Counting arcs in negative curvature

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Abstract

Let M be a complete Riemannian manifold with negative curvature, and let C_-, C_+ be two properly immersed locally convex subsets of M. We survey the asymptotic behaviour of the number of common perpendiculars of length at most s from C_- to C_+, giving error terms and counting with weights, starting from the work of Huber, Herrmann, Margulis and ending with the works of the authors. We describe the relationship with counting problems in circle packings of Kontorovich, Oh, Shah. We survey the tools used to obtain the precise asymptotics (Bowen-Margulis and Gibbs measures, skinning measures). We describe several arithmetic applications, in particular the ones by the authors on the asymptotics of the number of representations of integers by binary quadratic, Hermitian or Hamiltonian forms. [1]

1 Introduction

Let M be a complete connected Riemannian manifold with negative sectional curvature. Let C_- and C_+ be two properly immersed locally convex subsets of M. For instance, C_- and C_+ could be points, totally geodesic immersed submanifolds, Margulis cusp neighbourhoods, or images in M of convex hulls in a universal Riemannian cover of M of limit sets of subgroups of the fundamental group of M. A *common perpendicular* between C_- and C_+ is a locally geodesic path $c : [a, b] \to M$ such that $\dot{c}(a)$ is an outer unit normal vector to C_- and $-\dot{c}(b)$ is an outer unit normal vector to C_+ (see Section 3.1 and 3.2 for precise definitions).

The aim of this survey is to present several results on the asymptotic behaviour as s tends to $+\infty$ of the number $\mathscr{N}(s)$ of common perpendiculars (counted with multiplicities) between C_- and C_+ with length at most s.

We describe the first results on this problem by Huber [Hub], Herrmann [Herr] and Margulis [Mar1], see also the surveys [Bab2, Sha] and their references. We explain in Section 3 how several works of Duke-Rudnick-Sarnak on counting integral points on hyperboloids and of Kontorovich, Oh and Shah on counting problems in circle and sphere packings are related to counting problems of common perpendiculars. We particularly emphasize the arithmetic applications developped in [PP5, PP3, PP4] to the counting of the representations of integers by quadratic, Hermitian or quaternionic binary forms, see Section 5.

[1]**Keywords:** counting, geodesic arc, common perpendicular, convexity, equidistribution, mixing, rate of mixing, decay of correlation, negative curvature, convex hypersurfaces, skinning measure, Bowen-Margulis measure, Gibbs measure. **AMS codes:** 37D40, 37A25, 53C22, 20H10, 20G20, 11R52, 11N45, 11E39, 30F40

A few results are new, in particular the deduction of the equidistribution of the outer unit normal bundles of equidistant hypersurfaces of totally geodesic submanifolds in a hyperbolic manifold from Eskin-McMullen's [EM] work (see Section 4), and the computations of the constant relating the Bowen-Margulis measure and the Liouville measure in constant curvature and finite volume, as well as the one relating the skinning measure and the Riemannian measure on the unit normal bundle of a totally geodesic submanifold in constant curvature and finite volume (see Section 7).

We survey the main tools used for the counting results, the geometry of negatively curved manifolds in Section 2, and in Section 6, the various measures, as the Patterson-Sullivan densities, the Bowen-Margulis measures, the skinning measures (introduced by Oh-Shah in constant curvature for totally geodesic submanifolds, and developped in general in [PP6]), that are needed to explicit the multiplicative constant in front of the exponential term in the asymptotics of the number $\mathscr{N}(s)$ of common perpendiculars.

We give in Section 8 a sketch of the proof of the main counting result of [PP7], which seems to contain as particular cases all previous results on the asymptotics of $\mathscr{N}(s)$, and to give many new ones. We conclude the paper by studying the error term to this asymptotic equivalent under hypotheses of exponential mixing of the geodesic flow (see Section 9), and by giving counting results when weights have been added to the common perpendiculars, by means of a potential function, the main tools being then the Gibbs measures (see Section 10).

To keep this paper to a reasonable length, we have chosen not to develop the related counting problems of closed geodesics with lengths at most s, which have been studied extensively (see for instance [Bowe, PPo, Par, Rob2, PPS], and the surveys [Bab2, Sha] and their references).

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2 Geometry and dynamics in negative curvature

In this section, we survey briefly the required background on the geometry and dynamics of negatively curved Riemannian manifolds, considered as locally CAT($-\kappa$) spaces (using for instance [BH] as a general reference), with a particular emphasis on the metric aspects and the regularity properties.

For every $n \geq 2$, we denote by $\mathbb{H}^n_{\mathbb{R}}$ any model of the real hyperbolic space of dimension n and constant sectional curvature -1.

For every $\epsilon > 0$, we denote by $\mathscr{N}_\epsilon A$ the closed ϵ-neighbourhood of a subset A of a metric space, and by convention $\mathscr{N}_0 A = \overline{A}$. Recall that a map $f : X \to Y$ between two metric spaces is called *α-Hölder*, where $\alpha \in]0, 1]$, if there exists $c > 0$ such that for every x, y in X with $d(x, y) \leq 1$, we have $d(f(x), f(y)) \leq c\, d(x, y)^\alpha$, and *Hölder* if there exists $\alpha \in]0, 1]$ such that f is α-Hölder.

2.1 Geometry of the unit tangent bundle

We denote by $\pi : TN \to N$ the tangent bundle of any (smooth) Riemannian manifold N, and again by $\pi : T^1N \to N$ its unit tangent bundle. Recall that the Levi-Civita connection

∇ of N gives a decomposition $TTN = V \oplus H$ of the vector bundle $TTN \to TN$ into the direct sum of two smooth vector subbundles $V \to TN$ and $H \to TN$, called vertical and horizontal, such that if $\pi_V : TTN \to V$ is the linear projection of TTN onto V parallelly to H, if H_v and V_v are the fibers of H and V above $v \in TN$, then

- we have $V_v = \operatorname{Ker} T_v\pi = T_v(T_{\pi(v)}N) = T_{\pi(v)}N$;
- the restriction $T\pi_{|H_v} : H_v \to T_{\pi(v)}N$ of the tangent map of π to H_v is a linear isomorphism;
- for every smooth vector field $X : N \to TN$ on N, we have $\nabla_v X = \pi_V \circ TX(v)$.

The manifold TN has a unique Riemannian metric, called *Sasaki's metric*, such that for every $v \in TM$, the map $T\pi_{|H_v} : H_v \to T_{\pi(v)}N$ is isometric, the restriction to V_v of Sasaki's scalar product is the Riemannian scalar product on $T_{\pi(v)}N$, and the decomposition $T_vTN = V_v \oplus H_v$ is orthogonal. We endow the smooth submanifold T^1N of TN with the induced Riemannian metric, also called *Sasaki's metric.* The fiber T^1_xN of every $x \in N$ is then isometric to the standard unit sphere $\mathbb{S}^{n-1}$ of the standard Euclidean space $\mathbb{R}^n$, if n is the dimension of N.

The Riemannian measure $d\operatorname{Vol}_{T^1N}$ of T^1N, called *Liouville's measure*, disintegrates under the fibration $\pi : T^1N \to N$ over the Riemannian measure $d\operatorname{Vol}_N$ of N, as

$$d\operatorname{Vol}_{T^1N} = \int_{x\in N} d\operatorname{Vol}_{T^1_xN}\; d\operatorname{Vol}_N(x)\ ,$$

where $d\operatorname{Vol}_{T^1_xN}$ is the spherical measure on the fiber T^1_xN of π above $x \in N$. In particular,

$$\operatorname{Vol}(T^1N) = \operatorname{Vol}(\mathbb{S}^{n-1})\operatorname{Vol}(N)\ .$$

2.2 Hölder structure on the boundary at infinity

Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with dimension at least 2 and sectional curvature at most -1, and let $x_0 \in \widetilde{M}$. To shorten the exposition, we assume in this survey that $\widetilde{M}$ has pinched negative sectional curvature $-b^2 \le K \le -1$ (where $b \in\ ]0,+\infty[$), though this is not necessary except when working with Gibbs measures in Section 10, see [PP6, PP7] for the extensions.

We denote by $\partial_\infty\widetilde{M}$ the boundary at infinity of $\widetilde{M}$, with its usual Hölder structure and conformal structure, which we describe below. Recall that a *Hölder structure* on a topological manifold X is a maximal atlas of charts (U,φ), where $\varphi : U \to \varphi(U)$ is a homeomorphism between an open subset U of X and an open subset of a fixed smooth manifold, such that the transition maps are α-Hölder homeomorphisms for some $\alpha > 0$.

Two geodesic rays $\rho,\rho' : [0,+\infty[\ \to \widetilde{M}$ are *asymptotic* if their images are at finite Hausdorff distance, or equivalently if there exists $c > 0$ and $t_0 \in \mathbb{R}$ such that $d(\rho(t),\rho'(t+t_0)) \le c\, e^{-t}$ for all $t \in [\max\{0,-t_0\},+\infty[$. The *boundary at infinity* $\partial_\infty\widetilde{M}$ of $\widetilde{M}$ is the quotient topological space of the space of geodesic rays, endowed with the compact-open topology, by the equivalence relation "to be asymptotic to". The asymptotic class of a geodesic ray is called its *point at infinity.* For all $x \in M$ and $\xi \in \partial_\infty\widetilde{M}$, there exists a unique geodesic ray with origin x and point at infinity ξ, whose image we denote by $[x,\xi[$. Given two distinct points at infinity $\xi,\eta \in \partial_\infty\widetilde{M}$, there exists a unique (up to translation on the source) geodesic line $\rho : \mathbb{R} \to \widetilde{M}$ such that the points at infinity of the geodesic rays $t \mapsto \rho(-t)$ and $t \mapsto \rho(t)$, $t \in [0,+\infty[$, are ξ and η, respectively, and we denote its image by $]\xi,\eta[$.

For every $x \in \widetilde{M}$, the map θ_x from $T^1_x\widetilde{M}$ to $\partial_\infty\widetilde{M}$, which sends $v \in T^1_x\widetilde{M}$ to the point at infinity of the geodesic ray with tangent vector at the origin v, is a homeomorphism. We define the angle $\angle_x(y,z)$ of two geodesic segments or rays with the same origin x and endpoints $y, z \in (\widetilde{M} - \{x\}) \cup \partial_\infty\widetilde{M}$ as the angle of their tangent vectors at x. The disjoint union $\widetilde{M} \cup \partial_\infty\widetilde{M}$ has a unique compact metrisable topology, inducing the original topologies on $\widetilde{M}$ and on $\partial_\infty\widetilde{M}$, such that a sequence of points $(y_n)_{n\in\mathbb{N}}$ in $\widetilde{M}$ converges to a point $\xi \in \partial_\infty\widetilde{M}$ if and only if $\lim_{n\to+\infty} d(y_n, x_0) = +\infty$ and $\lim_{n\to+\infty} \angle_{x_0}(y_n, \xi) = 0$. An isometry γ of $\widetilde{M}$ uniquely extends to a homeomorphism of $\widetilde{M} \cup \partial_\infty\widetilde{M}$, and we will also denote by γ its extension to the boundary at infinity.

Since $\widetilde{M}$ has pinched negative curvature, an easy comparison argument shows that the maps $\theta_x^{-1} \circ \theta_{x'} : T^1_{x'}M \to T^1_x M$, for all $x, x' \in \widetilde{M}$, are α-Hölder homeomorphisms for some $\alpha > 0$. Hence there is a unique Hölder structure on the topological manifold $\partial_\infty\widetilde{M}$ such that θ_x is a Hölder homeomorphism, for every $x \in \widetilde{M}$. Furthermore, the isometries of $\widetilde{M}$ are α-Hölder homeomorphisms of $\partial_\infty\widetilde{M}$ for some $\alpha > 0$.

2.3 Conformal structure on the boundary at infinity

Let us now define the natural conformal structure on $\partial_\infty\widetilde{M}$. Recall that two distances d and δ on a set Z are called *conformally equivalent* if they induce the same topology and if for every $z_0 \in Z$, the limit $\lim_{x\to z_0, x\neq z_0} \frac{d(x,z_0)}{\delta(x,z_0)}$ exists and is strictly positive. The relation "to be conformally equivalent to" is an equivalence relation on the set of distances on Z, and a *conformal structure* on Z is an equivalence class thereof.

Let Z and Z' be two sets endowed with a conformal structure, and let d and d' be distances on Z and Z' representing them. A bijection $\gamma : Z \to Z'$ is *conformal* if the distances d and $\gamma^*d' : (x,y) \mapsto d'(\gamma x, \gamma y)$ are conformally equivalent, that is, if for every $z_0 \in Z$, the limit $\lim_{x\to z_0, x\neq z_0} \frac{d'(\gamma x,\gamma z_0)}{d(x,z_0)}$ exists and is strictly positive. This does not depend on the choice of representatives d and d' of the conformal structures of Z and Z'.

The *Busemann cocycle* of $\widetilde{M}$ is the Hölder continuous map $\beta\colon \widetilde{M} \times \widetilde{M} \times \partial_\infty\widetilde{M} \to \mathbb{R}$ defined by

$$(x, y, \xi) \mapsto \beta_\xi(x, y) = \lim_{t\to+\infty} d(\rho_t, x) - d(\rho_t, y) \;,$$

where $\rho : t \mapsto \rho_t$ is any geodesic ray with point at infinity ξ. The above limit exists and is independent of ρ. The Busemann cocycle satisfies the following equivariance and cocycle properties:

$$\beta_{\gamma\xi}(\gamma x, \gamma y) = \beta_\xi(x, y) \quad \text{and} \quad \beta_\xi(x, y) + \beta_\xi(y, z) = \beta_\xi(x, z) \;, \tag{1}$$

for all $\xi \in \partial_\infty\widetilde{M}$, all $x, y, z \in \widetilde{M}$ and every isometry γ of $\widetilde{M}$, and in particular $\beta_\xi(y, x) = -\beta_\xi(x, y)$. By the triangular inequality, we have

$$|\beta_\xi(x, y)| \leq d(x, y) \;. \tag{2}$$

If y is a point in the (image of the) geodesic ray from x to ξ, then $\beta_\xi(x, y) = d(x, y)$. For every $y \in \widetilde{M}$ and $\xi \in \partial_\infty\widetilde{M}$, the map $x \mapsto \beta_\xi(x, y)$ is smooth and 1-Lipschitz.

For every $\xi \in \partial_\infty\widetilde{M}$, the *horospheres centered at* ξ are the level sets of the map $y \mapsto \beta_\xi(y, x_0)$ from $\widetilde{M}$ to $\mathbb{R}$, and the (closed) *horoballs centered at* ξ are its sublevel sets. A horosphere centered at ξ is a smooth hypersurface of $\widetilde{M}$, orthogonal to the geodesic lines having ξ as a point at infinity. In the upper halfspace model of the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$, the horospheres are the horizontal affine hyperplanes therein or the Euclidean spheres therein tangent to the horizontal coordinate hyperplane, with the point of tangency removed. In the ball model of the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$, the horospheres (respectively horoballs) are the Euclidean spheres (respectively balls) tangent to the unit sphere and contained in the unit ball, with the point of tangency removed.

The horoballs are limits of big balls (and their centres the limits of the centres thereof, explaining the terminology). More precisely, if ρ is a geodesic ray in $\widetilde{M}$ with point at infinity ξ, if $B(t)$ is the ball of centre $\rho(t)$ and radius t, then the map $t \mapsto B(t)$ converges to the horoball HB of centre ξ containing $\rho(0)$ in its boundary, for the Hausdorff convergence on compact subsets (that is, for every compact subset K of $\widetilde{M}$, as $t \to +\infty$, the closed subset $(B(t) \cap K) \cup \overline{{}^c K}$ converges to the closed subset $(HB \cap K) \cup \overline{{}^c K}$ for the Hausdorff distance).

For every $x \in \widetilde{M}$, for all distinct $\xi, \eta \in \partial_\infty \widetilde{M}$, the *visual distance* between ξ and η seen from x is

$$d_x(\xi, \eta) = \lim_{t \to +\infty} e^{-\frac{1}{2}\big(d(x,\, \rho_\xi(t)) + d(x,\, \rho_\eta(t)) - d(\rho_\xi(t),\, \rho_\eta(t))\big)} ,$$

for any geodesic ray ρ_ξ and ρ_η with point at infinity ξ and η, respectively. Equivalently, with $t \mapsto \rho_t$ and $t \mapsto \rho'_t$ the geodesic rays with origin x converging to ξ and η, we have

$$d_x(\xi, \eta) = \lim_{t \to +\infty} e^{\frac{1}{2} d(\rho_t,\, \rho'_t) - t} .$$

Again equivalently, if u is any point on the geodesic line between ξ and η, then

$$d_x(\xi, \eta) = e^{-\frac{1}{2}(\beta_\xi(x,\, u) + \beta_\eta(x,\, u))} . \tag{3}$$

Define $d_x(\xi, \eta) = 0$ if $\xi = \eta$.

For every $x \in \widetilde{M}$, the above limits exist and the three formulas coincide. The map $d_x : \partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M} \to [0, +\infty[$ is a distance, inducing the original topology on $\partial_\infty \widetilde{M}$. For all $x, y \in \widetilde{M}$, for all distinct $\xi, \eta \in \partial_\infty \widetilde{M}$, and for every isometry γ of $\widetilde{M}$we have

$$e^{-d(x,\,]\xi,\, \eta[)} \leq d_x(\xi, \eta) \leq (1 + \sqrt{2})\, e^{-d(x,\,]\xi,\, \eta[)} ,$$

$$\frac{d_x(\xi, \eta)}{d_y(\xi, \eta)} = e^{-\frac{1}{2}(\beta_\xi(x,\, y) + \beta_\eta(x,\, y))} , \tag{4}$$

$$d_{\gamma x}(\gamma \xi, \gamma \eta) = d_x(\xi, \eta) . \tag{5}$$

It follows from Equation (4) that the visual distances d_x for $x \in \widetilde{M}$ belong to the same conformal structure on $\partial_\infty \widetilde{M}$. It follows from Equation (4) and Equation (5) that the (boundary extensions of) the isometries of $\widetilde{M}$ are conformal bijections for this conformal structure. Furthermore, these equations and Equation (2) imply that the isometries of $\widetilde{M}$ are bilipschitz homeomorphisms for any visual distance: for all $x \in \widetilde{M}$, for all distinct $\xi, \eta \in \partial_\infty \widetilde{M}$, we have

$$e^{-2\, d(x, \gamma x)}\ d_x(\xi, \eta) \leq d_x(\gamma \xi, \gamma \eta) \leq e^{2\, d(x, \gamma x)}\ d_x(\xi, \eta) .$$

2.4 Stable and unstable leaves of the geodesic flow

We now turn to the description of the dynamics of the geodesic flow on $\widetilde{M}$.

The unit tangent bundle T^1N of a complete Riemannian manifold N can be identified with the set of locally geodesic lines (parametrised by arclength) $\ell\colon \mathbb{R} \to N$ in N, endowed with the compact-open topology. More precisely, we identify a locally geodesic line ℓ and its (unit) tangent vector $\dot{\ell}(0)$ at time $t = 0$ and, conversely, any $v \in T^1N$ is the tangent vector at time $t = 0$ of a unique locally geodesic line. We will use this identification without mention in this survey. In particular, the basepoint projection $\pi\colon T^1N \to N$ is given by $\pi(\ell) = \ell(0)$.

The *geodesic flow* on T^1N is the smooth 1-parameter group $(g^t)_{t\in\mathbb{R}}$, where $g^t\ell\,(s) = \ell(s+t)$, for all $\ell \in T^1N$ and $s, t \in \mathbb{R}$. We denote by $\iota : T^1N \to T^1N$ the *antipodal (flip) map* $v \mapsto -v$. We have $\iota \circ g^t = g^{-t} \circ \iota$. The isometry group of N acts on the space of geodesic lines in N by postcomposition: $(\gamma, \ell) \mapsto \gamma \circ \ell$, and this action commutes with the geodesic flow and the antipodal map.

