RESONANCES ON SOME GEOMETRICALLY FINITE HYPERBOLIC MANIFOLDS

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Abstract. We first prove the meromorphic extension to $\mathbb{C}$ for the resolvent of the Laplacian on a class of geometrically finite hyperbolic manifolds with infinite volume and we give a polynomial bound on the number of resonances. This class notably contains the quotients $\Gamma\backslash\mathbb{H}^{n+1}$ with rational non-maximal rank cusps previously studied by Froese-Hislop-Perry.

1. Introduction

The purpose of this work is to prove the meromorphic extension of the resolvent as well as a polynomial bound of resonances for the Laplacian on some geometrically finite hyperbolic manifolds $\Gamma\backslash\mathbb{H}^{n+1}$ whose non-maximal rank cusps are 'rational'. This condition is essentially equivalent to suppose that the parabolic subgroups are conjugate to lattices of translations acting on $\mathbb{R}^n$.

Scattering theory, meromorphic continuation of the resolvent for the Laplacian, Eisenstein series and distribution of resonances have been deeply studied on geometrically finite hyperbolic surfaces (see [23, 18, 2, 9, 17, 10, 12, 24]). New geometric difficulties appear in higher dimension, notably the fact that a geometrically finite quotient $\Gamma\backslash\mathbb{H}^{n+1}$ is not, in general, a compact perturbation of explicitly computable models and the method used for surfaces can not be applied. However, when the manifold has a nice structure near infinity, say when it conformally compactifies, Mazzeo and Melrose [15] have found a powerful method to prove the meromorphic continuation of the resolvent and to describe it in details. Roughly, this conformal hypothesis is equivalent to taking groups without parabolic elements. Perry [20, 21], Joshi-Sa Barreto [13], Guillopé-Zworski [11], and Patterson-Perry [19] have studied the scattering matrix, Eisenstein series and distribution of resonances for these classes of manifolds and it is not difficult to see that parabolic elements with maximal rank can be added to the group without significative difficulties. Nevertheless, when the group has parabolic elements with non-maximal rank, some infinite volume cusps appear and most of those cases remain quite mysterious in general, at least in the point of view of the scattering theory and the meromorphic continuation of the resolvent. It is worth noting that the analysis of the spectrum of the Laplacian on forms and Hodge theory has been written down by Mazzeo and Phillips [16] on geometrically finite hyperbolic manifolds and later, Froese-Hislop-Perry [3, 4] have studied scattering theory and Eisenstein functions on these manifolds in dimension 3. Perry [22] also proved the meromorphic continuation of the resolvent in a small strip near the critical line for a class of manifolds with rational non-maximal rank cusps in all dimensions. Last but not least, Bunke and Olbrich [1] dealt with all cases of geometrically finite hyperbolic manifold using a very different approach, in particular they are able to extend the Eisenstein functions and scattering operator but they did not study the resolvent explicitly and have no result about the distribution of resonances.

In this work, we hope to give a rather simple way to prove the meromorphic continuation of the resolvent on a class of geometrically finite hyperbolic manifolds which allows to bound the number of resonances (poles of the resolvent) in a disc of radius R of $\mathbb{C}$. Our method is very

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similar from the approaches of Froese-Hislop-Perry [4], Perry [22] or Guillopé-Zworski [11] in the sense that it provides a precise analysis of the resolvent. Actually, the pseudo-differential structure of the resolvent and the scattering operator will be described in more details in [8], the purpose of the present work being essentially the estimate of the resonances distribution.

We thus consider an infinite volume hyperbolic manifold $\Gamma\backslash\mathbb{H}^{n+1}$ where Γ is a discrete group of isometries of $\mathbb{H}^{n+1}$ which admits a fundamental domain with finitely many sides (the manifold is said geometrically finite) and such that each parabolic subgroup of Γ is conjugate to a lattice of translations in $\mathbb{R}^n$. This is exactly the class of manifolds studied by Perry [22] and, as noticed in this work, it covers the case when no parabolic subgroup contains irrational rotations, possibly by passing to a finite cover. The manifold $X := \Gamma\backslash\mathbb{H}^{n+1}$ equipped with the hyperbolic metric is then complete and the spectrum of the Laplacian Δ_X splits into continuous spectrum $[\frac{n^2}{4},\infty)$ and a finite number of eigenvalues included in $(0,\frac{n^2}{4})$. Perry [22] proved that the modified resolvent

$$R(\lambda) := (\Delta_X - \lambda(n-\lambda))^{-1}$$

extends from $\{\Re(\lambda) > \frac{n}{2}\}$ to $\{\Re(\lambda) > \frac{n-1}{2}\}$ meromorphically with poles of finite multiplicity (i.e. the rank of the polar part in the Laurent expansion at each pole is finite) in weighted L^2 spaces.

We first show the

Theorem 1.1. *Let $X = \Gamma\backslash\mathbb{H}^{n+1}$ be a geometrically finite hyperbolic manifold with infinite volume and such that each parabolic subgroup of Γ is conjugate to a lattice of translations in $\mathbb{R}^n$. Then the modified resolvent for the Laplacian*

$$R(\lambda) = (\Delta_X - \lambda(n-\lambda))^{-1} : L^2_{comp}(X) \to H^2_{loc}(X)$$

extends from $\{\Re(\lambda) > \frac{n}{2}\}$ to $\mathbb{C}$ meromorphically with poles of finite multiplicity.

The poles of the resolvent are called resonances and are spectral data which correspond in a sense to the eigenvalues of the compact cases. They are closely related to the zeros and poles of Selberg's zeta function. Our second result involves the asymptotic distribution of resonances:

Theorem 1.2. *With the assumptions of Theorem 1.1, the number of resonances $N(R)$ (counted with multiplicities) contained in the complex disc $D(\frac{n}{2},R)$ or radius R satisfies*

$$N(R) \leq CR^{n+2} + C$$

for some constant $C > 0$.

Notice that in a less general case, the non-optimal power $n+2 = \dim(X)+1$ already appeared in the work of Guillopé-Zworski [11]. We also emphasize that our method extends to compact perturbations of the hyperbolic manifolds considered in Theorems 1.1 and 1.2.

These theorems are proved by using a parametrix construction of the resolvent. The main difficulty of the construction is that X can not be splitted into one compact set and one neighbourhood of infinity where a model resolvent is explicitly known. As for convex co-compact hyperbolic manifolds [20, 21, 11, 19], we use a finite covering of a neighbourhood of the infinity $X(\infty)$ of X by several model neighbourhoods with some exlicit formula for the resolvent of the laplacian on these models. Following the method of Perry [22], these model resolvents, cut-off by functions forming a partition of unity near infinity, give a first approximation of $R(\lambda)$ but it is not sufficient to obtain a compact error. The important property we then use is that the error terms resemble those which appear in the convex co-compact manifolds (and more generally on asymptotically hyperbolic ones), in the sense that their Schwartz kernels are smooth in $X \times X$ and have a polyhomogeneous asymptotic expansion at $X(\infty) \times X$ with support on the left factor which does not intersect the cusp points. Consequently the indicial equation used in [15, 11] allows to refine the parametrix of $R(\lambda)$ to get a residual term which is compact on all $\rho^N L^2(X)$

($N > 0$) with ρ a weight function converging to 0 at infinity.

We will use the following notations: $\langle z\rangle := (1+|z|^2)^{\frac{1}{2}}$; if A is a compact operator on a Hilbert space, $|A| := (A^*A)^{\frac{1}{2}}$ and $(\mu_l(A))_l$ are the eigenvalues of $|A|$ (called the singular values of A); C will denote a large constant, not necessarily always the same. We shall also often identify operators with their Schwartz kernels.

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2. Geometry of the manifold

This section is strongly inspired by Perry's paper [22], the reader can refer to it for more details (see also [16, 3, 4]). We consider a hyperbolic quotient $X = \Gamma\backslash\mathbb{H}^{n+1}$ where Γ is a geometrically finite discrete group of hyperbolic isometries with no elliptic elements, and such that all parabolic subgroups are conjugate to lattices of translations acting on $\mathbb{R}^n$. In this case there exists a compact K of X such that $X \setminus K$ is covered by a finite number of charts isometric to either a regular neighbourhood (M_r, g_r) or a rank-k cusp neighbourhood (M_k, g_k) (with $1 \leq k \leq n$), where

$$M_r = \{(x,y) \in (0,\infty)\times\mathbb{R}^n; x^2+|y|^2 < 1\}, \quad g_r = x^{-2}(dx^2+dy^2),$$

$$M_k = \{(x,y,z) \in (0,\infty)\times\mathbb{R}^{n-k}\times T^k; x^2+|y|^2 > 1\}, \quad g_k = x^{-2}(dx^2+dy^2+dz^2)$$

for $k < n$ with (T^k, dz^2) a k-dimensional flat torus and

$$M_n = \{(x,z) \in (0,\infty)\times T^n; x > 1\}, \quad g_n = x^{-2}(dx^2+dz^2)$$

with (T^n, dz^2) a n-dimensional flat torus. For notational simplicity, we will make as if there was one neighbourhood of each type. There exist some smooth functions $\chi^i, \chi^r, \chi^1, \dots, \chi^n$ on respectively $X, M_r, M_1, \dots, M_n$ which, through the isometric charts $I_r, I_1, \dots, I_n$, satisfy

$$I_r^*\chi^r + \sum_{k=1}^{n} I_k^*\chi^k + \chi^i = 1 \tag{2.1}$$

with χ^i having compact support in X.

