Local-global principle for zero-cycles of degree one and integral Tate conjecture for 1-cycles

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Let k be a global field, and Ω the set of its places.
Work of Cassels and Tate on curves and of CT, Sansuc, Swinnerton-Dyer, Salberger, ... on some special higher dimensional varieties has led to two general conjectures.

Conjecture 1 *Let X be a smooth, projective, geometrically integral variety X over k. Let $\ell > 0$ be a prime number. Let $\{z_v\}_{v\in\Omega}$ be a family of local zero-cycles on X. If for all $A \in \mathrm{Br}(X)[\ell^\infty]$,*

$$\sum_{v\in\Omega} \mathrm{inv}_v(A(z_v)) = 0 \in \mathbf{Q}/\mathbf{Z}$$

holds, then for any positive integer n there exists a global zero-cycle z_n on X such that for each place v and each A in $\mathrm{Br}(X)[\ell^n]$

$$A(z_v) = A(z_n) \in \mathrm{Br}(k_v).$$

Conjecture 2 *Let X be a smooth, projective, geometrically integral variety X over a global field k. Let $\ell > 0$ be a prime number. Let $\{z_v\}_{v\in\Omega}$ be a family of local zero-cycles of degree 1 on X. If for all $A \in \mathrm{Br}(X)[\ell^\infty]$,*

$$\sum_{v\in\Omega} \mathrm{inv}_v(A(z_v)) = 0 \in \mathbf{Q}/\mathbf{Z}$$

holds, then there exists a global zero-cycle of degree prime to ℓ on X.

The existence of a family of z_v's orthogonal to the whole group $\mathrm{Br}(X)$ is often referred to as : "There is no Brauer-Manin obstruction to the existence of a zero-cycle of degree 1 on X."

In this talk I want to discuss the case where the global field k is the function field $\mathbf{F}(C)$ of a curve C over a finite field $\mathbf{F}$.

Proposition (S. Saito 1989, CT 1999) *Let* $\mathbf{F}$ *be a finite field, let* $C/\mathbf{F}$ *be a smooth, projective, geometrically connected curve over* $\mathbf{F}$*. Let* $\mathbf{F}(C)$ *be its function field. Let* X *be a smooth, projective, geometrically integral variety over* $\mathbf{F}$ *of dimension* $d+1$*, equipped with a flat morphism* $X \to C$ *whose generic fibre* $X_\eta/\mathbf{F}(C)$ *is smooth and geometrically integral. Let* ℓ *be a prime number,* $\ell \neq \mathrm{char}(\mathbf{F})$*.*
a) If the étale cycle map $\mathrm{CH}^d(X) \otimes \mathbf{Z}_\ell \to H^{2d}(X, \mathbf{Z}_\ell(d))$ *is onto, then Conjecture 1 holds for* $X_\eta/\mathbf{F}(C)$*.*
b) If the étale cycle map $\mathrm{CH}^d(X) \otimes \mathbf{Z}_\ell \to H^{2d}(X, \mathbf{Z}_\ell(d))$ *is onto modulo torsion, then Conjecture 2 holds for* $X_\eta/\mathbf{F}(C)$*.*

Let $\mathbf{F}$ be a finite field, $\overline{\mathbf{F}}$ an algebraic closure of $\mathbf{F}$, $G = \mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$. Let X be a smooth, projective, geometrically integral variety over $\mathbf{F}$ of dimension d. Let ℓ be a prime, $\ell \neq \mathrm{char}(\mathbf{F})$. The cycle maps into étale cohomology lead to various cycle maps

$$CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2i}(X, \mathbf{Z}_\ell(i)) \tag{1}$$

$$CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2i}(\overline{X}, \mathbf{Z}_\ell(i))^G \tag{2}$$

$$CH^i(\overline{X}) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to \bigcup_U H^{2i}(\overline{X}, \mathbf{Z}_\ell(i))^U \tag{3}$$

where $\overline{X} := X \times_{\mathbf{F}} \overline{\mathbf{F}}$ and U runs through all open subgroups of G.