For every unit tangent vector $v \in T^1\widetilde{M}$, let $v_- = v(-\infty)$ and $v_+ = v(+\infty)$ be the two endpoints in the sphere at infinity of the geodesic line defined by v. Let $\partial^2_\infty \widetilde{M}$ be the open subset of $\partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M}$ which consists of pairs of distinct points at infinity, with the restriction of the product Hölder structure. *Hopf's parametrisation* (see [Hop]) of $T^1\widetilde{M}$ is the Hölder homeomorphism from $T^1\widetilde{M}$ to $\partial^2_\infty \widetilde{M} \times \mathbb{R}$ sending $v \in T^1\widetilde{M}$ to the triple $(v_-, v_+, t) \in \partial^2_\infty \widetilde{M} \times \mathbb{R}$, where t is the signed (algebraic) distance of $\pi(v)$ from the closest point p_{v,x_0} to x_0 on the (oriented) geodesic line defined by v. In this survey, we will identify an element of $T^1\widetilde{M}$ with its image by Hopf's parametrisation. The geodesic flow acts by $g^s(v_-, v_+, t) = (v_-, v_+, t+s)$ and, for every isometry γ of $\widetilde{M}$, the image of γv is $(\gamma v_-, \gamma v_+, t + t_{\gamma,v})$, where $t_{\gamma,v}$ is the signed distance from $\gamma p_{v,x_0}$ to $p_{\gamma v,x_0}$. Furthermore, in these coordinates, the antipodal map ι is $(v_-, v_+, t) \mapsto (v_+, v_-, -t)$.

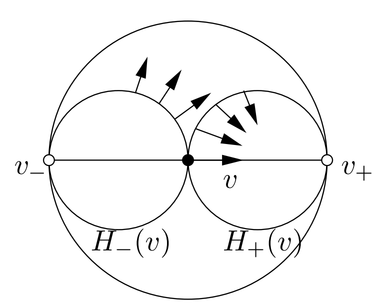
The *strong stable manifold* of $v \in T^1\widetilde{M}$ is

$$W^{\rm ss}(v) = \{v' \in T^1\widetilde{M} : d(\pi(g^t v), \pi(g^t v')) \to 0 \text{ as } t \to +\infty\},$$

and the *strong unstable manifold* of v is

$$W^{\rm ss}(v) = \{v' \in T^1\widetilde{M} : d(\pi(g^t v), \pi(g^t v')) \to 0 \text{ as } t \to -\infty\},$$

The projections in $\widetilde{M}$ of the strong unstable and strong stable manifolds of $v \in T^1\widetilde{M}$, denoted by $H_-(v) = \pi(W^{\rm su}(v))$ and $H_+(v) = \pi(W^{\rm ss}(v))$, are called, respectively, the *unstable and stable horospheres* of v, and are the horospheres containing $\pi(v)$ centered at v_- and v_+, respectively. The unstable horosphere of v coincides with the zero set of the map $x \mapsto f_-(x) = \beta_{v_-}(x, \pi(v))$, and, similarly, the stable horosphere of v coincides with the zero set of $x \mapsto f_+(x) = \beta_{v_+}(x, \pi(v))$. The corresponding sublevel sets $HB_-(v) = f_-^{-1}(]-\infty, 0])$ and $HB_+(v) = f_+^{-1}(]-\infty, 0])$ are called the *unstable and stable horoballs* of v.



The union for $t \in \mathbb{R}$ of the images under g^t of the strong stable manifold of $v \in T^1\widetilde{M}$ is the *stable manifold* $W^{\rm s}(v) = \bigcup_{t\in\mathbb{R}} g^t W^{\rm ss}(v)$ of v, which consists of the elements $v' \in T^1\widetilde{M}$

with $v'_+ = v_+$. Similarly, the union of the images under the geodesic flow at all times of the strong unstable manifold of v is the *unstable manifold* $W^{\rm u}(v)$ of v, which consists of the elements $v' \in T^1\widetilde{M}$ with $v'_- = v_-$.

The subspaces $W^{\rm ss}(v)$ and $W^{\rm su}(v)$ (which are the lifts by the inner and outer unit normal vectors of the unstable and stable horospheres of v, respectively), as well as $W^{\rm s}(v)$ and $W^{\rm u}(v)$, are smooth submanifolds of $T^1\widetilde{M}$. The maps from $\mathbb{R} \times W^{\rm ss}(v)$ to $W^{\rm s}(v)$ defined by $(t, v') \mapsto g^t v'$ and from $\mathbb{R} \times W^{\rm su}(v)$ to $W^{\rm u}(v)$ defined by $(t, v') \mapsto g^t v'$ are smooth diffeomorphisms. We have $W^{\rm ss}(\iota v) = \iota W^{\rm su}(v)$.

Hamenstädt's distance on stable and unstable leaves

For every $v \in T^1\widetilde{M}$, let $d_{W^{\rm ss}(v)}$ be *Hamenstädt's distance* on the strong stable leaf of v, defined as follows (see [Ham1], [HP1, Appendix], as well as [HP3, §2.2] for a generalisation when the horosphere $H_+(v)$ is replaced by the boundary of any nonempty closed convex subset): for all $w, w' \in W^{\rm ss}(v)$, we have

$$d_{W^{\rm ss}(v)}(w, w') = \lim_{t \to +\infty} e^{\frac{1}{2} d(w(-t),\, w'(-t)) - t} .$$

This limit exists, and Hamenstädt's distance is a distance inducing the original topology on $W^{\rm ss}(v)$. For all $w, w' \in W^{\rm ss}(v)$ and for every isometry γ of $\widetilde{M}$, we have

$$d_{W^{\rm ss}(\gamma v)}(\gamma w, \gamma w') = d_{W^{\rm ss}(v)}(w, w') .$$

For all $v \in T^1\widetilde{M}$, $s \in \mathbb{R}$ and $w, w' \in W^{\rm ss}(v)$, we have

$$d_{W^{\rm ss}(g^s v)}(g^s w, g^s w') = e^{-s} d_{W^{\rm ss}(v)}(w, w') .$$

For every horosphere H in $\widetilde{M}$ with center H_∞, we also have a distance d_H on the open subset $\partial_\infty \widetilde{M} - \{H_\infty\}$ defined by

$$d_H(\xi, \eta) = \lim_{t \to +\infty} e^t \, d_{\rho_t}(\xi, \eta) = \lim_{t \to +\infty} e^{\frac{1}{2} d(\xi_t,\, \eta_t) - t} ,$$

where $t \mapsto \rho_t$ is any geodesic ray with origin a point of H and point at infinity H_∞, and $t \mapsto \xi_t$ and $t \mapsto \eta_t$ are the geodesic lines in $\widetilde{M}$ with origin H_∞, passing at time $t = 0$ through H, and with endpoints ξ and η, respectively. Using the homeomorphism from $W^{\rm ss}(v)$ to $\partial_\infty \widetilde{M} - \{v_+\}$ defined by $w \mapsto w_-$, we have

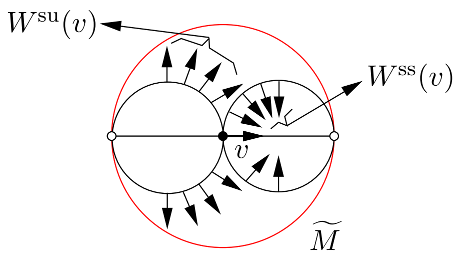
$$d_{W^{\rm ss}(v)}(w, w') = d_{H_+(v)}(w_-, w'_-) .$$

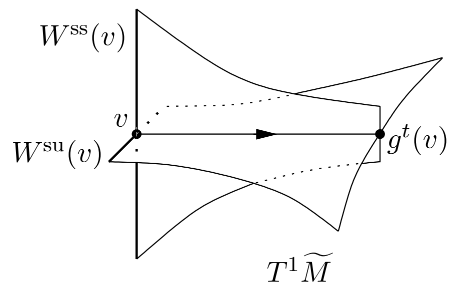
The distance d_H and the restriction of any visual distance to $\partial_\infty \widetilde{M} - \{H_\infty\}$ are conformally equivalent, since for all $x \in H$ and $\xi, \eta \in \partial_\infty \widetilde{M} - \{H_\infty\}$, with the above notation $t \mapsto \xi_t$ and $t \mapsto \eta_t$, we have

$$\frac{d_H(\xi, \eta)}{d_x(\xi, \eta)} = e^{-\frac{1}{2}(\beta_\xi(\xi_0, x) + \beta_\eta(\eta_0, x))} .$$

The Anosov property of the geodesic flow

The strong stable manifolds, stable manifolds, strong unstable manifolds and unstable manifolds are the (smooth) leaves of Hölder continuous foliations on $T^1\widetilde{M}$, invariant under the geodesic flow and the isometry group of $\widetilde{M}$, denoted by $\mathscr{W}^{\rm ss}$, $\mathscr{W}^{\rm s}$, $\mathscr{W}^{\rm su}$ and $\mathscr{W}^{\rm u}$, respectively. When $\widetilde{M}$ is a symmetric space (that is, up to homothety, when $\widetilde{M}$ is isometric to the real, complex, quaternionic hyperbolic n-space or to the octonionic hyperbolic plane), then the strong stable, stable, strong unstable and unstable foliations are smooth. But in general, the Hölder regularity cannot be much improved, as we will explain in Section 2.5.





Let $N = T^1\widetilde{M}$. The vector field $Z : N \to TN$ defined by $v \mapsto Z(v) = \frac{d}{dt}g^t(v)$ is called the *geodesic vector field*. The geodesic flow $(g^t)_{t\in\mathbb{R}}$ on the Riemannian manifold N is a *contact Anosov flow*. That is, the vector bundle $TN \to N$ is the direct sum of three topological vector subbundles $TN = E_{\rm su} \oplus E_0 \oplus E_{\rm ss}$ that are invariant under $(g^t)_{t\in\mathbb{R}}$, where $E_0 \cap T_vN = \mathbb{R}Z(v)$, $E_{su} \cap T_vN = T_vW^{su}(v)$, $E_{ss} \cap T_vN = T_vW^{ss}(v)$, and there exist two constants $c, \lambda > 0$ such that for every $t > 0$, we have (see the above picture on the right)

$$\|T_v\, g^t{}_{|E_{\rm ss}}\| \le c\, e^{-\lambda t} \;\text{ and }\; \|T_v\, g^{-t}{}_{|E_{\rm su}}\| \le c\, e^{-\lambda t}\;.$$

Furthermore, if α is the differential 1-form on N defined by $\alpha_{|E_{\rm su}\oplus E_{\rm ss}} = 0$ and $\alpha(Z) = 1$, called *Liouville's* 1*-form*, then $\alpha \wedge (d\alpha)^{n-1}$, where n is the dimension of M, is a volume form on N, which is invariant under the geodesic flow. Thus, the strong stable leaves are contracted by the geodesic flow, and the strong unstable leaves are dilated. See for instance [KH] for more information.

2.5 Discrete isometry groups

Let Γ be a discrete group of isometries of $\widetilde{M}$, which is *nonelementary*, that is, it does not preserve a set of one or two points in $\widetilde{M} \cup \partial_\infty \widetilde{M}$. To shorten the exposition, we will assume in this survey that Γ has no torsion, though this assumption is not necessary (see [PP5, PP6, PP7] for the extension), and is useful for some arithmetic applications.

Let us denote the quotient space of $\widetilde{M}$ under Γ by $M = \Gamma\backslash\widetilde{M}$, which is a smooth Riemannian manifold since Γ is torsion free. We also say that the manifold M is *nonelementary* if Γ is nonelementary.

We denote by $\Lambda\Gamma$ the *limit set* of Γ, that is, the set of accumulation points in $\partial_\infty\widetilde{M}$ of any orbit Γx of a point x of $\widetilde{M}$ under Γ. It is the smallest nonempty closed Γ-invariant subset of $\partial_\infty\widetilde{M}$. The *critical exponent* of Γ is

$$\delta_\Gamma = \lim_{n\to+\infty} \frac{1}{n} \ln \operatorname{Card}\{\gamma \in \Gamma : d(x_0, \gamma x_0) \le n\}.$$

The above limit exists (see [Rob1]), and the critical exponent is positive (since Γ is nonelementary), finite (since M has a finite lower bound on its sectional curvatures, see for instance [Bowd]), and independent of the basepoint x_0.

Since Γ acts without fixed points on $\widetilde{M}$, we have an identification $\Gamma\backslash T^1\widetilde{M} = T^1M$, and we again denote by $\mathscr{W}^{\rm ss}, \mathscr{W}^{\rm s}, \mathscr{W}^{\rm su}$ and $\mathscr{W}^{\rm u}$ the Hölder continuous foliations of T^1M induced by the corresponding ones in $T^1\widetilde{M}$. We use the notation $(g^t)_{t\in\mathbb{R}}$ also for the geodesic flow on T^1M. We again denote by $\iota : T^1M \to T^1M$ the antipodal (flip) map $v \mapsto -v$, which also anti-commutes with the geodesic flow.

Let us conclude this section by explaining some rigidity results on the regularity of the foliations $\mathscr{W}^{\rm ss}, \mathscr{W}^{\rm s}, \mathscr{W}^{\rm su}$ and $\mathscr{W}^{\rm u}$. Anosov has proved that if M is compact, then the vector subbundles $E_{\rm su}$ and $E_{\rm ss}$ are Hölder continuous. If M is a compact surface, Hurder and Katok [HK, Theo. 3.1, Coro. 3.5] have proved that these subbundles are $\mathrm{C}^{1,\alpha}$ for every $\alpha \in \,]0,1[$ (see also [HPu]), and that if they are $\mathrm{C}^{1,1}$, then they are C^∞. Ghys [Ghy, p. 267] has proved that if M is a compact surface, and if the stable foliation of T^1M is C^2, then the geodesic flow is C^∞-conjugated to the geodesic flow of a hyperbolic surface. In higher dimension, we have the following result.

Theorem 1 (Benoist-Foulon-Labourie [BFL]) *Let M be a compact negatively curved Riemannian manifold. If the stable foliation of T^1M is smooth, then the geodesic flow of M is C^∞-conjugated to the geodesic flow of a Riemannian symmetric manifold with negative curvature.* □

3 Common perpendiculars of convex sets

Let M be a complete nonelementary connected Riemannian manifold of dimension at least 2, with pinched negative curvature $-b^2 \le K \le -1$. Let $\widetilde{M} \to M$ be a universal Riemannian cover of M, so that $\widetilde{M}$ is complete simply connected with the same curvature bound, and let Γ be its covering group, so that Γ is a discrete, torsionfree, nonelementary group of isometries of $\widetilde{M}$.

3.1 Convex subsets

Let $\widetilde{C}$ be a nonempty closed convex subset of $\widetilde{M}$ (recall that a subset A of $\widetilde{M}$ is said to be *convex* if (the image of) any geodesic segment with endpoints in A is contained in A). We denote by $\partial\widetilde{C}$ the boundary of $\widetilde{C}$ in $\widetilde{M}$ and by $\partial_\infty\widetilde{C}$ its set of points at infinity (the set of endpoints of geodesic rays contained in $\widetilde{C}$). We say that the Γ-orbit of $\widetilde{C}$ is *locally finite* if, with $\Gamma_{\widetilde{C}}$ the stabiliser of $\widetilde{C}$ in Γ, for every compact subset K of $\widetilde{M}$, the number of right cosets $[\gamma] \in \Gamma/\Gamma_{\widetilde{C}}$ such that $\gamma\widetilde{C}$ meets K is finite.

Natural examples of convex subsets of $\widetilde{M}$ include the points, the balls, the horoballs, the totally geodesic subspaces of $\widetilde{M}$ and the convex hulls in $\widetilde{M}$ of the limit sets of nonelementary subgroups of Γ. Recall that the *convex hull* of a subset A of $\partial_\infty\widetilde{M}$ with at least two points is the smallest closed convex subset of $\widetilde{M}$ that contains A in its set of points at infinity.

Let $P_{\widetilde{C}} : \widetilde{M} \cup (\partial_\infty\widetilde{M} - \partial_\infty\widetilde{C}) \to \widetilde{C}$ be the closest point map: if $\xi \in \partial_\infty\widetilde{M} - \partial_\infty\widetilde{C}$, then $P_{\widetilde{C}}(\xi)$ is defined to be the unique point in $\widetilde{C}$ that minimises the map $x \mapsto \beta_\xi(x, x_0)$ from $\widetilde{C}$ to $\mathbb{R}$. For every isometry γ of $\widetilde{M}$, we have $P_{\gamma\widetilde{C}} \circ \gamma = \gamma \circ P_{\widetilde{C}}$. The closest point map is continuous in the topology of $\widetilde{M} \cup \partial_\infty\widetilde{M}$.

Let $\partial^1_+\widetilde{C}$ be the subset of $T^1\widetilde{M}$ consisting of the geodesic lines $v\colon \mathbb{R} \to \widetilde{M}$ with $v(0) \in \partial\widetilde{C}$, $v_+ \notin \partial_\infty\widetilde{C}$ and $P_{\widetilde{C}}(v_+) = v(0)$. Note that $\pi(\partial^1_+\widetilde{C}) = \partial\widetilde{C}$, and that for all isometries

γ of $\widetilde{M}$, we have $\partial^1_+(\gamma\widetilde{C}) = \gamma\,\partial^1_+\widetilde{C}$. In particular, $\partial^1_+\widetilde{C}$ is invariant under the isometries of $\widetilde{M}$ that preserve $\widetilde{C}$. When $\widetilde{C} = HB_-(v)$ is the unstable horoball of $v \in T^1\widetilde{M}$, then $\partial^1_+\widetilde{C}$ is the strong unstable manifold $W^{\rm su}(v)$ of v, and similarly, $W^{\rm ss}(v) = \iota\,\partial^1_+ HB_+(v)$.

The restriction of $P_{\widetilde{C}}$ to $\partial_\infty\widetilde{M} - \partial_\infty\widetilde{C}$ (which is not necessarily injective) has a unique lift to a homeomorphism

$$\nu P_{\widetilde{C}} : \partial_\infty\widetilde{M} - \partial_\infty\widetilde{C} \to \partial^1_+\widetilde{C}$$

such that $\pi\circ\nu P_{\widetilde{C}} = P_{\widetilde{C}}$. The inverse of $\nu P_{\widetilde{C}}$ is the map $v \mapsto v_+$ from $\partial^1_+\widetilde{C}$ to $\partial_\infty\widetilde{M} - \partial_\infty\widetilde{C}$. In particular, $\partial^1_+\widetilde{C}$ is a Hölder submanifold of $T^1\widetilde{M}$. For every $s \geq 0$, the geodesic flow induces a homeomorphism $g^s\colon \partial^1_+\widetilde{C} \to \partial^1_+\mathscr{N}_s\widetilde{C}$. For every isometry γ of $\widetilde{M}$, we have $\nu P_{\gamma\widetilde{C}}\circ\gamma = \gamma\circ\nu P_{\widetilde{C}}$. When $\widetilde{C}$ has nonempty interior and $\mathrm{C}^{1,1}$ boundary, then $\partial^1_+\widetilde{C}$ is the Lipschitz submanifold of $T^1\widetilde{M}$ consisting of the outer unit normal vectors to $\partial\widetilde{C}$, and the map $P_{\widetilde{C}}$ itself is a homeomorphism (between $\partial_\infty\widetilde{M} - \partial_\infty\widetilde{C}$ and $\partial\widetilde{C}$). This holds when $\widetilde{C}$ is the closed η-neighbourhood of any nonempty convex subset of $\widetilde{M}$ with $\eta > 0$) (see [Fed, Theo. 4.8(9)], [Wal, p. 272]).

We define a *properly immersed locally convex subset C of M* as the data of a nonempty proper closed convex subset $\widetilde{C}$ of $\widetilde{M}$, with locally finite Γ-orbit, and of the locally isometric proper immersion $C = \Gamma_{\widetilde{C}}\backslash\widetilde{C} \to M$ induced by the inclusion of $\widetilde{C}$ in $\widetilde{M}$ and the Riemannian covering map $\widetilde{M} \to M$. (To simplify the exposition, we don't allow in this survey the replacement of $\Gamma_{\widetilde{C}}$ by one of its finite index subgroup, but it is sometimes useful.) By abuse, when no confusion is possible, we will again denote by C the image of this immersion. We define $\partial^1_+C = \Gamma_{\widetilde{C}}\backslash\partial^1_+\widetilde{C}$, which comes with a proper immersion $\partial^1_+C \to T^1M$ induced by the inclusion of $\partial^1_+\widetilde{C}$ in $T^1\widetilde{M}$ and the covering map $T^1\widetilde{M} \to T^1M$. By abuse also, we will again denote by ∂^1_+C the image of this immersion.

