SPECIAL SURFACES IN CUBIC 4FOLDS

$/\mathbb{C}$

joint work in progress w/

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Def cubic 4fold $Y \subseteq \mathbb{P}^5$

ipersuperficie liscia di grado 3. $\mathbb{P}^4 \overset{\sim}{\dashrightarrow} Y$

Cong. (Harris, Hassett)

L'ipersup. cubica molto generale non è razionale
di dim 4

$$\mathcal{M} = U_{lis.} \subseteq \mathbb{P}^{55} \big/ PGL(6)$$

sp. moduli delle cubiche

Teoria di Hodge

Y cubic 4fold ⇝ $H^{odd}(Y,\mathbb{Z}) = 0$

$H^{ev}(Y,\mathbb{Z})$ priva di torsione

Diam. di Hodge:

$h = \mathcal{O}_{\mathbb{P}^5}(1)|_Y$

$$\begin{array}{ccc} & 1 & \\ & 1 & \\ 1 & 21 & 1 \end{array} \qquad \begin{array}{c} H^0 \\ H^2 \\ H^4 \end{array}$$

$h^{3,1}$

$$L := \left(H^4(Y,\mathbb{Z}), \langle -,- \rangle \right) \quad (\cong \langle 1 \rangle^{21} \oplus \langle -1 \rangle^{2})$$

$$L^0 := H^4(Y,\mathbb{Z})_{prim} = (h^2)^{\perp} \subseteq L \quad (\cong A_2 \oplus U^2 \oplus E_8^2)$$

+ str Hodge $H^{3,1} \subseteq L^0_{\mathbb{C}}$ di $\underline{\underline{\text{peso } 2}}$

Appl. periodi :

$$\wp : \mathcal{M} \longrightarrow \mathcal{D} = \tilde{\mathcal{D}}/\Gamma \leftsquigarrow \text{gr. aritm.}$$

$$Y \longmapsto [H^{3,1}(Y)]$$

$$\tilde{\mathcal{D}} = \left\{ [\lambda] \in \mathbb{P}(L^0_{\mathbb{C}}) : \langle \lambda, \lambda \rangle = 0 \right\}^{+}$$

Teo di Torelli (Voisin)

$\wp$ embedding aperto.

↑

Y molto gen.

$\Rightarrow H^4(Y,\mathbb{Z})_{alg} =$

$H^4(Y,\mathbb{Z}) \cap H^{2,2}(Y) =$

$= \mathbb{Z}\cdot h^2$

Divisori di Hassett:

K sottoret., rg 2, saturato

$\mathbb{Z}\cdot h^2 \subseteq K \subseteq L$

$\mathrm{disc}(K) = d \boxed{> 0}$

$$\tilde{\mathcal{C}}_K = \{ \omega \in \tilde{\mathcal{D}} : \langle \omega, K \rangle = 0 \}$$

Lem (Hassett)

- $\tilde{\mathcal{C}}_K \subseteq \tilde{\mathcal{D}}$ irred. ipersmf.
- immagine$(\tilde{\mathcal{C}}_K) \subseteq \mathcal{D} = \tilde{\mathcal{D}}/\Gamma$ dipende solo da d

$\rightsquigarrow$ **notaz.** $\mathcal{C}_d \subseteq \mathcal{D}$

- $\mathcal{C}_d \neq \emptyset$ sse $d \equiv 0, 2 \mod 6$.

Suriett. della mappa dei periodi :

$$\wp : \mathcal{M} \longrightarrow \mathcal{D}$$

(Laza, Looijenga) $\quad \operatorname{im} \wp = \mathcal{D} \setminus (\mathcal{C}_2 \cup \mathcal{C}_6)$

Notaz. : $\quad Y \in \mathcal{C}_d \quad :\Longleftrightarrow \quad \wp(Y) \in \mathcal{C}_d$.

Exs

(1) $Y \supseteq P$ piano $\rightsquigarrow$ $K = \begin{array}{cc} h^2 & P \\ \left[\begin{array}{c} 3 \\ 1 \end{array}\right. & \left.\begin{array}{c} 1 \\ 3 \end{array}\right] \end{array}$

$\rightsquigarrow Y \in \mathcal{C}_8$

Lem (Voisin) $Y \in \mathcal{C}_8$ sse $Y \supseteq P$ piano.

(2) $Y \supseteq T_4$ scroll quartico liscio $\sim \underset{|\mathcal{O}(1,2)|}{\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5}$

di grado 4

$\rightsquigarrow K = \begin{bmatrix} 3 & 4 \\ 4 & 10 \end{bmatrix} \rightsquigarrow Y \in \mathcal{C}_{14}$.

