IDENTIFICATION OF A CONNECTION FROM CAUCHY DATA ON A RIEMANN SURFACE WITH BOUNDARY

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ABSTRACT. We consider a connection ∇^X on a complex line bundle over a Riemann surface with boundary M_0, with connection 1-form X. We show that the Cauchy data space of the connection Laplacian (also called magnetic Laplacian) $L := \nabla^{X*}\nabla^X + q$, with q a complex valued potential, uniquely determines the connection up to gauge isomorphism, and the potential q.

1. INTRODUCTION

Let M_0 be a smooth Riemann surface with boundary, equipped with a metric g. A complex line bundle E on M_0 has a trivialization $E \simeq M_0 \times \mathbb{C}$, thus there is a non-vanishing smooth section $s : M_0 \to E$, and a connection ∇ on E induces a complex valued 1-form iX on M_0 (where $i = \sqrt{-1} \in \mathbb{C}$) defined by $\nabla s = s \otimes iX$, which means that $\nabla(fs) = s \otimes (d + iX)f$ if d is the exterior derivative. The associated connection Laplacian ($*$ is the Hodge operator with respect to g) is the operator

$$\Delta^X := \nabla^{X*}\nabla^X = -*(d*+iX \wedge *)(d + iX)$$

acting on complex valued functions (sections of E). When X is real valued, this operator is often called the *magnetic Laplacian* associated to the magnetic field dX, and the connection 1-form X can be seen as to a connection 1-form on the principal bundle $M_0 \times S^1$ by identifying $i\mathbb{R} \subset \mathbb{C}$ with the Lie algebra of S^1. This also corresponds to a Hermitian connection, in the sense that it preserves the natural Hermitian product on E. Let q be a complex valued function on M_0 and assume that the 1-form X is real valued, and consider the *magnetic Schrödinger Laplacian* associated to the couple (X, q)

$$(1) \qquad L := \nabla^{X*}\nabla^X + q = -*(d*+iX \wedge *)(d + iX) + q.$$

If $H^s(M_0)$ denotes the Sobolev space with s derivatives in L^2, we define the Cauchy data space of L to be

$$(2) \qquad \mathcal{C}_L := \{(u, \nabla_\nu^X u)|_{\partial M_0} \in H^{\frac{1}{2}}(M_0) \times H^{-\frac{1}{2}}(M_0); u \in H^1(M_0), Lu = 0\}$$

where ν is the outward pointing unit normal vector field to ∂M_0 and $\nabla_\nu^X u := (\nabla^X u)(\nu)$. The first natural inverse problem is to see if the Cauchy data space determines the connection form X and the potential q uniquely, and one easily sees that it is not the case since there are gauge invariances in the problem: for instance, conjugating L by e^f with $f = 0$ on ∂M_0, one obtains the same Cauchy data space but with a Laplacian associated to the connection ∇^{X+df}, therefore it is not possible to identify X but rather one should expect to recover its relative cohomology class.

In general in inverse problems for magnetic Laplacians, it is shown that if two couples (X_1, q_1) and (X_2, q_2) are such that the associated connection Laplacian L_1 and L_2 have same

Cauchy data space, then $d(X_1 - X_2) = 0$ and $q_1 = q_2$. In geometric terms, this means here that the curvature of the connections ∇^{X_1} and ∇^{X_2} agree. If the domain is simply connected, X_1 would then differ from X_2 by an exact form, and moreover $i^*_{\partial M_0}(X_1 - X_2) = 0$ by boundary determination . For these types of results in Euclidean domains of dimensions three and higher, we refer the readers to the works of Henkin-Novikov [11], Sun [21, 22], Nakamura-Sun-Uhlmann in [19], Kang-Uhlmann in [15], and for partial data Dos Santos Ferreira-Kenig-Sjöstrand-Uhlmann in [5]. For simply connected planar domains, Imanuvilov-Yamamoto-Uhlmann in [14] deal with the case of general second order elliptic operators for partial data measurement, and Lai [17] deals with the special case of magnetic Schrödinger operator for full data measurement.

However, on a general Riemann surface with boundary, the first cohomology space is non-trivial in general and $X_1 - X_2$ may not be exact and there is another gauge invariance. Indeed, if X_j are real valued and if there exists a unitary bundle isomorphism $F : E \to E$ (i.e. preserving the Hermitian product) such that $\nabla^{X_1} = F^* \nabla^{X_2} F$ with $F = \mathrm{Id}$ on $E|_{\partial M_0}$, then it is clear that the Cauchy data spaces $\mathcal{C}_{L_1} = \mathcal{C}_{L_2}$ agree. We shall say in this case that the connections are *related by a gauge isomorphism.* Such a bundle isomorphism corresponds to the multiplication by a function F on M_0 satisfying $|F| = 1$ everywhere and $F = 1$ on ∂M_0 and this is equivalent to have $i^*_{\partial M_0}(X_1 - X_2) = 0$ on ∂M_0, $d(X_1 - X_2) = 0$ and that $\int_\gamma (X_1 - X_2) \in 2\pi\mathbb{Z}$ for all closed loop γ in M_0. Another way of stating this isomorphism is the following: let $\gamma_1, \ldots, \gamma_M$ be some non-homotopically equivalent closed loops of M_0, non-homotopically equivalent to any boundary component, and let $\omega_1, \ldots, \omega_M$ be closed 1-forms which form a basis of the first relative cohomology group $H^1(M_0, \partial M_0)$, dual to $\gamma_1, \ldots, \gamma_M$ in the sense $\int_{\gamma_k} \omega_j = \delta_{ij}$, then there is a bundle isomorphism as above if and only if $X_1 = X_2 + 2\pi \sum_{m=1}^M n_m \omega_m + df$ with $n_m \in \mathbb{Z}$ and $f|_{\partial M_0} = 0$.

For $s \in \mathbb{N}, p \in [1, \infty]$, let us denote by $W^{s,p}(M_0)$ the Sobolev space consisting of functions with s derivatives in L^p. The following theorem provides a characterization of precisely when two Cauchy data spaces agree:

Theorem 1.1. *Let $X_1, X_2 \in W^{2,p}(M_0, T^*M_0)$ be real valued 1-forms and let $q_1, q_2 \in W^{1,p}(M_0)$ be complex valued functions, where $p > 2$. Let L_1, L_2 be the magnetic Schrödinger Laplacians defined in (1) for the couples (X_1, q_1) and (X_2, q_2). Then the Cauchy data spaces $\mathcal{C}_{L_1}$ and $\mathcal{C}_{L_2}$ coincide if and only if $q_1 = q_2$, and ∇^{X_1} is related to ∇^{X_2} by a gauge isomorphism.*

As far as we know, this is the first result with such a characterization in the case of elliptic equations at fixed frequency. There are results for time-dependent inverse problems, or equivalently when one knows the Cauchy data spaces at all frequencies, for instance by Kurylev-Lassas [16] and Eskin-Isozaki-O'Dell [6] (see the references therein for results about inverse scattering). The problem of recovering the fluxes $\int_\gamma X$ (modulo $2\pi\mathbb{Z}$) of the magnetic potential along closed loops is related to the so-called Aharonov-Bohm effect [1]. We notice that for the free case $X = 0$, the identification of q (or of an isotropic conductivity) on a Riemann surface with boundary from full data measurement was proved recently in [10, 8], and in [9] from partial data measurement, it was done in a domain of $\mathbb{C}$ in [13].

For a general complex valued connection form X and the associated operator L of (1), we can define its real and imaginary parts of iX by $X_{\mathbb{R}} := \frac{1}{2}(iX + \overline{iX})$ and $X_{i\mathbb{R}} := \frac{1}{2}(iX - \overline{iX})$

and by an easy computation we have

$$\nabla^{X*}\nabla^X = \nabla^{X_{i\mathbb{R}}*}\nabla^{X_{i\mathbb{R}}} + \delta(X_{\mathbb{R}}) + |X_{\mathbb{R}}|_g^2$$

with $\delta := d^* = -*d*$ and therefore, comparing to the case where X was real, it changes the operator L in (1) only through the addition of a potential, namely $\delta(X_{\mathbb{R}}) + |X_{\mathbb{R}}|^2$. Considering the Cauchy data space of L to be

$$\mathcal{C}_L := \{(u, \nabla_\nu^{X_{i\mathbb{R}}} u)|_{\partial M_0} \in H^{\frac{1}{2}}(M_0) \times H^{-\frac{1}{2}}(M_0); u \in H^1(M_0), Lu = 0\}$$

we see from Theorem 1.1 that the Cauchy data space $\mathcal{C}_L$ determines $\delta(X_{\mathbb{R}}) + |X_{\mathbb{R}}|^2 + q$, and the imaginary part of iX up to gauge isomorphism.

To prove the identification of the potential and the curvature of the connection, we shall reduce the problem to a first order system and we shall actually prove an inverse result for the following auxiliary problem: let us define the bundle $\Sigma := \Lambda^0(M_0) \oplus \Lambda^{0,1}(M_0)$ over M_0 where $\Lambda^k(M_0)$ denotes the bundle of complex valued k-forms on M_0 (for $k = 0, 1, 2$) and

$$\Lambda^{0,1}(M_0) = T_{0,1}^* M_0 = \ker_{\Lambda^1}(* - i\mathrm{Id}), \ \Lambda^{1,0}(M_0) = T_{0,1}^* M_0 = \ker_{\Lambda^1}(* + i\mathrm{Id})$$

and $*$ is the Hodge star operator on $\Lambda^k(M_0)$. Let $D := C^\infty(\Sigma) \to C^\infty(\Sigma)$ be the self adjoint operator defined by

$$D := \begin{pmatrix} 0 & \bar{\partial}^* \\ \bar{\partial} & 0 \end{pmatrix}$$

and V be a complex valued diagonal endomorphism, then we set $P := D + V$. The Cauchy data space is defined by

$$\mathcal{C}_V := \{U|_{\partial M_0}; U \in W^{1,p}(M_0, \Sigma), (D + V)U = 0\}.$$

and we prove

Theorem 1.2. *Let $V_1, V_2 \in W^{1,p}(M_0, \mathrm{End}(\Sigma))$ be two diagonal endomorphisms of Σ. Assume that they have the same Cauchy data spaces, ie. $\mathcal{C}_{V_1} = \mathcal{C}_{V_2}$, then $V_1 = V_2$.*

Such a result was recently proved by Bukhgeim [3] in the case of the disk in $\mathbb{C}$, and we use a similar approach to solve this problem in this geometric setting, together with some results on holomorphic Morse functions proved in our previous work [8]. This also provides a reconstruction procedure of a potential V at any given point z_0 of M_0 where there exists a Morse holomorphic function with a critical point at z_0, see Remark 3.4. Constructive methods have also been obtained by [10, 12] for isotropic conductivity on Riemann surfaces. The result in Theorem 1.2 is however not sufficient to identify the curvature connection and potential, and part of our work is to show that the Cauchy data space of L determines the Cauchy data space of $P = D + V$ for a certain V associated to (X, q).

The last part of Theorem 1.1 consist in showing that the integrals of $X_1 - X_2$ along closed loops are in $2\pi\mathbb{Z}$ if $\mathcal{C}_{L_1} = \mathcal{C}_{L_2}$. This is done using parallel transport and unique continuation.

The organisation of the paper is the following: in the first section, we construct some right inverses for the operators $\bar{\partial}$ and $\bar{\partial}^*$ on a manifold with boundary M_0. Then in the next section we prove identification results for first order systems $D + V$ as explained above. The following section is focused on boundary determination in the inverse problem for the operator L, and then we use this to reduce the problem on the magnetic Laplacian to the problem for a first

order system $D + V$. The last section deals with the holonomy identification and we end up with an appendix containing technical results.

2. The Cauchy-Riemann Operator on M

Let M be the interior of a compact Riemann surface with boundary $\overline{M}$. The surface is equipped with a complex structure. The Hodge star operator $*$ acts on the cotangent bundle T^*M, its eigenvalues are $\pm i$ and the respective eigenspace $T_{1,0}^*M := \ker(* + i\mathrm{Id})$ and $T_{0,1}^*M := \ker(* - i\mathrm{Id})$ are sub-bundle of the complexified cotangent bundle $\mathbb{C}T^*M$ and the splitting $\mathbb{C}T^*M = T_{1,0}^*M \oplus T_{0,1}^*M$ holds as complex vector spaces. In local complex coordinate z induced by the complex structure, $T_{1,0}M$ is spanned by dz while $T_{0,1}^*M$ is spanned by $d\bar{z}$. If $\pi_{1,0}, \pi_{0,1}$ are the respective projections from T^*M onto $T_{1,0}^*M$ and $T_{0,1}^*M$, the Cauchy-Riemann operators $\bar{\partial}, \partial$ mapping functions to 1-forms are defined by $\bar{\partial} := \pi_{0,1}d$ and $\partial := \pi_{1,0}d$ so that $d = \partial + \bar{\partial}$. The operators $\bar{\partial}, \partial$ mapping 1-forms to 2-forms are defined by $\bar{\partial} := d\pi_{1,0}$ and $\partial := d\pi_{0,1}$, and again $d = \partial + \bar{\partial}$. Our main goal in this subsection is to construct some right inverses of the Cauchy-Riemann operators.

