Local-global principle for zero-cycles of degree one and integral Tate conjecture for 1-cycles

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Let k be a global field, and Ω the set of its places.
Work of Cassels and Tate on curves and abelian varieties (in the 60's) and of CT, Sansuc, Swinnerton-Dyer 1984/87, Salberger 1988, ... on some special higher dimensional varieties has led to two general conjectures
(CT/Sansuc 1981, Kato/Saito 1986, Saito 1989, CT 1995, 1999)

Conjecture 1 *Let X be a smooth, projective, geometrically integral variety X over k. Let $\ell > 0$ be a prime number. Let $\{z_v\}_{v\in\Omega}$ be a family of local zero-cycles on X. If for all $A \in \mathrm{Br}(X)[\ell^\infty]$,*

$$\sum_{v\in\Omega} \mathrm{inv}_v(A(z_v)) = 0 \in \mathbf{Q}/\mathbf{Z}$$

holds, then for any positive integer n there exists a global zero-cycle z_n on X such that for each place v and each A in $\mathrm{Br}(X)[\ell^n]$

$$A(z_v) = A(z_n) \in \mathrm{Br}(k_v).$$

Conjecture 2 *Let X be a smooth, projective, geometrically integral variety X over a global field k. Let $\ell > 0$ be a prime number. Let $\{z_v\}_{v\in\Omega}$ be a family of local zero-cycles of degree 1 on X. If for all $A \in \mathrm{Br}(X)[\ell^\infty]$,*

$$\sum_{v\in\Omega} \mathrm{inv}_v(A(z_v)) = 0 \in \mathbf{Q}/\mathbf{Z}$$

holds, then there exists a global zero-cycle of degree prime to ℓ on X.

The existence of a family of z_v's orthogonal to the whole group $\mathrm{Br}(X)$ is often referred to as : "There is no Brauer-Manin obstruction to the existence of a zero-cycle of degree 1 on X."

The number field case has been the object of much study (CT, Sansuc, Swinnerton-Dyer, Salberger, Skorobogatov, Frossard, work in progress of Wittenberg)

In this talk I want to discuss the case where the global field k is the function field $\mathbf{F}(C)$ of a curve C over a finite field $\mathbf{F}$.

Proposition (S. Saito 1989, CT 1999) *Let* $\mathbf{F}$ *be a finite field, let* $C/\mathbf{F}$ *be a smooth, projective, geometrically connected curve over* $\mathbf{F}$*. Let* $\mathbf{F}(C)$ *be its function field. Let* X *be a smooth, projective, geometrically integral variety over* $\mathbf{F}$ *of dimension* d*, equipped with a flat morphism* $X \to C$ *whose generic fibre* $X_\eta/\mathbf{F}(C)$ *is smooth and geometrically integral. Let* ℓ *be a prime number,* $\ell \neq \mathrm{char}(\mathbf{F})$*.*
a) If the étale cycle map $\mathrm{CH}^{d-1}(X) \otimes \mathbf{Z}_\ell \to H^{2d-2}(X, \mathbf{Z}_\ell(d))$ *is onto, then Conjecture 1 holds for* $X_\eta/\mathbf{F}(C)$*.*
b) If the étale cycle map $\mathrm{CH}^{d-1}(X) \otimes \mathbf{Z}_\ell \to H^{2d-2}(X, \mathbf{Z}_\ell(d))$ *is onto modulo torsion, then Conjecture 2 holds for* $X_\eta/\mathbf{F}(C)$*.*

Let $\mathbf{F}$ be a finite field, $\overline{\mathbf{F}}$ an algebraic closure of $\mathbf{F}$, $G = \mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$. Let X be a smooth, projective, geometrically integral variety over $\mathbf{F}$ of dimension d. Let ℓ be a prime, $\ell \neq \mathrm{char}(\mathbf{F})$. The cycle maps into étale cohomology lead to various cycle maps

$$CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2i}(X, \mathbf{Z}_\ell(i)) \tag{1}$$

$$CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2i}(\overline{X}, \mathbf{Z}_\ell(i))^G \tag{2}$$

$$CH^i(\overline{X}) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to \bigcup_U H^{2i}(\overline{X}, \mathbf{Z}_\ell(i))^U \tag{3}$$

where $\overline{X} := X \times_{\mathbf{F}} \overline{\mathbf{F}}$ and U runs through all open subgroups of G.

