Zero-cycles (and rational points) over a global field : Per Salberger 1983 to 1993, and beyond

Jean-Louis Colliot-Thélène
(CNRS et Université Paris-Sud, Paris-Saclay)

Göteborg, 19 Juillet 2017

Some topics to which Per Salberger contributed during the years 1983 to 1993.

• Algebraic K-Theory of orders and their Severi-Brauer schemes
• Finiteness of certain Chow groups over finitely generated fields
• Over a number field, Hasse principle and weak approximation for rational points, Brauer-Manin obstruction, descent (torsors) and fibration methods
• Over a number field, analogues for zero-cycles
• Over a number field or a local field, finiteness for torsion in the codimension 2 Chow group under some geometric hypothesis
• Higher class field theory
• Chow group of zero-cycles representable implies vanishing of invariants (variations on Mumford, Bloch, Bloch-Srinivas)

In this talk, I shall say a few words on the third topic (rational points) and mainly discuss the fourth topic (arithmetic of zero-cycles) and its developments.

Rational points

Initial work : Classical work on homogeneous spaces of semisimple groups (Hasse, Eichler, Kneser, Harder), and of tori (Voskresenskii, reinterpretation of Poitou-Tate class field theory).
For X/k smooth, projective, basic remark (Manin 1970):
$X(k) \subset X(\mathbf{A}_k)^{Br}$, where $X(\mathbf{A}_k)^{Br}$ is the kernel of the pairing

$$X(\mathbf{A}_k) \times [\mathrm{Br}(X)/\mathrm{Br}(k)] \to \mathbb{Q}/\mathbb{Z}.$$

Further work on rational points of homogeneous spaces with connected stabilizers : Sansuc 1981, Borovoi 1996.
Various varieties which are birational to such spaces. For example, Hasse principle for del Pezzo surfaces of degree at least 5.

Conjecture RPRC (Rational points on Rationally Connected varieties)

Let X be a smooth, projective, geometrically connected variety over a number field k. If X is geometrically rationally connected, then $X(k)$ is dense in the Brauer-Manin set $X(\mathbf{A}_k)^{Br}$.
(Control of both Hasse principle and weak approximation)

(conjecture made by CT-Sansuc 1979 for surfaces, extended 1999 in arbitrary dimension)

For such X, $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite and the closed subset $X(\mathbf{A}_k)^{Br} \subset X(\mathbf{A}_k)$ is open in $X(\mathbf{A}_k)$.

Beyond the world of homogeneous spaces of linear algebraic groups, results on conjecture RPRC have been obtained by various methods.

• Circle method, mostly for complete intersections in projective space : here one counts points and as a corollary deduces their existence

• Descent method (universal torsors, CT-Sansuc) combined with fibration method (Hasse, .., CT-Sansuc-Swinnerton-Dyer)

• Descent method combined with circle method

• In some special situations, results on zero-cycles (below) imply results on rational points

• Theorem : *Conjecture RPRC holds for conic bundles over* $\mathbb{P}^1$ *with at most 4 geometric degenerate fibres.*

CT-Sansuc-Swinnerton-Dyer 1984-1987 for Châtelet surfaces $y^2 - az^2 = P(x)$, $P(x)$ of degree 4) ;
Salberger 1986-1987 for some more conic bundles;
CT 1989 and Salberger (unpublished) independently in the general case

C. R. Acad. Sc. Paris, t. 303, Série I, nº 7, 1986

THÉORIE DES NOMBRES. — *Sur l'arithmétique de*

Note de **Per Salberger,** présentée par Jean-Pierre Serre.

Pour certaines surfaces de Del Pezzo de degré 4 définies sur un
l'obstruction de Manin au principe de Hasse est la seule, et, lorsqu'elles
une conjecture de Colliot-Thélène et Sansuc [3] sur la taille du groupe

NUMBER THEORY. — On the arithmetic of certain Del Pezzo su

We prove that the Manin obstruction to the Hasse principle is the o
degree 4 over number fields. Further, when they have a rational point,
and Sansuc about the size of the Chow group of zero-cycles of degree ze

• *Conjecture RPRC holds for smooth projective models of cubic hypersurfaces with 3 conjugate singular points* (CT-Salberger 1988-89, descent + fibration). Parallel to work done in CT-Sansuc-SwD 84-87 for intersection of two quadrics containing a globally rational pair of skew lines.