Proposition 2.1. *There exists an operator $\bar{\partial}^{-1} : C_0^\infty(M, T_{1,0}^*M) \to C^\infty(M)$ which satisfies the following*
(i) $\bar{\partial}\bar{\partial}^{-1}\omega = \omega$ for all $\omega \in C_0^\infty(M, T_{1,0}^*M)$,
(ii) if $\chi_j \in C_0^\infty(M)$ are supported in some complex charts U_i bi-holomorphic to a bounded open set $\Omega \subset \mathbb{C}$ with complex coordinate z, and such that $\chi := \sum_j \chi_j$ equal 1 on M_0, then as operators

$$\bar{\partial}^{-1}\chi = \sum_j \hat{\chi}_j \bar{T}\chi_j + K$$

where $\hat{\chi}_j \in C_0^\infty(U_j)$ are such that $\hat{\chi}_j\chi_j = \chi_j$, K has a smooth kernel on $M \times M$ and $\bar{T}$ is given in the complex coordinate $z \in U_i \simeq \Omega$ by

$$\bar{T}(fd\bar{z}) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(z')}{z - z'} dz_1' dz_2'$$

where $dv_g(z) = \alpha^2(z)dz_1dz_2$ is the volume form of g in the chart.
(iii) $\bar{\partial}^{-1}$ is bounded from $L^p(T_{0,1}^*M)$ to $W^{1,p}(M)$ for any $p \in (1, \infty)$.

Proof. The existence of a right inverse $\bar{\partial}^{-1}$ is proven in [18, Th. C1.10] : by taking a totally real subbundle $F \subset \partial\overline{M} \times \mathbb{C}$ over the boundary $\partial\overline{M}$ with a large boundary Maslov index (see [9, Cor 2.2.]) for an explicit F having large Maslov index, the operator $\bar{\partial} = W_F^{1,2}(M) \to L^2(T_{0,1}^*M)$ is Fredholm if $W_F^{1,2}(M)$ denotes the space of L^2 functions with one derivative in L^2 and boundary value in the bundle F, and moreover $\bar{\partial}$ is surjective if the Maslov index is chosen larger than $-2\chi(M)$ where $\chi(M)$ is the Euler characteristic of M. Moreover, $\bar{\partial}^{-1}$ maps $W^{k,2}(M)$ to $W^{k+1,2}(M, T_{0,1}^*M)$ for all $k \in \mathbb{N}$ by elliptic regularity. Observe that $\bar{\partial}^{-1}\bar{\partial} - 1$ maps $W_F^{1,2}(M)$ into $\ker\bar{\partial} \cap W_F^{1,2}(M)$ which is a finite dimensional space spanned by some smooth functions $\psi_1, \dots \psi_n$ (by elliptic regularity) on M. Assuming that $(\psi_j)_j$ is an orthonormal

basis in L^2, this implies that, on $W_F^{1,2}(M)$

$$\bar{\partial}^{-1}\bar{\partial} = 1 - \Pi \ \text{where} \ \Pi = \sum_{k=1}^{n} \psi_k \langle \cdot, \psi_k \rangle_{L^2(M)}.$$

Now we also have

$$\bar{\partial} \sum_j \hat{\chi}_j \bar{T} \chi_j = \chi + \sum_j [\bar{\partial}, \chi_j] \bar{T} \chi_j$$

and the last operator on the right has a smooth kernel in view of $\chi_j \nabla \hat{\chi}_j = 0$ and the fact that T has a smooth kernel outside the diagonal $z = z'$. Now since $\hat{\chi}_j \in C_0^\infty(M) \subset W_F^{1,2}(M)$, we can multiply by $\bar{\partial}^{-1}$ on the left of the last identity and obtain

$$\bar{\partial}^{-1}\chi = \sum_j \hat{\chi}_j \bar{T} \chi_j - \Pi \sum_j \hat{\chi}_j \bar{T} \chi_j - \partial^{-1} \sum_j [\bar{\partial}, \chi_j] \bar{T} \chi_j.$$

The last two operator on the right have a smooth kernel on $M \times M$, in view of the smoothness of ψ_k and the kernel of $\sum_j [\bar{\partial}, \chi_j] \bar{T} \chi_j$, and since ∂^{-1} maps $C_0^\infty(M)$ to $C^\infty(M, T_{0,1}^*M)$. $\quad\square$

Lemma 2.1. *Let $\bar{\partial}^* = -i * \partial : W^{1,p}(T_{0,1}^*M) \to L^p(M)$, then there exists an operator $\bar{\partial}^{*-1}$ mapping $C_0^\infty(M)$ to $C^\infty(T_{0,1}^*M)$ which satisfies the following:*
(i) $\bar{\partial}^* \bar{\partial}^{*-1}\omega = \omega$ for all $\omega \in C_0^\infty(M)$,
(ii) if $\chi_j \in C_0^\infty(M)$ are supported in some complex charts U_i bi-holomorphic to a bounded open set $\Omega \subset \mathbb{C}$ with complex coordinate z, and such that $\chi := \sum_j \chi_j$ equal 1 on M_0, then as operators

$$\bar{\partial}^{*-1}\chi = \sum_j \hat{\chi}_j T \chi_j + K$$

where $\hat{\chi}_j \in C_0^\infty(U_j)$ are such that $\hat{\chi}_j \chi_j = \chi_j$, K has a smooth kernel on $M \times M$ and T is given in the complex coordinate $z \in U_i \simeq \Omega$ by

$$Tf(z) = \left(\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(z')}{\bar{z} - \bar{z}'} dv_g(z') \right) d\bar{z}$$

where $dv_g(z) = \alpha^2(z) dz_1 dz_2$ is the volume form of g in the chart.
(iii) $\bar{\partial}^{*-1}$ is bounded from $L^p(M)$ to $W^{1,p}(T_{0,1}^*M)$ for any $p \in (1, \infty)$.

Proof. Let G be the Green's kernel with Dirichlet condition on $\overline{M}$. Then one has $2\bar{\partial}^* \bar{\partial} G = 1$ and G maps $L^p(M)$ to $W^{2,p}(M)$ by elliptic regularity. Thus we shall set $\bar{\partial}^{*-1} := 2\bar{\partial} G$ which maps $L^p(M)$ to $W^{1,p}(T_{0,1}^*M)$. This proves (i) and (iii). In local complex coordinate z in each U_i, the metric has the local form $g = \alpha^2(z)|dz|^2$ for some positive function $\alpha(z)$, thus $\Delta_g = \alpha^{-2}(z)\Delta_z$ where $\Delta_z = -4\partial_z \partial_{\bar{z}}$ is the Euclidean Laplacian. Therefore

$$\Delta_g \sum_j \hat{\chi}_j G_0 \alpha^2 \chi_j = \chi + \sum_j [\Delta, \hat{\chi}_j] G_0 \alpha^2 \chi_j$$

if $G_0(z, z') := -(2\pi)^{-1} \log|z - z'|$ is the Green's function on $\mathbb{R}^2$. Since $\hat{\chi}_j \in C_0^\infty(M)$, we can multpliply this identity on the right by $(\bar{\partial}^*)^{-1} = 2\bar{\partial} G$ and we deduce

$$(\bar{\partial}^*)^{-1}\chi = 2\sum_j \hat{\chi}_j \partial_{\bar{z}} G_0 \alpha^2 \chi_j + 2\sum_j [\bar{\partial}, \hat{\chi}_j] G_0 \alpha^2 \chi_j - 2\sum_j \bar{\partial} G [\Delta, \hat{\chi}_j] G_0 \alpha^2 \chi_j.$$

Since $\chi_j \nabla \hat{\chi}_j = 0$ and G_0 is smooth outside the diagonal $z = z'$, we deduce that the last two terms have smooth kernel in view of the fact that $\bar{\partial}G$ preserves C^∞. Now the operator $2\partial_{\bar{z}} G_0 \alpha^2$ is equal to T and we have proved (ii). □

Let M_0 be a surface with boundary included strictly in M (for instance a deformation rectract of M) and for $q, p \in [1, \infty]$ let $\mathcal{E}$ be a linear extension operator from M_0 to M which maps continuously $W^{k,p}(M_0, T^*_{0,1}M_0)$ to the set $W^{k,p}_c(M, T^*_{0,1}M)$ of compactly supported functions in $W_{k,p}(M, T^*_{0,1}M)$, for $k = 0, 1$, with a range made of functions with support inside the region $M_\delta := \{m \in M; d(m, M_0) \leq \delta\}$ for some small $\delta > 0$. Finally, let $\mathcal{R} : L^q(M) \to L^q(M_0)$ be the restriction map from M to M_0.

Lemma 2.2. *Let ψ be a real valued smooth Morse function on M and let $\bar{\partial}^{-1}_\psi := \mathcal{R}\bar{\partial}^{-1}e^{-2i\psi/h}\mathcal{E}$ where $\bar{\partial}^{-1}$ is the right inverse of $\bar{\partial} : W^{1,p}(M) \to L^p(T^*_{0,1}M)$ constucted in Proposition 2.1. Let $q \in (1, \infty)$ and $p > 2$, then there exists $C > 0$ independent of h such that for all $\omega \in W^{1,p}(M_0, T^*_{0,1}M_0)$*

(3)
$$||\bar{\partial}^{-1}_\psi \omega||_{L^q(M_0)} \leq Ch^{2/3}||\omega||_{W^{1,p}(M_0, T^*_{0,1}M_0)} \text{ if } 1 \leq q < 2$$

(4)
$$||\bar{\partial}^{-1}_\psi \omega||_{L^q(M_0)} \leq Ch^{1/q}||\omega||_{W^{1,p}(M_0, T^*_{0,1}M_0)} \text{ if } 2 \leq q \leq p.$$

There exists $\epsilon > 0$ and $C > 0$ such that for all $\omega \in W^{1,p}_c(M_0, T^*_{0,1}M_0)$

(5)
$$||\bar{\partial}^{-1}_\psi \omega||_{L^2(M_0)} \leq Ch^{\frac{1}{2}+\epsilon}||\omega||_{W^{1,p}(M_0, T^*_{0,1}M_0)}.$$

Proof. Observe that the estimate (5) is a direct corollary of (4) and (3) by using interpolation. We recall the Sobolev embedding $W^{1,p}(M) \subset C^\alpha(M)$ for $\alpha \leq 1 - 2/p$ if $p > 2$, and we shall denote by T the Cauchy-Riemann inverse of $\partial_{\bar{z}}$ in $\mathbb{C}$:
$$Tf(z) := \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\xi)}{z - \xi} d\xi_1 d\xi_2$$

where $\xi = \xi_1 + i\xi_2$. If $\Omega, \Omega' \subset \mathbb{C}$ are bounded open sets, then the operator $\mathbb{1}_{\Omega'} T$ maps $L^p(\Omega)$ to $L^p(\Omega')$. Clearly, since $\mathcal{E}, \mathcal{R}$ are continuous operators, it suffices to prove the estimates for compactly supported forms $\omega \in W^{1,p}(T^*_{0,1}M)$ on M. Thus by partition of unity, it suffices to assume that ω is compactly supported in a chart biholomorphic to a bounded domain $\Omega \in \mathbb{C}$, and since the estimates will be localized, we can assume with no loss of generality that ψ has only one critical point, say $z_0 \in \Omega$ (in the chart). The expression of $\bar{\partial}^{-1}_\psi(f d\bar{z})$ in complex local coordinates in the chart Ω satisfies
$$\bar{\partial}^{-1}_\psi(f(z)d\bar{z}) = \chi(z)T(e^{-2i\psi/h}f) + K(e^{-2i\psi/h}f d\bar{z})$$

where K is an operator with smooth kernel and $\chi \in C^\infty_0(\mathbb{C})$.

