H_{α_j} at a time strictly between 0 and $t^+_{\alpha_{k-1}}(\gamma_{k-1})$, by the inductive assertions (3) if $k\neq 1$ and $j\leq k-2$, or (1) if $k=1$ or (2) if $j=k-1$, so does the geodesic ray or line γ_k; this allows, in particular, to define $t_k=t^+_{\alpha_{k-1}}(\gamma_k)$, which satisfies $t_k\leq t^-_{\alpha_k}(\gamma_k)<t^+_{\alpha_k}(\gamma_k)$ and

  (- 40 -) $$d\big(\gamma_k(t_k),\gamma_{k-1}(t^+_{\alpha_{k-1}}(\gamma_{k-1}))\big)\leq 10^{-5}/4$$

  if A is big enough; this proves the assertion (3) at rank k;
- for every α such that $\gamma_k(]0,+\infty[)$ meets H_α and $t^-_\alpha(\gamma_k)<t_k$, we may assume by continuity that $|f_\alpha(\gamma_k)-f_\alpha(\gamma_{k-1})|<\frac{1}{2^k}$.

Hence, to prove the assertion (4) at rank k, we consider $\alpha\in\mathscr{A}$ such that γ_k meets H_α with $t_k\leq t^-_\alpha(\gamma_k)<t^-_{\alpha_k}(\gamma_k)$, and we prove that $f_\alpha(\gamma_k)\leq c_*$. We may assume that $f_\alpha(\gamma_k)>0$. Let v be the highest point of γ_k in H_α, which, by disjointness, belongs to $]\gamma_k(t^-_\alpha(\gamma_k)),\gamma_k(t^-_{\alpha_k}(\gamma_k))[$. Let u be a point in $]\xi,\gamma'_{k-1}(s_k)]\cup[\gamma'_{k-1}(s_k),p_k]$ at distance at most $c'_1(\infty)+\log(1+\sqrt{2})=\frac{3}{2}\,c'_1(\infty)$ from v.

Assume first that $u\in[\gamma'_{k-1}(s_k),p_k]$. Note that by the minimality assumption on α_k and since $\alpha\neq\alpha_k$, the point u then does not belong to H_α. As $c_*\geq 3\,c'_1(\infty)$, this implies that $f_\alpha(\gamma_k)\leq 2\,d(u,v)\leq c_*$.

Assume now that $u=\gamma'_{k-1}(t)$ with $t\in[t^+_{\alpha_{k-1}}(\gamma_{k-1})+A,s_k[$. Then by the minimality of s_k, the point u again does not belong to H_α (it is in fact at distance at least $K+1$ from H_α). Hence similarly $f_\alpha(\gamma_k)\leq c_*$.

Finally, assume that $u=\gamma'_{k-1}(t)$ with $t\leq t^+_{\alpha_{k-1}}(\gamma_{k-1})+A$. Let $u'=\gamma_{k-1}(t)$, which satisfies $d(u,u')\leq 10^{-5}/4$ (as γ_{k-1} and γ'_{k-1} were assumed to be very close on that range). Assume by absurd that $f_\alpha(\gamma_k)>c_*$. In particular,

$$d(v,\gamma_k(t_k))\geq d\big(v,\gamma_k(t^-_\alpha(\gamma_k))\big)\geq f_\alpha(\gamma_k)/2>c_*/2\;.$$

Let ξ_α be the point at infinity of H_α and $x_\alpha=\gamma_k(t^-_k(\gamma_k))$ the entrance point of γ_k in H_α. Since $d(v,u')\leq d(v,u)+d(u,u')\leq\frac{3}{2}\,c'_1(\infty)+10^{-5}/4$, we hence have

$$\begin{aligned}2\beta_{\xi_\alpha}(u',x_\alpha)&=2\beta_{\xi_\alpha}(u',v)+2\beta_{\xi_\alpha}(v,x_\alpha)\\&\geq f_\alpha(\gamma_k)-2d(v,u')\geq c_*-3\,c'_1(\infty)+10^{-5}/2>c'\geq 0\;,\end{aligned}$$

by the definition of c_*. Hence $u' \in H_\alpha$ and $f_\alpha(\gamma_{k-1}) \ge 2\beta_{\xi_\alpha}(u', x_\alpha) > c'$. Furthermore, $t \ge t^+_{\alpha_{k-1}}(\gamma_{k-1})$, since otherwise, and by Equation (- 40 -),

$$d(\gamma_k(t_k)), v) \le d(v, u') + d\big(\gamma_k(t_k), \gamma_{k-1}(t^+_{\alpha_{k-1}}(\gamma_{k-1}))\big) \le \frac{3}{2}\, c_1'(\infty) + 10^{-5}/2 \le c_*/2$$

by the definition of c_*, a contradiction. Hence, by convexity and since H_α and $H_{\alpha_{k-1}}$ have disjoint interiors, we have $t^-_\alpha(\gamma_{k-1}) \ge t^+_{\alpha_{k-1}}(\gamma_{k-1})$. Since $f_\alpha(\gamma_{k-1}) > c'$, this contradicts the inductive hypothesis (4) on γ_{k-1}. This proves the result. □

Remark. There exists an analogous statement, with an analogous proof, when $(H_\alpha)_{\alpha\in\mathscr{A}}$ is a family of balls of radius $R > 0$, replacing $c_1'(\infty)$ by $c_1'(R)$, and assuming both in the hypothesis and in the conclusion that $h \le c''$ for some c''.

Corollary 5.14 *Let X be a complete simply connected Riemannian manifold with sectional curvature at most -1 and dimension at least 3, and let $\big(H_\alpha\big)_{\alpha\in\mathscr{A}}$ be a family of horoballs in X with disjoint interiors. Assume that there exists $K \in [0, +\infty[$ and a dense subset Y in $\partial_\infty X$ such that, for every geodesic ray γ in X with $\gamma(+\infty) \in Y$, we have $\liminf_{t\to+\infty} d(\gamma(t), \bigcup_{\alpha\in\mathscr{A}} H_\alpha) \le K$. Then, for every $\xi \in X \cup \partial_\infty X$ and*

$$h \ge c_1''(\infty, 0, 0) + 4\, c_1'(\infty) + 10^{-5} \approx 13.5542,$$

there exists a geodesic ray or line γ starting from ξ such that, with $(a_i(\gamma))_{n\in\mathscr{N}}$ the penetration sequence of γ with respect to $(H_\alpha, \mathfrak{ph}_{H_\alpha})_{\alpha\in\mathscr{A}}$, we have

$$\limsup_{i\to+\infty}\; a_i(\gamma) = h\ .$$

Proof. Let $c = c_1''(\infty, 0, 0)$, $c' = c_1''(\infty, 0, 0) + c_1'(\infty)$. We apply Theorem 5.1 with $\epsilon = \infty$, $\delta = 0$, $\kappa = c_1'(\infty)$, $\xi_0 = \xi$, $C_0 = H_\alpha$ where $\alpha \in \mathscr{A}$ satisfies $\xi \notin H_\alpha \cup H_\alpha[\infty]$, $f_0 = \mathfrak{ph}_{C_0}$, $(C_n)_{n\ge1}$ is $(H_\beta)_{\beta\in\mathscr{A}-\{\alpha\}}$ (up to indexing), $f_n = \mathfrak{ph}_{C_n}$ for every $n \in \mathbb{N}$ such that $\xi \notin C_n \cup \partial_\infty C_n$. Then the assumptions of Theorem 5.13 are satisfied. An easy computation of h_* in Theorem 5.13 then yields the result. □

Remark. Using Theorem 5.2 instead of Theorem 5.1, there is an analogous statement when $(H_\alpha)_{\alpha\in\mathscr{A}}$ is a family of balls of radius $R > 0$, for $h \in [c_1''(R, 0, 0) + 4\, c_1'(R) + 10^{-5}, 2R - c_1'(R)]$.

As in Section 5.1, we consider a complete, nonelementary, geometrically finite Riemannian manifold M, and e an end of M. The *asymptotic height spectrum* of the pair (M, e) is

$$\mathrm{LimsupSp}(M, e) = \{\, \limsup_{t\to\infty} \mathrm{ht}_e(\gamma(t)) : \gamma \in T^1 M\}\ .$$

In classical Diophantine approximation, the *Lagrange spectrum* is defined as the subset of $[0,+\infty[$ consisting of the *approximation constants* $c(x)$ of an irrational real number x by rational numbers p/q, defined by

$$c(x)=\liminf_{q\to\infty}|q|^2|x-\frac{p}{q}|.$$

Using the well known connection between the Diophantine approximation of real numbers by rational numbers and the action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on the upper halfplane model of the real hyperbolic plane, the asymptotic height spectrum of the modular orbifold $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$ is the image of the Lagrange spectrum by the map $t\mapsto -\log 2t$ (see for instance [HP3, Theo. 3.4]). Hall [Hal1, Hal2] showed that the Lagrange spectrum contains an interval $[0,c]$ for some $c>0$. The maximal such interval $[0,\mu]$ (which is closed as the Lagrange spectrum is closed, by Cusick's result, see for instance [CF]), called *Hall's ray*, was determined by Freiman [Fre] (see also [Slo] where the map $t\mapsto 1/t$ has to be applied). The geometric interpretation of Freiman's result in our context is that $\mathrm{LimsupSp}(\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}})$ contains the maximal interval $[c,+\infty]$ with

$$c=-\log(2\mu)=-\log 2\left(\frac{491993569}{2221564096+283748\sqrt{462}}\right)\approx 0.817\ .$$

The following result is the asymptotic analog of Corollary 5.4, and has a completely similar proof. Theorem 1.6 in the introduction follows, since $(c''_1(\infty,0,0)+4\,c'_1(\infty)+10^{-5})/2\approx 6.7771$. The result proves the existence of Hall's ray in our geometric context, which is much more general than the above particular case; there is no assumption of arithmetic nature, nor of constant curvature nature. Furthermore, we obtain a universal constant (though we do not know the optimal one) 6.7771 which is not too far from the geometric Freiman constant 0.817.

Corollary 5.15 *Let M be a complete, nonelementary, geometrically finite Riemannian manifold with sectional curvature at most -1 and dimension at least 3, and let e be a cusp of M. Then $\mathrm{LimsupSp}(M,e)$ contains the interval*

$$[(c''_1(\infty,0,0)+4\,c'_1(\infty)+10^{-5})/2,+\infty]\ .\quad\Box$$

6 Applications to Diophantine approximation in negatively curved manifolds

In this section, we consider a number of arithmetically defined examples, illustrating the last result, Corollary 5.15. But we need first to recall some properties and do some computations in the real and complex hyperbolic spaces.