Recall that there are exact sequences

$$0 \to H^1(\mathbf{F}, H^{2i-1}(\overline{X}, \mathbf{Z}_\ell(i))) \to H^{2i}(X, \mathbf{Z}_\ell(i)) \to H^{2i}(\overline{X}, \mathbf{Z}_\ell(i))^G \to 0,$$

where the groups $H^1(\mathbf{F}, H^{2i-1}(\overline{X}, \mathbf{Z}_\ell(i)))$ are finite (this is a consequence of Deligne's proof of the Weil conjectures)

One may consider the associated maps with $\mathbf{Q}_\ell$ coefficients.

$$CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \to H^{2i}(X, \mathbf{Q}_\ell(i)) \tag{4}$$

$$CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \to H^{2i}(\overline{X}, \mathbf{Q}_\ell(i))^G \tag{5}$$

$$CH^i(\overline{X}) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \to \bigcup_U H^{2i}(\overline{X}, \mathbf{Q}_\ell(i))^U \tag{6}$$

Surjectivity of one $\mathbf{Q}_\ell$-map (for all X over all $\mathbf{F}$) is equivalent to surjectivity of the others.

Tate's conjecture : these $\mathbf{Q}_\ell$-maps are surjective.

The case $i = 1$ (cycles of codimension 1, divisors) is the classical conjecture by Tate on surfaces, which is directly related with the finiteness conjecture of Tate–Shafarevich groups.

If the $\mathbf{Q}_\ell$-conjecture holds for X and $i = 1$, then the $\mathbf{Q}_\ell$-conjecture holds for X and $i = d - 1$. This is a known consequence of the hard Lefschetz theorem (proved by Deligne).
Thus for X of dimension 3, the whole $\mathbf{Q}_\ell$-conjecture reduces to the conjecture for divisors.

We are interested here in the $\mathbf{Z}_\ell$-maps.

For $i = d$, the $\mathbf{Z}_\ell$-map $CH^d(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2d}(X, \mathbf{Z}_\ell(d))$ is onto (consequence of Chebotarev's theorems)

For $i = 1$ (divisors), surjectivity of the $\mathbf{Q}_\ell$-maps is equivalent to surjectivity of the $\mathbf{Z}_\ell$-maps.

Using the standard formulas for the computation of Chow groups and of cohomology for a blow-up along a smooth projective subvariety, one shows

If the maps $T_j : CH^j(Y) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2j}(Y, \mathbf{Z}_\ell(j))$ are surjective for all smooth projective varieties Y of dimension at most $d-2$ and all $j < i$, then the cokernel of the map $T_i : CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2i}(X, \mathbf{Z}_\ell(i))$ is invariant under smooth blow-up.

(Analogous results with $\mathbf{Q}_\ell$-coefficients.)

For $d = \dim(X) = 3$, and $i = 2$ this implies : the cokernel of T_2 is invariant under blow-up of smooth projective subvarieties. Known results on the resolution of indeterminacies of rational maps (Abhyankar, Cossart) then imply that this cokernel is a birational invariant of smooth projective threefolds over a finite field, hence in particular that it is trivial for $\mathbf{F}$-rational, smooth projective threefolds.
Under the Tate conjecture for divisors, this cokernel is a finite group.

For $d = \dim(X)$ arbitrary, under the Tate conjecture for divisors, the cokernel of $T_2 : CH^2(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^4(X, \mathbf{Z}_\ell(2))$ is invariant under smooth blow-up.

For i arbitrary, the $\mathbf{Z}_\ell$-maps need not be onto. As pointed out by various people, in particular Burt Totaro, one may mimick the Atiyah-Hirzebruch counterexamples to the integral Hodge conjecture and produce varieties $X/\mathbf{F}$ for which not all $\mathbf{Z}_\ell$-maps are onto. More precisely, one produces examples where some *torsion classes* in integral cohomology are not in the image of the integral cycle class map.
There exist such examples in $H^4(X, \mathbf{Z}_\ell(2))$ but the dimension of X is rather high.

This leaves the following questions open:

1) Are the integral maps surjective modulo torsion?
2) For suitable i and d, do we have surjection for the integral maps?

The case $i = d - 1$ is precisely the hypothesis made in Saito's theorem.
Already for $d = 3$ and $i = 2$, the analogous questions in the framework of the Hodge conjecture have a negative answer, as shown by Kollár (1990).

