Construction of quasimodes for non-selfadjoint operators under finite-type dynamical conditions

Víctor Arnaiz Solórzano

CNRS and Université Paris-Saclay

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Problèmes Spectraux en Physique Mathématique

The spectral theory for selfadjoint operators on Hilbert spaces is quite confortable. We have the resolvent estimate

$$\|(P-\zeta)^{-1}\| = \big(\operatorname{dist}(\zeta,\sigma(P))\big)^{-1},$$

and the spectral theorem also gives very good control over functions of selfadjoint operators, so for instance if P is selfadjoint with $\sigma(P) \subset [\lambda_0, +\infty)$, then

$$\|e^{-tP}\| \leq e^{-\lambda_0 t}, \quad t \geq 0.$$

Motivation: Why non-selfadjoint operators?

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However, non-normal operators appear frequently in different problems; Scattering poles, Convection-diffusion problems, Kramers-Fokker-Planck equation, damped wave equations, linearized operators in fluid dynamics. Typically, $\|(P-\zeta)^{-1}\|$ may be very large even when ζ is far from the spectrum.

Let us consider the typical example

$$\widehat{P}_\hbar := -\hbar^2\Delta + V(x), \quad x \in \mathbb{R}^d.$$

The *semiclassical pseudo-spectrum* of $P(\hbar)$ is defined as

$$\Lambda(p) = \overline{\{p(x,\xi) = \xi^2 + V(x) \,:\, (x,\xi) \in \mathbb{R}^{2d},\ \Im\langle \xi, \partial V(x)\rangle \neq 0\}}.$$

Theorem (Davies, 1999; Zworski, 2001)

Suppose that $V \in \mathcal{C}^\infty(\mathbb{R}^d)$*. Then, for any* $\zeta \in \{\xi^2 + V(x) \,:\, (x,\xi) \in \mathbb{R}^{2d},\ \Im\langle \xi, \partial V(x)\rangle \neq 0\}$*, there exists* $(\psi_\hbar) \subset L^2(\mathbb{R}^d)$ *with the property*

$$\|(\widehat{P}_\hbar - \zeta)\psi_\hbar\|_{L^2(\mathbb{R}^d)} = O(\hbar^\infty)\|\psi_\hbar\|_{L^2}.$$

Moreover, $WF_\hbar(\psi_\hbar) = \{(x_0,\xi_0)\}$ *for some* (x_0,ξ_0) *with* $p(x_0,\xi_0) = \zeta$*.*

Take $(x_0, \xi_0) \in p^{-1}(\zeta)$ with $\Im\langle \xi_0, \partial V(x_0)\rangle < 0$. Then the submanifold $p^{-1}(\zeta) \subset \mathbb{R}^{2d}$ has codimension two. The symplectic form restricted to this submanifold is non-degenerate. One can then find a local canonical transformation $\kappa : (x_0, \xi_0) \mapsto (0, 0)$ such that

$$\kappa^*(\xi_1 - ix_1) = up,$$

for some smooth function u with $u(x_0, \xi_0) > 0$, and a Fourier integral operator T such that

$$\widehat{P}_\hbar = T^{-1} A(\hbar D_{x_1} - ix_1) T,$$

microlocally near $((x_0, \xi_0), (0, 0))$, where A is elliptic at $(0, 0)$. Then defining

$$\varphi_0^\hbar(x) := \frac{1}{(\pi\hbar)^{d/4}} e^{-\frac{|x|^2}{2\hbar}},$$

one can take $\psi_\hbar := T^{-1}\varphi_0^\hbar$.