Let us first prove (3). Let $\varphi \in C^\infty_0(\mathbb{C})$ be a function which is equal to 1 for $|z - z_0| > 2\delta$ and to 0 in $|z - z_0| \leq \delta$, where $\delta > 0$ is a parameter that will be chosen later (it will depend on h). Using Minkowski inequality, one can write when $q < 2$

(6)
$$||\chi T((1 - \varphi)e^{-2i\psi/h}f)||_{L^q(\mathbb{C})} \leq \int_\Omega \left|\left|\frac{\chi(\cdot)}{|\cdot - \xi|}\right|\right|_{L^q(\mathbb{C})} |(1 - \varphi(\xi))f(\xi)| d\xi_1 d\xi_2$$
$$\leq C||f||_{L^\infty} \int_\Omega |(1 - \varphi(\xi))| d\xi_1 d\xi_2 \leq C\delta^2 ||f||_{L^\infty}.$$

On the support of φ, we observe that since $\varphi = 0$ near z_0, we can use

$$T(e^{-2i\psi/h}\varphi f) = \frac{1}{2}ih[e^{-2i\psi/h}\frac{\varphi f}{\bar{\partial}\psi} - T(e^{-2i\psi/h}\bar{\partial}(\frac{\varphi f}{\bar{\partial}\psi}))]$$

and the boundedness of T on L^q to deduce that for any $q < 2$

$$(7) \qquad ||\chi T(\varphi e^{-2i\psi/h}f)||_{L^q(\mathbb{C})} \leq Ch\Big(||\frac{\varphi f}{\bar{\partial}\psi}||_{L^q} + ||\frac{f\bar{\partial}\varphi}{\bar{\partial}\psi}||_{L^q} + ||\frac{\varphi\bar{\partial}f}{\bar{\partial}\psi}||_{L^q} + ||\frac{f\varphi}{(\bar{\partial}\psi)^2}||_{L^q}\Big).$$

The first term is clearly bounded by $\delta^{-1}||f||_{L^\infty}$ due to the fact that ψ is Morse. For the last term, observe that since ψ is Morse, $\frac{1}{|\bar{\partial}\psi|} \leq \frac{c}{|z-z_0|}$ near z_0, therefore

$$||\frac{f\varphi}{(\bar{\partial}\psi)^2}||_{L^q} \leq C||f||_{L^\infty}(\int_\delta^1 r^{1-2q}dr)^{1/q} \leq C\delta^{\frac{2}{q}-2}||f||_{L^\infty}.$$

The second term can be bounded by $||\frac{f\bar{\partial}\varphi}{\bar{\partial}\psi}||_{L^q} \leq ||f||_{L^\infty}||\frac{\bar{\partial}\varphi}{\bar{\partial}\psi}||_{L^q}$. Observe that while $||\frac{\bar{\partial}\varphi}{\bar{\partial}\psi}||_{L^\infty}$ grows like δ^{-2}, $\bar{\partial}\varphi$ is only supported in a neighbourhood of radius 2δ. Therefore we obtain

$$||\frac{f\bar{\partial}\varphi}{\bar{\partial}\psi}||_{L^q} \leq \delta^{2/q-2}||f||_{L^\infty}.$$

The third term can be estimated by

$$||\frac{\varphi\bar{\partial}f}{\bar{\partial}\psi}||_{L^q} \leq C||\bar{\partial}f||_{L^p}||\frac{\varphi}{\bar{\partial}\psi}||_{L^\infty} \leq C\delta^{-1}||\bar{\partial}f||_{L^p}.$$

Combining these four estimates with (7) we obtain

$$||\chi T(\varphi e^{-2i\psi/h}f)||_{L^q(\mathbb{C})} \leq h||f||_{W^{1,p}}(\delta^{-1} + \delta^{2/q-2}).$$

Combining this and (6) and optimizing by taking $\delta = h^{1/3}$, we deduce that

$$(8) \qquad ||\chi T(e^{-2i\psi/h}f)||_{L^q(\mathbb{C})} \leq h^{2/3}||f||_{W^{1,p}}$$

if $q < 2$. We now move on to the smoothing part given by $K(e^{-2i\psi/h}f)$. Take χ to be a compactly supported function in Ω such that it is equal to 1 on the support of f, we see that $K(e^{2i\psi/h}f) = K(e^{-2i\psi/h}(f - \chi f(z_0)) + f(z_0)K(e^{-2i\psi/h}\chi)$. By applying stationary phase, we easily see that $||f(z_0)K(e^{-2i\psi/h}\chi)||_{L^q} \leq Ch||f||_{C^0}$ for any $q \in [1,\infty]$. For the first term, we write $\widetilde{f} := f - \chi f(z_0)$ and we integrate by parts to get, for some smoothing operator K'

$$K(e^{-2i\psi/h}\widetilde{f}) = hK'(e^{-2i\psi/h}\widetilde{f}) + \frac{h}{2i}K\Big(e^{-2i\psi/h}\partial_z\Big(\frac{\widetilde{f}}{\partial_z\psi}\Big)\Big).$$

By the fact that K and K' are smoothing, we see that for all $k \in \mathbb{N}$

$$||K(e^{2i\psi/h}\widetilde{f})||_{C^k} \leq hC\Big(||f||_{L^\infty} + \Big\|\partial_z\Big(\frac{\widetilde{f}}{\partial_z\psi}\Big)\Big\|_{L^1}\Big)$$

Using the fact that ψ is Morse, the Sobolev embedding $W^{1,p} \subset C^\alpha$ for $\alpha = 1 - 2/p$ and $\widetilde{f}(z_0) = 0$, we can estimate the last term by $C||f||_{W^{1,p}}$ if $p > 2$. Therefore,

$$(9) \qquad ||K(e^{2i\psi/h}f)||_{L^q} \leq Ch||f||_{W^{1,p}}$$

for any $q \in [1,\infty]$ and $p > 2$. Combining (9) and (8) we see that (3) is established.

Let us now turn our attention to the case when $\infty > q \geq 2$, one can use the boundedness of T on L^q and thus

$$(10) \qquad ||\chi T((1 - \varphi)e^{-2i\psi/h}f)||_{L^q(\mathbb{C})} \leq ||(1 - \varphi)e^{-2i\psi/h}f||_{L^q(\Omega)} \leq C\delta^{\frac{2}{q}}||f||_{L^\infty}.$$

Now since $\varphi = 0$ near z_0, we can use

$$T(e^{-2i\psi/h}\varphi f) = \frac{1}{2}ih[e^{-2i\psi/h}\frac{\varphi f}{\partial_{\bar{z}}\psi} - T(e^{-2i\psi/h}\partial_{\bar{z}}(\frac{\varphi f}{\partial_{\bar{z}}\psi}))]$$

and the boundedness of T on L^q to deduce that for any $q \leq p$, (7) holds again with all the terms satisfying the same estimates as before so that

$$||T(e^{-2i\psi/h}\varphi f)||_{L^q} \leq Ch||f||_{W^{1,p}}(\delta^{2/q-2} + \delta^{-1}) \leq Ch\delta^{2/q-2}||f||_{W^{1,p}}$$

since now $q \geq 2$. Now combine the above estimate with (10) and take $\delta = h^{\frac{1}{2}}$ we get

$$||T(e^{-2i\psi/h}f)||_{L^q} \leq h^{1/q}||f||_{W^{1,p}}$$

for $2 \leq q \leq p$. The smoothing operator K is controlled by (9) for all $q \in [1, \infty]$ and therefore we obtain (4). $\qquad \square$

Observe that the adjoint $\mathcal{R}^* : L^2(M_0) \to L^2(M)$ of $\mathcal{R} : L^2(M) \to L^2(M_0)$ is simply given by $\mathcal{R}^*f = \mathbb{1}_{M_0}f$ where $\mathbb{1}_{M_0}$ is the characteristic function of M_0 in M. In particular $\mathcal{R}^*V \in W^{1,p}(M)$ for $p > 2$ if $V|_{\partial M_0} = 0$ and $V \in W^{1,p}(M_0)$. By Proposition 2.1, the operator $(\bar{\partial}^{-1})^*$ satisfies

$$\chi(\bar{\partial}^{-1})^* = \sum_j \chi_j \bar{T}^* \hat{\chi}_j + K^*$$

where K^* has a smooth integral kernel on $M \times M$ and $\bar{T}^* = T$. The proof of Lemma 2.2 can then be applied in the same way to deduce that for $v \in W^{1,p}(M_0)$ with $v|_{\partial M_0} = 0$

$$(11) \qquad ||\mathcal{E}^*(\bar{\partial}^{-1})^*R^*(e^{-2i\psi/h}v)||_{L^2(T^*_{0,1}M_0)} \leq Ch^{\frac{1}{2}+\epsilon}||v||_{W^{1,p}(M_0)}$$

where we used that $\chi\mathcal{E} = \mathcal{E}$ if $\chi = 1$ on M_δ. The same following estimate also holds if $w \in W^{1,p}(M_0, T^*_{0,1}M_0)$ with $w|_{\partial M_0} = 0$

$$(12) \qquad ||\mathcal{E}^*(\bar{\partial}^{*-1})^*R^*(e^{2i\psi/h}w)||_{L^2(M_0)} \leq Ch^{\frac{1}{2}+\epsilon}||w||_{W^{1,p}(M_0, T^*_{0,1}M_0)}.$$

Similarly, let $\mathcal{R}$ also denotes the restriction of section of $T^*_{0,1}M$ to M_0 and if $\mathcal{E}$ is an extension map from M_0 to M which is continuous from $W^{k,p}(M_0)$ to $W^{k,p}_c(M)$ for $k = 0, 1$ and with range some functions having support in M_δ. One has

Lemma 2.3. *Let ψ be a smooth real valued Morse function on M and let $\bar{\partial}^{*-1}_\psi := \mathcal{R}\bar{\partial}^{*-1}e^{2i\psi/h}\mathcal{E}$ where $\bar{\partial}^{*-1}$ is the right inverse constucted in Proposition 2.1 for $\bar{\partial}^* : W^{1,p}(M, T^*_{0,1}M) \to L^p(M)$. Let $q \in (1, \infty)$ and $p > 2$, then there exists $C > 0$ independent of h such that for all $\omega \in W^{1,p}(M_0)$*

$$(13) \qquad ||\bar{\partial}^{*-1}_\psi \omega||_{L^q(M_0, T^*_{0,1}M_0)} \leq Ch^{2/3}||\omega||_{W^{1,p}(M_0)} \text{ if } 1 \leq q < 2$$

$$(14) \qquad ||\bar{\partial}^{*-1}_\psi \omega||_{L^q(M_0, T^*_{0,1}M_0)} \leq Ch^{1/q}||\omega||_{W^{1,p}(M_0)} \text{ if } 2 \leq q \leq p.$$

There exists $\epsilon > 0$ and $C > 0$ such that for all $\omega \in W^{1,p}_c(M_0)$

$$(15) \qquad ||\bar{\partial}^{*-1}_\psi \omega||_{L^2(M_0, T^*_{0,1}M_0)} \leq Ch^{\frac{1}{2}+\epsilon}||\omega||_{W^{1,p}(M_0)}.$$

Proof. The proof is exactly the same as the proof of Lemma 2.2. We do not repeat it. $\square$

3. Solutions to First Order Systems

Let M_0 be a surface with boundary included strictly in M (for instance a deformation retract of M) and let $\Phi = \phi + i\psi$ be a Morse holomorphic function on M. Such Φ exist by Corollary 2.2. in [8]. We shall denote respectively by

$$V = \begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix} \quad \text{and } D = \begin{pmatrix} 0 & \bar{\partial}^* \\ \bar{\partial} & 0 \end{pmatrix}$$

the matrix potential where $v, v' \in W^{1,p}(M_0)$ (with $p > 2$) are complex valued and the Dirac type operator, acting on sections of the bundle $\Sigma := \Lambda^0(M_0) \oplus \Lambda^{0,1}(M_0)$ over M_0. In this section, we will construct geometric optic solutions $F \in W^{1,p}(\Sigma)$ (also called Faddeev type solutions) which solve the equation

$$(D + V)F = 0$$

on M_0. It is clear that

$$D = \begin{pmatrix} e^{-\bar{\Phi}/h} & 0 \\ 0 & e^{-\Phi/h} \end{pmatrix} D \begin{pmatrix} e^{\Phi/h} & 0 \\ 0 & e^{\bar{\Phi}/h} \end{pmatrix}$$

and thus

$$\begin{pmatrix} e^{-\bar{\Phi}/h} & 0 \\ 0 & e^{-\Phi/h} \end{pmatrix} (D + V) \begin{pmatrix} e^{\Phi/h} & 0 \\ 0 & e^{\bar{\Phi}/h} \end{pmatrix} = D + V_\psi,$$

$$V_\psi := \begin{pmatrix} e^{2i\psi/h}v & 0 \\ 0 & e^{-2i\psi/h}v' \end{pmatrix}.$$