• Combination of descent and circle method, in a situation where it was advocated by Salberger : Heath-Brown and Skorobogatov 2002, Heath-Brown and Browning 2012, Schindler-Skorobogatov 2014

• A conditional technique using Schinzel's hypothesis (CT-Sansuc 1979-1982). Gives RPRC for $y^2 - az^2 = P(x)$, $a \in k^*$, $P(x) \in k[x]$.

• A conditional technique using finiteness of Sha of elliptic curves and (often) Schinzel's hypothesis. Invented by Swinnerton-Dyer 1995 and developed by him, CT, Skorobogatov, Wittenberg, Harpaz. The technique works for some families of varieties which are not rationally connected, e.g. some $K3$-surfaces.

• Over the rationals, an unconditional technique building on work of Green, Tao, Ziegler (2010, 2012) ensuring that finite systems of degree 1 polynomials in at least two variables over $\mathbb{Z}$ (with appropriate local conditions) simultaneously represent primes.
Browning, Matthiesen, Skorobogatov 2014;
Harpaz, Skorobogatov, Wittenberg 2014
Example : *Conjecture RPRC holds for equation* $y^2 - az^2 = P(x)$ *where* $P(x) \in \mathbb{Q}[x]$ *has all its roots in* $\mathbb{Q}$.

For arbitrary smooth, projective, geometrically irreducible varieties, from the very beginning it was unreasonable to expect $X(k)$ is dense in $X(\mathbf{A}_k)^{Br}$, or even that $X(\mathbf{A}_k)^{Br} \neq \emptyset$ implies $X(k) \neq \emptyset$.

The first unconditional example with $X(\mathbf{A}_k)^{Br} \neq \emptyset$ and $X(k) = \emptyset$. was produced by Skorogobatov in 1999. This has led to a series of papers analysing the obstruction or producing other types of obstruction (Harari, Skorobogatov, Poonen, CT-Pál-Skorobogatov, Smeets).

Beyond rationally connected varieties, the question whether $X(k)$ is dense in $X(\mathbf{A}_k)^{Br}$ (after getting rid of the connected components) is open for curves, for abelian varieties and for $K3$-surfaces.

Zero-cycles

Let X be a projective variety (of finite type) over a field k. A point P of the scheme X is closed if and only if its residue field $k(P)$ is a finite extension of k.
One associates to X the free abelian group $Z_0(X)$ on closed points. Given a proper k-morphism $f : C \to X$ from a projective, integral, smooth curve C to X and a rational function $g \in k(C)^*$, one associated the zero-cycle $f_*(div_C(g)) \in Z_0(X)$.
The Chow group of zero-cycles $CH_0(X)$ is the quotient of $Z_0(X)$ by the group spanned by all such zero-cycles when (C, f, g) vary.
There is a degree map $CH_0(X) \to \mathbb{Z}$. Its kernel is denoted $A_0(X)$.

Conjectures on zero-cycles for *geometrically rational surfaces* (CT/Sansuc 1981)
Let X be a smooth, projective, geometrically rational surface over a number field k. [Examples : conic bundles over the projective line; del Pezzo surfaces, in particular smooth cubic surfaces.]

Conjecture A : *There is an exact sequence of finite groups :*

$$0 \to Ш^1(k, S) \to A_0(X) \to \oplus_v A_0(X \times_k k_v) \to Hom(\mathrm{Br}(X)/\mathrm{Br}(k), \mathbb{Q}/\mathbb{Z}),$$

where S is the k-torus with character group $\hat{S} = \mathrm{Pic}(\overline{X})$.
The sequence is induced by a functorial map $A_0(X) \to H^1(k, S)$.

Conjecture C *If there exists a family $\{z_v\}_{v \in \Omega_k}$ of local zero-cycles of degree one, and if If $\mathrm{Br}(X) = \mathrm{Br}(k)$, then there exists a global zero-cycle of degree one.*

Zero-cycles on *arbitrary varieties*

Conjecture E_1 (Saito 89, CT 95)
Let X be a smooth, projective, geometrically connected variety over a number field k. If there exists a family $\{z_v\}_{v\in\Omega_k}$ of local zero-cycles of degree one orthogonal to the Brauer group of X, then there exists a global zero-cycle of degree one.