3.2 The general counting problem

Let C_+, C_- be two properly immersed locally convex subsets of M. A locally geodesic path $c : [0,T] \to M$ is a *common perpendicular* from C_- to C_+ if $\dot c(0) \in \partial^1_+C_-$ and $\dot c(T) \in \iota\,\partial^1_+C_+$. For every $s \geq 0$, we denote by $\mathrm{Perp}_{C_-,C_+}(s)$ the set of common perpendiculars from C_- to C_+ of length at most s. Each common perpendicular c from C_- to C_+ has a *multiplicity* $m(c)$, defined as follows. If C_- and C_+ are the images in M of two nonempty proper closed convex subsets $\widetilde{C}_-$ and $\widetilde{C}_+$ of $\widetilde{M}$ with locally finite Γ-orbits, respectively, then $m(c)$ is the number of (left) orbits under Γ of pairs $([\alpha],[\beta])$ in $\Gamma/\Gamma_{\widetilde{C}_-}\times\Gamma/\Gamma_{\widetilde{C}_+}$ such that the closed convex subsets $\alpha\widetilde{C}_-$ and $\beta\widetilde{C}_+$ have a common perpendicular (it is then unique) whose image by $\widetilde{M} \to M$ is c. (Multiplicities are also useful when Γ is allowed to have torsion and when the stabilizers $\Gamma_{\widetilde{C}_\pm}$ are replaced by finite index subgroups.)

In particular, any locally geodesic path is a common perpendicular of its endpoints (with multiplicity 1), since the outer unit normal bundle of a point equals its unit tangent sphere. If C_- and C_+ have nonempty interior and $\mathrm{C}^{1,1}$ smooth boundary (in the appropriate sense for immersed subsets), the above definition of common perpendicular agrees with the usual definition: a common perpendicular exits C_- perpendicularly to the boundary of C_- at its initial point and it enters C_+ perpendicularly to the boundary of C_+ at its terminal point.

We study in this survey the asymptotic behaviour, as $s \to +\infty$, of the number

$$\mathscr{N}(s) = \mathscr{N}_{C_-,\,C_+}(s) = \sum_{c \in \mathrm{Perp}_{C_-,\,C_+}(s)} m(c)$$

of common perpendiculars, counted with multiplicities, from C_- to C_+, of length at most s.

Problems of this kind have been studied in various forms in the literature since the 1950's and in a number of recent works, sometimes in a different guise, as demonstrated in the examples below. These examples indicate that the general form of the counting results is $\mathscr{N}(s) \sim \kappa\, e^{\delta s}$, where $\delta = \delta_\Gamma$ is the critical exponent of Γ and $\kappa > 0$ is a constant. The notation $f(s) \sim g(s)$ (as $s \to \infty$) means as usual that $g(s) \neq 0$ for s big enough, and that the ratio $\frac{f(s)}{g(s)}$ converges to 1 as $s \to \infty$.

Observing that for $t \geq 2\epsilon$, we have

$$\mathscr{N}_{\mathscr{N}_\epsilon(C_-),\,\mathscr{N}_\epsilon(C_+)}(t-2\epsilon) \leq \mathscr{N}_{C_-,\,C_+}(t) \leq \mathscr{N}_{\mathscr{N}_\epsilon(C_-),\,\mathscr{N}_\epsilon(C_+)}(t-2\epsilon) + \mathscr{N}_{C_-,\,C_+}(2\epsilon)\ ,$$

we could replace the convex sets C_- and C_+ by their ϵ-neighbourhoods for some fixed (small) positive ϵ, and then assume that C_- and C_+ have smooth enough boundaries and use the more conventional definition of common perpendicular. However, it is more natural to work directly with the given convex sets instead of, for example, replacing points by small balls.

3.3 Counting orbit points in a ball

If $C_- = \{\bar{p}\}$ and $C_+ = \{\bar{q}\}$ are singletons in M, then

$$\mathscr{N}(s) = \mathrm{Card}\big(B(p,s) \cap \Gamma q\big)\ ,$$

for any lifts p and q of $\bar{p}$ and $\bar{q}$ in $\widetilde{M}$. When $\widetilde{M} = \mathbb{H}^2_{\mathbb{R}}$ and M is compact and orientable, Huber [Hub, Satz 3] proved that

$$\mathscr{N}(s) \sim \frac{1}{4(g-1)} e^s,$$

where g is the genus of M. The proof uses the Dirichlet series $\sum_{\gamma\in\Gamma} \cosh^{-s} d(p,\gamma q)$ and the Tauberian theorem of Wiener-Ikehara [Wie].

Margulis [Mar1, Theo. 2] (see also [Pol]) generalised Huber's result for all compact connected negatively curved manifolds of arbitrary dimension $n \geq 2$. He showed that

$$\mathscr{N}(s) \sim c(p,q) e^{\delta s}$$

for some constant $c(p,q)$ which depends continuously on p and q, and if $\widetilde{M} = \mathbb{H}^n_{\mathbb{R}}$, then

$$\mathscr{N}(s) \sim \frac{\mathrm{Vol}(\mathbb{S}^{n-1})}{2^{n-1}(n-1)\,\mathrm{Vol}(M)}\ e^{(n-1)s}\ . \tag{6}$$

This agrees with Huber's result in dimension $n = 2$ because the area of a compact genus g surface is $4\pi(g-1)$. Margulis's proof of the above result established the approach

$$\text{mixing} \to \text{equidistribution} \to \text{counting}$$

that has been used in most of the subsequent results. Roblin [Rob2, p. 56] generalised Margulis's result when Γ is nonelementary, the sectional curvature of M is at most -1, and the Bowen-Margulis measure of T^1M is finite, and he has an expression for the constant $c(p,q)$ in terms of the Patterson density and the Bowen-Margulis measure, see Section 8 for more details. Lax and Phillips [LP] obtained an expression with error bounds for the asymptotic behaviour of $\mathscr{N}(s)$ in terms of the eigenvalues of the Laplacian on $\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$ when Γ is *geometrically finite* (that is, in constant curvature, when Γ is nonelementary and has a fundamental polyhedron with finitely many sides).

3.4 Counting common perpendiculars from a point to a totally geodesic submanifold

Herrmann [Herr, Theo. I] proved for $\widetilde{M}=\mathbb{H}^n_{\mathbb{R}}$, M compact, $C_-=\{p\}$ a singleton, C_+ a compact totally geodesic submanifold of dimension k, an asymptotic estimate

$$\mathscr{N}(s)\sim\frac{2}{n-1}\,\frac{\pi^{(n-k)/2}}{\Gamma(\frac{n-k}{2})}\,\frac{\operatorname{Vol}(C_+)}{\operatorname{Vol}(M)}\,\frac{e^{(n-1)s}}{2^{n-1}}=\frac{\operatorname{Vol}(\mathbb{S}^{n-k-1})\operatorname{Vol}(C_+)}{2^{n-1}\operatorname{Vol}(M)}\,\frac{e^{(n-1)s}}{n-1}\,, \tag{7}$$

as $s\to+\infty$. Furthermore, he showed that the endpoints of the common perpendiculars on the totally geodesic submanifold C_+ are evenly distributed in terms of the Riemannian measure of C_+. More precisely, if Ω_+ is a measurable subset of C_+ with (Lebesgue) measure 0 boundary, and $\mathscr{N}_{p,\,\Omega_+}(s)$ is the number of those common perpendiculars of $\{p\}$ and C_+ whose terminal endpoints are contained in Ω_+, then, as $s\to+\infty$,

$$\mathscr{N}_{p,\,\Omega_+}(s)\sim\frac{\operatorname{Vol}(\mathbb{S}^{n-k-1})\operatorname{Vol}(\Omega_+)}{2^{n-1}\operatorname{Vol}(M)}\,\frac{e^{(n-1)s}}{n-1}\,. \tag{8}$$

The method of proof was a generalisation of that used by Huber. The asymptotic (7) was also treated in [EM] as an illustration of their equidistribution result, see Theorem 3.

Oh and Shah [OS2] generalised Herrmann's result in dimension 3 for $\widetilde{M}=\mathbb{H}^3_{\mathbb{R}}$ and Γ geometrically finite, and showed that, as $s\to+\infty$,

$$\mathscr{N}(s)\sim c(p,C_+)e^{\delta s}$$

with a constant $c(p,C_+)$ generalising that of Roblin's result described above. Again, we postpone the description of the constant $c(p,C_+)$ until Section 8. This result is used in [OS2] to study Γ-invariant families $(P_i)_{i\in I}$ of possibly intersecting circles in $\mathbb{S}^2$, called "circle packings", that consist of a finite number of Γ-orbits such that the family of totally geodesic planes $(P_i^*)_{i\in I}$ in the ball model of $\mathbb{H}^3_{\mathbb{R}}$ with $\partial_\infty P_i^*=P_i$ is locally finite. They consider the counting function

$$N(T)=\operatorname{Card}\{i\in I\ :\ \operatorname{curv}_{\mathrm{S}}(P_i^*)<T\},$$

where the *spherical curvature* $\operatorname{curv}_{\mathrm{S}}P_i^*$ is the cotangent of the angle, at the origin 0 of the ball model, between the common perpendicular between $\{0\}$ and P_i^*, and any geodesic ray from 0 which is tangent to P_i^* at infinity. Elementary hyperbolic geometry (the angle of parallelism formula, see for instance [Bea, p. 147]) implies that

$$\operatorname{curv}_{\mathrm{S}}P_i^*=\sinh d(0,P_i^*)\ ,$$

and thus, the above asymptotic estimate of $\mathscr{N}(s)$ is equivalent to $N(T)\sim c(p,C_+)(2T)^\delta$ as $T\to+\infty$.

3.5 Counting common perpendiculars between horoballs

When $C_- = C_+ = \mathscr{H}$ is a *Margulis cusp neighbourhood* (that is, the image by $\widetilde{M} \to M$ of a Γ-orbit of horoballs, centered at fixed points of parabolic elements of Γ, with pairwise disjoint interiors), then results of [BHP, HP2, Cos, Rob2] show that if Γ is geometrically finite, then

$$\mathscr{N}(s) \sim c(\mathscr{H})\, e^{\delta s} \;,$$

as $t \to +\infty$, for some $c(\mathscr{H}) > 0$. Cosentino obtained explicit expressions for the constant $c(\mathscr{H})$ in special arithmetic cases: $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ acts on the upper halfplane model of $\widetilde{M} = \mathbb{H}^2_{\mathbb{R}}$ and the quotient space $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$ has a unique cusp that corresponds to the orbit of ∞. The orbit of the subset $\widetilde{\mathscr{H}}$ of $\mathbb{H}^2_{\mathbb{R}}$ consisting of points with imaginary part at least 1 maps under the quotient map to the maximal Margulis cusp neighbourhood of $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$. The Γ-orbit of $\widetilde{\mathscr{H}}$ consists of $\widetilde{\mathscr{H}}$ and of the horoballs based at all rational points $\frac{p}{q}$ with $\gcd(p,q) = 1$ and with Euclidean diameter q^{-2}. The number of such horoballs of diameter q^{-2} modulo the stabiliser of ∞ (consisting of translations by the integers) is $\phi(q)$, where ϕ is Euler's totient function. A classical result of Mertens on the average order of ϕ (see for example [HW, Theo. 330]) implies that

$$\mathscr{N}(s) = \frac{3}{\pi^2}\, e^s + O(se^{s/2}),$$

as $s \to +\infty$. Similarly, if $\mathscr{O}_K$ is the ring of integers in $K = \mathbb{Q}(\sqrt{d})$ with d a negative squarefree integer and D_K is the discriminant of K, then $\Gamma = \mathrm{PSL}(\mathscr{O}_K)$ acts on $\mathbb{H}^3_{\mathbb{R}}$ by homographies as a cofinite volume discrete group of isometries. A generalisation of the above argument gives, when $D_K \neq -3, -4$,

$$\mathscr{N}(s) = \frac{\pi}{\sqrt{|D_K|}\,\zeta_K(2)}\, e^{2s} + O(e^{3t/2}), \tag{9}$$

as $s \to +\infty$, where ζ_K is Dedekind's zeta function of K. See the subsections 6.1 and 6.2 in [Cos] for the proof.

3.6 Counting common perpendiculars of horoballs and totally geodesic submanifolds

When $\widetilde{M} = \mathbb{H}^n_{\mathbb{R}}$, M has finite volume, C_- is a Margulis cusp neighbourhood of M and C_+ is a finite volume totally geodesic immersed submanifold of M of dimension k with $1 \leq k < n$, we proved in [PP5] (see Theorem 1.1 and Lemma 3.1) the following result, announced in [PP2].

Theorem 2 (Parkkonen-Paulin [PP5]) *As $s \to +\infty$,*

$$\begin{aligned}\mathscr{N}(s) &\sim \frac{\mathrm{Vol}(\mathbb{S}^{n-k-1})\,\mathrm{Vol}(C_-)\,\mathrm{Vol}(C_+)}{\mathrm{Vol}(\mathbb{S}^{n-1})\,\mathrm{Vol}(M)}\; e^{(n-1)s}\\ &= \frac{\mathrm{Vol}(\mathbb{S}^{n-k-1})\,\mathrm{Vol}(\partial C_-)\,\mathrm{Vol}(C_+)}{\mathrm{Vol}(\mathbb{S}^{n-1})\,\mathrm{Vol}(M)}\;\frac{e^{(n-1)s}}{n-1}\;. \quad \square\end{aligned} \tag{10}$$

Oh and Shah [OS1] studied a counting problem analogous to the one described in Subsection 3.4 for Γ geometrically finite and $\mathscr{P}$ a family of circles in $\mathbb{R}^2$ that consists of a finite number of Γ-orbits such that the family $\mathscr{P}^*$ of totally geodesic hyperplanes in the upper halfspace model of $\mathbb{H}^3_{\mathbb{R}}$ whose boundaries are the circles of $\mathscr{P}$ is locally finite. For any circle $P \in \mathscr{P}$, let $\mathrm{curv_S}(P)$ be the reciprocal of the radius of P, that is, the curvature of the circle P. For any $T > 0$ and any bounded Borel set $E \subset \mathbb{R}^2$, let

$$N(T, E) = \mathrm{Card}\{P \in \mathscr{P} : P \cap E \neq \emptyset,\ \mathrm{curv_S}(P) < T\}\,.$$

Oh and Shah showed [OS1, Theo. 1.4] that

$$N(T) \sim c(\mathscr{P}, E)\, T^{\delta} \tag{11}$$

for a constant $c(\mathscr{P}, E) > 0$, as $T \to \infty$. Assume now that ∞ is a rank 2 parabolic fixed point of Γ and let $\widetilde{C}_-$ be the horoball that consists of points whose third coordinate in the upper half space model is at least 1 (this can be achieved by conjugating the group Γ by a Möbius transformation if necessary). Now,

$$d(\widetilde{C}_-, \widetilde{C}_+) = \ln \mathrm{curv_S}(\widetilde{C}_+)$$

for any hyperbolic plane $\widetilde{C}_+$ in $\mathscr{P}^*$. Hence, the result (11) on circle packings has an interpretation as a counting result for the common perpendiculars between the images of $\widetilde{C}_-$ and the images of the hyperplanes of $\mathscr{P}^*$ in M. If $\mathscr{N}$ is defined as above and E is a fundamental domain in $\mathbb{R}^2$ for the action of the stabiliser of ∞, then $\mathscr{N}(s) \sim c(\mathscr{P}, E)e^{\delta s}$. Furthermore, the endpoints of the common perpendiculars are evenly distributed on $\partial\widetilde{C}_-$ in the same sense as in Equation (8) in terms of a natural measure, which is the skinning measure pushed to the boundary, see Section 8.

3.7 The density of integer points on homogeneous varieties

Let us denote a generic element of the Euclidean space $\mathbb{R}^{n+1}$ by $x = (x_0, \bar{x})$, where $x_0 \in \mathbb{R}$ and $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, and consider the quadratic form

$$q(x) = -2x_0^2 + \|x\|^2 = -x_0^2 + \|\bar{x}\|^2 = -x_0^2 + x_1^2 + \cdots + x_n^2$$

of signature $(1, n)$. The identity component $G = \mathrm{SO}_0(1, n)$ of the special orthogonal group of the form q is a connected semisimple real Lie group with trivial center when $n \geq 2$.

Let $\mathbb{R}^{1,n} = (\mathbb{R}^{n+1}, \langle\cdot,\cdot\rangle)$ be the $(n+1)$-dimensional Minkowski space with the (indefinite) inner product $\langle x, y\rangle = -x_0y_0 + \sum_{i=1}^n x_iy_i$ for any $x = (x_0, x_1, \dots, x_n), y = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$. The hyperboloid model of the n-dimensional real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ is the upper half $\{x \in \mathbb{R}^{1,n} \ :\ q(x) = -1,\ x_0 > 0\}$ of the hyperboloid with equation $q = -1$, endowed with the Riemannian metric of constant sectional curvature -1 induced by the (positive definite) restriction of the indefinite inner product $\langle\cdot,\cdot\rangle$ to the tangent space of the hyperboloid. The hyperbolic distance of two points $x, y \in \mathbb{H}^n_{\mathbb{R}}$ has a simple expression in terms of the indefinite inner product: $\cosh d(x, y) = -\langle x, y\rangle$. The restriction to $\mathbb{H}^n_{\mathbb{R}}$ of the (left) linear action of G on $\mathbb{R}^{1,n}$ is the group of orientation-preserving isometries of $\mathbb{H}^n_{\mathbb{R}}$.

Oh and Shah [OS3, p. 4-5] proved the following counting result for orbits of (nonelementary discrete) geometrically finite subgroups Γ of G on varieties defined as level sets of the form q, generalising a special case of a result of Duke, Rudnick and Sarnak [DRS]:

For any $m \in \mathbb{R}$, let $V_m = \{x \in \mathbb{R}^{n+1} \ : \ q(x) = m\}$. Let $w \in V_m - \{0\}$ be a vector such that the linear orbit Γw is discrete. If $\delta > 1$, then by [OS3, p. 4-5]

$$\mathrm{Card}\{y \in \Gamma w : \|y\| < T\} \sim c(m) T^\delta \,, \tag{12}$$

with a constant $c(m) > 0$ similar to those in the previous cases.

The above counting result is equivalent to three results on counting common perpendiculars, depending on the sign of m, as we will now explain. For convenience, we will restrict to the three essential cases $m \in \{-1, 0, 1\}$. Consider first the case $q(w) = -1$. Now, the orbit of w is contained in $\mathbb{H}^n_\mathbb{R}$. For any $y \in \mathbb{H}^n_\mathbb{R}$, we have $\langle y, y\rangle = -y_0^2 + \|\bar{y}\|^2 = -1$. Thus, $\|y\|^2 = 2y_0^2 + 1$, and we have $\cosh d(y, (1, 0)) = -\langle (1, 0), y\rangle = y_0 \sim \frac{1}{\sqrt{2}}\|y\|$ as $\|y\| \to +\infty$. Therefore, the asymptotic (12) gives an asymptotic count of orbit points as in the results of Margulis and Roblin, see Section 3.3.

If $q(w) = 0$, then w lies in the light cone of q and it defines a horosphere

$$H_w = \{y \in \mathbb{H}^n_\mathbb{R} : \langle y, w\rangle = -1\} \,.$$

Using a rotation with fixed point at $(0, 1)$, we can assume that $w = (w_0, w_0, 0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1}$, with $w_0 \neq 0$. Now

$$H_w = \{y \in \mathbb{H}^n_\mathbb{R} : y_1 = y_0 - \frac{1}{w_0}\} = \{y \in \mathbb{R}^{1,n} : y_0 = \frac{1}{2}(w_0 + \frac{1}{w_0}),\ y_1 = \frac{1}{2}(w_0 - \frac{1}{w_0})\} \,.$$

By the symmetry of the situation, it is clear that

$$\begin{aligned} d((1,0), H_w) &= d((1,0), (\frac{1}{2}(w_0 + \frac{1}{w_0}), \frac{1}{2}(w_0 - \frac{1}{w_0}), 0)) = \mathrm{arcosh}(\frac{1}{2}(w_0 + \frac{1}{w_0})) \\ &= \ln w_0 = \ln \|w\| - \ln 2 \,, \end{aligned}$$

so the asymptotic for the norms of points in the orbit of a point in the light cone is equivalent to an asymptotic of the distance of an orbit of horoballs from a point. The same counting problem is also considered by Kontorovich and Oh in [KOh], and earlier in [Kon] in the two-dimensional case.

In the third case, when $q(w) = 1$, the vector w defines a totally geodesic hyperplane $w^\perp = \{y \in \mathbb{H}^n_\mathbb{R} : \langle y, w\rangle = 0\}$ in $\mathbb{H}^n_\mathbb{R}$. As in the two cases above, one can check that this asymptotic is equivalent to the asymptotic count of geodesic arcs starting at a fixed point and ending perpendicularly at an orbit of totally geodesic hyperplanes.

4 Using Eskin-McMullen's equidistribution theorem

In order to prove the kind of asymptotic results described in Section 3, following Margulis [Mar1, Mar2], one usually proves first an appropriate equidistribution result using mixing, and this result is then used to study the common perpendiculars.

Eskin and McMullen [EM] proved a very general equidistribution theorem for Lie groups orbits using mixing properties and a technical "wave front lemma" in affine symmetric spaces.

