ANTISYMPLECTIC INVOLUTIONS ON PROJECTIVE HYPERKÄHLER MANIFOLDS

work in progress with

Laure Flapan, Kieran O'Grady, Giulia Saccà

Setting: (X, λ)

X proj. smooth HK, $\dim X = 2g$, $g \geq 1$

K3-type

$(\mathrm{div}(\lambda)) = \lambda . H^2(X, \mathbb{Z}) \subseteq \mathbb{Z}$ ideal

λ prim. polarizat., $\lambda^2 = 2$, $\mathrm{div}(\lambda) = 1, 2$

wrt BBF form on $H^2(X, \mathbb{Z})$

$$\rho_\lambda(x) = -x + (\lambda, x)\cdot\lambda \quad \text{on } H^2(X,\mathbb{Z})$$

$$\rho_\lambda(\lambda) = \lambda$$

Verbitsky's Torelli Thm.

+

Markman's Monodromy Thm.

$\Longrightarrow$ $\exists\, \tau : X \xrightarrow{\sim} X$ s.t.

$\tau^2 = id$

$\tau_* = \rho_\lambda$.

Main Thm

(1) The number of connected components of Fix(τ) is equal to div(λ)

(2) If div(λ) $= 2$, then one cmpt. is Fano manifold (of dim 9, index 3)

Ex & Motivation 1

$\operatorname{div}(\lambda) = 2$, $g = 4$ $\rightsquigarrow$ $\dim X = 8$

Y cubic 4 fold

$\rightsquigarrow$

$F(Y) = \{\text{variety of lines} \subseteq Y\} \hookrightarrow Gr(2,6)$

h = Plücker pol.

$X(Y) = \left\{\begin{array}{l}\text{variety of equivalence} \\ \text{classes of twisted cubics} \subseteq Y\end{array}\right\}$ $\overset{f\ 2:1}{\dashrightarrow} Gr(4,6)$

λ = Plücker pol.

($Y \not\supseteq \mathbb{P}$ plane)

Duality: $\underbrace{H^2(F(Y),\mathbb{Z})_h \simeq H^2(X(Y),\mathbb{Z})_\lambda}$

$\exists$ "natural" involution $\tau : X(Y) \xrightarrow{\sim} X(Y)$

$\mathrm{Fix}(\tau) = Y \sqcup \tilde{Y}$

second comp.

cubic

Lehn-Lehn-Sorger
-Van Straten

$$\boxed{2h = C_1 + C_2}$$

Aim: • $(F, h) \rightsquigarrow (X, \lambda)$

F: HK4 K3type

h: pol. $h^2 = 2g-2$, $\mathrm{div}(h) = 2$

X: HK 2g K3type

$\lambda^2 = 2$, $\mathrm{div}(\lambda) = 2$

• $(F, h) \rightsquigarrow Y$ **Fano**

$$D^b Y = \langle \mathcal{D}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$$

Kuznetsov

Y cubic 4fold

- $\mathcal{D}_Y$ non-comm. K3 surface
- $\sigma \in \mathrm{Stab}(\mathcal{D}_Y)$ "canonical"

Bayer-Lahoz-M.
Stellari

Li-Pertusi-Zhao $\rightsquigarrow$

- $(F(Y), h) \cong (M_\sigma(\lambda), l_\sigma)$

Ex & Motivation 2

$\mathrm{div}(\lambda) = 1$, $g = 2$ $\rightsquigarrow$ (X, λ) 4 $\lambda^2 = 2$

V_6 v.space of dim 6

$A \subseteq \Lambda^3 V_6$ Lagr. subspace

dim 10

$$Y_A = \left\{ [v] \in \mathbb{P}V_6 : \begin{array}{c} A \xrightarrow{\varphi_v} \Lambda^4 V_6 \\ \alpha \mapsto v \wedge \alpha \end{array} \text{ has non-zero kernel} \right\}$$

