# SEMI-CLASSICAL MEASURES FOR GENERALIZED PLANE WAVES

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ABSTRACT. Following joint work with Dyatlov [DyGu], we describe the semi-classical measures associated with generalized plane waves for metric perturbation of $\mathbb{R}^d$, under the condition that the geodesic flow has trapped set $K$ of Liouville measure 0.

## 1. INTRODUCTION

Semi-classical measures associated to (approximate) high frequency solutions of elliptic or hyperbolic equations have been studied in many different settings. They describe the concentration in phase space of these solutions. To fix the ideas, let $(u_h)_{h\to 0}$ be a sequence of $L^2$ functions on a $d$-dimensional Riemannian compact manifold $(M, g)$, such that $||u_h||_{L^2} = 1$. We say that a measure $\mu$ on the cotangent bundle $T^*M$ is a semi-classical measure associated to $u_h$ if for any pseudo differential operator $A \in \Psi^0(M)$ of order 0, one has

$$\langle Au_h, u_h\rangle_{L^2} \to \int_{S^*M} \sigma(A)d\mu, \quad \text{as } h \to 0.$$

where $\sigma(A)$ is the principal symbol of $A$. One natural question to understand eigenfunctions of the Laplacian for large eigenvalues is to consider semi-classical measures associated to $u_{h_j}$ satisfying $(h_j^2\Delta_g - 1)u_{h_j} = 0$ and $||u_{h_j}||_{L^2} = 1$, with $h_j \to 0$ as $j \to \infty$. Except in particular cases, describing the semi-classical measures of such eigenfunctions is very difficult. However, when the geodesic flow is ergodic on $S^*M$, this reflects in semi-classical measures: it was shown by Schnirelman [Sh], Zelditch [Ze87] and Colin de Verdière [CdV] that for any orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of eigenfunctions of the Laplacian with eigenvalues $h_j^{-2}$, there exists a density one subsequence $(e_{j_k})$ that converges microlocally to the Liouville measure $\mu_L$ on $S^*M$. A more precise statement is possible to obtain (see [HeMaRo] or [DyGu, Appendix D]):

$$h^{d-1} \sum_{h^{-1}\leq z_j\leq h^{-1}+1} \left| \langle \mathrm{Op}_h(a)e_j, e_j\rangle_{L^2} - \frac{1}{\mu_L(S^*M)} \int_{S^*M} a\, d\mu_L \right| \to 0 \quad \text{as } h \to 0 \quad (1.1)$$

where $\mathrm{Op}_h$ is a semi-classical quantization mapping semi-classical symbols to semi-classical pseudo-differential operators.

For many non-compact manifolds, there are in general much more eigenfunctions for the Laplacian than in the compact setting, however those are not $L^2$. In scattering theory, the two natural families of eigenfunctions are the generalized (or distorted) plane waves and the resonant states. On $\mathbb{R}^d$, the plane waves are simply

$$E(z; \xi, m) := e^{iz\xi.m}, \quad m \in \mathbb{R}^d, \xi \in S^{d-1}, z > 0$$

and they solve $(\Delta - z^2)E(z; \xi, \cdot) = 0$. Semi-classically, i.e. setting $z = \lambda/h$ for $\lambda$ close to 1, they can be rewritten as $E_h(\lambda; \xi) = E(z/h; \xi, \cdot)$ and they solve $(h^2\Delta - \lambda^2)E_h(\lambda; \xi) = 0$. A simple stationary phase argument shows that they are microlocally concentrated on the Lagrangian $\mathcal{L}_\xi := \{(m, \lambda\xi) \in T^*M; m \in M\}$: if $a$ is a compactly supported symbol and the quantization is chosen so that $\mathrm{Op}_h(a)$ has compact Schwartz kernel, then as $h \to 0$

$$\langle \mathrm{Op}_h(a)E_h(\lambda; \xi), E_h(\lambda, \xi) \rangle = \int_{\mathbb{R}^d} a(m, \lambda\xi)dm + \mathcal{O}(h). \tag{1.2}$$