We will then construct solutions F_h of $(D + V_\psi)F_h = 0$ which have the form

$$F_h = \begin{pmatrix} a + r_h \\ b + s_h \end{pmatrix} =: A + Z_h$$

where a is some holomorphic functions on M, b some anti-holomorphic 1-form and $(r_h, s_h) \in W^{1,p}(\Sigma)$ which decays appropriately as $h \to 0$. In particular, we need to solve the system

$$(D + V_\psi)Z_h = -V_\psi A = - \begin{pmatrix} e^{2i\psi/h}va \\ e^{-2i\psi/h}v'b \end{pmatrix}.$$

Let us define the operators D^{-1} and D_ψ^{-1} acting on $\Lambda^0(M_0) \oplus \Lambda^{0,1}(M_0)$ by

$$D^{-1} := \begin{pmatrix} 0 & \mathcal{R}\bar{\partial}^{-1}\mathcal{E} \\ \mathcal{R}\bar{\partial}^{*-1}\mathcal{E} & 0 \end{pmatrix}, \quad D_\psi^{-1} := \begin{pmatrix} 0 & \bar{\partial}_\psi^{-1} \\ \bar{\partial}_\psi^{*-1} & 0 \end{pmatrix}$$

which satisfy $DD^{-1} = \text{Id}$ on $L^q(M_0)$ for all $q \in (1, \infty)$ and $DD_\psi^{-1}V = V_\psi$. To construct Z_h solving $(D + V_\psi)Z_h = -V_\psi A$ in M_0, it then suffices to solve

$$(I + D_\psi^{-1}V)Z_h = -D_\psi^{-1}VA.$$

Writing the components of this system explicitly we get

$$(16) \qquad \begin{cases} r_h + \bar{\partial}_\psi^{-1}(v's_h) = -\bar{\partial}_\psi^{-1}(v'b) \\ s_h + \bar{\partial}_\psi^{*-1}(vr_h) = -\bar{\partial}_\psi^{*-1}(va) \end{cases}.$$

Observe that since we are allowed to choose any holomorphic function a and anti-holomorphic 1-form b, we may set $a = 0$ in (16) and solve for r_h to get

$$(17) \qquad (I - S_h)r_h = -\bar{\partial}_\psi^{-1}(v'b) \quad \text{with } S_h := \bar{\partial}_\psi^{-1}v'\bar{\partial}_\psi^{*-1}v.$$

where v, v' are viewed as multiplication operators. We have the following lemma:

Lemma 3.1. *Let $p > 2$ and assume that $v \in L^\infty(M_0)$ and $v' \in W^{1,p}(M_0)$, then S_h is bounded on $L^r(M_0)$ for any $1 < r \le p$ and satisfies $||S_h||_{L^r \to L^r} = O(h^{1/r})$ if $r > 2$ and $||S_h||_{L^2 \to L^2} = O(h^{1/2-\epsilon})$ for any $0 < \epsilon < 1/2$ small.*

Proof. First, notice that $\bar{\partial}_\psi^{*-1}$ maps $L^r(M_0)$ to $W^{1,r}(M_0, T_{0,1}^* M_0)$ with norm $O(1)$ as $h \to 0$ by (iii) in Lemma 2.1 and the properties of $\mathcal{E}, \mathcal{R}$. Therefore, if $v' \in W^{1,p}(M_0)$ and $v \in L^\infty(M_0)$, the operator $v'\bar{\partial}_\psi^{*-1}v$ maps L^r to $W^{1,r}(M_0, T_{0,1}^* M_0)$ with norm $O(1)$ for $r \le p$ and Lemma 2.2 can be used to deduce that S_h maps L^r to L^r with norm $O(h^{1/r})$ if $r > 2$. If $r < 2$, $v'\bar{\partial}_\psi^{*-1}$ maps $L^r(M_0)$ to $L^r(T_{0,1}^* M_0)$ with norm $O(1)$, and $\bar{\partial}_\psi^{-1}$ maps $L^r(T_{0,1}^* M_0)$ to $L^r(M_0)$ with norm $O(1)$, and thus S_h is bounded on $L^r(M_0)$ with norm $O(1)$. For all $\epsilon > 0$ small, interpolating between $r = 1 + \epsilon$ and $r = 2 + \epsilon$, gives the desired result for $r = 2$. $\qquad\square$

In view of Lemma 3.1, equation (17) can be solved by using Neumann series by setting (for small $h > 0$)

$$(18) \qquad r_h := -\sum_{j=0}^\infty S_h^j \bar{\partial}_\psi^{-1} v'b$$

as an element of any $L^q(M_0)$ for $q \ge 2$. Substituting this expression for r into equation (16) when $a = 0$, we get that

$$(19) \qquad s_h = -\bar{\partial}_\psi^{*-1}vr_h.$$

We now derive the asymptotics in h for s_h and r_h.

Lemma 3.2. *If $v \in L^\infty(M_0)$ and $v' \in W^{1,p}(M_0)$ for some $p > 2$, then there exists $\epsilon > 0$ such that*

$$||s_h||_{L^2(M_0)} + ||r_h||_{L^2(M_0)} = O(h^{\frac{1}{2}+\epsilon})$$

Proof. The statement for r_h is an easy consequence of Lemma 2.2 and 3.1: indeed $||\bar{\partial}_\psi^{-1}v'b||_{L^2} = O(h^{\frac{1}{2}+\epsilon})$ by Lemma 2.2 and $||S_h||_{L^2 \to L^2} = O(h^{\frac{1}{2}-\epsilon})$ thus $||r_h||_{L^2} = O(h^{\frac{1}{2}+\epsilon})$. The estimate for s_h comes from the fact that $||\bar{\partial}_\psi^{*-1}||_{L^2 \to L^2} = O(1)$ and (19). $\qquad\square$

The same method can clearly be used by setting $b = 0$ and solving for s_h first. We summarize the results of this section into the following proposition

Proposition 3.1. *Let $\Phi = \phi + i\psi$ be a Morse holomorphic function on M, and b an anti-holomorphic 1-form on M. If $v \in L^\infty(M_0)$ and $v' \in W^{1,p}(M_0)$ for some $p > 2$, then there exist solutions to $(D + V)F = 0$ on M_0 of the form*

$$(20) \qquad F_h = \begin{pmatrix} e^{\Phi/h}r_h \\ e^{\bar{\Phi}/h}(b + s_h) \end{pmatrix}$$

where $\|s_h\|_{L^2(M_0)} + \|r_h\|_{L^2(M_0)} = O(h^{\frac{1}{2}+\epsilon})$ *for some* $\epsilon > 0$. *If conversely* $v' \in L^\infty(M_0)$ *and* $v \in W^{1,p}(M_0)$ *for some* $p > 2$, *then there exist solutions to* $(D+V)G = 0$ *on* M_0 *of the form*

$$(21) \qquad G_h = \begin{pmatrix} e^{\Phi/h}(a + r_h) \\ e^{\bar\Phi/h}s_h \end{pmatrix}$$

where $\|s_h\|_{L^2(M_0)} + \|r_h\|_{L^2(M_0)} = O(h^{\frac{1}{2}+\epsilon})$ *for some* $\epsilon > 0$.

As a corollary we obtain the

Theorem 3.3. *Let* $p > 2$ *and* $V_j \in L^\infty(M_0, \mathrm{End}(\Sigma))$ *be some complex diagonal endomorphisms of* Σ *for* $j = 1, 2$ *with diagonal entries* $v_j \in L^\infty(M_0), v'_j \in L^\infty(M_0)$. *Denote by*

$$\mathcal{C}_{V_j} := \{i^*_{\partial M_0} F \in H^{\frac{1}{2}}(\partial M_0; \Sigma); F \in H^1(M_0, \Sigma), (D+V_j)F = 0\}$$

the Cauchy data space of $D + V_j$, *where* $i_{\partial M_0} : \partial M_0 \to M_0$ *denotes the natural inclusion map.*
(i) If $v'_j \in W^{1,p}(M_0)$ *and* $\mathcal{C}_{V_1} = \mathcal{C}_{V_2}$ *then* $v'_1 = v'_2$.
(ii) if $v_j \in W^{1,p}(M_0)$ *and* $\mathcal{C}_{V_1} = \mathcal{C}_{V_2}$ *then* $v_1 = v_2$.
As a consequence, if $V_j \in W^{1,p}(M_0, \mathrm{End}(\Sigma))$, *and* $\mathcal{C}_{V_1} = \mathcal{C}_{V_2}$, *then* $V_1 = V_2$.

Proof. Let Φ be a Morse holormophic function with a critical point at z_0. The existence of such a function for a dense set of points z_0 of M_0 is insured by Proposition 2.1 of [8]. We start by defining the respective solutions

$$F_h^1 := \begin{pmatrix} e^{\Phi/h}r_h^1 \\ e^{\bar\Phi/h}(b + s_h^1) \end{pmatrix}, \quad F_h^2 := \begin{pmatrix} e^{-\Phi/h}r_h^2 \\ e^{-\bar\Phi/h}(b + s_h^2) \end{pmatrix}$$

of $(D + V_1)F_h^1 = 0$ and $(D + V_2^*)F_h^2 = 0$ where r_h^j, s_h^j are constructed in Proposition 3.1. Since $\mathcal{C}_1 = \mathcal{C}_2$, there exists F_h solution of $(D + V_2)F_h = 0$ such that $i^*_{\partial M_0}F_h = i^*_{\partial M_0}F_h^1$. In particular, $(D + V_2)(F_h^1 - F_h) = (V_2 - V_1)F_h^1$ and $i^*_{\partial M_0}(F_h^1 - F_h) = 0$. Then using Green's formula and the vanishing of $F_h^1 - F_h$ on the boundary

$$(22) \qquad 0 = \int_{M_0} \langle (D + V_2)(F_h^1 - F_h), F_h^2 \rangle = \int_{M_0} \langle (V_2 - V_1)F_h^1, F_h^2 \rangle.$$

where $\langle \cdot, \cdot \rangle$ denotes the natural Hermitian scalar product on Σ induced by g. This gives

$$(23) \qquad 0 = \int_{M_0} (v'_2 - v'_1)e^{-2i\psi/h}\Big(|b|^2 + \langle b, s_h^2 \rangle + \langle s_h^1, b \rangle\Big) + (v_2 - v_1)e^{2i\psi/h}r_h^1 \overline{r_h^2}.$$

First, notice that by Proposition 3.1,

$$(24) \qquad \int_{M_0} (v_2 - v_1)e^{2i\psi/h}r_h^1 \overline{r_h^2} = O(h^{1+\epsilon})$$

for some $\epsilon > 0$. Then we choose b to be an anti-holomorphic 1-form which vanishes at all critical points of Φ in M_0 except at the critical point $z_0 \in M_0$ of Φ. This can be done by using Riemann-Roch theorem (see Lemma 4.1 in [8]). We observe by using stationary phase that

$$(25) \qquad \int_{M_0} (v'_1 - v'_2)e^{2i\psi/h}|b|^2 = C_{z_0}he^{2i\psi(z_0)/h}(v'_1(z_0) - v'_2(z_0))|b(z_0)|^2 + o(h)$$

for some constant $C_{z_0} \neq 0$. More precisely, to show this, it suffices to decompose $v' := v'_1 - v'_2$ as $(v' - \chi v'(z_0)) + \chi v'(z_0)$ where $\chi \in C_0^\infty(M_0)$ is supported near p, then we apply stationary

phase to the term $\chi v'(z_0)$ and use integration by parts for the other term: if $\sum_j \chi_j = 1$ is a partition of unity on M_0 associated to charts U_j with complex coordinate z

$$\int_{M_0} e^{2i\psi/h} \chi_j (v' - \chi v'(z_0)) |b|^2 = \frac{h}{2i} \int_{U_j} e^{2i\psi/h} \partial_z \Big(\frac{\chi_j (v' - \chi v'(z_0)) |b|^2}{\partial_z \psi} \Big)$$

since $v'|_{\partial M_0} = 0$ by the boundary identification of Lemma 7.3, and we finally conclude using Riemann-Lebesgue to deduce that the right hand side term is $o(h)$ since $\partial_z (\chi_j (v' - \chi v'(z_0)) |b|^2 / \partial_z \psi) \in L^1(M_0)$ if $v' \in W^{1,p}$ for $p > 2$. Let us now consider the term with $\langle b, s_h^2 \rangle$ in (23): using (19)

$$\int_{M_0} \langle e^{2i\psi/h} v' b, s_h^2 \rangle = \int_{M_0} \langle \mathcal{E}^* (\bar{\partial}^{*-1})^* \mathcal{R}^* e^{2i\psi/h} v' b, v_2 r_h^2 \rangle.$$