6.1 On complex hyperbolic geometry and the Heisenberg group

To facilitate computations, we identify elements in $\mathbb{C}^{n-1}$ with their coordinate column matrices. We will denote by $A^* = {}^t\overline{A}$ the adjoint matrix of a complex matrix A. In particular, the standard Hermitian scalar product of $w, w' \in \mathbb{C}^{n-1}$ is $\overline{w^*\, w'} = \sum_{i=1}^{n-1} w_i \overline{w'_i}$. We also use the notation $|w|^2 = w^*\, w$.

Let $\mathbb{H}^n_{\mathbb{C}}$ be the Siegel domain model of the complex hyperbolic n-space, whose underlying set is

$$\mathbb{H}^n_{\mathbb{C}} = \{(w_0, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \;:\; 2\,\mathrm{Re}\; w_0 - |w|^2 > 0\}\ ,$$

and whose Riemannian metric is

$$ds^2_{\mathbb{C}} = \frac{4}{(2\,\mathrm{Re}\; w_0 - |w|^2)^2}\big((dw_0 - dw^*\, w)(\overline{dw_0} - w^*\, dw) + (2\,\mathrm{Re}\; w_0 - |w|^2)\, dw^*\, dw\big)\ ,$$

(see for instance [Gol, Sect. 4.1]). The complex hyperbolic space has constant holomorphic sectional curvature -1, hence its real sectional curvatures are bounded between -1 and $-\frac{1}{4}$. Its boundary at infinity is

$$\partial_\infty \mathbb{H}^n_{\mathbb{C}} = \{(w_0, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \;:\; 2\,\mathrm{Re}\; w_0 - |w|^2 = 0\} \cup \{\infty\}\ .$$

The horoballs centered at ∞ in $\mathbb{H}^n_{\mathbb{C}}$ are the subsets

$$\mathscr{H}_s = \{(w_0, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \;:\; 2\,\mathrm{Re}\; w_0 - |w|^2 \geq s\}\ ,$$

for $s > 0$. Note that the subset $\mathbb{H}^1_{\mathbb{C}} = \{(w_0, w) \in \mathbb{H}^n_{\mathbb{C}} \;:\; w = 0\}$ is the right halfplane model of the real hyperbolic plane with constant curvature -1, and it is totally geodesic in $\mathbb{H}^n_{\mathbb{C}}$. In particular, the (unit speed) geodesic line starting from ∞, ending at $(0,0) \in \partial_\infty \mathbb{H}^n_{\mathbb{C}}$ and meeting the horosphere $\partial\mathscr{H}_2$ at time $t = 0$ is the map $c_0 : \mathbb{R} \to \mathbb{H}^n_{\mathbb{C}}$ defined by $c_0 : t \mapsto (e^{-t}, 0)$.

Let q be the nondegenerate Hermitian form $-z_0\overline{z_n} - z_n\overline{z_0} + |z|^2$ of signature $(1,n)$ on $\mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}$ with coordinates (z_0, z, z_n). This is not the form considered in [Gol, page 67], hence we need to do some computations with it, but it is better suited for our purposes. The Siegel domain $\mathbb{H}^n_{\mathbb{C}}$ embeds in the complex projective n-space $\mathbb{P}_n(\mathbb{C})$ by the map (using homogeneous coordinates)

$$(w_0, w) \mapsto [w_0 : w : 1]\ .$$

Its image is the negative cone of q, that is $\{[z_0 : z : z_n] \in \mathbb{P}_n(\mathbb{C}) \;:\; q(z_0, z, z_n) < 0\}$. This embedding extends continuously to the boundary at infinity, by mapping $(w_0, w) \in \partial_\infty \mathbb{H}^n_{\mathbb{C}} - \{\infty\}$ to $[w_0 : w : 1]$ and ∞ to $[1 : 0 : 0]$, so that the image of $\partial_\infty \mathbb{H}^n_{\mathbb{C}}$ is the null cone of q, that is $\{[z_0 : z : z_n] \in \mathbb{P}_n(\mathbb{C}) \;:\; q(z_0, z, z_n) = 0\}$. We use matrices by blocks in the decomposition $\mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}$.

Let

(- 41 -)
$$Q = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

be the matrix of q. If

$$X = \begin{pmatrix} a & \gamma^* & b \\ \alpha & A & \beta \\ c & \delta^* & d \end{pmatrix},$$

then

$$Q^{-1}X^*Q = \begin{pmatrix} \overline{d} & -\beta^* & \overline{b} \\ -\delta & A^* & -\gamma \\ \overline{c} & -\alpha^* & \overline{a} \end{pmatrix}.$$

If U_Q is the group of $(n+1)\times(n+1)$ invertible matrices with complex coefficients preserving the Hermitian form q, then X belongs to U_Q if and only if X is invertible with inverse $Q^{-1}X^*Q$. In particular, if X belongs to U_Q, then

(- 42 -)
$$\left\{\begin{aligned} c\overline{d} - \delta^*\delta + d\overline{c} &= 0 \\ a\overline{b} - \gamma^*\gamma + b\overline{a} &= 0 \\ -\alpha\beta^* + AA^* - \beta\alpha^* &= I \\ c\overline{b} - \delta^*\gamma + d\overline{a} &= 1 \\ \overline{d}\alpha - A\delta + \overline{c}\beta &= 0 \\ \overline{b}\alpha - A\gamma + \overline{a}\beta &= 0\ . \end{aligned}\right.$$

The group U_Q acts projectively on $\mathbb{P}_n(\mathbb{C})$, preserving the negative cone of q, hence it acts on $\mathbb{H}^n_{\mathbb{C}}$. We will denote in the same way the action of U_Q on $\mathbb{H}^n_{\mathbb{C}}$ and the action of U_Q on the image of $\mathbb{H}^n_{\mathbb{C}}$ in $\mathbb{P}_n(\mathbb{C})$. It is well known (see for instance [Gol]) that U_Q preserves the Riemannian metric of $\mathbb{H}^n_{\mathbb{C}}$.

The Heisenberg group Heis_{2n-1} is the real Lie group with the underlying space $\mathbb{C}^{n-1}\times\mathbb{R}$ and the group law

$$(\zeta, v)(\zeta', v') = (\zeta+\zeta', v+v' - 2\ \mathrm{Im}\ \zeta^*\zeta')\ .$$

It has a Lie group embedding in U_Q, defined by

$$(\zeta, v)\mapsto u_{\zeta,v} = \begin{pmatrix} 1 & \zeta^* & \frac{|\zeta|^2}{2} - i\frac{v}{2} \\ 0 & I & \zeta \\ 0 & 0 & 1 \end{pmatrix},$$

whose image preserves the point ∞ as well as each horoball centered at ∞, as an easy computation shows.

The *Cygan distance* (see [Gol, page 160]) on Heis_{2n-1} is the unique left-invariant distance d_{Cyg} such that

$$d_{\mathrm{Cyg}}((0,0),(\zeta,v)) = (|\zeta|^4 + v^2)^{1/4}\ .$$

We introduce the *modified Cygan distance* d'_{Cyg} as the unique left-invariant distance d'_{Cyg} such that

$$d'_{\mathrm{Cyg}}((0,0),(\zeta,v)) = ((|\zeta|^4 + v^2)^{1/2} + |\zeta|^2)^{1/2}\ .$$

It is straightforward to check that d'_{Cyg} is indeed a distance, in the same way as the Cygan distance, see for instance [KR, page 320], and that it is equivalent to the Cygan distance,

$$d_{\mathrm{Cyg}} \le d'_{\mathrm{Cyg}} \le \sqrt{2}\, d_{\mathrm{Cyg}}\ .$$

Hence, its induced length distance is equivalent to the Carnot-Carathéodory distance on the Heisenberg group Heis_{2n-1} (see [Gol, page 161]).

As the action of Heis_{2n-1} on $\partial_\infty \mathbb{H}^n_{\mathbb{C}} - \{\infty\}$ is simply transitive, d_{Cyg} and d'_{Cyg} define distances on $\partial_\infty \mathbb{H}^n_{\mathbb{C}} - \{\infty\}$, which are invariant under the action of Heis_{2n-1}. We also call these distances the *Cygan distance* and the *modified Cygan distance*, and again denote them by d_{Cyg} and d'_{Cyg}. Explicitly, these distances are given by

$$d_{\mathrm{Cyg}}(u_{\zeta,v}(0,0), u_{\zeta',v'}(0,0)) = d_{\mathrm{Cyg}}((\zeta,v),(\zeta',v'))\ ,$$

and the similar expression for the modified Cygan distance.

Lemma 6.1 *The distance d_{Cyg} (respectively d'_{Cyg}) is the unique distance on $\partial_\infty \mathbb{H}^n_{\mathbb{C}} - \{\infty\}$ invariant under the action of Heis_{2n-1} such that $d_{\mathrm{Cyg}}((w_0,w),(0,0)) = \sqrt{2|w_0|}$ (respectively $d'_{\mathrm{Cyg}}((w_0,w),(0,0)) = \sqrt{2|w_0| + |w|^2}$).*

Proof. For every (w_0,w) in $\partial_\infty \mathbb{H}^n_{\mathbb{C}} - \{\infty\}$, note that $(w_0,w) = u_{\zeta,v}(0,0)$ if and only if $v = -2\ \mathrm{Im}\ w_0$ and $\zeta = w$, and that $2\ \mathrm{Re}\ w_0 = |w|^2$. Hence

$$d'_{\mathrm{Cyg}}(u_{\zeta,v}(0,0),(0,0)) = ((4\ \mathrm{Re}^2\ w_0 + 4\ \mathrm{Im}^2\ w_0)^{1/2} + |w|^2)^{1/2} = \sqrt{2|w_0| + |w|^2}\ .$$

A similar proof gives the result for the Cygan distance. □

In particular, if $n = 2$, then d'_{Cyg} is indeed defined as in the statement of Theorem 1.8 in the introduction.

Let $d_{\mathbb{H}^n_{\mathbb{C}}}$ be the Riemannian distance on $\mathbb{H}^n_{\mathbb{C}}$, and $d'_{\mathbb{H}^n_{\mathbb{C}}} = \frac{1}{2}\, d_{\mathbb{H}^n_{\mathbb{C}}}$ be the Riemannian distance of the Riemannian metric of $\mathbb{H}^n_{\mathbb{C}}$ renormalized to have maximal real sectional curvatures -1.