Estimates on the boundary of the pseudospectrum

Consider an operator

$$\widehat{P}_\hbar = \mathrm{Op}_\hbar(p), \quad p = V + iA, \quad V, A \in S^N(\mathbb{R}^{2d}; \mathbb{R}), \quad A \geq 0.$$

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Let $z_0 \in \mathbb{R}^{2d}$ such that $A(z_0) = 0$, $\nabla A(z_0) = 0$. Notice that $\{V, A\}(z_0) = 0$. Assume the following **finite-type** condition:

$$\langle X_V(z_0), \partial^2 A(z_0) X_V(z_0) \rangle > 0.$$

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Theorem (Dencker, Sjöstrand, Zworski, 2004)

$$\| (\widehat{P}_\hbar - p(z_0))^{-1} \|_{\mathcal{L}(L^2)} \leq \frac{C}{\hbar^{2/3}}, \quad \hbar \leq \hbar_0.$$

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Theorem (Sjöstrand, 2009)

There exists $C_0 > 0$ *such that,* $\forall C_1 > 0$*, if* $|p(z_0) - \zeta| < \left(C_1 \hbar \log \frac{1}{\hbar}\right)^{2/3}$ *then*

$$\|(\widehat{P}_\hbar - \zeta)^{-1}\|_{\mathcal{L}(L^2)} \leq \frac{C_0}{\hbar^{2/3}} \exp\left(\frac{C_0}{\hbar} |p(z_0) - \zeta|^{3/2}\right), \quad \hbar \leq \hbar_0(C_0, C_1) > 0.$$

Theorem (Dencker, Sjöstrand, Zworski, 2004)

Let $z_0 \in \mathbb{R}^{2d}$ *such that* $\{V, A\}(z_0) < 0$. *Then there exists a quasimode* $(\psi_\hbar, p(z_0))$ *for* $\widehat{P}_\hbar$ *of width* $O(\hbar^\infty)$ *such that* $WF_\hbar(\psi_\hbar) = \{z_0\}$.

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Theorem (A.; 2020)

Let $z_0 \in A^{-1}(0)$, assume that $\nabla A(z_0) = 0$, and

$$\left\langle X_V(z_0), \partial^2 A(z_0) X_V(z_0) \right\rangle > 0.$$

Then there exists a quasimode $(\psi_\hbar, \lambda_\hbar)$ for $\widehat{P}_\hbar$ of width $r_\hbar = O(\hbar^\infty)$ with quasi-eigenvalue

$$\lambda_\hbar = V(z_0) + i\beta_\hbar, \quad \hbar^{2/3-\epsilon} \gg \beta_\hbar \gg \left(\hbar \log \frac{1}{\hbar}\right)^{2/3},$$

such that

$$WF_\hbar(\psi_\hbar) = \{z_0\}.$$

Let us consider the harmonic oscillator

$$\widehat{H}_\hbar := \frac{1}{2}\sum_{j=1}^{d}\omega_j(-\hbar^2\partial_{x_j}^2 + x_j^2), \quad \hbar > 0,$$

acting on $L^2(\mathbb{R}^d)$, where $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}_+^d$ is called vector of frequencies.

Perturbations of the harmonic oscillator

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We consider perturbations of the form:

$$\widehat{P}_\hbar = \widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar + i\hbar \widehat{A}_\hbar,$$

where the symbols $A, V \in S^0(\mathbb{R}^d; \mathbb{R})$ are real valued and bounded together with all its derivatives. We assume that $A \geq 0$ and that $\hbar^2 \ll \varepsilon_\hbar \lesssim \hbar^\alpha$ for some $0 < \alpha < 2$.

Consider sequences of (pseudo-)eigenvalues $\lambda_\hbar = \alpha_\hbar + i\hbar\beta_\hbar$ such that

$$(\alpha_\hbar, \beta_\hbar) \to (1, \beta), \quad \text{as } \hbar \to 0^+,$$

and

$$\widehat{P}_\hbar \psi_\hbar = \lambda_\hbar \psi_\hbar + R_\hbar, \quad \|\psi_\hbar\|_{L^2} = 1, \tag{1}$$

where $r_\hbar = \|R_\hbar\|_{L^2}$ is the **width** of the quasimode, typically of order $o(\hbar)$.