Since $v'|_{\partial M_0} = 0$, we may use (12) to deduce that $||\mathcal{E}^*(\bar{\partial}^{*-1})^* \mathcal{R}^* e^{2i\psi/h} v' b||_{L^2} = O(h^{1/2+\epsilon})$ and thus combining with Proposition 3.1, we deduce that

$$\int_{M_0} \langle e^{2i\psi/h} v' b, s_h^2 \rangle = O(h^{1+\epsilon}).$$

the same argument gives that the term involving $\langle s_h^1, b \rangle$ in (23) is $O(h^{1+\epsilon})$. These last two estimates combined with (25) and (24) imply that $v_1'(z_0) = v_2'(z_0)$ by letting $h \to 0$. The same proof using the complex geometric optics G_h of Proposition 3.1 gives $v_1 = v_2$ if $v_j \in W^{1,p}(M_0)$ and $v_j \in L^\infty(M_0)$. $\qquad\square$

Remark 3.4. *As noted in section 4 of [3], this methods allows to get an inversion (or reconstruction) procedure to recover the value of a matrix potential V at a given $z_0 \in M_0$, provided we know a Morse holomorphic function Φ with a critical point at z_0 and $\Phi(z_0) = 0$. We do not give details since it is essentially the same idea as [3], but essentially the method is to compare to case $V_1 = V$ to the free case $V_2 = 0$ and use complex geometric optic (or Faddeev type) solutions F_h^1, F_h^2 for $h \to 0$, together with the Green formula as we did above for identification: the boundary term is not zero anymore but is the information we measure and therefore multiplying by h^{-1} and letting $h \to 0$, the boundary term converges to $V(z_0)$ times an explicit non-zero constant .*

4. BOUNDARY DETERMINATION

For smooth (X, q), it was shown by Nakamura-Sun-Uhlmann in [19] that the Cauchy data space determines the boundary values of X_j up to an exact form. This is relaxed to regularity $X \in C^1, q \in L^\infty$ by Brown-Salo [2]. We summarize it in

Proposition 4.1. *Let $X_1, X_2 \in W^{2,p}(M_0, T^*M_0)$ and $q_1, q_2 \in W^{1,p}(M_0)$ for some $p > 2$, then if $\mathcal{C}_{L_1} = \mathcal{C}_{L_2}$ then $i_{\partial M_0}^* X_1 = i_{\partial M_0}^* X_2$ and $q_1|_{\partial M_0} = q_2|_{\partial M_0}$, where $i_{\partial M_0}$ is the inclusion map of ∂M_0 into M_0.*

This statement was only shown in [19, 2] for $M = \Omega \subset \mathbb{R}^n$ but since the proof is only localized near a neighbourhood of the boundary, it adapts naturally on a general Riemann surface and we will not provide a proof here. Notice that by adding an exact form $d\zeta$ to X_1 with ζ a function vanishing on the boundary, we do not change the Cauchy data space $\mathcal{C}_{L_1}$. If x is a boundary defining function such that $|dx|_g = 1$ at ∂M_0, we can set $\zeta = x f(x)$ for some C^1 function f and we have $d\zeta|_{\partial M_0} = f|_{\partial M_0} dx$; therefore if ν is the unit interior pointing normal vector field to ∂M_0, we have $d\zeta(\nu)|_{\partial M_0} = f|_{\partial M_0}$ and choosing f accordingly, we can

choose ζ so that $X_1 + d\zeta = X_2$ at ∂M_0. By the gauge invariance, we can at best identify X up to exact forms, and therefore we may assume that $X_1 = X_2$ at ∂M_0 as forms on M_0, possibly by modifying X_1 through an exact form $d\zeta$.

For our purpose we will need additional information along the boundary in order to reduce our problem to a first order system. That is, we will show that the boundary value of certain primitives of the forms X_j agree with that of the boundary value of a holomorphic function. More precisely, let $X_j \in \Lambda^1(M_0)$ and $A_j := \pi_{0,1}X_j$ and $B_j := \pi_{1,0}X_j$ where $\pi_{0,1} : \Lambda^1(M_0) \to \Lambda^{0,1}(M_0)$, and $\pi_{1,0} : \Lambda^1(M_0) \to \Lambda^{1,0}(M_0)$ are the natural projections. The main result of this section is

Proposition 4.2. *Let α_1 and α_2 be smooth functions such that $\bar{\partial}\alpha_j = A_j$. Then $e^{-i(\alpha_1 - \alpha_2)}|_{\partial M}$ is the boundary value of a holomorphic function.*

In order to prove the Proposition, we shall need a few Lemma characterizing boundary values of holomorphic functions. Let us denote by $i_{\partial M_0} : \partial M_0 \to M_0$ the inclusion map.

Lemma 4.1. *Let $f \in H^{1/2}(\partial M_0)$ be a complex valued function. Then f is the restriction of a holomorphic function if and only if*

$$\int_{\partial M} f i^*_{\partial M_0}\eta = 0$$

for all 1-forms $\eta \in C^\infty(M_0; T^*_{1,0}M_0)$ satisfying $\bar{\partial}\eta = 0$. Similarly, $f \in H^{1/2}(\partial M_0)$ is the restriction of an anti-holomorphic function if and only if

$$\int_{\partial M_0} f i^*_{\partial M_0}\eta = 0$$

for all 1- forms $\eta \in C^\infty(M_0; T^*_{0,1}M_0)$ satisfying $\partial\eta = 0$.

Proof. We will only prove the holomorphic statement, the anti-holomorphic statement follows similarly. Suppose $f \in H^{1/2}(\partial M_0)$ is such that

$$\int_{\partial M} f i^*_{\partial M_0}\eta = 0.$$

and denote by $u \in H^1(M_0)$ its harmonic extension to M_0. We would like to show that u is actually holomorphic. We will do this by showing that

$$\langle \bar{\partial}u, \omega \rangle = 0, \ \forall\omega \in C^\infty(M_0, T^*_{0,1}M_0).$$

By the Hodge-Morrey decomposition given in [20, Th 2.4.2], a 1-form $\omega \in L^2$ can be decomposed as

$$\omega = d\alpha + *d\beta + \omega_0$$

where $\alpha, \beta \in H^1(M_0)$ satisfy Dirichlet condition $\alpha|_{\partial M_0} = \beta|_{\partial M_0} = 0$ and ω_0 is closed and co-closed $d\omega_0 = 0, d*\omega_0 = 0$. If $\omega \in C^\infty(M_0, T^*_{0,1}M_0)$, then

$$\omega = \pi_{0,1}\omega = \pi_{0,1}d\alpha + \pi_{0,1} * d\beta + \pi_{0,1}\omega_0$$

Since $\pi_{0,1}d = \bar{\partial}$ and $\pi_{0,1} * d = -i\bar{\partial}$ on functions, and $\eta := \pi_{0,1}\omega_0 \in C^\infty(M_0, T^*_{0,1}M_0)$ satisfies $\partial\eta = 0$, we can write $\omega = \bar{\partial}\gamma + \eta$ where γ has Dirichlet boundary condition and $\partial\eta = 0$. We now compute by Stoke's Theorem

$$\langle \bar{\partial}u, \omega \rangle = \langle \bar{\partial}u, \bar{\partial}\gamma \rangle + \langle \bar{\partial}u, \eta \rangle = \langle \bar{\partial}^*\bar{\partial}u, \gamma \rangle - i\int_{\partial M} f i^*_{\partial M_0}\bar{\eta}.$$

The first term on the right side vanishes since u is harmonic, the last term vanishes since $\bar\eta \in C^\infty(M_0, T_{1,0}^* M_0)$ satisfies $\bar\partial \bar\eta = 0$ and we have assumed that f is orthogonal to all such boundary values of sections of $T_{1,0}^* M_0$. Therefore we conclude that $\bar\partial u = 0$. Since we will not actually use the converse statement in this paper, it is left as an exercise.   □

Lemma 4.2. *Let $f \in H^{1/2}(\partial M_0)$ be a complex valued function. Then f is the restriction of a holomorphic function if and only if*

$$\int_{\partial M_0} f i_{\partial M_0}^* \partial \phi = 0$$

for all smooth real valued harmonic functions ϕ which have no critical point on the boundary. Similary, $f \in H^{1/2}(\partial M_0)$ is the restriction of an anti-holomorphic function if and only if

$$\int_{\partial M_0} f i_{\partial M_0}^* \bar\partial \phi = 0$$

for all smooth real valued harmonic functions ϕ which have no critical point on the boundary.

Proof. By applying [18, Th. C1.10] like in [9, Cor. 2.3] with a totally real subbundle boundary condition having high boundary Maslov index, one obtains that the operator $\partial : H^4(M_0) \to H^3(M_0; T_{1,0}^* M_0)$ is surjective. Using this we see that Lemma 4.1 implies that f is the restriction of a holomorphic function if and only if

$$\int_{\partial M_0} f i_{\partial M_0}^* \partial \phi = 0$$

for all harmonic functions ϕ. Now it remains to show that this statement is equivalent to the case where we consider only smooth harmonic functions with no critical points on the boundary. That is, we want to show that smooth harmonic functions with no critical points on the boundary form a dense subset of the harmonic functions in $C^k(M_0)$. Indeed, let ϕ be a harmonic function with smooth boundary value $g \in C^\infty(\partial M_0)$. Since Morse functions are generic on the circle, it suffices to consider the case where g is a Morse function with isolated critical points $\{x_1, .., x_N\}$. Clearly, the critical points of ϕ forms a subset of the set $\{x_1, .., x_N\}$. We will make a small perturbation to ϕ so that x_1 is guaranteed to not be a critical point. Let ϕ_1 be a smooth harmonic function with boundary value $g_1 \in C^\infty(\partial M_0)$ and such that and $d\phi_1(x_j) = d\nu$ for all $j = 1, \ldots, N$ if $d\nu$ is the unit conormal form to ∂M_0; the existence of such a function ϕ_1 is insured by applying Lemma 2.6 of [9] on the manifold M containing strictly M_0: indeed this Lemma says that there exist holomorphic functions with prescribed Taylor expansion to order 2 at $x_1, \ldots, x_N \in \text{int}(M)$ and therefore taking its real part one obtains the desired harmonic function. For all $\epsilon > 0$ small, the points $x_1, \ldots, x_N$ are not critical point of the function $\phi' := \phi + \epsilon \phi_1$. Furthermore, for all $\epsilon > 0$ small enough, $g' := \phi'|_{\partial M_0}$ is again a Morse function on ∂M_0 with critical points $\{x_1, .., x_N\}$, therefore the critical points of ϕ' on ∂M_0 are contained in $\{x_1, .., x_N\}$, which implies that ϕ' has no critical points on ∂M_0. Thus smooth harmonic functions with no critical points on ∂M_0 form a dense subset of C^k harmonic functions and we are done.   □

In view of Lemma 4.2 and the fact that Morse harmonic functions form a dense subset of harmonic functions (see [8, Lemma 2.2]) in $C^k(M_0)$, we have the following corollary:

Corollary 4.3. *Let $f \in H^{1/2}(\partial M_0)$ be a complex valued function. Then f is the restriction of a holomorphic function if and only if*

$$\int_{\partial M_0} f i_{\partial M_0}^* \partial \phi = 0$$

for all smooth real valued harmonic functions ϕ which is Morse up to the boundary and has no critical point on the boundary. Similary, $f \in H^{1/2}(\partial M_0)$ is the restriction of an anti-holomorphic function if and only if

$$\int_{\partial M_0} f i_{\partial M_0}^* \bar{\partial} \phi = 0$$

for all smooth real valued harmonic functions ϕ which is Morse up to the boundary and has no critical point on the boundary.

Let ϕ be a smooth real valued harmonic function defined on M. It is easy to see that ϕ has a harmonic conjugate if and only if

$$M_j(\phi) := \int_{\gamma_j} *d\phi = 0 \quad j = 1, .., n.$$

where $\{\gamma_1, .., \gamma_n\}$ is a family of generators of the fundamental group $\pi(M_0, m_0)$ (with $m_0 \in M_0$ fixed). However, if $M(\phi) = (M_1(\phi), .., M_n(\phi)) \in (2\pi\mathbb{Z})^n$ then one can construct a multivalued function ψ such that $F := e^{\phi + i\psi}$ is a (single-valued) holomorphic function with $|F| = e^\phi$ and $\partial F = (1 - i)F\partial\phi$. Indeed, it simply suffices to set $\psi(m) = -\int_{\gamma(m)} *d\phi$ where $\gamma(m)$ is a smooth path joining m_0 to m.