We will also use cutoff functions in what follows, thus we define

$$\phi, \phi_L \in C_0^\infty([0,2)), \quad \phi_L = 1 \text{ on } [0,1], \quad \phi = 1 \text{ on } \mathrm{supp}(\phi_L). \tag{2.2}$$

2.1. The non-maximal rank cusps neighbourhoods.

Let $X_k = \Gamma_k\backslash\mathbb{H}^{n+1}$ be the quotient of $\mathbb{H}^{n+1}$ by a rank-k parabolic subgroup Γ_k of Γ which fixes a single point at infinity of $\mathbb{H}^{n+1}$. Modulo conjugation by a hyperbolic isometry, one can suppose that the fixed point is the point at infinity of $\mathbb{H}^{n+1}$ in the half-space model $(0,\infty)\times\mathbb{R}^n$. Γ_k can then be considered as a rank-k lattice of translations acting on $\mathbb{R}^n$ (actually on a subspace of $\mathbb{R}^n$ isomorphic to $\mathbb{R}^k$), therefore it is the image of the lattice $\mathbb{Z}^k$ by a map $A_k \in GL_k(\mathbb{R})$ and the flat torus $T^k := \Gamma_k\backslash\mathbb{R}^k$ is well defined. X_k is isometric to $\mathbb{R}^+_x \times \mathbb{R}^{n-k}_y \times T^k_z$ equipped with the metric

$$g_k = \frac{dx^2+dy^2+dz^2}{x^2}$$

dz^2 being the flat metric on a k-dimensional torus T^k. M_k is the subset of X_k with $x^2+|y|^2 > 1$ and we will often consider $\mathbb{R}^+\times\mathbb{R}^{n-k}$ as the $n-k+1$-dimensional hyperbolic space $\mathbb{H}^{n-k+1}$. Hence,

$$\rho_k(x,y,z) = \rho_k(x,y) := \frac{x}{|y|^2+x^2+1} = (2\cosh(d_{\mathbb{H}^{n-k+1}}(x,y;1,0)))^{-1} \tag{2.3}$$

will be a natural weight function on M_k.

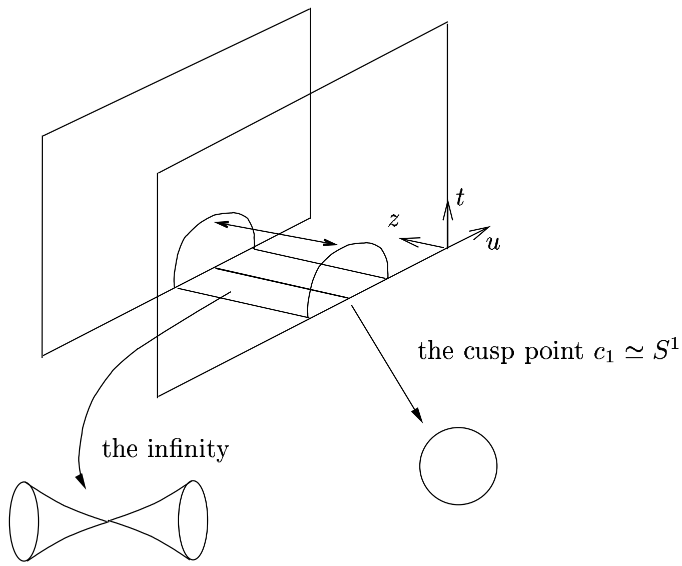

FIGURE 1. The compactified neighbourhood of the cusp point c_1 in dimension 3

Following Perry [22], we remark that the change of coordinates

$$t := \frac{x}{x^2+|y|^2}, \quad u := \frac{-y}{x^2+|y|^2} \tag{2.4}$$

is an isometry from M_k to

$$\{(t,u,z) \in (0,\infty) \times \mathbb{R}^{n-k} \times T^k; t^2+|u|^2 < 1\}$$

equipped with the metric

$$\frac{dt^2+du^2+(t^2+|u|^2)^2dz^2}{t^2} \tag{2.5}$$

and $\rho_k(t,u) = \rho_k(x,y)$. These coordinates compactify M_k and the infinity of X in this neighbourhood is then given by $\{\rho_k = 0\}$ or equivalently $\{t = 0\}$. The 'cusp point' here becomes a torus $c_k := \{t = u = 0\} \simeq T^k$.

Without loss of generality and possibly by adding regular charts in the covering of a neighbourhood of the infinity of X, we can choose the cut-off function for the rank-k cusp neighbourhood such that $\chi^k(x,y) := 1 - \phi(x)\psi^k(y)$ with $\psi^k(y) \in C_0^\infty(|y|_{\mathbb{R}^{n-k}} < 2)$ and $\psi^k(y) = 1$ on $\{|y| \leq 1\}$. We also set $\psi_L^k \in C_0^\infty(|y| < 2)$ such that $\psi^k = 1$ on $\mathrm{supp}(\psi_L^k)$, $\psi_L^k = 1$ on $\{|y| \leq 1\}$ and we define $\chi_L^k(x,y) := 1 - \phi_L(x)\psi_L^k(y)$, which satisfies $\chi_L^k = 1$ on $\mathrm{supp}(\chi^k)$.

2.2. **The maximal rank cusps.** Let $X_n = \Gamma_n\backslash\mathbb{H}^{n+1}$ be the quotient of $\mathbb{H}^{n+1}$ by a rank-n parabolic group subgroup Γ_n of Γ, it is then isometric to $\mathbb{R}_x^+ \times T_z^n$ equipped with the metric

$$g_n = \frac{dx^2+dz^2}{x^2}$$

dz^2 being the flat metric on the n-dimensional torus $T^n = \Gamma_n\backslash\mathbb{R}^n$. As before M_n is the subset of X_n with $x > 1$ and the weight function we choose on M_n is $\rho_n = x^{-1}$. Taking $u = x^{-1}$, M_n is also isometric to

$$M_n = \{(u,z) \in (0,\infty) \times T^n; u < 1\}$$

with the metric

$$u^{-2}(du^2 + u^4dz^2)$$

and $\rho_n = u$. The infinity of X in this neighbourhood is given by the 'cusp point' $c_n := \{\rho_n = 0\} \simeq T^n$ which is a torus.

Here χ^n can be taken depending only on x, for example $\chi^n(x,z) = 1 - \phi(x)$ and we set $\chi^n_L := 1 - \phi_L(x)$, hence $\chi^n_L = 1$ on $\text{supp}(\chi^n)$.

2.3. **The regular neighbourhoods.** The regular neighbourhoods are those treated in [11] and will not be discussed in details. We just recall that the weight function here is $\rho_r := x$ and that the infinity of X in this neighbourhood is given by $\rho_r = 0$.

The function χ^r can be chosen so that $\chi^r = \phi_L(\frac{x}{\epsilon})\psi^r(y)$ with $\psi^r \in C_0^\infty(|y| < 1)$ and $\epsilon > 0$ small. Let $\psi^r_L \in C_0^\infty(|y| < 1)$ with $\psi^r_L = 1$ on $\text{supp}(\psi^r)$ and $\chi^r_L(x,y) = \phi_L(\frac{x}{2\epsilon})\psi^r_L(y)$, we thus have $\chi^r_L = 1$ on $\text{supp}(\chi^r)$.

2.4. **Weight function, compactification.** We can then define as weight function the following

$$\rho := \chi^i + I_r^*(\chi^r\rho_r) + \sum_{k=1}^{n} I_k^*(\chi^k\rho_k). \tag{2.6}$$

X can be compactified in a compact manifold with boundary $\bar{X}$ such that ρ is a boundary defining function of $\bar{X}$. The boundary can be decomposed in the form $\partial\bar{X} = \bar{B} \sqcup T^n$ with $\bar{B}$ a smooth compact manifold and T^n the torus coming from the maximal rank cusp. From the discussion above, we see that the metric on X can be expressed by

$$g = \frac{H}{\rho^2}$$

with H a smooth non-negative symmetric 2-tensor on $\bar{X}$ which degenerates at cusps points $(c_k)_{k=1,\dots,n}$. Let $B := \bar{B} \setminus \{c_1, \dots, c_n\}$, the restriction $H|_B$ is then a smooth metric on the non-compact manifold B. In a sense B will be the geometric infinity where the scattering can occur.

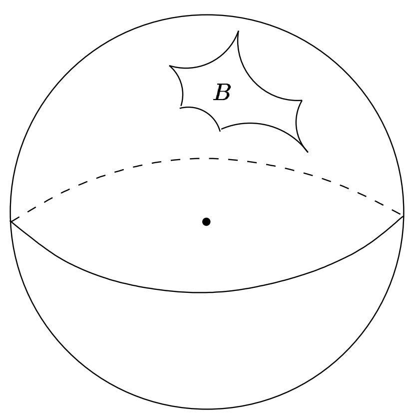

FIGURE 2. A fundamental domain in S^n for the infinity B of $X = \Gamma\backslash\mathbb{H}^{n+1}$

We need to construct a new weight function depending on a small paramater $\delta > 0$, which is equal to 1 on a big compact (depending on δ) and to ρ near infinity.

For $\alpha = 1, \dots, n, r$, we have $\rho_\alpha(w) = I_{\alpha *}\rho(w)$ when $\chi^\alpha(w) = 1$ and there exists $C > 1$ such that

$$C^{-1}\rho_\alpha \leq I_{\alpha *}\rho \leq C\rho_\alpha \text{ on } \text{supp}(\chi^\alpha). \tag{2.7}$$

Let $1 > \delta > 0$ be small, we define the new weight function on X

$$\rho_\delta(w) := (1 - \phi_L(4\rho(w)\delta^{-1})) + \rho(w)\phi_L(4\rho(w)\delta^{-1})$$

and using that $\rho \leq 1$ we can check that $\rho_\delta = \rho$ on $\{\rho \leq \frac{\delta}{4}\}$, $\rho_\delta = 1$ on $\{\rho > \frac{\delta}{2}\}$ and $\rho_\delta \geq \rho$ everywhere, hence

$$1 \leq \frac{\rho_\delta}{\rho} \leq \frac{4}{\delta}. \tag{2.8}$$

We also define the function

$$\chi_\delta^\alpha(w) := \phi_L(\rho(w)\delta^{-1})\chi^\alpha(w).$$

Thus $\chi_L^\alpha = 1$ on the support of χ_δ^α and

$$I_{\alpha *}\rho \leq 2\delta \text{ on } \mathrm{supp}(\chi_\delta^\alpha). \tag{2.9}$$

In view of (2.1), we deduce

$$I_r^* \chi_\delta^r + \sum_{k=1}^n I_k^* \chi_\delta^k + \chi_\delta^i = 1$$

with

$$\chi_\delta^i(w) := 1 - \phi_L(\rho(w)\delta^{-1}) + \chi^i(w)\phi_L(\rho(w)\delta^{-1})$$

having a compact support included in $\{\rho \geq \delta\}$ for δ sufficiently small. Finally let

$$\chi_{L,\delta}^i(w) := 1 - \phi_L(2\rho(w)\delta^{-1})$$

which is a smooth function with support in $\{\rho \geq \frac{\delta}{2}\}$ and $\chi_{L,\delta}^i = 1$ on $\mathrm{supp}(\chi_\delta^i)$, thus $\rho_\delta = 1$ on $\mathrm{supp}(\chi_{L,\delta}^i)$.