Proposition 6.2 *For every ξ, ξ' in $\partial_\infty \mathbb{H}^n_{\mathbb{C}} - \{\infty\}$, for every $s_0 > 0$, the distance ℓ' for the renormalized Riemannian distance $d'_{\mathbb{H}^n_{\mathbb{C}}}$ between the horoball $\mathscr{H}_{s_0}$ and the horoball centered at ξ and tangent to the geodesic line between ∞ and ξ' is, if these horoballs are disjoint,*

$$\ell' = -\log d'_{\mathrm{Cyg}}(\xi, \xi') + \frac{1}{2}\log(\frac{s_0}{2}) \; .$$

Proof. By invariance of the modified Cygan distance, of each horoball centered at ∞, and of the normalized Riemannian distance, under the action of the Heisenberg group, we may assume that $\xi = (0,0)$. Let $(\zeta, v) \in \mathrm{Heis}_{2n-1}$ such that $\xi' = u_{\zeta,v}(\xi)$. As $u_{\zeta,v}$ sends geodesic lines to geodesic lines, and fixes ∞, the geodesic lines (for $d_{\mathbb{H}^n_{\mathbb{C}}}$) starting from ∞ and ending at ξ' are time translates of $u_{\zeta,v} \circ c_0$, which by an easy computation is

$$u_{\zeta,v} \circ c_0 : t \mapsto (e^{-t} + (|\zeta|^2 - iv)/2, \zeta) \; .$$

The matrix

$$X_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{- 43 -}$$

belongs to U_Q, as $Q^{-1}X_0^*Q = X_0 = X_0^{-1}$, and the corresponding isometry of $\mathbb{H}^n_{\mathbb{C}}$ sends $\infty \in \partial_\infty \mathbb{H}^n_{\mathbb{C}}$ to $(0,0) \in \partial_\infty \mathbb{H}^n_{\mathbb{C}}$. Hence X_0 sends the horoballs centered at ∞ to the horoballs centered at $(0,0)$. Let $s > 0$, an easy computation shows that

$$X_0 \mathscr{H}_s = \{(w_0, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \;:\; 2\,\mathrm{Re}\; w_0 - |w|^2 \geq s|w_0|^2\} \; .$$

For every t in $\mathbb{R}$, the point $u_{\zeta,v} \circ c_0(t)$ belongs to the horosphere $X_0 \partial \mathscr{H}_s$ if and only if

$$2\,\mathrm{Re}\;(e^{-t} + (|\zeta|^2 - iv)/2) - |\zeta|^2 = s|e^{-t} + (|\zeta|^2 - iv)/2|^2 \; ,$$

that is, if and only if

$$s\, e^{-2t} + (s|\zeta|^2 - 2)e^{-t} + \frac{s}{4}\,(|\zeta|^4 + v^2) = 0 \; .$$

The horoball $X_0\mathscr{H}_s$ is hence tangent to the geodesic line $u_{\zeta,v} \circ c_0$ if and only if the above quadratic equation with unknown e^{-t} has a double solution, that is, if and only if its discriminant Δ is 0. An easy computation gives $-\Delta = s^2v^2 + 4s|\zeta|^2 - 4$. Thus, the horoball $X_0\mathscr{H}_s$ is tangent to $u_{\zeta,v} \circ c_0$ if and only if

$$s = \frac{2}{\sqrt{|\zeta|^4 + v^2} + |\zeta|^2} \; . \tag{- 44 -}$$

As the geodesic line c_0 passes through the point at infinity of both horoballs $\mathscr{H}_{s_0}$ and $X_0\mathscr{H}_s$ (which have disjoint interiors if s_0 is big enough), the Riemannian distance between them is the length of the subsegment of c_0 joining them. Note that c_0 meets $X_0\partial\mathscr{H}_s$ at $(\frac{2}{s},0)$. Hence, by an easy computation in $\mathbb{H}^1_{\mathbb{C}}$,

$$\begin{aligned}\ell' &= d'_{\mathbb{H}^n_{\mathbb{C}}}(\mathscr{H}_{s_0}, X_0\mathscr{H}_s) = \frac{1}{2}\, d_{\mathbb{H}^n_{\mathbb{C}}}(\mathscr{H}_{s_0}, X_0\mathscr{H}_s) = \frac{1}{2}\, d_{\mathbb{H}^n_{\mathbb{C}}}((\frac{s_0}{2},0),(\frac{2}{s},0))\\ &= \frac{1}{2}(\log\frac{s_0}{2} - \log\frac{2}{s})\ .\end{aligned}$$

By Equation (- 44 -), the result follows. □

For every X in U_Q, we will denote by $c = c(X)$ its $(3,1)$-coefficient in its matrix by blocks. Note that X fixes ∞ if and only if $c = 0$, by the equations (- 42 -). Equivalently, by the same set of equations, a matrix fixes ∞ if and only if it is upper triangular by blocks (this is the main reason why we chose the Hermitian form q rather than the one in [Gol]). The following lemma is completely analogous to Proposition 3.14 of [HP4], but as we are using a different quadratic form, we need to give a proof.

Lemma 6.3 *For every X in U_Q and every $s>0$ such that the horoballs $\mathscr{H}_s$ and $X\mathscr{H}_s$ have disjoint interiors, we have*

$$d'_{\mathbb{H}^n_{\mathbb{C}}}(\mathscr{H}_s, X\mathscr{H}_s) = \log|c| + \log\frac{s}{2}\ .$$

Proof. As $\mathscr{H}_s$ and $X\mathscr{H}_s$ have disjoint interiors, X does not fix ∞, hence $c\neq 0$. Left and right multiplication of X by an element $u_{\zeta,v}$ for some (ζ,v) in Heis_{2n-1} does not change the coefficient c of X, nor does it change $d'_{\mathbb{H}^n_{\mathbb{C}}}(\mathscr{H}_s, X\mathscr{H}_s) = \frac{1}{2}\, d_{\mathbb{H}^n_{\mathbb{C}}}(\mathscr{H}_s, X\mathscr{H}_s)$, as $u_{\zeta,v}$ preserves the distance $d_{\mathbb{H}^n_{\mathbb{C}}}$ and each horosphere centered at ∞. Hence, as Heis_{2n-1} acts transitively on $\partial_\infty\mathbb{H}^n_{\mathbb{C}} - \{\infty\}$, we may assume that $X\infty = (0,0)$ and that $X^{-1}\infty = (0,0)$. As $X\infty = (0,0)$, the coefficients a,α of X are 0, and hence by the second equation of (- 42 -), the coefficient γ is 0. As $X^{-1}\infty = (0,0)$, the coefficients d,δ of X are 0, and hence by the fifth equation of (- 42 -), the coefficient β is 0. Therefore, by the third and fourth equation of (- 42 -), the matrix X has the form $\begin{pmatrix} 0 & 0 & \frac{1}{\overline{c}}\\ 0 & A & 0\\ c & 0 & 0\end{pmatrix}$, with A unitary. An easy computation, similar to the one we already did with X_0, shows that

$$X\mathscr{H}_s = \{(w_0,w)\in\mathbb{C}\times\mathbb{C}^{n-1}\ :\ 2\,\mathrm{Re}\, w_0 - |w|^2 \geq s|c|^2|w_0|^2\}\ .$$

Hence, as above, since $\frac{s}{2}\geq\frac{2}{s|c|^2}$ as the horoballs $\mathscr{H}_s$ and $X\mathscr{H}_s$ have disjoint interiors,

$$d'_{\mathbb{H}^n_{\mathbb{C}}}(\mathscr{H}_s, X\mathscr{H}_s) = \frac{1}{2}\, d_{\mathbb{H}^n_{\mathbb{C}}}(\mathscr{H}_s, X\mathscr{H}_s) = \frac{1}{2}\, d_{\mathbb{H}^n_{\mathbb{C}}}((\frac{s}{2},0),(\frac{2}{s|c|^2},0)) = \log|c| + \log\frac{s}{2}\ .\ \square$$

Let m be a squarefree positive integer, let $K_{-m} = \mathbb{Q}(i\sqrt{m})$ be the corresponding imaginary quadratic number field, and let $\mathscr{O}_{-m}$ be the ring of integers of K_{-m}. An *order* $\mathscr{O}$ in K_{-m} is a unitary subring of $\mathscr{O}_{-m}$ which is a free $\mathbb{Z}$-module of rank 2. We use for instance [Cox, chap. 7] for a general reference on these objects. An example of an order in K_{-m} is $\mathbb{Z}[i\sqrt{m}]$, and $\mathscr{O}_{-m}$ is the maximal order of K_{-m}. In particular, $\mathscr{O}$ contains a $\mathbb{Q}$-basis of K_{-m}, and the field of fractions of $\mathscr{O}$ is K_{-m}. Let ω be an element of $\mathscr{O}$ with $\operatorname{Im}\omega > 0$ such that $\mathscr{O} = \mathbb{Z}[\omega] = \mathbb{Z} + \omega\mathbb{Z}$.

As $\mathscr{O}$ is stable by complex conjugation, the subset

$$\mathrm{SU}_Q(\mathscr{O}) = \mathrm{SU}_Q \cap \mathscr{M}_{n+1}(\mathscr{O})$$

is a discrete subgroup of the semi-simple connected real Lie group $\mathrm{SU}_Q = \mathrm{U}_Q \cap \mathrm{SL}_{n+1}(\mathbb{C})$.

Let $\mathscr{I}$ be a non-zero ideal of $\mathscr{O}$. We denote by $\Gamma_{\mathbb{C},\mathscr{I}}$ the preimage, by the group morphism $\mathrm{SU}_Q(\mathscr{O}) \to \mathrm{SL}_{n+1}(\mathscr{O}/\mathscr{I})$ of reduction modulo $\mathscr{I}$, of the parabolic subgroup of matrices whose first column has all its coefficients 0 except the first one. As $\mathscr{O}/\mathscr{I}$ is finite ($\mathscr{I}$ is nonzero), $\Gamma_{\mathbb{C},\mathscr{I}}$ is a finite index subgroup of $\mathrm{SU}_Q(\mathscr{O})$. In particular, if $\mathscr{I} = \mathscr{O}$, then $\Gamma_{\mathbb{C},\mathscr{I}} = \Gamma_{\mathbb{C},\mathscr{O}} = \mathrm{SU}_Q(\mathscr{O})$.

