Wall-crossing in algebraic geometry

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Vector bundles on curves

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60 years ago:

THE GROTHENDIECK RING IN GEOMETRY AND TOPOLOGY

By M. F. ATIYAH

§ 1. The Grothendieck ring in homotopy theory

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- $C =$ elliptic curve: [Atiyah, 1957] indecomposable bundles E on C of given rank and degree are parametrised by C.

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How to classify and parametrise bundles on curves of higher genus?

Stable vector bundles on curves

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C smooth projective curve.

E vector bundle on C $\rightsquigarrow$ $\begin{cases} \mathrm{rk}(E), & \textit{rank} \\ \deg(E), & \textit{degree} \end{cases}$

60 years ago (cont.):

DEFINITION. A vector bundle E is *stable* if for all sub-bundles F,

$$\mathrm{Deg}\ c_1(F) < \mathrm{Deg}\ c_1(\mathrm{E}) \cdot \frac{\mathrm{rank}\ F}{\mathrm{rank}\ E},$$

where c_1 denotes the first chern class.

In other words, a vector bundle is stable if all its subbundles are "less ample" than itself. To illustrate the stability condition, let me mention its simplest properties:

(i) If L is a line bundle, then E is stable if and only if $E \otimes L$ is stable; moreover, E is stable if and only if $\check{E}$ is stable.

(ii) If E_1 and E_2 are two vector bundles, $E_1 \oplus E_2$ is never stable.

(iii) A line bundle is always stable.

(iv) If a vector bundle E of rank 2 is not stable, then either E is isomorphic to $L_1 \oplus L_2$, *or* there is a unique sub-bundle L for which $\geqslant$ holds in the definition and E can be canonically described as an extension.

Then I can prove the following theorem:

THEOREM. *The set of all stable vector bundles of rank* r *over a fixed curve* C *in characteristic* 0 *is "naturally" isomorphic to the set of points of a non-singular quasi-projective variety* V_r(C).

[DAVID MUMFORD, ICM 1962]

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$$E \text{ vector bundle on } C \quad \rightsquigarrow \quad \begin{cases} \mathrm{rk}(E), & \textit{rank} \\ \deg(E), & \textit{degree} \end{cases} \rightsquigarrow \quad Z(E) := -\deg(E) + i\,\mathrm{rk}(E) \in \mathbb{C}.$$

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DEFINITION [Mumford, 1962]. E is semistable if, for all sub-bundles $F \subset E$,

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PROPERTIES. (i) Moduli spaces, parametrising semistable vector bundles on C of given rank and degree, exist as *projective* varieties.
(ii) Any vector bundle E admits a Harder–Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{m-1} \subset E_m = E$$

such that

- $A_k := E_k/E_{k-1}$ is semistable, for all $k = 1, \ldots, m$.
- $\arg(Z(A_1)) > \cdots > \arg(Z(A_m))$.

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APPLICATIONS. (i) Classification of vector bundles.
(ii) Moduli spaces relate theory of curves to higher-dimensional varieties, generalising the Torelli relation $C \mapsto \mathrm{Jac}(C)$.

Bridgeland stability on K3 surfaces

4. Stability conditions

The notion of a stability condition was introduced in [12] as a way to understand Douglas' work on π-stability for D-branes in string theory [18]. Here we wish to emphasise the purely mathematical aspects of this definition. For more on the connections with string theory see [15].

In the context of the present article stability conditions are relevant for three reasons. Firstly, the choice of a stability condition picks out classes of stable objects for which one can hope to form well-behaved moduli spaces. Secondly the space of all stability conditions Stab(D) allows one to bring geometric methods to bear on the problem of understanding t-structures on D. Finally, the space Stab(D) provides a complex manifold on which the group Aut(D) naturally acts.

[Tom Bridgeland, ICM 2006]

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  (in particular, $\mathcal{A}$ abelian category)

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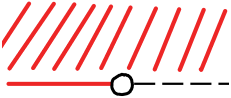

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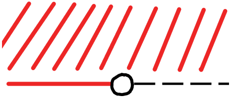

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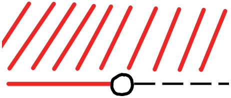

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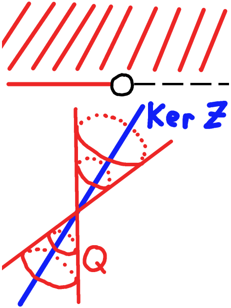
  (iii) there exists a quadratic form Q on $H^*_{\mathrm{alg}}(X, \mathbb{R})$ with
  - Q is negative definite on $\mathrm{Ker} Z_{\mathbb{R}}$
  - $Q(E) \geq 0$, for all E semistable



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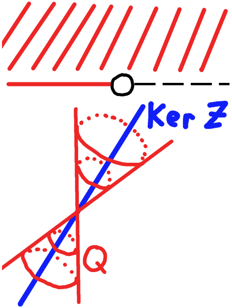
  (iv) openness and boundedness for semistable objects.