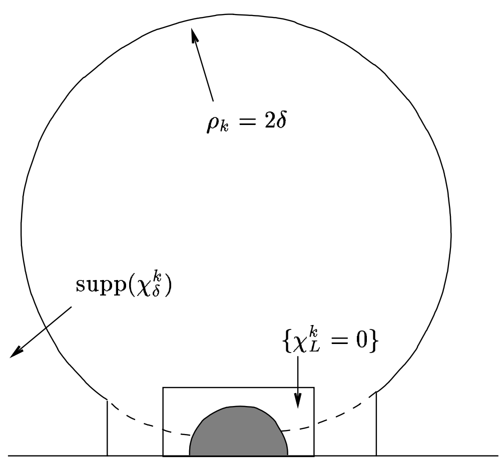

FIGURE 3. The support of cut-off functions in M_k

Later, this parameter δ will be chosen small enough to insure that our residual terms in the parametrix construction have a small norm for $\Re(\lambda) \gg \frac{n}{2}$. This is, in a sense, the idea used in [11]. For what follows and for simplicity, we will often write ρ_δ instead of $I_{k*}\rho_\delta$.

3. Parametrix and estimates

3.1. The non-maximal rank cusps. We recall that $X_k = \Gamma_k \backslash \mathbb{H}^{n+1}$ with Γ_k the image of the lattice $\mathbb{Z}^k$ by a map $A_k \in GL_k(\mathbb{R})$. By using a Fourier decomposition on the torus $T^k = \Gamma_k \backslash \mathbb{R}^k$ and conjugating by $x^{\frac{k}{2}}$, the operator $\Delta_{X_k} - \lambda(n-\lambda)$ acts on

$$L^2(X_k) = \bigoplus_{m \in \mathbb{Z}^k} \mathcal{H}_m, \quad \mathcal{H}_m \simeq L^2(\mathbb{R}_y^{n-k} \times \mathbb{R}_x^+, x^{-(n-k+1)}dydx) = L^2(\mathbb{H}^{n-k+1})$$

as a family of operators

$$P_m(\lambda) := -x^2\partial_x^2 + (n-k-1)x\partial_x + x^2(\Delta_y + |\omega_m|^2) - s(n-k-s)$$

where $\omega_m = 2\pi {}^t(A_k^{-1})m$ for $m \in \mathbb{Z}^k$ are the eigenvalues of the Laplacian on T_k (with eigenfunction $e^{i\omega_m.z}$) and $s := \lambda - \frac{k}{2}$ is a shifted spectral parameter. The resolvent $R_{X_k}(\lambda) = (\Delta_{X_k} - \lambda(n-\lambda))^{-1}$ for the Laplacian on X_k is computed by Perry [22] for $\Re(\lambda) > \frac{n}{2}$

$$R_{X_k}(\lambda) = \bigoplus_{m\in\mathbb{Z}^k} R_m(\lambda) \text{ on } L^2(X_k) = \bigoplus_{m\in\mathbb{Z}^k} \mathcal{H}_m \tag{3.1}$$

with

$$R_m(\lambda; x, y, x', y') = |A_k|^{-\frac{1}{2}}(xx')^{-\frac{k}{2}} \int_{\mathbb{R}^k} R_{\mathbb{H}^{n+1}}(\lambda; x, y, z; x', y', 0)e^{i\omega_m.z}dz \tag{3.2}$$

where $R_{\mathbb{H}^{n+1}}(\lambda) = (\Delta_{\mathbb{H}^{n+1}} - \lambda(n-\lambda))^{-1}$ is the resolvent for the Laplacian on $\mathbb{H}^{n+1}$ and $|A_k| := |\det(A_k)|$. We set

$$r = (|y-y'|^2 + x^2 + x'^2)^{\frac{1}{2}}, \quad d = \frac{xx'}{r^2}, \quad \tau = \frac{xx'}{r^2 + |z|^2} = d(1 + \frac{|z|^2}{r^2})^{-1} \tag{3.3}$$

and recall (see e.g. [11], [22]) that the resolvent on $\mathbb{H}^{n+1}$ can be written for all $J \in \mathbb{N} \cup \infty$

$$R_{\mathbb{H}^{n+1}}(\lambda; x, y, z; x', y', 0) = \tau^\lambda \sum_{j=0}^{J-1} \alpha_{j,n}(\lambda)\tau^{2j} + \tau^{\lambda+2J}G_{J,n}(\lambda, \tau) \tag{3.4}$$

$$\alpha_{j,n}(\lambda) := \frac{2^{-1}\pi^{-\frac{n}{2}}\Gamma(\lambda+2j)}{\Gamma(\lambda - \frac{n}{2} + 1 + j)\Gamma(j+1)}$$

with $G_{J,n}(\lambda, \tau)$ a smooth function in $\tau \in [0, \frac{1}{2})$ with a conormal singularity at $\tau = \frac{1}{2}$ and $G_{\infty,n}(\lambda, \tau) = 0$. Note that the sum (3.4) converges locally uniformly in $\tau \in [0, \frac{1}{2})$ if $J = \infty$. From (3.2) and (3.4) it is easy to see, by the change of variable $w = z/r$, that for $m \neq 0$ and setting $s := \lambda - \frac{k}{2}$

$$R_m(\lambda) = d^s \sum_{j=0}^{J-1} \alpha_{j,n}(\lambda)d^{2j}F_{j,\lambda}(r\omega_m) + d^{s+2J}\int_{\mathbb{R}^k} e^{-ir\omega_m.z}\frac{G_{J,n}(\lambda, d(1+|z|^2)^{-1})}{(1+|z|^2)^{\lambda+2J}}dz \tag{3.5}$$

(3.6)

$$F_{j,\lambda}(u) = |A_k|^{-\frac{1}{2}}\int_{\mathbb{R}^k} e^{-iu.w}(1+|w|^2)^{-\lambda-2j}dw = |A_k|^{-\frac{1}{2}}\frac{2^{-\lambda-2j+1}(2\pi)^{\frac{k}{2}}}{\Gamma(\lambda+2j)}|u|^{s+2j}K_{-s-2j}(|u|)$$

when $\Re(\lambda) > \frac{n}{2}$ (see e.g. [5] for the last formula), $K_s(z)$ being the Bessel function defined by

$$K_s(z) := \int_0^\infty \cosh(st)e^{-z\cosh(t)}dt, \quad z > 0. \tag{3.7}$$

It is easy to see (and will be studied later) that the sum (3.5) with $J = \infty$ converges uniformly for $r > 0$ and $d \in [0, \frac{1}{2})$. When $m = 0$, $R_0(\lambda)$ is the shifted Green kernel of the Laplacian on $\mathbb{H}^{n-k+1}$, that is

$$R_0(\lambda) = d^s \sum_{j=0}^{J-1} \alpha_{j,n-k}(s)d^{2j} + d^{s+2J}G_{J,n-k}(\lambda, d), \quad s = \lambda - \frac{k}{2}. \tag{3.8}$$

For simplicity, we will write (3.8) under the form (3.5) by defining

$$F_{j,\lambda}(0) := \frac{\alpha_{j,n-k}(s)}{\alpha_{j,n}(\lambda)}.$$

The representations (3.5) and (3.8) give a meromorphic extension of $R_m(\lambda)$ to $\mathbb{C}$, with poles on $\frac{k}{2} - \mathbb{N}_0$ of finite rank which only come from the case $m = 0$ when $n - k + 1$ is even. The continuity property of the extented operators on weighted L^2 spaces will be checked later.

We are now going to find a parametrix for $\Delta_X - \lambda(n-\lambda)$ on the neighbourhood $I_k^{-1}(M_k)$ of our manifold X, by using the explicit formulae given before for the cusp X_k. The constructions are very similar to those of Guillopé-Zworski [11] for the conformally compact ends. After the Fourier decomposition, the construction of a parametrix will be obtained on each $\mathcal{H}_m$ from the model resolvent and an iterative process as in [11] but for a $n-k+1$ dimensional hyperbolic Laplacian with the potential $x^2|\omega_m|^2$.

