THE RATIONALITY PROBLEM FOR FIELDS OF INVARIANTS UNDER LINEAR ALGEBRAIC GROUPS (WITH SPECIAL REGARDS TO THE BRAUER GROUP)

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Introduction

Let V be a vector space over an algebraically closed field k of characteristic zero. Let G be a reductive subgroup of $\mathrm{GL}(V)$. Assume that G acts almost freely on V, i.e. that for a general v in V, the stabilizer of G at v is trivial. Let $k(V)$ denote the field of rational functions on V.

Problem. *Is the field of invariants $k(V)^G$ purely transcendental over k?*

This is an old problem. In these notes, it will be referred to as the *purity problem*, or the *rationality problem*, indifferently. The question is open when G is a connected group. Even the case where G is the projective linear group PGL_n is not known, except for small values of n. However, when G is finite, the question has a negative answer, as was shown by D. Saltman in 1984. Saltman's paper was soon followed by a series of papers of F. Bogomolov, then by further papers of D. Saltman. Between 1986 and 1988 we ran seminars on their work, and in July 1988 one of us lectured on this topic at the IX Escuela Latinoamericana de Matemáticas, held in Santiago de Chile, and a set of notes was distributed. Over the years, versions of these notes were circulated and used as a complement to the original work of Saltman and Bogomolov. We are grateful to the organizers of the 2004 International Conference in Mumbai for giving us the opportunity to publish a revised version of our text. We have not tried to update the notes systematically, but we have added references to work done since 1988. The rationality problem may also be raised over a field k which is not algebraically closed. The interested reader is referred to [36, 37, 92, 117, 146, 147, 153].

Here is the list of sections. Sections 1 to 4 are devoted to general definitions and results, whereas Sections 5 to 9 concentrate on the computation of the unramified Brauer group.

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Section 1 introduce notions close to "rationality": unirationality, stable rationality, retract rationality. Definitions and properties are given in terms of function fields — the algebraic side — and in terms of algebraic varieties — the geometric side.

Section 2 deals with fields of invariants K^G under a linear algebraic group G acting on a function field K. It also deals with "geometric models" of such an action, i.e. a G-action on an integral k-variety X such that K coincides with the function field $k(X)$ of X and K^G coincides with the function field $k(X/G)$ of some "good" quotient X/G. The most relevant concept for these notes is that of an "almost free" action — sometimes also called "generically free" action.

Section 3 discusses two basic techniques: a "slice" technique and a lemma which has since then gone into the literature as the "no-name lemma". The "no-name lemma" says that for "almost-free" linear G-actions the (stable) purity question depends only on G. The "slice method" enables one to see the field of invariants for an action of a given group G as the field of invariants for another action of a smaller group H.

Section 4 considers linear actions of some particular groups for which the (stable) rationality problem has a positive answer: some finite groups ($\mathfrak{S}_n, \mathfrak{A}_5$), solvable groups, the so-called "special groups" ($\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}$), the orthogonal groups $\mathrm{O}_n, \mathrm{SO}_n$. For the spinor groups Spin_n and for PGL_n, there are results in low dimension.

The unramified Brauer group of a function field K/k is defined in Section 5 and some of its basic properties are given. A key property is that if K is a purely transcendental extension of k, then the unramified Brauer group of K (over k) is trivial.

In Section 6, a quite general formula of Bogomolov for the unramified Brauer group of the field of invariants of an (almost free) action of a group G is given, first for G finite then more generally for G reductive. The finite bicyclic subgroups of G play a key rôle in this computation.

In Section 7, this formula is applied to the case of a linear action of a finite group. It is further specialized to the case of nilpotent groups of class 2, where it yields concrete examples of fields of linear invariants which are not rational, among which one finds Saltman's original example.

Section 8 discusses (twisted) multiplicative invariants of a finite group G, after Saltman and others. In favourable circumstances, one may view the associated field of G-invariants as the field of invariants of a linear action of a finite group G' which is an extension of G. This is used to produce other types of fields of linear invariants which are not rational.

Finally, Section 9 gives in some detail Bogomolov's proof of the vanishing of the unramified Brauer group of the function field of a quotient G/H, where H is a connected subgroup of a simply connected group G. One first proves the vanishing of the unramified Brauer group of the field $k(V)^H$ for an almost free linear representation V of H.

Precise references to the papers of Saltman and Bogomolov are given in the text. Some of our proofs differ from the original ones.

Known examples of non-rationality. For the action of a finite group G on a purely transcendental extension K of $k = \mathbb{C}$, non-rational fields of invariants K^G of the following types have been exhibited :

(i) Linear action of a nilpotent group of order p^7 and class 2.
(ii) Multiplicative actions of $G = (\mathbb{Z}/p)^3$.
(iii) Twisted multiplicative actions of $G = (\mathbb{Z}/p)^2$.
(iv) For an arbitrary action, $G = \mathbb{Z}/2$ (Clemens-Griffiths [31], Artin-Mumford [3]).

Over a non-algebraically closed field k, there exist examples of non-rational fields of invariants K^G for an almost free linear action of the following kinds:

(v) $k = \mathbb{Q}$, $G = \mathbb{Z}/47$ (Swan [146], Voskresenskiĭ [153]).
(vi) $k = \mathbb{Q}$, $G = \mathbb{Z}/8$ (Saltman [117], Voskresenskiĭ [153]).
(vii) $k = \mathbb{Q}$, G a k-torus [36].
(viii) G a simply connected semisimple group (Merkurjev [92])

But for almost free linear actions of connected linear algebraic groups over the complex field, the rationality question for fields of invariants is open.

1. Rationality

Let k be a field. Let $\mathbb{A}^n$, resp. $\mathbb{P}^n$, denote the n-dimensional affine space, resp. projective space, over k. The function field of an integral k-variety X is denoted $k(X)$. The set of rational k-points of X is denoted $X(k)$.

One says that two integral k-varieties X and Y are *k-birationally equivalent* if the following equivalent conditions are satisfied:

(i) The function fields $k(X)$ and $k(Y)$ are isomorphic (over k).
(ii) There exist non-empty Zariski open sets $U \subset X$ and $V \subset Y$ which are isomorphic over k.

One says that two integral k-varieties X and Y are *stably k-birationally equivalent* if the following equivalent conditions are satisfied:

(i) For suitable integers r, s and independent variables $\{x_i\}_{i=1}^r$ and $\{y_j\}_{j=1}^s$, the fields $k(X)(x_1, \ldots, x_r)$ and $k(Y)(y_1, \ldots, y_s)$ are k-isomorphic (one then says that the fields $k(X)$ and $k(Y)$ are *stably equivalent*).
(ii) For suitable integers r, s, the k-varieties $X \times_k \mathbb{A}^r$ and $Y \times_k \mathbb{A}^s$ are k-birationally equivalent.

A k-variety X is said to be

- *k-rational* if it is integral and it is k-birational to an affine space (one then says that $k(X)$ is pure over k),
- *stably k-rational* if there exists an affine space $\mathbb{A}^n$ over k such that $X \times_k \mathbb{A}^n$ is k-rational (one then says that $k(X)$ is stably pure over k),

- a *direct factor of a k-rational variety* if there exists an integral k-variety Y such that $X \times_k Y$ is a k-rational variety.
- *k-unirational* if it is integral and it satisfies one of the equivalent properties in the following lemma.

Lemma 1.1. *Let X be an integral k-variety. The following conditions are equivalent:*

(i) *The function field $k(X)$ of X is a k-subfield of a pure extension K of k.*
(ii) *There exists a dominating k-morphism from a k-rational variety Y to X.*
(iii) *(Under the additional assumption that k is infinite) there exists a dominating k-morphism from a k-rational variety Y to X, with* $\dim Y = \dim X$.

Proof. We only need to show that (ii) implies (iii). Let $f\colon W \to X$ be a dominating k-morphism, where $W \subset \mathbb{A}^n$ is an open set of affine space over k. By linear subspace of W we shall mean the non-empty trace on W of a linear subspace of $\mathbb{A}^n$. Let us consider the closed, geometrically integral subvarieties $Y \subset X$ over k with the following property: there exists a linear subspace $V \subset W$ with $\dim V = \dim Y$ such that the k-morphism f restricts to a dominant k-morphism from V to Y. Any k-point in $f(W(k))$ is of this type. Let Y be such a variety. Assume $Y \neq X$. Since k is infinite and f is dominant, there exists a k-point $P \in W(k)$ such that $M = f(P) \in X(k)$ does not lie on Y. Since f is defined at the generic point of V, it is also defined at the generic point of the linear span $L \subset W$ of V and P. Moreover, the closure $Y_1 \subset X$ of the image of L under f contains Y and M, hence is of dimension strictly bigger than Y. Iterating this procedure, we find that X is covered by a k-linear space of dimension $\dim X$. □

Finally, following Saltman, one says that a k-variety X is

- *retract rational* (over k) if it satisfies one of the equivalent conditions in the following proposition (one then says that $k(X)$ is retract rational over k).

Proposition 1.2 (Saltman [118, Theorem 3.8])**.** *Let k be a field and X be an integral k-variety. The following conditions are equivalent:*

(i) *There exists a non-empty open set U of X such that the identity morphism of U factorizes through a Zariski open set Y of an affine space over k,* i.e. *there are maps $U \to Y \to U$ whose composite is identity on U.*
(ii) *There exists a non-empty open set V of X such that for any local k-algebra A with residue field κ, the natural map $V(A) \to V(\kappa)$ is onto.*

Proof. The surjectivity of the natural map $V(A) \to V(\kappa)$ means that any map

$$\operatorname{Spec} \kappa \to V$$

extends to $\operatorname{Spec} A$:

$$\operatorname{Spec} \kappa \to \operatorname{Spec} A \to V.$$

If U is a Zariski open set of V, it is stable by generisation, hence the surjectivity of $V(A) \to V(\kappa)$ implies that of $U(A) \to U(\kappa)$.

That (i) implies (ii) is now clear. For an affine space $Y = \mathbb{A}^n$ the map $Y(A) \to Y(\kappa)$ is the natural reduction $A^n \to \kappa^n$ and it is surjective. Thus this is also true for the open set Y of affine space, hence also for U.

In order to prove the converse assertion, we may assume that $X = V$ and that X is affine, say $X = \operatorname{Spec} B$ with $B = R/\mathfrak{p}$, with $\mathfrak{p}$ a prime ideal of the k-algebra $R = k[x_1, \ldots, x_n]$ of polynomials in n variables $x_1, \ldots, x_n$. Let $K = k(X)$ be the function field of X. This is also the residue class field of the local ring $A = R_{\mathfrak{p}}$. The generic point η of X defines a point in $X(K)$. The assumption implies that this point comes from an element of $X(R_{\mathfrak{p}})$, i.e. we have maps

$$\eta \to \operatorname{Spec} R_{\mathfrak{p}} \to X$$

whose composite is the natural inclusion of η into X. The map $\operatorname{Spec} R_{\mathfrak{p}} \to X$ factorizes through some $\operatorname{Spec} R_f$, where $R_f = R[\frac{1}{f}]$, i.e. we have an open subset $V \subset \mathbb{A}^n$ and maps

$$\eta \to V \xrightarrow{\pi} X$$

whose composite is the natural inclusion of η into X. Then this situation extends to an open set U of X, i.e. there exist a dense open set $U \subset X$ and maps

$$U \to V \xrightarrow{\pi} X$$

whose composite is the natural inclusion of U into X. Setting $Y = \pi^{-1}(U)$ gives rise to the announced factorization

$$U \to Y \to U. \qquad \square$$

Remark 1.3. Saltman's motivation for introducing the concept of retract rationality was to try to understand the relation between rationality and approximation properties (in a more arithmetical context). The criterion above, though nice, seems a priori to be of little value: for instance it seems impossible to prove that a smooth affine conic satisfies the lifting property (ii) without a priori proving that it is a rational curve! Nevertheless, in the particular case of an almost free (see §2) linear action of the projective linear group PGL_p with p prime, Saltman used (ii) to prove that the quotient is retract rational [118, Corollary 5.3] — but a direct proof that this variety satisfies (i) could later be given [37, Corollary 9.13].

Proposition 1.4. *Let X be an integral k-variety. In the list of properties:*

(i) *X is k-rational,*
(ii) *X is stably k-rational,*
(iii) *X is a direct factor of a k-rational variety,*
(iv) *X is retract rational over k,*
(v) *X is k-unirational,*

each property implies the following one.

Proof. Only the penultimate implication (iii) $\Longrightarrow$ (iv) requires a proof. Let Y be an integral k-variety such that $X \times_k Y$ is rational. Let $U \subset X \times_k Y$ be a non-empty open set which is isomorphic to an open set of affine space. Let $(x_0, y_0) \in U(k)$. Let $X_1 \subset X$ be the non-empty open set such that $X_1 \times \{y_0\} = U \cap (X \times \{y_0\})$. The open set $U_1 = U \cap (X_1 \times Y)$ is still isomorphic to an open set of affine space. Now the composite map $X_1 \to U_1 \to X_1$, where the first map is given by $x \mapsto (x, y_0)$ and the second map is induced by projection onto X, satisfies the requirements of Proposition 1.2 (i). □

Remarks. Over an algebraically closed field of characteristic zero, examples are known of:

(a) unirational varieties which are not rational (Artin-Mumford [3], Iskovskikh-Manin [57], Clemens-Griffiths [31]);
(b) stably rational varieties which are not rational [6].

The Artin-Mumford method for proving non-rationality uses the Brauer group. This method will be discussed and applied in §§5–9 of these notes. The method applies to varieties of arbitrary dimension. The associated invariant is insensitive to replacement of the variety X by $X \times_k \mathbb{A}^n$. Such is not the case for the two other methods mentioned.

The Clemens-Griffiths method is quite specific to threefolds. That method, in a variant due to Mumford, was used to produce the examples in (b).

The Iskovskikh-Manin method was originally developped to prove the non-rationality of the general quartic threefold. It has witnessed a strong development: birational rigidity and birational super-rigidity, work of Iskovskikh [56], Pukhlikov [112, 113], de Fernex-Ein-Mustaţă [38].

One should here also mention two further techniques for disproving rationality. One is a natural generalization of the Artin-Mumford technique: it uses higher dimensional unramified cohomology (with torsion coefficients). For this, see [32, 35, 92, 98–100, 128–130]. The other one, due to Kollár [75, 76], uses reduction to positive characteristic.

2. Quotients

In this section k denotes a field of characteristic zero and $\bar{k}$ an algebraic closure of k. We denote by $\mathfrak{g}$ the Galois group of $\bar{k}$ over k. Given a k-variety X we denote $\bar{X} = X \times_k \bar{k}$. If X is integral, resp. geometrically integral, we denote by $k(X)$, resp. $\bar{k}(X)$, the function field of X, resp. $\bar{X}$.

Let G be a linear algebraic group over k and let X be a geometrically integral k-variety with a G-action. Our interest will be in the field $k(X)^G$, which by definition is the fixed field of $\mathfrak{g}$ acting on the field of invariants $\bar{k}(X)^{G(\bar{k})}$.

2.1. **Elementary properties.**

The ring of regular functions on a k-variety X is denoted $k[X]$.

Lemma 2.1. *Let A be a domain, K its field of fractions, G a finite group acting on A. Then the field K^G is the field of fractions of A^G.*

Proof. Let $f \in K^G$. Write $f = a/b$ with $a, b \in A$. Let $e \in G$ be the identity element. Since A is commutative, $\beta := N(b) = \prod_{\sigma\in G} {}^{\sigma}b \in A^G$. Moreover, $\beta = bc$ with $c := \prod_{\sigma\in G,\sigma\neq e} {}^{\sigma}b \in A$. We have $f = \alpha/\beta$ with $\alpha := ac \in A$. Since f and β are G-invariants, $\alpha = f\beta$ is also G-invariant. Finally, $f = \alpha/\beta$ with $\alpha, \beta \in A^G$. □

Lemma 2.2. *Let $k = \bar{k}$. Let A be an integral k-algebra of finite type, equipped with an (algebraic) action of a linear algebraic k-group G. Let K be the field of fractions of A. Assume that A is a UFD and that the identity component of G has no nontrivial character. Then K^G is the field of fractions of A^G.*

See [95, Theorem 4.1; 108, Lemma 3.2 and Theorem 3.3].

Proof. (i) Let us first assume that G is connected. For each prime ideal of height one $\mathfrak{p}$ of A, fix a generator $f_{\mathfrak{p}} \in A$. Let $f \in K^G$. Write

$$f = uf_1^{r_1} \dots f_s^{r_s}$$

with $u \in A^*$, the $f_i = f_{\mathfrak{p}_i}$ among the generators chosen above and the $r_i \in \mathbb{Z}$, $r_i \neq 0$. For any $\sigma \in G(k)$, the identity ${}^{\sigma}f = f$ implies

$$f = uf_1^{r_1} \dots f_s^{r_s} = {}^{\sigma}u \cdot {}^{\sigma}f_1^{r_1} \dots {}^{\sigma}f_s^{r_s}.$$

Then, by unique factorization, for each $i = 1, \dots, s$ there exists a $j = \tau_\sigma(i)$ such that ${}^{\sigma}f_i = \varepsilon_i(\sigma)f_j$, with $\varepsilon_i(\sigma) \in A^*$. Since G is connected, the induced homomorphism $\tau\colon G \to \mathfrak{S}_s$ is trivial. For each i, we thus have ${}^{\sigma}f_i = \varepsilon_i(\sigma)f_i$. Let $\varepsilon_i\colon G(k) \to A^*$ be the induced homomorphism, and $\pi\colon A^* \to A^*/k^*$. Since G is connected and A^*/k^* is an abelian group of finite type (see [36]), $\pi\circ\varepsilon_i$ is trivial. Hence ε_i is a homomorphism $\varepsilon_i\colon G(k) \to k^*$. All data being algebraic, it is induced by a character $G \to \mathbb{G}_m$, hence trivial by assumption. Thus each f_i is G-invariant. This then implies $u \in A^G$, and f lies in the fraction field of A^G.

(ii) Let G be arbitrary. Let G° be the identity component of G. Let $f \in K^G$. By (i), $f = a/b$ with $a, b \in A^{G^\circ}$. The same arguments as in the proof of Lemma 2.1 for the finite group G/G° show that $f = \alpha/\beta$ with $\alpha \in A$ and $\beta \in A^G$, hence also $\alpha \in A^G$. □

Remarks. (1) The above arguments also shows that A^G is a UFD.

(2) The assumption that the identity component of G has no nontrivial character is a necessary one, as the example of the diagonal action of $\mathbb{G}_m$ on $\mathbb{A}^2_k$ shows.

Proposition 2.3. *Let $k = \bar{k}$. Let G be a linear algebraic group and X a factorial affine variety with a G-action.*

(i) *If $k[X]^* = k^*$, there exists a G-invariant affine open set $U \subset X$ such that $k(X)^G$ is the field of fractions of $k[U]^G$.*
(ii) *If G is finite, or if G has no nontrivial character $G \xrightarrow{\chi} \mathbb{G}_m$, then $k(X)^G$ is the field of fractions of $k[X]^G$.*

See also [108, Theorem 3.3].

Proof. (i) Let $k(X)^G = f(f_1, \ldots, f_s)$. Let Z be the union of the supports of the divisors of the functions f_i. Since X is factorial affine, there exists $g \in k[X]$ whose divisor is Z. The complement U of Z is therefore an affine open set with $k[U] = k[X][1/g]$. The divisor Z being G-invariant, U is G-invariant. Since $f_1, \ldots, f_s$ belong to $k[U]^G$, this implies $k(X)^G)$ is the field of fractions of $k[U]^G$.

Assertion (ii) is just a rephrasing of Lemmas 2.1 and 2.2. □

In order to describe a field of invariants $k(X)^G$, it will often be convenient to realize it as the function field of a "quotient" variety Y of X by G.

In the literature, several integral varieties Y equipped with a morphism $p \colon X \to Y$ such that $p(gx) = p(x)$ for $g \in G$ and $x \in X$ go under the name of quotient varieties. However, our main interest is in such quotients with the additional property $k(Y) = k(X)^G$.

Proposition 2.4. *Let $\pi \colon X \to Y$ be a dominant morphism of geometrically integral k-varieties. Let the algebraic k-group G act on X. Assume that for $x, y \in X(\bar{k})$, the following two conditions are equivalent:*

(i) *$\pi(x) = \pi(y)$;*

(ii) *there exists $g \in G(\bar{k})$ such that $gx = y$.*

Then $k(Y)$ may be identified with the field $k(X)^G$ of invariants:

$$k(Y) = k(X)^G.$$

Proof. For the proof, we may assume $k = \bar{k}$. In this case the proposition is an immediate consequence of the following elementary proposition — whose proof uses the characteristic zero hypothesis. □

Proposition 2.5 (see [83, AI.3.7 Satz 2])**.** *Let $k = \bar{k}$. Let $\phi \colon X \to Y$ be a dominant morphism of integral k-varieties. If $f \in k(X)$ is a rational function which is constant on the fibres of ϕ, then f may be identified with a rational function on Y.*

2.2. **Algebraic quotients, geometric quotients, torsors.**

Definition 2.6. If G is a reductive k-group and $X = \operatorname{Spec} A$ is an affine k-variety with a G-action, the *algebraic quotient* $X/\!\!/G$ of X by G is the affine scheme $\operatorname{Spec} A^G$. This actually is a k-variety, since G reductive implies (see [83, II.3.2; 94, Chap. 1, §2]) that A^G is a finitely generated k-algebra.