We however have :

Theorem (C. Schoen, 1998)
Assume that the Tate conjecture holds for divisors on surfaces. Then for any smooth, projective, geometrically connected variety $X/\mathbf{F}$ *of dimension* d*, the map*

$$CH^{d-1}(\overline{X}) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to \bigcup_U H^{2d-2}(\overline{X}, \mathbf{Z}_\ell(i))^U$$

is onto.
(There is a detailed version of Schoen's argument in a recent text by T. Szamuely and the speaker.)

Here are consequences of Schoen's theorem.

Theorem (CT and Szamuely 2008)
Let $f : X \to C$ be a proper surjective morphism of smooth, projective $\mathbf{F}$-varieties, where C is a curve. Let $X_\eta/\mathbf{F}(C)$ be its generic fibre. Assume it is smooth and geometrically integral.
Assume :
(i) There is no Brauer–Manin obstruction to the existence of a zero-cycle of degree 1 on the $\mathbf{F}(C)$-variety X_η.
(ii) Tate's conjecture holds for divisors on smooth projective surfaces over a finite field.
Then the gcd of the degrees of multisections of the $\overline{\mathbf{F}}$-morphism $\overline{X} \to \overline{C}$ is equal to a power of $p = \mathrm{char}(\mathbf{F})$.

A concrete application is the following theorem, which one may also establish directly from Schoen's result, without using the Brauer–Manin détour.

Theorem (CT and Szamuely 2008)
Let $f : X \to C$ be a proper surjective morphism of smooth, projective $\overline{\mathbf{F}}$-varieties, where C is a curve. Assume :
(i) The generic fibre of f is a smooth hypersurface of dimension at least 3 and of degree prime to $\mathrm{char}(\mathbf{F})$.
(ii) Each fibre of f contains a multiplicity one component.
(iii) Tate's conjecture holds for divisors on smooth projective surfaces over a finite field.
Then the gcd of the degrees of multisections of f is equal to 1.

This result should be confronted with the following result of C. Voisin (2009), who develops Kollár's 1990 idea :

Over the complex field $\mathbf{C}$, for d big enough, there exist very general hypersurfaces of bidegree $(4, d)$ in $\mathbf{P}^3_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}}$ defining Lefschetz fibrations $X \to \mathbf{P}^1_{\mathbf{C}}$ for which there are only even degree multisections.

Coming back to finite fields, let us observe that if instead of Schoen's theorem we had the surjectivity of the maps

$$CH^{d-1}(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to [H^{2d-2}(X, \mathbf{Z}_\ell(d-1))/\mathrm{tors}]$$

or of

$$CH^{d-1}(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to [H^{2d-2}(\overline{X}, \mathbf{Z}_\ell(d-1))^G/\mathrm{tors}]$$

then we would have the same theorems with $\mathbf{F}$ in place of $\overline{\mathbf{F}}$. We would thus have a proof of Conjecture 2.

For a certain class $B_{Tate}(\mathbf{F})$ of smooth projective varieties, B. Kahn (2003) has produced statements which are equivalent to the surjectivity of $CH^i(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to H^{2i}(X, \mathbf{Z}_\ell(i))$.

The class $B_{Tate}(\mathbf{F})$ roughly speaking contains the smooth projective varieties whose Chow motif is spanned by Artin motives and motives of abelian varieties, and which moreover satisfy Tate's conjecture with $\mathbf{Q}_\ell$-coefficients.

By a result of Soulé (1984), the class $B_{Tate}(\mathbf{F})$ contains all smooth, projective, dimension 3 varieties over $\mathbf{F}$ which after a finite extension of $\mathbf{F}$ are dominated by the product of a curve and a projective plane.

Given a smooth, projective integral variety X over a field F and a prime $\ell \neq \mathrm{char}(F)$, and integers r and s, for any point x of codimension 1 on X there is a residue map

$$H^r(F(X), \mathbf{Q}_\ell/\mathbf{Z}_\ell(s)) \to H^{r-1}(F(x), \mathbf{Q}_\ell/\mathbf{Z}_\ell(s-1)).$$

The unramified subgroup $H^r_{nr}(F(X)/F, \mathbf{Q}_\ell/\mathbf{Z}_\ell(s)) \subset H^r(F(X), \mathbf{Q}_\ell/\mathbf{Z}_\ell(s))$ is the group of classes with trivial residue at each codimension 1 point of X.