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If there exists a quasimode $(\psi_\hbar, \lambda_\hbar)$ for $\widehat{P}_\hbar$ of with $r_\hbar$, then

$$\|(\widehat{P}_\hbar - \lambda_\hbar)^{-1}\|_{\mathcal{L}(L^2)} \geq \frac{1}{r_\hbar}.$$

Geometric control (I)

Let

$$H(x,\xi) = \frac{1}{2} \sum_{j=1}^{d} \omega_j (\xi_j^2 + x_j^2).$$

For any $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$, we define

$$\langle a \rangle (x,\xi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T a \circ \phi_t^H (x,\xi) dt.$$

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Proposition

Let $\lambda_\hbar = \alpha_\hbar + i\hbar\beta_\hbar$ be a sequence of quasi-eigenvalues for (1) with $r_\hbar = o(\hbar)$. Then

$$\beta \in \left[\min_{(x,\xi)\in H^{-1}(1)} \langle A\rangle(z), \max_{z\in H^{-1}(1)} \langle A\rangle(z)\right].$$

In particular, if $\langle A\rangle \geq a_0 > 0$ (GC), then $\beta > 0$.

We assume that $\min_{(x,\xi)\in H^{-1}(1)} \langle A\rangle(x,\xi) = 0$ but one still has a *Weak Geometric Control* (WGC):

$$\forall z = (x,\xi) \in H^{-1}(1) \cap \langle A\rangle^{-1}(0), \quad \exists t \in \mathbb{R} : \quad \langle A\rangle \circ \phi_t^{\langle V\rangle}(z) > 0.$$

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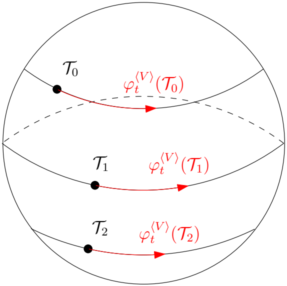

$d = 2, \quad \omega = (1,1)$

$H^{-1}(1)/\mathbb{S}^1 \simeq \mathbb{S}^2$

$\pi_H : H^{-1}(1) \to H^{-1}(1)/\mathbb{S}^1$

The dimension of the minimal invariant tori by ϕ_t^H depends on the arithmetic relations between components of the vector of frequencies $\omega = (\omega_1, \dots, \omega_d)$.

The **resonant set** $\Lambda_\omega = \{k \in \mathbb{Z}^d \, : \, k \cdot \omega = 0\}$ determines the maximal dimension d_ω of the Kronecker tori reached by ϕ_t^H. Precisely,

$$d_\omega = d - \operatorname{rk} \Lambda_\omega.$$

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In particular, in the case $d_\omega = d$, conditions (GC) and (WGC) are equivalent.

We will say that ω is **partially Diophantine** if

$$|k \cdot \omega| \geq \frac{C}{|k|^\nu}, \quad k \in \mathbb{Z}^d \setminus \Lambda_\omega, \quad \nu \geq d_\omega - 1.$$

Theorem (A., Rivière; 2018)

Assume (WGC) *and* $\varepsilon_\hbar \gg \hbar^2$.

Then, for every sequence $\lambda_\hbar = \alpha_\hbar + i\hbar\beta_\hbar$ *satisfying* (1) *with* $r_\hbar \ll \hbar\varepsilon_\hbar$,

$$\liminf_{\hbar \to 0^+} \frac{\beta_\hbar}{\varepsilon_\hbar} = +\infty. \tag{2}$$

As a consequence, for every $R > 0$, there exists $\delta_R > 0$ s.t.

$$\frac{\operatorname{Im}\zeta}{\hbar} \leq R\varepsilon_\hbar \implies \left\| \left(\widehat{P}_\hbar - \zeta \right)^{-1} \right\|_{L^2 \to L^2} \leq \frac{1}{\delta_R \hbar \varepsilon_\hbar}.$$

Some results

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Theorem (Asch, Lebeau, 2003; A., Rivière, 2018)

Let A, V *be* ***real analytic***, *and* ω *partially Diophantine. Assume* (WGC), $\varepsilon_\hbar \geq \hbar$ *and* $r_\hbar \equiv 0$. *Then*

$$\beta > 0.$$

Question: Under the hypothesis of the second theorem, there exist quasimodes of width $r_\hbar = o(\varepsilon_\hbar \hbar)$ such that $\varepsilon_\hbar \ll \beta_\hbar \to 0$?