Proposition 3.1. *For $N \in \mathbb{N}$ large, there exist some bounded operators*

$$\mathcal{E}_N^k(\lambda) : \rho_\delta^N L^2(M_k) \to \rho_\delta^{-N} L^2(M_k)$$

$$\mathcal{K}_N^k(\lambda) : \rho_\delta^N L^2(M_k) \to \rho_\delta^N L^2(M_k)$$

meromorphic in $\{\Re(\lambda) > -\frac{N+1}{2}\}$ with simple poles at $\frac{k}{2} - j$ (with $j \in \mathbb{N}_0$) of ranks uniformly bounded by $C(j+1)^{n-k+1}$ such that

$$(\Delta_{M_k} - \lambda(n-\lambda))\mathcal{E}_N^k(\lambda) = \chi_\delta^k + \mathcal{K}_N^k(\lambda),$$

$\mathcal{K}_N^k(\lambda)$ is trace class on $\rho_\delta^N L^2(M_k)$ and for $|\lambda| \leq \frac{N}{2}$, dist$(\lambda, \frac{k}{2} - \mathbb{N}_0) > \frac{1}{8}$ and $q > 0$ there exists $C_{\delta,q} > 0$ such that

$$\det(1 + q|\mathcal{K}_N^k(\lambda)|) \leq e^{C_{\delta,q}\langle N\rangle^{n+2}}, \tag{3.9}$$

the determinant being on $\rho_\delta^N L^2(M_k)$. Moreover for $\lambda_N = \frac{N}{4}$,

$$||\mathcal{K}_N^k(\lambda_N)||_{\mathcal{L}(\rho_\delta^N L^2(M_k))} \leq (C\delta)^{\frac{N}{4}}. \tag{3.10}$$

Proof: we first set

$$E_0(\lambda) := \chi_L^k R_{X_k}(\lambda)\chi_\delta^k, \quad K_0(\lambda) := \phi_L[\Delta_{X_k}, \psi_L^k] R_{X_k}(\lambda)\chi_\delta^k, \quad L^\sharp(\lambda) := \psi_L^k[\Delta_{X_k}, \phi_L] R_{X_k}(\lambda)\chi_\delta^k$$

and, since $\Delta_{M_k} = \Delta_{X_k}$ as a differential operator on M_k, we clearly have

$$(\Delta_{M_k} - \lambda(n-\lambda))E_0(\lambda) = \chi_\delta^k + K_0(\lambda) + L^\sharp(\lambda).$$

Since the functions in the range of $L^\sharp(\lambda)$ have compact support, $L^\sharp(\lambda)$ is compact on our weighted spaces but $K_0(\lambda)$ is not. However, it is important to note that the range of $K_0(\lambda)$ is composed of functions whose support does not intersect the cusp point, thus they can be included in a regular neighbourhood of infinity and the iterative method of Mazzeo-Melrose [15] (or [11]) can then be used to remove all the Taylor expansion of $K_0(\lambda; w, w')$ at the boundary $\{\rho(w) = 0\}$.

To achieve it, one can decompose $E_0(\lambda) = \oplus_{m\in\mathbb{Z}^k} E_{0,m}(\lambda)$ and $K_0(\lambda), L^\sharp(\lambda)$ similarly on the Fourier modes of T^k and using the new variables $v := x^2$ and $v' := x'^2$ as in [11] we can write

$$K_{0,m}(\lambda) = \phi_L \sum_{j=0}^{2N} K_{0,m}^j(\lambda) + K_{2N,m}^\sharp(\lambda)$$

where for $j = 0, \dots, 2N$, $w = (v, y)$, $w' = (v', y')$, $r = r(w, w')$ and $d = d(w, w')$

$$K_{0,m}^j(\lambda; w, w') = v^{\frac{s}{2}+j+1}[\Delta_y, \psi_L^k] r^{-2s-4j} \alpha_{j,n}(\lambda) F_{j,\lambda}(r\omega_m)\chi_\delta^k(w') v'^{\frac{s}{2}+j}$$

$$K_{2N,m}^\sharp(\lambda; w, w') = v\phi_L[\Delta_y, \psi_L^k] d^s \sum_{j=2N+1}^{\infty} \alpha_{j,n}(\lambda) d^{2j} F_{j,\lambda}(r\omega_m)\chi_\delta^k(w')$$

At last we complete the construction by mimicking [11, Prop. 3.1]

$$E_j(\lambda) = \bigoplus_{m\in\mathbb{Z}^k} E_{j,m}(\lambda), \quad K_j(\lambda) = \bigoplus_{m\in\mathbb{Z}^k} K_{j,m}(\lambda), \quad L_j(\lambda) = \bigoplus_{m\in\mathbb{Z}^k} L_{j,m}(\lambda) \tag{3.11}$$

with the induction formulae for $j = 1, \dots, 2N$

$$L_{0,m}(\lambda) = 0,$$

$$E_{j,m}(\lambda) := E_{j-1,m}(\lambda) + [2j(2\lambda + 2j - n)]^{-1} K_{j-1,m}^{j-1}(\lambda),$$

$$K_{j,m}(\lambda) := K_{j,m}^{j}(\lambda) + K_{j,m}^{j+1}(\lambda) + \cdots + K_{j,m}^{2N},$$

$$K_{j,m}^{j}(\lambda) := K_{j-1,m}^{j}(\lambda) + [2j(2\lambda+2j-n)]^{-1}Q_m(s/2+j)K_{j-1,m}^{j-1}(\lambda),$$

$$K_{j,m}^{l}(\lambda) := K_{j-1,m}^{l}(\lambda) \text{ for } l = j+1, \ldots, 2N$$

$$L_{j,m}(\lambda) := L_{j-1,m}(\lambda) + [2j(2\lambda+2j-n)]^{-1}[\Delta_{X_k}, \phi_L]K_{j-1,m}^{j-1}(\lambda),$$

where $Q_m(\zeta)$ is defined by

$$Q_m(\zeta) := 2(n-k-2-4\zeta)\partial_v + 4v\partial_v^2 + \Delta_y + |\omega_m|^2.$$

Using the crucial 'indicial relation' (see [11, Eq. 3.12])

$$(\Delta_{\mathbb{H}^{n-k+1}} + v|\omega_m|^2)v^{\zeta}f(v,y) = 2\zeta(n-k-2\zeta)v^{\zeta}f(v,y) + v^{\zeta+1}Q_m(\zeta)f(v,y),$$

we then obtain from the previous construction that

$$P_m(\lambda)E_{2N,m}(\lambda) = \chi_\delta^k + \phi_L K_{2N,m}(\lambda) + L_{2N,m}(\lambda) + K_{2N,m}^{\sharp}(\lambda) + L_m^{\sharp}(\lambda)$$

and a straightforward calculus shows that for $w = (v,y), w' = (v',y')$

$$K_{2N,m}(\lambda; w; w') = v^{\frac{s}{2}+2N+1}\sum_{j=0}^{2N-1}\beta_{j,2N}(\lambda)\prod_{k=1}^{2N-j}Q_m(\frac{s}{2}+j+k)_w H_{j,m}(\lambda; w, w'),$$

$$L_{2N,m}(\lambda; w, w') = [\Delta_{X_k}, \phi_L]\sum_{0\leq j\leq p\leq 2N}\beta_{j,p}(\lambda)v^{\frac{s}{2}+p}\prod_{k=1}^{p-j}Q_m(\frac{s}{2}+j+k)_w H_{j,m}(\lambda; w, w')$$

with

$$\beta_{j,p}(\lambda) := \frac{\pi^{-\frac{n}{2}}\Gamma(\lambda+2j)2^{-1}}{\Gamma(\lambda-\frac{n}{2}+1+p)\Gamma(p+1)}$$

$$H_{j,m}(\lambda; w, w') = [\Delta_y, \psi_L^k]r^{-2s-4j}F_{j,\lambda}(r\omega_m){v'}^{\frac{s}{2}+j}\chi_\delta^k(w').$$

The distributional kernels of $E_{2N,m}(\lambda), K_{2N,m}(\lambda), L_{2N,m}(\lambda)$ and $K_{2N,m}^{\sharp}(\lambda)$ are holomorphic in $\mathbb{C}$ when $m \neq 0$ and meromorphic with simple poles of finite rank at each $\frac{k}{2} - j$ $(j \in \mathbb{N}_0)$ when $\omega_m = 0$, the ranks being bounded by $C(1+j)^{n-k+1}$ (see again [11, Prop. 3.1] for details).

At this stage we can set

$$\mathcal{E}_N^k(\lambda) := \bigoplus_{m\in\mathbb{Z}^k} E_{2N,m}(\lambda),$$

$$\mathcal{K}_N^k(\lambda) := \bigoplus_{m\in\mathbb{Z}^k}\Big(\phi_L K_{2N,m}(\lambda) + L_{2N,m}(\lambda) + K_{2N,m}^{\sharp}(\lambda) + L_m^{\sharp}(\lambda)\Big) = \bigoplus_{m\in\mathbb{Z}}\mathcal{K}_{N,m}^k(\lambda).$$

Notice that the sums (3.11) are just formal so far, but we will show their convergence in the following lemmas.

We will first show that $\mathcal{K}_{N,m}^k(\lambda)$ is compact on $\rho_\delta^N\mathcal{H}_m$ and we will bound its singular values uniformly with respect to m. These estimates will prove that $\bigoplus_{m=0}^{M}\mathcal{K}_{N,m}^k(\lambda)$ converges to an operator $\mathcal{K}_N^k(\lambda)$ compact on $\rho_\delta^N L^2(X_k)$ when $M\to\infty$, whose singular values are $(\mu_l(\mathcal{K}_{N,m}^k(\lambda)))_{l\in\mathbb{N},m\in\mathbb{Z}^k}$ if $(\mu_l(\mathcal{K}_{N,m}^k(\lambda)))_{l\in\mathbb{N}}$ are the singular values of $\mathcal{K}_{N,m}^k(\lambda)$ on $\rho_\delta^N\mathcal{H}_m$.