Definition 2.7. Let X be an algebraic k-variety equipped with an action of an algebraic k-group G. A *geometric quotient* of X by G is a k-variety Y equipped with a k-morphism $\phi \colon X \to Y$ such that:

(i) ϕ is open, constant on G-orbits and induces a bijection of $X(\bar{k})/G(\bar{k})$ with $Y(\bar{k})$.

(ii) For any open subset $V \subset Y$, the natural morphism $k[V] \to k[\phi^{-1}(V)]^G$ is an isomorphism.

If such a variety Y exists, it is unique up to unique isomorphism. It will then be denoted $Y = X/G$.

By a classical theorem of Rosenlicht [115], given X an integral algebraic variety equipped with an action of a linear algebraic group G, there exists a nonempty G-stable open set $U \subset X$ admitting a geometric quotient $U \to U/G$.

Remark. A geometric quotient (Y, ϕ) of X by G has the following properties:
(a) The map $\phi\colon X(\bar{k}) \to Y(\bar{k})$ is onto, all $G(\bar{k})$-orbits on $X(\bar{k})$ are closed and they coincide with the fibres of ϕ over $\bar{k}$-points.
(b) Proposition 2.4 implies $k(Y) = k(X)^G$.
(c) The map $\phi\colon X \to Y$ has the obvious universal mapping property [23, p. 172]. This accounts for the unicity statement above.
(d) When X is affine and G is reductive, the algebraic quotient $X/\!\!/G$ need not be a geometric quotient (consider the diagonal action of $\mathbb{G}_m$ on $\mathbb{A}^2$).

Proposition 2.8. *Let $k = \bar{k}$. Let X be a normal algebraic k-variety equipped with an action of an algebraic k-group G. Then a k-morphism $\phi\colon X \to Y$ is a geometric quotient if and only if*
(1) *ϕ is constant on G-orbits.*
(2) *ϕ induces a bijection $X(k)/G(k) \xrightarrow{\cong} Y(k)$.*
(3) *Y is normal.*

Proof. The proof uses the $\operatorname{char} k = 0$ hypothesis and Zariski's main theorem, see [94, Prop. 0.2, p. 7]. □

Definition 2.9. Let G be a linear algebraic k-group. A k-variety X equipped with a G-action and a faithfully flat k-morphism $X \to Y$ is a *G-torsor — principal homogeneous space* under G — over Y if the map $(g, x) \mapsto (gx, x)$ induces an isomorphism $G \times_k X \cong X \times_Y X$.

Remark 2.10. If X is affine, then Y is affine and $Y = X/\!\!/G$.

Remark 2.11. If X is a G-torsor over Y the map $X \to Y$ makes Y into a geometric quotient X/G. Hence $k(Y) = k(X)^G$.

Remark 2.12. Let G be a k-algebraic group and H a subgroup. The geometric quotient G/H exists, and $G \to G/H$ makes G into an H-torsor over G/H. Hence $k(G/H) = k(G)^H$.

There is a vast literature on the various possible notions of quotient. We refer the reader to [13, 14, 40, 83, 108].

2.3. **Almost free actions.**

Definition 2.13. An action of an algebraic k-group G on a geometrically integral k-variety X will be called an *almost free action* if for all $x \in U(\bar{k})$ in a non-empty open set U of X the stabilizer $G_x \subset G(\bar{k})$ is trivial.

Theorem 2.14 (Luna [87])**.** *Let X be an affine k-variety and let the reductive k-group G act on X. If each stabilizer G_x, $x \in X(\bar{k})$, is trivial, then X is a G-torsor over $X/\!\!/G$. Hence, X/G exists, and $X/G = X/\!\!/G$.*

Remark. The hypothesis "X affine" cannot be ignored (see [94, p. 11]).

Theorem 2.15 (Luna [87]). *Let $k = \bar{k}$. Let the reductive group G act upon the affine variety X. Let $\pi\colon X \to X/\!\!/G$ be the natural projection. If there exists a point $x \in X(k)$ with trivial stabilizer and closed orbit, then there exists an affine open set $V \subset X/\!\!/G$ which contains $\pi(x)$ and is such that the projection $\pi^{-1}(V) \to V$ makes $\pi^{-1}(V)$ into a G-torsor over V.*

Proof. This is an immediate consequence of Luna's "slice theorem" [87]. $\square$

Corollary 2.16. *Let $k = \bar{k}$. Let the reductive group G act upon the variety X. The following conditions are equivalent:*

(i) *There exist a G-invariant non-empty open set $U \subset X$ and a variety Y with a morphism $U \to Y$ such that U is a G-torsor over Y.*
(ii) *There exists a G-invariant non-empty affine open set $U \subset X$ such that the projection map $U \to U/\!\!/G$ makes U into a G-torsor over $U/\!\!/G$.*
(iii) *There exist an* affine *G-invariant open set U and a point $x \in U(k)$ such that the stabilizer G_x is trivial and the orbit of x in U is closed.*

Proof. To show that (i) $\Longrightarrow$ (ii), choose an affine non-empty open set $V \subset Y$ and take its inverse image $\pi^{-1}(V)$ under $\pi\colon U \to Y$. Conversely, (ii) $\Longrightarrow$ (i) is obvious. It is also obvious that (ii) $\Longrightarrow$ (iii). Finally, (iii) $\Longrightarrow$ (ii) is a consequence of Theorem 2.15. $\square$

Theorem 2.17 (Popov [103], Luna-Vust [89], see [94, appendix p. 154]). *Let $k = \bar{k}$. Let the semisimple group G act on the factorial affine variety X. The following conditions are equivalent:*

(i) *There exists a G-invariant non-empty open set $U \subset X$ such that for each $x \in U(k)$, the stabilizer G_x is reductive.*
(ii) *There exists a G-invariant non-empty open set $V \subset X$ such that for each $x \in V(k)$, the orbit $G \cdot x$ of x is closed in X.*

As the diagonal action of the multiplicative group $\mathbb{G}_m$ on affine space reveals, Theorem 2.17 does not extend to reductive groups. We thank M. van den Bergh for showing us how to prove the next theorem for arbitrary reductive groups.

Theorem 2.18. *Let $k = \bar{k}$. Let X be a factorial affine variety, and let the reductive group G act on X. Assume that the open set consisting of points with finite stabilizer is non-empty.*

Then there exist non-empty G-invariant open sets $U_1 \subset U \subset X$ such that U is affine and G-orbits of points of U_1 are closed in U.

Proof. Let $G^\circ \subset G$ be the connected component of identity in G. Assume we have found suitable $U_1 \subset U$ for G°. Let V be the intersection of all σU, for σ running through a (finite) system of representatives in G for G/G°. The open set V is affine and G-stable. Let $V_1 = U_1 \cap V$. The G-orbits of points of V_1 are finite unions of orbits of G°, hence are closed in U, hence also in V. We may therefore assume that G is connected.

Let $T = Z(G)^\circ$ be the torus which is the connected centre of G, and let $G_0 = [G, G]$ be the derived subgroup of G. It is a connected semisimple group.

Then G is the almost direct product of T and G_0, i.e. it is the quotient of $T \times G_0$ by the finite group $= T \cap G_0$. Let $f\colon X \to X /\!\!/ G_0$ be the algebraic quotient of X by G_0. The torus T acts on $X /\!\!/ G_0$ and the map f is T-equivariant. Any point of X with finite stabilizer in G projects down to a point of $X /\!\!/ G_0$ with finite stabilizer in T. Hence there exists a non-empty T-stable open set $W_0 \subset X /\!\!/ G_0$ consisting of points with finite stabilizer. All orbits of T on W_0 are closed (if an orbit of T on W_0 were not closed, its closure would contain a smaller dimensional orbit, hence points with positive dimensional stabilizer).

Since T is a torus, a theorem of Sumihiro [145] asserts that there exists a non-empty affine T-invariant open set $U_0 \subset W_0$. Clearly, T-orbits are closed in U_0. Let $U = f^{-1}(U_0)$. Since f is affine, this is a G-invariant affine open set of X. Since X is factorial, so is U. By assumption, X and hence U contains a non-empty G-stable open set consisting of points with finite stabilizer. Thus according to Theorem 2.17 there exists a non-empty G_0-stable open set U_1 of U consisting of points with closed G_0-orbit in U and finite stabilizer. For any point $x \in U_1(k)$, G_0x is of maximal dimension in $f^{-1}(f(x))$. Since it is closed, it coincides with $f^{-1}(f(x))$. Now

$$Gx = TG_0x = T(f^{-1}(f(x))) = f^{-1}(Tf(x)))$$

is the inverse image of the closed set $Tf(x) \subset U_1$, hence is closed. □

Corollary 2.19. *Let X be an affine k-variety. Assume that $\bar{X}$ is factorial. Let the reductive k-group G act almost freely on X. Then there exists a G-invariant non-empty affine open set $U \subset X$ such that the projection map $U \to U /\!\!/ G$ makes U into a G-torsor over $U /\!\!/ G$.*

Proof. To prove this, we may assume $k = \bar{k}$. According to Theorem 2.18, there exist non-empty G-invariant open sets $U_1 \subset U \subset X$ such that U is affine and U_1 consists of points x with trivial stabilizer and closed orbit in U. It only remains to apply Corollary 2.16, (iii) $\Longrightarrow$ (i). □

3. General techniques

In this section, k denotes a field of characteristic zero and $\bar{k}$ an algebraic closure of k.

Several methods have been used to prove the rationality of some quotient spaces, in particular by Katsylo. A good survey of these methods is given by Dolgachev [41], to whom we refer for many examples from the theory of moduli spaces. In this section, these general methods will be reviewed; an attempt will be made to give complete proofs of some of the basic lemmas.

3.1. **The slice method.** The following very useful fact, which is known under the name "slice method" goes back at least to C. S. Seshadri [138]. See M. Nagata [96, Lemma 1 ("Lemma of Seshadri"), p. 37] and also V. L. Popov [106]. The following version of Seshadri's lemma comes from [46].

Theorem 3.1 (Slice lemma)**.** *Let G be an algebraic group over k and X a geometrically integral k-variety with a G-action. Let $Y \subset X$ be a closed* *geometrically integral sub-k-variety, and let $H \subset G$ be the normalizer of Y. Assume:*

(i) *The closure of $G \cdot Y$ is the whole of X.*
(ii) *There exists a non-empty H-stable open subset Y_0 of Y such that if $g \in G(\bar{k})$ and $y \in Y_0(\bar{k})$ satisfy $g \cdot y \in Y_0(\bar{k})$, then there exists $h \in H(\bar{k})$ such that $h \cdot y = g \cdot y$.*

Then there is a natural k-isomorphism of fields $k(X)^G \cong k(Y)^H$.

Proof. We immediately reduce to the case $k = \bar{k}$. Let $f \in k(X)^G$, $f \neq 0$. Since $G \cdot Y$ is dense in X, the subvariety Y is not contained in the support of $\operatorname{div} f$. Hence $f|_Y \in k(Y)$ and the restriction defines an injection $k(X)^G \subset k(Y)^H$.

Conversely, let $f \in k(Y)^H = k(Y_0)^H$. Let $\pi \colon G \times Y_0 \to X$ be the natural morphism. The rational function ϕ defined on $G \times Y_0$ by $\phi(g, y) = f(y)$ is constant on the fibres of π (use assumption (ii) and the H-invariance of f). Since π is a dominant morphism of integral varieties, ϕ may be identified — see Proposition 2.5 — with a rational function $\tilde{f}$ on X, and $\tilde{f}|_Y = f$. Hence $k(X)^G = k(Y)^H$. □

Remark 3.2. We refer to [107] for more historical remarks regarding this theorem, which may be consider as a birational analogue of the Chevalley restriction theorem [88].

For other reduction techniques, see [81, 82].

3.2. **The no-name lemma ([20, 41]).** In quite a few circumstances, the method to be described below enables one to deduce stable rationality and in some cases rationality of some fields of invariants once it is known for other fields. This method has been independently discovered by several people, hence the denomination "no-name lemma" given by Dolgachev [41] to one of the corollaries below.

We first give statements and proofs for finite groups actions. Then we give an independent proof for the general case of reductive group actions.

Theorem 3.3 (Speiser's lemma)**.** *Let G be a finite group. Let V and W be two faithful finite-dimensional linear representations of G over k. Then the fields $k(V)^G$ and $k(W)^G$ are stably equivalent.*

This means that there exist independent variables $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$ such that
$$k(V)^G(x_1, \ldots, x_r) \cong k(W)^G(y_1, \ldots, y_s).$$

Corollary 3.4. *Let V be a faithful finite-dimensional linear representation of a finite group G over k. The stable purity of the field $k(V)^G$ depends only on G.*

We have the well-known lemma:

Lemma 3.5. *Let K/k be a finite Galois extension. Let G be its Galois group. Then for any finite dimensional K-linear space E and any semi-linear action of G on E with respect to the extension K/k we have*
$$E = E^G \otimes_k K.$$

In particular, $K(E)^G/k = k(E^G)/k$ is pure.

Proof. Let us prove the last assertion. If $E = E^G \otimes_k K$, we are reduced to the case $E = K^n$ with the natural action of G. Hence $K(E) = K(t_1, \dots, t_n)$ with trivial action of G on the t_i's. So $K(E)^G/k = k(t_1, \dots, t_n)/k$.

For the first assertion, let $\{\omega_i\}_{i=1,\dots,d}$ be a linear basis of K/k and $G = \{\sigma_i\}_{i=1,\dots,d}$ with $\sigma_1 = \mathrm{id}_K$. Let $v \in E$. Consider, for every $i = 1, \dots, d$,

$$w_i := \sum_j \sigma_j(\omega_i v) \in E^G.$$

The group G acting semi-linearly on E with respect to K/k, we have

$$w_i = \sum_j \sigma_j(\omega_i)\sigma_j(v).$$

The matrix $(\sigma_j(\omega_i))$ being invertible, one can express $v = \sigma_1(v)$ as a linear combination of the w_j's with coefficients in K, i.e. $v \in E^G \otimes_k K$. Thus the natural map

$$E^G \otimes_k K \to E$$

is onto. It remains to show that this map is injective. Let $\sum_i \lambda_i v_i = 0$ be a non-trivial K-linear relation between elements $v_i \in E^G$. There exist i_0 and $\alpha \in K$ with $\mathrm{Tr}_{K/k}(\alpha\lambda_{i_0}) \neq 0$. Then $\sum_i \mathrm{Tr}_{K/k}(\alpha\lambda_i)v_i = 0$ is a non-trivial k-linear relation between the v_i's. □

Proof of Theorem 3.3. By hypothesis, the linear representation of G on V is faithful. As a consequence the extension $k(V)/k(V)^G$ is Galois of group G. Then we can apply Lemma 3.5 to the following data: the extension $k(V)/k(V)^G$ and the $k(V)$-linear space $E = W_{k(V)}$. Lemma 3.5 ensures the purity of the extension

$$k(V \oplus W)^G/k(V)^G = \big(k(V)(W)\big)^G/k(V)^G.$$

By exchange of V and W we also obtain the purity of $k(V \oplus W)^G/k(W)^G$. Hence $k(V \oplus W)^G$ is a common pure extension of $k(V)^G$ and of $k(W)^G$. □

We now give statements and proofs for the general case.

Theorem 3.6. *Let G be a reductive group over k. Let $Y = \mathrm{Spec}\, A$ be an affine k-variety with a G-action. Assume that all geometric stabilizers at closed points of X are trivial. Let $X \to Y$ be a vector bundle over Y equipped with an equivariant G-action. Then:*

(a) *$X/\!\!/G$ is a vector bundle over $Y/\!\!/G$.*

(b) *If Y and hence X are integral, the field $k(X)^G$ is pure over $k(Y)^G$.*

Remark 3.7. In the paper [20] by Bogomolov and Katsylo, no condition is imposed upon the linear group G. Also, the base variety Y is not assumed to be affine. Although this last extension is perfectly legitimate, one should be aware that there are actions of SL(2) on non-affine varieties such that:

(i) All stabilizers are trivial, i.e., the action is set-theoretically free.

(ii) There is a geometric quotient.

(iii) The action is not scheme-theoretically free (Mumford, [94, p. 11]), i.e., the morphism $G \times X \to X \times X$ given by $(g, x) \mapsto (gx, x)$ is not a closed immersion.

For a proof of part (b) for arbitrary G and Y, see [27, §4.3].

Proof of Theorem 3.6. Let M be the group of global sections of the vector bundle X over $Y = \operatorname{Spec} A$. This is a finitely generated projective A-module, and $X = V(M) = \operatorname{Spec} \mathcal{S}_A(M)$, where $\mathcal{S}_A(M)$ denotes the symmetric algebra of M over A. Let $B = A^G$. Our assumptions imply that $Y \to Y/\!\!/G$ is a G-torsor, see Theorem 2.14. Then, A is a faithfully flat extension of B. By assumption, there is a G-equivariant action on the A-module M. This assumption may be translated into descent data on the projective A-module M. Since such descent data are effective, there exists a projective B-module N such that $A \otimes_B N \cong M$, and this B-module N is none other but M^G. One then has $\mathcal{S}_A(M)^G = \mathcal{S}_B(N)$. Since $X/\!\!/G = \operatorname{Spec} \mathcal{S}_A(M)^G$ and $Y/\!\!/G = \operatorname{Spec} B$, it follows that $X/\!\!/G$ is a vector bundle over $Y/\!\!/G$, which is the first assertion. Now note that since all geometric stabilizers of G at a closed point of Y are trivial, certainly the same holds for X, and X is a principal homogeneous space over $X/\!\!/G$, just as Y is over $Y/\!\!/G$. If Y and hence X are integral, so are $X/\!\!/G$ and $Y/\!\!/G$, and $k(X)^G = k(X/\!\!/G)$ is purely transcendental over $k(Y/\!\!/G) = k(Y)^G$. □

Corollary 3.8. *Let G be a reductive group over k, let V be a finite-dimensional k-vector space with a linear G-action and let X be an affine k-variety with an almost free G-action. Assume that $\bar{X}$ is factorial. Then the field $k(X \times V)^G$ is pure over the field $k(X)^G$. In particular, if $k(X)^G$ is pure, so is $k(X \times V)^G$.*

Proof. By Corollary 2.19 there exists a non-empty G-invariant affine open set $U \subset X$ such that $U \to U/\!\!/G$ is a principal homogeneous space under G, and $k(U/\!\!/G) = k(X)^G$. The natural projection from $Z_1 = U \times V$ to U makes Z_1 into a (trivial) vector bundle over the affine variety U. This vector bundle is equipped with the diagonal G-action. According to Theorem 3.6, $Z_1/\!\!/G$ is a vector bundle over $U/\!\!/G$. Since such a bundle is locally trivial in the Zariski topology, the function field $k(Z_1/\!\!/G)$ is purely transcendental over $k(U/\!\!/G) = k(X)^G$. On the other hand, for the diagonal action of G on the affine variety $Z_1 = U \times Y$ all stabilizers are trivial, hence by Luna's Theorem 2.14, the map $Z_1 \to Z_1/\!\!/G$ makes Z_1 into a principal homogeneous space over Z_1 under G. In particular, $k(Z_1/\!\!/G) = k(Z_1)^G = k(X \times V)^G$. Thus $k(X \times V)^G$ is purely transcendental over $k(X)^G$. □

Corollary 3.9 (No-name lemma)**.** *Let G be a reductive group over k and let V and W be two finite-dimensional k-vector spaces with almost free linear G-actions. Then the fields $k(V)^G$ and $k(W)^G$ are stably equivalent. If one of them is pure, the other one is stably pure.*

Proof. The diagonal action of G on $V \times W$ gives a common extension:

$$\begin{tikzcd} & k(V \times W)^G \arrow[dl, dash, "\text{pure}"'] \arrow[dr, dash, "\text{pure}"] & \\ k(V)^G & & k(W)^G, \end{tikzcd}$$

where purity follows from Corollary 3.8. □

Corollary 3.10. *Let G be a reductive k-group and let V, V_1 and V_2 be finite-dimensional k-vector spaces with linear G-actions. Assume that the action on V is almost free. If the action of G on V_1 is almost free and $\dim V_2 \geq \dim V$, then $k(V_1 \oplus V_2)^G$ is pure over $k(V)^G$. If moreover $k(V)^G$ is pure, so is $k(V_1 \oplus V_2)^G$.*

Proof. Because the action of G on V_1 is almost free, the same argument as in the previous proof shows that $k(V_1 \oplus V)^G$ is pure over $k(V_1)^G$, of transcendence degree $\dim V$. Similarly, $k(V_1 \oplus V_2)^G$ is pure over $k(V_1)^G$, of transcendence degree $\dim V_2$. Thus $k(V_1 \oplus V_2)^G$ is pure over $k(V_1 \oplus V)^G$. Now since the action of G on V is almost free, $k(V_1 \oplus V)^G$ is pure over $k(V)^G$. □

Corollary 3.11. *Let the reductive k-group G act almost freely and linearly on the finite-dimensional k-vector space V. Let $G \subset \mathrm{GL}_n$ be a closed embedding of G as a subgroup of GL_n for some integer n. Then the field $k(V)^G$ is stably equivalent to the function field $k(\mathrm{GL}_n/G)$ of the homogeneous space GL_n/G.*

Proof. Let $\mathcal{M}_n$ denote be the set of $n \times n$ matrices. Via the natural (left) multiplication, there is a linear G-action on the vector space $\mathcal{M}_n$. On the G-stable affine open set $U = \mathrm{GL}_n \subset \mathcal{M}_n$, the group G acts with trivial stabilizers. Thus the action of G on $\mathcal{M}_n$ is almost free, and Corollary 3.9 implies that $k(V)^G$ is stably equivalent to the function field $k(\mathrm{GL}_n)^G$ which, by Remark 2.12, coincides with $k(\mathrm{GL}_n/G)$. □

See [107, Remark (1.5.7)] for a generalization of the above statement. There also exist projective versions of the no-name method.