Let us now take the case of compact elliptic perturbations of the flat Laplacian, say for instance that $(M, g)$ is a Riemannian manifold which contains a compact region $N$ so that $(M \setminus N, g)$ is isometric to $(\mathbb{R}^d \setminus B(0, R_0), g_{\mathrm{eucl}})$. For any $\lambda \in (1/2, 2)$, the semiclassical Laplacian $h^2\Delta_g$ associated to $(M, g)$ has a family of eigenfunctions $E_h(\lambda; \xi, m)$ which are, in $M \setminus N$, of the form

$$E_h(\lambda; \xi, m) = e^{i\frac{\lambda}{h}\xi.m} + E_{\mathrm{inc}}(\lambda; \xi, m)$$

where $E_{\mathrm{inc}}$ is incoming in the sense that it satisfies a Sommerfeld radiation condition near infinity, or equivalently, that it lies in the image of $C_0^\infty(\mathbb{R}^d)$ under the free (incoming) resolvent $R_0(\lambda/h)$ of the Laplacian on the Euclidean space $\mathbb{R}^d$. These $E_h$ are called generalized plane waves. Resonant states are more complicated, due to the fact that they correspond to eigenfunctions of a non self-adjoint problem, we shall not discuss their semiclassical limits here.

In this short note, we will explain the idea of [DyGu] to describe the semiclassical measures of $E_h(\lambda; \xi)$. Before we give the statement, we recall that the *trapped set* $K \in S^*M$ of the geodesic flow is the set of points $(m, \nu) \in S^*M$ such that the geodesic $(g^t(m, \nu))_{t \in \mathbb{R}}$ lies entirely in some compact subset of $S^*M$. Here $g^t$ detotes the geodesic flow on $S^*M$ or $T^*M$.

**Theorem 1.** *Let $(M, g)$ be a Riemannian manifold for which there exists a compact set $N$ so that $(M \setminus N, g)$ is isometric to $(\mathbb{R}^d \setminus B(0, R_0), g_{\mathrm{eucl}})$ for some $R_0 > 0$. Suppose that the trapped set has Liouville measure $\mu_L(K) = 0$. For Lebesgue almost every $\xi \in S^{d-1}$, there exists a Radon measure $\mu_\xi$ on $S^*M$ such that for each compactly supported $h$-semiclassical pseudodifferential operator $A \in \Psi^0(M)$, we have as $h \to 0$,*

$$\frac{1}{h}\int_1^{1+h}\int_{S^{d-1}} \left| \langle AE_h(\lambda, \xi), E_h(\lambda, \xi) \rangle_{L^2(M)} - \int_{S^*M} \sigma(A)\, d\mu_\xi \right| d\xi d\lambda \to 0. \tag{1.3}$$

*The measure $\mu_\xi$ is invariant under the geodesic flow on $S^*M$, and it is defined by*

$$\int_{S^*M} a \, d\mu_\xi := \lim_{t \to +\infty} \int_{M \setminus N} a \circ g^{-t}(m, \xi) dm.$$

*where we identify $S^*(M \setminus N)$ with $(\mathbb{R}^d \setminus B(0, R_0)) \times S^{d-1}$. It disintegrates the Liouville measure in the sense for the natural measure $d\xi$ on $S^{d-1}$, then*

$$\int_{S^{d-1}} \mu_\xi \, d\xi = \mu_L. \tag{1.4}$$

This result states that for almost any direction $\xi \in S^{d-1}$, the generalized plane wave directed by $\xi$ microlocally converges to $\mu_\xi$ in average in frequency in windows of semi-classical size $\mathcal{O}(h)$ (or equivalently classical windows of frequency of size $\mathcal{O}(1)$). In spirit, this can be compared to the quantum ergodicity statement of (1.2). In [DyGu, Th. 2], we actually get estimates for the speed of convergence in terms of the volume in phase space of points staying for roughly $|\log h|$ time near the trapped set. When the trapped set is partially uniformly hyperbolic for the flow, this gives a power of $h$ for the speed of convergence.

An application of the above result (with the speed of convergence) is a local Weyl law with several terms expansions. We shall not discuss it here, but we give a particular case of this (see [DyGu, Th. 3] for a general statement).

