BRIDGELAND STABILITY ON 3FOLDS

Plan:

1. The quintic 3fold
2. Bogomolov inequality for vector bundles
3. The support property and Bridgeland deformation thm.
4. Construction and generalized Bogomolov inequality.

1. The quintic 3 fold

$X := \{ W = 0 \} \subseteq \mathbb{P}^4_{\mathbb{C}}$ smooth quintic CY3

homog. poly in $\mathbb{C}[x_0, \dots, x_4]$
$\deg W = 5$

$D^b X :=$ bounded derived category

Interlude 1 : $D^b X$

V vector bundle on X

$$
f:\quad
\begin{tikzcd}
\cdots \arrow[r] & V^i \arrow[r, "d"] \arrow[d, "f^i"'] \arrow[dr, phantom, "\circlearrowright"] & V^{i+1} \arrow[r, "d"] \arrow[d, "f^{i+1}"] \arrow[dr, phantom, "\circlearrowright"] & V^{i+2} \arrow[r] \arrow[d, "f^{i+2}"] & \cdots \\
\cdots \arrow[r] & W^i \arrow[r, "d"] & W^{i+1} \arrow[r, "d"] & W^{i+2} \arrow[r] & \cdots
\end{tikzcd}
$$

V^i v. bdle
$V^n = 0$, if $|n| \gg 0$
$d^2 = 0$

<u>bounded complex</u> <u>of v. bdles on X</u>

$$\mathrm{Hom}_{D^b X}(V^\cdot, W^\cdot) = \left\{ \begin{matrix} \text{morph. of} \\ \text{complexes } f \end{matrix} \right\} \left[\begin{matrix} \text{invert} \\ \text{quasi-isomorphisms} \end{matrix} \right]$$

technical...

as above

Idea:

- $V^\bullet \rightsquigarrow V^\bullet[1]$ shifted complex (to the left)
- V, W v.bdles, $\mathrm{Hom}_{D^bX}(V, W[k]) = H^k(X, V^\vee \otimes W)$
- "generalized exact seq.": triangle

$$U^\bullet \to V^\bullet \to W^\bullet \to U^\bullet[1]$$

s.t. $\forall\, C^\bullet \in D^bX$, $\exists$ long ex. seq.:

$$\cdots \to \mathrm{Hom}(C^\bullet, U^\bullet[k]) \to \mathrm{Hom}(C^\bullet, V^\bullet[k]) \to \mathrm{Hom}(C^\bullet, W^\bullet[k]) \to \mathrm{Hom}(C^\bullet, U^\bullet[k+1]) \to \cdots$$

1. The quintic 3fold

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$D^bX :=$ bounded derived category

$\mathrm{Stab}(D^bX) :=$ space of Bridgeland stability conditions on D^bX

Interlude 2 : $\mathrm{Stab}(D^bX)$ [Bridgeland, 2003]

$\sigma = (Z, P)$

$\cong \mathbb{Z}^{\oplus 4}$, for X quintic

- $Z : H_{alg}(X, \mathbb{Z}) \longrightarrow \mathbb{C}$ gp. homom.

  central charge

- $P = \bigcup_{\phi \in \mathbb{R}} P(\phi) \subseteq D^bX$

  semistable objects of phase ϕ

st. :

$Z(E) := Z(ch(E))$

Chern character (we'll see later)

(i) $\forall E \in P(\phi)$, $m(E) := |Z(E)| \neq 0$ mass

and $Z(E) = m(E) \cdot e^{\pi i \phi}$

(ii) $P(\phi + 1) = P(\phi)[1]$

(iii) $\mathrm{Hom}(P(\phi_1), P(\phi_2)) = 0 \quad \forall \phi_1 > \phi_2$

(iv) $\forall E \in D^b X$, $\exists$ seq. of triangles

$$\begin{tikzcd}[column sep=small]
0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, "{[1]}"] & & & & A_n \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

st. $\quad A_i \in P(\phi_i)$

$\phi_1 > \dots > \phi_n$.

