

ANTISYMPLECTIC INVOLUTIONS

ON PROJECTIVE HYPERKÄHLER

MANIFOLDS

work in progress with

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Setting: (X, λ)

X proj. smooth HK, $\dim X = 2g$, $g \geq 1$

$K3$ -type

$$(\text{div}(\lambda)) = \lambda \cdot H^2(X, \mathbb{Z}) \subseteq \mathbb{Z}$$

↓
ident

λ prim. polarizat., $\lambda^2 = 2$, $\text{div}(\lambda) = 1, 2$

writ BBF form on $H^2(X, \mathbb{Z})$

$$f_\lambda(n) = -n + (\lambda, n) \cdot \lambda \quad \text{on } H^2(X, \mathbb{Z})$$

$$f_\lambda(\lambda) = \lambda$$

Verbitsky's Torelli Thm.

+

Markham's Monodromy
Thm.



$\exists \tau: X \xrightarrow{\sim} X$ s.t.

$$\tau^2 = \text{id}$$

$$\tau_* = f_\lambda.$$

Main Thm

(1) The number of connected components of $\text{Fix}(\tau)$ is equal to $\text{div}(\lambda)$

(2) If $\text{div}(\lambda) = 2$, then one comp. is

Fano manifold

(of dim g , index 3)

Ex & Motivation 1

$$\text{div}(\lambda) = 2, \quad g = 4 \quad \rightsquigarrow \quad \dim X = 8$$

Y cubic 4 fold

$$F(Y) = \{ \text{variety of lines } \subseteq Y \} \hookrightarrow \text{Gr}(2,6)$$

$h = \text{Plücker pol.}$

$$\mathbb{P}^2 \hookrightarrow \text{Gr}(4,6)$$

$$X(Y) = \{ \text{variety of equivalence classes of twisted cubics } \subseteq Y \}$$

$\lambda = \text{Plücker pol.}$

$(Y \not\subseteq \mathbb{P}^3 \text{ plane})$

Duality: $H^2(F(Y), \mathbb{Z})_h \simeq H^2(X(Y), \mathbb{Z})_\lambda$

\exists "natural" involution $\tau: X(Y) \xrightarrow{\sim} X(Y)$

$\text{Fix}(\tau) = Y \amalg \tilde{Y}$
↗ cubic ↖ second comp.

$$2h = C_1 + C_2$$

Lehm-Lehm-Sorger
 - Van Straten

Aim: $\bullet (F, h) \rightsquigarrow (X, \lambda)$
 \nearrow HK4 \nwarrow pol. $h^2 = 2g - 2$
 K_3 type $\text{div}(h) = 2$ $\text{HK } 2g$ $\lambda^2 = 2$
 K_3 type $\text{div}(\lambda) = 2$

$\bullet \underbrace{(F, h)} \rightsquigarrow \gamma \underbrace{\underline{\underline{\text{Fano}}}}$

$$D^b Y = \langle \mathcal{D}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$$

Y cubic 4-fold

• \mathcal{D}_Y non-comm. K3 surface

• $\sigma \in \text{Stab}(\mathcal{D}_Y)$ "canonical"

Bayen-Lahoz-M.
Stellari

Li-Pentusi-Zhao \longrightarrow • $(F(Y), h) \cong (M_\sigma(\lambda), l_\sigma)$

Ex & Motivation 2

$$\operatorname{div}(\lambda) = 1, \quad g = 2 \rightsquigarrow$$

$$(X, \lambda) \quad \begin{array}{l} \swarrow 4 \\ \searrow \lambda^2 = 2 \end{array}$$

V_6 v.space of dim 6

$A \subseteq \Lambda^3 V_6$ Lagr. subspace

\swarrow
dim 10

$$Y_A = \left\{ [v] \in \mathbb{P}V_6 : \begin{array}{l} A \xrightarrow{\psi_v} \Lambda^4 V_6 \\ d \mapsto v \wedge d \end{array} \text{ has non-zero kernel} \right\}$$

Thm (O'Grady) A generic

Then: $\cdot Y_A$ singular sextic hypersurface

$\text{sing}(Y_A) = W_A$ irred smooth surface
of gen type

$\cdot X_A \xrightarrow{2-1} Y_A$ ramif at W_A

X_A HK sm. proj. of K3 type

$\cdot \lambda_A := f^* \mathcal{O}_{\mathbb{P}^6}(1)$ ample, $\lambda_A^2 = 2$, $\text{div}(\lambda_A) = 1$

$\cdot \tau$ cov.
inv. assoc
to λ_A .