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Theorem (A.; 2020)

Let $\varepsilon_\hbar = \hbar$. Assume that ω is partially Diophantine, (WGC) *and let $\mathcal{T}_0$ be a minimal invariant torus for ϕ_t^H such that $\langle A \rangle|_{\mathcal{T}_0} = 0$. Suppose also that*

$$\big\langle X_{\langle V \rangle}(z_0), \partial^2 \langle A \rangle(z_0) X_{\langle V \rangle}(z_0) \big\rangle > 0, \quad z_0 \in \mathcal{T}_0.$$

Then there exists a quasimode $(\psi_\hbar, \lambda_\hbar)$ for $\widehat{P}_\hbar$ of width $r_\hbar = O(\hbar^\infty)$ with quasi-eigenvalue

$$\lambda_\hbar = 1 + \hbar \langle V \rangle(z_0) + i\hbar\beta_\hbar, \quad \hbar^{2/3-\epsilon} \gg \beta_\hbar \gg \left(\hbar \log \frac{1}{\hbar} \right)^{2/3},$$

such that

$$WF_\hbar(\psi_\hbar) = \mathcal{T}_0.$$

Strategy of proof

- We first conjugate the operator $\widehat{P}_\hbar = \widehat{H}_\hbar + \hbar\,\mathrm{Op}_\hbar(V) + i\hbar\,\mathrm{Op}_\hbar(A)$ into a normal form

$$\widehat{P}_\hbar^\dagger = \widehat{H}_\hbar + \hbar\,\mathrm{Op}_\hbar(\langle V\rangle) + i\hbar\,\mathrm{Op}_\hbar(\langle A\rangle) + O(\hbar^2).$$

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- We next construct (following A., Macià 2020) a sequence of eigenfunctions $(\Psi_\hbar, E_\hbar)_\hbar$ for $\widehat{H}_\hbar$ which concentrates on $\mathcal{T}_0$ of the form:

$$\Psi_\hbar(x) := \left(\frac{|dH(z_0)|}{\sqrt{\pi\hbar}T_\omega} \right)^{1/2} \int_0^{T_\omega} e^{\frac{it}{\hbar}E_\hbar} e^{\frac{-it|\omega|_1}{2}} \varphi^\hbar_{\phi_t^H(z_0)}(x)dt,$$

where $E_\hbar \to H(z_0) = 1$, and

$$\varphi_z^\hbar(x) = \frac{1}{(\pi\hbar)^{d/4}} e^{\frac{-|x-x_0|^2}{2\hbar}} e^{\frac{i}{\hbar}\xi_0\cdot\left(x - \frac{x_0}{2}\right)}, \quad z_0 = (x_0, \xi_0).$$

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- We then average the eigenmode $\Psi_\hbar$ also by the propagator of a suitable polynomial approximation of $\mathrm{Op}_\hbar(\langle V\rangle + i\langle A\rangle)$ near $\mathcal{T}_0$ to obtain our quasimode $\psi_\hbar$.

Self-adjoint operators:

- Hagedorn (1985).
- De Bièvre, Houard, Irac-Astaud (1992-1993).
- Paul, Uribe (1993).
- Combescure, Robert (1996).
- Eswarathasan, Nonnenmacher (2015).
- A., Macià (2020).

Non-selfadjoint operators:

- Graefe, Schubert (2011, 2012).
- Dietert, Keller, Troppmann (2016).
- Lasser, Schubert, Troppmann (2018).
- Pravda-Starov (2018).