(motivated by the work of Cassels and Tate on abelian varieties, by the work of CT-Sansuc 1981 on rational surfaces, and by higher class field theory of Bloch and Kato-Saito in the 80s)

It may sound strange to allow for arbitrary varieties in this conjecture, when RPRC does not extend too far. However, for a number of counterexemples to the analogue of RPRC for some varieties which are not rationally connected, one has managed to prove the existence of a zero-cycle of degree 1 (recently this was achieved by Creutz for Skorobogatov's 1999 example).

Conjecture E *Let X be a smooth, projective, geometrically connected variety over a number field k. The natural complex*

$$\widehat{CH_0(X)} \to \prod_v \widehat{CH_0'(X_v)} \to Hom(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z})$$

is exact.
Here $\widehat{A} = \operatorname{proj\,lim}_n A/n$.
Here $CH_0'(X_v) = CH_0(X \times_k k_v)$ for v a nonarchimedean place, and a modified version at the archimedean places.

(CT-Sansuc 81, Kato-Saito 86, CT 95 and 99, van Hamel 2003, Wittenberg 2012)

Subsumes earlier conjectures E_1 (on existence of zero-cycles of degree 1) and E_0 (on classes of zero-cycles of degree zero in the Chow group).

For the first type of nontrivial conic bundles over $\mathbb{P}^1$ (Châtelet surfaces, $y^2 - az^2 = P(x)$, with $P(x)$ of degree 4), conjecture *RPRC* proved by CT, Sansuc, Swinnerton-Dyer 1984-1987. In that case E follows from *RPRC*.

Fundamental result and new technique :

Theorem (Salberger 1988 + ϵ)
Conjectures A, C, E on zero-cycles holds for surfaces X with a conic bundle structure $X \to \mathbb{P}^1_k$.
These are surfaces birationally given by an affine equation

$$a(t)x^2 + b(t)y^2 + c(t) = 0$$

with $a(t), b(t), c(t) \in k[t]$, none of them zero.

Salberger 1988 if $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$.
$+\epsilon$: CT-Swinnerton-Dyer 1993-1994, Salberger 1993-2003.

Salberger's ideas in the 1988 paper

Algebraic part :
Let $\pi : X \to \mathbb{P}^1_k$ be a family of conics. The generic fibre is a conic over $k(\mathbb{P}^1)$. Let $D/k(\mathbb{P}^1)$ denote the associated quaternion algebra. Let $k(P^1)^*_{dn} \subset k(\mathbb{P}^1)^*$ be the multiplicative group of rational functions $f \in k(\mathbb{P}^1)^*$ whose divisor on $\mathbb{P}^1$ is the image of a zero-cycle on X under $\pi_* : Z_0(X) \to Z_0(\mathbb{P}^1)$.
These are precisely the elements $f \in k(\mathbb{P}^1)^*$ such that at any closed point $M \in \mathbb{P}^1_k$ one may write $f = u_P.Nred(\xi_P)$ for some $u_P \in k(\mathbb{P}^1)^*$ invertible at P and some $\xi_P \in D^*$.

One identifies $A_0(X) = Ker[\pi_* : CH_0(X) \to CH_0(\mathbb{P}^1) = \mathbb{Z}]$ with the quotient $k(\mathbb{P}^1)^*_{dn}/k^*.Nred(D^*)$.

There is a well defined homomorphism

$$k(\mathbb{P}^1)^*_{dn}/Nred(D^*) \to \oplus_{i \in I} k_i^*/Norm_{K_i/k_i} K_i^*$$

where i runs through the finite set I of bad reduction points for the conic bundle fibration $X \to \mathbb{P}^1$, where we may assume the conic reduces to a pair of secant lines defined over a quadratic extension K_i/k_i of the residue field k_i at the point i.
Quotienting by k^* on both sides induces a map on $A_0(X)$ which is essentially the map $A_0(X) \to H^1(k, S)$ (where $\hat{S} = \mathrm{Pic}(\overline{X})$) earlier defined via algebraic K-theory by Spencer Bloch and via universal torsors by CT/Sansuc.
The $k(\mathbb{P}^1)^*_{dn}$ point of view introduced by Salberger "reduces" questions on zero-cycles on X to questions on rational functions on $\mathbb{P}^1$.