Corollary 3.12. *Let G be a reductive k-group, let V be a finite-dimensional k-vector space with a linear G-action and let X be a geometrically integral k-variety with an almost free G-action. Then the field $k(X \times \mathbb{P}(V))^G$ is pure over the field $k(X)^G$. In particular, if $k(X)^G$ is pure, so is $k(X \times \mathbb{P}(V))^G$.*

Proof. By restricting X to a suitable open set, we may assume that X is affine and that the action of G on X is free. According to the proof of Theorem 3.6, the projection $(X \times V)/\!\!/G \to X/\!\!/G$ makes the first space into a vector bundle over the second one (the action on $X \times V$ being the diagonal one). Since the action of G commutes with that of $\mathbb{G}_m$ on the factor V, the action of $\mathbb{G}_m$ descends to an action on $(X \times V)/\!\!/G$, which respects the vector bundle structure. Let $U \subset X/\!\!/G$ be an open set over which the vector bundle has a section, i.e. $(X \times V)/\!\!/G \cong U \times V$. We now have:

$$\begin{aligned} k(X \times \mathbb{P}(V))^G &= k(X \times V)^{\mathbb{G}_m} = k((X \times V)/\!\!/G)^{G \times \mathbb{G}_m} \\ &= k(U \times V)^{\mathbb{G}_m} = k(U \times \mathbb{P}(V)) = k((X/\!\!/G) \times \mathbb{P}(V)) \end{aligned}$$

and this last field is clearly purely transcendental over $k(X/\!\!/G)$. □

There also exists a generalized version of the no-name lemma, due to Bogomolov [16], and which he has used to study fields of invariants $k(V)^G$ when G is a simply connected semisimple group.

3.3. **Combining both methods.** We now restrict attention to linear actions and prove a basic proposition which uses both the "no-name lemma" and the "slice method".

Proposition 3.13. *Let G be a reductive k-group and let W be a finite-dimensional k-vector space with a linear G-action. Let $v \in W$ be a point such that:*

(i) *the stabilizer of v is a reductive k-group H;*
(ii) *the orbit of v under G is dense in W.*

Then if V is a finite-dimensional k-vector space with an almost free linear G-action, the action of H on V is almost free and the field $k(V)^H$ is pure over $k(V)^G$. In particular, $k(V)^H$ is stably pure if and only if $k(V)^G$ is stably pure.

Proof. Consider the diagonal action of G on $W \times V$. This is an almost free action. Indeed, let $U \subset V$ be a G-invariant open set and let $x \in U$ be such that $G_x = 1$ and $G \cdot x$ is closed in U. Then the same properties hold for the open set $W \times U$ and the point $y = (0, x) \in V \times U$. Since the action of G on V is almost free, the no-name lemma implies that $k(W \oplus V)^G$ is pure over $k(V)^G$. On the other hand, the closed subvariety $Y = \{v\} \times V \subset X = W \times V$ satisfies the assumptions of Theorem 3.1 — the "slice lemma" — for the pair (H, G). Indeed, H is clearly the normalizer of Y and if x and y belong to $Y(\bar{k})$ and $g \in G(\bar{k})$ satisfies $gx = y$, then $g \in H(\bar{k})$. That the orbit of Y under G is dense in $W \times V$ follows from the density of $G \cdot x$ in W. Thus by the "slice lemma" $k(Y)^H$ is isomorphic to $k(V \oplus W)^G$. Also, the action of H on Y is isomorphic to the action of H on the vector space V, which is an almost free action since the action of G itself was almost free (use the same open set U and the same point x as above: since H is closed in G, $H \cdot x$ is closed in $G \cdot x$ hence in U). Thus $k(V)^H$ is pure over $k(V)^G$. □

There is a projective variant of the above theorem:

Proposition 3.14. *Let G be a reductive group over k and let W be a finite-dimensional k-vector space with a linear G-action. Let $v \in W$ be a point such that:*

(i) *the stabilizer of v is a reductive group H;*
(ii) *the orbit of v under the action of $G \times \mathbb{G}_m$ is dense in W.*

Then if V is a finite-dimensional k-vector space with an almost free linear G-action, the action of H on V is almost free and the field $k(\mathbb{P}(V))^H$ is stably pure over $k(\mathbb{P}(V))^G$.

3.4. **A criterion for retract rationality.**

Proposition 3.15. *Let G be a reductive group over k. Assume that for any local k-algebra A, with residue field κ, the reduction map on étale cohomology sets $H^1(A,G) \to H^1(\kappa,G)$ is onto. Let V be a k-vector space with an almost free linear G-action. Then $k(V)^G$ is the function field of a retract rational k-variety.*

Proof. One easily checks that if X and Y are two stably k-birationally equivalent geometrically integral k-varieties, then X is retract rational over k if and only if Y is. By the no-name lemma one is therefore reduced to proving that if $G \subset \mathrm{GL}_n$ is a closed embedding of algebraic groups, and $X = \mathrm{GL}_n/G$ is the quotient variety, then the variety X is retract rational. For A and κ as above, we have the natural compatible exact sequences of pointed sets

$$\begin{array}{ccccccc} \mathrm{GL}_n(A) & \longrightarrow & X(A) & \longrightarrow & H^1(A,G) & \longrightarrow & H^1(A,\mathrm{GL}_n) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{GL}_n(\kappa) & \longrightarrow & X(\kappa) & \longrightarrow & H^1(\kappa,G) & \longrightarrow & H^1(\kappa,\mathrm{GL}_n). \end{array}$$

Grothendieck's version of Hilbert's theorem 90 ensures $H^1(A,\mathrm{GL}_n)=0$, hence the map $X(A) \to H^1(A,G)$ is onto. The first arrow in each sequence is equivariant with respect to the left action of $\mathrm{GL}_n(A)$, resp. $\mathrm{GL}_n(\kappa)$. Two elements of $X(\kappa)$ have the same image in $H^1(\kappa,G)$ if and only if they are in the same orbit of $G(\kappa)$. If the map $H^1(A,G) \to H^1(\kappa,G)$ is onto, one concludes that the reduction map $X(A) \to X(\kappa)$ is onto. It remains to apply Proposition 1.2. □

4. Examples

In this section, k denotes a field of characteristic zero and $\bar{k}$ an algebraic closure of k.

4.1. **Solvable Groups.** Details upon the following results may be found in the notes by Kervaire and Vust [72].

Theorem 4.1 (T. Miyata [93], È. Vinberg [152])**.** *Let the abstract group G act linearly upon the finite dimensional vector space V over k, and assume that there is a complete flag of V which is invariant under the action of G. Then the field $k(V)^G$ is pure.*

See also [150].

Proof. The proof is easily reduced to the following nice lemma, whose proof only uses the euclidean division algorithm. □

Lemma 4.2. *Let K be a field, and let the abstract group G act upon the polynomial ring in one variable $K[t]$. If $K \subset K[t]$ is globally fixed, then there exists a G-invariant polynomial p such that $K(t)^G = K^G(p)$.*

Proof. See [72, II, Lemme 1.1, page 167]. ☐

Proposition 4.3. *Assume $k = \bar{k}$. If V is a finite dimensional vector space over k and $G \subset \mathrm{GL}(V)$ is an (abstract) abelian group consisting of semisimple elements, then $k(V)^G$ is pure.*

For G a finite group this is just Fischer's well known theorem [42].

Proof. Indeed, all elements of G may be simultaneously diagonalized, so that G actually acts through a maximal torus of $\mathrm{GL}(V)$. ☐

Proposition 4.4. *Assume $k = \bar{k}$. If G is a connected solvable group over k and if G acts linearly on the finite dimensional vector space V over k, then $k(V)^G$ is pure.*

Proof. This is a consequence of the above theorem and of the Lie-Kolchin theorem. ☐

If the (abstract group) G acts linearly on the vector space V, this action induces an action on the projective space $\mathbb{P}(V)$.

Proposition 4.5. *The field $k(V)^G$ is pure over $k(\mathbb{P}(V))^G$.*

Proof. This follows from Lemma 4.2. See [72, II, Corollaire 1.4, page 168]. ☐

For more general actions, we have the following

Theorem 4.6 (Rosenlicht [114, 116])**.** *Let G be a connected solvable algebraic group over an algebraically closed field k. If X is an integral variety with a G-action, and $f\colon X \to Y$ is a geometric quotient for this action, then f admits a section over a non-empty open set of Y.*

This theorem has the following consequences:

(i) If X is a rational variety, then Y is a retract rational variety.
(ii) If the action of G on X is free, i.e. if X is a G-torsor over Y, then X is birational to $G \times Y$; if moreover X is rational, then Y is stably rational.

4.2. **Special Groups.** Recall that an algebraic group G over a field k is called *special* if any principal homogeneous space (torsor) under G over a k-variety is locally trivial for the Zariski topology. This implies $H^1(K, G) = 0$ that for any field K containing k.

The following facts were proved by Serre [28]. Any special group G is linear and connected. If $G \subset \mathrm{GL}_n$ is some closed embedding, G is special if and only if the fibration $\mathrm{GL}_n \to \mathrm{GL}_n / G$ is locally trivial for the Zariski topology.

Special groups include

(i) the additive group $\mathbb{G}_a$,
(ii) the multiplicative group $\mathbb{G}_m$,
(iii) more generally, split connected solvable groups,

as well as some connected linear algebraic groups, e.g.,

(v) the general linear groups GL_n, more generally, $\mathrm{GL}(A)$ for A a central simple algebra over k,

(vi) the special linear groups SL_n,
(vii) the symplectic groups Sp_{2n},
(viii) the split spinor groups Spin_n for $n \leq 6$.

This last case follows from the existence of the exceptional isomorphisms in the classification of Lie groups: $\mathrm{Spin}_3 \cong \mathrm{SL}_2$, $\mathrm{Spin}_4 \cong \mathrm{SL}_2 \times \mathrm{SL}_2$, $\mathrm{Spin}_5 \cong \mathrm{Sp}_4$, $\mathrm{Spin}_6 \cong \mathrm{SL}_4$. The special orthogonal groups SO_n for $n \geq 3$ and the spinor groups for $n \geq 7$ are not special.

Over an algebraically closed field, any connected linear algebraic group is a rational variety. Over an arbitrary field k, if a semisimple group is special, it is a product of copies of Weil restriction of scalars of groups of type SL_n and Sp_{2n} (Grothendieck [28] over an algebraically closed field, one checks this extends to arbitrary k). Thus the underlying variety of a special semisimple k-group is a k-rational variety. This statement is not true for special k-tori.

Proposition 4.7. *Let G be a special group over k whose underlying variety is k-rational, and let X be a factorial affine k-variety with an almost free G-action. Then $k(X)$ is purely transcendental over $k(X)^G$. In particular, if X is a (stably) rational variety, then $k(X)^G$ is stably pure.*

Proof. Since the action is almost free, there exist a non-empty open set $U \subset X$ and a morphism $U \to Y$ which makes U into a G-torsor over Y (Corollary 2.19). In particular, the function field $k(Y)$ coincides with $k(X)^G$. Because G is special, any G-torsor is locally trivial for the Zariski topology and U, hence also X, is birational to the product $Y \times G$. Thus $k(X)$ is pure over $k(Y) = k(X)^G$. □

Corollary 4.8. *Let G be a semisimple special group over k, and let V be a finite dimensional k-vector space equipped with an almost free linear G-action. Then $k(V)^G$ is stably pure.*

Proof. Indeed, as mentioned above, G is a k-rational variety. □

Proposition 4.9. *Let G be a linear algebraic group over k. Let G_1, G_2 be two special groups which contain G as a closed subgroup. Assume that the underlying varieties of G_1 and G_2 are rational over k. Then G_1/G and G_2/G are stably k-birationally equivalent.*

Proof. Let $X = (G_1 \times_k G_2)/G$ be the quotient of $G_1 \times_k G_2$ under the diagonal embedding. The projection map $X \to G_1/G$ makes X into a G_2-torsor over G_1/G. Since G_2 is special, X is k-birational to the product $G_2 \times (G_1/G)$. Thus X is stably k-birational to G_1/G. The same argument applies to the projection $X \to G_2/G$. Hence G_1/G is stably k-birationally equivalent to G_2/G. □

Example 4.10. Let $G \subset \mathrm{GL}_p$ and $G \subset \mathrm{SL}_q$ be two algebraic k-group embeddings of a given algebraic group G. Then the function fields $k(\mathrm{GL}_p/G)$ and $k(\mathrm{SL}_q/G)$ are stably equivalent.

4.3. **The Orthogonal Group.**

Proposition 4.11. *Let q be a nondegenerate quadratic form over k of rank $n \geq 2$. Let $G = \mathrm{O}(q)$ be the orthogonal group of q. Let V be a finite-dimensional k-vector space with an almost free linear G-action. Then the field $k(V)^G$ is stably pure over k.*

1st Proof (by reduction to GL_n*).* Apply Proposition 3.13 to the group $G = \mathrm{GL}_n$ acting upon the vector space V of symmetric matrices M through $g \cdot M = {}^t gMg$, and take v in that proposition to be the matrix S of q. The orbit of v is the open set of non-singular symmetric matrices, hence it is dense. The stabilizer of v is $\mathrm{O}(q)$. Then Proposition 3.13 reduces the case of $\mathrm{O}(q)$ to that of GL_n which is a "special" group. Then we are done by Proposition 4.7. □

2nd Proof. Let $Q_n \subset \mathcal{M}_n$ be the vector space consisting of symmetric matrices. Consider the map $\mathcal{M}_n \to Q_n$ which sends the matrix A to ${}^t ASA$. This map induces a map f of affine varieties $X = \mathrm{GL}_n \to Y = Q_n^{\mathrm{ns}}$, where Q_n^{ns} denotes the open set of non-singular symmetric $n \times n$ matrices. Let $G = \mathrm{O}(q)$ act upon $\mathcal{M}_n$, hence also GL_n, by left translation, i.e. by the action $(g, A) \mapsto gA$. One easily checks that f makes X into a G-torsor over Y for this action, hence $k(X)^G = k(Y)$ (that the map f is onto over $\bar{k}$ reflects the fact that any non-degenerate quadratic form of rank n is equivalent to q over $\bar{k}$). Since $Y = Q_n^{\mathrm{ns}}$ is an open set in a vector space, $k(Y)$ is purely transcendental. On the other hand, the action of G on $\mathcal{M}_n$ is linear, and it is an almost free action. The theorem now follows from the no-name lemma (Corollary 3.9). □

Remark 4.12. Using Proposition 1.2, the retract rationality of $k(V)^G$ can already be seen from the following fact: *For any local k-algebra A with residue class field κ, the natural map $H^1(A, \mathrm{O}(q)) \to H^1(\kappa, \mathrm{O}(q))$ is surjective.* Indeed, $H^1(\kappa, \mathrm{O}(q))$ classifies isomorphism classes of non-degenerate rank n quadratic forms over κ. Any such class α contains a diagonal form $a_1x_1^2 + \cdots + a_nx_n^2$. Lifting the $a_i \in \kappa^*$ to elements of A^* produces a non-degenerate quadratic form, hence an element in $H^1(A, \mathrm{O}(q))$ whose image under the restriction map is α.

4.4. **The Special Orthogonal Group.** Let q be a nondegenerate quadratic form over k of rank $n \geq 2$, let $\mathrm{disc}(q) \in k^*/k^{*2}$ be its discriminant. Let $G = \mathrm{SO}(q)$ be the special orthogonal group of q. If A is a local k-algebra with $2 \in A^*$, then $H^1(A, \mathrm{SO}(q))$ classifies non-degenerate quadratic forms over A with the same discriminant as q in A^*/A^{*2}. Now, if A is a local ring with residue class field κ ($\mathrm{char}\, \kappa \neq 2$) and $\sum a_ix_i^2$ is a non-degenerate quadratic form over κ with discriminant $\mathrm{disc}(q)$, it can clearly be lifted to a non-degenerate quadratic form over A with the same property.

Just as in Remark 4.12, the lifting property just proven implies that for an almost free linear G-action of $G = \mathrm{SO}(q)$ on a finite dimensional vector space V over k the field $k(V)^G$ is retract rational. One can actually do better.

Proposition 4.13. *Let q be a nondegenerate quadratic form over k of rank $n \geq 2$. For any almost free finite dimensional linear representation V of the special orthogonal group* $\mathrm{SO}(q)$ *over k, the field $k(V)^{\mathrm{SO}(q)}$ is stably rational.*

Proof. Let $X = V^{n-1}$, let $Y = k^N$, $N = n(n-1)/2$, and let $f\colon X \to Y$ be the map which sends $(e_1, \dots, e_{n-1})$ to the point $\{\langle e_i, e_j\rangle\}_{1\le i\le j\le n-1}$, where $\langle x, y\rangle$ denotes the "scalar product" of two vectors x and y with respect to the bilinear form assoiated to q. Applying Proposition 2.4 to the open set U of X consisting of $(e_1, \dots, e_{n-1})$ with $e_1 \wedge \cdots \wedge e_{n-1} \neq 0$, one obtains $k(Y) = k(X)^G$. Moreover, the action of G on X is almost free. Since Y is rational, this produces an almost-free finite dimensional linear representation X over k such that $k(X)^G$ is pure over k. We conclude by the no-name lemma (Corollary 3.9). □

Remark 4.14. To prove this result, Bogomolov [16] uses a generalized version of the no-name lemma together with the slice method; see also Saltman's approach [124].

4.5. **The Spinor Group.** Let $\mathrm{Spin}(q)$ denote the spinor group attached to a nondegenerate quadratic form q of rank n. If $k = \bar{k}$, we write simply $\mathrm{Spin}(q) = \mathrm{Spin}_n$.

Assume $k = \bar{k}$. As we saw in §4.2 for $n \le 6$, the group Spin_n is special, hence if V is an almost free finite dimensional linear representation of Spin_n, the field $k(V)^{\mathrm{Spin}_n}$ is stably pure. In the literature, there is a proof [16] of the stable purity of $k(V)^{\mathrm{Spin}_n}$ for arbitrary n, but that proof is incomplete, it builds upon an incorrectly proved lemma [16, Lemma 3.1]. A proof of the stable rationality of Spin_n for $n = 7$ and $n = 10$ is given by V.È. Kordonskiĭ [80].

D. Shapiro suggested that the classification of forms of low degree given by A. Pfister in 1966 would immediately yield the retract rationality of $k(V)^{\mathrm{Spin}_n}$ for $n \le 12$. Here we shall only discuss the case $n = 12$.

Theorem 4.15. *Let $G = \mathrm{Spin}(q_0)$ be the spinor group of the nondegenerate hyperbolic quadratic form q_0 of rank 12. Let V be a finite-dimensional k-vector space with an almost free linear G-action. Then the field $k(V)^G$ is retract rational over k.*

Proof. Let us write $\mathrm{Spin} = \mathrm{Spin}(q_0)$ and $\mathrm{SO} = \mathrm{SO}(q_0)$. Let

$$1 \to \mu_2 \to \mathrm{Spin} \to \mathrm{SO} \to 1$$

be the natural sequence. For any k-algebra A, there is an an exact sequence of pointed sets in étale cohomology:

$$H^1(A, \mu_2) \to H^1(A, \mathrm{Spin}) \to H^1(A, \mathrm{SO}) \xrightarrow{\partial} H^2(A, \mu_2).$$

There is an action of the group $H^1(A, \mu_2)$ on $H^1(A, \mathrm{Spin})$, and this action is transitive on elements of $H^1(A, \mathrm{Spin})$ with the same image in $H^1(A, \mathrm{SO})$ [28; 137, I. 5.7, Prop. 42 p. 52]. The set $H^1(A, \mathrm{SO})$ classifies the non-degenerate quadratic forms over A of dimension $n = 12$ with discriminant $1 \in A^*/A^{*2}$. The map

$$H^1(A, \mathrm{SO}) \xrightarrow{\partial} H^2(A, \mu_2)$$

associates to the class of a such a form its Clifford invariant [74, p. 437; 137, pp. 147–148].

Let A be a local k-algebra, let κ be its residue field. Let $\xi \in H^1(\kappa, \mathrm{Spin})$. Its image in $H^1(\kappa, \mathrm{SO})$ is the isomorphy class of a quadratic form q of rank 12, trivial discriminant, and trivial Clifford invariant. According to a theorem of Pfister [101, Satz 14] (see also [59]) there exist elements a, b, c, d, e, f in k^* such that such a form may be written as $q = a\langle 1, b\rangle \otimes \langle -c, -d, cd, e, f, -ef\rangle$. Since the reduction map $A^*/A^{*2} \to \kappa^*/\kappa^{*2}$ is onto, one may lift this quadratic form to a class of the same shape over A, and the Clifford invariant of such a class is trivial, as may be checked by a direct computation. Indeed, such a class clearly lies in I^3A. One thus finds an element $\eta_A \in H^1(A, \mathrm{Spin})$ whose image $\eta_\kappa \in H^1(\kappa, \mathrm{Spin})$ has the same image as ξ in $H^1(\kappa, \mathrm{Spin})$. There thus exists $\rho_\kappa \in H^1(\kappa, \mu_2)$ such that $\rho_\kappa.\eta_\kappa = \xi$. Let $\rho \in H^1(A, \mu_2)$ be a lift of this element. Then $\rho \cdot \eta \in H^1(A, \mathrm{Spin})$ reduces to $\xi \in H^1(\kappa, \mathrm{Spin})$. □

4.6. **The Projective Linear Group.** Let V be a finite dimensional vector space over k, $n \geq 2$ an integer. Suppose the projective linear group PGL_n acts linearly and almost freely on V. The question whether the field of invariants $k(V)^{\mathrm{PGL}_n}$ is rational (pure), or at least stably rational, has come up in a variety of contexts. It is still open.