Lemma 4.4. *Let $f \in H^{1/2}(\partial M_0)$ be a complex valued function. Then f is the restriction of a holomorphic function if and only if*

$$(26) \qquad \int_{\partial M_0} f i_{\partial M_0}^* \partial \phi = 0$$

for all smooth real valued harmonic functions ϕ which is Morse up to the boundary and $M(\phi) \in (2\pi\mathbb{Z})^n$. Similarly, $f \in H^{1/2}(\partial M_0)$ is the restriction of an anti-holomorphic function if and only if

$$\int_{\partial M_0} f i_{\partial M_0}^* \bar{\partial} \phi = 0$$

for all smooth real valued harmonic functions ϕ which is Morse up to the boundary and $M(\phi) \in (2\pi\mathbb{Z})^n$.

Proof. By Corollary 4.3, we need to check that (26) is equivalent to the condition stated in Corollary 4.3. Observe that if $M(\phi) \in (2\pi\mathbb{Q})^n$, then $M(k\phi) \in (2\pi\mathbb{Z})^n$ for some integer k, thus the condition (26) is satisfied for all smooth harmonic Morse ϕ with $M(\phi) = (2\pi\mathbb{Q})^n$ and no boundary critical point if and only if it is satisfied for all smooth harmonic Morse ϕ with $M(\phi) \in (2\pi\mathbb{Z})^n$ and no boundary critical points. Therefore, by taking limits, it suffices to show that any Morse harmonic function ϕ with no critical point on the boundary can be approximated by a sequence of Morse harmonic functions with periods in $2\pi\mathbb{Q}$ in the $C^3(M_0)$ topology. Indeed, let $\{\phi_1, .., \phi_n\}$ be a set of real valued C^3 harmonic functions in M_0 such that $M_j(\phi_k) = \delta_{jk}$. Such harmonic functions exist due to equation (3.1) of [4] which states that the period matrix is invertible. Then given a real valued harmonic function ϕ, we consider small perturbations $\phi' := \phi + \sum_{j=1}^n \epsilon_j \phi_j$ of ϕ. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, ϵ_j can be chosen arbitrarily

small such that $M(\phi') \in (2\pi\mathbb{Q})^n$. If ϕ is Morse with no critical points on the boundary and being Morse is an open condition, ϕ' is also Morse with no critical points on the boundary if ϵ_j are taken small enough. $\qquad\square$

Lemma 4.5. *Let ϕ be a harmonic Morse function such that $M(\phi) \in (2\pi\mathbb{Z})^n$ and α_1, α_2 are functions such that $\bar{\partial}\alpha_1 = A_1$ and $\bar{\partial}\alpha_2 = A_2$. Then for all $k \in \mathbb{Z}$ large enough, there exists a solution $u_1 \in H^1(M_0)$ to*

$$L_1 u_1 = 0$$

such that

$$u_1 = F^k(e^{-i\alpha_1} + r_1) \ \text{ with } \ \sqrt{k}\|r_1\|_{L^2} + \|r_1\|_{H^1} \le C$$

where F is holomorphic, $\partial F = (1-i)F\partial\phi$ and $|F| = e^\phi$. Similarly, there exists a solution $u_2 \in H^1(M_0)$ to

$$L_2^* u_2 = 0$$

such that

$$u_2 = \bar{F}^{-k}(e^{-i\bar\alpha_2} + r_2) \ \text{ with } \ \sqrt{k}\|r_2\|_{L^2} + \|r_2\|_{H^1} \le C.$$

Proof. Since $M(\phi) \in (2\pi\mathbb{Z})^n$ we can construct a multivalued complex conjugate ψ such that $F := e^{\phi + i\psi}$ is a well defined holomorphic function satisfying $\partial F = (1-i)F\partial\phi$ and $|F| = e^\phi$. Since for any α_1 satisfying $\partial\bar\alpha_1 = \bar{A}_1$, L_1 can be written as

$$L_1 = -2i * (\partial + i\bar{A}_1 \wedge)(\bar\partial + iA_1) + Q_1 = -2i * e^{-i\bar\alpha_1}\partial e^{i\bar\alpha_1} e^{-i\alpha_1}\bar\partial e^{i\alpha_1} + Q_1$$

for some $Q_1 \in L^\infty(M_0)$, we have that $L_1 F^k e^{i\alpha_1} = F^k e^{i\alpha_1} Q_1$. By Corollary 7.2, for all $|k|$ large enough there exists a $\tilde{r}_1$ solving

$$e^{-k\phi}L_1 e^{k\phi}\tilde{r}_1 = |F^{-k}|L_1|F^k|\tilde{r}_1 = \tilde{Q}_1$$

such that

$$\sqrt{k}\|\tilde{r}_1\|_{L^2} + \|\tilde{r}_1\|_{H^1} \le C\|\tilde{Q}_1\|_{L^2}.$$

Setting $r_1 = \frac{F^k}{|F^k|}\tilde{r}_1$ we have that

$$L_1 F^k(e^{\alpha_1} + r_1) = 0$$

and that r_1 satisfies the desired estimates. The construction for u_2 follows similarly after factorizing

$$L_2 = -2i * e^{-i\alpha_1}\bar\partial e^{i\alpha_1} e^{-i\bar\alpha_2}\partial e^{i\bar\alpha_2} + \widetilde{Q}_2$$

for some $\widetilde{Q}_2 \in L^\infty(M_0)$ and using that $L_2^* - L_2$ is a zeroth order differential operator. $\qquad\square$

Proof of Proposition 4.2: We need to show that if α_1 and α_2 are functions such that $\bar\partial\alpha_j = A_j$, then $e^{-i(\alpha_1 - \alpha_2)}|_{\partial M_0}$ is the boundary value of a holomorphic function. By Lemma 4.4 this is equivalent to showing that

$$\int_{\partial M_0} f i_{\partial M_0}^* \partial\phi = 0$$

for all smooth real valued harmonic functions ϕ which is Morse up to the boundary and $M(\phi) \in (2\pi\mathbb{Z})^n$.

Let ϕ be such a harmonic function and let $u_1 = F^k(e^{-i\alpha_1} + r_1)$ and $u_2 = \bar{F}^{-k}(e^{-i\bar\alpha_2} + r_2)$ be

the solutions constructed in Lemma 4.5. Plugging these solutions into the boundary integral identity

$$\int_{M_0} \bar{u}_2(2(A_1 - A_2) \wedge \partial u_1 + 2(\bar{A}_1 - \bar{A}_2) \wedge \bar{\partial} u_1 + (Q_1 - Q_2)u_1) = 0$$

we have that

$$2k(1-i) \int_{M_0} e^{-i(\alpha_1 - \alpha_2)}(A_1 - A_2) \wedge \partial\phi + O(\sqrt{k}) = 0.$$

Using the fact that $-ie^{-i(\alpha_1 - \alpha_2)}(A_1 - A_2) = \bar{\partial} e^{-i(\alpha_1 - \alpha_2)}$ we can integrate by parts and take $k \to \infty$ to get that

$$\int_{\partial M_0} e^{-i(\alpha_1 - \alpha_2)} i^*_{\partial M_0} \partial\phi = 0.$$

Since this is true for any real valued harmonic Morse function ϕ with $M(\phi) \in (2\pi\mathbb{Z})^n$, $e^{-i(\alpha_1 - \alpha_2)}|_{\partial M_0}$ is the boundary value of a holomorphic function by Lemma 4.4. $\square$

In view of this proposition we will denote by $F_{-i(\alpha_1 - \alpha_2)}$ to be the unique holomorphic function with boundary value $e^{-i(\alpha_1 - \alpha_2)}|_{\partial M_0}$. Observe that one can reverse the indices 1 and 2 in the proof above, and this shows that $e^{i(\alpha_1 - \alpha_2)}|_{\partial M_0}$ is also the boundary value of a holomorphic function $F_{i(\alpha_1 - \alpha_2)}$. Therefore, writing $\alpha = \alpha_1 - \alpha_2$, it is clear after remarking that the product $F_{i\alpha}F_{-i\alpha}$ has boundary value 1, that

$$(27) \qquad\qquad\qquad\qquad F_{i\alpha}F_{-i\alpha} = 1.$$

5. Reduction to First Order Systems

In this section we use the boundary identity result we obtained in Proposition 4.2 to reduce the inverse problem for the second order equation to an inverse problem for the first order system of Dirac type on sections of the bundle $\Sigma = \Lambda^0 \oplus \Lambda^{0,1}$ introduced in Section 3. We will do this by factoring the operators L_1 and L_2 the appropriate way.

Let α_j satisfy $\bar{\partial}\alpha_j = A_j$ (this exists by Proposition 2.1), then we also have $\partial\bar{\alpha}_j = \bar{A}_j$. We set $\alpha := \alpha_1 - \alpha_2$, then by Proposition 4.2 and (27), we see that there exists a nonvanishing holomorphic function $F_{-i\alpha}$ such that $F_{-i\alpha}|_{\partial M_0} = e^{-i\alpha}|_{\partial M_0}$. In particular, if

$$(28) \qquad\qquad\qquad F_{A_2} := e^{i\alpha_2} \text{ and } F_{A_1} = F_{-i\alpha}e^{i\alpha_1}$$

then

$$(29) \qquad\qquad\qquad \bar{\partial}F_{A_j} = iA_j F_{A_j} \text{ and } F_{A_1}|_{\partial M_0} = F_{A_2}|_{\partial M_0}.$$

Similarly, there is a unique nonvanishing anti-holomorphic function $\bar{F}_{i\alpha}$ such that $F_{\bar{A}_1} := \bar{F}_{i\alpha}e^{i\bar{\alpha}_1}$ and $F_{\bar{A}_2} := e^{i\bar{\alpha}_2}$ satisfy

$$(30) \qquad\qquad\qquad \partial F_{\bar{A}_j} = iF_{\bar{A}_j}\bar{A}_j \text{ and } F_{\bar{A}_1}|_{\partial M_0} = F_{\bar{A}_2}|_{\partial M_0}.$$

We can then write

$$L_j = -2i * (\partial + i\bar{A}_j \wedge)(\bar{\partial} + iA_j) + Q_j = 2F_{\bar{A}_j}^{-1}\bar{\partial}^* F_{\bar{A}_j} F_{A_j}^{-1}\bar{\partial}F_{A_j} + Q_j.$$

where $Q_j = *dX_j + q_j$. Let $u_j \in H^2(M_0)$ and set $\omega_j := F_{A_j}^{-1}\bar{\partial}F_{A_j}u_j \in H^1(M_0, T^*_{0,1}M_0)$,

$$L_j u_j = 0 \iff \begin{pmatrix} 0 & \bar{\partial}^* \\ \bar{\partial} & 0 \end{pmatrix}\begin{pmatrix} F_{A_j} & 0 \\ 0 & F_{\bar{A}_j} \end{pmatrix}\begin{pmatrix} u_j \\ \omega_j \end{pmatrix} + \begin{pmatrix} F_{\bar{A}_j}Q_j/2 & 0 \\ 0 & -F_{A_j} \end{pmatrix}\begin{pmatrix} u_j \\ \omega_j \end{pmatrix} = 0.$$

Observe that (27) implies $\bar{F}_{\bar{A}_j} = F_{A_j}^{-1}$, therefore, if we set $(\tilde{u}_j, \tilde{\omega}_j) := (F_{A_j} u_j, F_{\bar{A}_j} \omega_j) = (F_{A_j} u_j, \overline{F}_{A_j}^{-1} \omega_j)$, then (u_j, ω_j) solves the above system of equations if and only if $(\tilde{u}_j, \tilde{\omega}_j)$ solves

$$(31) \qquad \begin{pmatrix} 0 & \bar{\partial}^* \\ \bar{\partial} & 0 \end{pmatrix} \begin{pmatrix} \tilde{u}_j \\ \tilde{\omega}_j \end{pmatrix} + \begin{pmatrix} \frac{1}{2} Q_j |F_{A_j}|^{-2} & 0 \\ 0 & -|F_{A_j}|^2 \end{pmatrix} \begin{pmatrix} \tilde{u}_j \\ \tilde{\omega}_j \end{pmatrix} = 0$$