($\rightsquigarrow$ existence of moduli spaces of semistable objects in $\mathcal{A}$ [TODA])

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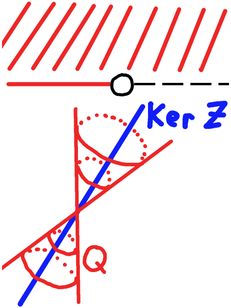

BRIDGELAND DEFORMATION THEOREM. The set $\mathrm{Stab}(\mathrm{D^b}(X))$ of stability conditions has the structure of complex manifold such that the forgetful morphism

$$\mathrm{Stab}(\mathrm{D^b}(X)) \longrightarrow H^*_{\mathrm{alg}}(X, \mathbb{C})^\vee, \quad (Z, \mathcal{A}) \mapsto Z$$

is a covering of an (explicitly defined) period domain.

Mukai's theory on K3 surfaces

ON THE MODULI SPACE OF BUNDLES ON $K3$ SURFACES, I

***By* S. MUKAI**

IN [12], WE have shown that the moduli space M_S of stable sheaves on a $K3$ or abelian surface S is smooth and has a natural symplectic structure. In this article, we shall study M_S more precisely in the case S is of type $K3$. We shall show that every compact 2 dimensional component of M_S is a $K3$ surface isogenous to S (Definition 1.7 and 1.8) and describe its period explicitly (Theorem 1.4). As an application of this result, we shall show that certain Hodge cycles on a product of two $K3$ surfaces are algebraic (Theorem 1.9). As a corollary, we have that two $K3$ surfaces with Picard number ≥ 11 are isogeneous in our sense if and only if their transcendental Hodge structures T_S and $T_{S'}$ are isogenous, i.e., isomorphic over $\mathbb{Q}$ (Corollary 1.10).

[Shigeru Mukai, 1987]

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Step 2. Use Fourier–Mukai transforms and wall-crossing.

Applications

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$\Rightarrow$ description of birational geometry (nef cones, birational models) of hyper-Kähler varieties of $\mathrm{K3}^{[n]}$ deformation type.

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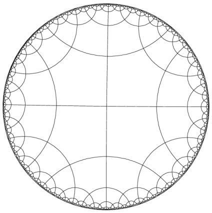

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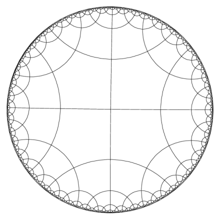
MUKAI'S DREAM. All birational maps between moduli spaces of sheaves are induced by Fourier–Mukai transforms. [B–M]



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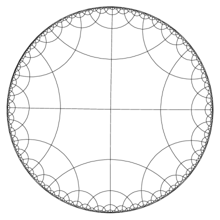

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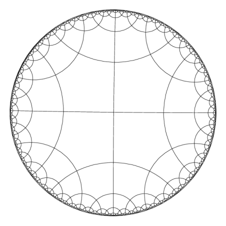

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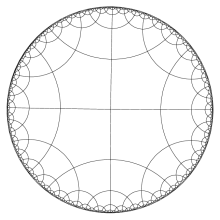

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THE PROJECTIVE PLANE. MMP via wall-crossing works also in this case [LI–ZHAO, ARCARA–BERTRAM–COŞKUN–HUIZENGA–WOOLF] $\Rightarrow$ applications to DT/GW theory. [BOUSSEAU]

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DIFFICULTIES.

- Step 2b depends on a Bogomolov-type conjecture governing the Chern character of tilt-stable objects in $\mathcal{A}$ in Step 1b. [B–M–TODA]
- No equivalent of Mukai theory governing non-emptiness of moduli spaces $\Rightarrow$ difficult to control wall-crossing.
- Moduli spaces badly behaved (singular, multiple components, ...).

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Existence.

