Online seminar Quadratic forms, linear algebraic groups and beyond (P. Gille, Z. Reichstein, K. Zainoulline)

Stable rationality and quadratic forms, a survey

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The topic of this series of two talks is precisely : quadratic forms *and beyond*. There was a time when quadratic forms over fields were deemed "too special" a topic. In these two talks I shall review how *quadratic forms, and more particularly Pfister forms*, have been used in an essential way to prove spectacular non-rationality results in complex algebraic geometry (something one will miss if one just reads the MR reports on works in the area).
The talks are not meant for the specialists.

Talk 1, 20th October 2021

An integral variety X of dimension d over a field k is called rational if it is birational to projective space $\mathbb{P}^d_k$, in other words if its function field $k(X)$ is purely transcendental over k.

An integral variety X over a field k is called stably rational if $X \times_k \mathbb{P}^r$ is birational to $\mathbb{P}^{r+d}$ for some $r \geq 0$.

Aim (generalized Lüroth problem) **to decide whether certain varieties which are very close to being rational, e.g. unirational varieties, rationally connected varieties, are rational, or at least stably rational.**

Around 1970, negative answer to Lüroth's problem over the complex field (are unirational varieties rational ?)

• Clemens-Griffiths (cubic threefolds) : intermediate jacobian. Seems special to dimension 3.

• Iskovskikh-Manin (quartic threefolds) : Noether, Fano, Segre, birational automorphisms, linear systems, canonical bundle, coherent cohomology. Later much developed (Ridigity). Applies to many Fano varieties.

• Artin-Mumford (conic bundle threefold) : torsion in $H^3(X, \mathbb{Z})$, Brauer group, étale cohomology. In contrast with the previous two methods, this method also dispoves stable rationality. Powerful to study BG for G finite group (Saltman, Bogomolov), but initially seemed useless for (smooth) Fano hypersurfaces.

• A further method was devised by Kollár in 1995

A list of birational invariants which are trivial on smooth, projective stably rational varieties X over a char. zero field k.

$H^0(X, \Omega^{\otimes r}) = 0$ for $r > 0$ and X rationally connected.
Over $k = \mathbb{C}$, $H^3_{Betti}(X, \mathbb{Z})_{tors} = 0$ for X stably rational.

Brauer group
$\mathrm{Br}(k) \simeq \mathrm{Br}(X) := H^2_{\mathrm{et}}(X, \mathbb{G}_m)$ for X/k stably rational.

Unramified cohomology
$H^i(k, \mathbb{Z}/n) \simeq H^i_{nr}(k(X)/k, \mathbb{Z}/n)$ (Galois cohomology of fields) X/k stably rational.
Also with other coefficients, e.g. $\mu_n^{\otimes j}$ or $\mathbb{Q}/\mathbb{Z}(j)$.

Artin-Mumford 1972

Several birational incarnations for these threefolds

• Double cover of $\mathbb{P}^3$ with ramification locus a quartic with 10 ordinary singular points

• Conic bundles over $\mathbb{P}^2_{\mathbb{C}}$ whose ramification locus is the union of two smooth cubic curves transversal to each other, and there exists a smooth conic which is tritangent to each of these cubics.

Artin and Mumford show $H^3_{Betti}(X(\mathbb{C}), \mathbb{Z})_{tors} \neq 0$ for an explicit smooth projective, unirational model $X/\mathbb{C}$.

All conic bundles over $\mathbb{P}^2_{\mathbb{C}}$ are rationally connected (exercise, use Tsen, but Max Noether is enough). It is an open question whether they are unirational.

Quite generally, for $X/\mathbb{C}$ a smooth, projective, connected variety $X/\mathbb{C}$, the group $\mathrm{Br}(X)$ is an extension of $H^3_{Betti}(X(\mathbb{C}), \mathbb{Z})_{tors}$ by $(\mathbb{Q}/\mathbb{Z})^{b_2-\rho}$ (Grothendieck), and $b_2 - \rho = 0$ if and only if $H^2(X, O_X) = 0$. If X is rationally connected, then $H^i(X, O_X) = 0$ for any $i > 0$.