Let us consider a wave packet centered at $z_0 = (x_0, \xi_0)$ and with *Lagrangian frame* $Z_0 = (P_0, Q_0) = (i\,\mathrm{Id}_d, \mathrm{Id}_d) \in \mathbb{C}^{2d\times d}$:

$$\varphi_0^\hbar[Z_0, z_0](x) = \frac{\det(Q_0)^{-1/2}}{(\pi\hbar)^{d/4}} e^{\frac{i}{2\hbar} P_0 Q_0^{-1}(x-x_0)\cdot(x-x_0)} e^{\frac{i}{\hbar}\xi_0\cdot\left(x-\frac{x_0}{2}\right)}.$$

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When we apply the propagator $e^{\frac{it\widehat{P}_\hbar}{\hbar}}$ of $\widehat{P}_\hbar$ to $\varphi_0^\hbar[Z_0, z_0]$, the center of the wave-packet evolves approximately according to the evolution equation (Graefe-Schubert, 2011):

$$\dot{z}_t = -\Omega\nabla V(z_t) - G_t^{-1}\nabla A(z_t),$$

$$\dot{G}_t = G_t\Omega\partial^2 V(z_t) - \partial^2 V(z_t)\Omega G_t - \partial^2 A(z_t) - G_t\Omega\partial^2 A(z_t)\Omega G_t.$$

This system is well-posed for $z_t|_{t=0} = z_0$, $G_0 = \mathrm{Id}_{2d}$, $0 \le t \le T$.

We split $p(z) = p_2(t,z) + R(t,z)$, where

$$p_2(t,z) = p(z_t) + (z - z_t) \cdot \nabla p(z_t) + \frac{1}{2}(z - z_t) \cdot \partial^2 p(z_t)(z - z_t),$$

$$R(t,z) = \sum_{|\beta|=3} \frac{|\beta|}{\beta!}(z - z_t)^\beta \int_0^1 (1-s)^{|\beta|-1} D^\beta p(z_t + s(z - z_t))ds.$$

The evolution of Hagedorn wave packets (and more generally excited states) by the propagator of $\mathrm{Op}_\hbar(p_2(t,z))$ has been characterized by Lasser, Schübert and Troppmann (2018). We also need to estimate the contribution of the remainder term $R(t,z)$.

Hagedorn wave-packets

A matrix $Z = (Q, P) \in \mathbb{C}^{2d\times d}$ is called *normalized Lagrangian frame* if:

$$Z^T \Omega Z = 0, \quad \frac{i}{2} Z^* \Omega Z = \mathrm{Id}_d .$$

If Z is a normalized Lagrangian frame, then $L = \operatorname{range} Z$ is a positive Lagrangian space, meaning that

$$L = \{(PQ^{-1}x, x) \ : \ x \in \mathbb{C}^d\}, \quad \Im\big(PQ^{-1}\big) > 0.$$

With Z one can associate ladder operators:

$$A[Z,z] = \frac{i}{\sqrt{2\hbar}} Z \cdot \Omega(\hat{z} - z), \quad A^\dagger[Z,z] = -\frac{i}{\sqrt{2\hbar}} \overline{Z} \cdot \Omega(\hat{z} - z).$$

Proposition

The set of states:

$$\varphi_k^\hbar[Z,z](x) = \frac{1}{\sqrt{k!}} A_k^\dagger[Z,z] \varphi_0^\hbar[Z,z](x), \quad k \in \mathbb{N}^d.$$

defines an orthonormal basis of $L^2(\mathbb{R}^d)$.

Quadratic evolution (selfadjoint case)

In the selfadjoint case the evolution of a Hagedorn excites state is determined by the system of differential equations

$$\dot{z}_t = -\Omega \nabla p(z_t), \qquad z_t\big|_{t=0} = z_0,$$

$$\dot{S}_t = \partial^2 p(z_t) S_t, \qquad S_0 = \mathrm{Id}\,.$$

We have:

$$e^{\frac{it\widehat{P}_{2,\hbar}}{\hbar}} \varphi_k[Z_0, z_0](x) = e^{\frac{i}{\hbar}\Lambda_t(z_0)} \varphi_k^{\hbar}[Z_t, z_t](x), \quad k \in \mathbb{N}^d,$$

where $Z_t = S_t Z_0$ and

$$\Lambda_t(z_0) = -\int_0^t \left(\frac{\dot{\xi}_s \cdot x_s - \dot{x}_s \cdot \xi_s}{2} - p(z_s) \right) ds.$$