The simplest linear representation of the above type is given by $V_n = \mathcal{M}_n(k) \oplus \mathcal{M}_n(k)$ with PGL_n acting by simultaneous conjugation.

The field of invariants $C_n := k(V_n)^{\mathrm{PGL}_n}$ coincides with the field of fractions of the centre of the generic division ring on two $n \times n$ generic matrices (Procesi [109, pp. 240–241]; M. Artin [2]).

At least for k the field of complex numbers, the field C_n may also be viewed as

— The function field of the moduli space $M(n; 0, n)$ of stable rank n vector bundles on $\mathbb{P}_2$ with Chern numbers $c_1 = 0$, $c_2 = n$ (see Le Bruyn [84, 85]).
— The function field of the generic Jacobian variety of plane curves of degree n (Van den Bergh [151]).

Katsylo and Schofield proved the following reduction result (see also [127]):

Proposition 4.16 ([66, 135])**.** *Suppose $n = rs$ with $\gcd(r, s) = 1$, then C_n is stably equivalent to the fraction field of $C_r \otimes C_s$.*

Saltman proved retract rationality of C_n for n prime [118, Corollary 5.3]. For an alternate proof, see also [37, Corollary 9.13].

Proposition 4.17. *Let n be squarefree or twice a squarefree number. Then for any almost free finite dimensional linear representation V of the projective linear group group PGL_n, the field $k(V)^{\mathrm{PGL}_n}$ is retract rational.*

The rationality for $n = 2$ was proved by Sylvester [148], and by Procesi [109]. Formanek solved the rationality problem for $n = 3$ [43] and $n = 4$ [44]. For n arbitrary, Formanek [43] gave a very useful description of C_n as a field of multiplicative invariants under an action of the symmetric group. For $k = \bar{k}$, Bessenrodt and Le Bruyn [12] proved the stable rationality for $n = 5, 7$. A simpler proof was later given by Beneish [7]. The question remains open for all other prime powers n. The quoted results combine to :

Proposition 4.18. *Let $k = \bar{k}$. Let $n = 2, 3, 4, 5, 7$, or more generally let n be any divisor of 420. Then for any almost free linear representation V of the projective linear group group PGL_n, the field $k(V)^{\mathrm{PGL}_n}$ is stably rational.*

For further rationality results on $k(V)^{\mathrm{PGL}_n}$, with connections with the rationality of some moduli varieties, see [15, 20, 22, 61–71, 140–143]. The case of some exceptional groups is considered in [54, 55].

4.7. Some Finite Groups.

Lemma 4.19. *Let X be a geometrically integral k-variety, and G a finite group acting on X. If the action of G is faithful, then the action is almost free.*

Proof. For any $g \in G$, $g \neq e$, the k-variety $X^g \subset X$ of fixed points has codimension at least 1. Let U be the complement of the union of the X^g, $g \neq e$. Then U is a G-stable nonempty open set and the action of G on U is free. □

Proposition 4.20. *For any faithful finite dimensional linear representation V of $\mathfrak{S}_n$ over k, the field $k(V)^{\mathfrak{S}_n}/k$ is stably pure over k.*

Proof. This is the most classical case. For the natural linear representation of $\mathfrak{S}_n$ in k^n by permutations on the canonical basis, the theorem on symmetric polynomials says that

$$k(t_1, \ldots, t_n)^{\mathfrak{s}_n} = k(s_1, \ldots, s_n)$$

is a pure extension of k. Here we denote by $s_1 = t_1 + \cdots + t_n, \ldots, s_n = t_1 \ldots t_n$ the fundamental symmetric polynomials. The proposition now follows from the no-name lemma. □

But the analogous question for $\mathfrak{A}_n$, $n \geq 6$ remains open!

Proposition 4.21. *Assume $k = \bar{k}$. Let G be a finite subgroup of $\mathrm{GL}(3, k)$. Then for any faithful finite dimensional linear representation V of G over k, the field $k(V)^G/k$ is stably pure over k.*

Proof. Let V_0 be a faithful representation, $\dim V \leq 3$. By Proposition 4.5 we know that the field $k(V_0)^G$ is pure over the field $k(\mathbb{P}(V_0))^G$. Since this last field is unirational of transcendence degree at most 2, it is purely transcendental by Castelnuovo's theorem. The result now follows from the no-name lemma. □

Remark 4.22. Let k be a field and $G \subset \mathrm{GL}(2, k)$ be any finite subgroup. Then for any faithful finite dimensional linear representation V of G over k, the field $k(V)^G/k$ is stably pure. This result applies to the dihedral group $G = D_n$ of order $2n$ and $k = \bar{k}$ or $k = \mathbb{R}$. For the proof we consider a faithful representation V_0, $\dim V_0 \leq 2$. By Proposition 4.5 and Lüroth's theorem, $k(V_0)^G/k$ is pure. Hence $k(V)^G/k$ is stably pure by the no-name lemma.

In some cases, one may use the no-name lemma to prove the purity of fields of invariants.

Theorem 4.23 (Maeda [90])**.** *Let V be the natural 5-dimensional representation of the alternating group $\mathfrak{A}_5$. Then $k(V)^{\mathfrak{A}_5}$ is pure over k.*

We here give a proof in the case $k = \bar{k}$.

Proof (for $k = \bar{k}$). There exists a faithful representation W of dimension 3 (as the group of automorphisms of the icosahedron). Also, there is a decomposition $V = V_0 \oplus V_1$ where V_0 is the trivial representation. Now $V/\mathfrak{A}_5$ is pure over $\mathbb{P}(V_1)/\mathfrak{A}_5$ and of transcendence degree 2. On the other hand, the action of $\mathfrak{A}_5$ on $\mathbb{P}(V_1)$ and $\mathbb{P}(W)$ is almost free (indeed, $\mathfrak{A}_5$ is simple). We may thus apply the projective no-name lemma, i.e. Corollary 3.12. This implies that $(\mathbb{P}(V_1) \times \mathbb{P}(W))/\mathfrak{A}_5$ is pure over $\mathbb{P}(V_1)/\mathfrak{A}_5$ of transcendence degree 2. Thus $\mathbb{P}(V_1)/\mathfrak{A}_5$ and $(\mathbb{P}(V_1) \times \mathbb{P}(W))/\mathfrak{A}_5$ are birational. A second application of the projective no-name lemma implies that $(\mathbb{P}(V_1) \times \mathbb{P}(W))/\mathfrak{A}_5$ is pure over $\mathbb{P}(W)/\mathfrak{A}_5$, which is unirational of dimension 2, hence pure by Castelnuovo's theorem since $\operatorname{char} k = 0$. ☐

We refer to [90] for the proof over $\mathbb{Q}$. In [102], B. Plans shows that for odd $n \geq 3$, the field $\mathbb{Q}(X_1, \ldots, X_n)^{\mathfrak{A}_n}$ is pure over $\mathbb{Q}(X_1, \ldots, X_{n-1})^{\mathfrak{A}_{n-1}}$, thus giving another proof of Maeda's theorem.

Remark 4.24. One may also prove that $k(\mathbb{P}(V))^{\mathfrak{A}_5}$ is pure, see [52, Lemma 5].

Remark 4.25. The binary icosahedral group is a subgroup of $\mathrm{SL}_2(\mathbb{C})$. It is a double cover of $\mathfrak{A}_5$, actually it is its "representation group" in the sense of Schur. It has a unique 4-dimensional faithful linear action. The analysis of the quotient requires a much more delicate analysis [77]. See also [78, 79] for some other finite subgroups of GL_4.

For further results on the rationality of the field of invariants for a linear action of other small finite groups, see [1, 29, 30, 111, 123].

5. The unramified Brauer group

In this section, we shall assume some knowledge of group cohomology, as well as some knowledge of local fields, all of which may be found in Serre's book [136]. For the sake of simplicity, all fields $(k, K, L, \kappa, \ldots)$ will be taken of characteristic zero. Let K be a field, let $\overline{K}$ be an algebraic closure of K and let $\mathfrak{g} = \mathrm{Gal}(\overline{K}/K)$.

Definition 5.1. The second (profinite) cohomology group $H^2(\mathfrak{g}, \overline{K}^*)$ of $\mathfrak{g}$ with values in the multiplicative group $\overline{K}^*$ of $\overline{K}$ is called the Brauer group of K and is denoted $\operatorname{Br} K$.

Given any field inclusion $K \subset L$, there is a natural map $\operatorname{Br} K \to \operatorname{Br} L$.

Definition 5.2. When K is the field of fractions of a discrete valuation ring A with residue field κ (of characteristic zero), there is a basic homomorphism

$$\partial_A\colon \operatorname{Br} K \to X(\kappa),$$

where $X(\kappa)$ denotes the group $\mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\bar{\kappa}/\kappa), \mathbb{Q}/\mathbb{Z})$ of continuous characters of $\mathrm{Gal}(\bar{\kappa}/\kappa)$ with values in the discrete group $\mathbb{Q}/\mathbb{Z}$.

Let us first assume that A is complete. Since the characteristic of κ is assumed to be zero, A, resp. K, may be then identified with the ring of power series $\kappa[[t]]$, resp. with its fraction field $\kappa((t))$. Let K_{nr} be the maximal unramified extension of K. Under the previous identifications, K_{nr} coincides with $\bar{k}((t))_{\mathrm{alg}}$ (the algebraic closure of $\kappa((t))$ in $\bar{k}((t))$), $\mathrm{Gal}(K_{\mathrm{nr}}/K) = \mathrm{Gal}(\bar{\kappa}/\kappa)$, and $\overline{K}$ is the union of all fields $\bar{\kappa}((t^{1/n}))_{\mathrm{alg}}$ ("Puiseux's theorem"). The Galois group $\mathrm{Gal}(\overline{K}/K_{\mathrm{nr}})$ can thus be identified with the profinite group $\varprojlim \mu_n$, which in a non canonical manner is isomorphic to the group $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n$. The cohomological dimension of such a group is 1, hence $H^2(\mathrm{Gal}(\overline{K}/K_{\mathrm{nr}}), \overline{K}^*) = 0$. Since also $H^1(\mathrm{Gal}(\overline{K}/K_{\mathrm{nr}}), \overline{K}^*) = 0$ according to Hilbert's Theorem 90, the restriction-inflation sequence gives rise to a natural isomorphism:

$$H^2(\mathrm{Gal}(K_{\mathrm{nr}}/K), K_{\mathrm{nr}}^*) \xrightarrow{\cong} H^2(\mathrm{Gal}(\overline{K}/K), \overline{K}^*) = \mathrm{Br}\, K. \tag{5.1}$$

The valuation $v\colon K^* \to \mathbb{Z}$ naturally extends to a valuation $v\colon K_{\mathrm{nr}}^* \to \mathbb{Z}$ which respects the action of $\mathrm{Gal}(K_{\mathrm{nr}}/K)$ ($\mathbb{Z}$ being taken with trivial action). It therefore induces a homomorphism:

$$H^2(\mathrm{Gal}(K_{\mathrm{nr}}/K), K_{\mathrm{nr}}^*) \to H^2(\mathrm{Gal}(K_{\mathrm{nr}}/K), \mathbb{Z}) = H^2(\mathrm{Gal}(\bar{\kappa}/\kappa), \mathbb{Z}). \tag{5.2}$$

Now the cohomology of the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ identifies the last group with $H^1(\mathrm{Gal}(\bar{\kappa}/\kappa), \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\bar{\kappa}/\kappa), \mathbb{Q}/\mathbb{Z})$. Combining the isomorphism (5.1) with the map (5.2) defines the map ∂_A in the case when A is complete.

In the general case, ∂_A will be defined as the composite map

$$\mathrm{Br}\, K \to \mathrm{Br}\, \widehat{K} \xrightarrow{\partial_{\hat{A}}} X(\kappa)$$

where $\widehat{A}$, resp. $\widehat{K}$, are the completions of A, resp. K — the residue class field κ being the same as that of A.

We are now in a position to define our basic invariant, first used efficiently by Saltman [119].

Definition 5.3. Let k be a field of characteristic zero. Let K/k be a function field, i.e. assume that the field K is finitely generated, as a field, over the field k. The subgroup $\bigcap_A \ker \partial_A$ of $\mathrm{Br}\, K$, where A runs through all discrete rank one valuation rings $A \subset K$ such that the field of fractions of A is K and k is included in A, is called the *unramified Brauer group* of K (with respect to k) and it is denoted $\mathrm{Br}_{\mathrm{nr}}(K/k)$, or $\mathrm{Br}_{\mathrm{nr}}\, K$ if there is no ambiguity on k.

Remark. In the case where k is algebraically closed, k^* is infinitely divisible, hence for any discrete valuation ring $A \subset K$ with fraction field K, we have $k \subset A$. Hence, in this case, we just write $\mathrm{Br}_{\mathrm{nr}}\, K$.

Lemma 5.4. *The natural map* $\mathrm{Br}\, k \to \mathrm{Br}\, K$ *sends* $\mathrm{Br}\, k$ *to* $\mathrm{Br}_{\mathrm{nr}}(K/k)$.

Proof. It is enough to prove the statement when K is the field of fractions of a complete discrete valuation ring A containing k, i.e. $A = \kappa[[t]]$ and $K = \kappa((t))$, with $k \subset \kappa$. The composite map

$$\bar{k}^* \to \bar{\kappa}((t))_{\mathrm{alg}}^* = K_{\mathrm{nr}}^* \to \mathbb{Z},$$

where the last map is given by the valuation, is zero. Hence the composite map

$$H^2(\mathrm{Gal}(\bar{k}/k), \bar{k}^*) \to H^2(\mathrm{Gal}(K_{\mathrm{nr}}/K), K_{\mathrm{nr}}^*) \to H^2(\mathrm{Gal}(\bar{\kappa}/\kappa), \mathbb{Z}) = X(\kappa)$$

is zero. □

Lemma 5.5. *Let $K \subset L$ be function fields over the field k. The natural map* $\mathrm{Br}\, K \to \mathrm{Br}\, L$ *induces a map* $\mathrm{Br}_{\mathrm{nr}}(K/k) \to \mathrm{Br}_{\mathrm{nr}}(L/k)$.

Proof. Let α be an element of $\mathrm{Br}_{\mathrm{nr}}(K/k)$, and let α_L be its image in $\mathrm{Br}\, L$. Let $B \subset L$ be a discrete valuation ring of rank one with field of fractions L and with $k \subset L$. Let $A = B \cap K$ be the trace of A on K. We have $k \subset A$. If $A = K$, then $\partial_B(\alpha_L) = 0$ according to the previous lemma. Otherwise A is a discrete valuation ring of rank one with field of fractions K, and there is a natural inclusion of the residue class field κ_A of A into the residue class field κ_B of B. Let π be a uniformizing parameter of A, and let $e = v_B(\pi) > 0$ be the valuation of π in B. The result now follows from the general fact that in such a situation, there is a commutative diagram:

$$\begin{array}{ccc} \mathrm{Br}\, L & \xrightarrow{\partial_B} & X(\kappa_B) \\ {\scriptstyle \mathrm{Res}_{K/L}}\uparrow & & \uparrow{\scriptstyle e\cdot \mathrm{Res}_{\kappa_A/\kappa_B}} \\ \mathrm{Br}\, K & \xrightarrow{\partial_A} & X(\kappa_A) \end{array} \tag{5.3}$$

where the L.H.S. vertical map is the natural map, and the R.H.S. vertical map is e times the map induced on character groups by the inclusion $\kappa_A \subset \kappa_B$. To prove this last fact, one simply goes over to completions, in which case the inclusion $K \subset L$ (resp. $K_{\mathrm{nr}} \subset L_{\mathrm{nr}}$) reads $\kappa_A((t)) \subset \kappa_B((u))$ (resp. $\bar{\kappa}_A((t))_{\mathrm{alg}} \subset \bar{\kappa}_B((u))_{\mathrm{alg}}$) with $u = \rho t^e$ for some $\rho \in \kappa_B^*$. □

Lemma 5.6. *If $K = k(t)$ is the rational field in one variable over the field k, the natural map* $\mathrm{Br}(k) \to \mathrm{Br}_{\mathrm{nr}}(K/k)$ *is an isomorphism.*

Proof. Let $\bar{k}$ be an algebraic closure of k. Let $G = \mathrm{Gal}(\bar{k}/k)$. Since $\bar{k}[t]$ is a unique factorization domain and its units are reduced to $\bar{k}^*$, there is an obvious exact sequence of G-modules:

$$1 \to \bar{k}^* \to \bar{k}(t)^* \xrightarrow{\{v_P\}} \oplus_P \mathbb{Z}_P \to 0. \tag{5.4}$$

Here P runs through all monic irreducible polynomials in $k[t]$. If P is such a polynomial, and $P(t) = (t - \alpha_1) \dots (t - \alpha_n)$ is its decomposition over $\bar{k}$, one lets $\mathbb{Z}_P = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_n$ be the permutation G-lattice (free $\mathbb{Z}$-module) with $\mathbb{Z}$-basis the roots $\{\alpha_1, \dots, \alpha_n\}$ of P in $\bar{k}$, the action of G being given by the permutation action on these roots. The map $\{v_P\}$ sends any function to its divisor, i.e. to the set of zeros and poles with multiplicities.

Sequence (5.4) is a split G-sequence: indeed, the map which associates to $f \in \bar{k}(t)^*$ the value at zero of $f/t^{v(f)}$, where v is the valuation associated to the polynomial t, defines a G-retraction of the embedding $k^* \to \bar{k}(t)^*$. Thus the G-cohomology of sequence (5.4) gives rise to a (split) short exact sequence

$$0 \to H^2(G, \bar{k}^*) \to H^2(G, \bar{k}(t)^*) \xrightarrow{\{v_P\}} \oplus_P H^2(G, \mathbb{Z}_P) \to 0. \tag{5.5}$$

If $k(\alpha) = k[t]/P$, the group $H^2(G, \mathbb{Z}_P)$ may be identified with

$$H^2(\mathrm{Gal}(\bar{k}/k(\alpha), \mathbb{Z}) = H^1(\mathrm{Gal}(\bar{k}/k(\alpha), \mathbb{Q}/\mathbb{Z}) = X(k(\alpha)).$$

Tsen's theorem asserts that the field $\bar{k}(t)$ is a C_1-field, and, as such, its Brauer group is trivial. Using Hilbert's theorem 90 and the restriction-inflation sequence we may conclude that the inflation map is an isomorphism

$$H^2(G, \bar{k}(t)^*) \xrightarrow{\cong} \mathrm{Br}(k(t)).$$

Let now P be a monic irreducible polynomial in $k[t]$. Such a polynomial defines a valuation on $k(t)$. Let A be the associated discrete valuation ring, whose residue class field may be identified with the field $k(\alpha) = k[t]/P$. By comparing with the above definition of ∂_A, one then easily checks that the following diagram is commutative:

$$\begin{tikzcd} H^2(G, \bar{k}(t)^*) \arrow[r, "v_P"] \arrow[d, "\cong"'] & H^2(G, \mathbb{Z}_P) \arrow[d, "\cong"] \\ \mathrm{Br}(k(t)) \arrow[r, "\partial_A"] & X(k(\alpha)). \end{tikzcd}$$

Putting this together with sequence (5.5) completes the proof. □

Proposition 5.7. *Let K be a function field over the field k. Let t be an indeterminate. The natural map from* $\mathrm{Br}\, K$ *to* $\mathrm{Br}(K(t))$ *induces an isomorphism* $\mathrm{Br}_{\mathrm{nr}}(K/k) \cong \mathrm{Br}_{\mathrm{nr}}(K(t)/k)$. *In particular, if K/k is rational,* i.e. *if K is purely transcendental over k, or more generally if K/k is stably rational,* i.e. *if $K(t_1, \dots, t_r)$ is purely transcendental over k for some suitable independent variables $t_1, \dots, t_r$, then the natural map from* $\mathrm{Br}(k)$ *to* $\mathrm{Br}_{\mathrm{nr}}(K/k)$ *is an isomorphism.*

Proof. It is enough to prove the first assertion. Let $L = K(t)$. According to Lemma 5.6 the map $\mathrm{Br}\, K \to \mathrm{Br}(K(t))$ is an injection and according to Lemma 5.5 it induces an inclusion $\mathrm{Br}_{\mathrm{nr}}(K/k) \subset \mathrm{Br}_{\mathrm{nr}}(L/k)$. By the very definition of the unramified Brauer group, any α in $\mathrm{Br}_{\mathrm{nr}}(L/k)$ clearly belongs to $\mathrm{Br}_{\mathrm{nr}}(L/K) = \mathrm{Br}_{\mathrm{nr}}(K(t)/K)$, and Lemma 5.6 says that this last group coincides with $\mathrm{Br}\, K$. Thus we only have to prove that if $\alpha \in \mathrm{Br}\, K$ becomes unramified over k when viewed in $\mathrm{Br}(K(t))$, then it already belongs to $\mathrm{Br}_{\mathrm{nr}}(K/k)$. Let A be a discrete rank one valuation ring with field of fractions K. The localization of $A[t]$ at the prime ideal of $A[t]$ spanned by a uniformizing parameter π of A is a discrete valuation ring B with uniformizing parameter π, and the induced map on residue class fields $\kappa_A \to \kappa_B$ may be identified with the inclusion $\kappa_A \to \kappa_A(t)$. Hence in diagram (5.3) applied to the present situation, the R.H.S. vertical map, i.e., $\mathrm{Res}_{\kappa_A/\kappa_B} \colon X(\kappa_A) \to X(\kappa_B) = X(\kappa_A(t))$ is an injection (this applies more generally as soon as $e = 1$ and κ_A is algebraically closed in κ_B). Thus for $a \in \mathrm{Br}\, K$, the equality $\partial_B(\alpha_L) = 0$ implies $\partial_A(\alpha) = 0$. Since A was an arbitrary rank one discrete valuation ring in K, the conclusion follows. □

Remark 5.8. More generally, one may show that if K is the function field of a k-variety which is retract rational (over k), then the natural map from $\mathrm{Br}(k)$

to $\mathrm{Br}_{\mathrm{nr}}(K/k)$ is an isomorphism. The proof requires functorial properties of the unramified Brauer group more elaborate than the one given in Lemma 5.5. For this, we refer to [32].