Saltman (1984), Bogomolov (1987)
Question : How to compute $\mathrm{Br}(X)$ for an unknown smooth projective model X/k of a given singular variety Y/k ?
Answer : Use all residues attached to discrete valuations on the function field of X (trivial on the ground field k).
For $X/\mathbb{C}$ smooth projective, we have :

- $\mathrm{Br}(X) = \mathrm{Ker}[\mathrm{Br}(\mathbb{C}(X)) \to \oplus_{x \in X^1} H^1(\mathbb{C}(x), \mathbb{Q}/\mathbb{Z})]$
- (unramified Brauer group)

$\mathrm{Br}(X) = \mathrm{Br}_{nr}(\mathbb{C}(X)) := \mathrm{Ker}[\mathrm{Br}(\mathbb{C}(X) \to \oplus_v H^1(\kappa(v), \mathbb{Q}/\mathbb{Z})]$
where v runs through all discrete valuations of $\mathbb{C}(X)$, and $\kappa(v)$ denotes the residue field

- $\mathrm{Br}(X) = \mathrm{Ker}[\mathrm{Br}(\mathbb{C}(X)) \to \oplus_x H^1(\kappa(x), \mathbb{Q}/\mathbb{Z})]$ where x runs through all codimension 1 points of all normal models of X.

One needs formulas to compute the residues. Such formula are well known on "symbols" such as quaternion algebras (a, b).
In some cases, this is enough to produce nontrivial elements in $\mathrm{Br}(X)$, which already proves that a variety X is not (stably) rational.
In some cases, one may obtain the exact value of $\mathrm{Br}(X)$.

Was used by Saltman and then Bogomolov to disprove rationality and give measure of lack of stable rationality of field of invariants $(\mathbb{C}(V))^G$ for various finite groups G and linear faithful linear action of G on a vector space V.

Artin–Mumford revisited (CT-Ojanguren 89)

Recall : For a smooth conic C over a field F, given by an equation $X^2 - aY^2 - bT^2 = 0$ (here $\mathrm{char}(F) \neq 2$), one has $\mathrm{Ker}[\mathrm{Br}(F) \to \mathrm{Br}(F(C)] = \mathbb{Z}/2.\alpha$ where $\alpha = (a, b)$ is the associated quaternion class (Witt, 1934).
[Moreover the image of $\mathrm{Br}(F)$ in $\mathrm{Br}(F(C)]$ coincides with $\mathrm{Br}(C) \subset \mathrm{Br}(F(C)]$.]

A conic bundle $X \to S$ over $S = \mathbb{P}^2$, with no section, is birationally given by its generic fibre C, which is a smooth conic over $F = \mathbb{C}(S)$ with no rational point. This corresponds to a quaternion algebra A/F with nontrivial class in $\mathrm{Br}(F)$.
In the Artin-Mumford context, one lets $A = (f, g_1 g_2)$ with functions f, g_1, g_2 in $\mathbb{C}(S)^\times$. Here g_1 et g_2 are the affine equations of the two smooth cubics and $f = 0$ is the affine equation of the conic tritangent to each of them. Let $\beta \in \mathrm{Br}(\mathbb{C}(X))$ be the image of (f, g_1), which is also the image of (f, g_2).

One shows :

(i) for $i = 1, 2$, there exists a discrete valuation w_i of $\mathbb{C}(\mathbb{P}^2)$ where $(f, g_i) \in \mathrm{Br}(\mathbb{C}(\mathbb{P}^2))$ has a nontrivial residue,
By Witt's result, this implies $\beta \neq 0$. [If $\beta = (f, g_1)_{\mathbb{C}(X)} = 0$ then either $(f, g_1) = 0$ or $(f, g_1) = (f, g_1 g_2)$ hence $(f, g_2) = 0$.]

(ii) for any discrete valuation w of $\mathbb{C}(\mathbb{P}^2)$, at least one of (f, g_i) has a trivial residue in w. A discussion of the possible points of $\mathbb{P}^2$ on which a valuation of $\mathbb{C}(X)$ is centered then gives that $\beta \in \mathrm{Br}(\mathbb{C}(X))$ is unramified.

Thus $\mathrm{Br}(X) = \mathrm{Br}_{nr}(\mathbb{C}(X)) \neq 0$, hence X is not stably rational.
QED

“Exercise”

Produce configuration of lines in $\mathbb{P}^2_{\mathbb{C}}$ and conic bundles over $\mathbb{P}^2_{\mathbb{C}}$, with ramification only within this set of lines, such that the total space of the fibration has nonzero unramified Brauer group. [CTOj89 contains an example with 10 lines.]