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In this case, the metric G_t, given by $G_t = S_t^{-T} S_t^{-1}$, satisfies the Ricatti equation

$$\dot{G}_t = G_t \Omega \partial^2 p(z_t) - \partial^2 p(z_t) \Omega G_t$$

Quadratic evolution (non-selfadjoint case)

Let S_t be the complex symplectic matrix satisfying:

$$\dot{S}_t = \Omega \partial^2 p(z_t)\, S_t, \quad S_0 = \mathrm{Id}_d,$$

and define

$$N_t := \left(\frac{1}{2i} (S_t Z_0)^* \Omega (S_t Z_0) \right)^{-1/2}.$$

Theorem (Lasser, Schubert, Troppmann (2018))

$$e^{\frac{it\widehat{P}_{2,\hbar}}{\hbar}} \varphi_k^\hbar[Z_0, z_0](x) = e^{\frac{i}{\hbar}\Lambda_t(z_0) + \varrho_t} \sum_{|l| \leq |k|} b_{kl}(t) \varphi_l^\hbar[Z_t, z_t], \quad 0 \leq t \leq T,$$

where $Z_t = S_t Z_0 N_t$ is a normalized Lagrangian frame, and

$$\Lambda_t(z_0) = -\int_0^t \left(\frac{\dot{\xi}_s \cdot x_s - \dot{x}_s \cdot \xi_s}{2} - p(z_s) \right) ds,$$

$$\varrho_t = -\frac{1}{4} \int_0^t \mathrm{Tr}\left(G_s^{-1} \partial^2 A(z_s) \right) ds.$$

We look at the equation:

$$i\hbar\partial_t \varphi_\hbar(t,x) + \widehat{P}_\hbar \varphi_\hbar(t,x) = 0, \quad \varphi_\hbar(0,x) = \varphi_0^\hbar[Z_0, z_0](x). \tag{3}$$

Making the ansatz

$$\varphi_\hbar(t,x) = \sum_{k\in\mathbb{N}^d} c_k(t,\hbar)\varphi_k^\hbar[Z_t, z_t](x),$$

equation (3) can be viewed as an equation on the coefficients:

$$\dot{c}_k(t,\hbar) = \sum_{l\in\mathbb{N}^d} \left(\frac{i\dot{\Lambda}_t}{\hbar} + \dot{\varrho}_t + p_{kl}(t) + r_{kl}(t,\hbar) \right) c_l(t,\hbar), \quad k \in \mathbb{N}^d.$$

The coefficients $p_{kl}(t)$ correspond to the quadratic part $p_2(t,z)$, while the coefficients $r_{kl}(t,\hbar)$ correspond to the remainder term $R(t,z)$.

Lemma

The matrix elements $p_{kl}(t)$ *corresponding with the quadratic part satisfy*

$$p_{0l}(t) = 0, \quad \forall l \in \mathbb{N}^d,$$

$$p_{kl}(t) = 0, \quad \text{if } |k - l| > 2,$$

$$\sup_{0 \leq t \leq T} |p_{kl}(t)| \leq C|k|.$$

Lemma

The matrix elements $r_{kl}(t,\hbar) = \hbar^{-1}\langle \varphi_k^\hbar[Z_t, z_t], \mathrm{Op}_\hbar(R(t))\varphi_l^\hbar[Z_t, z_t]\rangle$ *satisfy*

$$\sup_{0 \leq t \leq T} |r_{k,k+l}(t,\hbar)| \leq C\sqrt{\hbar}\big(1 + |k|\big)^{N/2}, \quad |l| \leq N$$

$$r_{k,k+l}(t,\hbar) = 0, \quad |l| > N.$$

Propagation of coefficients (I)

Let us consider the Banach spaces:

$$\ell_\rho^N(\mathbb{N}^d) = \left\{ \vec{c} = (c_k) \ : \ \|\vec{c}\|_\rho := \sum_{k \in \mathbb{N}^d} |c_k| e^{\rho |k|^{N/2}} < +\infty \right\}.$$