Denoting the Cauchy data space for L_j to be

$$\mathcal{C}_{L_j} = \{(u, \nabla_\nu^{X_j} u)|_{\partial M_0} \in H^{\frac{1}{2}}(\partial M_0) \times H^{-\frac{1}{2}}(\partial M_0); u \in H^1(M_0), L_j u = 0\}$$

where $\nabla_\nu^{X_j} u := du(\nu) + iX_j(\nu)u$ and ν the unit normal interior vector field to ∂M_0, we deduce from this discussion the

Proposition 5.1. *Assume that $X_1, X_2 \in W^{2,p}(M_0, T^* M_0)$ are real valued and $q_1, q_2 \in W^{1,p}(M_0)$ complex valued for some $p > 2$. If L_1 and L_2 have the same Cauchy data space $\mathcal{C}_{L_1} = \mathcal{C}_{L_2}$, then the first order system (31) with diagonal endomorphism $(\frac{1}{2} Q_1 |F_{A_1}|^{-2}, |F_{A_1}|^2)$ of Σ has the same Cauchy data space as the one with endomorphism $(\frac{1}{2} Q_2 |F_{A_2}|^{-2}, |F_{A_2}|^2)$ for $Q_j = -dX_j + q_j$ and F_{A_j} defined as above.*

Proof. By the boundary determination and the remark following Proposition 4.1, we can suppose that $X_1 = X_2$ on ∂M_0 as forms, and therefore $A_1 = A_2$ as well on ∂M_0. Combining the discussion above with equalities (29), (30), we have that 2 solutions u_j of $L_j u_j = 0$ satisfying $(u_1 - u_2)|_{\partial M_0} = 0$ are such that $(\tilde{u}_1 - \tilde{u}_2)|_{\partial M_0} = 0$ and

$$i_{\partial M_0}^*(\tilde{\omega}_1 - \tilde{\omega}_2) = i_{\partial M_0}^* \left(\bar{F}_{A_1}^{-1}[(\bar{\partial} + iA_1)u_1 - (\bar{\partial} + iA_2)u_2] \right) = i_{\partial M_0}^*(e^{i\bar{\alpha}_2} \bar{\partial}(u_1 - u_2))$$

where $(\tilde{u}_j, \tilde{\omega}_j) := (F_{A_j} u_j, |F_{A_j}|^{-2} \bar{\partial} F_{A_j} u_j)$. Now,

$$\nabla_\nu^{X_j} u_j|_{\partial M_0} = (du_j(\nu) + iX_j(\nu)u_j)|_{\partial M_0}$$

and since $X_1 = X_2$ on ∂M_0, $\mathcal{C}_{L_1} = \mathcal{C}_{L_2}$ implies that $d(u_1 - u_2)(\nu)|_{\partial M_0} = 0$, which together with $(u_1 - u_2)|_{\partial M_0} = 0$ gives $\tilde{\omega}_1 = \tilde{\omega}_2$ on ∂M_0. This achieves the proof. $\square$

We deduce from Proposition 5.1 and Theorem 3.3 the following

Corollary 5.1. *Let $X_j \in W^{2,p}(M_0, T^* M_0)$ be real valued and $q_j \in W^{1,p}(M_0)$ complex valued, and $p > 2$. If the Cauchy data spaces for L_1, L_2 satisfy $\mathcal{C}_{L_1} = \mathcal{C}_{L_2}$, then $d(X_1 - X_2) = 0$ and $q_1 = q_2$.*

Proof. Acoording to Proposition 5.1 and Theorem 3.3, we have that

$$(32) \qquad\qquad -dX_1 + q_1 = -dX_2 + q_2 \text{ and } |F_{A_2}|^2 = |F_{A_1}|^2$$

where $X_j = A_j + \bar{A}_j$ and $A_j \in \Lambda^{0,1}, \bar{A}_j \in \Lambda^{1,0}$. The functions F_{A_j} are defined in (28) and α_j satisfies $\bar{\partial}\alpha_j = A_j$. Since $|F_{A_2}| = e^{-\mathrm{Im}(\alpha_2)} = |F_{A_1}| = e^{-\mathrm{Im}(\alpha_1)}|F_{-i\alpha}|$ for $\alpha = \alpha_1 - \alpha_2$ and $F_{i\alpha}$ is a holomorphic function which does not vanish in M_0 in view of (27) (the log of its modulus is then harmonic), then setting $A = \bar{\partial}\alpha = A_1 - A_2$ we deduce that

$$0 = 2i\Delta\mathrm{Im}(\alpha) = \Delta(\alpha - \bar{\alpha}) = -2i * \partial\bar{\partial}\alpha - 2i * \bar{\partial}\partial\bar{\alpha} = -2i * (\partial A + \bar{\partial}\bar{A}) = -2i * d(X_1 - X_2)$$

and therefore $q_1 = q_2$ as well by (32). $\square$

6. Cauchy data determine the holonomy

For each $m \in M_0$ and each closed loop γ based at m_0, the parallel transport for the connection ∇^X on the bundle $M_0 \times \mathbb{C}$ defines an isomorphism $P_\gamma^X : \mathbb{C} \to \mathbb{C}$ of the fiber $\mathbb{C}$ at m, thus P_γ can be viewed as a non-zero complex number $P_\gamma^X \in \mathbb{C} \setminus \{0\}$. The holonomy group of ∇^X at m is given by

$$H_m(\nabla^X) := \{P_\gamma^X \in \mathbb{C} \setminus \{0\}; \gamma \text{ is a closed loop based at } m\}.$$

For the connection $\nabla^X = d + iX$, an easy computation shows that

$$P_\gamma^X = e^{-i \int_\gamma X}$$

where γ is an oriented closed curve. Notice that if X is real valued, the holonomy group $H_m(\nabla^X)$ is a subgroup of S^1. If X is a flat connections 1-forms, ie. with cuvature $dX = 0$, then the map $\gamma \to P_\gamma^X$ induces a natural group morphism $\rho_m^X : \pi_1(M_0, m) \to H_m(\nabla^X)$ where $\pi_1(M_0, m)$ is the fundamental group based at m, ie. the set of closed loop up to homotopy equivalence. The morphism ρ^X is called the holonomy representation into $GL(\mathbb{C})$ and it is trivial if and only if

$$e^{-i \int_\gamma X} = 1 \text{ for all closed loop } \gamma \text{ based at } m,$$

this condition is also independent of m. If X_1, X_2 are two connection 1-forms with same curvature $dX_1 = dX_2$, and if the holonomy representation of $X := X_2 - X_1$ is trivial, then there exist a unitary bundle isomorphism $F : E \to E$ (recall that $E = M_0 \times \mathbb{C}$), or equivalently a function $F : M_0 \to \mathbb{C}$ of modulus $|F| = 1$, defined by

$$F(m') = e^{i \int_{\gamma(m,m')} X}$$

where $\gamma(m, m')$ is any C^1 path joining m and m', this is well defined independently of the path since $dX = 0$ and thanks to the triviality of the holonomy representation. The connections X_1 and X_2 are related by $F^*(d + iX_1)F = d + iX_2$, and if moreover $i^*_{\partial M_0} X_1 = i^*_{\partial M_0} X_2$, then the isomorphism F is the identity when restricted to ∂M_0.

In view of this discussion, to prove Theorem 1.1, we need to prove

Theorem 6.1. *Let $X_1, X_2 \in W^{2,p}(M_0)$ and $q_1, q_2 \in W^{1,p}(M_0)$ for some $p > 2$. Then the Cauchy data spaces $\mathcal{C}_{L_1}$ and $\mathcal{C}_{L_2}$ coincide if and only if $q_1 = q_2$, ∇^{X_1} and ∇^{X_2} have same curvature $dX_1 = dX_2$, and the holonomy representation ρ_m^X is trivial for each $m \in M_0$, where we have set $X := X_1 - X_2$.*

Proof. We shall give two different proofs, the first one using directly the Cauchy data space, the other one using Proposition 5.1.

First Proof. We have already shown that $d(X_1 - X_2) = 0$ and $q_1 = q_2$. Furthermore, by boundary determination (Proposition 4.1 and the remark that follows), we can conclude that the tangential components of X_1 and X_2 agree along the boundary, ie. $i^*_{\partial M_0}(X_1 - X_2) = 0$, and that there exist a function ζ vanishing on the boundary such that $X_1 + d\zeta = X_2$ on ∂M_0. Since the addition of an exact form as above does not change the Cauchy data space, we may assume without loss of generality that $X_1 = X_2$ at ∂M_0. Let $[\gamma] \in \pi_1(M_0, p_1)$ be an equivalence class of loop and γ be a representative. Since $X := X_1 - X_2$ is closed, $\int_\gamma X$ is independent of the chosen representative. We choose a simple (non self-intersecting) representative γ based at p_1, made of 2 oriented pieces $[p_1, p_2]$ and $[p_2, p_1]$ such that $[p_2, p_1] \subset \partial M_0$. This is possible for each

primitive class (or generator) of $[\gamma] \in \pi_1(M_0, p_1)$, ie. every class which can not be expressed as a power of another class. All the other classes in $\pi_1(M_0, p_1)$ are obtained by products of primitive classes (in the group law) and therefore the integrals of X along these classes are obtained by linear combinations over $\mathbb{Z}$ of integrals of X on primitive classes, therefore it suffices to compute the integrals on X on simple representatives (primitive classes) in $\pi_1(M_0, p_1)$. To prove the statement about the trivial holonomy representation, it suffices to prove that $\int_\gamma X \in 2\pi\mathbb{Z}$.

Let $\gamma : [0, 2] \to M$ be a parametrization of this loop in such a way that $\gamma(0) = p_1$, $\gamma(1) = p_2$, $\gamma_1 := \gamma((0, 1)) \subset \text{int}(M_0)$, and $\gamma_2 := \gamma([1, 2]) \subset \partial M_0$. Now consider a thin tubular neighbourhood $\mathcal{O} := \{p \in \text{int}M_0; \text{dist}(p, \gamma_1) < \epsilon\}$ which is homeomorphic to $(0, 1) \times (-\epsilon, \epsilon)$ and therefore simply connected, in particular we need to take ϵ so that $\text{dist}(p_1, p_2) > 2\epsilon$. We define for points $p \in \bar{\mathcal{O}}$ the function $\alpha(p) := \int_{p_1}^p X$, so that $d\alpha = X$ in $\bar{\mathcal{O}}$ (since X is closed). Since $X = 0$ on ∂M_0, we also have $\alpha|_{U_1} = 0$ where $U_1 = \{p \in \partial M_0; \text{dist}(p, p_1) \leq \epsilon\}$. Now let f be a smooth function defined on the boundary such that $f(p_1) = f(p_2) = 1$ and let u_j solve (for $j = 1, 2$)

$$L_j u_j = 0, \quad u|_{\partial M_0} = f.$$

Let $u := e^{i\alpha} u_1$ be defined in the tubular neighbourhood $\mathcal{O}$ and we shall now show that $u = u_2$ in $\mathcal{O}$ by unique continuation if the Cauchy data spaces $\mathcal{C}_{L_j}$ agree. Indeed, using that $\nabla^{X_2} e^{i\alpha} = e^{i\alpha} \nabla^{X_1}$ in $\mathcal{O}$ by simple calculation, we deduce that u solves $L_2 u = 0$ in $\mathcal{O}$ since

$$e^{-i\alpha}(\nabla^{X_2*}\nabla^{X_2} + q_2)e^{i\alpha}u_1 = (\nabla^{X_1*}\nabla^{X_1} + q_1)u_1 = 0.$$

Furthermore, since $\alpha(p) = 0$ for all $p \in U_1$, one has $u(p) = u_1(p) = f(p) = u_2(p)$ for all $p \in U_1$. Moreover, since the normal components of X_1 and X_2 agree on U_1 as well, the normal derivatives satisfy

$$\partial_\nu u|_{U_1} = \partial_\nu u_1|_{U_1} = \nabla_\nu^{X_1} u|_{U_1} - iX_1(\nu)f = \nabla_\nu^{X_2} u|_{U_1} - iX_2(\nu)f = \partial_\nu u_2|_{U_1}$$

where in the third equality we used the fact that the Cauchy data spaces $\mathcal{C}_{L_1}$ and $\mathcal{C}_{L_2}$ agree. So we have that $(u - u_2)$ is the solution of a homogenous elliptic equation in $\mathcal{O}$, vanishing on U_1 and with normal derivative vanishing on U_1, therefore by standard unique continuation

$$e^{i\alpha(p)}u_1(p) = u(p) = u_2(p), \ \forall p \in \mathcal{O}.$$

Now letting p converging to $p_2 \in \bar{\mathcal{O}}$ and using the fact that $u_1|_{\partial M_0} = u_2|_{\partial M_0} = f$, we have that $e^{i\alpha(p_2)}f(p_2) = f(p_2)$. And since $f(p_2) = 1$ by assumption, we deduce that $e^{i\alpha(p_2)} = 1$ and consequently

$$\alpha(p_2) = \int_{\gamma_1} X \in 2\pi\mathbb{Z}.$$

Now since the tangential component of X vanishes along the boundary and $\gamma_2 \subset \partial M_0$, we deduce that $\int_{\gamma_2} X = 0$ and conclude that

$$\int_\gamma X = \int_{\gamma_1} X \in 2\pi\mathbb{Z}$$

and the proof is complete.