There exists such configuration of 6 lines which are degenerate versions of the two Artin-Mumford cubics.
Show that one needs at least 6 lines.

One advantage of the CT-Ojanguren version is that it is birational, hence does not require the construction of any smooth or mildly singular model for the total space of the conic bundle.
There are however situations where one is interested in flat projective models of quadric bundles over projective space, for instance for deformation arguments, as we shall later see. It is then worth considering quadric bundles over $\mathbb{P}^n$ given by a vector bundle $\mathcal{E}$ over $\mathbb{P}^n$, often $\mathcal{E} = \oplus_i O_{\mathbb{P}^n}(d_i)$, an invertible sheaf $O(m)$ on $\mathbb{P}^n$, a nonzero element in $H^0(\mathbb{P}^n, S^2(\mathcal{E})(m))$, and the vanishing locus of this section.
In the simplest case, one considers a bidegree $(d, 2)$ and a symmetric square matrix made up of homogeneous polynomials all of the same degree d. But other choices for the family $\{d_i\}$ may be more efficient.

Higher unramified cohomology

Let M be a finite Galois module over a field k. For X/k smooth, projective, integral with fraction field $k(X)$, et $i \geq 1$, let us define

$$H^i_{nr}(k(X), M) := \mathrm{Ker}[H^i(k(X), M) \to \oplus_v H^{i-1}(\kappa(v), M(-1))].$$

Here v runs through all discrete valuations of $k(X)$ which are trivial on k, the field $\kappa(v)$ is the residue field and the RHS maps are residue maps on Galois cohomology of discretely valued fields.
By definition, these groups are birational k-invariants. They are stable invariants, i.e. coincide on X and on $X \times_k \mathbb{P}^1_k$.
Bloch-Ogus theory (Gersten's conjecture) shows that one could restrict attention to the v's associated to codimension 1 points of a single smooth projective model X. It also shows that these groups are contravariant functorial for arbitrary k-morphisms of smooth, proper k-varieties.

For ℓ prime to the characteristic, the basic standard coefficients are $M = \mu_{\ell^n}^{\otimes j}$ and $\mathbb{Q}_\ell/\mathbb{Z}_\ell(j)$, the direct limit of the $\mu_{\ell^n}^{\otimes j}$ for all n.
As a consequence of Voevodsky's theorems, for $j \geq 1$ and any field F with char. $\neq \ell$,

$$H^j(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j-1)) = \bigcup_n H^j(F, \mu_{\ell^n}^{\otimes j-1}).$$

The group $H^1_{nr}(k(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell) = H^1_{et}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ classifies ℓ-primary cyclic étale covers of X.

For X/k smooth, projective, we have

$$H^2_{nr}(k(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) = \mathrm{Br}(X)\{\ell\}.$$

CT-Ojanguren 89 used H^3_{nr} to produce an analogue of Artin-Mumford with H^3.
We produced 6-dimensional unirational varieties, as a matter of fact quadric bundles of relative dimension 3 over $\mathbb{P}^3_{\mathbb{C}}$ for which the Artin-Mumford invariant vanishes but H^3_{nr} shows the varieties are not stably rational.

[Further work along these lines, with higher unramified cohomology, was done by E. Peyre and later by A. Asok.]

Let me describe the method.

First tool : **a higher analogue of Witt's theorem**.
Let F be a field, $\mathrm{car}(F) \neq 2$. Let $a_1, \ldots, a_n \in F^\times$. Let Q/F lbe the smooth quadric defined by the vanishing of the diagonal quadratic form (Pfister form) of rank 2^n

$$<< a_1, \ldots, a_n >>:=< 1, -a_1 > \otimes \cdots \otimes < 1, -a_n > .$$

Theorem. *The kernel of the map of Galois cohomology groups* $H^n(F, \mathbb{Z}/2) \to H^n(F(Q), \mathbb{Z}/2)$ *it at most* $\mathbb{Z}/2$*, it is spanned by the element* $(a_1) \cup \ldots (a_n)$*, where* (a_i) *is the class of* a_i *in* $F^\times/F^{\times 2} = H^1(F, \mathbb{Z}/2)$*.*
($n = 2$, Witt 1934 ; $n = 3$, Arason 1975 ; $n = 4$ Jacob-Rost 1989 ; any n, Orlov-Vishik-Voevodsky 2007)
This also holds for the quadric Q' defined by the Pfister neighbour of rank $2^{n-1} + 1$:

$$[< 1, -a_1 > \otimes \cdots \otimes < 1, -a_{n-1} >] \perp < -a_{n>}$$

Second tool. **A delicate configuration of 40 planes in $\mathbb{P}^3$.**
Various quotients of their equations enable one to produce rational functions $f, g, h_1, h_2 \in \mathbb{C}(\mathbb{P}^3)^\times$ whose residues at valuations of $\mathbb{C}(\mathbb{P}^3)$ have good properties.
One then lets $X/\mathbb{C}$ be a quadric bundle over $\mathbb{P}^3$ whose generic fibre is given by the quadric in $\mathbb{P}^4_{\mathbb{C}(\mathbb{P}^3)}$ defined by a Pfister neighbour of rank 5 of the Pfister form $<< f, g, h_1 h_2 >>$.
Arason's theorem and the good properties of (f, g, h_1, h_2) then ensure :
(a) The image of the cup-product $(f, g, h_1) \in H^3(\mathbb{C}(\mathbb{P}^3), \mathbb{Z}/2)$ in $H^3(\mathbb{C}(X), \mathbb{Z}/2))$ is unramified.
(b) This image does not vanish.
This yields $H^3_{nr}(\mathbb{C}(X), \mathbb{Z}/2) \neq 0$, hence X is not stably rational.
QED

An outcome of this example ("quadratic forms and beyond").

Claire Voisin (2006) had proved that *an integral version of the Hodge conjecture holds for codimension 2 cycles on complex rationally connected varieties* of dimension 3 and had asked (2007) whether this still holds for rationally connected varietes of any dimension. In a joint paper (2012), using results from algebraic K-theory, she and I related this question to that of the vanishing of the third unramified cohomology group with torsion coefficients. From the CT-Oj89 example, we thus got a negative answer to her question for rationally connected varieties of dimension at least 6. Later progress by Schreieder (2019) along similar lines has enabled him to get a negative answer in any dimension at least 4.

Kollár 1995, Specialization method.

Starting point : Matsusaka's specialization theorem (1968)
Let A be a discrete valuation ring, K its field of fractions, k its residue field. Let $X \to \mathrm{Spec}\,(A)$ be a flat projective morphism with geometrically integral fibres. Assume X is normal. If the genreric fibre is ruled (birational to a product with $\mathbb{P}^1$), the geometric special fibre is ruled.

Best result in some sense, think of a smooth cubic surface specializing to a cone over an elliptic curve.

$$x^3 + y^3 + z^3 + \lambda.t^3 = 0.$$

For integers $d \leq n$, smooth hypersurfaces of degree d in $\mathbb{P}^n_{\mathbb{C}}$ are rationally connected (Kollár-Miyaoka-Mori, Campana).
Kollár (1995) produces mildly singular projective varieties in *positive characteristic* (they are purely inseparable cyclic covers of projective space), which after desingularisation admit nontrivial global differentials of a certain type, which implies that these varieties are not ruled.
Using specialisation in unequal characteristic to such varieties, Kollár proves : *very general* smooth complex Fano hypersurfaces of degree d in $\mathbb{P}^n$ (Fano : $d \leq n$) with roughly $d \geq 2n/3$, more precisely $d \geq 2\lceil (n+2)/3 \rceil$, are not ruled, in particular not rational.

Voisin 2014, Inventiones 2015

New type of specialisation argument

Theorem (Voisin 2014). *Let C be a smooth connected curve over $\mathbb{C}$. Let $X \to C$ be a projective family of complex varieties of relative dimension at least 2, smooth away from a point $0 \in C$. If the singularities of the special fibre X_0 are all ordinary double points, and there is a resolution of singularities $\tilde{X}_0 \to X_0$ with $H^3_{Betti}(\tilde{X}_0, \mathbb{Z})_{tors} \neq 0$, then a very general fibre X_t is not stably rational.*

Proof uses : rationality implies Bloch-Srinivas decomposition of the diagonal, then Fulton's specialization for Chow groups.

Corollary : *A double cover of $\mathbb{P}^3$ ramified only along a very general quartic surface in $\mathbb{P}^3$ is not stably rational.*

Proof : ultimately reduces to the Artin-Mumford example, in the avatar of a double cover of $\mathbb{P}^3$ ramified along a quartic surface whose singular locus consists of 10 ordinary double points.

