Exponential mixing of geodesic flows for geometrically finite manifolds with cusps

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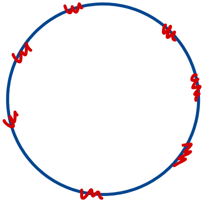
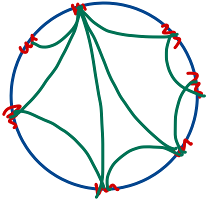
Setting and some basics

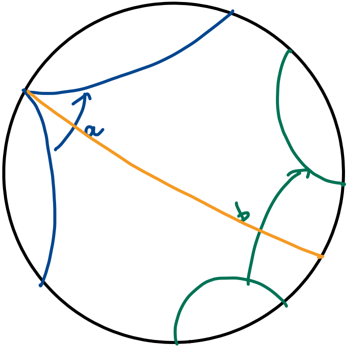
- $\mathbb{H}^n$ hyperbolic n-space
- $\Gamma < \mathrm{Isom}_+(\mathbb{H}^n)$ torsion-free discrete subgroup
- $\Lambda(\Gamma)$ limit set of Γ
  - the set of accumulation points of $\Gamma \cdot o$ ($o \in \mathbb{H}^n$)
  - $\subset \partial\mathbb{H}^n$
- $\mathrm{Hull}(\Gamma)$ smallest convex subset in $\mathbb{H}^n$ which contains all the geodesics connecting any two points in $\Lambda(\Gamma)$
- convex core C_Γ of $\Gamma = \Gamma\backslash\,\mathrm{Hull}(\Gamma) \subset \Gamma\backslash\mathbb{H}^n$

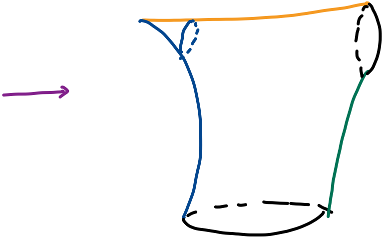
Γ is called geometrically finite if $\mathrm{Vol}(1$-nbhd of $C_\Gamma) < \infty$

Example

$\mathbb{H}^2$, $\quad Isom_+(\mathbb{H}^2) = PSL_2(\mathbb{R})$

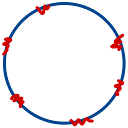


$\Gamma = \langle a, b \rangle$

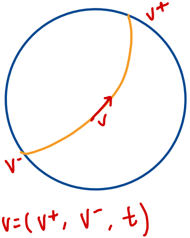


$\Gamma \backslash \mathbb{H}^2$

Assume Γ geometrically finite



- Patterson-Sullivan measures (PS measures) $\{\mu_x\}_{x\in\mathbb{H}^n}$ a family of finite measures on $\Lambda(\Gamma)$
- Bowen-Margulis-Sullivan measure on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ (Hopf parametrization)



$$(\partial\mathbb{H}^n \times \partial\mathbb{H}^n\backslash\Delta)\times\mathbb{R}\to\mathrm{T}^1(\mathbb{H}^n)$$

$T^1(\mathbb{H}^n)$: $d\tilde{m}^{BMS} = \dfrac{d\mu_x(v^+)\,d\mu_x(v^-)\,dt}{D(v^+,v^-)^{\delta_\Gamma}}$

μ_x is Γ-quasi-inv

$T^1(\Gamma\backslash\mathbb{H}^n)$: dm^{BMS}

δ_Γ: critical exponent of Γ

$D(v^+,v^-)$: visual distance

Thm(Sullivan, Otal-Peigné) m^{BMS} is the unique measure supported on the nonwandering set for the geodesic flow which has the maximal entropy.