Ferretti : W_A does not move
(O'grady) $2 W_A$ does "move"

\rightsquigarrow covering family of Lagrangian
cycles on X_A

Q: In general, $m \cdot \text{Fix}(\tau)$ "moves" ?

Sketch of pf of Main Thm

(X, λ)
 \nearrow HK 2g
 \nwarrow K3 type
 $\lambda^2 = 2 \rightsquigarrow \tau: X \xrightarrow{\sim} X \quad \tau^2 = \text{id}$

Thm (1) $\text{Fix}(\tau)$ has exactly $\text{div}(\lambda)$ comm. pts.

(2) If $\text{div}(\lambda) = 2$, then one cmt. is Fano.

Recall

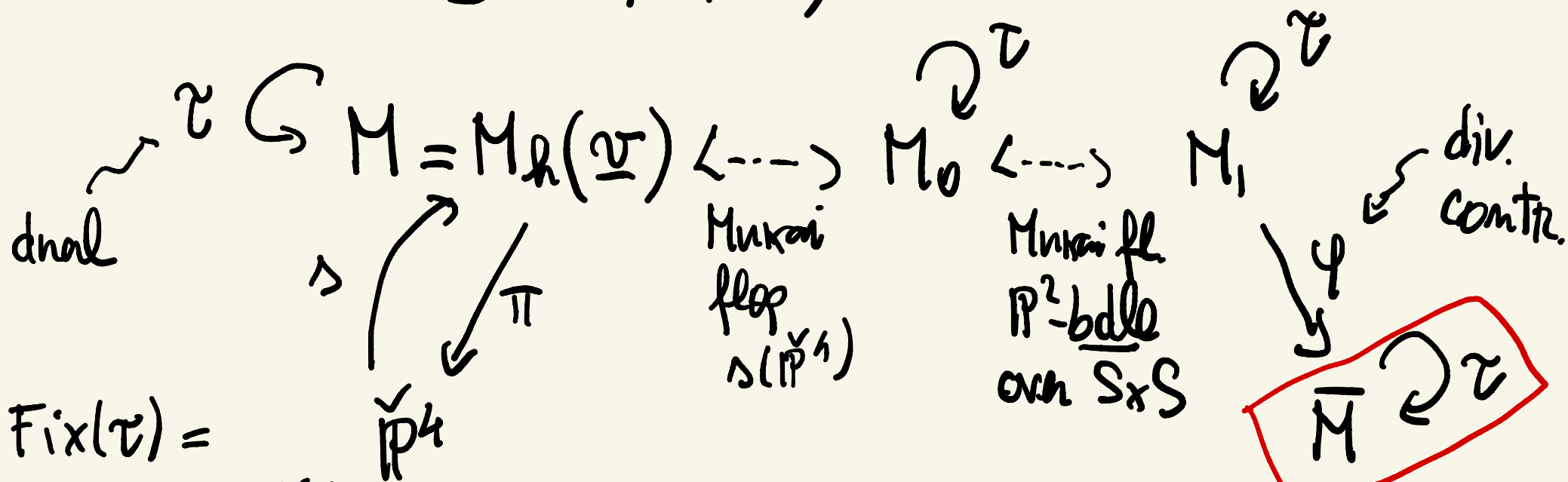
Idea: Specialize to (X, λ) where
we can understand the involution τ

Issue: need to consider singular X

Ex Y cubic $\begin{cases} \nearrow \text{nodal cubic (A)} \\ \searrow \text{chordal cubic (B)} \end{cases}$