Recall there that there are exact sequences

$$0 \to H^1(\mathbf{F}, H^{2i-1}(\overline{X}, \mathbf{Z}_\ell(i))) \to H^{2i}(X, \mathbf{Z}_\ell(i)) \to H^{2i}(\overline{X}, \mathbf{Z}_\ell(i))^G \to 0,$$

where the groups $H^1(\mathbf{F}, H^{2i-1}(\overline{X}, \mathbf{Z}_\ell(i)))$ are finite (this is a consequence of Deligne's proof of the Weil conjectures)

One may consider the associated maps with $\mathbf{Q}_\ell$ coefficients.

$$CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \to H^{2i}(X, \mathbf{Q}_\ell(i)) \tag{4}$$

$$CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \to H^{2i}(\overline{X}, \mathbf{Q}_\ell(i))^G \tag{5}$$

$$CH^i(\overline{X}) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \to \bigcup_U H^{2i}(\overline{X}, \mathbf{Q}_\ell(i))^U \tag{6}$$

Surjectivity of one $\mathbf{Q}_\ell$-map (for all X over all $\mathbf{F}$) is equivalent to surjectivity of the others.
Tate's conjecture : these $\mathbf{Q}_\ell$-maps are surjective.

The case $i = 1$ (cycles of codimension 1, divisors) is the classical conjecture by Tate on surfaces, which is directly related with the finiteness conjecture of Tate–Shafarevich groups.

If the $\mathbf{Q}_\ell$-conjecture holds for X and $i = 1$, then the $\mathbf{Q}_\ell$-conjecture holds for X and $i = d - 1$. This is a known consequence of the hard Lefschetz theorem (proved by Deligne).
Thus for X of dimension 3, the whole $\mathbf{Q}_\ell$-conjecture reduces to the conjecture for divisors.

We are interested here in the $\mathbf{Z}_\ell$-maps.

For $i = d$, the $\mathbf{Z}_\ell$-map $CH^d(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2d}(X, \mathbf{Z}_\ell(d))$ is onto (consequence of Chebotarev's theorems)

For $i = 1$ (divisors), surjectivity of the $\mathbf{Q}_\ell$-maps is equivalent to surjectivity of the $\mathbf{Z}_\ell$-maps.

Using the standard formulas for the computation of Chow groups and of cohomology for a blow-up along a smooth projective subvariety, one shows

If the maps $T_j : CH^j(Y) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2j}(Y, \mathbf{Z}_\ell(j))$ are surjective for all varieties of dimension at most $d-2$ and all $j < i$, then the cokernel of the map $T_i : CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2i}(X, \mathbf{Z}_\ell(i))$ is invariant under smooth blow-up.

(Analogous results with $\mathbf{Q}_\ell$-coefficients.)

For $d = \dim(X) = 3$, and $i = 2$ this implies : the cokernel of T_2 is invariant under blow-up of smooth projective subvarieties (it is thus presumably a birational invariant). Under the Tate conjecture for divisors, this cokernel is a finite group.

For $d = \dim(X)$ arbitrary, under the Tate conjecture for divisors, the cokernel of $T_2 : CH^2(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^4(X, \mathbf{Z}_\ell(2))$ is invariant under smooth blow-up.

For i arbitrary, the $\mathbf{Z}_\ell$-maps need not be onto. As pointed out by various people, in particular Burt Totaro, one may mimick the Atiyah-Hirzebruch counterexamples to the integral Hodge conjecture and produce varieties $X/\mathbf{F}$ for which not all $\mathbf{Z}_\ell$-maps are onto. More precisely, one produces examples where some *torsion classes* in integral cohomology are not in the image of the integral cycle class map.
There exist such examples in $H^4(X, \mathbf{Z}_\ell(2))$ but the dimension of X is rather high.

This leaves the following questions open:

1) Are the integral maps surjective modulo torsion?
2) For suitable i and d, do we have surjection for the integral maps?

The case $i = d - 1$ is precisely the hypothesis made in Saito's theorem.
Already for $d = 3$ and $i = 2$, the analogous questions in the framework of the Hodge conjecture have a negative answer, as shown by Kollár (1990).

We however have :

Theorem (C. Schoen, 1998)
Assume that the Tate conjecture holds for divisors on surfaces. Then for any smooth, projective, geometrically connected variety $X/\mathbf{F}$ *of dimension* d*, the map*

$$CH^{d-1}(\overline{X}) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to \bigcup_U H^{2d-2}(\overline{X}, \mathbf{Z}_\ell(i))^U$$

is onto.
(There is a detailed version of Schoen's argument in a recent text by T. Szamuely and the speaker.)

Here are consequences of Schoen's theorem.

Theorem (CT and Szamuely 2008)

Let $f : X \to C$ be a proper surjective morphism of smooth, projective $\mathbf{F}$-varieties, where C is a curve. Let $X_\eta/\mathbf{F}(C)$ be its generic fibre. Assume it is smooth and geometrically integral.

Assume :

(i) There is no Brauer–Manin obstruction to the existence of a zero-cycle of degree 1 on the $\mathbf{F}(C)$-variety X_η.