Voisin's method was generalized by CT-Pirutka in 2014. Use of the Chow group of zero-cycles over an arbitrary field rather than the Bloch-Srinivas decomposition of the diagonal (though both points of view are essentially equivalent).

Given a variety X over a field k, one lets $Z_0(X)$ be the free abelian group on closed points of X. A zero-cycle rationally equivalent to zero is by definition a linear combination of zero-cycles of the following type. One is given a proper k-morphism $f : C \to X$, where C/k is a normal integral curve, and $g \in k(C)^\times$ is a rational function on C ; one considers the zero-cycle $f_*(div_C(g))$.
The quotient of $Z_0(X)$ by such linear combinations is the Chow group $CH_0(X)$ of zero-cycles.
If X/k is proper, the map $\sum_i n_i P_i \mapsto \sum_i n_i [k(P_i) : k] \in \mathbb{Z}$ induces a homomorphism $deg : CH_0(X) \to \mathbb{Z}$ whose kernel is denoted $A_0(X)$.

Definition (CT-Pirutka). Let k be a field. A proper morphism of k-varieties $f : Z \to Y$ is called *universally* CH_0*-trivial* if for *any overfield* F *of* k the map $f_* : CH_0(Z_F) \to CH_0(Y_F)$ is an isomorphism.

If $Y = \mathrm{Spec}\,(k)$, the proper k-variety Z is called CH_0-trivial. For such a Z, for any overfield F of k, the degree map $\deg_F : CH_0(Z_F) \to \mathbb{Z}$ is an isomorphism.

Lemma. To prove that $f : Z \to Y$ is universally CH_0-trivial, it is enough to prove that for any schematic point $M \in Y$, the fibre $Z_M/\kappa(M)$ over the residue field $\kappa(M)$ is a universally CH_0-trivial $\kappa(M)$-variety.

Examples of CH_0-trivial k-varieties.

• For $n \geq 1$, et $F(x_1, \ldots, x_n)$ homogeneous of any degree, the cone defined by $F(x_1, \ldots, x_n) = 0$ in $\mathbb{P}^n_k$.

• For $n \geq 2$, a smooth quadric $Q \subset \mathbb{P}^n_k$ with $Q(k) \neq \emptyset$.

• A smooth, projective, geometrically integral k-variety which is k-rational ; i.e. k-birational to a projective space $\mathbb{P}^n_k$. More generally, a smooth, projective, geometrically integral k-variety which is retract rational.

• (CT, 2017) The Fermat cubic hypersurface $\sum_{i=0}^n x_i^3 = 0$ over $\mathbb{C}$ for any $n \geq 3$. [For n odd, the Fermat cubic hypersurface is rational. Already for $n = 4$, it is unknown whether the 3-fold Fermat cubic is stably rational or not.]

Beware : There exist CH_0-trivial surfaces over $\mathbb{C}$ which are surfaces of general type, hence are very far from being rational.

Specialisation method, baby/basic case.

Proposition (CT-Pirutka)

Let A be a henselian discrete valuation ring, K its field of fractions, k its residue field.
Let $\mathcal{X}/A$ be proper and faithfully flat with geometrically integral fibres, and smooth generic fibre X/K.
Assume the special fibre $Y = \mathcal{X} \times_A k$ admits a desingularization $f : Z \to Y$ such that $f_* =: CH_0(Z) \to CH_0(Y)$ is an isomorphism.
Suppose X is a stably K-rational variety.
Then the degree map $CH_0(Z) \to \mathbb{Z}$ is injective.

Proof : Moving lemma, Hensel, and Fulton's specialisation homomorphism.
Pick up $U \subset Y$ nonempty smooth over k, and such that $V := f^{-1}(U) \to U$ is an isomorphism. Moving lemma : any zero-cycle z_a of degree 0 on Z is rationally equivalent to a zero-cycle z_b of degree 0 supported on $V \subset Z$. Since A is henselian and U/k is smooth, may lift this to a zero-cycle z_c of degree 0 on X/K. Since X/K is stably K-rational, we know z_c is rationally equivalent to zero on X. For the proper flat $\mathcal{X}/A$ we have Fulton's specialization homomorphism $CH_0(X) \to CH_0(Y)$. Thus the 0-cycle z_b is rationally equivalent to 0 on Y. This implies that the image of z_a under $f_* : CH_0(Z) \to CH_0(Y)$ is zero. The hypothesis on f_* then implies that z_a is rationally equivalent to zero. QED