Recall that a horoball H centered at a point ξ in a CAT(-1) metric space X is *precisely invariant* under a group of isometries Γ if for every $g \in \Gamma$ that does not fix ξ, the intersection $g\,\overset{\circ}{H} \cap \overset{\circ}{H}$ is empty.

Lemma 6.4 *For $v = 2 \operatorname{Im}\omega$ if $\operatorname{Re}\omega \in \mathbb{Z}$, and $v = 4 \operatorname{Im}\omega$ otherwise, the horoball $\mathscr{H}_v$ is precisely invariant under $\Gamma_{\mathbb{C},\mathscr{I}}$. Furthermore, if $\mathscr{I} = \mathscr{O} = \mathscr{O}_{-1}$, then $\mathscr{H}_2$ is the maximal horoball centered at ∞ which is precisely invariant under $\Gamma_{\mathbb{C},\mathscr{I}}$.*

Proof. With v as in the statement, the element $-iv/2$ belongs to $\mathscr{O}$, as $i \operatorname{Im}\omega = \omega - \operatorname{Re}\omega$ belongs to $\mathscr{O}$ if $\operatorname{Re}\omega \in \mathbb{Z}$, and $2i \operatorname{Im}\omega = \omega - \overline{\omega}$ belongs to $\mathscr{O}$ (which is stable by conjugation). Hence $u_{0,v}$ belongs to $\Gamma_{\mathbb{C},\mathscr{I}}$. It follows for instance from [HP1, Prop. 5.7] (which is an easy consequence of the complex hyperbolic Shimizu inequality of Kamiya [Kam] and Parker [Par]) that the horoball $\mathscr{H}_v$ is precisely invariant (the Hermitian form q in [HP1] is not the same one as the one above, but it is equivalent by a permutation of coordinates, hence we may indeed apply [HP1, Prop. 5.7]).

If $\mathscr{I} = \mathscr{O} = \mathscr{O}_{-1}$, then X_0 defined in Equation (- 43 -) belongs to $\Gamma_{\mathbb{C},\mathscr{I}}$ and $\operatorname{Im}\omega = 1$, $\operatorname{Re}\omega \in \mathbb{Z}$. By Lemma 6.3, we have $d(\mathscr{H}_2, X_0\mathscr{H}_2) = 0$, hence the last assertion follows. □

For every $(a, \alpha, c) \in \mathscr{O} \times \mathscr{O}^{n-1} \times \mathscr{O}$, let $\langle a, \alpha, c\rangle$ be the ideal of $\mathscr{O}$ generated by a, c and the components of α.

Proposition 6.5 *If $n = 2$ and $\mathscr{O} = \mathscr{O}_{-m}$, then*

(1) *for every $\mathscr{I}$, the set of parabolic fixed points of $\Gamma_{\mathbb{C},\mathscr{I}}$ is exactly the set of points in $\partial_\infty \mathbb{H}^n_{\mathbb{C}}$ having homogeneous coordinates in $\mathbb{P}_n(\mathbb{C})$ that are elements in K_{-m};*

(2) *the orbit $\Gamma_{\mathbb{C},\mathscr{I}} \cdot \infty$ is exactly the set of points in $\partial_\infty \mathbb{H}^n_{\mathbb{C}}$ having homogeneous coordinates in $\mathbb{P}_n(\mathbb{C})$ of the form $[a : \alpha : c]$ with $(a, \alpha, c) \in \mathscr{O} \times \mathscr{I}^{n-1} \times \mathscr{I}$, $2\ \mathrm{Re}\ a\overline{c} = |\alpha|^2$ and $\langle a, \alpha, c\rangle = \mathscr{O}$;*

(3) *if $m = 1, 2, 3, 7, 11, 19, 43, 67, 163$ and $\mathscr{I} = \mathscr{O}$, then $\Gamma_{\mathbb{C},\mathscr{I}}$ has only one orbit of parabolic fixed points.*

Proof. (1) If $\mathscr{I} = \mathscr{O}$, then the first result is due to Holzapfel [Hol1], see [Hol2, page 280]. As $\Gamma_{\mathbb{C},\mathscr{I}}$ has finite index in $\mathrm{SU}_Q(\mathscr{O})$, and as a discrete group and a finite index subgroup have the same set of parabolic fixed points, the first claim follows.

(2) A result of Feustel [Feu] (see [Hol2, page 280], [Zin]) says that the map which associates to a parabolic fixed point of $\mathrm{SU}_Q(\mathscr{O})$ the fractional ideal generated by its homogeneous coordinates in $\mathscr{O}_{-m}$ induces a bijection from the set of orbits under $\mathrm{SU}_Q(\mathscr{O})$ of parabolic fixed points of $\mathrm{SU}_Q(\mathscr{O})$ to the set of ideal classes of K_{-m}. As ∞ corresponds to $[1 : 0 : 0]$ whose coordinates generate the trivial fractional ideal, and as K_{-m} has class number one if and only if $m = 1, 2, 3, 7, 11, 19, 43, 67, 163$ (see [Cox, Thm. 7.30]), the second claim follows if $\mathscr{I} = \mathscr{O}$, as well as claim (3).

If $M \in \Gamma_{\mathbb{C},\mathscr{I}}$ and $M\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}a\\ \alpha\\ c\end{pmatrix}$, then $\begin{pmatrix}a\\ \alpha\\ c\end{pmatrix}$ is the first column of the matrix M, so that the second claim if $\mathscr{I} \neq \mathscr{O}$ follows by the definition of $\Gamma_{\mathbb{C},\mathscr{I}}$. $\square$

6.2 Quaternions and 5-dimensional real hyperbolic geometry

Let $\mathbb{H}$ be Hamilton's quaternion algebra over $\mathbb{R}$, generated as a real vector space by the standard basis $1, i, j, k$, with products $k = ij = -ji$, $i^2 = -1$, $j^2 = -1$ and unit 1. Recall that the *conjugate* of the quaternion $z = x_1 + x_2 i + x_3 j + x_4 k$ is $\overline{z} = x_1 - x_2 i - x_3 j - x_4 k$, which satisfies $\overline{z\,w} = \overline{w}\,\overline{z}$, and that the *absolute value* of z (or the square root of its reduced norm) is

$$|z| = \sqrt{N(z)} = \sqrt{z\overline{z}} = \sqrt{\overline{z}z} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}\,.$$

The *Dieudonné determinant* (see [Die] and [Asl]) Δ is the group morphism from the group $\mathrm{GL}_2(\mathbb{H})$ of invertible 2×2 matrices with coefficients in $\mathbb{H}$ to $\mathbb{R}_+^*$, given by

$$\Delta\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right) = \begin{cases} |\,ad - aca^{-1}b\,| & \text{if } a \neq 0\\ |\,cb - cac^{-1}d\,| & \text{if } c \neq 0\end{cases}.$$

We will denote by $\mathrm{SL}_2(\mathbb{H})$ the group of 2×2 quaternionic matrices with Dieudonné determinant 1. We refer for instance to [Kel] for more information on $\mathrm{SL}_2(\mathbb{H})$. Note that this notation is different from, hence should not be confused with, the notation $\mathrm{SL}(2, C_n)$ for $n = 3$ and $C_3 = \mathbb{H}$ of Vahlen and Ahlfors [Ahl], see also [MWW], giving a description of the isometry group of the real hyperbolic $(n+1)$-space using the 2^n-dimensional real Clifford algebra C_n.

The group $\mathrm{SL}_2(\mathbb{H})$ acts on the Alexandrov compactification $\mathbb{H} \cup \{\infty\}$ of $\mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \begin{cases} (az+b)(cz+d)^{-1} & \text{if } z \neq \infty, -c^{-1}d \\ ac^{-1} & \text{if } z = \infty, c \neq 0 \\ \infty & \text{otherwise.} \end{cases}$$

It is well known (see for instance [Kel]) that $\mathrm{PSL}_2(\mathbb{H}) = \mathrm{SL}_2(\mathbb{H})/\{\pm \mathrm{Id}\}$ is the orientation preserving conformal group of the 4-sphere $\mathbb{H} \cup \{\infty\}$ with its standard conformal structure defined by the 4-dimensional Euclidean space $(\mathbb{H}, |\cdot|)$. In the upper half-space model $\mathbb{H}^5_{\mathbb{R}}$ of the 5-dimensional real hyperbolic space with constant curvature -1, consider the coordinates (z,t) with $z \in \mathbb{H}$ and $t > 0$ (called the *vertical coordinate*), so that $\partial_\infty \mathbb{H}^5_{\mathbb{R}}$ identifies with the union of $\mathbb{H}$ (for $t = 0$) and of $\{\infty\}$. By the Poincaré extension procedure (see for instance [Bea, Sect. 3.3]), the group $\mathrm{PSL}_2(\mathbb{H})$ hence identifies with the group of orientation preserving isometries of $\mathbb{H}^5_{\mathbb{R}}$. We will denote the Riemannian distance on $\mathbb{H}^5_{\mathbb{R}}$ by $d_{\mathbb{H}^5_{\mathbb{R}}}$.