Theorem 3 (Eskin-McMullen [EM]) *Let G be a connected semisimple real Lie group with finite center. Let $\sigma : G \to G$ be an involutive Lie group automorphism, and H its fixed* *subgroup. Let Γ be a lattice in G and let m be the unique G-invariant probability measure on $\Gamma\backslash G$. Assume that the projection of Γ to G/G' is dense for all noncompact connected normal Lie subgroups G' of G, and that $\Gamma \cap H$ is a lattice in H. Let $Y = (\Gamma \cap H)\backslash H$ and let μ_g be the image by the right multiplication by g of the unique H-invariant probability measure on Y. Then, for every $f : \Gamma\backslash G \to \mathbb{R}$ which is continuous with compact support,*

$$\int_{Yg} f(h)\, d\mu_g(h) \to \int_{\Gamma\backslash G} f(x)\, dm(x),$$

as g goes to infinity in $H\backslash G$. □

This result is used in [EM] to prove a result of Duke, Rudnick and Sarnak [DRS] on counting integral points on homogeneous varieties, see Subsection 3.7.

In [PP5], we proved the following equidistribution result using mixing and hyperbolic geometry, as a tool to prove the asymptotic estimate (10). A modification of the proof in [PP5] enabled us to prove the general equidistribution result in variable curvature in [PP6], whose tools are also used for the counting result of [PP7] that we describe in Section 8. Here, at the instigation of Hee Oh, we present a different proof using Theorem 3, which also serves as an illustration of the use of Theorem 3.

Theorem 4 (Parkkonen-Paulin [PP5]) *Let M be a complete connected hyperbolic manifold with finite volume. Let C be a nonempty proper totally geodesic immersed submanifold of M with finite volume. The induced Riemannian measure on $g^t\partial^1_+ C$ equidistributes to the Liouville measure as $t \to +\infty$:*

$$\mathrm{Vol}_{g^t\partial^1_+ C} / \| \mathrm{Vol}_{g^t\partial^1_+ C} \| \;\overset{*}{\rightharpoonup}\; \mathrm{Vol}_{T^1 M} / \| \mathrm{Vol}_{T^1 M} \| .$$

More general versions of the above result appear in [OS3, Theo. 1.8] and [PP6, Theo. 17].

We use below the notation introduced in Section 3.7. Before giving the proof of this result, let us review some preliminaries on the action on $T^1\mathbb{H}^n_{\mathbb{R}}$ of the orientation-preserving isometry group $G = \mathrm{SO}_0(1, n)$ of the hyperboloid model of $\mathbb{H}^n_{\mathbb{R}}$, where $n \geq 2$. Let $(e_0, e_1, \dots, e_n)$ be the canonical basis of $\mathbb{R}^{1,n}$, and let $w_0 = (1, 0, \dots, 0) \in \mathbb{H}^n_{\mathbb{R}}$. For any $1 \leq k < n$, we embed $\mathbb{H}^k_{\mathbb{R}}$ isometrically in $\mathbb{H}^n_{\mathbb{R}}$ as the intersection of $\mathbb{H}^n_{\mathbb{R}}$ with the linear subspace given by the equations $x_{k+1} = x_{k+2} = \cdots = x_n = 0$. For any $p \in \mathbb{N}$, let I_p be the $p \times p$ identity matrix. Let H_k be the subgroup of G that consists of the fixed points of the involution $\sigma_k \colon G \to G$ defined by $\sigma_k(g) = J_k g J_k^{-1}$, where $J_k = \begin{pmatrix} I_{k+1} & 0 \\ 0 & -I_{n-k} \end{pmatrix}$. Note that H_k is isomorphic to $(\mathrm{O}(1, k) \times \mathrm{O}(n-k)) \cap G$, hence contains $\mathrm{SO}_0(1, k) \times \mathrm{SO}(n-k)$ with index 2. Let us identify $\mathrm{SO}(n-1)$ with its image in $\mathrm{SO}(n)$, and similarly $\mathrm{SO}(n)$ with its image in G, by the maps $x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$. Let λ_G and λ_{H_k} be fixed left Haar measures on G and H_k.

The group G acts transitively on $T^1\mathbb{H}^n_{\mathbb{R}}$ and the action commutes with the geodesic flow, the stabiliser of $e_1 \in T^1\mathbb{H}^n_{\mathbb{R}}$ being $\mathrm{SO}(n-1)$. Note that H_k is the subgroup of G which preserves $\mathbb{H}^k_{\mathbb{R}}$. It acts transitively on the unit normal bundle $\partial^1_+\mathbb{H}^k_{\mathbb{R}}$.

The orbital map $g \mapsto ge_1$ from G to $T^1\mathbb{H}^n_{\mathbb{R}}$ induces a diffeomorphism $\overline{\varphi} \colon G/\mathrm{SO}(n-1) \to T^1\mathbb{H}^n_{\mathbb{R}}$ which is equivariant for the left actions of G. The commutativity of the diagram

$$\begin{array}{ccc} G/\mathrm{SO}(n-1) & \longrightarrow & G/\mathrm{SO}(n) \\ \downarrow\simeq \overline{\varphi} & & \downarrow\simeq \\ T^1\mathbb{H}^n_{\mathbb{R}} & \longrightarrow & \mathbb{H}^n_{\mathbb{R}} \end{array}$$

and the fact that the Riemannian measure of $\mathbb{S}^{n-1} \simeq \mathrm{SO}(n)/\mathrm{SO}(n-1)$ is the unique (up to multiplication by a positive constant) positive Borel measure which is invariant under rotations imply that the image of λ_G by the smooth map $g \mapsto \bar{\varphi}(g\,\mathrm{SO}(n-1))$ is a multiple of $\mathrm{Vol}_{T^1\mathbb{H}^n_\mathbb{R}}$.

Consider the one-parameter subgroup $(a_t)_{t\in\mathbb{R}}$ of G, where

$$a_t = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix}.$$

The action $g \mapsto g a_t$ of a_t by right translations on G commutes with that of $\mathrm{SO}(n-1)$. A calculation in hyperbolic geometry shows that $a_t e_1$ is the image of the unit tangent vector $e_1 \in T^1_{w_0}\mathbb{H}^n_\mathbb{R}$ under the geodesic flow g^t in $T^1\mathbb{H}^n_\mathbb{R}$. Thus, by equivariance, $\overline{\varphi}(g a_t\,\mathrm{SO}(n-1)) = g^t\overline{\varphi}(g\,\mathrm{SO}(n-1))$ for all $g \in G$ and all $t \in \mathbb{R}$. Let us fix a group element $r \in \mathrm{SO}(n)$ which maps e_1 to e_n. As the measures under consideration are induced by differential forms, homogeneity arguments imply that the image measure of λ_{H_k} by the smooth map $h \mapsto \overline{\varphi}(hra_t\,\mathrm{SO}(n-1))$ is a multiple of $\mathrm{Vol}_{g^t\partial^1_+\mathbb{H}^k_\mathbb{R}}$.

Proof of Theorem 4. By additivity, we can assume that C is connected. Let $\overline{M} \to M$ be the Riemannian orientation cover of M (which is the identity map if M is orientable), and let $\overline{C} \to C$ be the one of C, so that $\overline{C}$ is a connected immersed totally geodesic submanifold of $\overline{M}$. As the image measures by the finite cover $T^1\overline{M} \to T^1M$ of $\mathrm{Vol}_{g^t\partial^1_+\overline{C}}$ and $\mathrm{Vol}_{T^1\overline{M}}$ are $\mathrm{Vol}_{g^t\partial^1_+C}$ and Vol_{T^1M}, respectively, it suffices to show that the Riemannian measure of $g^t\partial^1_+\overline{C}$ equidistributes to the Liouville measure of $T^1\overline{M}$ as $t \to +\infty$. We may, therefore, assume that M and C are oriented.

Let us fix a universal Riemannian cover $\mathbb{H}^n_\mathbb{R} \to M$. Its covering group Γ is a lattice in G, since M has finite volume. We may assume that the image of $\mathbb{H}^k_\mathbb{R}$ under this covering map is C. We define $H = H_k$ as above. Since C has finite volume, $\Gamma \cap H$ is a lattice in H. Since r fixes e_0 and sends e_1 to e_n which is perpendicular to $\mathbb{H}^k_\mathbb{R}$, the map $t \mapsto \pi(ra_te_1)$ from $[0, +\infty[$ to $\mathbb{H}^n_\mathbb{R}$ is a geodesic ray starting perpendicularly to $\mathbb{H}^k_\mathbb{R}$. Since H is the stabiliser of $\mathbb{H}^k_\mathbb{R}$, the map $t \mapsto Hra_te_1$ from $[0, +\infty[$ to $H\backslash\mathbb{H}^n_\mathbb{R}$ tends to infinity. Hence the map $t \mapsto Hra_t$ from $[0, +\infty[$ to $H\backslash G$ tends to infinity.

Since the connected semi-simple real Lie group G has trivial center, and only one noncompact factor, the projection of Γ to G/G' is dense for all noncompact connected normal Lie subgroups G' of G. (In fact, G has no compact factor, and any lattice in G is irreducible, see for instance [Mos]).

We can now use Theorem 3 to conclude that as t tends to $+\infty$, the measure μ_t on $\Gamma\backslash G$ with support ΓHra_t which is defined to be the translate by ra_t of the unique H-invariant probability measure on $(\Gamma \cap H)\backslash H$, equidistributes towards the probability measure m on $\Gamma\backslash G$ induced by λ_G. Let $p : \Gamma\backslash G \to \Gamma\backslash G/\mathrm{SO}(n-1)$ be the canonical projection. The G-equivariant diffeomorphism $\overline{\varphi} : G/\mathrm{SO}(n-1) \to T^1\mathbb{H}^n_\mathbb{R}$ induces a diffeomorphism $\varphi : \Gamma\backslash G/\mathrm{SO}(n-1) \to T^1M$ such that $\varphi(\Gamma g a_t\,\mathrm{SO}(n-1)) = g^t\varphi(\Gamma g\,\mathrm{SO}(n-1))$ for all $g \in G$ and $t \in \mathbb{R}$. By the homogeneity argument just before the beginning of the proof and covering arguments, and since direct images of measures preserve the total masses, we have $\varphi_*(p_*\mu_t) = \frac{1}{\mathrm{Vol}(g^t\partial^1_+C)}\mathrm{Vol}_{g^t\partial^1_+C}$ and $\varphi_*(p_*m) = \frac{1}{\mathrm{Vol}(T^1M)}\mathrm{Vol}_{T^1M}$. As taking direct images of measures by a given continuous map is continuous in the weak-* topology, the measures $\frac{1}{\mathrm{Vol}(g^t\partial^1_+C)}\mathrm{Vol}_{g^t\partial^1_+C} = (\varphi\circ p)_*\mu_t$ equidistribute to $(\varphi\circ p)_*m = \frac{1}{\mathrm{Vol}(T^1M)}\mathrm{Vol}_{T^1M}$ in T^1M as t tends to $+\infty$, which is what we wanted to prove. □

5 Arithmetic applications

If the manifold M is arithmetically defined, many counting results for common perpendiculars have an arithmetic interpretation. In this section, we will review some of these arithmetic applications. The arithmetically defined groups in this section will, in general, have torsion. This is not a problem however, as the geometric counting result used in the various cases below is indeed valid in this more general context. In certain cases, the interaction has also worked in the opposite direction, as evidenced by the results of Cosentino in Section 3.5.

5.1 Counting representations of integers by binary quadratic forms

Let $Q(X,Y) = aX^2 + bXY + cY^2$ be an integral binary quadratic form with *discriminant* $\Delta = b^2 - 4ac$. An element $x \in \mathbb{Z}^2$ is a *representation* of an integer n by Q if $Q(x) = n$, and the representation is *primitive* if the components of x are relatively prime. If Q is positive definite (equivalently, if $\Delta < 0$), then the number $N(t)$ of representations of integers that are at most t equals the number of lattice points of $\mathbb{Z}^2$ inside the ellipse defined by the equation $Q(x) = t$. The asymptotics of this number (Gauss' circle problem) have been studied extensively, and the best known result with an error bound

$$N(t) = \frac{2\pi}{\sqrt{-\Delta}}\, t + O(t^{131/416})$$

is due to Huxley [Hux]. Gauss already had a solution with a worse bound on the error term.

The modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on the right by precomposition on the set of binary quadratic forms, preserving the discriminant, and linearly on the left on $\mathbb{Z}^2$. Let us assume that Q is *primitive* (that is, the coefficients a, b and c are relatively prime), indefinite and not the product of two integral linear forms. The stabiliser of a form f in $\mathrm{SL}_2(\mathbb{Z})$, called the *group of automorphs* of f, is

$$\begin{aligned}\mathrm{SO}(Q,\mathbb{Z}) =& \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : Q \circ \gamma = Q\}\\ =& \left\{\gamma_{Q,t,u} = \begin{pmatrix} \frac{t-bu}{2} & -cu \\ au & \frac{t+bu}{2}\end{pmatrix} \;:\; t,u \in \mathbb{Z},\;\; t^2 - \Delta u^2 = 4\right\},\end{aligned}$$

see for instance [Lan, Theo. 202]. This group is infinite and thus any nonzero integer that is represented by the form Q is represented infinitely many times. Accordingly, in the generalisation of the circle problem for these forms, one counts the number of orbits of lattice points under the linear action of $\mathrm{SO}(Q,\mathbb{Z})$ between the hyperbolas defined by the equations $|Q(x)| = t$. Let

$$\tilde{\Psi}_Q(t) = \mathrm{Card}\big(\,\mathrm{SO}(Q,\mathbb{Z})\backslash\{x \in \mathbb{Z}^2 \;:\; |Q(x)| \le t\}\big)$$

and

$$\Psi_Q(t) = \mathrm{Card}\big(\,\mathrm{SO}(Q,\mathbb{Z})\backslash\{x \in \mathscr{P} \;:\; |Q(x)| \le t\}\big)$$

be the counting functions of all the representations and of the primitive representations by Q. The asymptotics of $\tilde{\Psi}_Q(t)$ are also known, see for example [Coh, p. 164] for a proof.

It turns out that the asymptotic result on the counting function $\mathscr{N}(s)$ for a horoball and a totally geodesic subspace can be used to give a different proof of this result.

Let us first observe that an asymptotic result for the primitive representations implies one for all representations. Assume that $\Psi_Q(t) = ct + O(t^{1-\epsilon})$ for some $c, \epsilon > 0$. For any $k \in \mathbb{N}$, let

$$\Psi_{Q,k}(t) = \operatorname{Card}\big(\operatorname{SO}(Q,\mathbb{Z})\backslash\{x = (x_1, x_2) \in \mathbb{Z} \;:\; \gcd(x_1, x_2) = k,\ |Q(x)| \leq t\}\big)\,.$$

Now, $\Psi_{Q,k}(t) = \Psi_Q(k^{-2}t)$ and

$$\tilde{\Psi}_Q(t) = \sum_{k=1}^{\infty} \Psi_{Q,k}(t) = \sum_{k=1}^{\infty} \Psi_Q(k^{-2}t) = \sum_{k=1}^{\infty} ck^{-2}t + O(k^{-2}t^{1-\epsilon}) = c\,\zeta(2)t + O(t^{1-\epsilon}).$$

We will now explain how to obtain an asymptotic estimate for $\Psi_Q(t)$ from the solution of the geometric counting problem of Section 3.6. We use the upper halfplane model of $\mathbb{H}^2_{\mathbb{R}}$. The subgroup $\operatorname{PSO}(Q,\mathbb{Z})$ of $\operatorname{PSL}_2(\mathbb{Z})$ is a cyclic group generated by a hyperbolic element. Its index i_Q in the stabiliser Γ_Q of the geodesic line C_Q invariant under $\operatorname{PSO}(Q,\mathbb{Z})$ is either 1 or 2 depending on whether the corresponding locally geodesic line on the modular surface $\operatorname{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$ passes through the cone point of order 2. Let (t_Q, u_Q) be the fundamental solution of the Pell-Fermat equation $t^2 - \Delta u^2 = 4$, and let $R_Q = \ln \frac{t_Q + u_Q\sqrt{\Delta}}{2}$ be the *regulator* of Q. It is easy to check that the length of the closed geodesic $\Gamma_Q\backslash C_Q$ is $\frac{2R_Q}{i_Q}$.

The stabiliser U of $(1,0) \in \mathbb{Z}^2$ for the linear action of $\operatorname{SL}_2(\mathbb{Z})$ is the subgroup that consists of integral upper triangular unipotent matrices. Geometrically, the image Γ_∞ of U in $\operatorname{PSL}_2(\mathbb{Z})$ is the stabiliser of the horoball $\mathscr{H} = \{z \in \mathbb{C} \;:\; \operatorname{Im} z \geq 1\}$ in $\mathbb{H}^2_{\mathbb{R}}$. The horoball $\mathscr{H}$ is *precisely invariant*, that is, each element of $\operatorname{PSL}_2(\mathbb{Z})$ either preserves $\mathscr{H}$ or maps $\mathscr{H}$ to a horoball whose interior is disjoint from $\mathscr{H}$. As $\mathscr{H}$ is the maximal such horoball at ∞, it corresponds to the maximal Margulis cusp neighbourhood of the unique cusp of the quotient space $\operatorname{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$.

The length of the common perpendicular of $\mathscr{H}$ and the hyperbolic line C_Q stabilised by $\operatorname{SO}(Q,\mathbb{Z})$ is $\ln \frac{2|a|}{\sqrt{\Delta}}$. For all $\gamma = \pm\begin{pmatrix} A & B \\ C & D\end{pmatrix}$ in $\operatorname{PSL}_2(\mathbb{Z})$, a simple computation (see [PP5, Lem. 5.2]) shows that the length of the common perpendicular of $\mathscr{H}$ and the image under γ of C_Q is

$$\ln \frac{2}{\sqrt{\Delta}}\, |Q(D, -C)|\,.$$

Thus, Corollaire 4.9 of [PP5], which generalises the result of Equation (10) to the case of groups with torsion, gives

$$\begin{aligned}\Psi_Q(s) &= i_Q \operatorname{Card}\Big\{[\gamma] \in \Gamma_\infty\backslash\Gamma/\Gamma_Q \;:\; d(\mathscr{H}_\infty, \gamma C_Q) \leq \ln\Big(\frac{2}{\sqrt{\Delta}}\, s\Big)\Big\}\\ &\sim i_Q\, \frac{\operatorname{Vol}(\mathbb{S}^0)\operatorname{Vol}(\Gamma_\infty\backslash\mathscr{H}_\infty)\operatorname{Vol}(\Gamma_Q\backslash C_Q)}{\operatorname{Vol}(\mathbb{S}^1)\operatorname{Vol}(\Gamma\backslash\mathbb{H}^2_{\mathbb{R}})}\Big(\frac{2}{\sqrt{\Delta}}\, s\Big) = \frac{12\,R_Q}{\pi^2\sqrt{\Delta}}\, s\,,\end{aligned}$$

see [PP5, §5] for more details and more applications, in particular to counting representations satisfying congruence relations.

5.2 Counting representations of integers by binary Hermitian forms

A function $f : \mathbb{C}^2 \to \mathbb{R}$ is a *binary Hermitian form* if there are constants $a, c \in \mathbb{R}$ and $b \in \mathbb{C}$, called the *coefficients* of f, such that for all $u, v \in \mathbb{C}$,

$$f(u,v) = a|u|^2 + 2\,\mathrm{Re}(bu\overline{v}) + c|v|^2 = \begin{pmatrix}\bar{u} & \bar{v}\end{pmatrix}\begin{pmatrix}a & b\\ \bar{b} & c\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}. \tag{13}$$

Let K be an imaginary quadratic number field, with discriminant D_K and ring of integers $\mathscr{O}_K$. If the coefficients of the Hermitian form f satisfy $a, c \in \mathbb{R} \cap \mathscr{O}_K = \mathbb{Z}$ and $b \in \mathscr{O}_K$, then we say that f is *integral*. It is easy to check that the values of the restriction of an integral binary Hermitian form to $\mathscr{O}_K \times \mathscr{O}_K$ are rational integers. If the *discriminant* $\Delta(f) = |b|^2 - ac$ of f is positive, then we say that f is *indefinite*, which is equivalent to saying that f takes both positive and negative values.

A binary Hermitian form naturally gives rise to a quaternary quadratic form. The representations of integers by positive definite quaternary quadratic forms have been studied for a long time (including Lagrange's four square theorem, see also the work of Ramanujan as in [Klo]).