Theorem (CT-Pirutka 2014)

Let A be a discrete valuation ring, K its field of fractions, k its residue field. Assume k algebraically closed.
Let $\mathcal{X}/A$ be proper and faithfully flat with geometrically integral fibres and smooth generic fibre $X = \mathcal{X} \times_A K$.
Assume the special fibre $Y = \mathcal{X} \times_A k$ admits a desingularization $f : Z \to Y$ such that f is universally CH_0-trivial.
Let $\overline{K}$ be an algebraic closure of K. Let $\overline{X} := X \times_K \overline{K}$.
/...

Each of the following statements implies the next one :
(i) The $\overline{K}$-variety $\overline{X}$ is stably rational.
(i) The $\overline{K}$-variety $\overline{X}$ is retract rational.
(iii) The $\overline{K}$-variety $\overline{X}$ is universally CH_0-trivial.
(iv) The (smooth) k-variety Z is universally CH_0-trivial.
This last property implies :
(a) For any $i \geq 0$ and any $n > 0$ prime to $\mathrm{char}.k$, for any overfield L of k, the map $H^i(L, \mathbb{Z}/n) \to H^i_{nr}(L(Z)/L, \mathbb{Z}/n)$ is an isomorphism.
(b) For any overfield L of k, the natural map $\mathrm{Br}(L) \to \mathrm{Br}(Z_L)$ is an isomorphism.
(c) $\mathrm{Br}(Z) = 0$.

Idea of proof of (iii) implies (iv) . Proceed as in the proposition, and use reductions :
– replace the dvr A by its henselization
– do finite extensions of K
– replace the residue field k by the (big) field extension given by the function field of Y, replace A by the local ring of the generic point of Y on $\tilde{X}$, replace K by $K(X)$.

Proof of consequences (a), (b), (c). Special case of general fact : if X/k is integral smooth, projective with a k-point P, if the generic point of X is rationally equivalent to $P_{k(X)}$ on $X_{k(X)}$, then for any Rost cycle modules M over k we have $M(k) = M_{nr}(k(X)/k)$ (Merkurjev).

Application : Smooth quartic hypersurfaces in $\mathbb{P}^4_{\mathbb{C}}$ (CT–Pirutka)

One starts with a quartic surface $Y \subset \mathbb{P}^4$ birational to an Artin-Mumford threefold, simply given by stupid homogeneisation of the affine equation $w^2 - f_4(x, y, z) = 0$ corresponding to the double cover of $\mathbb{P}^3$ ramified along a quartic surface with 10 singular points.

The hypersurface Y has 9 ordinary singular points and a line of singular points. One explicitly desingularizes $f : Z \to Y$ and one checks that f is a universally CH_0-trivial morphism. One the other hand one knows that $\mathrm{Br}(Z) \neq 0$ by Artin-Mumford. The theorem then gives :

Theorem : *A very general, smooth quartic hypersurface in* $\mathbb{P}^4_{\mathbb{C}}$ *is not stably rational. It is not even retract rational.*
[That smooth quartic hypersurfaces in $\mathbb{P}^4_{\mathbb{C}}$ are not rational is a celebrated result of Iskovskikh and Manin.]

There are at least two versions for the result : one for varieties of a given type over a discrete valuation ring (as above) and a "more geometric" version for "very general" members of a family $X \to S$ of complex varieties.

The discrete valuation method enables one to handle unequal characteristic.
Using such specialization, one proves that there exist smooth quartic hypersurfaces over $\mathbb{C}$ which are defined over a number field, and are not stably rational.

Theorem (Totaro 2015). *Let $n \geq 3$. For $d \geq 2\lceil (n+2)/3 \rceil$ a very general hypersurface $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ of degree d is not universally CH_0-trivial, hence is not stably rational.*

Slightly better range than Kollár for nonrationality (but Kollár disproves uniruledness).