Lemma 3.2. *The operators $\phi_L K_{2N,m}(\lambda)$ and $L_{2N,m}(\lambda)$ are trace class on $\rho_\delta^N \mathcal{H}_m$ for $|\lambda|\leq \frac{N}{2}$, dist$(\lambda, \frac{k}{2}-\mathbb{N}_0) > \frac{1}{8}$ and their singular values satisfy*

$$\mu_l\Big(\phi_L K_{2N,m}(\lambda)+L_{2N,m}(\lambda)\Big)\leq e^{-\frac{\epsilon_0\langle\omega_m\rangle}{4}}\delta^{\Re(s)}\Big(Cl^{-\frac{1}{n-k+1}}N\Big)^{2N}\max\Big(1,(\frac{\langle\omega_m\rangle}{N})^{\Re(s)-|\Re(s)|}\Big)$$

there for some $\epsilon_0>0$. Moreover, for $\lambda_N=\frac{N}{4}$, we have

$$||\phi_L K_{2N}(\lambda_N)+L_{2N}(\lambda_N)||_{\mathcal{L}(\rho_\delta^N L^2(M_k))}\leq (C\delta)^{\frac{N}{4}} \tag{3.12}$$

Proof: to begin, we give for $\epsilon>0$ an estimate on the Bessel function in $\Re(z)>2\epsilon$

$$e^{\Re(z)}|K_s(z)|\leq C_\epsilon \sup_{t\geq 0}[e^{-(\Re(z)-\epsilon)(e^t-1)+|\Re(s)|t)}]\leq C_\epsilon\max\left(1,\left(\frac{\langle\Re(s)\rangle}{|\Re(z)-\epsilon|}\right)^{|\Re(s)|}\right). \tag{3.13}$$

To see that $\phi_L K_{2N,m}(\lambda)$ is trace class on $\rho_k^N\mathcal{H}_m\simeq\rho_\delta^N\mathcal{H}_m$, we use a standard trick. Let $\Omega\subset\mathbb{R}^{n-k+1}$ be an open ball containing $\{x^2+|y|^2\leq 4\}$ and Δ_Ω the Dirichlet realization of the Laplacian on Ω. Since $(\Delta_\Omega+1)^{-M}$ is trace class for $M>\frac{n-k+1}{2}$ on $L^2(\Omega)$, it suffices to show that $(\Delta_\Omega+1)^N\rho_k^{-N}\phi_L K_{2N,m}(\lambda)\rho_k^N$ can be extended as a bounded operator on $L^2(\Omega)$, and a uniform bound on its norm together with a comparison with the singular values of $(\Delta_\Omega+1)^{-N}$ will give an estimate for the singular values $(\mu_l(\phi_L K_{2N,m}(\lambda)))_{l\in\mathbb{N},m\in\mathbb{Z}^k}$ on $\rho_k^N\mathcal{H}_m$. The same method can be applied for $L_{2N,m}(\lambda)$.

By Stirling's formula and the complement formula, we check that for $p\geq 0$ and $|\lambda|<\frac{N}{2}$

$$\left|\beta_{j,p}(\lambda)\frac{2^{-\lambda-2j+1}(2\pi)^{\frac{n}{2}}}{\Gamma(\lambda+2j)}\right|\leq C^{N+p}p^{-p}\langle p+\lambda\rangle^{-p-\Re(\lambda)}\leq C^{N+p}N^{-2p-\Re(\lambda)}. \tag{3.14}$$

A straightforward estimate for $|\lambda|\leq\frac{N}{2}, |\alpha|\leq 2N$ shows that

$$|\partial_w^\alpha(v^{\frac{s}{2}+2N+1}\rho_k(v,y)^{-N})|\leq C^N(|\alpha|+N)^{|\alpha|}$$

for $w=(v,y)\in\{0\leq v\leq 2,|y|\leq 2\}$. We now choose the cut-off functions ψ_L^k,ϕ_L quasi-analytic of order $5N$, that is

$$||\partial_y^\alpha\psi_L^k(y)||_\infty\leq (CN)^{|\alpha|},\quad ||\partial_x^l\phi_L(x)||_\infty\leq (CN)^l\quad \text{for } |\alpha|\leq 5N, l\leq 5N. \tag{3.15}$$

Therefore, for all smooth function $f(v,y)$ with support in $\{v\in[0,2],|y|\leq 2\}$, we have for $M\leq N$ and $p\leq 2N$

$$\begin{aligned}&\left|(\Delta_w+1)^M\phi_L(w)\rho_k(w)^{-N}v^{\frac{s}{2}+2N+1}\prod_{k=1}^{2N-j}Q_m(\frac{s}{2}+j+k)f(v,y)\right|_\infty\\ &\leq C^N\sum_{l_0+l_1+|l_2|+|l_3|=2N+M-j}\langle\omega_m\rangle^{2l_0}(CN)^{2l_1+|l_3|}|\partial_w^{2l_2+l_3}f(w)|_\infty\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}&\left|(\Delta_w+1)^M[\Delta_{X_k},\phi_L]\rho_k(w)^{-N}v^{\frac{s}{2}+p+1}\prod_{k=1}^{p-j}Q_m(\frac{s}{2}+j+k)_w f(v,y)\right|_\infty\\ &\leq C^N\sum_{l_0+l_1+|l_2|+|l_3|=p+M-j}\langle\omega_m\rangle^{2l_0}(CN)^{2l_1+|l_3|}|\partial_w^{2l_2+l_3}f(w)|_\infty\end{aligned} \tag{3.17}$$

(recall that $[\Delta_{X_k},\phi_L]$ has compact support). For w' fixed, we want to extend $r(.,w')$ in a complex neighbourhood of $\mathbb{R}^{n-k+1}$ in $\mathbb{C}^{n-k+1}$ to obtain bounds on its derivatives by Cauchy formula. Using the fact that $r(w,w')>\epsilon_0$ for some $\epsilon_0>0$ when $w\in\text{supp}\nabla\chi_L^k$, $w'\in\text{supp}(\chi_\delta^k)$ we argue that for $(v_0,y_0)\in\mathbb{R}\times\mathbb{R}^n$ such that $|v_0|+|y_0|\leq\epsilon<\frac{1}{2}\epsilon_0$ then

$$r^2=r(v+iv_0,y+iy_0,w')^2=(v+v'+|y-y'|^2-|y_0|^2)+i(v_0+2(y-y').y_0)$$

satisfies in $w=(v,y)\in \mathrm{supp}\nabla\chi_L^k$, $w'=(v',y')\in \mathrm{supp}(\chi_\delta^k)$

$$\Re(r^2)>\frac{1}{2}\epsilon_0^2,\quad \arg(r^2)\leq C\epsilon. \tag{3.18}$$

This implies, with (3.13), that for $|\lambda|\leq \frac{N}{2}$, $\mathrm{dist}(\lambda,\frac{k}{2}-\mathbb{N}_0)>\frac{1}{8}$, the function

$$\theta_{j,\lambda,p,m}(w,w'):=\beta_{j,p}(\lambda)r^{-2s-4j}F_{j,\lambda}(r\omega_m){v'}^{\frac{s}{2}+j}\rho_k^N(w')\chi_\delta^k$$

is analytic for $p\geq 2N$ in w in some complex neighbourhood $U_\epsilon=\{w\in\mathbb{C}^{n-k+1};\mathrm{dist}(w,U)\leq\epsilon\}$ of $U=\mathrm{supp}\nabla\chi_L^k$ and can be bounded there, for $m\neq 0$ and some $C>0$ independent of w', by

$$|\theta_{j,\lambda,p,m}(w,w')|\leq C^{N+p+j}\frac{e^{-\frac{\epsilon_0}{2}\langle\omega_m\rangle}}{N^{2p+\Re(\lambda)}}\rho_k(w')^{N+\Re(s)+2j}(N+2j)^{\Re(s)+2j} \tag{3.19}$$

if $\Re(s)+2j\geq 0$ and by

(3.20)
$$|\theta_{j,\lambda,p,m}(w,w')|\leq C^{N+p+j}\frac{e^{-\frac{\epsilon_0}{2}\langle\omega_m\rangle}}{N^{2p+\Re(\lambda)}}\rho_k(w')^{N+\Re(s)+2j}\max\left(\langle\omega_m\rangle^{\Re(s)+2j},\left(\frac{\langle\omega_m\rangle^2}{N+2j}\right)^{\Re(s)+2j}\right)$$

if $\Re(s)+2j<0$. Indeed, we have from (3.18)

$$C^{-1}{v'}^{\frac{1}{2}}\rho_k(w')^{-1}\leq\Re(r^2)\leq|r|^2\leq C\Re(r^2)\leq C{v'}^{\frac{1}{2}}\rho_k(w')^{-1}$$

$$C^{-1}{v'}^{\frac{1}{4}}\rho_k(w')^{-\frac{1}{2}}\leq(\Re(r^2))^{\frac{1}{2}}\leq\Re(r)\leq(\Re(r^2))^{\frac{1}{2}}+C{v'}^{\frac{1}{4}}\rho_k(w')^{-\frac{1}{2}}\leq C{v'}^{\frac{1}{4}}\rho_k(w')^{-\frac{1}{2}}$$

for $w\in U_\epsilon, w'\in\mathrm{supp}(\chi_\delta^k)$, hence

$$\rho_k(w')^N|{v'}^{\frac{s}{2}+j}r^{-2s-4j}|\leq C^{N+j}\rho_k(w')^{N+\Re(s)+2j}$$

$$\rho_k(w')^N\left|\frac{{v'}^{\frac{s}{2}+j}r^{-s-2j}}{(C\Re(r))^{|\Re(s)+2j|}}\right|\leq C^{N+j}\rho_k(w')^{N+\Re(s)+2j}({v'}^{-\frac{1}{2}}\rho_k(w'))^{|\frac{\Re(s)}{2}+j|-(\frac{\Re(s)}{2}+j)}$$

and (3.19),(3.20) are obtained in view of (3.13), (3.14), the bound ${v'}^{-\frac{1}{2}}\rho_k(w')\leq 1$ and the uniform estimate

$$(\Re(r)|\omega_m|)^{\Re(s)+2j}e^{-\frac{\Re(r)|\omega_m|}{2}}\leq(C\langle\Re(s)+2j\rangle)^{|\Re(s)+2j|}.$$

By Cauchy formula and (2.9), we deduce for $|\alpha|\leq 4N$, $w\in U$ and $\Re(s)+2j\geq 0$

$$\frac{|\partial_w^\alpha\theta_{j,\lambda,p,m}(w,w')|}{\rho_k(w')^n}\leq C^{N+j+p}\frac{\delta^{N+\Re(s)+2j}}{N^{2p-|\alpha|-2j}}e^{-\frac{\epsilon_0\langle\omega_m\rangle}{2}} \tag{3.21}$$

whereas for $\Re(s)+2j<0$

(3.22)
$$\frac{|\partial_w^\alpha\theta_{j,\lambda,p,m}(w,w')|}{\rho_k(w')^n}\leq\frac{C^{N+j+p}\delta^{N+\Re(s)+2j}}{N^{2p+\Re(\lambda)-|\alpha|}}e^{\frac{-\epsilon_0\langle\omega_m\rangle}{2}}\max\left(\langle\omega_m\rangle^{\Re(s)+2j},\left(\frac{\langle\omega_m\rangle^2}{N+2j}\right)^{\Re(s)+2j}\right)$$