Second Proof. Consider the functions F_{A_j} of (28), then by Proposition 5.1 and using $\bar{F}_{\bar{A}_j} = F_{A_j}^{-1}$, we know that $\Theta := F_{A_1}/F_{A_2} = \bar{F}_{\bar{A}_2}/\bar{F}_{\bar{A}_1}$ is a function mapping M_0 to the unit

circle $S^1 \subset \mathbb{C}$, and by (29), (30) we also have

$$\bar{\partial}\Theta/\Theta = i(A_1 - A_2), \quad \partial\Theta/\Theta = i(\bar{A}_1 - \bar{A}_2)$$

and thus $d\Theta/\Theta = i(X_1 - X_2)$. Let $\gamma, \mathcal{O}, p_1, p_2$ be like in the First Proof just above, we want to prove that $\int_\gamma d\Theta/\Theta \in 2i\pi\mathbb{Z}$. In the tubular neighbourhood $\mathcal{O}$ of γ, we define $G(p) := i\int_{p_1}^p (X_1 - X_2) = \int_{p_1}^p d\Theta/\Theta$, which is well defined in the simply connected domain $\mathcal{O}$ since $d(X_1 - X_2) = 0$, and $dG = d\Theta/\Theta$ in $\mathcal{O}$ with $G(p_1) = 0$. The function e^G takes value in S^1 and we have then proved that $e^G = \Theta$ since $\Theta(p_1) = 1$. Then, to conclude, it suffices to notice that $\Theta(p_2) = 1$ and so $G(p_2) \in 2\pi i\mathbb{Z}$. $\qquad\square$

7. Appendix

In this appendix, we gather a couple of technical results which are essentially already proved in the literature. First, we give a Carleman estimate

Lemma 7.1. *Let $X \in W^{2,\infty}(M_0, T^1 M_0)$ be real valued and $q \in W^{1,\infty}(M_0)$ complex valued and set $L = \nabla^{X*}\nabla^X + q$. Let ϕ be some harmonic real valued Morse function. Then there exists $C > 0$ such that for all large $k \in \mathbb{N}$ and all $u \in H^2(M_0)$*

$$||e^{-\phi/h} L e^{\phi/h} u||_{L^2}^2 \geq C(\frac{1}{h}||u||_{L^2}^2 + ||du||_{L^2}^2).$$

Proof. We observe that L is a first order perturbation of the Laplacian Δ_g on M_0, therefore the Carleman estimate obtained for Δ_g in Lemma 3.2 of [8] with a convexified weight ϕ_ϵ allows to absorb the first order terms by taking $\epsilon > 0$ small and this shows the result for L for the convexified weight. Then the argument of Proposition 3.1 in [8] shows that the desired estimate for L holds for the weight ϕ. $\qquad\square$

As a corollary

Corollary 7.2. *With the same assumptions as in Lemma 7.1, there exists $h_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$ and for all $f \in L^2(M_0)$ there exist a solution $u \in H^2(M_0)$ of $e^{-\phi/h} L e^{\phi/h} u = f$ with norms $||u||_{L^2} \leq C\sqrt{h}||f||_{L^2}$ and $||du||_{L^2} \leq C||f||_{L^2}$.*

Proof. The proof is a standard application of Lax-Milgram theorem (or Riesz representation) with the estimate of Lemma 7.1, in exactly the same way as Lemma 4.4 of [8]. $\qquad\square$

The boundary determination is standard, but since there seem to be no proof in the case of the system $D + V$ studied in section 3, we provide a sketch of proof, based essentially on the arguments of [9, Appendix].

Lemma 7.3. *With the notations of Section 3, let $V_1, V_2 \in W^{1,p}(M_0, \mathrm{End}(\Sigma))$ be two diagonal complex valued potentials endomorphisms of Σ. Assume that the Cauchy data spaces $\mathcal{C}_{V_1}$ of $D + V_1$ and $\mathcal{C}_{V_2}$ of $D + V_2$ agree, then $i^*_{\partial M_0} V_1 = i^*_{\partial M_0} V_2$.*

Proof. Let $H_0^1(M_0, \Sigma)$ be the completion of $C_0^\infty(\mathrm{int}(M_0))$ for the $H^1(M_0, \Sigma)$ topology, and $H^{-1}(M_0, \Sigma)$ the dual space. By standard arguments (for instance Carleman estimates and Lax-Milgram theorem), we have that for all $W \in L^2(M_0, \Sigma)$, there exists a $U \in H^1(M_0, \Sigma)$ such that $(D + V_j)U = W$ and $||U||_{L^2} \leq C||W||_{H^{-1}}$. Now let $A_h = (a_h, 0) \in C^\infty(M_0, \Sigma)$ with a_h a function supported in a chart $\mathcal{U}_p$ near a boundary point p and defined as follows: if $z = x + iy$ are complex coordinates near p with $\{y = 0\} = \partial M_0 \cap \mathcal{U}_p$, $M_0 \cap \mathcal{U}_p = \{y \geq 0\}$

and $p = \{z = 0\}$ in this chart, then we set $a_h(z) := \eta(zh^{-\alpha})e^{iz/h}$ with $\eta \in C_0^\infty(\mathbb{C})$ supported in the chart and equal to 1 at $z = 0$, and $\alpha \in (0, 1/2)$. Notice that $||A_h||_{L^2} = O(h^{\frac{1}{2}(1+\alpha)})$ and $DA_h = (0, h^{-\alpha}\partial_{\bar{z}}\eta(zh^{-\alpha})e^{iz/h})$ has L^2 norm $O(h^{\frac{1}{2}(1-\alpha)})$. Let $= (f_1, f_2 d\bar{z}) \in H_0^1(M_0, \Sigma)$, then one has (the metric g is of the form $e^{2\rho}|dz|^2$ for some smooth function ρ)

$$\langle DA_h, F \rangle = -ih^{-\alpha+1}\int_{\mathcal{U}_p} \partial_z(e^{iz/h})\langle \partial_{\bar{z}}\eta(zh^{-\alpha})d\bar{z}, f_2\rangle e^{2\rho}dxdy$$

and integrating by parts, we loose at most a power $h^{-\alpha}$ when the derivative hits $\eta(zh^{-\alpha})$. Using Cauchy-Schwartz, we deduce $|\langle DA_h, F\rangle| \le Ch^{\frac{3}{2}(1-\alpha)}||F||_{H^1}$ and therefore $||DA_h||_{H^{-1}} = O(h^{\frac{3}{2}(1-\alpha)})$. Adding a potential is harmless and thus $||(D+V)A_h||_{H^{-1}} = O(h^{\frac{3}{2}(1-\alpha)})$. Taking $\alpha = \frac{1}{3}$ for instance, we obtain that there exists $Z_h \in H^1(M_0, \Sigma)$, with norm $||Z_h||_{L^2} = O(h)$ such that $(D+V)F_h = 0$ with $F_h = A_h + Z_h$. We get these solutions F_h^1, F_h^2 for the diagonal potentials $V_1 = \begin{pmatrix} v_1 & 0 \\ 0 & v_1' \end{pmatrix}$ and $V_1 = \begin{pmatrix} v_2 & 0 \\ 0 & v_2' \end{pmatrix}$ and plug them into the integral identity (22), giving then by elementary computations (and using $v_j, v_j' \in W^{1,p}(M_0)$)

$$0 = \int_{M_0}(v_1 - v_2)\eta^2(zh^{-\frac{1}{3}})e^{-2y/h}e^{2\rho}dxdy + o(h^{\frac{4}{3}}) = C(v_1(p) - v_2(p))h^{\frac{4}{3}} + o(h^{\frac{4}{3}}).$$

for some $C \ne 0$. This proves that $v_1 = v_2$ at p, the same argument can be used to prove that $v_1' = v_2'$ at p and since p is arbitrarily chosen, we have achieved the proof of the Lemma. □

REFERENCES

[1] Y. Aharonov, D. Bohm, *Significance of electromagnetic potentials in the quantum theory.* Phys. Rev. (2) **115**, (1959) 485–491.

[2] R.M. Brown, M. Salo, *Identifiability at the boundary for first-order terms.* Appl. Anal. **85** (2006), no. 6-7, 735–749.

[3] A.L. Bukhgeim, *Recovering a potential from Cauchy data in the two-dimensional case.* J. Inverse Ill-Posed Probl. **16** (2008), no. 1, 19–33.

[4] M. Cerne, M. Flores, *Generalized Ahlfors functions*, Trans. Amer. Math. Soc. **359** (2007), no. 2, 671-686

[5] D. Dos Santos Ferreira, J. Sjöstrand, C. Kenig, G. Uhlmann, *Determining the magnetic field for the magnetic Schrödinger operator from partial Cauchy data*, Comm. Math. Phys. **271** (2007), no. 2, 467-488

[6] G. Eskin, H. Isozaki, S. O'Dell *Gauge Equivalence and Inverse Scattering for Aharonov-Bohm Effect*, arXiv:0809.3291.

[7] G. Eskin, J. Ralston, *Inverse Scattering Problem for the Schrödinger Equation with Magnetic Potential at a Fixed Energy*, Comm. Math. Phys. **173**, Number 1 (1995), 199-224.

[8] C. Guillarmou, L. Tzou, *Calderón inverse problem for Schrödinger operator on Riemann surfaces.* Proceedings of the Centre for Mathematics and its Applications, Vol **44** (2010) - proceedings of the AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis.

[9] C. Guillarmou, L. Tzou, *Calderón inverse Problem with partial data on Riemann Surfaces*, arXiv:0908.141.

[10] G. Henkin, V. Michel, *Inverse conductivity problem on Riemann surfaces.* J. Geom. Anal. **18** (2008), no. 4, 1033–1052.

[11] R.G. Novikov, G.M. Khenkin, *The ∂-equation in the multidimensional inverse scattering problem*, Russ. Math. Surv., Vol **42**, (1987), 109180.

[12] G. Henkin, R.G. Novikov, *On the reconstruction of conductivity of bordered two-dimensional surface in $\mathbb{R}^3$ from electrical currents measurements on its boundary*, arXiv:1003.4897. To appear in J. Geom. Anal.

[13] O.Y. Imanuvilov, G. Uhlmann, M. Yamamoto, *Global uniqueness from partial Cauchy data in two dimensions*, J. Amer. Math. Soc. **23** (2010), 655-691.

[14] O. Imanuvilov, G. Uhlmann, M. Yamamoto, *Partial Cauchy data for general second order operators in two dimensions.*, preprint 2010.

[15] H. Kang, G. Uhlmann, *Inverse problem for the Pauli Hamiltonian in two dimensions*, J. Fourier Anal. Appl. **10** (2004), no. 2, 201-215

[16] Y. Kurylev, M. Lassas, *Inverse problems and index formulae for Dirac operators*, Adv. Math. **221** (2009), no. 1, 170-216.

[17] R-Y. Lai *Global uniqueness for an inverse problem for the Pauli Hamiltionian*, preprint.

[18] D. Mc Duff, D. Salamon, *J-Holomorphic curves and symplectic topology*, AMS Colloquium publications Vol 52.

[19] G. Nakamura, Z.Q. Sun, G. Uhlmann, *Global identifiability for an inverse problem for the Schrödinger equation in a magnetic field.*, Math. Ann. **303** (1995), no. 3, 377-388

[20] G. Schwarz, *Hodge Decomposition - a Method for Solving Boundary Value Problems*, Springer-Verlag, Berlin Heidelberg, (1995)

[21] Z. Sun ,*An Inverse Boundary Value Problem for Schrödinger Operators with Vector Potentials*, Trans. Amer. Math. Soc. (1993), 953-969.

[22] Z. Sun, *An inverse boundary value problem for the Schrödinger operator with vector potentials in two dimensions*, Comm. Partial Differential Equations **18** (1993), no. 1-2, 83-124.

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