In the case of indefinite forms, the counting problem is again complicated by the presence of an infinite group of automorphs: The group $\mathrm{SL}_2(\mathscr{O}_K)$ acts on the right by precomposition on the set of (indefinite) integral binary Hermitian forms, and the stabiliser of such a form under this action is, analogously to the case of binary quadratic forms treated in Section 5.1, called the *group of automorphs* of the form and denoted by $\mathrm{SU}_f(\mathscr{O}_K)$. The Bianchi group $\mathrm{PSL}_2(\mathscr{O}_K)$ acts discretely on the upper halfspace model of $\mathbb{H}^3_\mathbb{R}$, with finite covolume. Now, the image in $\mathrm{PSL}_2(\mathscr{O}_K)$ of the group of automorphs of a fixed indefinite integral binary Hermitian form f is a Fuchsian subgroup that preserves a real hyperbolic plane $\mathscr{C}(f)$ whose boundary at infinity is the circle

$$\mathscr{C}_\infty(f) = \{[u:v] \in \mathbb{P}^1(\mathbb{C}) = \partial_\infty \mathbb{H}^3_\mathbb{R} \;:\; f(u,v) = 0\}.$$

The group of automorphs $\mathrm{SU}_f(\mathscr{O}_K)$ is an arithmetic group that acts on $\mathscr{C}(f)$ with finite covolume.

The cusps of a Bianchi group $\mathrm{PSL}_2(\mathscr{O}_K)$ are in a natural bijective correspondence with the ideal classes of K, see for example Theorem 2.4 in Chapter 7 of [EGM]. Let $x, y \in \mathscr{O}_K$ be not both zero, so that $[x:y] \in \mathbb{P}^1(\mathbb{C})$ is a cusp of $\mathrm{PSL}_2(\mathscr{O}_K)$. Then, if $y = 0$, the horoball $\mathscr{H}$ that consists of those points in the upper halfspace model $\mathbb{H}^3_\mathbb{R}$ whose third coordinate is at least 1 is precisely invariant, and if $y \neq 0$, then there is some $\tau > 0$ such that the horoball $\mathscr{H}$ centered at $\frac{x}{y}$ of Euclidean height τ is precisely invariant. Analogously with the case of indefinite binary quadratic forms, for any $g \in \mathrm{SL}_2(\mathscr{O}_K)$, the distance between $\mathscr{H}$ and $\mathscr{C}_\infty(f \circ g) = g^{-1}\mathscr{C}_\infty(f)$ is $\ln \frac{|f \circ g(x,y)|}{\tau |y|^2 \sqrt{\Delta(f)}}$, and, as in the case of binary quadratic forms, we find a connection between representing integers by f and the counting problem of Section 3.6.

We define counting functions of the representation of integers for each nonzero fractional ideal $\mathfrak{m}$ of K. For every $u, v \in K$, let $\langle u, v\rangle$ be the $\mathscr{O}_K$-module they generate. For every $s > 0$, we consider the integer

$$\psi_{f,\mathfrak{m}}(s) = \mathrm{Card}\ {}_{\mathrm{SU}_f(\mathscr{O}_K)}\backslash\big\{(u,v) \in \mathfrak{m} \times \mathfrak{m} \;:\; (N\mathfrak{m})^{-1}|f(u,v)| \leq s, \quad \langle u, v\rangle = \mathfrak{m}\big\}.$$

Generalising the argument used for binary quadratic forms (see Section 5.1), we can again use the generalisation of Equation (10) to obtain an asymptotic expression for $\psi_{f,\mathfrak{m}}(s)$.

Theorem 5 (Parkkonen-Paulin [PP3]) *As s tends to $+\infty$, we have the equivalence*

$$\psi_{f,\mathfrak{m}}(s)\ \sim\ \frac{\pi\ \operatorname{Covol}(\operatorname{SU}_f(\mathscr{O}_K))}{2\ |D_K|\ \zeta_K(2)\ \Delta(f)}\quad s^2\ .\quad \Box$$

Here $\operatorname{Covol}(\operatorname{SU}_f(\mathscr{O}_K))$ is the area of the quotient of the hyperbolic plane $\mathscr{C}(f)$ in $\mathbb{H}^3_{\mathbb{R}}$ by the group of automorphs of f, ζ_K is Dedekind's zeta function of K. In the proof, after applying Equation (10), we use the fact that there is an explicit formula (essentially due to Humbert) for the volume

$$\operatorname{Vol}(\operatorname{PSL}_2(\mathscr{O}_K)\backslash\mathbb{H}^3_{\mathbb{R}}) = \frac{1}{4\pi^2}|D_K|^{3/2}\zeta_K(2)\,.$$

See [Sar] for a proof of this formula using Eisenstein series, and Sections 8.8 and 9.6 of [EGM] for further proofs. The following corollary follows immediately by taking $\mathfrak{m}=\mathscr{O}_K$: If $\mathscr{P}_K$ is the set of relatively prime pairs of integers of K, then

$$\operatorname{Card}\ {}_{\operatorname{SU}_f(\mathscr{O}_K)}\backslash\big\{(u,v)\in\mathscr{P}_K\ :\ |f(u,v)|\le s\big\}\ \sim\ \frac{\pi\ \operatorname{Covol}(\operatorname{SU}_f(\mathscr{O}_K))}{2\ |D_K|\ \zeta_K(2)\ \Delta(f)}\quad s^2\ ,$$

as s tends to $+\infty$.

In general, one could compute the covolume of the group of automorphs $\operatorname{SU}_f(\mathscr{O}_K)$ with the aid of Prasad's formula in [Pra]. Maclachlan and Reid [MR] computed the covolumes of all stabilisers in $\operatorname{PSL}(\mathbb{Q}(i))$ of discs centered at 0 with radius $\sqrt{D}$ with D a rational integer. This result can be used to obtain an even more explicit expression of the asymptotic formula of Theorem 5 when $K=\mathbb{Q}(i)$: A constant $\iota(f)\in\{1,2,3,6\}$ is defined as follows. If $\Delta(f)\equiv 0 \mod 4$, let $\iota(f)=2$. If the coefficients a and c of the form f as in Equation (13) are both even, let $\iota(f)=3$ if $\Delta(f)\equiv 1 \mod 4$, and let $\iota(f)$ be the remainder modulo 8 of $\Delta(f)$ if $\Delta(f)\equiv 2 \mod 4$. In all other cases, let $\iota(f)=1$. The class number of $\mathbb{Q}(i)$ is 1, and there is just one counting function to be considered. We prove in [PP3, Coro. 3] that, as s tends to $+\infty$, we have

$$\psi_{f,\,\mathscr{O}_K}(s)\sim\frac{\pi^2}{8\ \iota(f)\ \zeta_{\mathbb{Q}(i)}(2)}\prod_{p|\Delta(f)}\Big(1+\Big(\frac{-1}{p}\Big)p^{-1}\Big)\ \ s^2\ .$$

Here p ranges over the odd positive rational primes and $\big(\frac{-1}{p}\big)$ is the Legendre symbol of -1 modulo p. We refer to [PP3] for more details and more applications, including counting representations satisfying congruence conditions.

5.3 Counting quadratic irrational in orbits of modular groups

The group $\operatorname{PSL}_2(\mathbb{Z})$ acts transitively on the rational real numbers, but not transitively on the irrational algebraic real numbers of a given degree. Hence, counting results (for appropriate complexities) of algebraic irrationals within an orbit of $\operatorname{PSL}_2(\mathbb{Z})$ is an interesting problem, and we give some solutions in [PP5] in the quadratic case. Similar problems occur for quadratic irrational complex numbers under the action of (congruence subgroups of) Bianchi groups, and we illustrate them by the following result.

Let $\phi=\frac{1+\sqrt{5}}{2}$ be the Golden Ratio, and $\phi^\sigma=\frac{1-\sqrt{5}}{2}$ its Galois conjugate. Let K be an imaginary quadratic number field, with discriminant $D_K\neq -4$ (to simplify the statement in this survey), Dedekind's zeta function ζ_K and ring of integers $\mathscr{O}_K$. We define as the complexity of a quadratic irrational α with Galois conjugate α^σ the quantity

$$h(\alpha) = \frac{2}{|\alpha - \alpha^\sigma|} \;.$$

(See [PP5, §4.1] for algebraic versions and explanations). Let $\mathfrak{a}$ be a nonzero ideal in $\mathscr{O}_K$, and let $\Gamma_0(\mathfrak{a})$ be the congruence subgroup $\Big\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathscr{O}_K) \;:\; c \in \mathfrak{a} \Big\}$. Assume (to simplify the statement in this survey) that ϕ^σ is not in the $\Gamma_0(\mathfrak{a})$-orbit of ϕ.

Corollary 6 (Parkkonen-Paulin [PP5, Coro. 4.7]) *As s tends to $+\infty$, the cardinality of $\{\alpha \in \Gamma_0(\mathfrak{a}) \cdot \{\phi, \phi^\sigma\} \bmod \mathscr{O}_K \;:\; h(\alpha) \leq s\}$ is equivalent to*

$$\frac{4\pi^2 \; k_{\mathfrak{a}} \; \ln \phi}{|D_K| \; \zeta_K(2) \; N(\mathfrak{a}) \prod_{\mathfrak{p}|\mathfrak{a}} \big(1 + \frac{1}{N(\mathfrak{p})}\big)} \;\; s^2 \;,$$

where $k_{\mathfrak{a}}$ is the smallest $k \in \mathbb{N} - \{0\}$ such that the $2k$-th term of Fibonacci's sequence belongs to $\mathfrak{a}$, and $\mathfrak{p}$ ranges over the prime ideals in $\mathscr{O}_K$. □

5.4 Counting representations of integers by binary Hamiltonian forms

A *quaternion algebra* over a field F is a four-dimensional central simple algebra over F. A real quaternion algebra (that is, a quaternion algebra over $\mathbb{R}$) is isomorphic either to the algebra of real 2×2 matrices over $\mathbb{R}$ or to Hamilton's quaternion algebra $\mathbb{H}$ over $\mathbb{R}$, with basis elements $1, i, j, k$ as a $\mathbb{R}$-vector space, with unit element 1 and $i^2 = j^2 = -1$, $ij = -ji = k$. We define the *conjugate* of $x = x_0 + x_1 i + x_2 j + x_3 k$ in $\mathbb{H}$ by $\overline{x} = x_0 - x_1 i - x_2 j - x_3 k$, its *reduced trace* by $\mathrm{tr}(x) = x + \overline{x}$, and its *reduced norm* by $\mathrm{n}(x) = x\,\overline{x} = \overline{x}\,x$. We refer for instance to [Vig] for generalities on quaternion algebras.

A *binary Hamiltonian form* f is a map $\mathbb{H} \times \mathbb{H} \to \mathbb{R}$ with

$$f(u, v) = a\,\mathrm{n}(u) + \mathrm{tr}(\overline{u}\,b\,v) + c\,\mathrm{n}(v) \;,$$

whose *coefficients* a and c are real, and b lies in $\mathbb{H}$. The *matrix* $M(f)$ of f is the Hermitian matrix $\begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix}$, so that $f(u, v) = (\,\overline{u} \quad \overline{v}\,) \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$. The *discriminant* of f is

$$\Delta(f) = \mathrm{n}(b) - ac \;.$$

In this section, we will describe the results in [PP4] on the representation of integers by indefinite binary Hamiltonian forms. The proof follows the same ideas as in the previous two sections but the noncommutativity of the quaternions adds several new features.

In order to generalise the results of the previous two subsections to the context of Hamiltonian forms, we have to introduce the correct analogs of the ring of integers and of Bianchi groups for quaternion algebras. We say that a quaternion algebra A over $\mathbb{Q}$ is *definite* (or ramified over $\mathbb{R}$) if the real quaternion algebra $A \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $\mathbb{H}$. We fix an identification between $A \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathbb{H}$, so that A is a $\mathbb{Q}$-subalgebra of $\mathbb{H}$. The *reduced discriminant* D_A of A is the product of the primes $p \in \mathbb{N}$ such that the quaternion algebra $A \otimes_{\mathbb{Q}} \mathbb{Q}_p$ over $\mathbb{Q}_p$ is a division algebra. For example, the $\mathbb{Q}$-vector space $\mathbb{H}_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$ generated by $1, i, j, k$ in $\mathbb{H}$ is Hamilton's quaternion algebra over $\mathbb{Q}$. It is the unique definite quaternion algebra over $\mathbb{Q}$ (up to isomorphism) with discriminant $D_A = 2$.

A $\mathbb{Z}$-*lattice* I in A is a finitely generated $\mathbb{Z}$-module generating A as a $\mathbb{Q}$-vector space. An *order* in a quaternion algebra A over $\mathbb{Q}$ is a unitary subring $\mathscr{O}$ of A which is a $\mathbb{Z}$-lattice, and the order is *maximal* if it is maximal with respect to inclusion among all orders of A. The *Hurwitz order* $\mathscr{O} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\frac{1+i+j+k}{2}$ in $\mathbb{H}_\mathbb{Q}$ is maximal, and it is the unique maximal order in $\mathbb{H}_\mathbb{Q}$ up to conjugacy.

The Dieudonné determinant (see [Die, Asl]) Det is the group morphism from the group $\mathrm{GL}_2(\mathbb{H})$ of invertible 2×2 matrices with coefficients in $\mathbb{H}$ to $\mathbb{R}_+^*$, defined by

$$\mathrm{Det}\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big)^2 = \mathrm{n}(a\,d) + \mathrm{n}(b\,c) - \mathrm{tr}(a\,\overline{c}\,d\,\overline{b}) = \begin{cases} \mathrm{n}(ad - aca^{-1}b) & \text{if } a \neq 0 \\ \mathrm{n}(cb - cac^{-1}d) & \text{if } c \neq 0 \\ \mathrm{n}(cb - db^{-1}ab) & \text{if } b \neq 0\,. \end{cases} \tag{14}$$

We will denote by $\mathrm{SL}_2(\mathbb{H})$ the group of 2×2 matrices with coefficients in $\mathbb{H}$ with Dieudonné determinant 1, which equals the group of elements of (reduced) norm 1 in the central simple algebra $\mathscr{M}_2(\mathbb{H})$ over $\mathbb{R}$, see [Rei, §9a]. We refer for instance to [Kel] for more information on $\mathrm{SL}_2(\mathbb{H})$.

The group $\mathrm{SL}_2(\mathbb{H})$ acts linearly on the left on the right $\mathbb{H}$-module $\mathbb{H} \times \mathbb{H}$. Let $\mathbb{P}^1_r(\mathbb{H}) = (\mathbb{H} \times \mathbb{H} - \{0\})/\mathbb{H}^\times$ be the right projective line of $\mathbb{H}$, identified as usual with the Alexandrov compactification $\mathbb{H} \cup \{\infty\}$ where $[1 : 0] = \infty$ and $[x : y] = xy^{-1}$ if $y \neq 0$. The projective action of $\mathrm{SL}_2(\mathbb{H})$ on $\mathbb{P}^1_r(\mathbb{H})$, induced by its linear action on $\mathbb{H} \times \mathbb{H}$, is then the action by homographies on $\mathbb{H} \cup \{\infty\}$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \begin{cases} (az+b)(cz+d)^{-1} & \text{if } z \neq \infty, -c^{-1}d \\ ac^{-1} & \text{if } z = \infty, c \neq 0 \\ \infty & \text{otherwise}\,. \end{cases}$$

The linear action on the left on $\mathbb{H} \times \mathbb{H}$ of the group $\mathrm{SL}_2(\mathbb{H})$ induces an action on the right on the set of binary Hermitian forms f by precomposition.

The above action of $\mathrm{SL}_2(\mathbb{H})$ on $\mathbb{H} \cup \{\infty\}$ induces a faithful left action of $\mathrm{PSL}_2(\mathbb{H}) = \mathrm{SL}_2(\mathbb{H})/\{\pm\,\mathrm{id}\}$ on $\mathbb{H} \cup \{\infty\} = \partial_\infty \mathbb{H}^5_\mathbb{R}$. By Poincaré's extension procedure (see for instance [PP1, Lem. 6.6]), this action extends to a left action of $\mathrm{SL}_2(\mathbb{H})$ on the upper halfspace model of $\mathbb{H}^5_\mathbb{R}$, by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, r) = \Big(\frac{(az+b)\,\overline{(cz+d)} + a\,\overline{c}\,r^2}{\mathrm{n}(cz+d) + r^2\,\mathrm{n}(c)}, \frac{r}{\mathrm{n}(cz+d) + r^2\,\mathrm{n}(c)} \Big)\,. \tag{15}$$

In this way, the group $\mathrm{PSL}_2(\mathbb{H})$ is identified with the group of orientation preserving isometries of $\mathbb{H}^5_\mathbb{R}$.

Given an order $\mathscr{O}$ in a definite quaternion algebra over $\mathbb{Q}$, a binary Hamiltonian form f is *integral* over $\mathscr{O}$ if its coefficients belong to $\mathscr{O}$. Note that such a form f takes integral values on $\mathscr{O} \times \mathscr{O}$. The *Hamilton-Bianchi group* $\Gamma_\mathscr{O} = \mathrm{SL}_2(\mathscr{O}) = \mathrm{SL}_2(\mathbb{H}) \cap \mathscr{M}_2(\mathscr{O})$ preserves the set of indefinite binary Hamiltonian forms f that are integral over $\mathscr{O}$. The stabiliser in $\Gamma_\mathscr{O}$ of such a form f is its *group of automorphs* $\mathrm{SU}_f(\mathscr{O})$.

The Hamilton-Bianchi group $\Gamma_\mathscr{O}$ is a (nonuniform) arithmetic lattice in the connected real Lie group $\mathrm{SL}_2(\mathbb{H})$ (see for instance [PP1, p. 1104] for details). The volume of the quotient real hyperbolic orbifold $\Gamma_\mathscr{O}\backslash\mathbb{H}^5_\mathbb{R}$ has a nice expression in terms of the discriminant D_A, generalising Humbert's formula.

Theorem 7 (Emery, Parkkonen-Paulin [PP4])

$$\mathrm{Covol}(\mathrm{SL}_2(\mathscr{O})) = \frac{\zeta(3)\prod_{p|D_A}(p^3-1)(p-1)}{11520}\ . \quad \square$$

This result is proved in [PP4] using two different methods: In the Appendix of that paper, Emery (who was the first to prove the theorem in full generality) uses Prasad's formula and we give a different proof using the theory of Eisenstein series for quaternions developped in [KOs], following Sarnak's proof in [Sar] for Bianchi groups.

With a, b, c the coefficients of f, let

$$\mathscr{C}_\infty(f) = \{[u:v] \in \mathbb{P}^1_r(\mathbb{H})\ :\ f(u,v) = 0\} \quad \text{and}$$
$$\mathscr{C}(f) = \{(z,r) \in \mathbb{H}\times\,]0,+\infty[\ :\ f(z,1) + a\,r^2 = 0\}\ .$$

In $\mathbb{P}^1_r(\mathbb{H}) = \mathbb{H} \cup \{\infty\}$, the set $\mathscr{C}_\infty(f)$ is the 3-sphere of center $-\frac{b}{a}$ and radius $\frac{\sqrt{\Delta(f)}}{|a|}$ if $a \neq 0$, and it is the union of $\{\infty\}$ with the real hyperplane $\{z \in \mathbb{H}\ :\ \mathrm{tr}(\overline{z}b) + c = 0\}$ of $\mathbb{H}$ otherwise. The arithmetic group $\mathrm{SU}_f(\mathscr{O})$ acts with finite covolume on $\mathscr{C}(f)$.

The action by homographies of $\Gamma_{\mathscr{O}}$ preserves the right projective space $\mathbb{P}^1_r(\mathscr{O}) = A \cup \{\infty\}$, which is the set of fixed points of the parabolic elements of $\Gamma_{\mathscr{O}}$ acting on $\mathbb{H}^5_{\mathbb{R}} \cup \partial_\infty \mathbb{H}^5_{\mathbb{R}}$. In order to describe the orbits of parabolic fixed points, we recall some basic definitions and facts on ideals in a quaternion algebra, see [Vig]. The *left order* $\mathscr{O}_\ell(I)$ of a $\mathbb{Z}$-lattice I is $\{x \in A\ :\ xI \subset I\}$. A *left fractional ideal* of $\mathscr{O}$ is a $\mathbb{Z}$-lattice of A whose left order is $\mathscr{O}$. A *left ideal* of $\mathscr{O}$ is a left fractional ideal of $\mathscr{O}$ contained in $\mathscr{O}$. Two nonzero left fractional ideals $\mathfrak{m}$ and $\mathfrak{m}'$ of $\mathscr{O}$ are isomorphic as left $\mathscr{O}$-modules if and only if $\mathfrak{m}' = \mathfrak{m}c$ for some $c \in A^\times$. A (left) *ideal class* of $\mathscr{O}$ is an equivalence class of nonzero left fractional ideals of $\mathscr{O}$ for this equivalence relation. We will denote by ${}_{\mathscr{O}}\mathscr{I}$ the set of ideal classes of $\mathscr{O}$. The *class number* h_A of A is the number of ideal classes of a maximal order $\mathscr{O}$ of A. It is finite and independent of the maximal order $\mathscr{O}$ (see for instance [Vig, p. 87-88]).