**Theorem 2.** *Let $(M, g)$ be as in Theorem 1 and assume that the curvature of $g$ is $-1$ near the projection of the trapped set $K$ on $M$. Let $2\delta + 1$ be the Hausdorff dimension of $K \subset S^*M$. Then there exist differential operators $L_j$ of order $2j$ on $T^*M$, with $L_0 = 1$, such that for each compactly supported zeroth order classical symbol $a$, we have for each $s > 0$ and $N \in \mathbb{N}$ such that $N > d - \delta$*

$$\mathrm{Tr}(\mathrm{Op}_h(a) \, \mathbb{1}_{[0,s]}(h^2\Delta)) = (2\pi h)^{-d} \sum_{j=0}^{N} h^j \int_{p \leq s} L_j a \, d\mu_\omega + \mathcal{O}(h^{-\delta-}) \tag{1.5}$$

*where $\mu_\omega$ is the standard volume form on $T^*M$, $p$ is the principal symbol of $\Delta_g$ and $\mathbb{1}_{[0,s]}(h^2\Delta)$ denotes the spectral projector of $h^2\Delta_g$ onto the frequency window $[0, s]$. The remainder is uniform in $s$ when $s$ varies in a compact subset of $(0, \infty)$.*

## 2. Proof of Theorem 1

We shall use some tools in semi-classical analysis, we thus refer to [Ma, Zw] for basics of semi-classical calculus and quantization.

**Generalized plane waves.** We start with the construction of the generalized plane waves. The continuous spectrum of the Laplacian $\Delta_g$ associated to the metric $g$ is the

half-line $[0, \infty)$. We will take the resolvent of $h^2\Delta_g$ to be the $L^2$-bounded operator

$$R_h(\lambda) := (h^2\Delta - \lambda^2)^{-1} \text{ in } \mathrm{Im}(\lambda) < 0.$$

The resolvent admits a continuous extension to $\{\lambda \neq 0, \mathrm{Im}(\lambda) \leq 0\}$ as a bounded operator from $L^2_{\mathrm{comp}}$ to $L^2_{\mathrm{loc}}$ (this is called limiting absorbtion principle), and when $\lambda > 0$ we call $R_h(\lambda)$ the incoming resolvent, while $R_h(-\lambda)$ is the outgoing resolvent

We define $r$ to be a smooth function on $M$, equal to $|m|$ in $\{m \in M \setminus N; |m| > 2R_0\}$ and equal to 1 in $N$, where we used the Euclidean coordinate in $M \setminus N$ induced by the isometry between $M \setminus N$ and $\mathbb{R}^d \setminus B(0, R_0)$. Following for instance Melrose [Me], the generalized plane wave is defined for $\xi \in \mathbb{S}^{d-1}$ by

$$E_h(\lambda; \xi, m) := 2i\lambda h \left(\frac{2\pi h}{i\lambda}\right)^{\frac{d-1}{2}} \lim_{r \to \infty} [r^{\frac{d-1}{2}} e^{\frac{i\lambda r}{h}} R_h(\lambda; m, r\xi)], \qquad (2.1)$$

where $R_h(\lambda; m, m')$ is the Schwartz kernel of $R_h(\lambda)$ and $r\xi \in N$. As a function of $(m, \xi) \in M \times \mathbb{S}^{d-1}$, it is smooth and solves $(h^2\Delta - \lambda^2)E_h(\lambda, \xi) = 0$ in $M$ where $E_h(\lambda, \xi) := E_h(\lambda; \xi, \cdot)$. In the case of $\mathbb{R}^d$ it is given by the usual plane wave $e^{i\frac{\lambda}{h}\xi.m}$. An alternative expression for $E_h$ in $M$ can be found as follows: set $E_h^0(\lambda; \xi, m) := e^{i\frac{\lambda}{h}\xi.m}$, let $(1 - \chi_0) \in C_0^\infty(M)$ with $\chi_0 = 1$ on $r \geq 1/\varepsilon$ and $\mathrm{supp}(\nabla\chi_0) \subset \{r \in [\frac{1}{2\varepsilon}, \frac{1}{\varepsilon}]\}$ for some small $\varepsilon > 0$; then we claim that

$$E_h(\lambda; \xi, m) = \chi_0(m)E_h^0(\lambda; \xi, m) + E_h^1(\lambda; \xi), \qquad (2.2)$$

where

$$E_h^1(\lambda; \xi) := -R_h(\lambda)F_h(\lambda; \xi), \ \ F_h(\lambda, \xi) = (h^2\Delta - \lambda^2)(\chi_0 E_h^0(\lambda, \xi)) = [h^2\Delta, \chi_0]E_h^0(\lambda, \xi). \tag{2.3}$$

Note that we can apply $R_h(\lambda)$ to $F_h(\lambda, \xi) \in C_0^\infty(M)$; in fact, $\mathrm{supp}\, F_h \subset \{\varepsilon < \frac{1}{r} < 2\varepsilon\}$. The proof of (2.2) is easy.