Lemma 6.6 *[Hel, Theo. 1.-2)] For every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{H})$, and (z,t) in $\mathbb{H}^5_{\mathbb{R}}$, the vertical coordinate of $g(z,t)$ is*

$$\frac{t}{|cz+d|^2 + |c|^2 t^2} \;.$$

Proof. As [Hel] is an announcement, we give a proof for the sake of completeness. The proof is an adaptation of the proof for $\mathrm{SL}_2(\mathbb{C})$ in [Bea, page 58], the main problem consists of being careful with the noncommutativity of $\mathbb{H}$. We may assume that $c \neq 0$, as the map $z \mapsto \alpha z \beta + \gamma$ for α, β, γ in $\mathbb{H}^* \times \mathbb{H}^* \times \mathbb{H}$ is a Euclidean similitude of ratio $|\alpha\beta|$. Define the *isometric sphere* of g to be the sphere S_g of center $-c^{-1}d$ and radius $\frac{1}{|c|}$ in the Euclidean space $(\mathbb{H}, |\cdot|)$. By the definition of a Euclidean reflection with respect to a sphere in this Euclidean space, the map

$$\sigma : z \mapsto -c^{-1}d + \frac{1}{|c|^2} \frac{z + c^{-1}d}{|z + c^{-1}d|^2}$$

is the Euclidean reflection with respect to the sphere S_g. An easy computation shows that the map $\varphi = g \circ \sigma$ is

$$z \mapsto (b - ac^{-1}d)(\overline{z}\,\overline{c} + \overline{d}) + ac^{-1} \ ,$$

which is a Euclidean isometry, as $z \mapsto \overline{z}$ is, and $|cb - cac^{-1}d| = 1$. The Poincaré extension of φ preserves the vertical coordinates, and the Poincaré extension of σ is the Euclidean reflection with respect to the sphere in $\mathbb{H}^5_{\mathbb{R}}$ whose equator is S_g. As $g = \varphi \circ \sigma$, the result follows by an easy computation. $\square$

The horoballs centered at ∞ in $\mathbb{H}^5_{\mathbb{R}}$ are the subsets $\mathscr{H}_s$ for $s > 0$, where

$$\mathscr{H}_s = \{(z,t) \in \mathbb{H} \times]0, +\infty[\ :\ t \geq s\} \ .$$

Lemma 6.7 *For every* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *in* $\mathrm{SL}_2(\mathbb{H})$*, and every* $s > 0$ *such that the horoballs* $\mathscr{H}_s$ *and* $g\mathscr{H}_s$ *have disjoint interiors, we have*

$$d(\mathscr{H}_s, g\mathscr{H}_s) = 2 \log |c| + 2 \log s \ .$$

Proof. As $\mathscr{H}_s$ and $g\mathscr{H}_s$ have disjoint interiors, we have $c \neq 0$. The map g sends the geodesic line between $-c^{-1}d$ and ∞ to the geodesic line between ∞ and ac^{-1}, hence the point $(-c^{-1}d, s)$ of intersection of the first line with $\mathscr{H}_s$ is sent to $g(-c^{-1}d, s)$, which is the point of intersection of the second line with $g\mathscr{H}_s$. The vertical coordinate of $g(-c^{-1}d, s)$ is $\frac{1}{|c|^2 s}$ by the previous lemma 6.6. Hence the result follows by an easy computation of hyperbolic distances. $\square$

We will use [Vig] and [MR, Section 2] as general references on quaternion algebras. Let $A(\mathbb{Q})$ be a quaternion algebra over $\mathbb{Q}$, which is *ramified* over $\mathbb{R}$, that is, the real algebra $A(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to Hamilton's algebra $\mathbb{H}$. We identify $A(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathbb{H}$ by any such isomorphism. Let $\mathscr{O}'$ be an *order* of $A(\mathbb{Q})$, that is an unitary subring which is a finitely generated $\mathbb{Z}$-module generating the $\mathbb{Q}$-vector space $A(\mathbb{Q})$. For instance, if

$$A(\mathbb{Q}) = \{x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H}\ :\ x_1, x_2, x_3, x_4 \in \mathbb{Q}\},$$

we can take

$$\mathscr{O}' = \{x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H}\ :\ x_1, x_2, x_3, x_4 \in \mathbb{Z}\},$$

or the Hurwitz ring

$$\mathscr{O}' = \{x_1 \frac{1+i+j+k}{2} + x_2 i + x_3 j + x_4 k \in \mathbb{H}\ :\ x_1, x_2, x_3, x_4 \in \mathbb{Z}\} \ ,$$

which is a maximal order. Let $\mathscr{I}'$ be a non-zero two-sided ideal in the ring $\mathscr{O}'$.

We denote by $\Gamma_{\mathscr{I}'}$ the preimage in the group morphism $\mathrm{SL}_2(\mathscr{O}') \to \mathrm{GL}_2(\mathscr{O}'/\mathscr{I}')$ of reduction modulo $\mathscr{I}'$ of the subgroup of upper triangular matrices. As $\mathscr{O}'/\mathscr{I}'$ is finite ($\mathscr{I}'$ is nonzero), $\Gamma_{\mathscr{I}'}$ is a finite index subgroup of $\mathrm{SL}_2(\mathscr{O}')$.

Lemma 6.8 *The horoball $\mathscr{H}_1$ is precisely invariant under $\Gamma_{\mathscr{I}'}$. Furthermore, if $\mathscr{I}' = \mathscr{O}'$, then $\mathscr{H}_1$ is the maximal horoball centered at ∞ which is precisely invariant under $\Gamma_{\mathscr{I}'}$.*

Proof. The element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ belongs to $\Gamma_{\mathscr{I}'}$. It follows 1091]Kel2 that the horoball $\mathscr{H}_1$ is precisely invariant.

If $\mathscr{I}' = \mathscr{O}'$, then $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ belongs to $\Gamma_{\mathscr{I}'}$, and by Lemma 6.7, $d(\mathscr{H}_1, g\mathscr{H}_1) = 0$, hence the last assertion follows. □

6.3 On arithmetic lattices

The following result follows from the work of Borel and Harish-Chandra [BHC] and of Borel [Bor2, Theo. 1.10] (see [Bor1] for an elementary presentation of semisimple algebraic groups). Two subgroups A and B of a group C are said to be *commensurable* in this theorem if $A \cap B$ has finite index in both A and B.

Theorem 6.9 *[BHC, Bor2] Let $\underline{G}$ be a connected semisimple algebraic group defined over $\mathbb{Q}$ of $\mathbb{R}$-rank one and $\underline{P}$ be a minimal parabolic subgroup of $\underline{G}$ defined over $\mathbb{Q}$, let $G = \underline{G}(\mathbb{R})_0$ and $P = G \cap \underline{P}(\mathbb{R})$, let Γ be a subgroup of G commensurable to $\underline{G}(\mathbb{Z}) \cap G$, then Γ is a lattice in G, and the set of parabolic fixed points of Γ on G/P is $\underline{G}(\mathbb{Q})P$.* □

Such a subgroup Γ will be called an *arithmetic lattice* in G. Note that the $\mathbb{R}$-rank assumption is equivalent to the fact that for every (or equivalently any) maximal compact subgroup K of the Lie group G, the associated symmetric space $X = G/K$ may be endowed with a G-invariant Riemannian metric with sectional curvature at most -1. Such a metric is then unique up to multiplication by a positive constant, and P is the stabilizer of a point in the boundary at infinity $\partial_\infty X$. The orbital map at this point hence induces a G-equivariant homeomorphism between G/P and $\partial_\infty X$. Note that there is a terminology problem: by a parabolic element, we mean an isometry of X having a unique fixed point (called a *parabolic fixed point*) on $X \cup \partial_\infty X$, that belongs to $\partial_\infty X$, but the set of real points of a parabolic subgroup of $\underline{G}$ also contains nonparabolic elements !

Examples. In (1) and (2) below, let m be a squarefree positive integer, and let $\mathscr{I}$ be a non-zero ideal in an order $\mathscr{O}$ in the ring of integers $\mathscr{O}_{-m}$ of the imaginary quadratic number field $K_{-m} = \mathbb{Q}(i\sqrt{m})$. Let $(1, \omega)$ be a basis of $\mathscr{O}$ as a $\mathbb{Z}$-module. It is also a basis of K_{-m} as a $\mathbb{Q}$-vector space, and of $\mathbb{C}$ as an $\mathbb{R}$-vector space. If $x_1, \dots, x_n, y_1, \dots, y_n$ are real numbers, as ω is a quadratic integer, note that

$$\prod_{i=1}^{n} (x_i + \omega\, y_i) = P(x_1, \dots, x_n, y_1, \dots, y_n) + \omega\, Q(x_1, \dots, x_n, y_1, \dots, y_n)$$

where P and Q are polynomials in $x_1, \dots, x_n, y_1, \dots, y_n$ with integer coefficients.

(1) By writing each coefficient of a $n \times n$ complex matrix X in the basis $(1, \omega)$ over $\mathbb{R}$, the equation $\det X = 1$ gives a system of two polynomial equations with integer coefficients, with unknown the coordinates of the coefficients of X in $(1, \omega)$.

Hence, there exists an algebraic group $\underline{G}$ defined over $\mathbb{Q}$ such that $\underline{G}(\mathbb{Z}) = \mathrm{SL}_2(\mathscr{O})$, $\underline{G}(\mathbb{Q}) = \mathrm{SL}_2(K_{-m})$ and $\underline{G}(\mathbb{R}) = \mathrm{SL}_2(\mathbb{C})$. As the Lie group $\underline{G}(\mathbb{R})$ is connected and semisimple, with associated symmetric space the real hyperbolic 3-space, the algebraic group $\underline{G}$ is connected, semisimple with $\mathbb{R}$-rank one. Let $\underline{P}$ be the algebraic subgroup of $\underline{G}$ corresponding to the upper triangular subgroup of 2×2 matrices, so that P is the stabilizer of the point at infinity ∞ in the upper halfspace model of $\mathbb{H}^3_{\mathbb{R}}$.

Let $\Gamma_{\mathbb{R},\mathscr{I}}$ be the finite index subgroup of the group $\mathrm{SL}_2(\mathscr{O})$, which is the preimage, by the group morphism $\mathrm{SL}_2(\mathscr{O}) \to \mathrm{SL}_2(\mathscr{O}/\mathscr{I})$ of reduction modulo $\mathscr{I}$, of the subgroup of upper triangular matrices. By Theorem 6.9, the subgroup $\Gamma_{\mathbb{R},\mathscr{I}}$ is a lattice in $\mathrm{SL}_2(\mathbb{C})$, and its set of parabolic fixed points is

$$\mathscr{P}_{\Gamma_{\mathbb{R},\mathscr{I}}} = \mathrm{SL}_2(K_{-m}) \cdot \infty = K_{-m} \cup \{\infty\},$$

as $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} x & 0 \\ 1 & x^{-1} \end{pmatrix}$, for every x in $K_{-m} - \{0\}$, are elements of $\mathrm{SL}_2(K_{-m})$ sending ∞ to 0 and x respectively.

Note that if $\mathscr{I} = \mathscr{O} = \mathscr{O}_{-m}$, then $\Gamma_{\mathbb{R},\mathscr{I}} = \mathrm{PSL}_2(\mathscr{O}_{-m})$ is a Bianchi group, which is well-known to be a lattice in $\mathrm{PSL}_2(\mathbb{C})$ (see for instance [MR]). The fact that $\mathscr{P}_{\Gamma_{\mathbb{R},\mathscr{O}_{-m}}} = K_{-m} \cup \{\infty\}$ is also proven in [EGM, Prop. 2.2, page 314].