Given the two evolution equations

$$(1) \quad \frac{d}{dt}\vec{c}(t) = \left(\frac{i\dot{\Lambda}_t}{\hbar} + \dot{\varrho}_t \right) \vec{c}(t) + (p_{kl}(t))\vec{c}(t),$$

$$(2) \quad \frac{d}{dt}\vec{c}(t) = \left(\frac{i\dot{\Lambda}_t}{\hbar} + \dot{\varrho}_t \right) \vec{c}(t) + (p_{kl}(t) + r_{kl}(t,\hbar))\vec{c}(t),$$

there exist propagators satisfying, for $0 \leq s \leq t \leq T$ small enough,

$$U_j(t,s) : \ell_\rho^N(\mathbb{N}^d) \to \ell_{\rho-\sigma}^N(\mathbb{N}^d), \quad j = 1, 2.$$

This allows us to compare the coefficients of the whole solution with those of the quadratic evolution; by Duhamel's principle:

$$U_1(t,s) - U_2(t,s) = \int_s^t U_2(t,\tau)(r_{kl}(\tau,\hbar))U_1(\tau,s)d\tau = O(\sqrt{\hbar}).$$

Therefore, given $\vec{c}(0) := (1, 0, \dots) \in \ell_\rho^N(\mathbb{N}^d)$, we obtain, for $0 \le t \le T$,

$$c_0(t,\hbar) = e^{\frac{i}{\hbar}\Lambda_t + \varrho_t}(1 + O(\sqrt{\hbar})),$$

$$c_k(t,\hbar) = e^{\frac{i}{\hbar}\Lambda_t + \varrho_t} O\left(\sqrt{\hbar}\, e^{-(\rho-\sigma)|k|^{N/2}}\right), \quad k \neq 0.$$

Averaging

We define our quasimode as:

$$\psi_\hbar(x) := \sqrt{C_\hbar} \int_{\mathbb{R}} \chi_\hbar(t) e^{-\frac{it\lambda_\hbar}{\hbar}} \varphi_\hbar(t,x)dt$$

$$= \sqrt{C_\hbar} \sum_{k \in \mathbb{N}^d} \int_{\mathbb{R}} \chi_\hbar(t) e^{-\frac{it\lambda_\hbar}{\hbar}} c_k(t,\hbar) \varphi_k^\hbar[Z_t, z_t](x)dt,$$

where $\chi_\hbar \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfies

$$\operatorname{supp} \chi_\hbar \subset \big\{ -\hbar^{1/3} \leq t \leq \hbar^{1/3-\delta} \big\},$$

$$\operatorname{supp} \chi_\hbar' \subset \{-\hbar^{1/3} \leq t \leq -\hbar^{1/3}(1-c)\} \cup \{\hbar^{1/3}(\hbar^{-\delta} - c) \leq t \leq \hbar^{1/3-\delta}\},$$

and $\lambda_\hbar = V(z_0) + i\beta_\hbar$.

Normalization

We compute, for $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$:

$$\left\langle \psi_\hbar, \mathrm{Op}_\hbar(a)\psi_\hbar \right\rangle_{L^2(\mathbb{R}^d)}$$

$$= C_\hbar \int_{\mathbb{R}^2} \int_{\mathbb{R}^{2d}} \chi_\hbar(t)\chi_\hbar(t')e^{-\frac{i}{\hbar}(t\lambda_\hbar - t'\overline{\lambda}_\hbar)} W_\hbar[\varphi_\hbar(t), \varphi_\hbar(t')](z)a(z)dzdtdt'.$$

The cross-Wigner functions $W_\hbar\left[\varphi_k^\hbar[Z_t, z_t], \varphi_{k'}^\hbar[Z_{t'}, z_{t'}]\right]$ can be computed explicitely. Using a stationary-phase argument near $|t - t'| = 0$, and expanding by Taylor in t near $t = 0$, we obtain the leading term:

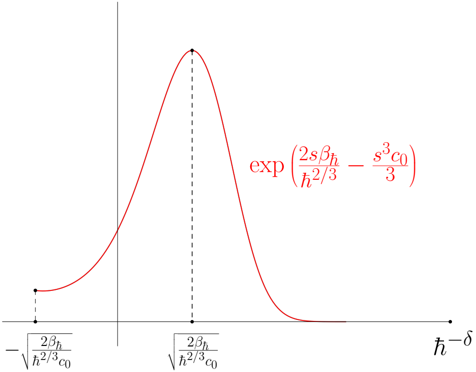
$$\left\langle \psi_\hbar, \mathrm{Op}_\hbar(a)\psi_\hbar \right\rangle_{L^2(\mathbb{R}^d)} \sim \frac{C_\hbar \sqrt{\pi}\hbar^{1/2}}{|V(z_0)|} a(z_0) \int_{\mathbb{R}} \chi_\hbar(t)^2 e^{\frac{1}{\hbar}\left(2t\beta_\hbar - \frac{c_0 t^3}{3}\right)} dt,$$

where

$$c_0 = \left\langle X_V(z_0), \partial^2 A(z_0) X_V(z_0) \right\rangle > 0.$$

Making the change $t = \hbar^{1/3} s$, we get

$$\left\langle \psi_\hbar, \mathrm{Op}_\hbar(a)\psi_\hbar \right\rangle_{L^2(\mathbb{R}^d)} \sim \frac{C_\hbar \sqrt{\pi}\hbar^{5/6}}{|V(z_0)|} a(z_0) \int_{\mathbb{R}} \chi_\hbar \left(\hbar^{1/3} s\right)^2 e^{\frac{2s\beta_\hbar}{\hbar^{2/3}} - \frac{s^3 c_0}{3}} ds.$$



With of the quasimode

We take $C_\hbar := \frac{\hbar^N |V(z_0)|}{\sqrt{\pi}\hbar^{5/6}}$ and $\beta_\hbar$ so that the above integral converges to $a(z_0)$. We obtain

$$\left(C_N \hbar \log \frac{1}{\hbar} \right)^{2/3} \leq \beta_\hbar \ll \hbar^{2/3-\epsilon}, \quad \forall \epsilon > 0.$$

Similarly, applying $\widehat{P}_\hbar$, integrating by parts in t, and repeating the argument, we obtain

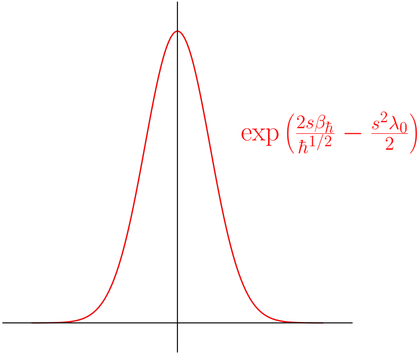
$$\begin{aligned} \left\langle \psi_\hbar, \widehat{P}_\hbar \psi_\hbar \right\rangle_{L^2(\mathbb{R}^d)} &= \lambda_\hbar \|\psi_\hbar\|^2 + C_\hbar \int_{\mathbb{R}} \chi_\hbar'(t) \chi_\hbar(t) e^{\frac{1}{\hbar}\left(2t\beta_\hbar - \frac{c_0 t^3}{3}\right)} dt \\ &= \lambda_\hbar \|\psi_\hbar\|^2 + O(\hbar^N). \end{aligned}$$

Therefore, we get a quasimode of width $O(\hbar^N)$.

Assuming that $\lambda_0 = \{A, V\}(z_0) < 0$, we obtain the leading term:

$$\int_{\mathbb{R}} \chi_\hbar \left(\hbar^{1/2} s \right)^2 e^{\frac{2s\beta_\hbar}{\hbar^{1/2}} - \frac{s^2 \lambda_0}{2}} a(z_0) ds.$$

Then it is sufficient to take $\beta_\hbar \equiv 0$ to obtain a quasimode of with $O(\hbar^\infty)$.



Thank you for your attention!