**Relation between plane wave and spectral projectors.** One has $\overline{E_h(\lambda; m, \xi)} = E_h(-\lambda; m, \xi)$ since $R_h(\lambda)^* = R(-\lambda)$ for $\lambda \in \mathbb{R}$, and the decomposition of the spectral measure in terms of these functions is given as follows: by Stone formula, the spectral measure of $h^2\Delta$ is given by

$$d\Pi_h(\lambda) = \frac{i\lambda}{\pi}(R_h(\lambda) - R_h(-\lambda)) \, d\lambda \quad \text{ for } \lambda \in (0, \infty) \qquad (2.4)$$

in the sense that $F(h^2\Delta) = \int_0^\infty F(\lambda^2)d\Pi_h(\lambda)$ for any bounded function $F$. Using Green's formula in balls or radius going to $\infty$, one can show that

$$d\Pi_h(\lambda; m, m') = \lambda^n(2\pi h)^{-d}\int_{\mathbb{S}^{d-1}} E_h(\lambda; m, \xi)\overline{E_h(\lambda, m', \xi)} \, d\xi d\lambda \qquad (2.5)$$

where $d\xi$ corresponds to the standard volume form on the sphere.

**Estimates on the local norms of** $E_h$. As we see from their construction, the norm $||E_h||_{L^2(K_0)}$ on a compact set $K_0 \subset M$ is related to the norm of the truncated incoming resolvent $\chi_1 R_h(\lambda)\chi_2$ where $\chi_j \in C_0^\infty(M)$. In general, Burq [Bu] proved that

$$||\chi_1 R_h(\lambda)\chi_2||_{L^2 \to L^2} = \mathcal{O}(e^{C/h}), \quad 0 < h < h_0$$

for some $C > 0$, and when the metric is non trapping one obtains $\mathcal{O}(h^{-1})$ instead. When the trapped set is non-empty, it is in general difficult to obtain polynomial bounds in $h$.

Let $\chi \in C_0^\infty(M)$. Then we claim that $\chi\Pi_{[1,1+h]}$ is a Hilbert–Schmidt operator and there exists a global constant $C$ such that for each bounded operator $A : L^2(M) \to L^2(M)$, we have ($|| \cdot ||_{\mathrm{HS}}$ denotes Hilbert-Schmidt norm)

$$h^{-1}\|A\chi E_h(\lambda,\xi)\|^2_{L^2_{m,\xi,\lambda}(M \times S^{d-1} \times [1,1+h])} \leq Ch^{d-1}\|A\chi\Pi_{[1,1+h]}\|^2_{\mathrm{HS}}. \tag{2.6}$$

From this, we can prove that if $A \in \Psi^0(M)$ is a compactly supported semi-classical pseudo-differential operator, the functions $AE_h$ are bounded in $L^2$ on average in the following sense: there exists a constant $C$ independent of $A$ such that for any $h > 0$

$$h^{-1}\|AE_h(\lambda,\xi)\|^2_{L^2_{m,\xi,\lambda}(M \times S^{d-1} \times [1,1+h])} \leq C\|\sigma(A)\|^2_{L^2(S^*M)} + \mathcal{O}(h). \tag{2.7}$$