Up till now, we have sticked to a down-to-earth definition of the unramified Brauer group. In some circumstances it is necessary to use a high-brow definition.

Definition 5.9 (Grothendieck [48])**.** The Brauer group $\mathrm{Br}\, X$ of a scheme X is the second étale cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)$.

Remark 5.10. Note that $H^2_{\text{ét}}(X, \mathbb{G}_m)$ is called "cohomological Brauer group" and denoted $\mathrm{Br}'\, X$ in [48], whereas $\mathrm{Br}\, X$ denotes the "Brauer group" defined in [47] as classes of Azumaya algebras over X.

Using Grothendieck's theorems [47, 48], and in particular the "purity theorem" [39, p. 63, §V, Théorème (3.4); 49, §6], one may show (see [32, §3.4 and Prop. 4.2.3]):

Theorem 5.11. *Let X be a smooth connected variety over a field k of characteristic zero. Let $k(X)$ be the function field of X.*

(i) *There is a natural inclusion* $\mathrm{Br}\, X \subset \mathrm{Br}\, k(X)$ *— in particular* $\mathrm{Br}\, X$ *is a torsion group.*
(ii) *The subgroup* $\mathrm{Br}_{\mathrm{nr}}(k(X)/k) \subset \mathrm{Br}\, k(X)$ *lies in* $\mathrm{Br}\, X$*.*
(iii) *If moreover X is proper,* $\mathrm{Br}\, X = \mathrm{Br}_{\mathrm{nr}}(k(X)/k)$*.*

Remark. For examples of computations of $\mathrm{Br}_{\mathrm{nr}}(k(X)/k)$ over a non-algebraically closed field k, see [24, 33, 34].

For later use, let us recall that given any smooth variety X over a field k and any integer n prime to $\mathrm{char}\, k$, the Kummer sequence in étale cohomology induces a short exact sequence:

$$0 \to \mathrm{Pic}\, X/n \to H^2_{\text{ét}}(X, \mu_n) \to {}_n\mathrm{Br}\, X \to 0,$$

where $\mathrm{Pic}\, X$ denotes the Picard group of X, μ_n the group of n-th roots of unity and ${}_n\mathrm{Br}\, X$ the n-torsion subgroup of $\mathrm{Br}\, X$.

6. A general formula

The results of this section are mainly due to F.A. Bogomolov [17, 18]. Here k denotes an algebraically closed field of characteristic zero.

We shall first consider the case of a finite group G.

Theorem 6.1. *Let G be a finite group of automorphisms of a function field L over the algebraically closed field k of characteristic zero. One then has:*

$$\mathrm{Br}_{\mathrm{nr}}\, L^G = \{\alpha \in \mathrm{Br}\, L^G \mid \alpha_H \in \mathrm{Br}_{\mathrm{nr}}\, L^H \textit{ for all } H \in \mathcal{B}_G\},$$

where $\mathcal{B}_G$ denotes the set of finite bicyclic subgroups of G (a bicyclic group is an abelian group generated by two elements) and α_H denotes the restriction of $\alpha \in \mathrm{Br}\, L^G$ to $\mathrm{Br}\, L^H$.

Proof. Let $K = L^G$ and let $\alpha \in \operatorname{Br} K$ be such that $\partial_A(\alpha) \neq 0$ for some discrete rank one valuation ring $A \subset K$ with fraction field K. We must show that there exists a subgroup $H \in \mathcal{B}_G$ such that

$$\alpha_H \notin \operatorname{Br}_{\mathrm{nr}} L^H.$$

The following facts may be read off from Serre ([136, I, §7]). Let $\mathfrak{p}$ be a prime ideal in the semi-local Dedekind ring $\tilde{A}$ which is the integral closure of A in L, and let $D \subset G$ be the associated decomposition group, and let $I \subset D$ be the inertia group, which is a normal subgroup of G. The localization $B = \tilde{A}_{\mathfrak{p}} \subset L$ is a discrete valuation ring. There is a tower of fields: $K \subset L^D \subset L^I \subset L$ and a corresponding tower of discrete valuation rings obtained by taking the traces $A = B^G \subset B^D \subset B^I$ of B on the subfields. The corresponding residue field extensions read: $F = F \subset E = E$, and we have $D/I = \operatorname{Gal}(E/F) = \operatorname{Gal}(L^I/L^D)$. The Galois extension L^I/K is unramified, i.e. a uniformizing parameter of A is still a uniformizing parameter in B^I.

Moreover, since the residue characteristic is zero, the inertia group I may be identified with a cyclic group, namely a group μ of roots of unity in F ([136, IV, §2, Corollaires 1 and 2]). Furthermore, the conjugacy action of D on the normal subgroup I is then trivial, since this action may be identified with the action of $D/I = \operatorname{Gal}(E/F)$ on $\mu \subset F$, and all roots of unity are in $k \subset F$. Thus I is central in D.

If $\alpha_I \notin \operatorname{Br}_{\mathrm{nr}} L^I$, we are done, since I is a cyclic subgroup of G. We may thus assume that $\alpha_I \in \operatorname{Br}_{\mathrm{nr}}(L^I)$. Since B^D/A is an unramified extension of discrete valuation rings which induces an isomorphism on the residue class fields, the assumption $\partial_A(\alpha) \neq 0$ implies $\partial_{B^D}(\alpha) \neq 0 \in \mathsf{X}(F)$. On the other hand, $\partial_{B^I}(\alpha) = 0 \in \mathsf{X}(E)$. Since B^I/B^D is unramified, the commutative diagram:

$$\begin{array}{ccc}
\operatorname{Br} K^I & \xrightarrow{\partial_{B^I}} & \mathsf{X}(E) \\
\uparrow & & \uparrow{\scriptstyle \operatorname{Res}_{F/E}} \\
\operatorname{Br} K^D & \xrightarrow{\partial_{B^D}} & \mathsf{X}(F)
\end{array}$$

implies that $\partial_{B^D}(\alpha)$ may be identified with a nontrivial character of $D/I = \operatorname{Gal}(E/F)$. Let $g \in D$ be an element of D whose class $\bar{g}$ in D/I satisfies $\partial_{B^D}(\alpha)(\bar{g}) \neq 0 \in \mathbb{Q}/\mathbb{Z}$, let $H = \langle I, g\rangle \subset D$ be the subgroup spanned by I and g, and let F_1 be the residue class field of B^H. Inserting $\operatorname{Br}(K^H) \to \mathsf{X}(F_1)$ in the above diagram, one immediately sees that $\partial(\alpha_H) \neq 0$, since $\partial(\alpha_H)$ may be identified with a character of $\operatorname{Gal}(E/F_1) = D/H$ which does not vanish on $\bar{g}$. This is enough to conclude, since H is an extension of the cyclic group $\langle \bar{g} \rangle$ by the central cyclic subgroup I (see above), hence is an abelian group spanned by two elements. $\square$

We now wish to extend Theorem 6.1 to almost free actions of reductive algebraic groups.

Lemma 6.2. *Let $p\colon X \to Y$ be a dominant morphism of smooth integral varieties. If this morphism admits a section over a non-empty open set of Y,* *then this section induces a cartesian diagram, where the horizontal maps are injective:*

$$\begin{tikzcd}
\operatorname{Br} Y \arrow[r, hook] & \operatorname{Br} X \\
\operatorname{Br}_{\mathrm{nr}} k(Y) \arrow[r, hook] \arrow[u, hook] & \operatorname{Br}_{\mathrm{nr}} k(X). \arrow[u, hook]
\end{tikzcd}$$

Proof. Let the section s be defined over the open set $V \subset Y$, and let $U = p^{-1}(V) \subset X$. Because of the functorial behaviour of the Brauer group and of the unramified Brauer group we have a commutative diagram:

$$\begin{tikzcd}
\operatorname{Br} V \arrow[r, "p^*"] & \operatorname{Br} U \arrow[r, "s^*"] & \operatorname{Br} V \\
\operatorname{Br} Y \arrow[r, "p^*"] \arrow[u, hook] & \operatorname{Br} X \arrow[u, hook] & \\
\operatorname{Br}_{\mathrm{nr}} k(Y) \arrow[r, "p^*"] \arrow[u, hook] & \operatorname{Br}_{\mathrm{nr}} k(X) \arrow[r, "s^*"] \arrow[u, hook] & \operatorname{Br}_{\mathrm{nr}} k(Y) \arrow[uu, hook]
\end{tikzcd}$$

where the top composite map is identity and where all vertical maps are inclusions.

The right hand part of the diagram requires an explanation, the notation s^* being a slight abuse of language. Since the characteristic of k is zero, Hironaka's theorem guarantees that the morphism $p\colon X \to Y$ extends to a morphism $p^{\mathrm{c}}\colon X^{\mathrm{c}} \to Y^{\mathrm{c}}$, where X^{c}, resp. Y^{c}, is a smooth, projective integral variety containing X, resp. Y, as a dense open set. Since p^{c} is a proper morphism and Y^{c} is smooth, the section $s\colon V \to X$ extends to a section $s^{\mathrm{c}}\colon W \to X^{\mathrm{c}}$ of p^{c}, where $W \subset Y^{\mathrm{c}}$ is an open set which contains all codimension 1 points of Y^{c}:

$$\begin{tikzcd}
 & U \arrow[d, hook] & V \arrow[l, "s"'] \arrow[d, equal] & \\
Y \arrow[d, hook] & X \arrow[l, "p"'] \arrow[d, hook] & V \arrow[l, "s"'] \arrow[r, hook] \arrow[d, hook] & Y \arrow[d, hook] \\
Y^{\mathrm{c}} & X^{\mathrm{c}} \arrow[l, "p^{\mathrm{c}}"'] & W \arrow[l, "s^{\mathrm{c}}"'] \arrow[r, hook] & Y^{\mathrm{c}}.
\end{tikzcd}$$

By purity of the Brauer group (Theorem 5.11), the restriction map $\operatorname{Br} Y^{\mathrm{c}} \to \operatorname{Br} W$ is an isomorphism, and both groups coincide with $\operatorname{Br}_{\mathrm{nr}} k(Y)$. We thus have the commutative diagram:

$$\begin{tikzcd}
 & \operatorname{Br} U \arrow[r, "s^*"] & \operatorname{Br} V & & \\
 & \operatorname{Br} X \arrow[r, "s^*"] \arrow[u, hook] & \operatorname{Br} V \arrow[u, equal] & & \\
\operatorname{Br}_{\mathrm{nr}} k(X) \arrow[r, equal] & \operatorname{Br} X^{\mathrm{c}} \arrow[r, "s^{\mathrm{c}*}"] \arrow[u, hook] & \operatorname{Br} W \arrow[u, hook] & \operatorname{Br} Y^{\mathrm{c}} \arrow[l, "\cong"'] \arrow[r, equal] & \operatorname{Br}_{\mathrm{nr}} k(Y).
\end{tikzcd}$$

The composite map $\mathrm{Br}\, X^{\mathrm{c}} \to \mathrm{Br}\, X \to \mathrm{Br}\, U \to \mathrm{Br}\, V$ coincides with the composite map $\mathrm{Br}\, X^{\mathrm{c}} \to \mathrm{Br}\, W \to \mathrm{Br}\, V$, which may be rewritten $\mathrm{Br}_{\mathrm{nr}}\, k(X) \to \mathrm{Br}_{\mathrm{nr}}\, k(Y) \to \mathrm{Br}\, V$.

Now if $\alpha \in \mathrm{Br}\, Y$ is such that $p^*(\alpha) \in \mathrm{Br}\, X$ actually lies in $\mathrm{Br}_{\mathrm{nr}}\, k(X)$, the above diagram shows that $\mathrm{Res}_{Y/V}(\alpha) = s^* p^*(\mathrm{Res}_{Y/V}(\alpha))$ belongs to $\mathrm{Br}_{\mathrm{nr}}\, k(Y)$, hence also $\alpha \in \mathrm{Br}_{\mathrm{nr}}\, k(Y)$. □

Lemma 6.3 (Bogomolov [17]). *Let k be an algebraically closed field,* $\mathrm{char}\, k = 0$*, and let G be a reductive algebraic group over k which is an extension of a finite group W by a torus T. Let X be an integral affine k-variety with an action of G. Assume that all stabilizers are trivial. Then, there exists a finite subgroup A of G such that the natural map $X/A \to X/G$ has a section over a non-empty open set of X/G.*

Proof. First af all, for any algebraic subgroup H of G, the natural map $X \to X/\!\!/H$ makes X into a H-torsor over $X/\!\!/H$. Hence X/H exists, and $X/H = X/\!\!/H$. By assumption, the group G defines an extension:

$$1 \to T \to G \to W \to 1,$$

hence an algebraic action of W on T, and a class in $H^2(W, T)$. Let ${}_nT$ be the subgroup of n-torsion points of T for $n \geq 1$, and let $T_{\mathrm{tors}} = \bigcup_{n \geq 1} {}_nT$ be the whole torsion subgroup of T. The exact sequence

$$1 \to T_{\mathrm{tors}} \to T \to T \otimes_{\mathbb{Z}} \mathbb{Q} \to 1$$

and the vanishing of $H^i(W, T \otimes_{\mathbb{Z}} \mathbb{Q})$ for $i \geq 1$ shows that this class comes from a (unique) class in $H^2(W, T_{\mathrm{tors}})$, hence from some class in $H^2(W, {}_nT)$ for some n. Thus there is a finite group H_n and a commutative diagram of extensions:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & W & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & {}_nT & \longrightarrow & H_n & \longrightarrow & W & \longrightarrow & 1.
\end{array}$$

This diagram gives rise to the following fibre product:

$$\begin{array}{ccc}
X/{}_nT & \longrightarrow & X/H_n \\
{\scriptstyle q_n}\downarrow & & \downarrow{\scriptstyle p_n} \\
X/T & \longrightarrow & X/G.
\end{array}$$

Moreover, the map q_n makes $X/{}_nT$ into a torsor over X/T under the torus $T = T/{}_nT$. Note that the group W acts upon $X/{}_nT$ and X/T (indeed, both horizontal maps in the above diagram are Galois coverings with group W). The map q_n is W-equivariant. Since torsors under tori are locally trivial for the Zariski topology, the set of rational sections of q_n is not empty. The group W acts upon this set.

If we may find such a section which is W-invariant (for the obvious action induced by the action of W on both spaces $X/{}_nT$ and X/T), then this section descends to a section of p_n and we get the conclusion of the lemma.

We shall show that at the cost of changing n into nm for a suitable $m \geq 1$, there exists such a section. Let $E = k(X)^T = k(X/T)$ and let $F = k(X)^G = k(X/G)$. The extension E/F is a Galois extension with Galois group W. Thus to the action of W on the k-torus T we may associate a unique (twisted) F-torus R which becomes isomorphic to the torus $T_E = T \times_k E$ over E. Namely, we take

$$R = \operatorname{Spec} F[X(T)]^W,$$

where $F[X(T)]$ is the group algebra over the character group $X(T)$ of T (which is a free abelian group of finite type), and the W-action on $F[X(T)]$ is simultaneous on F and $X(T)$. The generic fibre of q_n is a principal homogenous space under the action of T_E. Since W acts equivariantly on the whole situation, the generic fibre P_n of p_n, which is an F-variety, inherits a structure of principal homogeneous space under the F-torus R. The isomorphy class of this principal homogeneous space in the étale (= Galois) cohomology group $H^1(F, R)$ is killed by some integer $m \geq 1$. Let us then consider the commutative diagram of exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & {}_{mn}R \arrow[r] & R \arrow[r, "mn"] & R \arrow[r] & 1 \\
1 \arrow[r] & {}_{n}R \arrow[r] \arrow[u] & R \arrow[r, "n"] \arrow[u, equal] & R \arrow[r] \arrow[u, "m"'] & 1,
\end{tikzcd}$$

which reflects the W-action on the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & {}_{mn}T \arrow[r] & T \arrow[r, "mn"] & T \arrow[r] & 1 \\
1 \arrow[r] & {}_{n}T \arrow[r] \arrow[u] & T \arrow[r, "n"] \arrow[u, equal] & T \arrow[r] \arrow[u, "m"'] & 1.
\end{tikzcd}$$

We also have the fibre product:

$$\begin{tikzcd}
X/{}_{mn}T \arrow[r] \arrow[d, "q_{mn}"'] & X/H_{mn} \arrow[d, "p_{mn}"] \\
X/T \arrow[r] & X/G,
\end{tikzcd}$$

where H_{nm} is the finite group obtained from H_n by pushing out through ${}_nT \to {}_{mn}T$, thus yielding the commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & T \arrow[r] & G \arrow[r] & W \arrow[r] & 1 \\
1 \arrow[r] & {}_{mn}T \arrow[r] \arrow[u] & H_{mn} \arrow[r] \arrow[u] & W \arrow[r] \arrow[u, equal] & 1 \\
1 \arrow[r] & {}_{n}T \arrow[r] \arrow[u] & R \arrow[r] \arrow[u, equal] & R \arrow[r] \arrow[u, equal] & 1.
\end{tikzcd}$$

Both the generic fibre P_n of q_n and P_{nm} of q_{nm} are principal homogeneous spaces under the same F-torus R. Indeed, on $X/_nT$, the torus which acts is the quotient $T = T/_nT$, and on $X/_nT$, it is $T = T/_{nm}T$. Now as the previous diagrams reveal, $r\colon X/_nT \to X/_{nm}T$ is the obvious projection, t is in T and x is in $X/_nT$, then $r(t.x) = t^m \cdot r(x)$. Thus the natural map from $X/H_n \to X/H_{nm}$ induces on generic fibres a map $r_1 : P_n \to P_{nm}$ which satisfies

$$r_1(t \cdot x) = t^m \cdot r_1(x) \text{ for } t \in R.$$

This implies that the class of P_{nm} in $H^1(F, R)$ is m times the class of P_n. Since the class of P_n is killed by m, we conclude that the class of P_{nm} is trivial, which completes the proof. □

Theorem 6.4 (Bogomolov [18, Theorem 2.1])**.** *Let k be an algebraically closed field of characteristic zero, let G be a reductive group over k, and let X be an integral affine k-variety with a G-action. Assume that all stabilizers are trivial. One then has*

$$\mathrm{Br}_{\mathrm{nr}}\, k(X)^G = \{\alpha \in \mathrm{Br}\, k(X)^G \mid \alpha_A \in \mathrm{Br}_{\mathrm{nr}}\, k(X)^A \textit{ for all } A \in \mathcal{B}_G\},$$

where $\mathcal{B}_G$ denotes the set of finite bicyclic subgroups of $G(k)$ and α_A denotes the restriction of $\alpha \in \mathrm{Br}\, k(X)^G$ to $\mathrm{Br}\, k(X)^A$.

Proof. Let G° be the connected component of identity, and let $\mathfrak{g}$ be its Lie algebra. Let the group G act on $\mathfrak{g}$ via the adjoint representation and let G act upon the product $X \times \mathfrak{g}$ via the diagonal action. Let $\mathfrak{t} \subset \mathfrak{g}$ denote a fixed Cartan subalgebra, and let $N \subset G$ be the normalizer of $\mathfrak{t}$ for the adjoint action. The group N is a reductive group whose identity component is a (maximal) torus $T \subset G$. It is a classical fact that any regular semisimple element in $\mathfrak{g}$ is conjugate under G to an element of $\mathfrak{t}$. Moreover, on the open set $\mathfrak{t}^\circ \subset \mathfrak{t}$ consisting of regular semisimple elements, the following property holds: if $g \in G$, $x, y \in \mathfrak{t}^\circ$ and $g \cdot x = y$, then $g \in N$. Thus the closed subvariety $X \times \mathfrak{t} \subset X \times \mathfrak{g}$ is a (G, N)-slice, and by the slice lemma 3.1 we have $k(X \times \mathfrak{t})^N \cong k(X \times \mathfrak{g})^G$. On the other hand, the no-name lemma 3.9 implies that $k(X \times \mathfrak{t})^N$, resp. $k(X \times \mathfrak{g})^G$, is purely transcendental over $k(X)^N$, resp. $k(X)^G$. According to Proposition 5.7, we thus have: $\mathrm{Br}_{\mathrm{nr}}\, k(X)^N \cong \mathrm{Br}_{\mathrm{nr}}\, k(X \times \mathfrak{t})^N$ and $\mathrm{Br}_{\mathrm{nr}}\, k(X)^G \cong \mathrm{Br}_{\mathrm{nr}}\, k(X \times \mathfrak{g})^G$. The functoriality of the unramified Brauer group now implies that the natural inclusion $k(X)^G \subset k(X)^N$ induces an isomorphism $\mathrm{Br}_{\mathrm{nr}}\, k(X)^N \cong \mathrm{Br}_{\mathrm{nr}}\, k(X)^G$. According to Lemma 6.3, there exists a finite group $H \subset N$ such that the map between quotient spaces: $X/H \to X/N$ has a rational section. Now Lemma 6.2 shows that an element $\alpha \in \mathrm{Br}(k(X)^N)$ is unramified if and only if its restriction α_H is unramified. By Theorem 6.1, α_H itself is unramified if and only if α_A is unramified for all bicyclic subgroups A of H. The theorem follows. □

7. Linear action of a finite group

In this section, k denotes an algebraically closed field of characteristic zero.