Method. Specialisation in unequal characteristic. Disproves CH_0-triviality of desingularisation Y of the special fibre using $H^0(Y, \Omega^j) \neq 0$, for some $j > 0$. Uses specialisation to residue characteristic 2. For even degree $d = 2a$ degenerates to a slightly singular variety, proves Kollár's desingularisation morphism is CH_0-trivial, then uses $H^0(Y, \Omega^j) \neq 0$ for some j (consequence of Kollár's computations), which implies that Y is not CH_0-trivial.

For odd degree $d = 2a + 1$, use of a double specialisation. First a degeneration to the union of a hyperplane and a hypersurface of degree $2a$, then use of a variant with *reducible special fibre* of the CT-Pirutka argument, then use of the $d = 2a$ specialisation result.

These results were followed by a series of papers, using these methods to disprove stable rationality of very general varieties in classical families of rationally connected varieties :

Conic bundles over $\mathbb{P}^n$ of certain types.

Cyclic covers of $\mathbb{P}^n$ of certain types (degree of cover, degree of the ramification locus).

Complete intersections of certain types, also in weighted projective spaces.

Families of hypersurfaces in $\mathbb{P}^r \times \mathbb{P}^s$ of various bidegrees.

The work of Voisin and CT–Pirutka used the Brauer group. Most of the papers that immediately followed, for instance those by T. Okada, used reduction to positive characteristic and differentials in positive characteristic. This seemed to give access to the largest classes of Fano varieties. As we shall see in the second talk, use of higher unramified cohomology and Pfister forms of any rank later enabled Schreieder to handle many more classes of Fano hypersurfaces.

For any smooth, projective, rationally connected variety $X/\mathbb{C}$, there exists an integer $N = N(X) \geq 1$ such that for any overfield F of $\mathbb{C}$, one has $N.A_0(X_F) = 0$. One is interested in getting lower bounds for this integer. If X is unirational, this gives a lower bound for the unirationality degree.

Kollár 1995 already gave a few results in this direction.

Chatzistamatiou-Levine ANT 2017 systematically give such lower bounds for Fano hypersurfaces, roughly in the same range (degree versus dimension) as Kollár, resp. Totaro, disprove uniruledness, resp. stable rationality. They use reduction to char. p and differentials.

Next week :

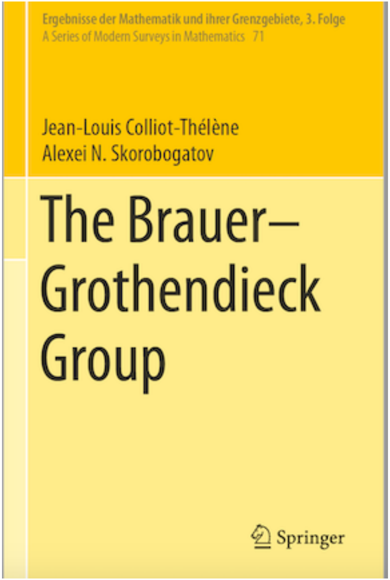
Back to the Brauer group and higher unramified cohomology

Work of Hassett-Pirutka-Tschinkel 2018
Smooth proper families of complex varieties with some fibres rational but the very general fibres not stably rational.

Work of Schreieder :
Using Pfister forms and related forms to extend the (approximate) statement
"A very general hypersurface $X \subset \mathbb{P}^n_{\mathbb{C}}$ of degree $d \geq 2n/3$ is not stably rational" (Kollár 1995, Totaro 2016)
to the (approximate) statement
"A very general hypersurface $X \subset \mathbb{P}^n_{\mathbb{C}}$ of degree $d \geq log_2 n$ is not stably rational" (Schreieder 2019)

References for this talk

Artin-Mumford Proc London Math Soc 1972 (Brauer group)
Saltman Inv math 1984 (Brauer group)
Bogomolov 1987 (Brauer group)
CT-Ojanguren Inv math 1989 : au-delà de l'exemple d'Artin et Mumford (Brauer group, higher unramified cohomology)
Kollár JAMS 1995 (specialization, differentials in positive characteristic)
Voisin 2013/2014, Inv math 2015 (specialization, Brauer group)
CT-Pirutka 2014, Ann Sc ENS 2016 (specialization, Brauer group)
Totaro JAMS 2016 (specialization, differentials in positive characteristic)
Chatzistamatiou-Levine ANT 2017 Minimal degree for unirationality (specialization, differentials in positive characteristic)
Surveys in *Birational Geometry of Hypersurfaces* (Gargnano del Garda 2018), Springer LNUMI 26 (2019)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