For the existence of a zero-cycle of degree 1, on a conic bundle $X/\mathbb{P}^1_k$ with say a smooth fibre above the point at infinity, there is a similar discussion using the group $k(\mathbb{A}^1)^*_{dn}$ with conditions only at closed points of $\mathbb{A}^1$.
If one finds a polynomial $P(t) \in k[t]$ of degree N which is in $k(\mathbb{A}^1)^*_{dn}$, then there is an (effective) zero-cycle of degree N on X.
If one does this for N and $N+1$ then there is a zero-cycle of degree 1 on X.

Arithmetic part, in the simple case $y^2 - az^2 = P(x)$, with $P(x)$ irreducible (cf. Salberger STN Paris 87-88). The hypothesis $P(x)$ irreducible ensures that the Brauer group of a smooth compactification is reduced to $\mathrm{Br}(k)$. Thus there is no Brauer-Manin obstruction.

Let us write $k[t]_{dn}$ for $k[t] \cap k(\mathbb{A}^1)^*_{dn}$.
Fix $N \geq deg(P)$. From local zero-cycles of degree one one gets polynomials $R_v[t] \in k_v[t]_{dn}$, of degree N, for all v, with zeros outside of $P(t) = 0$.

Use Euclidean division, strong approximation, Dirichlet's theorem on primes in arithmetic progression in the field $E = k[t]/P(t)$ to approximate the polynomials $R_v[t] \in k_v[t]_{dn}$ by an irreducible polynomial $R(t) \in k[t]$ such that, either by a good reduction argument or by the approximation at bad places v, the conic $y^2 - az^2 = P(\rho)$ over the field $F := k(\rho) = k[t]/R(t)$ has points in all completions of F except possibly at the Dirichlet place (here use is made of the formula for the resultant of two polynomials).

One concludes with the reciprocity fact : a conic over a number field F which has points in all completions F_v except possibly for one v has a rational point over F (used in Hasse's proof of the Hasse principle for zeros of four-dimensional quadratic forms.) Thus there is a closed point of degre N on X. And N is any integer greater than $deg(P)$. This proves conjecture C for such surfaces.

The construction may be viewed as a successful substitute for Schinzel's hypothesis. One is given an irreducible polynomial $P(t)$ over a number field k. It is hard to find an almost integral value $\alpha \in k$ such that $P(\alpha)$ is almost a prime. However for any degree $N \geq deg(P)$ one may easily produce a field extension L/k of degree N and an almost integral element $\beta \in L$ such that $P(\beta)$ is almost a prime in L. Similarly for a finite set of polynomials (e.g. for the twin primes problem).
[Here "almost" means away from the finite set of obvious bad reduction places and from the primes small with respect to the degree of the polynomial $R(t)$.]

From zero-cycles to rational points

For conic bundles over $\mathbb{P}^1_k$ with at most 5 geometric degenerate fibres, any zero-cycle of degree 1 is rationally equivalent to a rational point (CT-Coray, vastly and cleverly generalized by Salberger in an unpublished part of his 1985 thesis).

Used by Salberger to study conic bundles with 4 degenerate fibres. Using Salberger 1988 together with the descent method, Salberger-Skorobogatov 1991 then prove Conjecture RPRC for del Pezzo surfaces of degree 4 with a rational point (which are birrational to conic bundles with 5 degenerate fibres).

From rational points to zero-cycles

Theorem (Yongqi Liang 2013). *Let X/k be a smooth, projective, geometrically rationally connected variety over a number field k. If conjecture RPRC holds for X_K/K for all finite extensions K/k, then conjecture E holds for zero-cycles on X.*

Application : *Conjecture E holds for X smooth compactification of a homogeneous space of a connected linear algebraic group with connected stabilizers.*
(uses Borovoi's result for rational points).