(A) (C. Lehn) (S, h) SK3, $h^2 = 6$

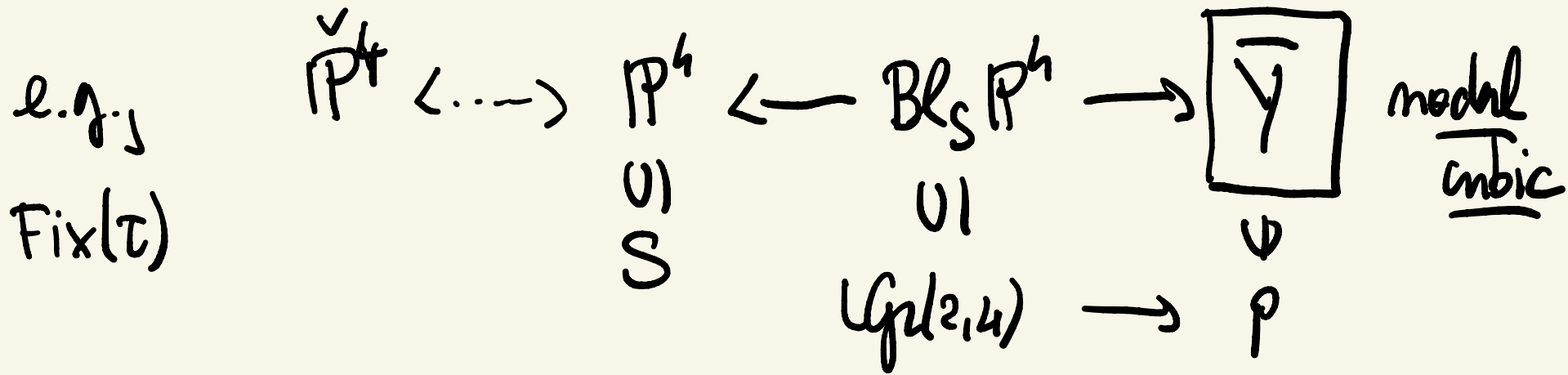
$$\underline{v} = (0, h, -3)$$



$$\text{Fix}(\tau) =$$

$$\mathcal{S}(\check{\mathbb{P}}^4) \perp \tilde{Y}'$$

\mathcal{L}_c on $C \in |h|$ st. $\mathcal{L}_c^2 \cong \mathcal{O}_C$, $\mathcal{L}_c \not\cong \mathcal{O}_C$



Pf Thm :

St.1 : Defo theory of (singular) HK

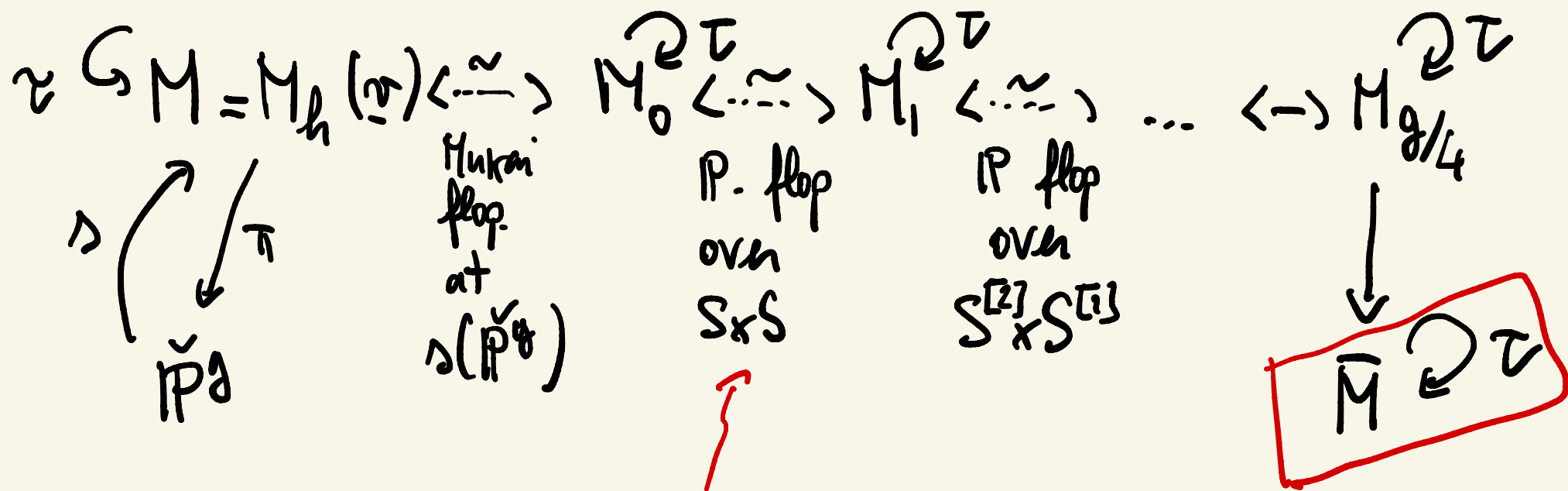
w/ involution.