For every (u,v) in $\mathscr{O} \times \mathscr{O} - \{(0,0)\}$, consider the two left ideals of $\mathscr{O}$

$$I_{u,v} = \mathscr{O}u + \mathscr{O}v\ ,\quad K_{u,v} = \left\{\begin{array}{cl} \mathscr{O}u \cap \mathscr{O}v & \text{if } uv \neq 0\ ,\\ \mathscr{O} & \text{otherwise.}\end{array}\right.$$

The map

$$\Gamma_{\mathscr{O}}\backslash\mathbb{P}^1_r(\mathscr{O}) \to ({}_{\mathscr{O}}\mathscr{I} \times {}_{\mathscr{O}}\mathscr{I})\ ,$$

which associates, to the orbit of $[u:v]$ in $\mathbb{P}^1_r(\mathscr{O})$ under $\Gamma_{\mathscr{O}}$, the couple of ideal classes $([I_{u,v}], [K_{u,v}])$ is a bijection by [KOs, Satz 2.1, 2.2]. In particular, the number of cusps of $\Gamma_{\mathscr{O}}$ (or the number of ends of $\Gamma_{\mathscr{O}}\backslash\mathbb{H}^5_{\mathbb{R}}$) is the square of the class number h_A of A.

The *norm* $\mathrm{n}(\mathfrak{m})$ of a nonzero left deal $\mathfrak{m}$ of $\mathscr{O}$ is the greatest common divisor of the norms of the nonzero elements of $\mathfrak{m}$. In particular, $\mathrm{n}(\mathscr{O}) = 1$. The *norm* of a nonzero left fractional ideal $\mathfrak{m}$ of $\mathscr{O}$ is $\frac{\mathrm{n}(c\mathfrak{m})}{\mathrm{n}(c)}$ for any $c \in \mathbb{N} - \{0\}$ such that $c\mathfrak{m} \subset \mathscr{O}$.

Let $\mathscr{O}$ be a maximal order in A, and let $\mathfrak{m}$ be a nonzero left fractional ideal of $\mathscr{O}$, with norm $\mathrm{n}(\mathfrak{m})$. For every $s > 0$, we consider the integer

$$\psi_{f,\mathfrak{m}}(s) = \mathrm{Card}\ {}_{\mathrm{SU}_f(\mathscr{O})}\backslash\big\{(u,v) \in \mathfrak{m}\times\mathfrak{m}\ :\ \mathrm{n}(\mathfrak{m})^{-1}|f(u,v)| \leq s,\quad \mathscr{O}u + \mathscr{O}v = \mathfrak{m}\big\}\ ,$$

which is the number of nonequivalent $\mathfrak{m}$-primitive representations by f of rational integers with absolute value at most s. Analogously to the cases of binary quadratic and Hermitian forms, we have an explicit asymptotic result for this counting function.

Theorem 8 (Parkkonen-Paulin [PP4]) *As s tends to $+\infty$, we have the equivalence, with p ranging over positive rational primes,*

$$\psi_{f,\mathfrak{m}}(s) \sim \frac{540\; h_A\;\operatorname{Covol}(\mathrm{SU}_f(\mathscr{O}))}{\pi^2\;\zeta(3)\;\Delta(f)^2\;\prod_{p|D_A}(p^3-1)(1-p^{-1})}\;\; s^4\;. \quad \Box$$

The proof of the above result again uses Corollaire 4.9 of [PP5]. One considers the h_A different orbits of the parabolic fixed points xy^{-1} of $\Gamma_{\mathscr{O}}$ for which $\mathscr{O}x+\mathscr{O}y=\mathfrak{m}$, and connects the counting functions

$$\psi_{f,x,y}(s) = \operatorname{Card}\;{}_{\mathrm{SU}_f(\mathscr{O})}\backslash\big\{(u,v)\in\Gamma_{\mathscr{O}}(x,y)\;:\;\mathrm{n}(\mathscr{O}x+\mathscr{O}y)^{-1}|f(u,v)|\leq s\big\}$$

with the geometric counting function that counts the common perpendiculars between a Margulis cusp neighbourhood of the cusp corresponding to xy^{-1} and the totally geodesic immersed hypersurface corresponding to $\mathscr{C}(f)$. The counting function $\psi_{f,x,y}$ depends (besides f) only on the $\Gamma_{\mathscr{O}}$-orbit of $[x:y]$ in $\mathbb{P}^1_r(\mathscr{O})$, and summing over all such orbits gives the result. We refer to [PP4] for more details and more general results that cover finite index subgroups of $\Gamma_{\mathscr{O}}$.

6 Patterson, Bowen-Margulis and skinning measures

Let M be a complete nonelementary connected Riemannian manifold of dimension at least 2, with pinched negative curvature $-b^2\leq K\leq -1$. Let $\widetilde{M}\to M$ be a universal Riemannian cover, and let Γ be its covering group. Let $x_0\in\widetilde{M}$ and let $\delta=\delta_\Gamma\in\,]0,+\infty[$ be the critical exponent of Γ.

6.1 Patterson densities and Bowen-Margulis measures

Let $r>0$. A family $(\mu_x)_{x\in\widetilde{M}}$ of nonzero finite measures on $\partial_\infty\widetilde{M}$ whose support is the limit set $\Lambda\Gamma$ is a *Patterson density of dimension* r for Γ if it is Γ-equivariant, that is, if it satisfies

$$\gamma_*\mu_x=\mu_{\gamma x} \tag{16}$$

for all $\gamma\in\Gamma$ and $x\in\widetilde{M}$, and if the pairwise Radon-Nikodym derivatives of the measures μ_x for $x\in\widetilde{M}$ exist and satisfy

$$\frac{d\mu_x}{d\mu_y}(\xi)=e^{-r\beta_\xi(x,y)} \tag{17}$$

for all $x,y\in\widetilde{M}$ and $\xi\in\partial_\infty\widetilde{M}$.

If *Poincaré's series*

$$\mathscr{P}(s)=\sum_{\gamma\in\Gamma}e^{-sd(x_0,\gamma x_0)}$$

diverges at $s=\delta$, then Γ is said to be of *divergence type*. In particular, this holds when $\widetilde{M}$ is a symmetric space and Γ is geometrically finite by [Sul2, CI], see [DOP] for many more general results. For groups of divergence type, there exists (see for instance [Rob2, Coro. 1.8]), up to multiplication by a constant, one and only one Patterson density $(\mu_x)_{x\in\widetilde{M}}$ of dimension δ for Γ: For every $x\in\widetilde{M}$, the measure μ_x is the weak-* limit of

$$\frac{1}{\mathscr{P}(s)}\sum_{\gamma\in\Gamma}e^{-s\,d(x,\gamma x_0)}\Delta_{\gamma x_0}$$

as $s \to \delta$, see [Pat, Kai], where Δ_y is the unit mass Dirac measure at any point $y \in \widetilde{M}$.

Let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension δ for Γ. The *Bowen-Margulis measure* $\widetilde{m}_{\rm BM}$ for Γ on $T^1\widetilde{M}$ is defined, using Hopf's parametrisation, by

$$d\widetilde{m}_{\rm BM}(v) = \frac{d\mu_{x_0}(v_-)d\mu_{x_0}(v_+)dt}{d_{x_0}(v_-,v_+)^{2\delta}} = e^{-\delta(\beta_{v_-}(\pi(v),x_0)+\beta_{v_+}(\pi(v),x_0))}d\mu_{x_0}(v_-)d\mu_{x_0}(v_+)dt\,,$$

see [Sul1, Sul2, Kai]. The Bowen-Margulis measure is independent of the basepoint x_0, and its support is (in Hopf's parametrisation) $(\Lambda\Gamma\times\Lambda\Gamma-\Delta)\times\mathbb{R}$, where Δ is the diagonal in $\Lambda\Gamma\times\Lambda\Gamma$. It is invariant under the geodesic flow, the antipodal map and the action of Γ, and thus it defines a measure $m_{\rm BM}$ on T^1M which is invariant under the geodesic flow of M and the antipodal map.

When the Bowen-Margulis measure $m_{\rm BM}$ is finite, the group Γ is of divergence type (see for instance [Rob2, p. 19]), hence denoting the total mass of a measure m by $\|m\|$, the probability measure $\frac{m_{\rm BM}}{\|m_{\rm BM}\|}$ is then uniquely defined, and is the unique probability measure of maximal entropy of the geodesic flow (see [OP]). When finite, the Bowen-Margulis measure $m_{\rm BM}$ on T^1M is mixing for the geodesic flow, under the mild assumption conjecturally always satisfied, that the geodesic flow is topologically mixing (or that the set of the lengths of the closed geodesic in M is not contained in a discrete subgroup of $\mathbb{R}$), see [Bab1]. This condition holds for instance if M is locally symmetric or if Γ contains a parabolic element, see for instance [Dal]. In this review, we assume that $m_{\rm BM}$ is finite.

6.2 Skinning measures

Let $\widetilde{C}$ be a nonempty closed convex subset of $\widetilde{M}$. We define in [PP6] the *skinning measure* $\widetilde{\sigma}_{\widetilde{C}}$ of Γ on $\partial^1_+\widetilde{C}$, using the homeomorphism $w\mapsto w_+$ from $\partial^1_+\widetilde{C}$ to $\partial_\infty\widetilde{M}-\partial_\infty\widetilde{C}$, by

$$\begin{aligned} d\widetilde{\sigma}_{\widetilde{C}}(w) &= e^{-\delta\,\beta_{w(+\infty)}(\pi(w),\,x_0)}\, d(\nu P_{\widetilde{C}})_*(\mu_{x_0}|_{\partial_\infty\widetilde{M}-\partial_\infty\widetilde{C}})(w)\\ &= e^{-\delta\,\beta_{w_+}(P_{\widetilde{C}}(w_+),\,x_0)}\, d\mu_{x_0}(w_+)\,. \end{aligned} \tag{18}$$

We also consider $\widetilde{\sigma}_{\widetilde{C}}$ as a measure on $T^1\widetilde{M}$ with support contained in $\partial^1_+\widetilde{C}$. The skinning measure $\widetilde{\sigma}_{\widetilde{C}}$ is independent of the basepoint x_0, satisfies $\widetilde{\sigma}_{\gamma\widetilde{C}}=\gamma_*\widetilde{\sigma}_{\widetilde{C}}$ for every isometry γ of $\widetilde{M}$ and its support is $\{w\in\partial^1_+\widetilde{C}: w_+\in\Lambda\Gamma\}=\nu P_{\widetilde{C}}(\Lambda\Gamma-\Lambda\Gamma\cap\partial_\infty\widetilde{C})$. For any $x\in\widetilde{M}$, up to identifying the unit tangent sphere $T^1_x\widetilde{M}$ at x with the boundary at infinity $\partial_\infty\widetilde{M}$ by the map $v\mapsto v_+$, we have $\sigma_{\{x\}}=\mu_x$.

The skinning measure has been defined by Oh and Shah [OS3, §1.2] for the outer unit normal bundles of spheres, horospheres and totally geodesic subspaces in real hyperbolic spaces, see also [HP3, Lem. 4.3] for a closely related measure. The terminology comes from McMullen's proof of the contraction of the skinning map (capturing boundary information for surface subgroups of 3-manifold groups) introduced by Thurston to prove his hyperbolisation theorem.

When $\widetilde{C}$ is a horoball, the skinning measure of $\widetilde{C}$ is well known. In fact, the outer unit normal bundle $\partial^1_+\widetilde{C}$ of $\widetilde{C}$ is a leaf of the strong unstable foliation of the geodesic flow and the skinning measure $\widetilde{\sigma}_{\widetilde{C}}$ is the conditional measure of the Bowen-Margulis measure on this leaf, see for example [Mar2]. The skinning measure of a horoball has also appeared as a measure on $\partial_\infty\widetilde{M}$ with the point at infinity ξ of the horoball removed in [Cos, Tuk, AM] in the constant curvature case and in [HP2] under the name *Patterson measure on* $\partial_\infty\widetilde{M}-\{\xi\}$ in the general case. Furthermore, using the upper halfspace model of $\mathbb{H}^n_{\mathbb{R}}$, Oh and Shah consider in [OS1] a measure ω_Γ defined in $\mathbb{R}^{n-1}=\partial_\infty\mathbb{H}^n_{\mathbb{R}}-\{\infty\}$ by

$$d\omega_\Gamma(\xi)=e^{\delta\beta_\xi(x,(\xi,1))}d\mu_x(\xi).$$

Noticing that $(\xi,1)=P_{\widetilde{C}}(\xi)$ if $\widetilde{C}$ is the horoball in $\mathbb{H}^n_{\mathbb{R}}$ that consists of the points whose third coordinate is at least 1, it follows that ω_Γ is the image of the skinning measure of $\widetilde{C}$ under the map $P_{\widetilde{C}}^{-1}:(\xi,1)\mapsto\xi$.

For later use in Section 8, we introduce some convenient notation. Let $w\in T^1\widetilde{M}$. When $\widetilde{C}=HB_-(w)$ is the unstable horoball of w, the conditional measure of the Bowen-Margulis measure on the strong unstable leaf $W^{\rm su}(w)$ of w is denoted by

$$\mu^{\rm su}_w=\widetilde{\sigma}_{HB_-(w)},$$

and similarly, we denote by

$$\mu^{\rm ss}_w=\iota_*\widetilde{\sigma}_{HB_+(w)}$$

the conditional measure of the Bowen-Margulis measure on the strong stable leaf $W^{\rm ss}(w)$ of w. These two measures are independent of the element w of a given strong unstable leaf and given strong stable leaf, respectively. We also define the conditional measure of the Bowen-Margulis measure on the stable leaf $W^{\rm s}(w)$ of w, using the homeomorphism $(v',t)\mapsto v=g^tv'$ from $W^{\rm ss}(w)\times\mathbb{R}$ to $W^{\rm s}(w)$, by

$$d\mu^{\rm s}_w(v)=e^{-\delta_\Gamma t}\,d\mu^{\rm ss}_w(v')dt\;. \tag{19}$$

Let $\widetilde{C}$ be a proper nonempty closed convex subset of $\widetilde{M}$ such that the Γ-orbit of $\widetilde{C}$ is locally finite, and let C be its image in M. Since $\widetilde{\sigma}_{\widetilde{C}}$ is invariant under the stabiliser $\Gamma_{\widetilde{C}}$ of $\widetilde{C}$ in Γ, the measure $\widetilde{\sigma}=\sum_{\gamma\in\Gamma/\Gamma_{\widetilde{C}}}\gamma_*\widetilde{\sigma}_{\widetilde{C}}$ is a Γ-invariant locally finite Borel positive measure on $T^1\widetilde{M}$ (independent on the choice of representatives of elements of $\Gamma/\Gamma_{\widetilde{C}}$), whose support is contained in the Γ-orbit of $\partial^1_+\widetilde{C}$. Hence $\widetilde{\sigma}$ induces a locally finite Borel positive measure σ_C on $T^1M=\Gamma\backslash T^1\widetilde{M}$, called the *skinning measure* of the properly immersed closed convex subset C, whose support is contained in ∂^1_+C.

Oh and Shah proved in particular that $\|\sigma_C\|$ is finite if $\widetilde{M}$ is geometrically finite with constant curvature -1 and either $\widetilde{C}$ is a horoball centered at a parabolic fixed point or $\delta_\Gamma>1$ and wtC is a codimension 1 totally geodesic submanifold. See [OS3, §5] for a precise, more general statement in higher codimension. Extending this result in variable curvature (with a different proof), we give a sharp criterion in [PP6, Theo. 9] for the finiteness of the skinning measure, by studying its decay in the cusps of M. This decay is analogous to the decay of the Bowen-Margulis measure in the cusps, which was first studied by Sullivan [Sul2] who called it the fluctuating density property (see also [SV] and [HP2, Theo. 4.1]). The criterion, as in the case of the Bowen-Margulis measure in [DOP], is a separation property of critical exponents.

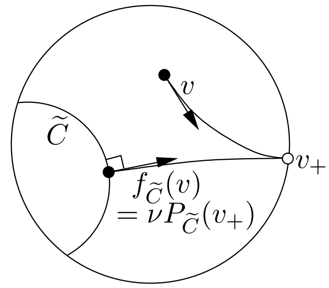
6.3 Disintegration of the Bowen-Margulis measure

Let $\widetilde{C}$ be a proper nonempty closed convex subset of $\widetilde{M}$. Define

$$U_{\widetilde{C}}=\{v\in T^1\widetilde{M}:\ v_+\notin\partial_\infty\widetilde{C}\}\;, \tag{20}$$

which is an open subset of $T^1\widetilde{M}$, invariant under the geodesic flow.

Let $f_{\widetilde{C}} : U_{\widetilde{C}} \to \partial^1_+\widetilde{C}$ be the composition of the map from $U_{\widetilde{C}}$ onto $\partial_\infty\widetilde{M} - \partial_\infty\widetilde{C}$ sending v to v_+ and the homeomorphism $\nu P_{\widetilde{C}}$ from $\partial_\infty\widetilde{M} - \partial_\infty\widetilde{C}$ to $\partial^1_+\widetilde{C}$. The map $f_{\widetilde{C}}$ is a Hölder continuous fibration, invariant under the geodesic flow. The fiber of $f_{\widetilde{C}}$ above $w \in \partial^1_+\widetilde{C}$ is exactly the stable leaf $W^{\rm s}(w) = \{v \in T^1\widetilde{M} \;:\; v_+ = w_+\}$. See [PP6] for further properties of $f_{\widetilde{C}}$.



The following disintegration result of the Bowen-Margulis measure over the skinning measure of $\widetilde{C}$ is the crucial tool for the proof in [PP7] of our general counting result, see Section 8.

Proposition 9 (Parkkonen-Paulin [PP6]) *Let $\widetilde{C}$ be a proper nonempty closed convex subset of $\widetilde{M}$. The restriction to $U_{\widetilde{C}}$ of the Bowen-Margulis measure $\widetilde{m}_{\rm BM}$ disintegrates by the fibration $f_{\widetilde{C}} : U_{\widetilde{C}} \to \partial^1_+\widetilde{C}$, over the skinning measure $\widetilde{\sigma}_{\widetilde{C}}$ of $\widetilde{C}$, with conditional measure $e^{\delta\, \beta_{w_+}(\pi(w),\,\pi(v))}\, d\mu^{\rm s}_w(v)$ on the fiber $f^{-1}_{\widetilde{C}}(w) = W^s(w)$ of $w \in \partial^1_+\widetilde{C}$:*

$$d\widetilde{m}_{\rm BM}(v) = \int_{w\in\partial^1_+\widetilde{C}} e^{\delta\, \beta_{w_+}(\pi(w),\,\pi(v))}\; d\mu^{\rm s}_w(v)\; d\widetilde{\sigma}_{\widetilde{C}}(w)\;. \quad \Box$$

7 Finite volume hyperbolic manifolds

In this section, we consider the special case when $\widetilde{M} = \mathbb{H}^n_{\mathbb{R}}$, Γ is a discrete group of isometries of $\widetilde{M}$ and $M = \Gamma\backslash\widetilde{M}$ has finite volume, and we relate the measures defined in Section 6 with more classical measures. For every $p \in \mathbb{N}$, we denote by λ_p the standard Lebesgue measure of $\mathbb{R}^p$.

Under the assumptions of this section, there exists a unique Patterson density $(\mu_x)_{x\in\mathbb{H}^n_{\mathbb{R}}}$ of dimension $n-1$ for Γ normalised to have total mass $\mathrm{Vol}(\mathbb{S}^{n-1})$ for every $x \in \mathbb{H}^n_{\mathbb{R}}$, which we call the *spherical density* and we now describe.

In the unit ball model of $\mathbb{H}^n_{\mathbb{R}}$ with origin 0, the measure μ_0 of the *spherical density* $(\mu_x)_{x\in\mathbb{H}^n_{\mathbb{R}}}$ is the Lebesgue measure of $\mathbb{S}^{n-1} = \partial_\infty\mathbb{H}^n_{\mathbb{R}}$ and (see for instance [BH, p. 273])

$$\frac{d\mu_x}{d\mu_0}(\xi) = e^{-(n-1)\beta_\xi(x,0)} = \left(\frac{1-\|x\|^2}{\|x-\xi\|^2}\right)^{n-1}.$$

In the upper halfspace model with point at infinity ∞, using the standard inversion mapping the ball model to the upper halfspace model, the *spherical density* $(\mu_x)_{x\in\mathbb{H}^n_{\mathbb{R}}}$ has the expression, for every $\xi \in \mathbb{R}^{n-1} = \partial_\infty\mathbb{H}^n_{\mathbb{R}} - \{\infty\}$,

$$d\mu_x(\xi) = \Big(\frac{2x_n}{\|x-\xi\|^2}\Big)^{n-1}\, d\lambda_{n-1}(\xi)\;, \tag{21}$$

where x_n is the last (vertical) coordinate of any $x \in \mathbb{H}^n_{\mathbb{R}}$.