<u>Thm</u> (O'Grady) A <u>generic</u>

Then: • Y_A singular sextic hypersurface

$\mathrm{sing}(Y_A) = W_A$ irred <u>smooth</u> surface

of gen <u>type</u>

• $X_A \xrightarrow[f]{2:1} Y_A$ ramif at W_A

X_A HK sm. proj. of K3 type

• $\lambda_A := f^*\mathcal{O}_{\mathbb{P}^5}(1)$ ample, $\lambda_A^2 = 2$, $\mathrm{div}(\lambda_A) = 1$

$\boxed{\begin{array}{l}\cdot\ \tau\ \underline{\text{cov.}}\\ \underline{\text{inv.}}\ \text{assoc.}\\ \text{to } \lambda_A.\end{array}}$

Ferretti : W_A does not move

(O'Grady)

$2W_A$ does "move"

$\rightsquigarrow$ covering family of Lagrangian cycles on X_A

Q: In general, $m \cdot \mathrm{Fix}(\tau)$ "moves" ?

Sketch of pf of Main Thm

(X, λ)

HK 2g K3 type

$\lambda^2 = 2 \rightsquigarrow \tau : X \xrightarrow{\sim} X \quad \tau^2 = \mathrm{id}$

Recall

Thm (1) $\mathrm{Fix}(\tau)$ has exactly $\mathrm{div}(\lambda)$ conn. cpts.

(2) If $\mathrm{div}(\lambda) = 2$, then one cmpt. is Fano.

<u>Idea</u>: Specialize to (X, λ) where we can understand the involution τ

<u>Issue</u>: need to consider <u>singular</u> X

<u>Ex</u> Y cubic

$$\begin{tikzcd} & \text{nodal cubic (A)} \\ Y \text{ cubic} \arrow[ur] \arrow[dr] & \\ & \text{chordal cubic (B)} \end{tikzcd}$$

(A) (C. Lehn) (S,h) $S\underline{K3}$, $h^2 = 6$

$$\underline{v} = (0, h, -3)$$

$$\begin{tikzcd}
\tau \circlearrowright M = M_h(\underline{v}) \arrow[d, "\pi"] \arrow[r, dashed, leftrightarrow] & M_0 \arrow[r, dashed, leftrightarrow] & M_1 \arrow[d, "\varphi"] \\
\check{\mathbb{P}}^4 \arrow[u, bend left, "s"] & & \boxed{\overline{M} \circlearrowleft \tau}
\end{tikzcd}$$

$\tau \circlearrowright M_0$, $\tau \circlearrowright M_1$

dual

Mukai flop $s(\check{\mathbb{P}}^4)$

Mukai fl. $\mathbb{P}^2$-$\underline{\text{bdle}}$ over $S \times S$

div. contr.

(!)

$\operatorname{Fix}(\tau) = s(\check{\mathbb{P}}^4) \amalg \tilde{Y}'$

L_C on $C \in |h|$ s.t. $L_C^2 \simeq \mathcal{O}_C$, $L_C \not\simeq \mathcal{O}_C$

e.g.,

Fix(τ)

$$\begin{array}{ccccccc} \check{\mathbb{P}}^4 & \dashrightarrow & \mathbb{P}^4 & \longleftarrow & Bl_S\,\mathbb{P}^4 & \longrightarrow & \boxed{\overline{Y}} \\ & & \cup & & \cup & & \psi \\ & & S & & LGr(2,4) & \longrightarrow & p \end{array}$$

nodal cubic

Pf Thm :