← Namikawa

St.2 : Specialization

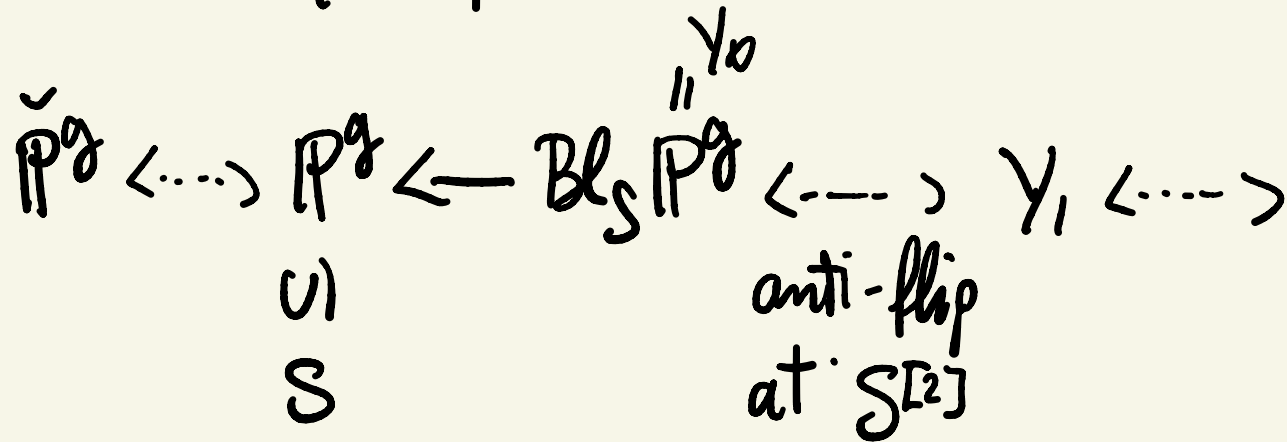
(S, h) S K3, $h^2 = 2m$, $g = m+1$

$4|g$, $\underline{v} = (0, h, -m)$



- $\text{Fix}(\tau)$ has 2 cpts on M
- flops do not destroy / create new cpts except first one.

• Fano cpt. of $\text{Fix}(T)$:



degen.
of Fano
cpt

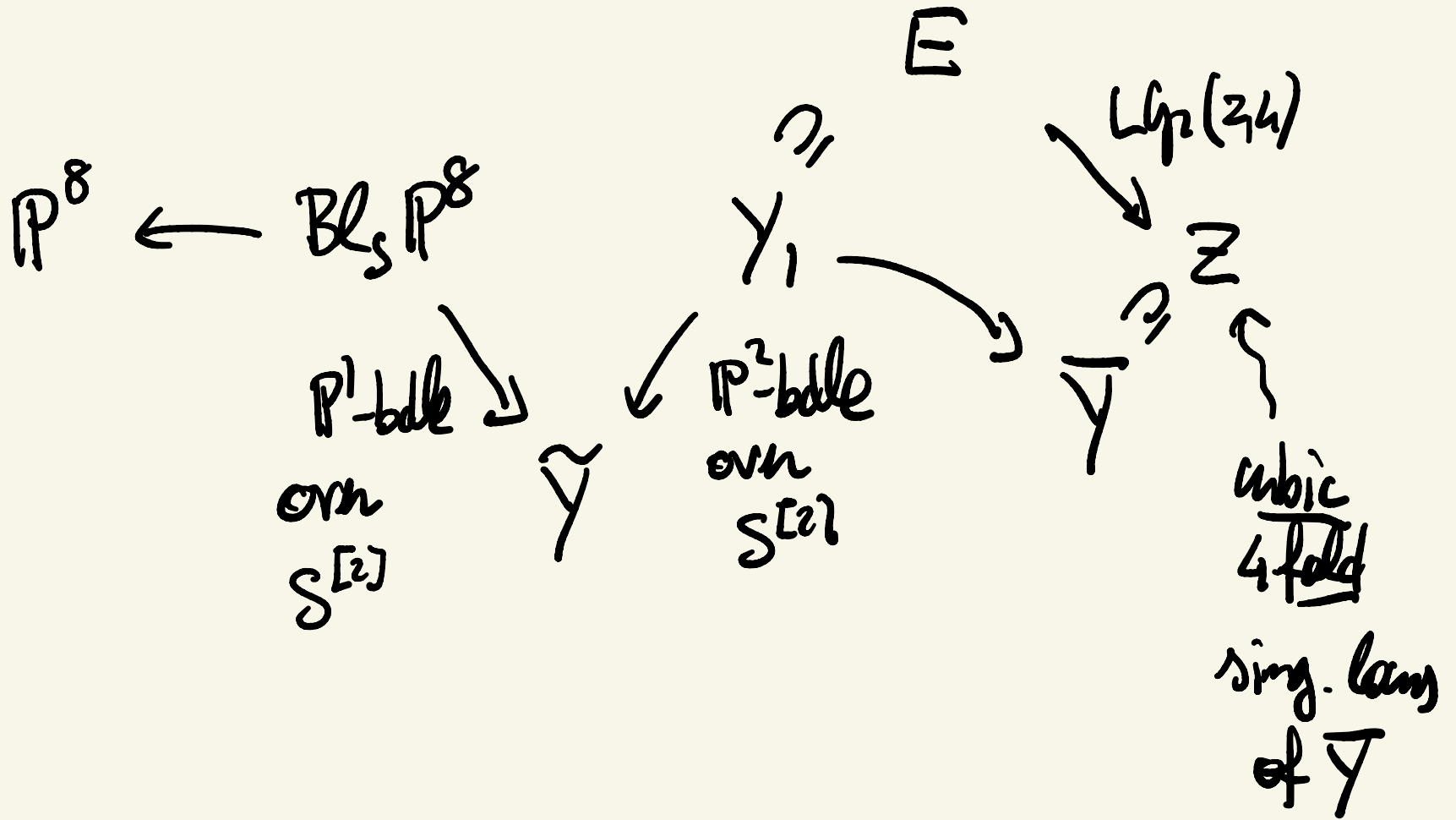
St.3 : Fixed loci in family

Conj Y Fano smpt. , $m = g/4$

Then

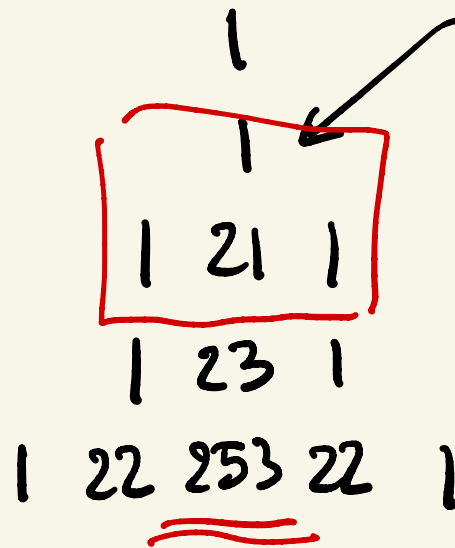
$$\begin{aligned} \mathcal{D}^b Y = & \langle \mathcal{D}^{[m]} \xrightarrow{\text{K3 category}} \mathcal{Q}, \mathcal{D}, \mathcal{D}^{[2]}, \dots, \mathcal{D}^{[m-1]}, \\ & \mathcal{Q}(1), \mathcal{D}(1), \mathcal{D}^{[2]}(1), \dots, \mathcal{D}^{[m-1]}(1), \\ & \mathcal{Q}(2), \mathcal{D}(2), \dots, \mathcal{D}^{[m-1]}(2) \rangle \end{aligned}$$

$g=8$:



Thm $g=8$

Hodge diamond is:



$$\Sigma = \text{Fix}(\sigma)$$

$$N = \binom{g+2}{2} - 1$$

$$0 \rightarrow T_\Sigma \rightarrow T_{\mathbb{P}^N}|_\Sigma \rightarrow N_\Sigma/\mathbb{P}^N \rightarrow 0$$

double EPW:

$$2 \cdot K_\Sigma = 6 \cdot \lambda|_\Sigma$$

$$\begin{array}{c} \hookrightarrow \\ \text{Sym}^2 N_\Sigma/\mathbb{P}^N \\ \text{"} \\ \text{Sym}^2 \Omega_\Sigma \end{array}$$