The $L^2$ norm of $\sigma(A)$ on the energy surface $S^*M$ is with respect to the Liouville measure $\mu_L$. The bound (2.6) follows rather directly from the expression (2.5) of the spectral measure in terms of $E_h(\lambda,\xi)$ and the identity $\mathrm{Tr}(BB^*) = ||B||^2_{\mathrm{HS}}$ for $B$ Hilbert-Schmidt. The bound (2.7) is a bit more involved. We won't give details (see Lemma 3.11 in [DyGu] for instance), but essentially the argument is to prove that for $\varphi \in C_0^\infty(\mathbb{R})$ equal to 1 near 0, the operator $\varphi((h^2\Delta_g - 1)/h)$ is a semi-classical Fourier Integral Operator, then we bound $||A\Pi_{[1,1+h]}A^*||_{\mathrm{HS}}$ by $C||\chi\varphi((h^2\Delta_g - 1)/h)\chi||_{\mathrm{HS}}$, and this amounts to estimate the $L^2(M \times M)$ norm of the Schwartz kernel: from its integral representation as a Lagrangian distribution, this reduces to $h^{\frac{1-d}{2}}$ times the $L^2$ norm of $\sigma(A)$ on the energy surface $S^*M$ up to lower order terms in $h$.

**Wave-front sets.** We will only work with semi-classical pseudo-differential operators with compactly supported Schwartz kernel in $M \times M$. For us, a semi-classical symbol of order $k$ will be a compactly supported in $m$ function $a(m,\nu;h)$ on $T^*M$ such that

$$\forall \alpha, \beta, \quad |\partial_m^\alpha \partial_\nu^\beta a(m,\nu;h)| \leq C_{\alpha,\beta}\langle\nu\rangle^{k-|\beta|}.$$

The microsupport or wave-front set $\mathrm{WF}_h(A)$ of a semi-classical pseudo-differential operator can be defined as follows. We denote by $\overline{T}^*M$ the radial compactification in the fibers of $T^*M$, this amounts to glue a sphere bundle isomorphic to $S^*M$ at fiber infinity $|\nu| = \infty$; open neighbourhoods of a point $(m,\mu)$ at fiber infinity are simply given by conic neighbourhoods with angle near $\mu \in S^{d-1}$. A point $(m_0,\nu_0) \in \overline{T}^*M$

does not belong to $\mathrm{WF}_h(A)$ if there exists a neighbourhood $V$ of $(m_0, \nu_0)$ such that $A$ can be written under the form

$$A = \mathrm{Op}_h(a) + A', \quad |\partial_m^\alpha \partial_\nu^\beta a(m, \nu)| = \mathcal{O}(h^\infty \langle \nu \rangle^{-\infty}) \text{ in } V, \quad K_{A'} \in h^\infty C_0^\infty(M \times M)$$

for all $\alpha, \beta$ and where $K_{A'}$ is the Schwartz kernel of $A'$. Let $u_h \in H_{\mathrm{comp}}^{-L}(M)$ for some $L \geq 0$ such that for any compact set $K_0$, $||u_h||_{H^{-L}(K_0)} = \mathcal{O}(h^{-L'})$ for some $L'$ (we say that $u_h$ is $h$-tempered in that case). We define the wave-front set $\mathrm{WF}_h(u_h)$ to be the complement of the set of points $(m, \nu) \in T^*M$ such that there is a neighbourhood $V$ of $(m, \nu)$ for which $||Au_h||_{L^2} = \mathcal{O}(h^\infty)$ for all semi-classical pseudo-differential operators $A$ microsupported in $V$.

The wave front set of $\chi_0 E_h^0(\lambda; \xi)$ is straightforward to obtain by simple stationary phase, it is given by a Lagrangian

$$\mathrm{WF}_h(\chi_0 E_h^0(\lambda; \xi)) = \{(m, \lambda\xi) \in T^*M; m \in \mathrm{supp}(\chi_0)\}$$

and $\chi E_h^0(\lambda; \xi)$ is called a semi-classical Lagrangian distribution. The wave front set of $E_h^1(\lambda; \xi)$ is more difficult to obtain, but we can still use some propagation of singularities properties of the incoming resolvent $R_h(\lambda)$. We can show the following statement. Define

$$W_\xi := \{(m, \xi) \in T^*M; m \in \mathrm{supp}(\nabla \chi_0)\}, \quad \widetilde{E}_h^1(\lambda; \xi) := \frac{E_h^1(\lambda; \xi)}{1 + ||E_h(\lambda, \xi)||_{L^2(r < \varepsilon^{-1})}}.$$