1. The quintic 3fold

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$D^b X :=$ bounded derived category

it's a manifold!
(if we add the support property)

$\mathrm{Stab}(D^b X) :=$ space of Bridgeland stability conditions on $D^b X$

Questions :

- $\mathrm{Stab}(D^b X) \neq \phi$? $\longleftarrow$ this talk !

- What is it ? $\longleftarrow$ Aspinwall conj.

- Why do we care ? $\longleftarrow$ moduli spaces
  counting invariants
  wall-crossing formulas
  symplectic geometry

Conj. [Aspinwall, 2004]

$$\exists\ I : \mathcal{M}_K \hookrightarrow \left[{}_{\mathrm{Aut}(D^bX)} \backslash^{\mathrm{Stab}(D^bX)} / {}_{\mathbb{C}} \right]$$

closed embedding

$$\left[{}^{\{\psi \in \mathbb{C}\,:\,\psi^5 \neq 1\}} / {}_{\mu_5} \right]$$

stringy Kähler moduli space

where, if we write $I(\psi) = (Z_\psi, P_\psi)$

we have :

$u_i \leftrightarrow H^{2i}(X,\mathbb{Z}) \cong \mathbb{Z}$, $i=0,\dots,3$

$$Z_\psi(u_0,\dots,u_3) = \sum \Phi_i(\psi)\cdot u_i$$

$$\Phi_0 = \frac{1}{5}\left(\varpi_0 - \varpi_1\right)$$

$$\Phi_1 = \frac{1}{30}\left(16\cdot\varpi_0 - 9\cdot\varpi_1 + 3\,\varpi_3\right)$$

$$\Phi_2 = \frac{1}{5}\left(\varpi_0 - 3\,\varpi_1 - 2\,\varpi_2 - \varpi_3\right)$$

$$\Phi_3 = \varpi_0$$

where

$$\varpi_j(\psi) := -\frac{1}{5}\sum_{m=1}^{\infty} \frac{\Gamma(m/5)}{\Gamma(m)\cdot\Gamma(1-m/5)^4}\cdot\left(5\left(e^{2\pi i/5}\right)^{2+j}\cdot\psi\right)^m$$

$$j = 0, \ldots, 3$$

basis of solution of Picard-Fuchs eqn associated to periods of the mirror.

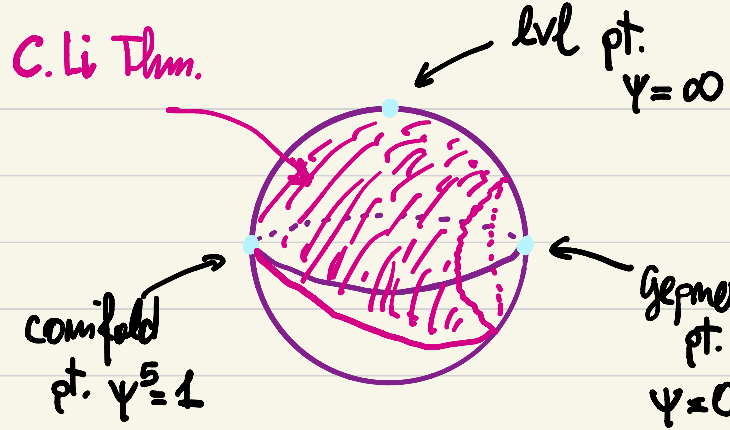
$\blacksquare$

Thm [Chunyi Li, 2019]

X quintic 3fold.

Then:

$$\mathrm{Stab}(D^bX) \neq \emptyset$$



and Aspinwall Conj. true near large volume limit. ∎

2. The Bogomolov inequality

X surface, H ample divisor

V v.bdle on X $\rightsquigarrow$ $ch(V)$ <u>Chern character</u>

$ch_0(V) := rk(V)$
$ch_1(V) := c_1(V)$
$ch_2(V) := \frac{c_1(V)^2}{2} - c_2(V)$

$$ch(V[1]) = -ch(V)$$

Interlude 3 : Chern classes

V v.bdle $\leftsquigarrow$ assume : V globally generated
$r := \mathrm{rk}(V)$

$s_0, \dots, s_{r-i} \in H^0(X, V)$
general sections

$D_i :=$ locus in X where $s_0, \dots, s_{r-i}$ linearly dependent

$\rightsquigarrow \quad c_i(V) := [D_i] \in H^{2i}(V, \mathbb{Z}).$

2. The Bogomolov inequality

X surface, H ample divisor

V v.bdle on X

Def V μ_H-<u>semistable</u> if $\forall\, W \subseteq V$,

$$\mu_H(W) := \frac{H.c_1(W)}{\operatorname{rk}(W)} \leq \mu_H(V)$$

Thm [Bogomolov inequality, ~1978]

V μ_H-semistable.

Then

$$\Delta(V) := c_1(V)^2 - 2\cdot \mathrm{rk}(V)\cdot ch_2(V) \geq 0$$

$\exists$ at least 5 different proofs

[Bogomolov, Reid, Gieseker, Le Potier, Lübke, Kobayashi, Langer, Lazarsfeld, ...]

$\blacksquare$

Rmk • $\exists$ version in higher dimension :

$$H^{n-2} \cdot \Delta(V) \geq 0 \quad , \quad n = \dim X$$

• For special surfaces, $\exists$ stronger ineq.

e.g., del Pezzo surfaces

K3 surfaces

$$\Delta(V) \geq \frac{3}{2} (\operatorname{rk} V)^2$$
if $\operatorname{rk} V \geq 2$

uses Euler characteristics and Serre duality :
$\chi(V,V) = \sum_{k=0}^{2} (-1)^k h^k(X, V \otimes V^\vee) \leq 2$, if V stable

Conj [Toda, 2013] based on [Douglas-Reinbacher-Yau, 2006]

X quintic 3fold, $H = \mathcal{O}_X(1)$

V <u>stable</u> v.bdle allow sing's ... V torsion-free sheaf

Assume: $\dfrac{c_1(V)}{rk(V)} = -\dfrac{H}{2}$

Then $H.\Delta(V) \geq 1.5139\ldots \cdot (rk(V))^2$

irrat'l number in $\mathbb{Q}(e^{2\pi i/5})$

$\blacksquare$

Thm [Chunyi Li, 2019]

$$H.\Delta(V) \geq \frac{5}{4} \cdot \operatorname{rk}(V)^2$$

$\blacksquare$

Key pt:

- Toda Conj. $\Rightarrow$ Aspinwall Conj
- Pf. of Li Thm uses Bridgeland stability cond. on surfaces

we will present main ideas in the remaining part of the seminar.

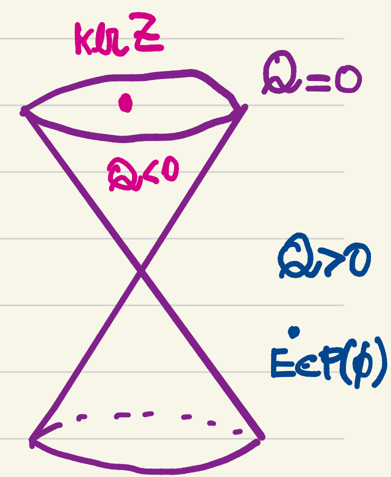
3. The support property [Kontsevich-Soibelman, 2008]

$\sigma = (Z, P) \in \mathrm{Stab}(D^b X)$

Def σ satisfies the support property if

$\exists$ real quadratic form Q st.

- $Q|_{\ker Z}$ negative definite
- $\forall E \in P(\phi)$, $Q(E) \geq 0$.