Main result

- $\mathbb{H}^n$
- $\Gamma < \mathrm{Isom}_+(\mathbb{H}^n)$ geometrically finite with parabolic elements
- $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n) \circlearrowleft$ geodesic flow $\mathcal{G}_t$, m^{BMS}

Thm (L-Pan) There exists $\eta > 0$ such that for any $u, v \in C^1(\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n))$, we have

$$\int_{\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)} u(\mathcal{G}_t x)v(x)dm^{\mathrm{BMS}}(x)$$
$$= \int_{\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)} udm^{\mathrm{BMS}}(x) \int_{\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)} vdm^{\mathrm{BMS}} + O(\|u\|_{C^1}\|v\|_{C^1}e^{-\eta t}).$$

Some history

- Rudolph proved the geodesic flow is mixing
- Γ convex cocompact: Naud, Stoyanov, Sarkar-Winter; built on Dolgopyat's framework
- Γ geometrically finite and $\delta_\Gamma > \frac{n-1}{2}$: Mohammadi-Oh (frame flow), Edwards-Oh

Application: resonance free region

Lax-Phillips: Δ negative of the Laplace operator on $\Gamma\backslash\mathbb{H}^n$

- $\delta_\Gamma > \frac{n-1}{2}$: there are finitely many eigenvalues of Δ on $L^2(\Gamma\backslash\mathbb{H}^n)$ in the interval $[\delta_\Gamma(n-1-\delta_\Gamma), (n-1)^2/4) \to$ representation theory
- $\delta_\Gamma \leq \frac{n-1}{2}$: L^2-spectrum of Δ is purely continuous

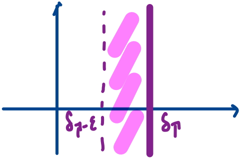
$\delta_\Gamma \leq \frac{n-1}{2}$: Resolvent $\mathcal{R}_s$ of Δ

$$\mathcal{R}_s = (\Delta - s(n-1-s))^{-1},$$

for $s \in \mathbb{C}$ with $\Re s > \frac{n-1}{2}$

- $\mathcal{R}_s$ has a meromorphic continuation to $\mathbb{C}$: convex-compact (Mazzeo-Melrose), geometrically finite (Guillarmou-Mazzeo)

- (Patterson) $\Gamma(s - \frac{n-1}{2} + 1)\mathcal{R}_s$ has a simple pole at δ_Γ and no further poles on $\operatorname{Re} s = \delta_\Gamma$



- Using exponential mixing of the geodesic flow, $\mathcal{R}_s$ has no poles in the strip $\delta_\Gamma - \epsilon < \operatorname{Re} s < \delta_\Gamma$
  - Effective orbit counting $\#\{\gamma \in \Gamma : d(x, \gamma y) < T\}$ (mixing $\rightarrow$ orbit counting: Margulis; Roblin (geo. fin.), Oh-Winter+Mohammadi-Oh)
  - Meromorphic extension of the Poincaré series $P(s, x, y) = \sum_\gamma e^{-sd(x,\gamma y)}$
  - (Guillarmou-Mazzeo) relate $P(s, x, y)$ with $\mathcal{R}_s$

Ideas of the proof

- Code the geodesic flow
- Prove a Dolgopyat-like spectral estimate for the corresponding transfer operator: Dolgopyat, Baladi-Vallée, Avila-Gouëzel-Yoccoz, Araújo-Melbourne, Naud, Stoyanov (non-wandering set of the geodesic flow is a fractal set: non-integrability condition; how to get the contraction of transfer operator)

Coding

Γ geometrically finite

- $\Lambda(\Gamma) = \Lambda_r \sqcup \Lambda_{bp}$
- A parabolic fixed point $\xi \in \Lambda(\Gamma)$ is said to be bounded if

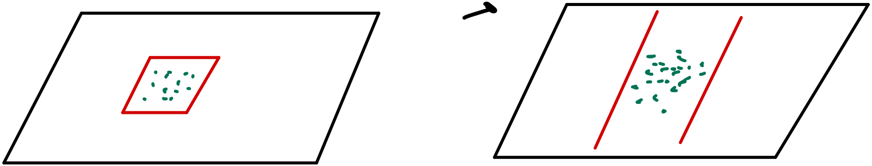
$$\mathrm{Stab}_\Gamma(\xi)\backslash\Lambda(\Gamma) - \{\xi\}$$

  is compact.