For the case $m=0$, we obtain the same bound as (3.21) by using

$$\left|\beta_{j,p}(\lambda)\frac{\alpha_{j,n-k}(\lambda-\frac{k}{2})}{\alpha_{j,n}(\lambda)}\right|\leq C^{N+p+j}N^{-2p+2j}$$

for $|\lambda|\leq\frac{N}{2}$ and $\mathrm{dist}(\lambda,\frac{k}{2}-\mathbb{N}_0)>\frac{1}{8}$. Using (3.16), (3.17) with $M=N$ and $p\leq 2N$, (3.21), (3.22), (3.15) and again the bound

$$\langle\omega_m\rangle^\Lambda e^{-\frac{\epsilon_0\langle\omega_m\rangle}{4}}\leq(C\Lambda)^\Lambda$$

for all $\Lambda>0$, we can conclude that

$$||(\Delta_\Omega+1)^N\rho_k^{-N}\phi_LK_{2N,m}(\lambda)\rho_k^N||_{\mathcal{L}(\mathcal{H}_m,L^2(\Omega))}\leq\frac{\delta^{\Re(s)+N}(CN)^{2N}}{e^{\frac{\epsilon_0\langle\omega_m\rangle}{4}}}\max\left(1,(\langle\omega_m\rangle N^{-1})^{\Re(s)-|\Re(s)|}\right)$$

and the same estimate for $L_{2N,m}(\lambda)$. We just recall that the singular values of $(1+\Delta_\Omega)^{-N}$ on $L^2(\Omega)$ satisfy

$$\mu_l((1+\Delta_\Omega)^{-N}) \leq (Cl)^{\frac{2N}{n-k+1}}$$

and that

$$\mu_l(AB) \leq \mu_l(A)||B|| \tag{3.23}$$

if A is trace class and B bounded to show that

$$\mu_l\Big(\phi_L K_{2N,m}(\lambda) + L_{2N,m}(\lambda)\Big) \leq e^{-\frac{\epsilon_0\langle\omega_m\rangle}{4}}\delta^{\Re(s)+N}\Big(Cl^{-\frac{1}{n-k+1}}N\Big)^{2N}\max\Big(1,\Big(\frac{\langle\omega_m\rangle}{N}\Big)^{\Re(s)-|\Re(s)|}\Big)$$

on $\rho_k^N\mathcal{H}_m$. In view of (2.7), (2.8) and (3.23) this gives

$$\mu_l\Big(\phi_L K_{2N,m}(\lambda) + L_{2N,m}(\lambda)\Big) \leq e^{-\frac{\epsilon_0\langle\omega_m\rangle}{4}}\delta^{\Re(s)}\Big(Cl^{-\frac{1}{n-k+1}}N\Big)^{2N}\max\Big(1,\Big(\frac{\langle\omega_m\rangle}{N}\Big)^{\Re(s)-|\Re(s)|}\Big)$$

on $\rho_\delta^N\mathcal{H}_m$.

By taking $M=0$ in (3.16), the previous estimates also show that for $\Re(\lambda) > \frac{n}{2}$

$$||\phi_L K_{2N,m}(\lambda) + L_{2N,m}(\lambda)||_{\mathcal{L}(\rho_\delta^N\mathcal{H}_m)} \leq C^N\delta^{\Re(s)}e^{-\frac{\epsilon_0\langle\omega_m\rangle}{4}}$$

and (3.12) is then easily deduced. □

In a second step, we are going to control the singular values of the terms $K^\sharp_{2N,m}(\lambda)$ and $L^\sharp_m(\lambda)$ on $\rho_\delta^N\mathcal{H}_m$.

Lemma 3.3. *$K^\sharp_{2N,m}(\lambda)$ and $L^\sharp_m(\lambda)$ are trace class on $\rho_\delta^N\mathcal{H}_m$ if $\delta>0$ is chosen small enough and $|\lambda|\leq\frac{N}{2}$, $dist(\lambda,\frac{k}{2}-\mathbb{N}_0)>\frac{1}{8}$. Moreover, their singular values satisfy*

$$\mu_l\Big(K^\sharp_{2N,m}(\lambda) + L^\sharp_m(\lambda)\Big) \leq e^{-\frac{\epsilon_0\langle\omega_m\rangle}{4}}\delta^{\Re(s)}\Big(Cl^{-\frac{1}{n-k+1}}N\Big)^{2N}\max\Big(1,\Big(\frac{\langle\omega_m\rangle}{N}\Big)^{\Re(s)-|\Re(s)|}\Big)$$

there for some $\epsilon_0>0$ and if $\lambda_N=\frac{N}{4}$ we have

$$\Big|\Big|K^\sharp_{2N}(\lambda_N) + L^\sharp(\lambda_N)\Big|\Big|_{\mathcal{L}(\rho_\delta^N L^2(M_k))} \leq (C\delta)^{\frac{N}{4}} \tag{3.24}$$

Proof: we recall that

$$\rho_k^N(w)K^\sharp_{2N,m}(\lambda;w,w')\rho_k^N(w') = \rho_k^N v^{\frac{s}{2}+2N+1}\phi_L[\Delta_{X_k},\psi_L^k]\sum_{j\geq 2N+1}v^{j-2N-1}\theta_{j,\lambda,j,m}(w,w')$$

provided the sum converges. Taking advantage of the estimates (3.21), we find for $w\in U_\epsilon$, $|\lambda|\leq\frac{N}{2}$, $\Re(s)+2j\geq 0$ and $p\geq 0$

$$|v^p\theta_{j,\lambda,j,m}(w,w')| \leq C^{N+p}\delta^{\Re(s)+N}e^{-\frac{\epsilon_0\langle\omega_m\rangle}{2}}(C\delta)^j\rho_k(w')^n \tag{3.25}$$

which proves that the sum converges if δ is chosen small enough and we obtain

$$|\rho_k^N(w)K^\sharp_{2N,m}(\lambda;w,w')\rho_k^N(w')| \leq (C\delta)^N\delta^{\Re(s)+N}e^{-\frac{\epsilon_0\langle\omega_m\rangle}{4}}\rho_k(w')^n.$$

Using the arguments of Lemma 3.2, it is straightforward to check that

$$\mu_l(K^\sharp_{2N,m}(\lambda)) \leq (C\delta)^N e^{-\frac{\epsilon_0\langle\omega_m\rangle}{4}}l^{-2}.$$

For $L^\sharp(\lambda)$ the method is similar, we recall that

$$\rho_k^{-N}(w)L^\sharp_m(\lambda;w,w')\rho_k^N(w') = \rho_k^{-N}\psi_L^k[\Delta_{X_k},\phi_L]v^{\frac{s}{2}}\sum_{j=0}^{\infty}v^j\theta_{j,\lambda,j,m}(w,w').$$

Using (3.25) for $\Re(s)+2j\geq 0$ and (3.22) for $\Re(s)+2j\leq 0$, we find that

$$|\partial_w^N(v^j\theta_{j,\lambda,j,m}(w,w'))| \leq e^{-\frac{\epsilon_0\langle\omega_m\rangle}{2}}(CN)^{2N}\delta^{\Re(s)+N}(C\delta)^j\max(1,(\langle\omega_m\rangle N^{-1})^{\Re(s)-|\Re(s)|})\rho_k(w')^n$$

for w in the complex neighbourhood U_ϵ of $\mathrm{supp}(\nabla\chi_L^k)$. One deduces that the sum $L_m^\sharp(\lambda)$ converges for small δ and the arguments of Lemma 3.2 yield the bound

$$\mu_l(L_m^\sharp(\lambda)) \leq e^{-\frac{\epsilon_0\langle\omega_m\rangle}{4}}\delta^{\Re(s)}\left(Cl^{-\frac{1}{n-k+1}}N\right)^{2N}\max\left(1,(N^{-1}\langle\omega_m\rangle)^{\Re(s)-|\Re(s)|}\right)$$

on $\rho_\delta^N\mathcal{H}_m$. These estimates on the singular values also imply (3.24). □

Lemmas 3.2 and 3.3 clearly prove that

$$\sum_{m\in\mathbb{Z}^k}\sum_{l\in\mathbb{N}}\mu_l(\mathcal{K}_{N,m}^k(\lambda))<\infty$$

and we have

$$\prod_{m\in\mathbb{Z}^k}\prod_{l\in\mathbb{N}}(1+q\mu_l(\mathcal{K}_{N,m}^k(\lambda)))\leq\exp\left(\sum_{m\in\mathbb{Z}^k}\sum_{l\in\mathbb{N}}\log\left(1+(\Lambda l^{-\frac{1}{n-k+1}})^{2N}\right)\right).$$

with $\Lambda:=Cq\delta^{-1}e^{-\frac{\epsilon_0\langle\omega_m\rangle}{8N}}N\max(1,(N\langle\omega_m\rangle^{-1})^{\frac12})$. Now, we use

$$\begin{aligned}\sum_l\log\left(1+(\Lambda l^{-\frac{1}{n-k+1}})^{2N}\right) &\leq \int_0^\infty\log\left(1+(\Lambda t^{-\frac{1}{n-k+1}})^{2N}\right)dt\\ &\leq \Lambda^{n-k+1}\int_0^\infty\log(1+t^{-\frac{2N}{n-k+1}})dt\\ &\leq CN\Lambda^{n-k+1}\int_1^\infty t^{-\frac32}dt\\ &\leq C_{q,\delta}e^{-\frac{\epsilon_0\langle\omega_m\rangle}{8N}}N^{n-k+2}\max(1,(N\langle\omega_m\rangle^{-1})^{\frac12}).\end{aligned}$$

for some $C_{q,\delta}>0$. Finally, since $\langle\omega_m\rangle\geq C|m|$ for some $C>0$ depending on $A_k\in GL_k(\mathbb{R})$ we have

$$\sum_{\substack{m\in\mathbb{Z}^k\\|\omega_m|\geq N}}e^{-\frac{\epsilon_0\langle\omega_m\rangle}{8N}}\leq\sum_{j\in\mathbb{N}_0}\sum_{j\leq|m|\leq j+1}e^{-\frac{j}{CN}}\leq C\int_0^\infty t^{k-1}e^{-\frac{t}{CN}}dt\leq CN^k$$

and

$$\sum_{\substack{m\in\mathbb{Z}^k\\\langle\omega_m\rangle\leq N}}e^{-\frac{\epsilon_0\langle\omega_m\rangle}{8N}}\sqrt{\frac{N}{\langle\omega_m\rangle}}\leq\sum_{j\in\mathbb{N}_0}\sum_{j\leq|m|\leq j+1}e^{-\frac{j}{CN}}\sqrt{\frac{N}{\langle j\rangle}}\leq C\int_0^\infty t^{k-1}\sqrt{\frac{N}{t}}e^{-\frac{t}{CN}}dt\leq CN^k.$$

This proves that we can find $C_{\delta,q}>0$ such that

$$\prod_{m\in\mathbb{Z}^k}\prod_{l\in\mathbb{N}}(1+q\mu_l(\mathcal{K}_{N,m}^k(\lambda)))\leq C_{\delta,q}e^{C_{\delta,q}N^{n+2}}$$

for $|\lambda|\leq\frac{N}{2}$, $\mathrm{dist}(\lambda,\frac{k}{2}-\mathbb{N}_0)>\frac18$, thus (3.9) is obtained.