First $\widetilde{E}_h^1(\lambda, \xi)$ is $h$-tempered in the sense that for any compact set $K_0 \subset M$, there exists $C_{K_0}$ such that $||\widetilde{E}_h^1(\lambda, \xi)||_{L^2(K_0)} \leq C_{K_0}$. Secondly,

$$(m, \lambda\nu) \in \mathrm{WF}_h(\widetilde{E}_h^1(\lambda, \xi)) \implies \begin{cases} (m, \nu) \in S^*M \text{ and} \\ g^t(m, \nu)_{t \to +\infty} \nrightarrow \infty \text{ or } \exists t \geq 0, g^t(m, \nu) \in W_\xi. \end{cases} \tag{2.8}$$

To show this, we can use the free resolvent $R_h^0(\lambda)$ on $\mathbb{R}^d$: if $\chi_1 \in C^\infty(M)$ equal 1 in $r > \varepsilon^{-1}$, supported inside $M \setminus N$, we write ($F_h$ is defined in (2.3))

$$\chi_1 E_h^1(\lambda; \xi) = -R_h^0(\lambda) F_h^0(\lambda; \xi), \quad F_h^0(\lambda, \xi) := F_h(\lambda; \xi) - [h^2\Delta, \chi_1] E_h^1(\lambda; \xi) \tag{2.9}$$

which holds since $E_h^1(\lambda, \xi)$ is incoming at infinity. For any $\chi \in C_0^\infty(M \setminus N)$, one has $||\chi R_h^0(\lambda)\chi|| = \mathcal{O}(h^{-1})$ as a map from the semi-classical Sobolev space $H_h^{-1}(M)$ to $L^2(M)$, and $||F_h^0(\lambda, \xi)||_{H_h^{-1}(M)} = \mathcal{O}(h(1 + ||E_h||_{L^2(r < \varepsilon^{-1})}))$, thus $\widetilde{E}_h^1(\lambda, \xi)$ is $h$-tempered. The $(m, \nu) \in S^*M$ statement in (2.8) comes from ellipticity. The second statement is an application of the classical propagation of singularities and the following property for the incoming free resolvent $R_h^0(\lambda)$: if $f$ is a compactly supported $h$-tempered family of distributions, then if $(m', \nu') \in \mathrm{WF}(R_h^0(\lambda)f)$, there exists $t \geq 0$ such that $g^t(m', \nu') \in \mathrm{supp}(f)$. This implies that if $(m, \nu) \in \mathrm{WF}_h(\widetilde{E}_h^1(\lambda, \xi))$ and $(\cup_{t \geq 0} g^t(m, \nu)) \cap W_\xi = \emptyset$, then $g^t(m, \nu)$ is trapped in forward time.

**End of the proof.** Since $u(m) = E_h(\lambda; m, \xi)$ solves $(h^2\Delta_g - \lambda^2)u = 0$, then by using the fact that the propagator $U(t) = e^{ith\Delta/2}$ is a Fourier Integral Operator associated to geodesic flow, we obtain that

$$\chi E_h(\lambda; \xi) = \chi e^{-itz/(2h)}U(t)\chi_t E_h(\lambda; \xi) + \mathcal{O}(h^\infty \|E_h(\lambda; \xi)\|_{L^2(K_t)})$$

where $\chi \in C_0^\infty(M)$ and for $t \in \mathbb{R}$, $\chi^t \in C_0^\infty(M)$ is supported in the interior of a compact set $K_t \subset M$ and satisfies ($d_g$ denotes Riemannian distance on $M$)

$$d_g(\text{supp}\,\chi, \text{supp}(1 - \chi_t)) > |t|. \tag{2.10}$$

Therefore, for $A \in \Psi^0(M)$ compactly supported and $\chi \in C_0^\infty(M)$ such that $\chi A\chi = A$

$$\langle AE_h(\lambda; \xi), E_h(\lambda; \xi)\rangle = \langle U(-t)AU(t)\chi^t E_h(\lambda; \xi), \chi^t E_h(\lambda; \xi)\rangle + \mathcal{O}(h^\infty\|E_h(\lambda; \xi)\|_{L^2(K_t)}^2).$$