(2) Recall that $N(\omega) = \omega\overline{\omega}$ and $\mathrm{Tr}(\omega) = \omega + \overline{\omega} = 2\,\mathrm{Re}\,\omega$ are integers, as ω is an algebraic integer. If x, y, x', y' are real numbers, note that

$$(x + \overline{\omega} y)(x' + \omega y') = (xx' + N(\omega)\, yy' + \mathrm{Tr}(\omega)\, yx') + \omega(xy' - yx')\ .$$

Recall that the matrix Q (introduced in Equation (- 41 -)) has integer coefficients. Hence by writing each coefficient of a $(n+1)\times(n+1)$ complex matrix X in the basis $(1,\omega)$ over $\mathbb{R}$, the system of equations given by $\det X = 1$ and $X^*\,QX = Q$ becomes a system of $2((n+1)^2+1)$ polynomial equations with integer coefficients, with unknown the coordinates of the coefficients of X in $(1,\omega)$.

Therefore there exists an algebraic group $\underline{G}$ defined over $\mathbb{Q}$ such that $\underline{G}(\mathbb{Z}) = \mathrm{SU}_Q(\mathscr{O})$, $\underline{G}(\mathbb{Q}) = \mathrm{SU}_Q(K_{-m})$ and $\underline{G}(\mathbb{R}) = \mathrm{SU}_Q$. As the Lie group $\underline{G}(\mathbb{R})$ is connected and semisimple, with associated symmetric space the complex hyperbolic n-space, the algebraic group $\underline{G}$ is connected, semisimple with $\mathbb{R}$-rank one. Let $\underline{P}$ be the algebraic subgroup of $\underline{G}$ corresponding to the upper triangular by blocks subgroup of $(n+1)\times(n+1)$ matrices, so that P is the stabilizer of the point at infinity ∞ in the Siegel domain model of $\mathbb{H}^n_{\mathbb{C}}$, or of the point $[1:0:0]$ in the projective model.

By Theorem 6.9, the group $\Gamma_{\mathbb{C},\mathscr{I}}$ defined in Section 6.1 is a lattice in SU_Q, and its set of parabolic fixed points is $\mathscr{P}_{\Gamma_{\mathbb{C},\mathscr{I}}} = \mathrm{SU}_Q(K_{-m})\cdot\infty$. By Witt's theorem, $\mathrm{SU}_Q(K_{-m})$ acts transitively on the isotropic lines in ${K_{-m}}^{n+1}$ for the Hermitian form q. Hence $\mathscr{P}_{\Gamma_{\mathbb{C},\mathscr{I}}} = \{[z_0 : z : z_n] \in \mathbb{P}_n(K_{-m}) \;:\; q(z_0,z,z_n) = 0\}$. If $n=2$ and $\mathscr{O} = \mathscr{O}_{-m}$, we recover Proposition 6.5 (1).

(3) Let $\mathscr{I}'$ be a non-zero two-sided ideal in an order $\mathscr{O}'$ of a quaternion algebra $A(\mathbb{Q})$ over $\mathbb{Q}$ such that $A(\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{R} = \mathbb{H}$. For every field K containing $\mathbb{Q}$, define $A(K) = A(\mathbb{Q})\otimes_{\mathbb{Q}} K$. Let (e_1,e_2,e_3,e_4) be a basis of $\mathscr{O}'$ as a $\mathbb{Z}$-module. It is also a basis of $A(K)$ as a K-vector space for every field K containing $\mathbb{Q}$.

If $x = x_1 + x_2 i + x_3 j + x_4 k$ is an element in $\mathbb{H}$, written in the standard basis $(1,i,j,k)$, let $\mathrm{Tr}\,x = 2x_1$ be its reduced trace, and $N(x) = x_1^2+x_2^2+x_3^2+x_4^2$ be its reduced norm. A 2×2 matrix $X = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ with coefficients in $\mathbb{H}$ has Dieudonné determinant 1 if and only if

$$N(ad) + N(bc) - \mathrm{Tr}(a\overline{c}d\overline{b}) = 1 \tag{- 45 -}$$

(see for instance [Kel, page 1085]). The maps $\mathbb{R}^4 \to \mathbb{R}$ defined by $(x_1,x_2,x_3,x_4)\mapsto N(x_1e_1+x_2e_2+x_3e_3+x_4e_4)$ and $(x_1,x_2,x_3,x_4)\mapsto \mathrm{Tr}(x_1e_1+x_2e_2+x_3e_3+x_4e_4)$ are polynomial maps in x_1,x_2,x_3,x_4 with rational coefficients.

By writing each coefficient a,b,c,d of a 2×2 matrix X with coefficients in $A(K)$ in the basis (e_1,e_2,e_3,e_4) for any field K, the equation (- 45 -) becomes a polynomial equation with coefficients in $\mathbb{Q}$, with unknown the coordinates of the coefficients of X in (e_1,e_2,e_3,e_4).

Hence there exists an algebraic group $\underline{G}$ defined over $\mathbb{Q}$ such that $\underline{G}(\mathbb{Z}) = \mathrm{SL}_2(\mathscr{O}')$ and $\underline{G}(K) = \mathrm{SL}_2(A(K))$ for every field K containing $\mathbb{Q}$. As the Lie group $\underline{G}(\mathbb{R}) = \mathrm{SL}_2(\mathbb{H})$ is connected and semisimple, with associated symmetric space the real hyperbolic 5-space, the algebraic group $\underline{G}$ is connected, semisimple with $\mathbb{R}$-rank one. Let $\underline{P}$ be the algebraic subgroup of $\underline{G}$ corresponding to the upper triangular matrices, so that P is the stabilizer of the point at infinity ∞ in the upper halfspace model of $\mathbb{H}^5_{\mathbb{R}}$.

Let $\Gamma_{\mathscr{I}'}$ be the group introduced in Section 6.2, which has finite index in $\mathrm{SL}_2(\mathscr{O}')$. By Theorem 6.9, the subgroup $\Gamma_{\mathscr{I}'}$ is a lattice in $\mathrm{SL}_2(\mathbb{H})$, and its set of parabolic fixed points is $\mathscr{P}_{\Gamma_{\mathscr{I}'}} = \mathrm{SL}_2(A(\mathbb{Q}))\cdot\infty$. As $A(\mathbb{Q})$ is a division algebra, the same argument as for example (1) shows that $\mathscr{P}_{\Gamma_{\mathscr{I}'}} = A(\mathbb{Q}) \cup \{\infty\}$.

6.4 The ubiquity of Hall rays

In this subsection, we give applications of our geometric results from Section 5 to the framework of Diophantine approximation in negatively curved manifolds, introduced in [HP3, HP4], to which we refer for notation and background. In particular, we will consider arithmetically defined examples. See also the previous works of [For, Ser], among many others.

Let M be a complete, nonelementary, geometrically finite Riemannian manifold with sectional curvature at most -1 and dimension at least 3. Let $\pi : \widetilde{M} \to M$ be a universal Riemannian covering, with covering group Γ. Let e be a cusp of M, and, as in Section 5.1, let V_e be a fixed Margulis neighbourhood of e, H_e a horoball in $\widetilde{M}$ with $\pi(H_e) = V_e$ and ξ_e the point at infinity of H_e. Note that V_e is a Margulis neighbourhood if H_e is precisely invariant under Γ. In the previous works [HP3, HP4], it was required that V_e is the maximal Margulis neighbourhood, as this makes the constructions independent of the choice of V_e. But since it is not always easy to determine the maximal Margulis neighbourhood of a cusp, and as it is not necessary for the statements, we will fix some choice of V_e (or equivalently H_e) which is not necessarily maximal.

Three (classes of) examples. Many of these examples are in fact orbifolds rather than manifolds, but the extension to this context is obvious. We use the same notation as in the examples of Subsection 6.3.

(1) Let $\Gamma_{\mathbb{R},\mathscr{I}}$ be the finite index subgroup of the group $\mathrm{SL}_2(\mathscr{O})$, which is the preimage, by the group morphism $\mathrm{SL}_2(\mathscr{O}) \to \mathrm{SL}_2(\mathscr{O}/\mathscr{I})$ of reduction modulo $\mathscr{I}$, of the subgroup of upper triangular matrices. The quotient $M = \Gamma_{\mathbb{R},\mathscr{I}}\backslash\mathbb{H}^3_{\mathbb{R}}$ is a finite volume

real hyperbolic orbifold. Let $\pi : \mathbb{H}^3_{\mathbb{R}} \to M$ be the canonical projection, e the cusp of M corresponding to $\xi_e = \infty$, and let H_e be the horoball of points of Euclidean height at least 1. As $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ belongs to $\Gamma_{\mathbb{R},\mathscr{I}}$, it is well known that H_e is precisely invariant under $\Gamma_{\mathbb{R},\mathscr{I}}$. Furthermore, if $\mathscr{I} = \mathscr{O}$, then $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ belongs to $\Gamma_{\mathbb{R},\mathscr{I}}$, hence H_e is maximal. For more details, see [HP3], end of Section 5.

(2) Let $\Gamma_{\mathbb{C},\mathscr{I}}$ be the finite index subgroup of $\mathrm{SU}_Q(\mathscr{O})$ introduced in Section 6.1, that acts by isometries on the Siegel domain model $\mathbb{H}^n_{\mathbb{C}}$ of the complex hyperbolic n-space with (constant) holomorphic sectional curvature -1. Let M be the finite volume complex hyperbolic orbifold $\Gamma_{\mathbb{C},\mathscr{I}} \backslash \mathbb{H}^n_{\mathbb{C}}$, which is endowed with the quotient of the renormalized Riemannian distance $d'_{\mathbb{H}^n_{\mathbb{C}}}$ in order for its sectional curvatures to be at most -1. Let $\pi : \mathbb{H}^n_{\mathbb{C}} \to M$ be the canonical projection, e be the cusp of M corresponding to $\xi_e = \infty$, and let H_e be the horoball $\mathscr{H}_{2\,\mathrm{Im}\,\omega}$ if $\mathrm{Re}\,\omega \in \mathbb{Z}$, and $\mathscr{H}_{4\,\mathrm{Im}\,\omega}$ otherwise, which is precisely invariant under $\Gamma_{\mathbb{C},\mathscr{I}}$ by Lemma 6.4 (and maximal if $\mathscr{I} = \mathscr{O} = \mathscr{O}_{-1}$).