The bound (3.10) is a consequence of Lemmas 3.2 and 3.3.

To conclude the proof of Theorem 3.1, it remains to prove the

Lemma 3.4. *The operator $\mathcal{E}_N^k(\lambda)$ is continuous from $\rho_\delta^NL^2(M_k)$ to $\rho_\delta^{-N}L^2(M_k)$.*

Proof: except $\rho_\delta^NE_0(\lambda)\rho_\delta^N$, we have seen that the other terms in the expression of $\rho_\delta^N\mathcal{E}_N^k(\lambda)\rho_\delta^N$ have Schwartz kernels in $L^2(M_k\times M_k)$ and thus are bounded on $L^2(M_k)$. To deal with $\rho_\delta^NE_0(\lambda)\rho_\delta^N$, we take $J=2N$ in (3.5) and first show that

$$(\rho_\delta(w)\rho_\delta(w'))^Nd^{s+2N}\int_{\mathbb{R}^k}e^{-ir\omega_m\cdot z}(1+|z|^2)^{-s-2N}G_{2N,n}(s,d(1+|z|^2)^{-1})dz \tag{3.26}$$

are the kernels of bounded operators on $\mathcal{H}_m$ with norms uniformly bounded with respect to m when $|s| \leq \frac{N}{2}$. We know that $G_{2N,n}(s,\tau)$ is smooth for $\tau \in [0, \frac{1}{2})$ thus (3.26) is square integrable in $(M_k \times M_k) \setminus \{d > \frac{1}{8}\}$ with norm bounded by C_N for $|s| \leq \frac{N}{2}$. Now using [22, Prop. A.1], we deduce that (3.26) is bounded for $\{d > \frac{1}{8}\}$ by $\varphi(d_{\mathbb{H}^{n-k+1}}(w,w'))$ for a function $\varphi > 0$ satisfying

$$\int_0^1 \varphi(\tau)\tau^{n-k} d\tau \leq C_N$$

for $|s| \leq \frac{N}{2}$. Therefore, Proposition B.1 of [22] allows to conclude that (3.26) is bounded on $\mathcal{H}_m$ uniformly with respect to m. Now it just remains to show the boundedness of the operator on $\mathcal{H}_m$ whose kernel is $\rho_k(w)^N\rho_k(w')^N d^{s+2j}F_{j,\lambda}(r|\omega_m|)$ with a uniform bound with respect to m. First we use that

$$\rho_k(w)^N\rho_k(w')^N d^{\Re(s)+2j} \leq C_N \rho_k(w)^{\frac{N}{2}}\rho_k(w')^{\frac{N}{2}}$$

if $|s| \leq \frac{N}{2}$, which is straightforward for $\Re(s)+2j > 0$ and comes easily from the estimate

$$d^{-1} < C\left(\frac{1}{\rho_k(w)x'} + \frac{1}{\rho_k(w')x}\right)$$

if $\Re(s)+2j < 0$. Now to bound $F_{j,\lambda}(r|\omega_m|)$ we see that if $r|\omega_m| > 1$ then

$$(r|\omega_m|)^{s+2j}K_{-s-2j}(r|\omega_m|)| \leq C_N$$

in view of the bound (3.13), whereas if $r|\omega_m| < 1$ we can use that $K_s(z) = z^s\phi_s(z^2) + z^{-s}\phi_{-s}(z^2)$ for some function $\phi_s(z^2)$ smooth on $z \in [0,1]$ and observe that

$$|F_{j,\lambda}(r|\omega_m|)| \leq |\phi_{-s-2j}(r|\omega_m|)| + |(r|\omega_m|)^{2s+4j}\phi_{s+2j}(r|\omega_m|)|$$

is bounded by C_N if $\Re(s)+4j > 0$ and by $C_N r^{2\Re(s)+4j} \leq C_N d^{-\Re(s)-2j}(\rho_k(w)\rho_k(w'))^{-\frac{N}{2}}$ if $\Re(s)+2j < 0$. This proves that in all cases we have

$$|\rho_k(w)^N\rho_k(w')^N d^{s+2j}F_{j,\lambda}(r|\omega_m|)| \leq C_N(\rho_k(w)\rho_k(w'))^{\frac{N}{2}}$$

and we conclude that $\rho_k(w)^N\rho_k(w')^N d^{s+2j}F_{j,\lambda}(r|\omega_m|)$ is the kernel of a Hilbert Schmidt operator on $\mathcal{H}_m$ with norm uniformly bounded with respect to m, this achieves the proof of the Lemma. □

The Proposition 3.1 is then proved. □

3.2. **The maximal rank cusps.** We recall that $X_n = \Gamma_n\backslash\mathbb{H}^{n+1}$ is a quotient by a group of translations acting on $\mathbb{R}^n$. The lattice of translations Γ_n acting on $\mathbb{R}^n$ is the image of the lattice $\mathbb{Z}^k$ by a map $A_n \in GL_n(\mathbb{R})$. By using a Fourier decomposition on the torus $T^n = \Gamma_n\backslash\mathbb{R}^n$ and conjugating by $x^{\frac{n}{2}}$, the operator $\Delta_{X_n} - \lambda(n-\lambda)$ acts on

$$L^2(X_n) = \bigoplus_{m\in\mathbb{Z}^n} \mathcal{H}_m, \quad \mathcal{H}_m \simeq L^2(\mathbb{R}^+, x^{-1}dx)$$

as a family of operators

$$P_m(\lambda) := -(x\partial_x)^2 + x^2|\omega_m|^2 + s^2$$

where $\omega_m = 2\pi{}^t(A_n^{-1})m$ for $m \in \mathbb{Z}^n$ and $s := \lambda - \frac{n}{2}$ the shifted spectral parameter. By elementary Sturm-Liouville theory (see [7, Lem. 3.1]), we find that the resolvent $R_{X_n}(\lambda) = (\Delta_{X_n} - \lambda(n-\lambda))^{-1}$ for the Laplacian on X_n is for $\Re(\lambda) > \frac{n}{2}$

$$R_{X_n}(\lambda) = \bigoplus_{m\in\mathbb{Z}^n} R_m(\lambda) \text{ on } L^2(X_n) = \bigoplus_{m\in\mathbb{Z}^n}\mathcal{H}_m \tag{3.27}$$

with

$$R_m(\lambda; x, x') = -K_{-s}(|\omega_m|x)I_s(|\omega_m|x')H(x-x') - K_{-s}(|\omega_m|x')I_s(|\omega_m|x)H(x'-x), \quad m \neq 0$$

$$R_0(\lambda; x, x') = (2s)^{-1}e^{-s|\log(x/x')|}$$

where H is the Heaviside function, K_s is defined in (3.7) and I_s is the modified Bessel function.

Now we construct a parametrix for $\Delta_X - \lambda(n-\lambda)$ on the end $I_n^{-1}(M_n)$ of our manifold X. Notice however that better estimates could be obtained for this part (see e.g. [10]) since the problem is essentially reduced to the one-dimensional case.

Proposition 3.5. *There exist some bounded operators*

$$\mathcal{E}_N^n(\lambda) : \rho_\delta^N L^2(M_n) \to \rho_\delta^{-N} L^2(M_n)$$

$$\mathcal{K}_N^n(\lambda) : \rho_\delta^N L^2(M_n) \to \rho_\delta^N L^2(M_n)$$

holomorphic in $\{\Re(\lambda) > -\frac{N+1}{2}, \lambda \neq \frac{n}{2}\}$ *with at most a simple pole at* $\frac{n}{2}$ *and such that*

$$(\Delta_{M_n} - \lambda(n-\lambda))\mathcal{E}_N^n(\lambda) = \chi_\delta^n + \mathcal{K}_N^n(\lambda),$$

$\mathcal{K}_N^n(\lambda)$ *is trace class on* $\rho_\delta^N L^2(M_n)$ *and for* $q>0$ *there exists* $C_{\delta,q}>0$ *such that for* $|\lambda| \leq \frac{N}{2}$

$$\det(1+q|\mathcal{K}_N^n(\lambda)|) \leq e^{C_{\delta,q}\langle N\rangle^{n+2}},$$

the determinant being on $\rho_\delta^N L^2(M_n)$. *Moreover, for* $\lambda_N = \frac{N}{4}$,

$$||\mathcal{K}_N^n(\lambda_N)||_{\mathcal{L}(\rho_\delta^N L^2(M_n))} \leq (C\delta)^{\frac{N}{4}}.$$