By Egorov's theorem, for each $t$ (independent of $h$), there exists a compactly supported operator $A^t \in \Psi^0(M)$ such that

$$U(t)AU(-t) = A^t + \mathcal{O}(h^\infty)_{L^2 \to L^2}. \tag{2.11}$$

Moreover, $\text{WF}_h(A^t) \subset g^{-t}(\text{WF}_h(A))$ and $\sigma(A^t) = \sigma(A) \circ g^t + \mathcal{O}(h)$. We get

$$\langle AE_h(\lambda; \xi), E_h(\lambda; \xi)\rangle = \langle A^{-t}\chi^t E_h(\lambda; \xi), \chi^t E_h(\lambda; \xi)\rangle + \mathcal{O}(h^\infty\|E_h(\lambda; \xi)\|_{L^2(K_t)}^2).$$

For some $r_0$ large, let $\varphi \in C_0^\infty(\{r < 2r_0\})$ be equal to 1 in a large region $\{r \leq r_0\}$ containing $N$ and $\text{supp}(\nabla\chi_0)$, and we split

$$\chi^t A^{-t}\chi^t = A_0 + A_1, \quad A_0 := \varphi\chi^t A^{-t}\chi^t, \quad A_1 := (1 - \varphi)\chi^t A^{-t}\chi^t.$$

This yields

$$\begin{aligned}\langle AE_h(\lambda; \xi), E_h(\lambda; \xi)\rangle =&\langle A_1\chi_0 E_h^0(\lambda; \xi), \chi_0 E_h^0(\lambda; \xi)\rangle + \langle A_0 E_h(\lambda; \xi), E_h(\lambda; \xi)\rangle \\ &+ \langle A_1\chi_0 E_h^0(\lambda; \xi), E_h^1(\lambda; \xi)\rangle + \langle A_1 E_h^1(\lambda; \xi), \chi_0 E_h^0(\lambda; \xi)\rangle \\ &+ O(h^\infty\|E_h(\lambda; \xi)\|_{L^2(K_t)}^2).\end{aligned} \tag{2.12}$$

Taking first $h \to 0$ and then $t \to +\infty$, the first term in the right hand side will give the limiting measure by using the fact that $E_h^0$ is a Lagrangian distribution; the second term will be estimated using (2.7) after averaging in $(\lambda, \xi)$, and we will show using propagation of singularities that the last 3 terms will not contribute to the limit $h \to 0$ as they are $\mathcal{O}(h^\infty\|E_h(\lambda; \xi)\|_{L^2(K_t)}^2)$ (thus by averaging in $(\lambda, \xi)$ they become an $\mathcal{O}(h^\infty)$ by (2.7)). Indeed, we have since $A_0$ has compact support in some $\{r \leq 2r_0\}$,

$$|\langle A_0 E_h(\lambda; \xi), E_h(\lambda; \xi)\rangle| \leq \|A_0 E_h(\lambda; \xi)\|_{L^2(M)}\|E_h(\lambda; \xi)\|_{L^2(\{r \leq 2r_0\})}$$

and integrating in $\lambda \in [1, 1 + h], \xi \in S^{d-1}$, we get for $h$ small by (2.7)

$$\frac{1}{h}\int_1^{1+h}\int_{S^{d-1}} |\langle A_0 E_h(\lambda; \xi), E_h(\lambda; \xi)\rangle|^2 d\xi d\lambda \leq C\|\sigma(A_0)\|_{L^2(S^*M)}^2 + \mathcal{O}(h).$$

The principal symbol of $A_0$ is $\varphi \chi^t \sigma(A) \circ g^{-t}$, and by the assumption that the trapped set has Liouville measure 0, we get

$$\lim_{t \to +\infty} \lim_{h \to 0} \frac{1}{h} \int_1^{1+h} \int_{S^{d-1}} |\langle A_0 E_h(\lambda; \xi), E_h(\lambda; \xi) \rangle|^2 d\xi d\lambda = 0. \tag{2.13}$$