Theorem 7.1 (Bogomolov [17])**.** *Let $G \subset \mathrm{GL}(V)$ be a finite group of automorphisms of a finite dimensional k-vector space V. One then has:*

$$\begin{aligned}\mathrm{Br}_{\mathrm{nr}}\, k(V)^G &\cong \ker\Big[H^2(G,k^*) \xrightarrow{\mathrm{Res}} \prod_{A\in\mathcal{B}_G} H^2(A,k^*)\Big]\\ &\cong \ker\Big[H^2(G,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{Res}} \prod_{A\in\mathcal{B}_G} H^2(A,\mathbb{Q}/\mathbb{Z})\Big]\\ &\cong \ker\Big[H^3(G,\mathbb{Z}) \xrightarrow{\mathrm{Res}} \prod_{A\in\mathcal{B}_G} H^3(A,\mathbb{Z})\Big],\end{aligned}$$

where $\mathcal{B}_G$ denotes the set of all bicyclic subgroups of G (in the last two formulas, G acts trivially upon $\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$).

The same formulas hold if one replaces the set $\mathcal{B}_G$ of bicyclic subgroups by the set $\mathcal{A}_G$ of all abelian subgroups of G.

Proof. If $A \subset G$ is any abelian group, the field $k(V)^A$ is pure over k by Fischer's theorem (see Proposition 4.3), hence $\mathrm{Br}_{\mathrm{nr}}\, k(V)^A = 0$. On the other hand, given any subgroup $H \subset G$, the restriction-inflation sequence for the faithful action of H on $k(V)^*$ yields an exact sequence:

$$0 \to H^2(H, k(V)^*) \to \mathrm{Br}\, k(V)^H \to \mathrm{Br}\, k(V).$$

Using the functoriality of the unramified Brauer group 5.5 and Theorem 6.1, one gets the formula:

$$\mathrm{Br}_{\mathrm{nr}}\, k(V)^G \cong \ker\Big[H^2(G,k(V)^*) \xrightarrow{\mathrm{Res}} \prod_{A\in\mathcal{B}_G} H^2(A,k(V)^*)\Big]. \tag{7.1}$$

Since $k[V]$ is a UFD and $k[V]^* = k^*$, one has a short exact sequence of G-modules:

$$1 \to k^* \to k(V)^* \to \mathrm{Div}\, V \to 0,$$

where the G-module $\mathrm{Div}\, V$ is a direct sum of permutation modules $\mathbb{Z}[G/H]$ for various subgroups H. Such a module satisfies the two properties:

$$\begin{aligned}&H^1(G,M) = 0,\\ &\ker\Big[H^2(G,M) \to \prod_{g\in G} H^2(\langle g\rangle, M)\Big] = 0,\end{aligned}$$

as one easily checks by reducing to the case $M = \mathbb{Z}$ with trivial action. The first formula now easily follows. The second formula is obtained by identifying the group μ of roots of unity in k^* with $\mathbb{Q}/\mathbb{Z}$, and using the unique divisibility, hence cohomological triviality of k^*/μ. As for the third formula, it is obtained by shifting via the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$.

The last statement now follows from the vanishing $\mathrm{Br}_{\mathrm{nr}}\, k(V)^A = 0$ for any abelian subgroup A of G. ∎

7.1. **The case of a nilpotent group of class 2.** Bogomolov [17] made a thorough application of the previous theorem when G is a nilpotent group, particularly of class 2. Such a group is a central extension

$$1 \to C \to G \to \Gamma \to 1$$

of a finite abelian group Γ by another finite abelian group C and classes of such extensions are classified by the group $H^2(\Gamma, C)$ where C is viewed as a trivial Γ-module. We denote by $[G] \in H^2(\Gamma, C)$ the class of G.

It is well known [26, §V.6, Theorem 6.4] that for an abelian group Γ we have $H_2(\Gamma, \mathbb{Z}) = \Lambda^2\Gamma$. Then, for any Γ-module M, the universal coefficient sequence [26, §V.6, exercise 5] yields

$$0 \to \operatorname{Ext}^1(\Gamma, M) \to H^2(\Gamma, M) \xrightarrow{\omega_M} \operatorname{Hom}(\Lambda^2\Gamma, M) \to 0.$$

Then for $M = \mathbb{Q}/\mathbb{Z}$ with trivial action, the map

$$\omega_{\mathbb{Q}/\mathbb{Z}} \colon H^2(\Gamma, \mathbb{Q}/\mathbb{Z}) \to \operatorname{Hom}(\Lambda^2\Gamma, \mathbb{Q}/\mathbb{Z})$$

is an isomorphism. For $M = C$ with trivial action, the map

$$\omega_C \colon H^2(\Gamma, C) \to \operatorname{Hom}(\Lambda^2\Gamma, C)$$

sends the class of $[G]$ to

$$\lambda_G \colon \Lambda^2\Gamma \to [G, G] \subset C$$

defined by

$$\lambda_G(\gamma_1 \wedge \gamma_2) = [g_1, g_2]$$

where $g_1, g_2 \in G$ are lifts of $\gamma_1, \gamma_2 \in \Gamma$. From the definition $\operatorname{Im} \lambda_G = [G, G]$. So λ_G is surjective if and only if $[G, G] = C$.

For any subgroup G' of G, we denote by $S_{G'}$ the kernel of $\lambda_{G'} \colon \Lambda^2\Gamma' \to C'$, where Γ' is the image of G' in Γ and $C' = G' \cap C$.

Definition 7.2. Let G be a central extension of a finite abelian group Γ by another one C. To the canonical homomorphism $\lambda_G \colon \Lambda^2\Gamma \to C$ one may attach the following subgroups

$$S_{\text{bic}} = S_{G,\text{bic}} \subset S_G := \ker \lambda_G \subset \Lambda^2\Gamma$$

where S_{bic} is the subgroup of S_G generated by all the $\gamma_1 \wedge \gamma_2$ which belong to S_G.

In other words, S_G is defined by the exact sequence

$$0 \to S_G \to \Lambda^2\Gamma \xrightarrow{\lambda_G} C$$

and S_{bic} is the subgroup generated by the images of the S_A's for all bicyclic subgroups A of G.

Theorem 7.3 (Bogomolov [17])**.** *Let G be a central extension of a finite abelian group Γ by another one C. Then*

$$\operatorname{Br}_{\text{nr}} k(V)^G = \ker(\widehat{S_G} \to \widehat{S_{\text{bic}}}) = (\widehat{S_G/S_{\text{bic}}})$$

where S_G is the kernel of the canonical morphism $\lambda_G \colon \Lambda^2\Gamma \to C$, and

$$S_{\text{bic}} \subset S_G \subset \Lambda^2\Gamma$$

is the subgroup generated by the S_A's for all bicyclic subgroups A of G.

Proof. The lower terms exact sequence of the Hochschild-Serre spectral sequence

$$H^p(\Gamma, H^q(C, \mathbb{Q}/\mathbb{Z})) \implies H^{p+q}(G, \mathbb{Q}/\mathbb{Z})$$

yields the top exact sequence in the following diagram:

$$\begin{tikzcd}
\widehat{C} \arrow[r] \arrow[d, equal] & H^2(\Gamma, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "\omega_{\mathbb{Q}/\mathbb{Z}}"', "\cong"] & \ker[H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(C, \mathbb{Q}/\mathbb{Z})] \arrow[r] & H^1(\Gamma, \widehat{C}) \\
\widehat{C} \arrow[r] & \mathrm{Hom}(\Lambda^2\Gamma, \mathbb{Q}/\mathbb{Z}) \arrow[r] & \mathrm{Hom}(S_G, \mathbb{Q}/\mathbb{Z}) = \widehat{S_G} \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

This gives the following exact sequence:

$$0 \to \widehat{S_G} \to \ker\bigl[H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(C, \mathbb{Q}/\mathbb{Z})\bigr] \to H^1(\Gamma, \widehat{C}).$$

We want to calculate

$$B_G := \mathrm{Br}_{\mathrm{nr}}\, \mathbb{C}(V)^G \subset H^2(G, \mathbb{Q}/\mathbb{Z}).$$

This inclusion is functorial and covariant in G. Fischer's theorem yields $B_A = 0$ for any abelian group A. In particular, $B_C = 0$. So by restriction B_G maps to 0 in $H^2(C, \mathbb{Q}/\mathbb{Z})$:

$$B_G \subset \ker\bigl[H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(C, \mathbb{Q}/\mathbb{Z})\bigr].$$

Then we prove that B_G maps to 0 in $H^1(\Gamma, \widehat{C}) = \mathrm{Hom}(\Gamma, \widehat{C})$. By functoriality, it is sufficient to prove this result by restriction to any cyclic subgroup Γ' of Γ. Let G' the restriction of the extension G to Γ'. A central extension of a cyclic group is an abelian group. Thus Γ' is abelian and again by Fisher's theorem $B_{G'} = 0$.

We finally get the inclusion

$$B_G \subset \widehat{S_G}.$$

Theorem 7.1 says that

$$B_G \cong \ker\Bigl[H^2(G, \mathbb{Q}/\mathbb{Z}) \to \prod_{G' \in \mathcal{B}_G} H^2(G', \mathbb{Q}/\mathbb{Z})\Bigr]$$

where G' is any bicyclic subgroup of G. Then by functoriality B_G is the subgroup of $\widehat{S_G}$ which maps to 0 in each of the $H^2(G', \mathbb{Q}/\mathbb{Z})$'s, hence to 0 in each of the $\widehat{S_{G'}}$'s for any bicyclic subgroup G' of G. Let Γ' be the image in Γ of such a subgroup G'. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S_G \arrow[r] & \Lambda^2\Gamma \arrow[r, "\lambda_G"] & C \\
0 \arrow[r] & S_{G'} \arrow[r] \arrow[u] & \Lambda^2\Gamma' \arrow[r, "\lambda_{G'} = 0"] \arrow[u] & C. \arrow[u, equal]
\end{tikzcd}$$

Let $g_1, g_2 \in G$ and $\gamma_1, \gamma_2 \in \Gamma$ their images. Then the two conditions

(i) $g_1 g_2 = g_2 g_1$,
(ii) $\gamma_1 \wedge \gamma_2 \in S_G$,

are equivalent.

Hence the cyclic subgroups of S_G generated by elements of the form $\gamma_1 \wedge \gamma_2$ are exactly the images of the $S_{G'}$'s for G' a bicyclic subgroup of G. $\square$

Corollary 7.4. *Let Γ be a finite abelian group. There is a bijection between the classes of central extensions G of Γ such that $B_G \neq 0$ and the subgroups*

$$S \subset \Lambda^2\Gamma$$

such that

$$S_{\mathrm{bic}} \neq S.$$

Examples 7.5. Let us find simple examples of finite groups G such that $B_G \neq 0$. We consider only nilpotent groups G of class 2.

We are searching G among central extensions of a fixed finite abelian group Γ by using Corollary 7.4.

Consider the case $\Gamma = (\mathbb{Z}/p)^4$. We find $\Lambda^2\Gamma = \mathbb{F}_p^6$. It is simpler to work in the projective space $\mathbb{P}(\Lambda^2\Gamma) = \mathbb{P}^5(\mathbb{F}_p)$. For $S \subset \Lambda^2\Gamma = \mathbb{F}_p^6$ the following conditions are equivalent:

(i) $S \neq S_{\mathrm{bic}}$.
(ii) $\mathbb{P}(S) \cap Q$ does not generate $\mathbb{P}(S)$ as a linear space, where Q denotes the quadric whose equation is $x_1x_2 + x_3x_4 + x_5x_6 = 0$ which consists of all undecomposable tensors $\gamma_1 \wedge \gamma_2$.

Here is the list of all possibilities for condition (ii):

1. $\mathbb{P}(S) =$ a point $\notin Q$ (this is Saltman's original example [119]; here $\#G = p^9$).
2. $\mathbb{P}(S) =$ a line tangent to Q (here $\#G = p^8$).
3. $\mathbb{P}(S) =$ a line which does not intersect Q (on $\mathbb{F}_p$) (here $\#G = p^8$).
4. $\mathbb{P}(S) =$ a 2-plane which intersects Q along a line (here $\#G = p^7$).
5. $\mathbb{P}(S) =$ a 2-plane which intersects Q in a point (here $\#G = p^7$).

More generally, for $\Gamma = (\mathbb{Z}/p)^{2m}$, the subvariety of $\mathbb{P}(\Lambda^2\Gamma)$ which consists of tensors in $\Lambda^2\Gamma$ of rank $< 2m$ is a hypersurface of degree m which is the zero locus of the Pfaffian. Among different examples, the simplest one is obtained by taking $\mathbb{P}(S)$ to be a point outside this hypersurface. One may also consider the subspace S which consists of all matrices

$$\begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix}$$

whose 4 blocks are $m \times m$ matrices of the following type:

$$M_{1,1} = M_{2,2} = 0$$
$$M_{1,2} \text{ trigonal with diagonal entries } \{\lambda, \dots, \lambda\}.$$

in which case S_{bic} is the subspace $\lambda = 0$.

Remark. In [17] Bogomolov also gives an example of a group G of order p^6 with $B_G \neq 0$ and proves that $B_G = 0$ for any G of order p^n for $n \leq 5$. As a matter of fact, for $n \leq 4$, the fields $k(V)^G$ are stably rational (Chu and Kang [30]). The stable rationality for $n = 5$ remains open. Recent computations of B_G for finite Chevalley groups of type A_n may be found in [21].

8. Multiplicative action of a finite group

Most results in the present section are due to D. Saltman. Here k denotes an algebraically closed field of characteristic zero.

Let G be a group. A G-lattice M is a $\mathbb{Z}$-free module M of finite rank equipped with a G-action. We denote $k[M]$ the group algebra of M over k. If M is of rank r, then $k[M] \cong k[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}]$. The associated field of fractions is denoted $k(M) \cong k(t_1, \ldots, t_r)$.

Definition 8.1. Given a G-lattice M consider the action of G on $k[M]$ which is trivial on k and coincides with the given one on M. The induced action of G on $k(M)$ is called the *multiplicative action* of G associated to the G-lattice M.

Such actions are sometimes referred to as “purely monomial actions”.

Definition 8.2. For a given G-lattice M we may also consider the *twisted multiplicative action* of G associated to a given crossed homomorphism of the finite group G in $\widehat{M} = \mathrm{Hom}(M, k^*)$, i.e. to a 1-cocycle $\alpha \in Z^1(G, \widehat{M})$. If we denote by t^m the canonical image of $m \in M$ in $k[M]$ the α-twisted action of G on the k-algebra $k[M]$ is given by:

$$g \cdot_\alpha t^m = \alpha_{g^{-1}}(m) t^{g \cdot m}.$$

with trivial action on the coefficients. We denote by $k[M]_\alpha$ this twisted G-k-algebra and by $k(M)_\alpha$ the field of fractions. Let α and α' be in the same cohomology class. Then the two G-k-algebras $k[M]_\alpha$ and $k[M]_{\alpha'}$ are isomorphic: if $\alpha'/\alpha = d\beta$ where $\beta : M \to k^*$ then $t^m \mapsto \beta(m)t^m$ defines a morphism of k-algebras “β” : $k[M]_\alpha \to k[M]_{\alpha'}$ which is G-equivariant and which is clearly an isomorphism. Note that the formula

$$g \cdot_\alpha t^m = \alpha_g(g \cdot m) t^{g \cdot m}$$

defines the opposite twisted action on $k[M]$, since we have

$$\alpha_g(g \cdot m) = ({}^{g^{-1}}\alpha_g)(m) \text{ and } 1 = \alpha_e = \alpha_{g^{-1}}({}^{g^{-1}}\alpha_g).$$

A good reason for studying such actions is provided by the following theorem, which extends early work of Procesi [109] and Formanek [43] for $G = \mathrm{PGL}_n$:

Theorem 8.3 (Saltman [124])**.** *Let G be a reductive connected linear algebraic group over k. Let V be a finite-dimensional k-vector space with an almost free linear G-action. Let $T \subset G$ be a maximal torus, $W = N_G(T)/T$ the Weyl group and $X(T)$ the character group of T, viewed as a W-lattice. Choose a surjective map $f\colon P \to X(T)$ with P a permutation W-lattice. Let M be the kernel of f. Then $k(V)^G$ is stably isomorphic over k to $k_\alpha(M)^W$ for suitable $\alpha \in Z^1(W, \mathrm{Hom}(M, k^*))$.*

For further work in this direction, see [9, 11, 132, 133].

Lemma 8.4. *Every twisted G-k-algebra $k[M]_\alpha$ defines an extension of G-modules*
$$1 \to k^* \to k[M]_\alpha^* \to M \to 0$$
whose class is precisely the class of $\alpha \in Z^1(G, \mathrm{Hom}(M, k^*))$.

Lemma 8.5. *Any twisted multiplicative action of a group G on a permutation G-lattice P extends to a G-linear action on a vector space V such that $k(M)^G = k(V)^G$.*

Proof. Let $\{e_x\}$ a basis of the $\mathbb{Z}$-lattice P which is G-stable. Then G acts linearly on $V = \oplus_x k e_x$. As $k[P]$ is obtained from $k[V]$ by inverting the x's we have the natural G-equivariant inclusions
$$k[V] \subset k[P] \subset k(V) = k(P)$$
and the same fields of invariants $k(P)^G = k(V)^G$. □

Lemma 8.6. *Given a multiplicative action of a group G on a lattice M one may find a G-lattice N, a finite group Γ which is a semi-direct extension of G by an abelian group and a faithful linear action of Γ on a finite dimensional vector space V such that $k(M \oplus N)^G = k(V)^\Gamma$.*

Proof. Up to changing M by $M \oplus N$, where N is an auxiliary G-lattice, one may find an exact sequence of G-modules
$$0 \to M \to P \xrightarrow{\chi} F \to 0,$$
where P is a permutation G-lattice and F is finite. The extension $k(P)/k(M)$ is Galois with group $\widehat{F} = \mathrm{Hom}(F, \mu)$ (where $\mu \subset k^*$ denotes the group of roots of unity in k). The G-action on $k(M)$ extends to $k(P)$. Then the extension $k(P)/k(M)^G$ is also Galois with Galois group Γ which is the semi-direct product of $\widehat{F}$ with G:
$$\Gamma = \widehat{F} \rtimes G.$$
The group Γ acts linearly on $k(P)$. This is clear for $g \in G$. One can verify that each $\gamma \in \widehat{F}$ acts on $P \subset k(P)$ through multiplication by roots of unity:
$$\gamma \cdot p = \langle \gamma, \chi(p) \rangle p \text{ for } \gamma \in \widehat{F} \text{ and } p \in P.$$
Thus Γ acts linearly on $k \otimes_{\mathbb{Z}} P$. □

For $M = \mathbb{Z}[G]$ the following theorem reduces to Theorem 7.1.

Theorem 8.7 (Saltman [126, Theorem 12])**.** *Let G be a finite group, let M be a faithful G-lattice, and let $k(M)$ denote the field of fractions of the group algebra $k[M]$. One then has:*
$$\mathrm{Br}_{\mathrm{nr}}\, k(M)^G \cong \ker\Big[H^2(G, k^* \oplus M) \xrightarrow{\mathrm{Res}} \prod_{A \in \mathcal{B}_G} H^2(A, k^* \oplus M)\Big].$$

Proof. The variety $X = \mathrm{Spec}\, k[M]$ is isomorphic to a product of copies of the multiplicative group $\mathbb{G}_m$. Hence it is factorial ($\mathrm{Pic}\, X = 0$). Note that $\mathrm{Div}\, X$ is always a direct sum of permutation modules. Moreover, the group of units

$k[M]^*$ may be identified with the group $k^* \oplus M$ (as G-modules). Thus one has an exact sequence of G-modules:

$$1 \to k^* \oplus M \to k(M)^* \to \operatorname{Div} X \to 0.$$

If we argue as in Theorem 7.1, the theorem will follow from the following propositions:

Proposition 8.8. *Let A be a cyclic group, and let M be an A-lattice. Then the field of invariants $k(M)^A$ is retract rational and* $\operatorname{Br}_{\mathrm{nr}} k(M)^A = 0$.

Proof. Replacing A by its image in $\operatorname{Aut} M$ allows us to assume that M is a faithful A-module. Let $0 \to M \to P \to F \to 0$ be a flasque resolution of M. Here P is a permutation G-lattice and F is a flasque A-lattice, i.e. $H^1(H, \operatorname{Hom}(F, \mathbb{Z})) = 0$ for all subgroups H of A. Such resolutions always exist [36]. If A is cyclic, a basic result of Endo and Miyata says that any flasque A-lattice is a direct factor of a permutation A-lattice, see [36]. Quite generally, given such an exact sequence as above, the field $k(P)^A/k(M)^A$ is the function field of a principal homogeneous space E over $k(M)^A$ under the $k(M)^A$-torus whose character group over $k(M)$ is the A-lattice F. Since F here is a direct factor of a permutation module, it follows from Hilbert's theorem 90 that any such principal homogenous space is trivial. In particular, E has a $k(M)^A$-rational point, and by Lemma 6.2 $\operatorname{Br}_{\mathrm{nr}} k(M)^A$ injects into $\operatorname{Br}_{\mathrm{nr}} k(P)^A$. This last group is trivial, because A is abelian and P is a permutation module, hence $k(P)^A$ is pure over k by Fischer's theorem. $\square$

Proposition 8.9. *Let A be a bicyclic group, and let M be an A-lattice. Then* $\operatorname{Br}_{\mathrm{nr}} k(M)^A = 0$.