Salberger's 1988 fundamental work on zero-cycles on conic bundles over $\mathbb{P}^1$ was extended over the years
(CT, Swinnerton-Dyer, Skorobogatov, 1994-1998; Frossard 2003; van Hamel 2003)

A most general result in that direction (CT-Sk-SwD 1998)

Theorem *Let k be a number field. Let $f : X \to \mathbb{P}^1_k$ be a family (X smooth, projective, geometrically connected, f proper, dominant, generic fibre geometrically irreducible). Assume*
(a) Hasse principle and weak approximation hold for the smooth fibres
and
(b) each fibre X_P above a closed point P contains a component of multiplicity one which is split by an abelian extension of $k(P)$
Then conjecture E_1 holds for X.

Given any degree $d >> 0$, one produces a closed point M of degree d on P^1 such that fibre $X_P/k(P)$ has points over each completion of $k(P)$.

A weak version of conjecture on $A_0(X)$ is also proved there. For families of Severi-Brauer varieties of square-free index, upon use of the Merkurjev-Suslin theorem, Conjecture E is proved by Frossard 2003 and van Hamel 2003. The square-free index condition is removed by Wittenberg (2012).

The case of fibrations $X \to C$ where C is an arbitrary curve rather than $\mathbb{P}^1$, and one accepts the finiteness of the Tate-Shafarevich group of the Jacobian of C, under the same assumptions (a) and (b), was handled in a series of papers (CT 2000, Frossard 2003, van Hamel 2003, Wittenberg 2012, Yongqi Liang).

Wittenberg 2012 introduced a new point of view which gets rid of the appeal to Merkurjev-Suslin. It involves results of Saito and Sato on Chow groups of varieties over a p-adic field, and uses properties of (geometrically) rationally connected varieties (à la Kollár, Miyaoka, Mori). His paper then ultimately proceeds by reduction to CT/Sk/SwD 98.

Hypotheses (a) and (b) are satisfied when the generic fibre is either a quadric or a Severi-Brauer variety.
If the generic fibre is a del Pezzo surface of degree 6, then (a) holds but (b) need not hold.

An ideal theorem would replace (a) by conjecture E for the fibres above smooth closed points, and would get rid of abelianity in condition (b), i.e., as a first step simply assume that all closed fibres contain a component of multiplicity one.

In a simple new case, (b) (abelianity) was dispensed with by Dasheng Wei 2014.

In the parallel, harder context of the search for rational points, some tools were developed, in particular by D. Harari (1994) to replace hypothesis (a) for the fibres by conjecture *RPRC*. Success could be achieved in some cases, under a very stringent assumption on the number of bad geometric fibres.

For zero-cycles, it has basically taken twenty years for a general result to be achieved, for the general class of varieties X with a fibration $X \to \mathbb{P}^1_k$ whose fibres are rationally connected varieties. Here is now the vast generalisation of Salberger's 1988 theorem.

Theorem (Harpaz and Wittenberg 2016)
Let k be a number field and $f : X \to \mathbb{P}^1_k$ be a family of rationally connected varieties. If conjecture E holds for the smooth fibres of f over closed points, then it holds for X.

[A celebrated theorem of Harris, Graber and Starr 2000 ensures that all fibres of such a fibration have a component of multiplicity one.]

There is a version with E replaced by E_1.
There is also a version with a fibration over a curve of arbitrary genus whose Jacobian has finite Ш.

One immediately deduces :

Corollary. *Let k be a number field and $f : X \to \mathbb{P}^n_k$ be a family of rationally connected varieties. If conjecture E holds for the smooth fibres of f over closed points, then it holds for X.*

Thus conjecture E holds for any $f : X \to \mathbb{P}^n_k$ such that the generic fibre is birational to a homogeneous space of a connected linear algebraic group. (A result of Harpaz and Wittenberg 2017 – also building on the fibration method for zero-cycles – allows one to dispense with the connected condition for stabilizers.)

Thus conjecture E holds for :

- Conic bundles over $\mathbb{P}^n_k$
- Smooth projective models of affine varieties with equation

$$Norm_{K/k}(\Xi) = P(t_1, \ldots, t_n)$$

with K/k finite field extension, Ξ variable in K, P polynomial in n variables over k.