$\mathbb{H}^3$, ∞ bounded parabolic fixed pt, $\Gamma_\infty = \mathrm{Stab}_\Gamma(\infty)$

(i) rank 2, $\Gamma_\infty \cong \mathbb{Z}^2$ up to a finite index subgp

(ii) rank 1, $\Gamma_\infty \cong \mathbb{Z}$

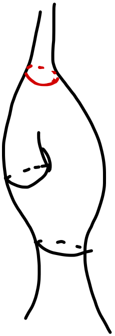


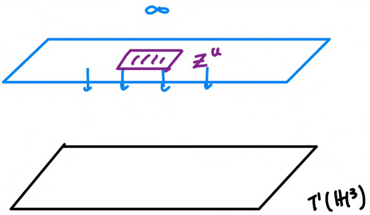
Coding

- $\mathbb{H}^3$
- $\Gamma \backslash \mathbb{H}^3$ has one full rank cusp
- ∞: a representative
- Intuitive idea: Poincaré section Λ for the geodesic flow

suspension flow

suspension space $\Lambda \times \mathbb{R}/\langle r \rangle \to \mathrm{T}^1(\Gamma \backslash \mathbb{H}^3)$ ↺ geodesic flow.





- unstable horosphere based at ∞
- $\Gamma_\infty = \mathrm{Stab}_\Gamma(\infty)$

- Poincaré section Λ: thickening of Z^u in the stable direction
- Reduction:

$$(Z^u \times \mathbb{R}/\langle R \rangle, \text{suspension flow}, \nu^R)$$

Avila-Gouëzel-Yoccoz, Araújo-Melbourne ↑

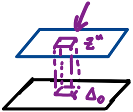
$$(\Lambda \times \mathbb{R}/\langle R \rangle, \text{suspension flow}, \hat{\nu}^R)$$

factor ↓ Φ: $\Phi_* \hat{\nu}^R = m^{\mathrm{BMS}}$, $\Phi \circ$ suspension flow $= \mathcal{G}_t \circ \Phi$

$$(\mathrm{T}^1(\Gamma \backslash \mathbb{H}^3), \mathcal{G}_t, m^{\mathrm{BMS}})$$

Proposition There are constants $C > 0$, $\lambda \in (0,1)$, a countable collection of disjoint, open subsets $\Delta_j \subset \Delta_0$ and an expanding map T defined on the union $\sqcup_j \Delta_j$ such that:

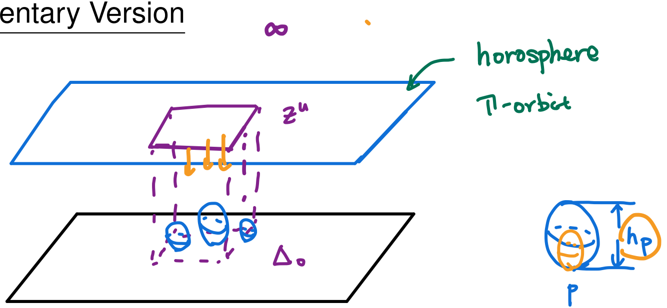
1. $\sum_j \mu(\Delta_j) = \mu(\Delta_0)$.
2. For each j, there exists $\gamma_j \in \Gamma$ such that $\Delta_j = \gamma_j \Delta_0$ and $T|_{\Delta_j} = \gamma_j^{-1}$.
3. For each γ_j, it is a uniform contraction: $|\gamma_j'(x)| \le \lambda$ for all $x \in \Delta_0$.
4. For each γ_j, $|(\log|\gamma_j'|)'(x)|_\infty < C$ for all $x \in \Delta_0$.



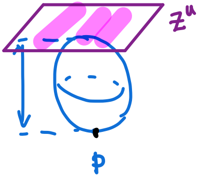
5. (Exponential tail) Let R be the roof function given by $R(x) = \log|T'(x)|$ for $x \in \Delta_0$. There exists $\epsilon_0 > 0$ such that

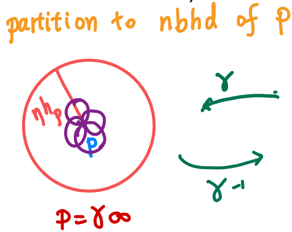
$$\int_{\Delta_0} e^{\epsilon_0 R} d\mu < \infty.$$