Ex X surface, H ample

$$Z = (-ch_2 + rk) + \sqrt{-1} \cdot H.c_1$$

$$Q := \Delta = c_1^2 - 2\, rk \cdot ch_2$$

$$\rightsquigarrow \quad \ker Z = \begin{cases} ch_2 = rk \\ H.c_1 = 0 \end{cases}$$

$$\rightsquigarrow \quad Q|_{\ker Z} \leq -2\, rk^2 < 0 .$$

$\blacktriangle$

Thm [Bridgeland Deformation Thm ; Bridgeland 2003]
[Bayer-M.-Stellari, 2016]

$\sigma = (Z, P_Z) \in \mathrm{Stab}(D^bX)$ w/ support property

Then $\forall\, W$ st. $Q|_{\ker W} < 0$

$\exists\, \tau = (W, P_W) \in \mathrm{Stab}(D^bX)$ w/ support property. $\square$

In particular, $\mathrm{Stab}(D^bX)$ w/ supp. prop. and an extra condition on existence of moduli spaces of semistable objs.

is complex manifold of dimension $\mathrm{rk}(H_{alg}(X, \mathbb{Z}))$.

4. Generalized Bogomolov inequality

$\sigma = (Z, P) \in \mathrm{Stab}(D^b X)$

$\xi \in \mathbb{C}$

e.g., $n \cdot \sigma = \sigma[n]$ $\forall n \in \mathbb{Z} \subseteq \mathbb{C}$

Def $\xi \cdot \sigma := \left(e^{-i\pi\xi} \cdot Z , P' \right)$, $P'(\phi) := P(\phi + \mathrm{Re}(\xi))$

$\rightsquigarrow$ $\mathbb{C} \curvearrowright \mathrm{Stab}(D^b X)$ "<u>rotations</u>"

Construction of Bridgeland stability conditions:

- Start w/ weak stability condition

  not all semistable objs. have a phase.
  e.g., μ_H-stability for sheaves
  (only torsion-free ones have a phase/slope)

- Rotate

  to match an eventual Bogomolov type inequality
  for the weak stability condition
  e.g., the actual Bogomolov inequality for stable sheaves.

- Deform

  by using an analogous
  result as Bridgeland Deformation Thm
  for weak stability.

- Iterate, if necessary

  e.g., it is needed in dim ≥ 3

Ex X surface, H ample.

- $\sigma_1 = (Z_1, P_1)$, $Z_1 = -H.c_1 + \sqrt{-1} \cdot \mathrm{rk}$

  (usual μ_H-stability) P_1 = μ_H-semist. v. bdles & **all** torsion sheaves

  they have no phase if codim = 2.

- Rotate by $\frac{\pi}{2}$: $\sigma_1' = (Z_1', P_1')$

- Deform by using $\Omega = \Delta$: $\sigma_2 = (Z_2, P_2) \in \mathrm{Stab}(D^b X)$ (!)

  $$Z_2 = (-\alpha \cdot ch_2 + \mathrm{rk}) + \sqrt{-1} \cdot H.c_1, \quad \forall \alpha > 0.$$

  Δ

Ex X 3fold, H ample

$$\sigma_1 \rightsquigarrow \sigma_2 \rightsquigarrow \sigma_3$$

σ_1: μ_H-stability

σ_2: tilt-stability (depends on 2 real parameters $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$; $\nu_{\alpha,\beta}$-stability)

σ_3: conjectural Bridgeland stability

Conj [Bayer–M.–Toda, 2011]

X 3fold, $\rho(X) = 1$ (e.g., X quintic 3fold.)

E $\nu_{\alpha,\beta}$-semistable

Then: $$\alpha^2 \Delta(E) + \underbrace{\nabla_\beta(E)}_{4(ch_2^\beta(E))^2 - 6 \cdot c_1^\beta(E) \cdot ch_3^\beta(E)} \geq 0$$

Idea of the pf.: X quintic 3fold.

similar to K3 surfaces case: recall that Homs in D^bX correspond to cohomology!

- [M, 2014; BMS, 2016] reduction to Euler characteristics estimates
- [C. Li, 2018] reduction to strong Bogomolov ineq. for μ_H-stable v.bdles

  Toda's Conjecture

- [Feyzbakhsh, 2016] reduction to cohomology estimates on curves of deg $(2,2,5) \subseteq \mathbb{P}^4$
- [C. Li, 2019] get such estimates by embedding curve in surface of deg $(2,2)$ (del Pezzo!) and use Bridgeland stab & strong Bogomolov there.

  by improving this, it might lead to proof of full Toda Conjecture and so to Aspinwall Conjecture.