Proof. The proof of this proposition is much more technical. Indeed it is the core of the proof of Theorem 8.7. We shall refer to Saltman's original paper [126] and to Barge's more natural proof [4]. $\square$

Remark 8.10. That the kernel $\ker[H^2(G, k^* \oplus M) \xrightarrow{\mathrm{Res}} \prod_{A \in \mathcal{B}_G} H^2(A, k^* \oplus M)]$ is a subgroup of $\operatorname{Br}_{\mathrm{nr}} k(M)^G$ does not rely on the last two propositions. This accounts for Saltman's early counter-examples to the "Noether problem" over an algebraically closed field [119, 122].

Proposition 8.11 (Saltman [122])**.** *Let G be a finite group of order p^n, p prime, and assume that G is neither cyclic nor bicyclic – which implies $n \geq 3$. Let M be a faithful G-lattice and assume* $\exp H^2(G, M) = p^n$. *Then* $\operatorname{Br}_{\mathrm{nr}} k(M)^G \neq 0$.

Proof. By hypothesis there exists an element $\alpha \in H^2(G, M)$ of order p^n. Then

$$0 \neq \beta := p^{n-1}\alpha \in H^2(G, M).$$

If A is a cyclic or a bicyclic subgroup subgroup of G, then $A \neq G$ hence $\#A = p^m$ with $m \leq n-1$. In particular, the group $H^2(A, M)$ is killed by p^{n-1} and the restriction of $\beta = p^{n-1}\alpha$ to $H^2(A, M)$ is trivial. Hence

$$0 \neq \beta \in \ker\Big[H^2(G, M) \xrightarrow{\mathrm{Res}} \prod_{A \in \mathcal{B}_G} H^2(A, M)\Big] \subset \operatorname{Br}_{\mathrm{nr}} k(M)^G$$

where the last inclusion is part of Theorem 8.7. □

Example 8.12 (multiplicative example [122]). Let G be as above, i.e. of order $\#G = p^n$, p prime and G not bicyclic which implies $n \geq 3$. In the standard resolution of the trivial G-lattice $\mathbb{Z}$, let M denote the kernel of the map $\pi\colon \mathbb{Z}[G \times G] \to \mathbb{Z}[G]$ given on generators of $\mathbb{Z}[G \times G]$ by $(g,h) \mapsto (g-h)$. From the exact sequence

$$0 \to M \to \mathbb{Z}[G \times G] \xrightarrow{\pi} \mathbb{Z}[G] \to \mathbb{Z} \to 0$$

we deduce $H^2(G,M) \cong \widehat{H}^0(G,\mathbb{Z}) \cong \mathbb{Z}/p^n$ where $\#G = p^n$. Then

$$\exp H^2(G,M) = p^n.$$

This gives for every such group G a faithful G-lattice M such that

$$\mathrm{Br}_{\mathrm{nr}}\, k(M)^G \neq 0.$$

For a given prime p the simplest group is $G = (\mathbb{Z}/p)^3$.

Example 8.13 (linear example). Using Lemma 8.6, for any prime p this example leads to a linear example for a group Γ of order p^{3p^3}.

Corollary 8.14 (Barge [4]). *For a finite group G, the following conditions are equivalent:*

(i) *For every faithful G-lattice M, we have* $\mathrm{Br}_{\mathrm{nr}}\, k(M)^G = 0$.
(ii) *All Sylow subgroups of G are bicyclic.*

Proof. Let us prove that (ii) $\Longrightarrow$ (i). Let p be a prime and $G_p \subset G$ a p-Sylow subgroup. The field extension $k(M)^{G_p}/k(M)^G$ is of degree prime to p. It therefore induces an imbedding on p-primary components of Brauer groups, and also of unramified Brauer groups:

$$\mathrm{Br}_{\mathrm{nr}}(k(M)^G)(p \subset \mathrm{Br}_{\mathrm{nr}}(k(M)^{G_p})(p,$$

and the later group vanishes according to Proposition 8.9 applied to the bicyclic group G_p.

For the converse, assume there exists an ℓ-Sylow subgroup $G_\ell \subset G$ which is not bicyclic. Let ℓ^n be its order. Let M be the faithful G-lattice defined by the exact sequence from Example 8.12:

$$0 \to M \to \mathbb{Z}[G \times G] \xrightarrow{\pi} \mathbb{Z}[G] \to \mathbb{Z} \to 0.$$

Then

$$H^2(G,M) \cong \oplus_p \mathbb{Z}/p^{n_p}$$

where $\prod_p p^{n_p}$ is the order of the group G. Let $\beta \in H^2(G,M)$ be the element defined in $\oplus_p \mathbb{Z}/p^{n_p}$ by

$$\beta_p = \begin{cases} \ell^{n-1} & \text{if } p = \ell, \\ 0 & \text{if } p \neq \ell. \end{cases}$$

The same argument as in Proposition 8.11 shows that $\beta \neq 0$ is unramified, hence $\mathrm{Br}_{\mathrm{nr}}\, k(M)^G \neq 0$. □

Theorem 8.15 (Saltman [126]). *Let G be a finite group, let M be a faithful G-lattice and $\alpha \in Z^1(G, \widehat{M})$. Let $k(M)_\alpha$ denote the field of fractions of the twisted G-k-algebra $k[M]$. One then has:*

$$\mathrm{Br}_{\mathrm{nr}}(k(M)^G_\alpha) \cong \ker\Big[H^2(G, k[M]^*_\alpha) \xrightarrow{\mathrm{Res}} \prod_{A \in \mathcal{B}'_G} H^2(A, k^* \oplus M)\Big]$$

where $\mathcal{B}'_G$ denotes the set of bicyclic subgroups A of G such that the map

$$H^2(A, k^*) \to H^2(A, k[M]^*_\alpha)$$

is injective.

Proof. Once more we refer to Saltman's original paper [126] and to Barge's more natural proof [5]. □

Theorem 8.16 (Barge [5]). *For a finite group G, the following conditions are equivalent:*

(i) *For every faithful G-twisted multiplicative action (M, α), we have*

$$\mathrm{Br}_{\mathrm{nr}}\, k(M)^G_\alpha = 0.$$

(ii) *All Sylow subgroups of G are cyclic.*

Example 8.17 (twisted multiplicative example). Let p be a prime and let $G \cong \mathbb{Z}/p \times \mathbb{Z}/p$. There exists a twisted multiplicative action of G on a lattice M such that $\mathrm{Br}_{\mathrm{nr}}\, k(M)^G_\alpha \neq 0$. In particular there exists a twisted multiplicative action of $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ on a lattice M such that the field of invariants $k(M)^G_\alpha$ is not pure.

Remark 8.18. Given a G-lattice M one may show that the field $k(M)^G$ is stably equivalent to the function field of a torus defined on the field $k(\mathbb{Z}[G])^G$.

For some computations on multiplicative invariants, see [7–10, 50, 51, 53, 125]. For a recent general report on multiplicative actions, we refer to the forthcoming book by M. Lorenz [86].

9. Homogeneous spaces

In this section, the ground field k is the field $\mathbb{C}$ of complex numbers. We fix an isomorphism between the group of all roots of unity in $\mathbb{C}$ and $\mathbb{Q}/\mathbb{Z}$. The tools used are topological. However standard arguments show that once the statements of Theorems 9.1 and 9.13 have been proved over $\mathbb{C}$, they hold over any algebraically closed field Ω of characteristic zero: reduction to an algebraically closed field k which may be embedded both in $\mathbb{C}$ and Ω, then invariance of étale cohomology with finite coefficients under extension of algebraically closed ground fields.

9.1. **The case of an almost free linear representation.**

Theorem 9.1 (Bogomolov [17, Lemma 5.7]). *Let G be a connected algebraic group over the complex field $k = \mathbb{C}$, and let V be an almost free finite dimensional linear representation of G. Then* $\mathrm{Br}_{\mathrm{nr}}\, k(V)^G = 0$.

For $G = \mathrm{PGL}_n$, this is a theorem of Saltman [121, Theorem 2.9], for which alternate proofs are given in [37, Theorem 9.7] and [127]. This answered a question of Procesi [110].

Lemma 9.2. *For any linear algebraic group G over a field k and any positive integer s, there exists a k-linear representation V of G and a non-empty G-stable open set $U \subset V$ such that*

(i) *the complement $V \setminus U$ is of codimension $\geq s$,*
(ii) *there is a morphism $U \to U/G$ of k-varieties making U into a G-torsor over U/G.*

Proof. See Totaro [149, Remark 1.4, p. 252]. Let $G \subset \mathrm{GL}_n$ be any faithful k-linear representation of G. Let $N \geq 1$ be an integer. Let $W = \mathcal{M}_{N+n}$ be the vector space of $(N+n) \times (N+n)$ matrices M over k:

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\begin{array}{l} \updownarrow N \\ \updownarrow n \end{array}$$

Let $V = \mathcal{M}_{n,N+n}$ be the vector space of $n \times (N+n)$ matrices over k and let $\pi\colon W \to V$ be the linear projection given by

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \longmapsto \left(\begin{array}{c|c} C & D \end{array}\right).$$

Let $\Omega = \mathrm{GL}_{N+n} \subset W$ be the group of invertible matrices. We denote by $\widetilde{G} \cong G$ the subgroup of matrices of the form

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & D \end{array}\right)$$

where $D \in G$ and $I = I_N$ is the $N \times N$ identity matrix. We denote by $H \lhd \mathrm{GL}_{N+n}$ the invariant subgroup consisting of matrices of the form

$$\left(\begin{array}{c|c} A & B \\ \hline 0 & I \end{array}\right)$$

where $A \in \mathrm{GL}_N$ and $I = I_n$ is the $n \times n$ identity matrix. The group H is clearly an extension of GL_N by a unipotent invariant subgroup whose underlying k-variety is an affine space $\mathbb{A}^{Nn}$. The product $\Gamma = H\widetilde{G} \cong H \rtimes G$ is a subgroup of GL_{N+n}. Let $U \subset V$ denote the open dense subset consisting of matrices of rank n. Let Z be its complement. The projection $\pi\colon W \to V$ induces a surjective morphism of G-k-varieties

$$\pi_\Omega\colon \Omega \to U.$$

This map induces a G-isomorphism of k-varieties $\mathrm{GL}_{N+n}/H \cong U$. Since $\Gamma = H \rtimes G$ is a closed subgroup of GL_{N+n}, the canonical morphism

$$\mathrm{GL}_{N+n} \to \mathrm{GL}_{N+n}/\Gamma$$

factorizes through π_Ω and induces a morphism ϖ of k-varieties which makes GL_{N+n}/H into a G-torsor over GL_{N+n}/Γ. Thus U/G exists and U is a G-torsor over U/G:

$$\begin{tikzcd} \mathrm{GL}_{N+n}/H \arrow[r, "\varpi"] \arrow[d, "\cong"', "\bar\pi_\Omega"] & \mathrm{GL}_{N+n}/\Gamma \arrow[d, "\cong"'] \\ U \arrow[r] & U/G. \end{tikzcd}$$

Since $\operatorname{codim}_V Z$ goes to infinity as N goes to infinity, we obtain $\operatorname{codim}_V Z \geq s$ for N large. □

Proof of Theorem 9.1. In view of the no-name lemma and Proposition 5.7, we may replace the linear representation V by a "better one". Namely, we take V as in Lemma 9.2 for $s = 3$. Thus we may now assume that we are given a linear representation V of G and a G-stable open set U with complement Z such that $\operatorname{codim}_V Z \geq 3$ and such that there is a morphism $U \to U/G$ which makes U into a G-torsor over U/G.

The connected group G is an extension

$$1 \to G' \to G \to T \to 1$$

of a torus T by a connected group G' without characters: $X(G') = 0$. Now we may first quotient U by the action of G', and then by the action of T, getting a morphism $U/G' \to U/G$ which one checks makes U/G' into a T-torsor over U/G. Since torsors under tori are locally trivial, U/G' is birational to $T \times U/G$. The stability of the unramified Brauer group under pure extensions (Proposition 5.7) now implies $\mathrm{Br}_{\mathrm{nr}}(k(V)^G) \cong \mathrm{Br}_{\mathrm{nr}}(k(V)^{G'})$. It is thus enough to prove the theorem when G is connected and satisfies $X(G) = 0$, which we now assume.

Claim. For such G we now claim that

$$\mathrm{Br}_{\mathrm{nr}}\, k(V)^G \cong \ker\Big[H^2(BG, \mathbb{Q}/\mathbb{Z}) \to \prod_{A\in\mathcal{B}_G} H^2(BA, \mathbb{Q}/\mathbb{Z})\Big]. \tag{9.1}$$

Here BG, resp. BA, denotes the classifying space of the topological group $G = G(\mathbb{C})$, resp. of the finite group A. Recall that $\mathcal{B}_G$ denotes the set of finite bicyclic subgroups of $G(\mathbb{C})$.

Theorem 6.4 implies

$$\mathrm{Br}_{\mathrm{nr}}(k(V)^G) = \{\alpha \in \mathrm{Br}\, k(V)^G \mid \alpha_A \in \mathrm{Br}_{\mathrm{nr}}(k(V)^A) \text{ for all } A \in \mathcal{B}_G\}.$$

Since A is abelian and the action on V is linear, $k(V)^A$ is pure by Fischer's Theorem (Proposition 4.3), hence $\mathrm{Br}_{\mathrm{nr}}\, k(V)^A = 0$ (Proposition 5.7). The field $k(V)^G$, resp. $k(V)^A$, is the function field of the smooth variety U/G, resp. U/A. Using Theorem 5.11, we get:

$$\mathrm{Br}_{\mathrm{nr}}\, k(V)^G \cong \ker\Big[\mathrm{Br}(U/G) \to \prod_{A\in\mathcal{B}_G} \mathrm{Br}(U/A)\Big]. \tag{9.2}$$

Lemma 9.3. *Let X be a smooth algebraic variety over $\mathbb{C}$. If $\operatorname{Pic} X$ is torsion, then we have a canonical isomorphism*

$$H^2_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Br}\, X.$$

Proof. This is a consequence of "Kummer's exact sequence":

$$0 \to \operatorname{Pic} X \otimes \mathbb{Q}/\mathbb{Z} \to H^2_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Br}\, X \to 0. \qquad \square$$

Lemma 9.4. *Let X be a smooth connected variety over $k = \mathbb{C}$. Let G be a linear algebraic group with character group $\mathsf{X}(G) = \mathrm{Hom}(G, \mathbb{G}_m)$. Let $X \to X/G$ be a G-torsor.*

(i) *If G is connected, there is a natural exact sequence*

$$1 \to k[X/G]^* \to k[X]^* \to \mathsf{X}(G) \to \operatorname{Pic}(X/G) \to \operatorname{Pic} X.$$

(ii) *If G is a finite constant group A, there is a natural exact sequence*

$$1 \to \mathsf{X}(A) \to \operatorname{Pic}(X/A) \to \operatorname{Pic} X.$$

Proof. For (i), see [134, Proposition 6.10]. For (ii), the same arguments yield an exact sequence

$$1 \to \mathrm{Hom}(A, k[X]^*) \to \operatorname{Pic}(X/A) \to \operatorname{Pic} X.$$

Then use the fact that $k[X]^*/k^*$ is torsionfree. $\square$

Lemma 9.5. *Let X be a smooth connected variety over $\mathbb{C}$. Let G be a linear algebraic group. Let $X \to X/G$ be a G-torsor. Assume that $\operatorname{Pic} X$ is a torsion group and that G is either finite, or connected and characterfree. Then we have a canonical isomorphism*

$$\mathrm{Br}(X/G) \cong H^2_{\text{ét}}(X/G, \mathbb{Q}/\mathbb{Z}).$$

Proof. Under our assumptions, Lemma 9.4 implies that $\operatorname{Pic}(X/G)$ is torsion. Then Lemma 9.3 gives the isomorphism $\mathrm{Br}(U/G) \cong H^2(U/G, \mathbb{Q}/\mathbb{Z})$. $\square$

Proof of Theorem 9.1 (continued). Lemma 9.5 applied to $X = U$ together with (9.2) implies

$$\mathrm{Br}_{\mathrm{nr}}\, k(V)^G \cong \ker\Big[H^2(U/G, \mathbb{Q}/\mathbb{Z}) \to \prod_{A\in\mathcal{B}_G} H^2(U/A, \mathbb{Q}/\mathbb{Z})\Big].$$

Since $U \to U/G$ is a G-torsor, there is a map of topological spaces $U/G \to BG$ such that $U \to U/G$ is the pull-back of the universal covering space $EG \to BG$. Similarly, there is a map $U/A \to BA$. Now the assumption on the codimension of the complement of U in V, which had not been used yet, implies that the maps $U/G \to BG$ and $U/A \to BA$ induce isomorphisms on the cohomology groups $H^i_{\text{ét}}(\,\cdot\,,\mathbb{Q}/\mathbb{Z})$ for $i \leq 2$ (the projections $U \to U/G$ and $U \to U/A$ are algebraic approximations of the universal covering spaces). The "claim" (9.1) now follows.

Lemma 9.6. *If $\widetilde{G}$ is a connected, simply connected group, then*

$$H^1(B\widetilde{G}, \mathbb{Q}/\mathbb{Z}) = H^2(B\widetilde{G}, \mathbb{Q}/\mathbb{Z}) = 0.$$

Proof. The fibration $E\widetilde{G} \to B\widetilde{G}$ yields isomorphisms $\pi_1 B\widetilde{G} = \pi_0\widetilde{G} = 0$ (since $\widetilde{G}$ is connected) and $\pi_2 B\widetilde{G} = \pi_1\widetilde{G} = 0$ (since $\widetilde{G}$ is simply connected). Thus $B\widetilde{G}$ is 2-connected. The Hurewicz theorem now yields $H_1(B\widetilde{G}) = H_2(B\widetilde{G}) = 0$. The universal coefficient theorem now yields the statement of the lemma. □

Lemma 9.7. *If G is a connected group without characters, the universal covering $\widetilde{G} \to G$ defines a natural isomorphism of finite abelian groups*

$$d_{2,G}\colon \operatorname{Hom}(\pi_1 G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} H^2(BG, \mathbb{Q}/\mathbb{Z}).$$

Proof. Since G is a connected group without characters, $G = R_{\mathrm{u}}(G) \rtimes G_{\mathrm{ss}}$ with $R_{\mathrm{u}}(G)$ unipotent and G_{ss} semisimple. Thus its fundamental group $\pi_1 G$ is a finite abelian group. We have the Leray-Serre spectral sequence [91, §5, Theorem 5.2] for the fibration $\pi_1 G \to B\widetilde{G} \to BG$:

$$E_2^{p,q} = H^p(BG, \mathcal{H}^q(\pi_1 G, \mathbb{Q}/\mathbb{Z})) \implies H^{p+q}(B\widetilde{G}, \mathbb{Q}/\mathbb{Z}).$$

Since BG is 1-connected, each local system $\mathcal{H}^q(\pi_1 G, \mathbb{Q}/\mathbb{Z})$ is constant. Then the associated exact sequence of terms of lower degree

$$H^1(B\widetilde{G}, \mathbb{Q}/\mathbb{Z}) \to H^1(\pi_1 G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{d_{2,G}} H^2(BG, \mathbb{Q}/\mathbb{Z}) \to H^2(B\widetilde{G}, \mathbb{Q}/\mathbb{Z})$$

together with Lemma 9.6 yields a natural isomorphism of finite abelian groups $d_{2,G}\colon \operatorname{Hom}(\pi_1 G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} H^2(BG, \mathbb{Q}/\mathbb{Z})$. □

Lemma 9.8. *Let A be a finite group and C be a trivial A-module. Let*

$$1 \to \pi \to \widetilde{A} \to A \to 1 \qquad (\mathcal{E}_A)$$

be a central extension. The associated Lyndon-Hochschild-Serre spectral sequence defines a natural morphism

$$d_{2,\mathcal{E}_A}\colon \operatorname{Hom}(\pi, C) \to H^2(A, C)$$

sending $\phi\colon \pi \to C$ to the class of the extension $\phi_*(\mathcal{E}_A)$ obtained from $(\mathcal{E}_A)$ by push-out along ϕ.