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- Calabi–Yau cases: quintic threefold, finite quotients of abelian threefolds, double/triple covers of $\mathbb{P}^3$, $(2, 4)$-complete intersections; [Chunyi Li, Koseki, Shengxuan Liu]
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$$\mathcal{K}u(X) \subset \mathrm{D}^{\mathrm{b}}(X), \text{ the Kuznetsov component of } \mathrm{D}^{\mathrm{b}}(X).$$

The Kuznetsov component of a cubic fourfold

Theorem 5.2 ([K10]). *Let $Y \subset \mathbb{P}^5$ be a cubic 4-fold. Then there is a semiorthogonal decomposition*

$$\mathbf{D}^b(\mathsf{coh}(Y)) = \langle \mathscr{A}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle,$$

and its nontrivial component $\mathscr{A}_Y$ is a Calabi–Yau category of dimension 2. Moreover, $\mathscr{A}_Y$ is equivalent to the derived category of coherent sheaves on a K3 surface, at least if Y is a Pfaffian cubic 4-fold, or if Y contains a plane Π and a 2-cycle Z such that $\deg Z + Z \cdot \Pi \equiv 1 \bmod 2$.

To establish this result for Pfaffian cubics one can use HP duality for $\mathsf{Gr}(2,6)$. The associated K3 is then a linear section of this Grassmannian. For cubics with a plane a quadratic bundle formula for the projection of Y from the plane Π gives the result. The K3 surface then is the double covering of $\mathbb{P}^2$ ramified in a sextic curve, and the cycle Z gives a splitting of the requisite Azumaya algebra on this K3.

For generic Y the category $\mathscr{A}_Y$ can be thought of as the derived category of coherent sheaves on a noncommutative K3 surface. Therefore, any smooth moduli space of objects in $\mathscr{A}_Y$ should be hyperkähler, and the Fano scheme of lines can be realized in this way, see [KM09].

[Alexander Kuznetsov, ICM 2014]

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$X \subset \mathbb{P}^5$ smooth cubic fourfold.

DEFINITION [Kuznetsov].

$$\mathcal{K}u(X) := \left\{ E \in \mathrm{D}^{\mathrm{b}}(X) \colon \operatorname{Hom}^\bullet(\mathcal{O}_X, E) = \operatorname{Hom}^\bullet(\mathcal{O}_X(1), E) = \operatorname{Hom}^\bullet(\mathcal{O}_X(2), E) = 0 \right\}.$$

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PROPERTIES:

- admissible subcategory: $i \colon \mathcal{K}u(X) \hookrightarrow \mathrm{D^b}(X)$ has left and right adjoints $i^*, i^!$
- $i^*, i^!$ act quite naturally and geometrically
- Calabi–Yau 2-category: $\mathrm{Hom}(E, F) = \mathrm{Hom}(F, E[2])^\vee$
- K3 category: CY2 and has a Mukai vector $E \in \mathcal{K}u(X) \rightsquigarrow v(E) \in \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$, where $\widetilde{H}(\mathcal{K}u(X), \mathbb{Z})$ is a Hodge structure on K3 lattice $H^*(\mathrm{K3})$ [ADDINGTON–THOMAS, PERRY]
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REMARK. For X very general, $\mathcal{K}u(X) \ncong \mathrm{D^b}(S)$.

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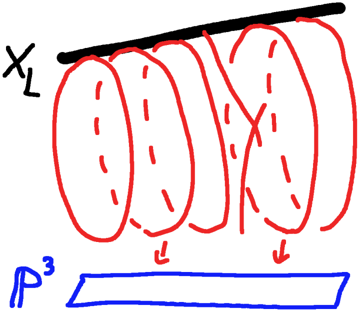

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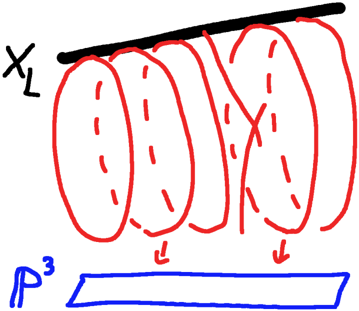

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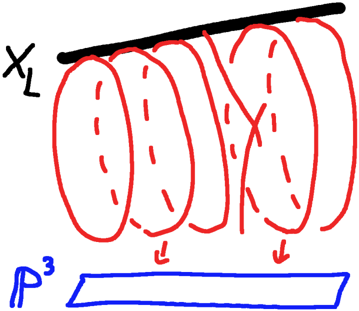

THEOREM [B–Lahoz–M–Stellari].

- Tilt-stability exists for the non-commutative threefold $(\mathbb{P}^3, \mathcal{C}_0)$.
- This restricts to a Bridgeland stability condition on $\mathcal{K}u(X)$.