In the unit ball model of $\mathbb{H}^n_{\mathbb{R}}$, the visual distance d_0 seen from the origin 0 (see Section 2.3) coincides with half the chordal distance (see for example [Bou]). In the upper halfspace model, an easy computation shows that the Busemann cocycle of $\mathbb{H}^n_{\mathbb{R}}$ is

$$\beta_\xi(x,y) = \ln\big(\frac{y_n}{x_n}\frac{\|x-\xi\|^2}{\|y-\xi\|^2}\big) \tag{22}$$

for all $x, y \in \mathbb{H}^n_{\mathbb{R}}$ and all $\xi \in \mathbb{R}^{n-1}$. By Equation (3), for any basepoint $x \in \mathbb{H}^n_{\mathbb{R}}$ and all $\xi, \eta \in \mathbb{R}^{n-1}$, we get an expression for the visual distance seen from x:

$$d_x(\xi,\eta) = \frac{x_n\|\xi-\eta\|}{\|x-\xi\|\|x-\eta\|}\ .$$

Thus, in the upper halfspace model, for any $v \in T^1\mathbb{H}^n_{\mathbb{R}}$ such that $v_\pm \neq \infty$, we have

$$d\widetilde{m}_{\rm BM}(v) = \frac{2^{2(n-1)}d\lambda_{n-1}(v_-)\ d\lambda_{n-1}(v_+)\ dt}{\|v_+ - v_-\|^{2(n-1)}}\ , \tag{23}$$

where t is the signed distance from the closest point to ∞ on the geodesic line $]v_-, v_+[$ to $\pi(v)$.

It is known that the Liouville measure, normalised to be a probability measure, is the probability measure of maximal entropy for the geodesic flow in constant curvature and finite volume. Thus, the Bowen-Margulis measure coincides (up to a positive multiplicative constant) with the Liouville measure. We now determine the proportionality constant.

Proposition 10 *Let M be a finite volume complete hyperbolic manifold of dimension $n \geq 2$, $d\,\mathrm{Vol}_{T^1M}$ its Liouville measure, and $dm_{\rm BM}$ its Bowen-Margulis measure, constructed using the spherical Patterson density. Then*

$$dm_{\rm BM} = 2^{n-1} d\,\mathrm{Vol}_{T^1M}\ .$$

In particular,

$$\|m_{\rm BM}\| = 2^{n-1}\,\mathrm{Vol}(\mathbb{S}^{n-1})\,\mathrm{Vol}(M)\ . \tag{24}$$

Proof. We use the upper halfspace model

$$\mathbb{H}^n_{\mathbb{R}} = \{x = (\overline{x}, x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}\ :\ x_n > 0\}\ .$$

We parametrise the unit tangent sphere at any point $x \in \mathbb{H}^n_{\mathbb{R}}$ by the positive endpoint $v_+ \in \mathbb{R}^{n-1}\cup\{\infty\}$ of a unit tangent vector $v \in T^1_x\mathbb{H}^n_{\mathbb{R}}$. This gives a parametrisation of $T^1\mathbb{H}^n_{\mathbb{R}}$ by pairs $(x, v_+) \in \mathbb{H}^n_{\mathbb{R}} \times (\mathbb{R}^{n-1}\cup\{\infty\})$. Recall that the Liouville measure disintegrates as

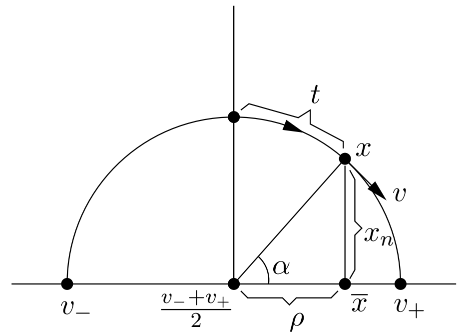
$$d\,\mathrm{Vol}_{T^1\mathbb{H}^n_{\mathbb{R}}} = \int_{x\in\mathbb{H}^n_{\mathbb{R}}} d\,\mathrm{Vol}_{T^1_x\mathbb{H}^n_{\mathbb{R}}}\ d\,\mathrm{Vol}_{\mathbb{H}^n_{\mathbb{R}}}(x)\ .$$

Hence in the full-measure subset where $v_+ \neq \infty$, the Liouville measure may be written

$$d\,\mathrm{Vol}_{T^1\mathbb{H}^n_{\mathbb{R}}} = \frac{(2x_n)^{n-1}\ d\lambda_{n-1}(v_+)}{\|x-v_+\|^{2(n-1)}}\ \frac{d\lambda_n(x)}{x_n^n} = \frac{2^{n-1}d\lambda_{n-1}(\overline{x})\,d\lambda_{n-1}(v_+)dx_n}{\|x-v_+\|^{2(n-1)}\ x_n}\ . \tag{25}$$

In order to relate the formulas (23) and (25), let us give the expression of the coordinates $(\overline{x}, x_n, v_+)$ in terms of the coordinates (v_-, v_+, t).

Let α be the angle between the segments $[\frac{v_-+v_+}{2}, v_+]$ and $[\frac{v_-+v_+}{2}, x]$. Let ρ be the algebraic distance from $\frac{v_-+v_+}{2}$ to $\overline{x}$ on the line through v_- and v_+ oriented from v_- to v_+. We have



$$\overline{x} = \frac{v_- + v_+}{2} + \rho \frac{v_+ - v_-}{\|v_+ - v_-\|}$$

and by a formula of [Bea, p. 147],

$$\sinh t = \frac{1}{\tan \alpha} = \frac{\rho}{x_n}\;.$$

Since $\rho^2 + x_n^2 = \|\frac{v_+-v_-}{2}\|^2$, we hence have

$$x_n = \frac{\|v_+ - v_-\|}{2\cosh t} \quad \text{and} \quad \overline{x} = \frac{v_- + v_+}{2} + \frac{v_+ - v_-}{2}\;\tanh t\;.$$

Writing $\overline{x} = (\overline{x}^1, \dots, \overline{x}^{n-1})$ and $v_\pm = (v_\pm^1, \dots, v_\pm^{n-1})$ and differentiating the above equations with v_+ constant, we have, for $i = 1\dots, n-1$,

$$dx_n = -\frac{\sinh t}{2\cosh^2 t}\,\|v_+ - v_-\|\;dt - \frac{1}{2\cosh t}\sum_{j=1}^{n-1} \frac{v_+^j - v_-^j}{\|v_+ - v_-\|}\;dv_-^j$$

and

$$d\overline{x}^i = \frac{1 - \tanh t}{2}\;dv_-^i + \frac{v_+^i - v_-^i}{2\cosh^2 t}\;dt\;.$$

Therefore an easy computation, using the facts that $\overline{x} - v_+ = \frac{1-\tanh t}{2}(v_- - v_+)$ and $\|x - v_+\|^2 = \|\overline{x} - v_+\|^2 + x_n^2 = \frac{1-\tanh t}{2}\|v_+ - v_-\|^2$, shows that

$$\begin{aligned} d\overline{x}^1 \wedge \dots \wedge d\overline{x}^{n-1} \wedge dx_n &= \frac{\|v_+ - v_-\|}{2\cosh t}\Big(\frac{1 - \tanh t}{2}\Big)^{n-1}\;dv_-^1 \wedge \dots \wedge dv_-^{n-1} \wedge dt \\ &= x_n \Big(\frac{\|x - v_+\|^2}{\|v_+ - v_-\|^2}\Big)^{n-1}\;dv_-^1 \wedge \dots \wedge dv_-^{n-1} \wedge dt\;. \end{aligned}$$

The result then follows from the formulas (23) and (25). □

Let now C be either a Margulis cusp neighbourhood in M or a totally geodesic immersed submanifold of M with finite volume, and let us relate the skinning measure of C to the usual Riemannian measure on the outer unit normal bundle of C. Note that the Riemannian measure $d\operatorname{Vol}_{\partial_+^1 C}$ disintegrates with respect to the basepoint fibration $\partial_+^1 C \to \partial C$ over the Riemannian measure of ∂C, with measure on the fiber of $x \in \partial C$ the spherical measure on the outer unit normal vectors to C at x:

$$d\operatorname{Vol}_{\partial_+^1 C} = \int_{x \in \partial C} d\operatorname{Vol}_{\partial_+^1 C \cap T_x^1 M}\;\;d\operatorname{Vol}_{\partial C}(x)\;. \tag{26}$$

Homogeneity considerations show that the skinning measure σ_C coincides up to a multiplicative constant with the Riemannian measure $\operatorname{Vol}_{\partial_+^1 C}$. We now compute the constant.

Proposition 11 *Let M be a finite volume complete hyperbolic manifold of dimension $n \geq 2$. We use the spherical Patterson density to define the skinning measures.*

(1) If C is a Margulis cusp neighbourhood, then

$$\sigma_C = 2^{n-1}\,\mathrm{Vol}_{\partial^1_+ C}\ .$$

(2) If C is a finite volume totally geodesic properly immersed submanifold of M, then

$$\sigma_C = \mathrm{Vol}_{\partial^1_+ C}\ .$$

In particular, if C is a Margulis cusp neighbourhood of M, then (see for instance [Hers, p. 473] for the last equality)

$$\|\sigma_C\| = 2^{n-1}\,\mathrm{Vol}(\partial^1_+ C) = 2^{n-1}\,\mathrm{Vol}(\partial C) = 2^{n-1}(n-1)\,\mathrm{Vol}(C)\ ,$$

and if C is a finite volume totally geodesic properly immersed submanifold of dimension $k \in \{1, \dots, n-1\}$ of M, then

$$\|\sigma_C\| = \mathrm{Vol}(\mathbb{S}^{n-k-1})\,\mathrm{Vol}(C)\ .$$

Proof. (1) Consider the horoball $\widetilde{C}$ in the upper halfspace model of $\mathbb{H}^n_{\mathbb{R}}$ that consists of the points whose last (vertical) coordinate is at least 1. Fix a basepoint $x_0 = (0, 1) \in \mathbb{R}^{n-1}\times\ ]0, +\infty[$. Note that the closest point to $\xi \in \mathbb{R}^{n-1}$ in $\widetilde{C}$ is $P_{\widetilde{C}}(\xi) = (\xi, 1) \in \mathbb{R}^{n-1}\times\ ]0, +\infty[$. Using the definition of the skinning measure for the first equality and the formulas (21) and (22) for the second one, we hence have

$$\begin{aligned} d\widetilde{\sigma}_{\widetilde{C}}(w) &= e^{-(n-1)\beta_{w_+}(P_{\widetilde{C}}(w_+),\, x_0)}\, d\mu_{x_0}(w_+) \\ &= (\|x_0 - w_+\|^2)^{n-1}\Big(\frac{2}{\|x_0 - w_+\|^2}\Big)^{n-1}\, d\lambda_{n-1}(w_+) = 2^{n-1} d\lambda_{n-1}(w_+)\,. \end{aligned}$$

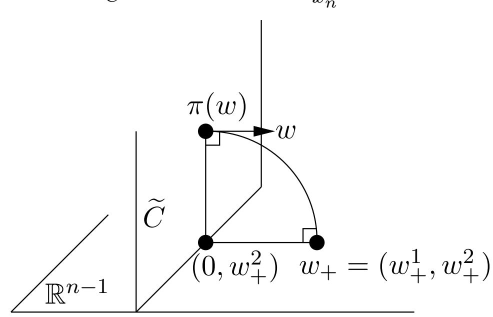
Since $\partial\widetilde{C} = \{(\overline{x}, 1)\ :\ \overline{x} \in \mathbb{R}^{n-1}\}$ is a codimension one submanifold of $\mathbb{H}^n_{\mathbb{R}}$, whose induced Riemannian metric is isometric to the Euclidean metric on $\mathbb{R}^{n-1}$ by the map $(\overline{x}, 1) \mapsto \overline{x}$, the result follows.

(2) Let $1 \leq k \leq n-1$. In the upper halfspace model of $\mathbb{H}^n_{\mathbb{R}}$ with basepoint $x_0 = (0, \dots, 0, 1)$, consider the k-dimensional totally geodesic subspace,

$$\widetilde{C} = \{x = (x_1, \dots, x_n) \in \mathbb{H}^n_{\mathbb{R}} : x_1 = \dots = x_{n-k} = 0\}\ ,$$

which is isometric to $\mathbb{H}^k_{\mathbb{R}}$ and has Riemannian volume $d\,\mathrm{Vol}_{\widetilde{C}} = \frac{d\lambda_{k-1}(x_{n-k+1},\dots,x_{n-1})\,d\lambda_1(x_n)}{x_n^k}$.

For any $\xi = (\xi^1, \xi^2) \in \mathbb{R}^{n-k} \times \mathbb{R}^{k-1} = \mathbb{R}^{n-1} = \partial_\infty \mathbb{H}^n_{\mathbb{R}} - \{\infty\}$, the closest point to ξ in $\widetilde{C}$ is $P_{\widetilde{C}}(\xi) = (0, \xi^2, \|\xi^1\|) \in \mathbb{R}^{n-k} \times \mathbb{R}^{k-1}\times\ ]0, +\infty[\ = \mathbb{H}^n_{\mathbb{R}}$. Note that $\pi(w)_n = \|w^1_+\|$ and $\|\pi(w) - w_+\|^2 = 2\,\pi(w)_n \|w^1_+\|$ for every $w \in \partial^1_+ \mathbb{H}^k_{\mathbb{R}}$. Recall that the map $w \mapsto w_+$ from $\partial^1_+ \widetilde{C}$ to $\partial_\infty \mathbb{H}^n_{\mathbb{R}} - \partial_\infty \widetilde{C} = \mathbb{R}^{n-1} - (\mathbb{R}^{k-1} \times \{0\})$ is a homeomorphism.



Using the definition of the skinning measure for the first equality and the formulas (21) and (22) for the second one, we hence get

$$\begin{aligned} d\widetilde{\sigma}_{\widetilde{C}}(w) &= e^{-(n-1)\beta_{w_+}(P_{\widetilde{C}}(w_+),\,x_0)}\,d\mu_{x_0}(w_+) \\ &= \Big(\frac{\pi(w)_n}{1}\,\frac{\|x_0-w_+\|^2}{\|\pi(w)-w_+\|^2}\Big)^{n-1}\Big(\frac{2}{\|x_0-w_+\|^2}\Big)^{n-1}\,d\lambda_{n-1}(w_+) = \frac{d\lambda_{n-1}(w_+)}{\|w_+^1\|^{n-1}}\;. \end{aligned}$$

On the other hand, by Equation (26), we have

$$d\,\mathrm{Vol}_{\partial^1_+\widetilde{C}}(w) = d\,\mathrm{Vol}_{\mathbb{S}^{n-k-1}}\big(\frac{w_+^1}{\|w_+^1\|}\big)\;\frac{d\lambda_{k-1}(w_+^2)\;d\lambda_1(\|w_+^1\|)}{\|w_+^1\|^k}\;.$$

Using spherical coordinates on the first factor of $\mathbb{R}^{n-1}=\mathbb{R}^{n-k}\times\mathbb{R}^{k-1}$, we have

$$d\lambda_{n-1}(w_+) = \|w_+^1\|^{n-k-1}\;d\,\mathrm{Vol}_{\mathbb{S}^{n-k-1}}\big(\frac{w_+^1}{\|w_+^1\|}\big)\;d\lambda_1(\|w_+^1\|)\;d\lambda_{k-1}(w_+^2)\;.$$

Hence $\widetilde{\sigma}_{\widetilde{C}}=\mathrm{Vol}_{\partial^1_+\widetilde{C}}$, and the result follows by taking quotients. □

8 The main counting result of common perpendiculars

Let M be a nonelementary complete connected Riemannian manifold with dimension at least 2 and pinched sectional curvature at most -1. Let $\widetilde{M}\to M$ be a universal Riemannian cover of M, with covering group Γ. Let δ be the critical exponent of Γ. We assume that the Bowen-Margulis measure $m_{\rm BM}$ of M is finite and mixing for the geodesic flow.

Theorem 12 (Parkkonen-Paulin [PP7]) *Let C_- and C_+ be two properly immersed closed convex subsets of M. Assume that their skinning measures σ_{C_-} and σ_{C_+} are finite and nonzero. Then, as $s\to+\infty$,*

$$\mathscr{N}_{C_-,\,C_+}(s)\sim\frac{\|\sigma_{C_-}\|\,\|\sigma_{C_+}\|}{\|m_{\rm BM}\|}\;\frac{e^{\delta s}}{\delta}\;.$$

As in Hermann's result (see Equation (8) in Section 3.4) or Oh-Shah's result (see the end of Section 3.6), the endpoints of the common perpendiculars are evenly distributed simultaneously on C_- and on C_+, in the following sense.

Theorem 13 (Parkkonen-Paulin [PP7]) *Let C_- and C_+ be two properly immersed closed convex subsets of M. Let Ω^- and Ω^+ be relatively compact subsets of $\partial^1_+C_-$ and $\partial^1_+C_+$, respectively. Assume that $\sigma_{C_-}(\Omega^-)\neq 0$, $\sigma_{C_+}(\Omega^+)\neq 0$ and $\sigma_{C_-}(\partial\Omega^-)=\sigma_{C_+}(\partial\Omega^+)=0$. Then, as $s\to+\infty$, the number $\mathscr{N}_{\Omega^-,\,\Omega^+}(s)$ of common perpendiculars of C_- and C_+, with lengths at most s, and with initial vector in Ω^- and terminal vector in $\iota\,\Omega^+$, satisfies*

$$\mathscr{N}_{\Omega^-,\,\Omega^+}(s)\sim\frac{\sigma_{C_-}(\Omega^-)\;\sigma_{C_+}(\Omega^+)}{\|m_{\rm BM}\|}\;\frac{e^{\delta s}}{\delta}\;.$$

Let us give a brief sketch of proof of these results, which uses directly the mixing property of the geodesic flow (and avoids the equidistribution step in Margulis's scheme of proof). This will, in particular, allow us in Section 9 to give estimates on the error terms in the presence of exponential decay of correlations. We refer to [PP7] for complete proofs.

By definition, C_- and C_+ are the images in M of two proper nonempty closed convex subsets $\widetilde{C}_-$ and $\widetilde{C}_+$ in $\widetilde{M}$, whose Γ-orbits are locally finite.

We introduce dynamical neighbourhoods of $\partial^1_+ C_-$ and $\partial^1_+ C_+$, and we define bump functions supported in them, to which we will apply the mixing property. We fix $\eta > 0$ small enough and $R > 0$ big enough.

For every $w \in T^1\widetilde{M}$, let $V_{w,R}$ be the ball of center w and radius R for Hamenstädt's distance $d_{W^{ss}(w)}$ on the strong stable leaf $W^{ss}(w)$ of w (see Section 2.4). For every proper nonempty closed convex subset $\widetilde{D}$ in $\widetilde{M}$ whose Γ-orbit is locally finite, let $\mathscr{V}_{\eta,R}(\widetilde{D})$ be the union for all $w \in \partial^1_+\widetilde{D}$ and $s \in \,]-\eta,\eta[$ of the sets $g^s V_{w,R}$. These dynamical neighbourhoods $\mathscr{V}_{\eta,R}(\widetilde{D})$ are natural under isometries, hence, with D is the image of $\widetilde{D}$ in M, they allow to define nice neighbourhoods $\mathscr{V}_{\eta,R}(D)$ of $\partial^1_+ D$, that scale nicely under the geodesic flow: $g^t\mathscr{V}_{\eta,R}(\widetilde{D}) = \mathscr{V}_{\eta,e^{-t}R}(\mathscr{N}_t\widetilde{D})$ for every $t \geq 0$.