Lem (Beauville, Tregub)

$Y \in \mathcal{C}_{14}$ molto gen. $\Rightarrow Y \simeq T_4$

⚠

∃ K3 associata

Osservazione di Hassett:

$Y \in \mathcal{C}_d \rightsquigarrow K^\perp \subseteq H^4(Y,\mathbb{Z})$

Then $\exists\, (S,f)$ sup. K3 polariz., $f^2 = d$ tc.

$\underset{\text{isom. di Hodge}}{K^\perp \cong} H^2(S,\mathbb{Z})_{prim.}$ sse $4 \nmid d$, $9 \nmid d$, $p \nmid d$

$\forall p \geq 5$ primo, $q \equiv 2 \bmod 3$.

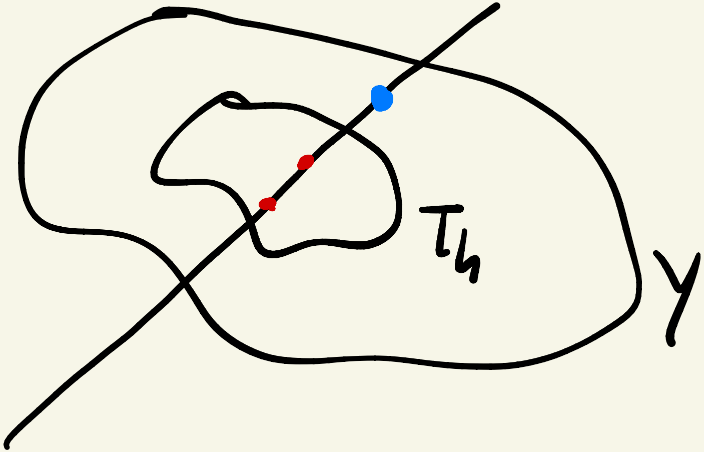
Domanda: Y cubic 4fold

Y razionale sse $\exists$ $K3$ associata. ?

Es $d = 14, 26, 38, 42$

(1) (Beauville - Donagi) $Y \in \mathcal{C}_{14} \Rightarrow Y$ razionale
Fano, Morin

Idea: Usa $T_4 \subseteq Y$ & rette secanti



$$\begin{tikzcd}
\mathrm{Sym}^2 T_4 \arrow[r, dashed, "\simeq"] \arrow[d, dashed, "\simeq"] & Y \\
\mathbb{P}^4 &
\end{tikzcd}$$

(2) (Russo - Staglianò)

$Y \in \mathcal{C}_{26} \cup \mathcal{C}_{38} \cup \mathcal{C}_{42} \Rightarrow Y$ <u>razionale</u>

<u>Idea</u>: $Y \in \mathcal{C}_d$, $d = 26, 38, 42$

$\Rightarrow \exists\ \Sigma_d \subseteq Y$ "speciale"

$+\ \exists$ curve razionali di grado e

$(3e-1)$-secanti a Σ_d

$(e = 2, 3)$

$\triangle$

Domanda (Hassett): Possiamo caratterizzare

$\underline{\underline{\forall d}}$

$Y \in \mathcal{C}_d$ molto gen. $\iff$ $\exists$ superficie "speciale"

$\Sigma_d \subseteq Y$?

Nostro risultato:

- costruz. cong. di Σ_d
- OK, per ∞ - cubiche

Categorie derivate

Y cubic 4fold $\rightsquigarrow$ $D^bY := D^b(\mathrm{coh}\, X)$

$$\mathcal{D}_Y := \langle \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle^{\perp}$$

$$\begin{tikzcd} = \left\{ E \in D^bY : \mathrm{Ext}^\bullet(\mathcal{O}_Y(\ell), E) = 0 \;\; \forall \ell = 0,1,2 \right\} \arrow[r, hook, "i_*"'] & D^bY \arrow[l, bend right, "i^*"'] \end{tikzcd}$$

componente di Kuznetsov

Lem (Kuznetsov, Addington-Thomas)

$\mathcal{D}_Y$ **categoria K3** :

- dualità di Serre : $\mathrm{Ext}^{\bullet}(E,F) = \mathrm{Ext}^{2-\bullet}(F,E)^{\vee}$
- $H(\mathcal{D}_Y, \mathbb{Z}) \cong H^*(\underset{\substack{\wr \\ K3}}{S}, \mathbb{Z}) \cong U^4 \oplus E_8(-1)^2$
- $HH_{\bullet}(\mathcal{D}_Y) \rightsquigarrow$ struttura di Hodge di peso 2
- polarizzazione ; $\underset{\substack{\wr \\ \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}}{A_2} \subseteq H_{alg}(\mathcal{D}_Y, \mathbb{Z})$ λ_1, λ_2 base

Thm (Bayer-Lahoz-M-Nuer-Perry-Stellari)

(1) teoria di Mukai degli sp. di moduli funziona per $\mathcal{D}_Y$: ∃ "canonica" σ_0

- $\mathrm{Stab}(\mathcal{D}_Y) \neq \emptyset$
- $v \in H_{alg}(\mathcal{D}_Y, \mathbb{Z})$ prim, $v^2+2 \geqslant 0$

  $\sigma \in \mathrm{Stab}(\mathcal{D}_Y)$ gen.