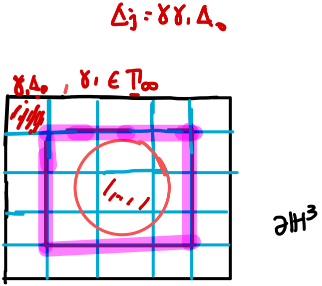
Elementary Version



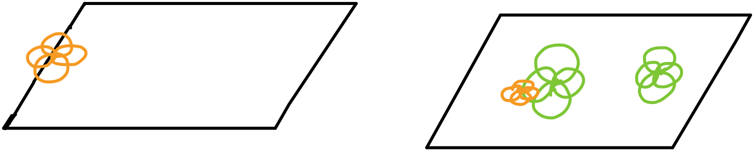
"nice" partition to Z^u (Lai-Sang Young, Burns-Masur-Matheus-Wilkinson)







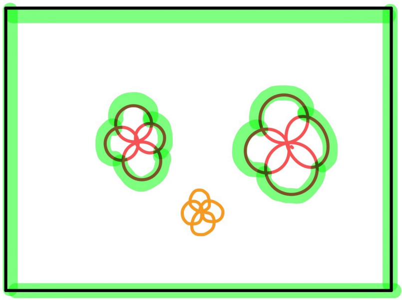
Issue about the boundary



Not be too greedy, need to wait for
the right time to eat the “flower”

Refined Version

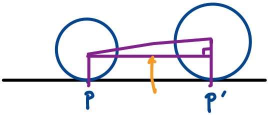
- $\Omega_0 = Z^u$, Ω_n
- $\Omega_{n+1} = \Omega_n - \bigcup_{p \in P_{n+1}}$ 
- $P_{n+1} = \{p$ parabolic fixed pts in $\Delta_0 : \eta h_p \in (h_{n+1}, h_n]$, $B(p, \eta h_p) \subset \Omega_n$, $d(p, \partial\Omega_n) > h_n/4\eta\}$

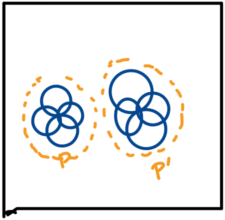


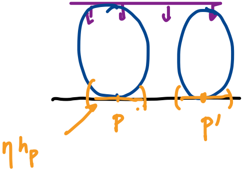
Separation between parabolic fixed points

<u>Lemma</u> For any two different parabolic fixed points p, p', we have

$$d(p, p') > \sqrt{h_p h_{p'}}$$







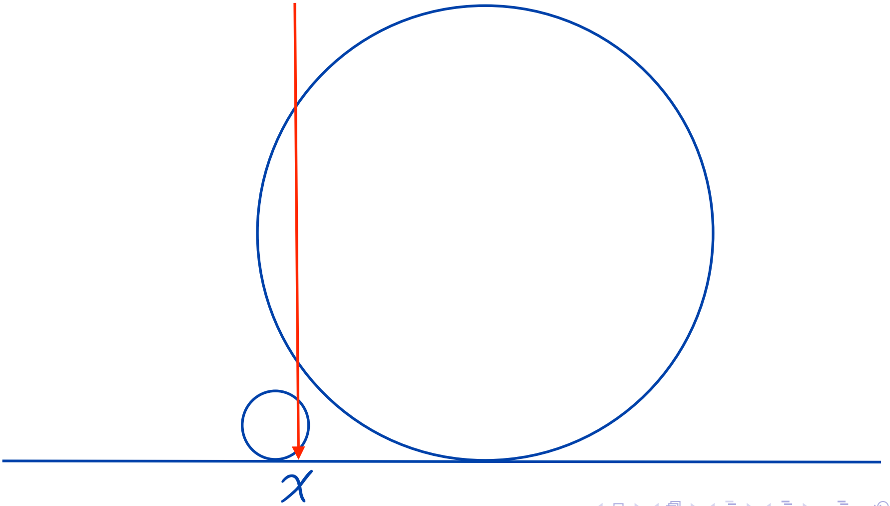
$h_p \approx h_{p'}$

$\geq \sqrt{h_p h_{p'}} - \eta h_p - \eta h_{p'}$

$\approx h_p - 2\eta h_p$

Recurrence of the geodesic flow

For a point x in $\Lambda_\Gamma \cap \Delta_0$, you can always find a flower containing it, even after zooming in.



There are three main ingredients in the construction of the coding:

- recurrence of the geodesic flow
- separation between parabolic fixed points
- doubling and friendliness of Patterson-Sullivan measure

Thank you!