(3) Let $\mathscr{I}'$ be a non-zero two-sided ideal in an order $\mathscr{O}'$ of a quaternion algebra $A(\mathbb{Q})$ over $\mathbb{Q}$ such that $A(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{H}$, and $\Gamma_{\mathscr{I}'}$ be the finite index subgroup of $\mathrm{SL}_2(\mathscr{O}')$ introduced in Section 6.2, that acts by isometries on the upper halfspace model $\mathbb{H}^5_{\mathbb{R}}$ of the real hyperbolic 5-space with (constant) sectional curvature -1. Let M be the finite volume real hyperbolic orbifold $\Gamma_{\mathscr{I}'} \backslash \mathbb{H}^5_{\mathbb{R}}$, $\pi : \mathbb{H}^5_{\mathbb{R}} \to M$ the canonical projection, e the cusp of M corresponding to $\xi_e = \infty$, and let H_e be the horoball $\mathscr{H}_1$, which is precisely invariant under $\Gamma_{\mathscr{I}'}$ by Lemma 6.8 (and maximal if $\mathscr{I}' = \mathscr{O}'$).

Let the *link* of e in M, $\mathrm{Lk}_e = \mathrm{Lk}_e(M)$, be the space of locally geodesic lines (up to translation at the source) starting from e in M that are nonwandering (i.e. such that each of them accumulates in some compact subset of M). Let Rat_e be the space of locally geodesic lines starting from e and converging to e. Normalize the locally geodesic lines in $\mathrm{Lk}_e \cup \mathrm{Rat}_e$ so that their first intersection with ∂V_e is at time 0. Endow $\mathrm{Lk}_e \cup \mathrm{Rat}_e$ with the compact-open topology. Let $\Lambda\Gamma \subset \partial_\infty \widetilde{M}$ be the limit set of Γ, $\mathscr{P}_\Gamma \subset \Lambda\Gamma$ the set of parabolic fixed points of Γ, and let Γ_∞ be the stabilizer of ξ_e in Γ. Then the maps $\Gamma\xi_e - \{\xi_e\} \to \mathrm{Rat}_e$ and $\Lambda\Gamma - \mathscr{P}_\Gamma \to \mathrm{Lk}_e$, which associate to x the projection in M by π of the geodesic line starting from ξ_e and ending at x, induce a bijection $\Gamma_\infty \backslash (\Gamma\xi_e - \{\xi_e\}) \to \mathrm{Rat}_e$ and a homeomorphism

$$\Gamma_\infty \backslash (\Lambda\Gamma - \mathscr{P}_\Gamma) \to \mathrm{Lk}_e\ .$$

We identify these spaces by these maps. Note that $\mathrm{Lk}_e \cup \mathrm{Rat}_e$ is compact if and only if M has only one cusp, and that Rat_e is dense in $\mathrm{Lk}_e \cup \mathrm{Rat}_e$ (as $\Gamma\xi_e$ is dense in

$\Lambda\Gamma$). Diophantine approximation in M (see [HP3, HP4, HP5]) studies the rate of convergence of sequences of points in Rat_e to given points in Lk_e.

For every r in Rat_e, let $D(r)$, called the *depth* of r, be the length of the subsegment of r between the first and the last meeting points with ∂V_e.

Examples. (1) Consider $M = \Gamma_{\mathbb{R},\mathscr{I}}\backslash\mathbb{H}^3_{\mathbb{R}}$. Then $\mathscr{P}_{\Gamma_{\mathbb{R},\mathscr{I}}} \subset \mathbb{C}\cup\{\infty\}$ is exactly $K_{-m}\cup\{\infty\}$, by the example (1) of Section 6.3. Thus, $\mathrm{Lk}_e(M) = (\Gamma_{\mathbb{R},\mathscr{I}})_\infty\backslash(\mathbb{C}-K_{-m})$. In a commutative unitary ring R, we denote by $\langle p_1,\dots,p_k\rangle$ the ideal generated by $p_1,\dots,p_k\in R$. It is easy to prove (see for instance [EGM, Lem. 2.1, page 314]) that Rat_e is the set of elements $r = p/q$ (modulo $(\Gamma_{\mathbb{R},\mathscr{I}})_\infty$) with $(p,q)\in\mathscr{O}\times\mathscr{I}$ such that $\langle p,q\rangle = \mathscr{O}$. Furthermore (see [HP3, Lem. 2.10])

$$D(r) = 2\log|q|\ .$$

(2) Consider $M = \Gamma_{\mathbb{C},\mathscr{I}}\backslash\mathbb{H}^n_{\mathbb{C}}$. Let $\mathscr{Q}(\mathbb{R})$ be the real quadric $\partial_\infty\mathbb{H}^n_{\mathbb{C}} - \{\infty\}$. By considering a basis of K_{-m} over $\mathbb{Q}$, it is easy to see that $\mathscr{Q}(\mathbb{R})$ is the set of $\mathbb{R}$-points of a quadric $\mathscr{Q}$ defined over $\mathbb{Q}$ (which depends on m), whose set $\mathscr{Q}(\mathbb{Q})$ of $\mathbb{Q}$-points is $\mathscr{Q}(\mathbb{R})\cap(K_{-m}\times K_{-m}^{n-1})$. We have $\mathrm{Lk}_e = (\Gamma_{\mathbb{C},\mathscr{I}})_\infty\backslash(\mathscr{Q}(\mathbb{R}) - \mathscr{P}_{\Gamma_{\mathbb{C},\mathscr{I}}})$. By the example (2) of Section 6.3, we have $\mathscr{P}_{\Gamma_{\mathbb{C},\mathscr{I}}} = \mathscr{Q}(\mathbb{Q})\cup\{\infty\}$.

Then Rat_e is the quotient modulo $(\Gamma_{\mathbb{C},\mathscr{I}})_\infty$ of the subset of $\mathscr{Q}(\mathbb{Q})$ of points of the form $(a/c,\alpha/c)$ with $(a,\alpha,c)\in\mathscr{O}\times\mathscr{I}^{n-1}\times\mathscr{I}$ such that there exist $b,d,\beta,\gamma,\delta,A$ matrices of the appropriate size such that $\begin{pmatrix} a & \gamma^* & b\\ \alpha & A & \beta\\ c & \delta^* & d\end{pmatrix}$ belongs to $\Gamma_{\mathbb{C},\mathscr{I}}$. By Proposition 6.5 (2), this existence requirement is equivalent to the requirement that $q(a,\alpha,c) = 0$ and $\langle a,\alpha,c\rangle = \mathscr{O}$, if $n=2$ and $\mathscr{O} = \mathscr{O}_{-m}$. By Proposition 6.5 (3), $\mathrm{Rat}_e = (\Gamma_{\mathbb{C},\mathscr{I}})_\infty\backslash\mathscr{Q}(\mathbb{Q})$ if $n=2$, $\mathscr{I}=\mathscr{O}=\mathscr{O}_{-m}$ and $m = 1,2,3,7,11,19,43,67,163$.

If $r\in\mathrm{Rat}_e$ is of the form $(a/c,\alpha/c)$ (modulo $(\Gamma_{\mathbb{C},\mathscr{I}})_\infty$) as above, then by Lemma 6.3, we have

$$D(r) = \log|c| + \begin{cases} \log\ \mathrm{Im}\,\omega & \text{if }\ \mathrm{Re}\,\omega\in\mathbb{Z},\\ \log(2\ \mathrm{Im}\,\omega) & \text{otherwise.}\end{cases}$$

(3) Consider $M = \Gamma_{\mathscr{I}'}\backslash\mathbb{H}^5_{\mathbb{R}}$. We have $\mathscr{P}_{\Gamma_{\mathscr{I}'}} = A(\mathbb{Q})\cup\{\infty\}$, by the example (3) of Section 6.3. Hence $\mathrm{Lk}_e = (\Gamma_{\mathscr{I}'})_\infty\backslash(\mathbb{H} - A(\mathbb{Q}))$.

It is easy to see that Rat_e is the set of elements $r = pq^{-1}$ (modulo $(\Gamma_{\mathscr{I}'})_\infty$) with $(p,q)\in\mathscr{O}'\times(\mathscr{I}'-\{0\})$ such that there exists $r,s\in\mathscr{O}'$ with $|qr - qpq^{-1}s| = 1$. Furthermore, by Lemma 6.7, we have

$$D(r) = 2\log|q|\ .$$

The *cuspidal distance* $d'_e(\gamma,\gamma')$ of γ,γ' in $\mathrm{Lk}_e\cup\mathrm{Rat}_e$ is the minimum of the $\widetilde{d'_e}(\widetilde{\gamma},\widetilde{\gamma'})$ for $\widetilde{\gamma},\widetilde{\gamma'}$ two lifts of γ,γ' to $\widetilde{M}$ starting from ξ_e, where $\widetilde{d'_e}(\widetilde{\gamma},\widetilde{\gamma'})$ is the greatest lower bound of $r>0$ such that the horosphere centered at $\widetilde{\gamma}(+\infty)$, at signed distance $-\log 2r$ from ∂H_e on the geodesic line $]\xi_e,\widetilde{\gamma}(+\infty)[$, meets $\widetilde{\gamma'}$ (see [HP3, Sect. 2.1]). Though not necessarily an actual distance, $\widetilde{d'_e}$ is equivalent to the Hamenstädt distance (see Subsection 3.1 and [HP3, Rem. 2.6]).

Examples. (1) If M has constant curvature -1, if one identifies $\mathrm{Lk}_e\cup\mathrm{Rat}_e$ with a subset of ∂H_e by the first intersection point, then d'_e is the induced Riemannian distance on ∂H_e, which is Euclidean (see [HP3, Sect. 2.1]); in particular, if $M=\Gamma_{\mathbb{R},\mathscr{I}}\backslash\mathbb{H}^3_{\mathbb{R}}$ or $M=\Gamma_{\mathscr{I}'}\backslash\mathbb{H}^5_{\mathbb{R}}$, then d'_e is the quotient of the standard Euclidean distance on $\mathrm{Lk}_e\cup\mathrm{Rat}_e$ identified with a subset of $(\Gamma_{\mathbb{R},\mathscr{I}})_\infty\backslash\mathbb{C}$ or $(\Gamma_{\mathscr{I}'})_\infty\backslash\mathbb{H}$.