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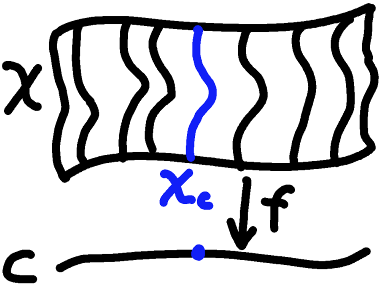

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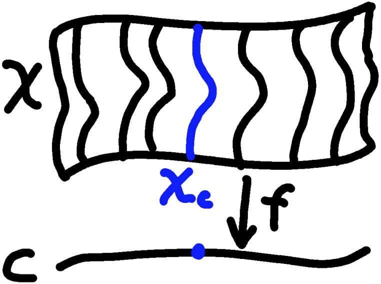

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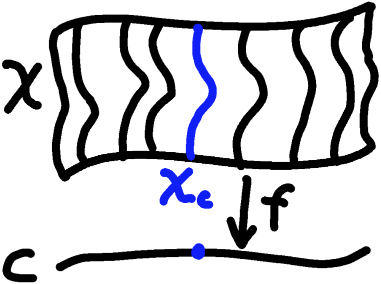

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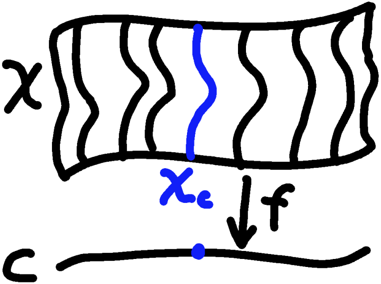

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THEOREM [BLMNPS]. Assume $v \in \widetilde{H}_{\mathrm{alg}}(\mathcal{K}u(X), \mathbb{Z})$ is primitive and $\sigma \in \mathrm{Stab}(\mathcal{K}u(X))$ is *generic* with respect to v. Then $M := M_\sigma(v)$ is non-empty if and only if

$$2d := v^2 + 2 \geq 0.$$

In this case, M is a smooth projective hyper-Kähler manifold of dimension $2d$, which comes with a canonical polarization $\ell := \ell_\sigma(v)$.

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- $Z(X) =$ Lehn–Lehn–Sorger–van Straten eightfold.
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- $V(X) =$ singular O'Grady type Voisin tenfold, twist of Laza–Saccà–Voisin tenfold.
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Theorem [Addington–Thomas, BLMNPS].
$\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{K3}) \Leftrightarrow X$ has *Hodge-theoretically associated* K3 surface S.

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- $F(X) =$ Beauville–Donagi fourfold (Fano variety of lines).
  Objects: $i^* I_L$ for $L \subset X$ line.
- $Z(X) =$ Lehn–Lehn–Sorger–van Straten eightfold.
  Objects: $i^* I_C$, for $C \subset X$ twisted cubic. Contains $X \subset M_\sigma(v)$ via $i^* I_x$.
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  Objects: $i^* I_E$, for $E \subset X$ elliptic quintic.

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THEOREM [Addington–Thomas, BLMNPS].
$\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{K3}) \Leftrightarrow X$ has *Hodge-theoretically associated* K3 surface S.
(in the sense of Hassett, i.e., $H^2_{\mathrm{prim}}(S, \mathbb{Z}) \subset H^4_{\mathrm{prim}}(X, \mathbb{Z})$)

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TORELLI THEOREM - ALTERNATIVE PROOF:
$H^4(X) \rightsquigarrow \widetilde{H}(\mathcal{K}u(X)) \rightsquigarrow$ [Markman, Verbistky] $F(X)$ with Plücker polarisation $\rightsquigarrow$ [Charles] X

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Torelli theorem - alternative proof:
$H^4(X) \rightsquigarrow \widetilde{H}(\mathcal{K}u(X)) \rightsquigarrow_{\text{[Markman, Verbistky]}} F(X)$ with Plücker polarisation $\rightsquigarrow_{\text{[Charles]}} X$

Construction of rational maps from special cubics: [B–Bertram–M–Perry]
E.g., when $Z(X) = S^{[4]}$, can combine $X \subset Z(X) = S^{[4]} \dashrightarrow S^{[5]}$ and MMP for $S^{[5]}$

Open questions

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Since the development of derived algebraic geometry by Grothendieck in the 1960s, the study of algebraic geometry has been deeply intertwined with the study of sheaves and their cohomology.

In recent years, there has been a grown interest in the study of derived categories of coherent sheaves on algebraic varieties.

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