Let $h_{\eta,\,R} : T^1\widetilde{M} \to [0,+\infty]$ be the measurable map defined by $w \mapsto \frac{1}{2\eta\,\mu^{\rm ss}_w(V_{w,\,R})}$. The constant $R > 0$ is chosen big enough so that the above denominator is nonzero if $w \in \partial^1_+\widetilde{C}_\pm$ (see [PP7, Lem. 7]). We denote by χ_A the characteristic function of a subset A. Let $\widetilde{\phi}_{\eta,\widetilde{D}} : T^1\widetilde{M} \to [0,+\infty]$ be the map defined by (using the convention $\infty \times 0 = 0$)

$$\widetilde{\phi}_{\eta,\widetilde{D}}(v) = \sum_{\gamma\in\Gamma/\Gamma_{\widetilde{D}}} h_{\eta,\,R}\circ f_{\gamma\widetilde{D}}(v)\ \chi_{\mathscr{V}_{\eta,\,R}(\gamma\widetilde{D})}(v)\ ,$$

wher $\Gamma_{\widetilde{D}}$ is the stabiliser of $\widetilde{D}$ in Γ. The function $\widetilde{\phi}_{\eta,\widetilde{D}}$ is invariant under Γ, hence defines by taking the quotient by Γ a test function $\phi_{\eta,D} : T^1M \to [0,+\infty]$. Now define $\phi^-_\eta = \phi_{\eta,C_-}$ and $\phi^+_\eta = \phi_{\eta,C_+}\circ\iota$. The invariance of the Bowen-Margulis measure by the antipodal map and the disintegration result of Proposition 9 allow to prove (see [PP7, Prop. 16]) that

$$\int_{T^1M} \phi^\pm_\eta\; dm_{\rm BM} = \|\sigma_{C_\pm}\|\ . \tag{27}$$

The main trick in the proof is to estimate in two ways the integral

$$\mathscr{I}_\eta(t) = \int_{T^1M} \phi^-_\eta\circ g^{t/2}\;\phi^+_\eta\circ g^{-t/2}\; dm_{\rm BM}\ .$$

On one hand, by Equation (27) and the mixing property of the geodesic flow, the integral $\mathscr{I}_\eta(t)$ converges, for every fixed $\eta > 0$, to $\frac{\|\sigma_{C_-}\|\,\|\sigma_{C_-}\|}{\|m_{\rm BM}\|}$ as $t \to +\infty$.

On the other hand, a vector $v \in T^1M$, with a fixed lift $\widetilde{v}$ to $T^1\widetilde{M}$, belongs to the support of $\phi^-_\eta\circ g^{t/2}\,\phi^+_\eta\circ g^{-t/2}$ if and only if $g^{-t/2}v$ belongs to the support of ϕ^-_η and $g^{t/2}v$ belongs to the support of ϕ^+_η, that is, if and only if there exist $\gamma^\pm \in \Gamma$, $s^\pm \in \,]-\eta,\eta[$, $w^\pm \in \gamma^\pm\partial^1_+\widetilde{C}_\pm$ and $v^\pm \in V_{w^\pm,R}$ such that $\widetilde{v} = g^{\frac{t}{2}+s^-}v^- = g^{-\frac{t}{2}-s^+}\iota v^+$. For every $\epsilon > 0$, by the properties of negative curvature, this implies, if η is small enough, and uniformly in t big enough, that $\pi(\widetilde{v})$ is not far from the midpoint of a common perpendicular arc between $\gamma^-\widetilde{C}_-$ and $\gamma^+\widetilde{C}_+$, of length close to $]t-\eta,t+\eta[$, and that $g^{t/2}\gamma^-\partial^1_+\widetilde{C}_-$ is close to a piece of strong unstable leaf at $\widetilde{v}$, and $g^{-t/2}\gamma^+\iota\,\partial^1_+\widetilde{C}_+$ is close to a piece of strong stable leaf at $\widetilde{v}$. Furthermore, each such midpoint contributes to the integral $\mathscr{I}_\eta(t)$, by an amount which is, as η is small and uniformly in t big enough, almost $\frac{e^{-\delta t}}{2\eta}$. By a Cesaro type of argument, the results follows, by integrating $e^{\delta t}$.

To pass from Theorem 12 to Theorem 13, we replace $\partial^1_+ C_-$ and $\partial^1_+ C_+$ by Ω^- and Ω^+, the endpoints of the common perpendicular constructed above being close to $\gamma_-\Omega^-$ and $\gamma_+\Omega^+$, which have measure 0 boundary.

We end this section by completing the list of examples given in Section 3, adding the following two cases. They follow (see [PP7]) by applying the main Theorem 12, the remarks following the statement of Proposition 11, and Equation (24).

Corollary 14 *Let M be a finite volume complete hyperbolic manifold of dimension $n \geq 2$.*
(1) If C_- and C_+ are properly immersed finite volume totally geodesic submanifolds of M of dimensions k_- and k_+ in $[1, n-1]$, respectively, then

$$\mathscr{N}(s) \sim \frac{\operatorname{Vol}(\mathbb{S}^{n-k_--1})\operatorname{Vol}(\mathbb{S}^{n-k_+-1})}{2^{n-1}\operatorname{Vol}(\mathbb{S}^{n-1})}\,\frac{\operatorname{Vol}(C_-)\operatorname{Vol}(C_+)}{\operatorname{Vol}(M)}\,\frac{e^{(n-1)s}}{n-1}\,,$$

as $s \to +\infty$.
(2) If $C_- = \mathscr{H}_-$ and $C_+ = \mathscr{H}_+$ are Margulis cusp neighbourhoods in M, then

$$\mathscr{N}(s) \sim \frac{2^{n-1}(n-1)\operatorname{Vol}(\mathscr{H}_-)\operatorname{Vol}(\mathscr{H}_+)}{\operatorname{Vol}(\mathbb{S}^{n-1})\operatorname{Vol}(M)}\; e^{(n-1)s}\,. \quad \square$$

In particular, if C_- and C_+ are closed geodesics of M of lengths ℓ_- and ℓ_+, respectively, then the number $\mathscr{N}(s)$ of common perpendiculars (counted with multiplicity) between C_- and C_+ of length at most s satisfies, as $s \to +\infty$,

$$\mathscr{N}(s) \sim \frac{\operatorname{Vol}(\mathbb{S}^{n-2})^2}{2^{n-1}\operatorname{Vol}(\mathbb{S}^{n-1})}\,\frac{\ell_-\,\ell_+}{\operatorname{Vol}(M)}\,\frac{e^{(n-1)s}}{n-1}\,.$$

9 Spectral gaps, exponential decay of correlations and error terms

Let M be a nonelementary complete connected Riemannian manifold with dimension at least 2 and pinched sectional curvature at most -1. Let $\widetilde{M} \to M$ be a universal Riemannian cover of M, with covering group Γ. Let δ be the critical exponent of Γ. We assume that the Bowen-Margulis measure $m_{\rm BM}$ of M is finite and mixing for the geodesic flow. We denote by $\overline{m}_{\rm BM} = \frac{m_{\rm BM}}{\|m_{\rm BM}\|}$ its normalisation to a probability measure.

In this section, we give error terms in our main counting result, when the geodesic flow is exponentially mixing, Recall that there are two types of exponential mixing results.

Firstly, when M is locally symmetric with finite volume, then the boundary at infinity of $\widetilde{M}$, the strong unstable, unstable, stable, and strong stable foliations of M are smooth. Hence talking about $\mathscr{C}^\ell$-smooth leafwise defined functions on T^1M makes sense. We will denote by $\mathscr{C}^\ell_c(T^1M)$ the vector space of $\mathscr{C}^\ell$ smooth functions on T^1M with compact support and by $\|\psi\|_\ell$ the Sobolev $W^{\ell,2}$-norm of any $\psi \in \mathscr{C}^\ell_c(T^1M)$. Note that now the Bowen-Margulis measure of T^1M is the unique (up to a multiplicative constant) locally homogeneous smooth measure on T^1M (hence it coincides, up to a multiplicative constant, with the Liouville measure).

Given $\ell \in \mathbb{N}$, we will say that the geodesic flow on T^1M is *exponentially mixing for the Sobolev regularity* ℓ (or that it has *exponential decay of* ℓ*-Sobolev correlations*) if there exist $c, \kappa > 0$ such that for all $\phi, \psi \in \mathscr{C}_c^\ell(T^1M)$ and all $t \in \mathbb{R}$, we have

$$\Big|\int_{T^1M} \phi \circ g^{-t}\, \psi\, d\overline{m}_{\rm BM} - \int_{T^1M} \phi\, d\overline{m}_{\rm BM} \int_{T^1M} \psi\, d\overline{m}_{\rm BM}\Big| \leq c\, e^{-\kappa|t|}\ \|\psi\|_\ell\ \|\phi\|_\ell\ .$$

When Γ is a torsion free arithmetic lattice in the isometry group of $\widetilde{M}$, this property, for some $\ell \in \mathbb{N}$, follows from [KM1, Theo. 2.4.5], with the help of [Clo, Theo. 3.1] to check its spectral gap property, and of [KM2, Lem. 3.1] to deal with finite cover problems.

Secondly, when $\widetilde{M}$ is only assumed to be as in the beginning of this section, then the boundary at infinity, the strong unstable, unstable, stable, and strong stable foliations are only Hölder smooth, hence the appropriate regularity on functions on $\widetilde{M}$ is the Hölder one. We denote by $\mathrm{C}_\mathrm{c}^\alpha(X)$ the space of α-Hölder continuous real-valued functions with compact support on a metric space (X, d), endowed with the Hölder norm

$$\|f\|_\alpha = \|f\|_\infty + \sup_{x,\, y \in X,\ x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}\ .$$

Assuming the Bowen-Margulis measure $m_{\rm BM}$ on T^1M to be finite, given $\alpha \in\,]0, 1[$, we will say that the geodesic flow on T^1M is *exponentially mixing for the Hölder regularity* α (or that it has *exponential decay of* α*-Hölder correlations*) if there exist $\kappa, c > 0$ such that for all $\phi, \psi \in \mathrm{C}_\mathrm{c}^\alpha(T^1M)$ and all $t \in \mathbb{R}$, we have

$$\Big|\int_{T^1M} \phi \circ g^{-t}\, \psi\, d\overline{m}_{\rm BM} - \int_{T^1M} \phi\, d\overline{m}_{\rm BM} \int_{T^1M} \psi\, d\overline{m}_{\rm BM}\Big| \leq c\, e^{-\kappa|t|}\ \|\phi\|_\alpha\ \|\psi\|_\alpha\,.$$

This holds if $\widetilde{M}$ has dimension 2 by the work of Dolgopyat [Dol] or if $\widetilde{M}$ is locally symmetric by [Sto, Coro. 1.5] (see also [Liv] when M is compact, the result stated for the Liouville measure should extend to the Bowen-Margulis measure).

Using smoothening processes of the functions $\phi_\eta^\pm$ introduced in the sketch of proof of Section 8, we obtain the following error terms in our main counting result Theorem 12.

Theorem 15 (Parkkonen-Paulin [PP7]) *Let C_- and C_+ be two properly immersed closed convex subsets of M. Assume that their skinning measures σ_{C_-} and σ_{C_+} are finite and nonzero. Assume that the geodesic flow is exponentially mixing (for the Hölder regularity or for the Sobolev regularity). Then there is some $\kappa > 0$ such that, as $s \to +\infty$,*

$$\mathscr{N}_{C_-, C_+}(s) \sim \frac{\|\sigma_{C_-}\|\, \|\sigma_{C_+}\|}{\|m_{\rm BM}\|}\, \frac{e^{\delta s}}{\delta}\big(1 + \mathrm{O}(e^{-\kappa s})\big)\ .$$

This error term is also valid for the effective counting Theorem 13. This result gives in particular an exponential control in the error terms in the list of examples given in Section 3, as well as in Corollary 14.

As an application of Theorem 15, using Humbert's formula and the area of the fundamental domain of $\mathscr{O}_K$ in $\mathbb{R}^2$ (see for example [EGM, p. 318]), we get a version of Cosentino's asymptotic estimate (9) on the number of common perpendiculars from the Margulis cusp neighbourhood corresponding to the horoball of points with vertical coordinates at most 1 to itself in $\mathrm{PSL}(\mathscr{O}_K)\backslash\mathbb{H}^3_{\mathbb{R}}$, valid for all discriminants:

$$\mathscr{N}(s) = \frac{\pi\,|\mathscr{O}_K^\times|^2}{4\sqrt{|D_K|}\,\zeta_K(2)}\; e^{2s}\big(1+\mathrm{O}(e^{-\kappa s})\big),$$

when $s \to +\infty$.

10 Gibbs measures and counting arcs with weights

Let M be a nonelementary complete connected Riemannian manifold with dimension at least 2 and pinched sectional curvature at most -1. Let $\widetilde{M} \to M$ be a universal Riemannian cover of M, with covering group Γ.

When counting geodesic arcs, it is sometimes useful to give them a higher weight if they are passing through a given region of M, and even more precisely, through a given region in position and direction. The trick is to introduce a *potential*, that is a Hölder continuous map $F : T^1M \to \mathbb{R}$. To shorten the exposition, we will assume in this survey that F is bounded, nonnegative and *reversible*, that is, that $F\circ\iota = F$. Neither assumption is necessary, up to the appropriate modifications, see [PP7]. Given a piecewise smooth path $c : [a,b] \to M$, one defines its *weighted length* for the potential F as

$$\int_c F = \int_a^b F\circ \dot c\,(t)\; dt\ .$$

We are now going to adapt the material of Section 6 to the weighted case, see for instance [Led, Ham2, Cou, Sch, Moh, PPS, PP7] with an emphasis on the last two ones for more information.

Let $\widetilde{F} = F\circ Tp : T^1\widetilde{M} \to \mathbb{R}$ be the lift of F by the universal cover $p : \widetilde{M} \to M$. For every x,y in $\widetilde{M}$, if $c : [0, d(x,y)] \to \widetilde{M}$ is the geodesic path from x to y, let $\int_x^y \widetilde{F} = \int_0^{d(x,y)} \widetilde{F}\circ\dot c\,(t)\; dt$. The *critical exponent* of the potential F is

$$\delta_F = \lim_{n\to+\infty}\frac{1}{n}\ln \sum_{\gamma\in\Gamma,\, d(x,\gamma y)\le n} e^{\int_x^{\gamma y}\widetilde{F}}\ ,$$

see [PPS, Theo. 3.2] for the existence and finiteness of the above limit and its independence on $x, y \in \widetilde{M}$. Replacing the previous critical exponent δ, the critical exponent δ_F of the potential F will give the exponential growth rate in the counting of weighted common perpendiculars.

Similarly, the Busemann cocycle $\beta_\xi(x,y)$ needs to be replaced. The *Gibbs cocycle* associated to the potential F is the function $C = C^F : \partial_\infty\widetilde{M}\times\widetilde{M}\times\widetilde{M} \to \mathbb{R}$ defined by

$$(\xi,x,y)\mapsto C_\xi(x,y) = \lim_{t\to+\infty}\int_y^{\xi_t}(\widetilde{F}-\delta_F) - \int_x^{\xi_t}(\widetilde{F}-\delta_F)\,,$$

where $t\mapsto\xi_t$ is any geodesic ray with endpoint $\xi\in\partial_\infty\widetilde{M}$. The Gibbs cocycle satisfies obvious equivariance and cocycle properties: For all $x,y,z\in\widetilde{M}$, and for every isometry γ of $\widetilde{M}$, we have

$$C_{\gamma\xi}(\gamma x,\gamma y) = C_\xi(x,y)\ \text{ and }\ C_\xi(x,z)+C_\xi(z,y) = C_\xi(x,y)\ . \tag{28}$$

Similarly, the Bowen-Margulis measure needs to be replaced. Let $r > 0$. A family $(\mu_x)_{x\in\widetilde{M}}$ of finite Borel measures on $\partial_\infty\widetilde{M}$ whose support is the limit set $\Lambda\Gamma$ of Γ is a *Patterson density of dimension* r *for the potential* F if

$$\gamma_*\mu_x = \mu_{\gamma x}$$

for all $\gamma \in \Gamma$ and all $x \in \widetilde{M}$, and if the following Radon-Nikodym derivative exists for all $x, y \in \widetilde{M}$ and satisfies, for all $\xi \in \partial_\infty\widetilde{M}$,

$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{-C_\xi(x,y)} \; .$$

Let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension δ_F for the potential F. The *Gibbs measure* on $T^1\widetilde{M}$ for the potential F is the measure $\widetilde{m}_F$ on $T^1\widetilde{M}$ given by

$$d\widetilde{m}_F(v) = e^{C_{v_-}(x_0,\pi(v))+C_{v_+}(x_0,\pi(v))}\, d\mu_{x_0}(v_-)\, d\mu_{x_0}(v_+)\, dt \; ,$$

using Hopf's parametrisation. The Gibbs measure $\widetilde{m}_F$ is independent of the basepoint $x_0 \in \widetilde{M}$ used in its definition, and it is invariant under the actions of the group Γ and the geodesic flow. Thus, it defines a measure m_F on T^1M which is invariant under the geodesic flow, called the *Gibbs measure* on T^1M for the potential F. When the Gibbs measure m_F is finite, there exists a unique (up to a multiplicative constant) Patterson density of dimension δ_F for the potential F; the probability measure $\frac{m_F}{\|m_F\|}$ is uniquely defined; it is the unique probability measure of maximal pressure for the geodesic flow and the potential F; see [PPS] for proofs of these claims. When finite, the Gibbs measure on T^1M is mixing if the geodesic flow is topologically mixing, see [Bab1].

Let D be a properly immersed closed convex subset of M, and let $\widetilde{D}$ be a proper nonempty closed convex subset of $\widetilde{M}$, whose Γ-orbit is locally finite and whose image in M is D. We also need to adapt the skinning measures to the presence of the potential F. The *skinning measure* of $\widetilde{D}$ for the potential F is the measure $\widetilde{\sigma}^F_{\widetilde{D}}$ on $\partial^1_+\widetilde{D}$, defined, using the homeomorphism $v \mapsto v_+$ from $\partial^1_+\widetilde{D}$ to $\partial_\infty\widetilde{M} - \partial_\infty\widetilde{D}$, by

$$d\widetilde{\sigma}^F_{\widetilde{D}}(v) = e^{C_{v_+}(x_0,\, P_{\widetilde{D}}(v_+))}\, d\mu_{x_0}(v_+) \; .$$

It is independent of the basepoint x_0, and satisfies $\gamma_*(\widetilde{\sigma}^F_{\widetilde{D}}) = \widetilde{\sigma}^F_{\gamma\widetilde{D}}$ for every $\gamma \in \Gamma$. Let $\Gamma_{\widetilde{D}}$ be the stabiliser in Γ of $\widetilde{D}$. The Γ-invariant locally finite Borel positive measure $\sum_{\gamma\in\Gamma/\Gamma_{\widetilde{D}}} \gamma_*\widetilde{\sigma}^F_{\widetilde{D}}$ defines, through the covering $T^1\widetilde{M} \to T^1M$, a locally finite measure σ^F_D, called the *skinning measure* of D for the potential F. See [PP7] for further information on the skinning measures with potential.

Let D_-, D_+ be two properly immersed closed convex subsets of M. For every $s \geq 0$, recall that $\mathrm{Perp}_{D_-,D_+}(s)$ is the set of the common perpendiculars from D_- to D_+ having lengths at most s. The *weighted counting function* of common perpendiculars between D_- and D_+ (counted with multiplicities) for the potential F is

$$\mathscr{N}_{D_-,D_+,F}(s) = \sum_{c\in\mathrm{Perp}_{D_-,D_+}(s)} m(c)\; e^{\int_c F} \; .$$

Using the same scheme of proof as explained in Section 8, we have the following asymptotic result.

Theorem 16 (Parkkonen-Paulin [PP7]) *Let M be a nonelementary complete connected Riemannian manifold with pinched sectional curvature $-b^2 \leq K \leq -1$, and let $F : T^1M \to \mathbb{R}$ be a (bounded, nonnegative, reversible) Hölder continuous map. Let δ_F be the critical exponent of the potential F. Assume that the Gibbs measure m_F is finite and mixing for the geodesic flow. Let D_- and D_+ be two properly immersed closed convex subsets of M. Assume that $\sigma^F_{D_-}$ and $\sigma^F_{D_+}$ are finite and nonzero. Then, as $s \to +\infty$,*

$$\mathscr{N}_{D_-,D_+,F}(s) \sim \frac{\|\sigma^F_{D_-}\| \, \|\sigma^F_{D_+}\|}{\|m_F\|} \, \frac{e^{\delta_F s}}{\delta_F} \, .$$

We have error terms in the presence of exponential decay of correlations, and the endpoints of the common perpendiculars are evenly distributed, that is, we may restrict to counting the common perpendiculars with endpoints in measurable subsets Ω_- and Ω_+, with finite nonzero skinning measures for the potential F and negligible boundary, of $\partial^1_+ D_-$ and $\partial^1_+ D_+$, respectively, as in the two previous sections, see [PP7] for precise statements and proofs.

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