(ii) Tate's conjecture holds for divisors on smooth projective surfaces over a finite field.

Then the gcd of the degrees of multisections of the geometric map $\overline{X} \to \overline{C}$ is equal to a power of $p = \mathrm{char}(\mathbf{F})$.

A concrete application is the following theorem, which one may also establish directly from Schoen's result, without using the Brauer–Manin détour.

Theorem (CT and Szamuely 2008)
Let $f : X \to C$ be a proper surjective morphism of smooth, projective $\overline{\mathbf{F}}$-varieties, where C is a curve. Assume :
(i) The generic fibre of f is a smooth hypersurface of dimension at least 3 and of degree prime to $\mathrm{char}(\mathbf{F})$.
(ii) Each fibre of f contains a multiplicity one component.
(iii) Tate's conjecture holds for divisors on smooth projective surfaces over a finite field.
Then the gcd of the degrees of multisections of f is equal to 1.

Remarks

1) It is unlikely that such a local-global theorem holds over the complex field $\mathbf{C}$ in place of $\overline{\mathbf{F}}$.

2) If instead of Schoen's theorem we had the surjectivity modulo torsion of the maps

$$CH^{d-1}(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2d-2}(X, \mathbf{Z}_\ell(d-1))$$

or of

$$CH^{d-1}(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2d-2}(\overline{X}, \mathbf{Z}_\ell(d-1))^G$$

then we would have the same theorems with $\mathbf{F}$ in place of $\overline{\mathbf{F}}$. We would thus have a proof of Conjecture 2.

For a certain class $B_{Tate}(\mathbf{F})$ of smooth projective varieties, B. Kahn (2003) has produced statements which are equivalent to the surjectivity of $CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2i}(X, \mathbf{Z}_\ell(i))$.

The class $B_{Tate}(\mathbf{F})$ roughly speaking contains the smooth projective varieties whose Chow motif is spanned by Artin motives and motives of abelian varieties, and which moreover satisfy Tate's conjecture with $\mathbf{Q}_\ell$-coefficients.

By a result of Soulé (1984), the class $B_{Tate}(\mathbf{F})$ contains all smooth, projective, dimension 3 varieties over $\mathbf{F}$ which after a finite extension of $\mathbf{F}$ are dominated by the product of a curve and a projective plane.

Given a smooth integral variety X over a field F and a prime $\ell \neq \mathrm{char}(F)$, and integers r and s, for any point x of codimension 1 on X there is a residue map

$$H^r(F(X), \mathbf{Q}_\ell/\mathbf{Z}_\ell(s)) \to H^{r-1}(\mathbf{F}(x), \mathbf{Q}_\ell/\mathbf{Z}_\ell(s-1)).$$

The unramified subgroup $H^r_{nr}(F(X), \mathbf{Q}_\ell/\mathbf{Z}_\ell(s)) \subset H^r(F(X), \mathbf{Q}_\ell/\mathbf{Z}_\ell(s))$ is the group of classes with trivial residue at each codimension 1 point of X.

It is an F-birational invariant of smooth, projective, geometrically integral F-varieties (this follows from the Gersten conjecture for étale cohomology, as proved by Bloch and Ogus.)

For $i = 2$, Kahn's result (2003) reads :

Theorem *For* $X/\mathbf{F}$ *in the class* $B_{Tate}(\mathbf{F})$, *the map* $CH^2(X) \otimes \mathbf{Z}_\ell \to H^4(X, \mathbf{Z}_\ell(2))$ *is onto if and only if* $H^3_{nr}(\mathbf{F}(X), \mathbf{Q}_\ell/\mathbf{Z}_\ell(2)) = 0$.

Question For smooth, projective threefolds $X/\mathbf{F}$ do we have $H^3_{nr}(\mathbf{F}(X), \mathbf{Q}_\ell/\mathbf{Z}_\ell(2)) = 0$?

(Note : This is known for a surface X. For a threefold X, one knows $H^4_{nr}(\mathbf{F}(X), \mathbf{Q}_\ell/\mathbf{Z}_\ell(3)) = 0$.)

Except in the rather trivial case where the threefold X is $\mathbf{F}$-rational (see below), it seems very hard to get the surjectivity of the map $CH^2(X) \otimes \mathbf{Z}_\ell \to H^4(X, \mathbf{Z}_\ell(2))$ or the vanishing of the group $H^3_{nr}(\mathbf{F}(X), \mathbf{Q}_\ell/\mathbf{Z}_\ell(2))$.