Proof: let us set

$$\mathcal{E}_N^n(\lambda) := \chi_L^n R_{X_n}(\lambda)\chi_\delta^n, \quad \mathcal{K}_N^n(\lambda) := [\Delta_{X_n}, \chi_L^n] R_{X_n}(\lambda)\chi_\delta^n$$

and check that this choice satisfies the announced properties. The boundedness of $\mathcal{E}_N^n(\lambda)$ is obtained by Schur's lemma. To show that $\mathcal{K}_N^n(\lambda)$ is trace class on $\rho_\delta^N L^2(M_n)$ and to estimate its singular values, we analyze each $[x\partial_x, \phi_L]R_m(\lambda)\chi_\delta^n$ on $\rho_\delta^N \mathcal{H}_m$ and use the same arguments as for the non-maximal rank cusps. Since we do not need the optimal estimates for the singular values, it suffices to control the derivatives of $R_m(\lambda; x, x')$ outside the diagonal. This is easily obtained from the formulae

$$I_k(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos(u)}\cos(ku)du - \frac{\sin(k\pi)}{\pi}\int_0^\infty e^{-z\cosh(u)-ku}du$$

$$K_{-k}(z) = \int_0^\infty \cosh(ku)e^{-z\cosh(u)}du.$$

the analyticity in z and Cauchy's formula as for the non-maximal rank cusps. Then a straightforward calculus shows that the singular values of $[x\partial_x, \phi_L]R_m(\lambda)\chi_\delta^n$ on $\rho_\delta^N\mathcal{H}_m$ satisfy

$$\mu_l([x\partial_x, \phi_L]R_m(\lambda)\chi_\delta^n) \leq e^{-\frac{\epsilon_0\langle\omega_m\rangle}{4}}\delta^{\Re(s)}\left(Cl^{-1}N\right)^{2N}\max\left(1, (N^{-1}\langle\omega_m\rangle)^{\Re(s)-|\Re(s)|}\right)$$

for some $\epsilon_0 > 0$ and the arguments of Proposition 3.1 allow to complete the proof. □

3.3. **The regular neighbourhoods.** For this part of the parametrix, we use the work of Guillopé-Zworski [11] and deduce the following

Proposition 3.6. *There exists some bounded operators*

$$\mathcal{E}_N^r(\lambda) : \rho_\delta^N L^2(M_r) \to \rho_\delta^{-N} L^2(M_r)$$

$$\mathcal{K}_N^r(\lambda) : \rho_\delta^N L^2(M_r) \to \rho_\delta^N L^2(M_r)$$

meromorphic in $\Re(\lambda) > -\frac{N+1}{2}$ *with simple poles at* $-j$ *(with* $j \in \mathbb{N}_0$*) of ranks uniformly bounded by* $C(j+1)^{n+1}$ *such that*

$$(\Delta_{M_r} - \lambda(n-\lambda))\mathcal{E}_N^r(\lambda) = \chi_\delta^r + \mathcal{K}_N^r(\lambda),$$

$\mathcal{K}_N^r(\lambda)$ *is trace class on* $\rho_\delta^N L^2(M_r)$ *and for* $q>0$ *and* $|\lambda| \leq \frac{N}{2}$ *and* $dist(\lambda, -\mathbb{N}_0) > \frac{1}{8}$ *there exists* $C_{\delta,q} > 0$ *such that*

$$\det(1+q|\mathcal{K}_N^r(\lambda)|) \leq e^{C_{\delta,q}\langle N\rangle^{n+2}},$$

the determinant being on $\rho_\delta^N L^2(M_r)$*. Moreover, for* $\lambda_N = \frac{N}{4}$,

$$||\mathcal{K}_N^r(\lambda_N)||_{\mathcal{L}(\rho_\delta^N L^2(M_r))} \leq (C\delta)^{\frac{N}{4}}.$$

Proof: we begin by defining a new cut-off function $\chi_{L,\delta}^r(x,y) = \phi_L(Cx\delta^{-1})\psi_L^r(y)$ with C chosen so that $\chi_{L,\delta}^r = 1$ on $\mathrm{supp}(\chi_L^r)$. It suffices now to use the construction of [11, Prop. 3.1] with the cut-off functions $\chi_{L,\delta}^r$ and χ_δ^r (i.e. the functions $\chi_1^\delta, \chi_2^\delta$ of [11] are $\chi_{L,\delta}^r, \chi_\delta^r$ here). Note that in their construction, Guillopé and Zworski used for χ_2^δ a function having a product structure like $\chi_{L,\delta}^r$ but it is not difficult to see that our χ_δ^r suits as well by using that $C^{-1}x \leq \rho_\delta \leq Cx\delta^{-1}$ in M_r for some $C > 0$. The end of the proof is given by Proposition 4.1 of [11]. □

4. Bounds on resonances

Combining the Propositions 3.1-3.5-3.6 and the same kind of arguments used by Guillopé-Zworski [11], we can prove the Theorems.

Proof of Theorem 1.1: the first thing is to construct the final parametrix. We define for $\Re(\lambda) > -\frac{N+1}{2}$

$$\mathcal{E}_N(\lambda) := \chi_{L,\delta}^i R(\lambda_N)\chi_\delta^i + \sum_{k=1}^n (I_k)^*\mathcal{E}_N^k(\lambda)(I_k)_* + (I_r)^*\mathcal{E}_N^r(\lambda)(I_r)_*$$

$$\mathcal{K}_N(\lambda) := \mathcal{K}_N^i(\lambda) + \sum_{k=1}^n (I_k)^*\mathcal{K}_N^k(\lambda)(I_k)_* + (I_r)^*\mathcal{K}_N^r(\lambda)(I_r)_*$$

$$\mathcal{K}_N^i(\lambda) := [\Delta_X, \chi_{L,\delta}^i]R(\lambda_N)\chi_\delta^i + (\lambda(n-\lambda) - \lambda_N(n-\lambda_N))\chi_{L,\delta}^i R(\lambda_N)\chi_\delta^i$$

and we get by construction

$$(\Delta_X - \lambda(n-\lambda))\mathcal{E}_N(\lambda) = 1 + \mathcal{K}_N(\lambda).$$

Moreover we deduce from the Propositions 3.1, 3.5 and 3.6 that $\mathcal{K}_N(\lambda)^{n+2}$ is trace class on $\rho_\delta^N L^2(X)$ such that $||\mathcal{K}_N(\lambda_N)||_{\mathcal{L}(\rho_\delta^N L^2(X))} \leq \frac{1}{2}$ if δ is chosen small and N large, and $\mathcal{E}_N(\lambda)$ is bounded from $\rho_\delta^N L^2(X)$ to $\rho_\delta^{-N} L^2(X)$. Consequently, $(1 + \mathcal{K}_N(\lambda_N))$ is invertible on $\rho_\delta^N L^2(X)$ and Fredholm analytic theory allows to invert $(1+\mathcal{K}_N(\lambda))$ meromorphically with finite rank poles on the same Hilbert space, which gives the analytic continuation of $R(\lambda)$ to $\{\Re(\lambda) > -\frac{N+1}{2}\}$ as a family of operators in $\mathcal{L}(\rho^N L^2(X), \rho^{-N} L^2(X))$, thus to $\mathbb{C}$ from $L^2_{comp}(X)$ to $L^2_{loc}(X)$ since N can be chosen arbitrarily large. □

Proof of Theorem 1.2: we define the determinant

$$D_N(\lambda) := \det(1 + \mathcal{K}_N(\lambda)^{n+2})$$

on $\rho_\delta^N L^2(X)$. This is a meromorphic function in $\Re(\lambda) > \frac{N+1}{2}$ such that the resonances of Δ_X are contained in the set of zeros of D_N with multiplicities and $\frac{1}{2}(n-\mathbb{N})$ with multiplicity of $\frac{1}{2}(n-j)$ bounded by $v_j(D_N) + C(j+1)^{n+1}$ where $v_j(D_N)$ is the order of the zero (or pole) j for D_N (see the appendix of [11] for the multiplicity). Moreover by fixing δ small enough, it is clear that $|\det(1 + \mathcal{K}_N(\lambda_N)^{n+2})| > \frac{1}{2}$ for N large.

δ is now fixed as before, we then use Lemma 6.1 of [10] and deduce that

$$|D_N(\lambda)| \leq q \sum_{k=1,\dots,n,r} \det(1 + q|\mathcal{K}_N^k(\lambda)|)^q + q\det(1 + q|\mathcal{K}_N^i(\lambda)|^{n+2})^q$$

for some $q>0$ independant of N. Moreover, it is straightforward to see that there exists $C>0$ such that

$$\mu_l(\mathcal{K}^i_N(\lambda)) \leq C(|\lambda-\lambda_N|+1)l^{-\frac{1}{n+1}}$$

which combined with the Propositions 3.1, 3.5 and 3.6 shows that

$$|D_N(\lambda)| \leq Ce^{CN^{n+2}}$$

for $|\lambda| \leq \frac{N}{2}$ and $\mathrm{dist}(\lambda, \frac{1}{2}(n-\mathbb{N})) > \frac{1}{8}$. To complete the proof, it suffices to multiply $D_N(\lambda)$ by the function

$$g_P(\lambda) := \lambda^{P2^{n+1}} \prod_{\omega\in U_{2n+4}\cdot\frac{1}{2}(n-\mathbb{N})} \left(E\left(\frac{\lambda}{\omega}, n+2\right)\right)^{P(|2\omega|)^{n+1}}$$

as defined in Section 5 of [11] (U_m being the set of m-th root of the unity and $E(z,p) := (1-z)\exp(z+\cdots+p^{-1}z^p)$ are the elementary Weierstrass functions) and use the Lemmas 5.1 and 5.3 of this article to prove that for P big enough (independant of N), $D_N(\lambda)g_P(\lambda)$ is a holomorphic function bounded by $Ce^{CN^{n+2}}$ in the disc $|\lambda| \leq \frac{N}{2}$ (note that the maximum principle is used to control the norms near the points $\frac{n-j}{2}$ for $j \in \mathbb{N}$). In view of the discussion about the relation between resonance multiplicity and the valuation of determinant, this completes the proof of the Theorem by applying Jensen's lemma to g_pD_N in the disc centered in $\lambda_N = \frac{N}{4}$ with radius $\frac{N}{2}$. □

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