$\Rightarrow M_\sigma(v) \neq \emptyset$ liscia proiett. HK

di dim v^2+2

(2) $(AT + \varepsilon)$ $\exists$ K3 assoc. sse

$\mathcal{D}_Y \cong D^b S$ sse $U \subseteq H_{alg}(\mathcal{D}_Y, \mathbb{Z})$

sse $4 \nmid d$, $9 \nmid d$, ...

Exs (1) $\nu = \lambda_1 \rightsquigarrow M_{\gamma_0}(\lambda_1) \cong F(Y)$ varietà delle rette $\subseteq Y$

(2) $\nu = \lambda_1 + 2\lambda_2 \rightsquigarrow M_{\gamma_0}(\lambda_1) \cong X(Y)$ HK 8fold LLSvS

P.to chiave: $Y \hookrightarrow X(Y)$

canonicamente

Supponiamo che: $d \equiv 2 \mod 6 \rightsquigarrow d = 6m + 2$

$Y \in \mathcal{C}_d \rightsquigarrow \exists\, u_a \in H_{alg}(\mathcal{A}_Y, \mathbb{Z})$ tc.

$a > 0$

- $u_a^2 + 2 = a^2 + a + (1 - m) \geq 0$
- $u_a \cdot (\lambda_1 + 2\lambda_2) = 1$

Def $F \in M_{\sigma_0}(u_a)$

$BN_F := \{ E \in Y \hookrightarrow M_{\sigma_0}(\lambda_1 + 2\lambda_2) : \mathrm{Ext}^\ell(F,E) \neq 0$
per un solo $\ell \}$

Lem Se $\boxed{BN_F \neq \emptyset} \longleftarrow$

Allora $i_* F \simeq \mathcal{I}_{\Sigma_F}((a+1)h)$

dove $\Sigma_F \subseteq Y$ **superficie**

$\deg(\Sigma_F) = 1 + \frac{3}{2}a(a+1)$

$\Sigma_F^2 = \dfrac{d + \deg^2}{3}$

Exs (1) $d=8 \rightsquigarrow \Sigma \cong \mathbb{P}^2 = P \subseteq Y$

(2) $d=14 \rightsquigarrow \Sigma = T_4 \subseteq Y$

(3) $d=26 \rightsquigarrow$ sup. Nuer $\rightsquigarrow$ utilizzata
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da Russo
Staglianò.

Prop $\boxed{d = 6a^2+6a+2} \Rightarrow BN_F \neq \emptyset$

$\underline{\underline{\forall}}\, F \in M_\sigma(v_a)$.

$$\begin{tikzcd} \mathcal{Z}_\Sigma(\ldots) & D^bY \\ i_*F \arrow[u, phantom, sloped, "\cong"] \arrow[r, phantom, "\in"] & \mathcal{D}_Y \arrow[u, phantom, sloped, "\subseteq"] \\ & X^{\emptyset} \arrow[u, squiggly] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{D}_Y \arrow[r, squiggly] & X = M_{\sigma_0}(\overbrace{\lambda_1+2\lambda_2}^{v}) \arrow[r, leftrightarrow] & u.v=1 \\ \text{defo di una K3} \arrow[u] & Y \arrow[u, hook] & u^2+2 \geq 0 \end{tikzcd}$$

$\underline{HK}$

$F \in M_{\sigma_0}(u) \rightsquigarrow X \ni E$ tc. $\mathrm{Ext}^{\cdot}(F,E)$ più $\underline{\text{piccoli}}$

$$M(\lambda_1+2\lambda_2) = \mathrm{Hilb}^4 S$$

$$u = v(\mathcal{K}(n)) \qquad M(u) = S$$

$$BN_{n\in S} := \{ \mathcal{I}_{4pts} : \mathrm{Ext}^{\cdot}(\mathcal{K}(n), \mathcal{I}_{4pts}) \text{ più grande} \}$$

$$\neq \emptyset \qquad 4pts \not\ni n$$

$$\rightsquigarrow \Gamma \subseteq \mathrm{Hilb}^4 S \quad \dim 6 \rightsquigarrow \Gamma \cap Y = \Sigma$$

$$\begin{tikzcd} M(2\lambda_1) \arrow[dd, two heads] & \\ {} & JY_H \arrow[l] \\ \mathbb{P}^5 & \end{tikzcd}$$

<u>cubic 3fold</u>

$$\underbrace{X(Y) = Bl_0 \textcircled{H} \supseteq Y}$$