(2) If M is Hermitian with constant holomorphic sectional curvature -1, then d'_e is no longer Riemannian, but by Proposition 6.2, it is a multiple of the modified Cygan distance d'_{Cyg}. In particular, if $M=\Gamma_{\mathbb{C},\mathscr{I}}\backslash\mathbb{H}^n_{\mathbb{C}}$, then d'_e is the quotient by $(\Gamma_{\mathbb{C},\mathscr{I}})_\infty$ acting on $\partial_\infty\mathbb{H}^n_{\mathbb{C}}$ of the distances

$$\begin{cases} \frac{1}{2\sqrt{\mathrm{Im}\;\omega}}\;d'_{\mathrm{Cyg}} & \text{if }\;\mathrm{Re}\;\omega\in\mathbb{Z}\\ \frac{1}{2\sqrt{2}\;\mathrm{Im}\;\omega}\;d'_{\mathrm{Cyg}} & \text{otherwise}\;. \end{cases}$$

Remark. The claim in the first paragraph of Section 3.11 in [HP4] (where the authors only considered the case $m=1$ and $\mathscr{I}=\mathscr{O}=\mathscr{O}_{-1}$) that the cuspidal distance coincides with the Hamenstädt distance is incorrect. But every statement remains correct. Since the Cygan distance and the modified Cygan distance are equivalent, this does not change the statement of the main results, Theorems 3.1, 3.2, 3.4, and 3.5 of [HP4]. Since $d'_{\mathrm{Cyg}}\le\sqrt{2}\;d_{\mathrm{Cyg}}$ and $d'_e=\frac{1}{2}d'_{\mathrm{Cyg}}$ by the above displayed formula with $\omega=i$, so that $d_{\mathrm{Cyg}}\ge\sqrt{2}\;d'_e$, Theorem 3.6 of [HP4] also remains correct, using in its proof the inequality $d_{\mathrm{Cyg}}\ge\sqrt{2}\;d'_e$ instead of the equality $d_{\mathrm{Cyg}}=\sqrt{2}\;d_\infty$ mentionned there.

With M as in the beginning of this subsection, for every x in Lk_e, define the *approximation constant* $c(x)$ of x as

$$c(x)=\liminf_{r\in\mathrm{Rat}_e\,,\;D(r)\to\infty} d'_e(x,r)\;e^{D(r)}\;.$$

The *Lagrange spectrum* of M with respect to e is the subset $\mathrm{Sp}_{\mathrm{Lag}}(M,e)$ of $\mathbb{R}$ consisting of the constants $c(x)$ for x in Lk_e. It is shown in [HP3] that

- $c(x)$ is well defined for any x in Lk_e (as Rat_e is dense in $\mathrm{Lk}_e\cup\mathrm{Rat}_e$ and $\{D(r)\;:\;r\in\mathrm{Rat}_e\}$ is a discrete subset of $\mathbb{R}$ with finite multiplicities),

- $c(x)$ is finite for any x in Lk_e (as x is nonwandering). (Note that if y is a locally geodesic line starting from e in M that converges into a cusp of M, then the same formula would yield $c(y) = +\infty$.)
- the least upper bound of $\mathrm{Sp}_{\mathrm{Lag}}(M,e)$, denoted by $K_{M,e}$ and called the *Hurwitz constant* of (M,e), is finite.

In particular, $\mathrm{Sp}_{\mathrm{Lag}}(M,e) \subset [0, K_{M,e}]$. The following result tells us that the Lagrange spectrum contains a nontrivial initial interval $[0,c]$, with a universal lower bound on c (whose optimal value we do not know).

Theorem 6.10 *Let M be a complete, nonelementary, geometrically finite Riemannian manifold with sectional curvature at most -1 and dimension at least 3, and let e be a cusp of M. The Lagrange spectrum $\mathrm{Sp}_{\mathrm{Lag}}(M,e)$ contains the interval $[0\,,\,0.00057]$. In particular, $K_{M,e} \geq 0.00057$.*

Proof. By [HP3], the map $h \mapsto \frac{1}{2}e^{-h}$ maps the asymptotic height spectrum bijectively onto the Lagrange spectrum. We apply Corollary 5.15 and the computation above it. □

A precise version of this theorem is stated as corollaire 5 in [PP2] when M is a real or complex hyperbolic manifold. Note that the constant c_* of [PP2], which satisfies $c_* = e^{-h_*}/2$ with $h_* \approx 6.7771$, is approximatively 0.00057, and not 0.0337 as indicated by mistake in [PP2]. Theorem 1.7 in the introduction follows immediately, by the first example discussed in this section. By varying the (nonuniform) arithmetic lattices in the isometry group of a negatively curved symmetric space (see for instance [MR, MWW]), other arithmetic applications are possible. We only state two of them in what follows, see also [PP2].

Let $\mathscr{I}'$ be a non-zero two-sided ideal in an order $\mathscr{O}'$ of a quaternion algebra $A(\mathbb{Q})$ over $\mathbb{Q}$ ramifying over $\mathbb{R}$, and let N be the reduced norm on $A(\mathbb{R}) = A(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$ (see for instance [Vig], and Section 6.2). For every $x \in A(\mathbb{R}) - A(\mathbb{Q})$, define the *approximation constant* of x by

$$c(x) = \liminf_{(p,q)\,\in\,\mathscr{O}'\times\mathscr{I}'\,:\,\exists\, r,s\in\mathscr{O}'\ N(qr-qpq^{-1}s)=1\,,\ N(q)\to\infty} N(q)N(x-pq^{-1})^{\frac{1}{2}}\ ,$$

and the *Hamilton-Lagrange spectrum* for the approximation of elements of $\mathbb{H}$ by elements of $\mathscr{O}'\mathscr{I}'^{-1}$ as the subset of $\mathbb{R}$ consisting of the $c(x)$ for $x \in A(\mathbb{R}) - A(\mathbb{Q})$. Note that $c(x)$ is finite if $x \notin A(\mathbb{Q})$, as then x is not a parabolic fixed point of $\Gamma_{\mathscr{I}'}$. Apply Theorem 6.10 to $M = \Gamma_{\mathscr{I}'}\backslash\mathbb{H}^5_{\mathbb{R}}$ with the above discussions of the third example to get the following result.

Theorem 6.11 *The Hamilton-Lagrange spectra contain the interval* $[0\,,\,0.00057]$. $\square$

In the case when $\mathscr{I}' = \mathscr{O}'$ and $\mathscr{O}'$ is the Hurwitz maximal order in Hamilton's quaternion algebra $A(\mathbb{Q}) \subset \mathbb{H}$ (see Subsection 6.2), A. Schmidt [Sch2] proved that the Hamilton-Lagrange spectrum contains $\sqrt{2}\,\mathrm{Sp}_{\mathbb{Q}}$ where $\mathrm{Sp}_{\mathbb{Q}}$ is the classical Lagrange spectrum for the approximation of real numbers by rational numbers. As $\mathrm{Sp}_{\mathbb{Q}}$ contains $[0,\mu]$ where μ is Freiman's constant (see the end of Subsection 5.4), this proves that the Hamilton-Lagrange spectrum in this case contains the interval $[0, 0.312]$. Note that the fact that our approximation constant coincides with the inverse of A. Schmidt's approximation constant follows from [Sch1, Thm.5]

Let m be a squarefree positive integer, let $\mathscr{I}$ be a non-zero ideal in an order $\mathscr{O}$ in the ring of integers $\mathscr{O}_{-m}$ of the imaginary quadratic number field $\mathbb{Q}(i\sqrt{m})$, and let ω be an element of $\mathscr{O}_{-m}$ with $\mathrm{Im}\,\omega > 0$ such that $\mathscr{O} = \mathbb{Z} + \omega\mathbb{Z}$. Let $\mathscr{E}_{\mathscr{O},\mathscr{I}}$ be the set of (a,α,c) in $\mathscr{O}\times\mathscr{I}^{n-1}\times\mathscr{I}$ such that there exists a matrix of the form $\begin{pmatrix} a & \gamma^* & b\\ \alpha & A & \beta\\ c & \delta^* & d\end{pmatrix}$ that belongs to $\Gamma_{\mathbb{C},\mathscr{I}}$. If $n=2$ and $\mathscr{O}=\mathscr{O}_{-m}$, then, as seen previously,

$$\mathscr{E}_{\mathscr{O},\mathscr{I}} = \{(a,\alpha,c)\in\mathscr{O}\times\mathscr{I}^{n-1}\times\mathscr{I}\;:\;q(a,\alpha,c)=0,\ \langle a,\alpha,c\rangle=\mathscr{O}\}\,.$$

We do not know if this is the case for every $n, \mathscr{I}, \mathscr{O}$ as above. For every $x\in\mathscr{Q}(\mathbb{R})-\mathscr{Q}(\mathbb{Q})$, define the *approximation constant* of x by

$$c(x) = \liminf_{(a,\alpha,c)\in\mathscr{E}_{\mathscr{O},\mathscr{I}}\,,\ |c|\to\infty} |c|\ d'_{\mathrm{Cyg}}(x,(a/c,\alpha/c))\ ,$$

and the *Heisenberg-Lagrange spectrum*, for the approximation of elements of $\mathscr{Q}(\mathbb{R})$ by elements of $\{(a/c,\alpha/c)\;:\;(a,\alpha,c)\in\mathscr{E}_{\mathscr{O},\mathscr{I}}\}\subset\mathscr{Q}(\mathbb{Q})$, as the subset of $\mathbb{R}$ consisting of the $c(x)$ for $x\in\mathscr{Q}(\mathbb{R})-\mathscr{Q}(\mathbb{Q})$. Note that $c(x)$ is finite if $x\notin\mathscr{Q}(\mathbb{Q})$, as then x is not a parabolic fixed point of $\Gamma_{\mathbb{C},\mathscr{I}}$. Our last result follows from Theorem 6.10 and the previous discussions of the second example.

Theorem 6.12 *The Heisenberg-Lagrange spectra contain the interval* $[0\,,\,\frac{0.0011}{\sqrt{\mathrm{Im}\,\omega}}]$ *if* $\mathrm{Re}\,\omega\in\mathbb{Z}$ *and* $[0\,,\,\frac{0.0008}{\sqrt{\mathrm{Im}\,\omega}}]$ *otherwise.* $\square$

Theorem 1.8 in the introduction follows from this one (and from Lemma 6.1) by taking $m=1$, $n=2$, $\mathscr{I}=\mathscr{O}=\mathscr{O}_{-1}$, as then $\mathrm{Im}\,\omega=1$.

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