When integrated in $\xi, \lambda$, the last term in (2.12) is an $\mathcal{O}(h^\infty)$, uniformly for $t$ in a compact set (independent of $h$), by using (2.7). To deal with the terms in the second line of (2.12), we use wavefront sets: since $||E_h^0(\lambda; \xi)||_{L^2(K_t)} \le C_t$ for some constant $C_t$ depending only on $t$, we write

$$|\langle \chi_0 E_h^0(\lambda; \xi), A_1 E_h^1(\lambda; \xi) \rangle| \le C_t ||A_1 E_h^1(\lambda; \xi)||_{L^2(M)}$$

and we will show that $||A_1 E_h^1(\lambda; \xi)||_{L^2(M)} = \mathcal{O}(h^\infty ||E_h(\lambda, \xi)||_{L^2(K_t)})$. Since $\mathrm{WF}_h(A_1) \subset \{(m, \nu) \in T^*M; \varphi(m) \ne 1, g^{-t}(m, \nu) \in \mathrm{supp}(\sigma(A))\}$ and $\varphi(m) = 1$ on a large region containing $\mathrm{supp}(\nabla \chi_0) \cup N$, we see that for a point $(m, \nu)$ in the wave-front set of $A_1$, the geodesic $g^t(m, \nu)$ for $t \ge 0$ escape to infinity without entering $W_\xi$: indeed the regions $S^*M \cap \{r < r_0\}$ are geodesically convex and since $\mathrm{supp}(\sigma(A)) \cup W_\xi \subset \{r < r_0\}$, the condition $\sigma(A)(g^{-t}(m, \nu)) \ne 0$ with $\varphi(m) \ne 1$ implies that $g^t(m, \nu) \subset \{r \ge r_0\}$ for all $t \ge 0$. Consequently, by (2.8), a point $(m, \nu) \in \mathrm{WF}_h(A_1)$ cannot be in $\mathrm{WF}_h(\widetilde{E}_h^1(\lambda; \xi))$ and we deduce that

$$|\langle \chi_0 E_h^0(\lambda; \xi), A_1 E_h^1(\lambda; \xi) \rangle| = \mathcal{O}(h^\infty ||E_h(\lambda, \xi)||_{L^2(r < \varepsilon^{-1})})$$

Integrating in $\xi \in S^{d-1}$ and in $\lambda \in [1, 1+h]$ this becomes an $\mathcal{O}(h^\infty)$ by (2.7). The same argument works for $\langle A_1 \chi_0 E_h^0(\lambda; \xi), E_h^1(\lambda; \xi) \rangle$ by writing it as $\langle \chi_0 E_h^0(\lambda; \xi), A_1^* E_h^1(\lambda; \xi) \rangle$ and using that the wave-front set of $A_1^*$ is that of $A_1$.

As a conclusion, we have obtained

$$\lim_{t \to 0} \lim_{h \to 0} \frac{1}{h} \int_1^{1+h} \int_{S^{d-1}} |\langle A E_h(\lambda; \xi), E_h(\lambda; \xi) \rangle - \langle A_1 \chi_0 E_h^0(\lambda; \xi), \chi_0 E_h^0(\lambda; \xi) \rangle| d\xi d\lambda = 0.$$

Since now $E_h^0(\lambda, \xi)$ is an explicit Lagrangian semiclassical distribution (just a plane wave), we immediatly get by stationary phase

$$\langle A_1 \chi_0 E_h^0(\lambda; \xi), \chi_0 E_h^0(\lambda; \xi) \rangle = \int_M \chi_0^2 (1 - \varphi)(m) \sigma(A)(g^{-t}(m, \xi)) dm + \mathcal{O}(h)$$

where the remainder is uniform for $t$ in a compact interval independent of $h$. One easily sees that

$$a \in C^0(S^*M) \mapsto \lim_{t \to +\infty} \int_M \chi_0^2(m)(1 - \varphi(m)) a(g^{-t}(m, \xi)) dm$$

$$= \lim_{t \to +\infty} \int_{M \setminus N} a(g^{-t}(m, \xi)) dm$$

defines a Radon measure on $S^*M$, supported on the closure of the set of points $(m, \nu)$ for which the forward geodesics $g^t(m, \nu)$ is of the form $(m_0 + (t - t_0)\xi, \xi)$ for all $t \ge t_0$

where $m_0$ is a point in $M \setminus N$ and $t_0 \geq 0$. This achieves the proof of Theorem 1. The fact that the Louville measure disintegrates into $\mu_L = \int_{S^{d-1}} \mu_\xi d\xi$ is quite straightforward using the assumption that the trapped set has Liouville measure 0.

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