Proof. The Leray-Serre spectral sequence [91, §5, Theorem 5.2] for the fibration $\pi \to B\widetilde{A} \to BA$ associated to $(\mathcal{E}_A)$:

$$E_2^{p,q} = H^p(BA, \mathcal{H}^q(\pi, C)) \implies H^{p+q}(B\widetilde{A}, C)$$

coincides with the Lyndon-Hochschild-Serre spectral sequence (see [91, §8^{bis}.2, p. 342])

$$E_2^{p,q} = H^p(A, H^q(\pi, C)) \implies H^{p+q}(\widetilde{A}, C)$$

associated to $(\mathcal{E}_A)$. This extension being central, the action of A on $H^q(\pi, C)$ is trivial and the local system $\mathcal{H}^q(\pi, C)$ is constant on BA. Hence both morphisms $H^0(BA, \mathcal{H}^1(\pi, C)) \to H^2(BA, \mathcal{H}^0(\pi, C))$ and $H^0(A, H^1(\pi, C)) \to H^2(A, H^0(\pi, C))$ coincide, yielding the natural morphism

$$d_{2,\mathcal{E}_A} \colon \operatorname{Hom}(\pi, C) \to H^2(A, C).$$

It is well known (see [26, §IV.3, Theorem 3.12]) that $H^2(A, C)$ classifies abstract central extensions of A by C. Let $C = \pi$. One can check that $d_{2,\mathcal{E}_A}(\mathrm{id}_\pi)$ is the class of the central extension $\mathcal{E}_A$. Now for any C, the functoriality of the LHS spectral sequence implies that $d_{2,\mathcal{E}_A}(\phi)$ is the class of $\phi_*(\mathcal{E}_A)$. $\square$

Proof of Theorem 9.1 (continued). Let $R_{\mathrm{u}}(G)$ be the unipotent radical of G. We have the exact sequence

$$1 \to R_{\mathrm{u}}(G) \to G \xrightarrow{\varrho} G_1 \to 1, \tag{9.3}$$

where G_1 is semisimple. Pulling back the universal cover $\widetilde{G}_1 \to G_1$ over G, we get the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R_{\mathrm{u}}(G) & \longrightarrow & \widetilde{G} & \longrightarrow & \widetilde{G}_1 & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & R_{\mathrm{u}}(G) & \longrightarrow & G & \longrightarrow & G_1 & \longrightarrow & 1,
\end{array}$$

where $\widetilde{G} \to G$ is the universal cover of G. From Lemma 9.7 we get the commutative diagram

$$\begin{array}{ccc}
\operatorname{Hom}(\pi_1 G, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\cong} & H^2(BG, \mathbb{Q}/\mathbb{Z}) \\
\uparrow & & \uparrow \\
\operatorname{Hom}(\pi_1 G_1, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\cong} & H^2(BG_1, \mathbb{Q}/\mathbb{Z}).
\end{array}$$

Since $R_{\mathrm{u}}(G)$ is contractible, the map $\pi_1 G \to \pi_1 G_1$ induced by (9.3) is an isomorphism. Hence the map $H^2(BG_1, \mathbb{Q}/\mathbb{Z}) \to H^2(BG, \mathbb{Q}/\mathbb{Z})$ is an isomorphism. For each finite subgroup $A \subset G$, the exact sequence (9.3) induces an isomorphism $A \cong \varrho(A)$. Moreover it induces a bijection $A \mapsto \varrho(A)$ between the finite subgroups of G and those of G_1. We conclude that

$$\ker\Big[H^2(BG, \mathbb{Q}/\mathbb{Z}) \to \prod_{A \in \mathcal{B}_G} H^2(BA, \mathbb{Q}/\mathbb{Z})\Big]$$

and

$$\ker\Big[H^2(BG_1, \mathbb{Q}/\mathbb{Z}) \to \prod_{A \in \mathcal{B}_{G_1}} H^2(BA, \mathbb{Q}/\mathbb{Z})\Big]$$

are isomorphic. To prove that the first kernel is trivial we may therefore assume that G is a semisimple group.

Consider now the natural isogeny $\widetilde{G} \to G$ where $\widetilde{G}$ denotes the simply connected cover of G. Each $A \in \mathcal{B}_G$ gives rise to a commutative diagram of exact sequences of groups

$$\begin{tikzcd} 1 \arrow[r] & \pi_1 G \arrow[r] & \widetilde{G} \arrow[r] & G \arrow[r] & 1 \\ 1 \arrow[r] & \pi_1 G \arrow[r] \arrow[u, equal] & \widetilde{A} \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 1. \end{tikzcd}$$

and to a commutative square of fibrations

$$\begin{tikzcd} B\widetilde{G} \arrow[r] & BG \\ B\widetilde{A} \arrow[r] \arrow[u] & BA. \arrow[u] \end{tikzcd} \tag{9.4}$$

Then (9.4) gives a morphism of Leray-Serre spectral sequences:

$$\begin{array}{ccc} E_2^{p,q} = H^p(BG, \mathcal{H}^q(\pi_1 G, \mathbb{Q}/\mathbb{Z})) & \Longrightarrow & H^{p+q}(B\widetilde{G}, \mathbb{Q}/\mathbb{Z}) \\ \downarrow & & \downarrow \\ E_2^{p,q} = H^p(BA, \mathcal{H}^q(\pi_1 G, \mathbb{Q}/\mathbb{Z})) & \Longrightarrow & H^{p+q}(B\widetilde{A}, \mathbb{Q}/\mathbb{Z}), \end{array}$$

hence $d_2 \colon E_2^{0,1} \to E_2^{2,0}$ defines a commutative square (see Lemmas 9.7 and 9.8)

$$\begin{tikzcd} \operatorname{Hom}(\pi_1 G, \mathbb{Q}/\mathbb{Z}) \arrow[r, "d_{2,G}"] \arrow[d, equal] & H^2(BG, \mathbb{Q}/\mathbb{Z}) \arrow[d] \\ \operatorname{Hom}(\pi_1 G, \mathbb{Q}/\mathbb{Z}) \arrow[r, "d_{2,\mathcal{E}_A}"] & H^2(BA, \mathbb{Q}/\mathbb{Z}). \end{tikzcd} \tag{9.5}$$

Any element $\alpha \in H^2(BG, \mathbb{Q}/\mathbb{Z})$ may be interpreted as a homomorphism ϕ of $\pi_1 G = \pi_1^{\mathrm{alg}} G$ to $\mathbb{Q}/\mathbb{Z}$, hence to some $\mathbb{Z}/n = \operatorname{Im} \phi$. Hence by push-out from the natural extension

$$1 \to \pi_1 G \to \widetilde{G} \to G \to 1$$

it defines an extension

$$1 \to \mathbb{Z}/n \to G_1 \to G \to 1,$$

where G_1 is a connected semisimple algebraic group over $\mathbb{C}$. This extension itself defines an extension of G by $\mathbb{Q}/\mathbb{Z}$. By Lemma 9.8, the natural restriction map $r \colon H^2(BG, \mathbb{Q}/\mathbb{Z}) \to H^2(BA, \mathbb{Q}/\mathbb{Z})$ may be interpreted in terms of restrictions of extensions. More precisely, if $\alpha \in H^2(BG, \mathbb{Q}/\mathbb{Z})$ corresponds to the isogeny

$$1 \to \mathbb{Z}/n \to G_1 \to G \to 1$$

(notations as above), and if $A \subset G$ is a finite subgroup, then the restriction of the above isogeny to this subgroup gives rise to a central extension

$$1 \to \mathbb{Z}/n \to A_1 \to A \to 1$$

whose class in $H^2(A, \mathbb{Z}/n)$ restricts to the class of $r(\alpha) \in H^2(A, \mathbb{Q}/\mathbb{Z}) \cong H^2(BA, \mathbb{Q}/\mathbb{Z})$.

To prove that

$$\ker\Big[H^2(BG, \mathbb{Q}/\mathbb{Z}) \to \prod_{A \in \mathcal{B}_G} H^2(BA, \mathbb{Q}/\mathbb{Z})\Big] = 0,$$

which will complete the proof of the theorem, all we now need to prove is that if we are given a nontrivial central extension

$$1 \to \mathbb{Z}/n \to G_1 \to G \to 1 \tag{9.6}$$

with G_1 connected, then there exists a finite bicyclic subgroup A in the semisimple group G such that the restriction

$$1 \to \mathbb{Z}/n \to A_1 \to A \to 1 \tag{9.7}$$

of (9.6) to A is a nontrivial extension. Since (9.7) is a central extension, it is nontrivial if A_1 is not commutative.

Let us postpone the proof of the:

Proposition 9.9. *Any element in the centre of a connected semisimple group is a commutator of elements of finite order.*

Proof of Theorem 9.1 (end). Let c be a generator of the subgroup $\mathbb{Z}/n$ of the centre of the connected semisimple group G_1. Using Proposition 9.9 we can write c as a commutator $a_1 b_1 a_1^{-1} b_1^{-1}$, with a_1 and b_1 of finite order in G_1, let a and b be the images of a_1 and b_1 in G, and let $A \subset G$ be the finite, abelian, bicyclic group which they generate. Since a_1 and b_1 do not commute, the group A_1 is not commutative and the proof of Theorem 9.1 is complete. □

Proof of Proposition 9.9. It is enough to give the proof when the connected semisimple group G is simply connected, and then it is enough to prove it when G is simple and simply connected.

First proof (suggested by O. Gabber). To each choice of a basis of roots in the character group $X(T)$ of a maximal torus $T \subset G$ one may associate an element c of the Weyl group W of G, known as the Coxeter element [25, V.6.2 and VI.1.11].

It is known (loc. cit.) that 1 is not an eigenvalue of c for its action on $\mathfrak{t} = X(T) \otimes \mathbb{C}$. Let d in the normalizer $N(T)$ of T be a representant of c. Such a representant may be chosen of finite order. Conjugacy by d on T induces an automorphism whose tangent linear map is $c \in \operatorname{Aut} \mathfrak{t}$. Now the tangent map to the homomorphism $\lambda \colon T \to T$ given by $\lambda(t) = dtd^{-1}t^{-1}$ is $c - \mathrm{id} \in \operatorname{End} \mathfrak{t}$, hence is invertible. Thus λ is an isogeny, and this isogeny induces a surjection on torsion points $\lambda \colon T_{\mathrm{tors}} \to T_{\mathrm{tors}}$. We conclude that any element $x \in T_{\mathrm{tors}}$ may be written as $x = dyd^{-1}y^{-1}$ in G, with $y \in T_{\mathrm{tors}}$. Since the centre of G is contained in any maximal torus of G, the conclusion follows. □

Second proof (suggested by J-P. Serre). One first proves the result for $G = \mathrm{SL}_n$ (a slick proof being as follows: the algebra generated by a and b with the relations $a^n = b^n = 1$ and $ab = \zeta ba$ with $\zeta^n = 1$ is none other than $\mathcal{M}_n(\mathbb{C})$). Now inspection of the root systems reveals that any simply connected semisimple group G contains a subgroup $H \cong \mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_r}$ with $\operatorname{rank} H = \operatorname{rank} G$, hence with $\operatorname{centre}(G) \subset \operatorname{centre}(H)$. □

9.2. **The case of a homogeneous space.** For Theorem 9.13, we need some preparation. Let us first recall a theorem of Steinberg (cf. [144, II, 3.9, p. 197]).

Theorem 9.10 (Steinberg)**.** *If G is a semisimple simply connected group, the centralizer of a semisimple element of G is a connected reductive group.*

Remark. This is no longer true if G is not simply connected.

Corollary 9.11. *Given two commuting semisimple elements in a semisimple, simply connected group G, there is a maximal torus of G which contains them both.*

Proof. Let x, y be two such elements. Let $H = Z_G(x)$ be the centralizer of x. Then H is a connected reductive group (Theorem 9.10), and x is in its center. Hence x belongs to any maximal torus of H. The element y belongs to H. Since it is semisimple, it belongs to a maximal torus T of H. There exists a maximal torus of G which contains T. Such a torus contains both x and y. □

Proposition 9.12. *Let A be a bicyclic finite subgroup of a simply connected group G. Then G/A is a rational variety, and the unramified Brauer group of G/A is trivial.*

Proof. We have the natural exact sequence

$$1 \to U \to G \xrightarrow{\pi} G_{\mathrm{ss}} \to 1,$$

where U is the unipotent radical of G and G_{ss} is a connected, semisimple, simply connected group. Since $U \cap A = \{1\}$, the map $A \to \pi(A)$ is an isomorphism and the natural projection $G/A \to G_{\mathrm{ss}}/A$ defines a U-torsor. Hence the varieties G/A and $G_{\mathrm{ss}}/A \times U$ are isomorphic. To prove the proposition we may therefore assume that G is semisimple and simply connected.

Let $A \subset G$ be a finite bicyclic group. By the previous corollary, there is a maximal torus T which contains A. Let $T' = T/A$. The map $G \to G/T$ makes G into a T-torsor over G/T, hence G is birational to $T \times G/T$. On the other hand, the map $G/A \to G/T$ makes G/A into a T'-torsor over G/T, hence G/A is birational to $T' \times G/T$. Since T and T' are birational to each other, we conclude that G/A is birational to G, hence is a rational variety. □

The proof of the following Theorem uses Theorem 9.1. In turn, Proposition 4.9 shows that Theorem 9.13 generalizes Theorem 9.1.

Theorem 9.13 (Bogomolov [18, Theorem 2.4])**.** *Let G be a connected, simply connected group over $\mathbb{C}$, and let $H \subset G$ be a connected closed subgroup. Then the unramified Brauer group of G/H vanishes.*

Proof. There is a closed normal subgroup $H_1 \subset H$ which is connected and characterfree, such that H/H_1 is a torus T. Thus the map $G/H_1 \to G/H$ makes the first variety into a T-torsor over the second one. Since torsors under tori are locally trivial, G/H_1 is birational to $T \times G/H$. By Proposition 5.7 it is thus enough to prove the theorem when H is connected and characterfree, which we now assume.

Just as in the proof of Theorem 9.1, we have an exact sequence:

$$0 \to \operatorname{Pic}(G/H) \otimes \mathbb{Q}/\mathbb{Z} \to H^2_{\text{ét}}(G/H, \mathbb{Q}/\mathbb{Z}) \to \operatorname{Br}(G/H) \to 0$$

and similar exact sequences for any finite subgroup $A \subset H$:

$$0 \to \operatorname{Pic}(G/A) \otimes \mathbb{Q}/\mathbb{Z} \to H^2_{\text{ét}}(G/A, \mathbb{Q}/\mathbb{Z}) \to \operatorname{Br}(G/A) \to 0.$$

We also have exact sequences (Lemma 9.4)

$$X(H) \to \operatorname{Pic}(G/H) \to \operatorname{Pic} G$$

and

$$X(A) \to \operatorname{Pic}(G/A) \to \operatorname{Pic} G.$$

Now $\operatorname{Pic} G = 0$ because G is connected and simply connected, $X(H) = 0$, and $X(A)$ is finite since A is finite. We thus get a commutative diagram:

$$\begin{array}{ccc} H^2_{\text{ét}}(G/A, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\cong} & \operatorname{Br}(G/A) \\ \uparrow & & \uparrow \\ H^2_{\text{ét}}(G/H, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\cong} & \operatorname{Br}(G/H). \end{array} \tag{9.8}$$

Proposition 9.12 together with the general formula for the unramified Brauer group (Theorem 6.4) and the above diagram imply that

$$\operatorname{Br}_{\text{nr}} G/H \cong \ker\Big[H^2(G/H, \mathbb{Q}/\mathbb{Z}) \to \prod_{A \in \mathcal{B}_H} H^2(G/A, \mathbb{Q}/\mathbb{Z})\Big]. \tag{9.9}$$

From the universal property of the topological H-fibration $EH \to BH$ there is a cartesian diagram of topological morphisms

$$\begin{array}{ccc} G & \longrightarrow & EH \\ \downarrow & & \downarrow \\ G/H & \longrightarrow & BH. \end{array}$$

This induces a map of spectral sequences

$$\begin{array}{ccc} E_2^{p,q} = H^p(G/H, \mathcal{H}^q(H, \mathbb{Q}/\mathbb{Z})) & \Longrightarrow & H^{p+q}(G, \mathbb{Q}/\mathbb{Z}) \\ \uparrow & & \uparrow \\ E_2^{p,q} = H^p(BH, \mathcal{H}^q(H, \mathbb{Q}/\mathbb{Z})) & \Longrightarrow & H^{p+q}(EH, \mathbb{Q}/\mathbb{Z}), \end{array}$$

hence a commutative diagram:

$$\begin{array}{ccc} H^0(G/H, \mathcal{H}^1(H, \mathbb{Q}/\mathbb{Z})) & \xrightarrow{\partial} & H^2(G/H, \mathcal{H}^0(H, \mathbb{Q}/\mathbb{Z})) \\ \uparrow & & \uparrow \\ H^0(BH, \mathcal{H}^1(H, \mathbb{Q}/\mathbb{Z})) & \xrightarrow{\partial} & H^2(BH, \mathcal{H}^0(H, \mathbb{Q}/\mathbb{Z})). \end{array}$$

Since G is simply connected, $\pi_1 G = 0$. It is also known (E. Cartan's theorem) that $\pi_2 G = 0$. Thus $H^1(G, \mathbb{Q}/\mathbb{Z}) = 0$ and the universal coefficient theorem also gives $H^2(G, \mathbb{Q}/\mathbb{Z}) = 0$. We clearly have the same vanishing properties for EH, which is contractible. This implies that the maps ∂ are isomorphisms.

Both G/H and BH are connected and simply connected, as follows from the long sequence of homotopy groups deduced from the two fibrations (H

is connected), hence each local system $\mathcal{H}^q(H,\mathbb{Q}/\mathbb{Z})$ is constant and equal to $H^q(H,\mathbb{Q}/\mathbb{Z})$. Since BH and G/H are connected, the L.H.S. vertical map is identity on $H^1(H,\mathbb{Q}/\mathbb{Z})$, hence the R.H.S. vertical map is an isomorphism. But this map is the natural map $H^2(BH,\mathbb{Q}/\mathbb{Z}) \to H^2(G/H,\mathbb{Q}/\mathbb{Z})$.

Remark. A shorter proof runs as follows. We may regard the map $G/H \to BH$ as a topological fibration with fibre G. Indeed the map $G/H \times EG \to (EG)/H$ makes $G/H \times EG$ into a principal G-bundle over $(EG)/H$. The total space $G/H \times EG$ has the same homotopy type as G/H and the base $(EG)/H$ has the same homotopy type as BH. Associated to this fibration there is a spectral sequence

$$E_2^{p,q} = H^p(BH, H^q(G,\mathbb{Q}/\mathbb{Z})) \implies H^{p+q}(G/H,\mathbb{Q}/\mathbb{Z}).$$

and this spectral sequence yields an isomorphism

$$H^2(BH,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} H^2(G/H,\mathbb{Q}/\mathbb{Z})$$

which is induced by the map $G/H \to BH$.

From the universal property of the topological A-covering $EA \to BA$ there is a cartesian diagram of topological morphisms

$$\begin{array}{ccc} G & \longrightarrow & EA \\ \downarrow & & \downarrow \\ G/A & \longrightarrow & BA. \end{array}$$

This induces a map of spectral sequences

$$\begin{array}{ccc} E_2^{p,q} = H^p(A, H^q(G,\mathbb{Q}/\mathbb{Z})) & \implies & H^{p+q}(G/A,\mathbb{Q}/\mathbb{Z}) \\ \uparrow & & \uparrow \\ E_2^{p,q} = H^p(A, H^q(EA,\mathbb{Q}/\mathbb{Z})) & \implies & H^{p+q}(BA,\mathbb{Q}/\mathbb{Z}). \end{array}$$

Since $H^1(G,\mathbb{Q}/\mathbb{Z})$ and $H^2(G,\mathbb{Q}/\mathbb{Z})$ are zero, and also all $H^q(EA,\mathbb{Q}/\mathbb{Z})$ for $q > 0$, we get the commutative diagram:

$$\begin{array}{ccc} H^2(A, H^0(G,\mathbb{Q}/\mathbb{Z})) & \xrightarrow{\cong} & H^2(G/A,\mathbb{Q}/\mathbb{Z})) \\ \uparrow & & \uparrow \\ H^2(A, H^0(EA,\mathbb{Q}/\mathbb{Z})) & \xrightarrow{\cong} & H^2(BA,\mathbb{Q}/\mathbb{Z})). \end{array}$$

Since G and EA are connected, the L.H.S. vertical map is identity on the group $H^2(A,\mathbb{Q}/\mathbb{Z})$, hence the R.H.S. vertical map is an isomorphism:

$$H^2(BA,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} H^2(G/A,\mathbb{Q}/\mathbb{Z}).$$

Using the diagram

$$\begin{array}{ccc} G/A & \longrightarrow & BA \\ \downarrow & & \downarrow \\ G/H & \longrightarrow & BH \end{array}$$

we get the commutative diagram

$$\begin{array}{ccc} H^2(G/H,\mathbb{Q}/\mathbb{Z}) & \longrightarrow & H^2(G/A,\mathbb{Q}/\mathbb{Z})) \\ \uparrow & & \uparrow \\ H^2(BH,\mathbb{Q}/\mathbb{Z}) & \longrightarrow & H^2(BA,\mathbb{Q}/\mathbb{Z})). \end{array}$$

We have proved that both vertical maps are isomorphisms. Formula (9.9) now yields:

$$\operatorname{Br}_{\mathrm{nr}} G/H \cong \ker\Big[H^2(BH,\mathbb{Q}/\mathbb{Z}) \to \prod_{A\in\mathcal{B}_H} H^2(BA,\mathbb{Q}/\mathbb{Z})\Big],$$

and we saw in the proof of Theorem 9.1 that this kernel is zero. □

Remark 9.14. It would be interesting to know whether Theorem 9.13 extends to quotients G/H with G and H connected.

Note that if G is connected, semisimple but not simply connected, and if $\widetilde{G}$ is its simply connected cover, the inverse image of H in $\widetilde{G}$ need not be connected, so that one may not reduce to the situation of the theorem. For example, if we consider in $\widetilde{G} = \mathrm{SL}(3)$ the subgroup H_1 which is the product of $\{1\} \times \mathrm{SL}(2)$ by the diagonal μ_3, the image H of that group in the projective special linear group PSL(3) is clearly connected whereas H_1 is not connected.

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