St. 1 : Defo theory of (singular) HK
w/ involution. ← Namikawa

St. 2 : Specialization

(S, h) S K3 , $h^2 = 2n$, $g = n+1$

$4 | g$, $\underline{v} = (0, h, -n)$

$$
\begin{tikzcd}
\tau \circlearrowright M = M_h(v) \arrow[d, shift left, "\pi"] \arrow[r, dashed, leftrightarrow, "\sim", "\substack{\text{Mukai}\\ \text{flop}\\ \text{at}\\ s(\check{\mathbb{P}}^g)}"'] & M_0 \circlearrowleft \tau \arrow[r, dashed, leftrightarrow, "\sim", "\substack{\mathbb{P}\text{-flop}\\ \text{over}\\ S\times S}"'] & M_1 \circlearrowleft \tau \arrow[r, dashed, leftrightarrow, "\sim", "\substack{\mathbb{P}\text{ flop}\\ \text{over}\\ S^{[2]}\times S^{[1]}}"'] & \cdots \arrow[r, dashed, leftrightarrow] & M_{g/4} \circlearrowleft \tau \arrow[d] \\
\check{\mathbb{P}}^g \arrow[u, bend left, "s"] & & & & \boxed{\bar{M} \circlearrowleft \tau}
\end{tikzcd}
$$

- $\mathrm{Fix}(\tau)$ has 2 cmpts on M
- flops do not destroy / create new cmpts except first one.

- Fano cmpt. of $\mathrm{Fix}(\tau)$:

$$
\begin{array}{ccccccccc}
 & & & & Y_0 & & & & \\
 & & & & \| & & & & \\
\check{\mathbb{P}}^g & \dashrightarrow & \mathbb{P}^g & \longleftarrow & \mathrm{Bl}_S \mathbb{P}^g & \dashrightarrow & Y_1 & \dashrightarrow & Y_{g/4} \\
 & & \cup & & & \text{anti-flip} & & & \downarrow \varphi \\
 & & S & & & \text{at } S^{[2]} & & & \boxed{\overline{Y}}
\end{array}
$$

degen. of Fano cmpt

St. 3: Fixed loci in family

Conj Y Fano cmpt., $m = g/4$

Then

K3 category

$$D^b Y = \langle \mathcal{D}^{[m]}, \mathcal{O}_Y, \mathcal{D}, \mathcal{D}^{[2]}, \ldots, \mathcal{D}^{[m-1]}, \mathcal{O}_Y(1), \mathcal{D}(1), \mathcal{D}^{[2]}(1), \ldots, \mathcal{D}^{[m-1]}(1), \mathcal{O}_Y(2), \mathcal{D}(2), \ldots, \mathcal{D}^{[m-1]}(2) \rangle$$

$g = 8$:

$$\begin{tikzcd}
 & & & E \arrow[dr, "LGr(2,4)"] & \\
\mathbb{P}^8 & \mathrm{Bl}_S \mathbb{P}^8 \arrow[l] \arrow[dr, "\mathbb{P}^1\text{-bdle over } S^{[2]}"'] & & Y_1 \arrow[u, phantom, "\cong"] \arrow[dl, "\mathbb{P}^2\text{-bdle over } S^{[2]}"] \arrow[dr] & Z \arrow[d, phantom, "\cong"] \\
 & & \tilde{Y} & & \overline{Y}
\end{tikzcd}$$

Z: cubic 4fold

sing. locus of $\overline{Y}$

Thm $g=8$

Hodge diamond is:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & 1 & & & & \\
 & & 1 & & 21 & & 1 & & \\
 & & 1 & & 23 & & 1 & & \\
1 & & 22 & & \underline{\underline{253}} & & 22 & & 1
\end{array}
$$

$\Sigma = \mathrm{Fix}(\tau)$ $\qquad N = \binom{g+2}{2} - 1$

$$0 \to T_\Sigma \to T_{\mathbb{P}^N}|_\Sigma \to N_{\Sigma/\mathbb{P}^N} \to 0$$

$$N_{\Sigma/\mathbb{P}^N} \cong \mathrm{Sym}^2 N_{\Sigma/X} = \mathrm{Sym}^2 \Omega_\Sigma$$

double EPW:

$$2 \cdot K_\Sigma = 6 \cdot \lambda|_\Sigma$$