For instance, can one handle

Threefold which are fibred into quadrics over a curve ? This case is related to the Hasse principle for 2-dimensional quadrics over $\mathbf{F}(C)$.

Threefolds which are geometrically rational ? No idea.

For X a threefold which admits a conic bundle structure over a surface birational to the product of $\mathbf{P}^1_{\mathbf{F}}$ and a curve C over $\mathbf{F}$, the vanishing of $H^3_{nr}(\mathbf{F}(X), \mathbf{Q}_\ell/\mathbf{Z}_\ell(2))$ (for $l = 2$, the only problem here) is very likely.
Indeed, if we replace $\mathbf{F}(C)$ by a number field k, a closely connected result is Salberger's theorem (1987 + later ε) that Conjectures 1 and 2 hold for conic bundles over $\mathbf{P}^1_k$.

Let us finish the talk by a discussion of the "trivial case" of **F**-rational threefolds

Theorem *Let C be a smooth, projective, g. i. curve over a finite field* **F**. *Let $f : X \to C$ be a dominant* **F***-morphism of smooth, projective, g. i.* **F***-varieties. Assume the generic fibre X_η of f is a smooth, g. i. surface over* $\mathbf{F}(C)$. *Assume that $X/\mathbf{F}$ is an* **F***-rational variety. Let ℓ be a prime, $\ell \neq \mathrm{char}(\mathbf{F})$. Then on the* $\mathbf{F}(C)$*-variety X_η, the Brauer–Manin obstruction to the existence of a zero-cycle of degree prime to ℓ is the only obstruction.*

Proof. By S. Saito's result, the conclusion holds if the map $CH^2(X) \otimes \mathbf{Z}_\ell \to H^4(X, \mathbf{Z}_\ell(2))$ is onto. If we accept the best resolution of singularities of rational maps between smooth projective threefolds over a finite field (not available), then the result reduces to the case $X = \mathbf{P}^3_{\mathbf{F}}$. If we do not accept this, we note that the variety X certainly belongs to $B_{Tate}(\mathbf{F})$. Since X is $\mathbf{F}$-rational, we have $H^3_{nr}(\mathbf{F}(X), \mathbf{Q}_\ell/\mathbf{Z}_\ell(2)) = 0$, because the group $H^3_{nr}(\mathbf{F}(X), \mathbf{Q}_\ell/\mathbf{Z}_\ell(2))$ is a birational invariant which is trivial on projective space. Then we resort to B. Kahn's result.

Corollary *Let* $\mathbf{F}$ *be a finite field of characteristic* p*, let* f *and* g *be two homogeneous forms of degree* d *in 4 variables. Assume* f *and* g *have no common divisor, and assume* d *prime to* p*. Assume that the homogeneous form* $f + tg$ *defines a smooth surface* $Y \subset \mathbf{P}^3_{\mathbf{F}(t)}$*. Then the Brauer–Manin obstruction for the* $\mathbf{F}(t)$*-variety* Y *is the only obstruction to the existence of a zero-cycle of degree 1 on* Y*.*

As pointed out to me by Swinnerton-Dyer, this corollary is trivial if the (geometrically) connected curve $Z \subset \mathbf{P}^3_{\mathbf{F}}$ defined by $f = g = 0$ is smooth. More generally, if the curve Z contains a geometrically integral component over $\mathbf{F}$, then by the Weil estimates for curves over a finite field, this component contains a (smooth) zero-cycle of degree 1 over $\mathbf{F}$, hence $Y/\mathbf{F}(t)$ also contains such a zero-cycle of degree 1. Better, for any prime q, the curve Z then contains a point in a field extension of $\mathbf{F}$ of degree a power of q, hence the same holds for Y.

In the particular case $d = 3$ (cubic surfaces) it would be nice to test on this example the conjecture (CT/Sansuc) that the Brauer-Manin obstruction to the existence of a *rational* point is the only obstruction.

A general result (CT/Levine) implies that if $Y/\mathbf{F}(t)$ as above contains a zero-cycle of degree 1 then the curve $Z/\mathbf{F}$ contains a zero-cycle of degree 1.

If there is no Brauer-Manin obstruction to the existence of a rational point, the above theorem guarantees the existence of a zero-cycle of degree 1 on the curve $Z/\mathbf{F}$. If $Z/\mathbf{F}$ has a point in an extension of degree a power of 2, then the cubic surface $Y/\mathbf{F}(t)$ has a rational point over $\mathbf{F}(t)$. The question whether this is always the case is under scrutiny by P. Swinnerton-Dyer.