One important idea in HW2016 is to shift attention from the exact sequence

$$\mathrm{Br}(k) \to \oplus_v \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$$

to the (more general) Poitou-Tate exact sequences

$$H^1(k, T) \to \oplus_v H^1(k_v, T) \to Hom(H^1(k, \hat{T}), \mathbb{Q}/\mathbb{Z})$$

for T an algebraic k-torus with geometric character group $\hat{T}$.

The abelianity condition on fibrations in earlier result was somehow forced by the need to model bad fibres of $X \to \mathbb{P}^1_k$ by bad fibres of a relative Severi-Brauer family $Y \to \mathbb{P}^1_k$: for such families, bad fibres are split by a cyclic extension.

Suppose the family $X \to \mathbb{P}^1_k$ of rationally connected varieties has a bad fibre over say a k-point $P \in \mathbb{A}^1_k = Speck[t]$ with coordinate $t = c \in k$. One considers a reduced, irreducible component Z/k of multiplicity one of the fibre X_P (there is one, Graber-Harris-Starr), then the integral closure K of k in the function field $k(Z)$ and then one models that component by the equation

$$t - c = Norm_{K/k}(\Xi),$$

where Ξ is a "variable" in K. One introduces the k-torus $T = R^1_{K/k}\mathbb{G}_m$ (elements of norm 1) and one uses torsors under T over open sets of $\mathbb{P}^1$ as a substitute (in fact an extension) of Severi-Brauer schemes over open sets of $\mathbb{P}^1$.

The use of the zero-cycle variant (CT-SwD1994) of Harari's formal lemma (1994 for rational points) for the Brauer group of open varieties in conjunction with the sequence

$$\mathrm{Br}(k) \to \oplus_v \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$$

is then replaced by a similar formal lemma for torsors under a torus T, in conjunction with the exact sequence

$$H^1(k, T) \to \oplus_v H^1(k_v, T) \to Hom(H^1(k, \hat{T}), \mathbb{Q}/\mathbb{Z}).$$

One is here slightly reminded of Salberger's 1993 paper (published in 2003).

Going back and forth between torsors unter tori and Brauer classes is not uncommon in the study of rational points (already done in CT-Sansuc 1977-1987).

Some results in the function field case

Over $k = \mathbb{F}(C)$ (function field of a curve over a finite field).

Use of connection with the integral Tate conjecture for 1-cycles over a finite field :

The analogue of Salberger's 1988 result on conic bundles over $\mathbb{P}^1$ was established by Parimala and Suresh 2016.

(CT-Swinnerton-Dyer 2011) For any degree $d > 0$, Conjecture E_1 over $k = \mathbb{F}(t)$ holds for smooth surfaces given by

$$\sum_{i=0}^{3} a_i(t) X_i^d = 0$$

with each $a_i(t) \in \mathbb{F}[t]$ linear. For $d \geq 5$, these are surfaces of general type.

Some results for cubic hypersurfaces (CT via Swinnerton-Dyer's method and a known case of the Tate conjecture as a substitute for finiteness of Ш; Browning-Vishe).

Over $k = \mathbb{F}(C)$, deformation techniques have enabled Zhiyu Tian 2015 to establish the Hasse principle for mooth cubic hypersurfaces in $\mathbb{P}^n_{\mathbb{F}(C)}$ for $n \geq 5$ ($char(\mathbb{F}) \geq 7$).

Back to the number field case, what we still do not know

Smooth cubic hypersurfaces in $\mathbb{P}^n_k$, $n \geq 3$
For $n = 3$: Conjecture RPRC for rational points, Conjecture E (or already E_1) for zero-cycles.
Hasse principle for $n \geq 4$ (over $k = \mathbb{Q}$, Heath-Brown and Hooley have established this for $n \geq 8$.)

Smooth intersections of two quadrics in $\mathbb{P}^n_k$, $n \geq 4$
Conjecture RPRC for $n = 4$ (conditional positive answer in most cases, Wittenberg 2007)
Hasse principle in $\mathbb{P}^n_k$, $n = 5, 6$ (for $n \geq 8$, proved 1987, for $n = 7$, Heath-Brown 2015; conditional positive answer for $n \geq 5$, Wittenberg 2007)

Conjecture E_1 for some surfaces of general type, e.g. for twisted Fermat surfaces of degree $d \geq 5$.

Grattis på fődelsedagen, Per