It is an F-birational invariant of smooth, projective, geometrically integral F-varieties (this follows from the Gersten conjecture for étale cohomology, as proved by Bloch and Ogus.)

For $i = 2$, Kahn's result (2003) reads :

Theorem *For $X/\mathbf{F}$ in the class $B_{Tate}(\mathbf{F})$, the map $CH^2(X) \otimes \mathbf{Z}_\ell \to H^4(X, \mathbf{Z}_\ell(2))$ is onto if and only if $H^3_{nr}(\mathbf{F}(X)/\mathbf{F}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(2)) = 0$.*

For a smooth, projective surface $X/\mathbf{F}$,
$H^3_{nr}(\mathbf{F}(X)/\mathbf{F}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(2)) = 0$ (CT/Sansuc/Soulé 1983).

For a smooth projective threefold $X/\mathbf{F}$,
$H^4_{nr}(\mathbf{F}(X)/\mathbf{F}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(3)) = 0$ (CT, S. Saito 1993).

Questions

1) For an arbitrary smooth projective threefold $X/\mathbf{F}$, is the group $H^3_{nr}(\mathbf{F}(X)/\mathbf{F}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(2))$ a divisible group ?

2) For a smooth projective threefold $X/\mathbf{F}$, can the group $H^3_{nr}(\mathbf{F}(X)/\mathbf{F}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(2)) = 0$ be nonzero ?
[With $\mathbf{C}$ in place of $\mathbf{F}$, examples of threefolds are known where $H^3_{nr}(\mathbf{C}(X)/\mathbf{C}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(2))$ is nonzero (Bloch-Esnault), and even examples where the l-torsion of that group is infinite (C. Schoen).]

3) Give geometric conditions on smooth, projective threefolds $X/\mathbf{F}$ which ensure $H^3_{nr}(\mathbf{F}(X)/\mathbf{F}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(2)) = 0$.

Except in the rather trivial case where the threefold X is $\mathbf{F}$-rational (see below), it seems hard to get the surjectivity of the map $CH^2(X) \otimes \mathbf{Z}_\ell \to H^4(X, \mathbf{Z}_\ell(2))$ or the vanishing of the group $H^3_{nr}(\mathbf{F}(X)/\mathbf{F}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(2))$.

For instance, can one handle threefolds which are geometrically rational ?

Theorem *Assume* $\mathrm{char}(\mathbf{F}) \neq 2$. *Let* $X \to C$ *be a quadric fibration, where* X*, resp.* C*, is a smooth projective threefold, resp. curve over the finite field* $\mathbf{F}$*. Then for any* $\ell \neq \mathrm{char}(\mathbf{F})$*, we have* $H^3_{nr}(\mathbf{F}(X)/\mathbf{F}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(2)) = 0$*.*

Proof. The nontrivial case is $\ell = 2$. Let $F = \mathbf{F}(C)$ and let Q/F be the generic fibre of $X \to C$. According to K-theoretical (motivic) work of Kahn, Rost and Sujatha, for any field F of char. not 2, and any 2-dimensional quadric Q/F, the map

$$H^3(F, \mathbf{Q}_2/\mathbf{Z}_2(2)) \to H^3_{nr}(F(Q)/F, \mathbf{Q}_2/\mathbf{Z}_2(2))$$

is onto. But $H^3(\mathbf{F}(C), \mathbf{Q}_2/\mathbf{Z}_2(2)) = 0$ since the cohomological dimension of $\mathbf{F}(C)$ is 2.

The variety X is in $B_{Tate}(\mathbf{F})$. If we now combine S. Saito's 1989 theorem, Kahn's 2003 result and an old theorem of Springer, we get a "new" and to say the least rather far fetched proof of the Hasse principle for quadratic forms in 4 variables over $\mathbf{F}(C)$.

For X a threefold which admits a conic bundle structure over a smooth projective surface over $\mathbf{F}$, the vanishing of $H^3_{nr}(\mathbf{F}(X)/\mathbf{F}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(2))$ (for $\ell = 2$, the only problem here) is very likely, as I pointed out in a talk in Cambridge (UK) in August 2009.

Indeed, for conic bundles over curves over a number field, there is a series of closely connected results starting with Salberger's theorem (1988 + later ε) that Conjectures 1 and 2 hold for conic bundles over $\mathbf{P}^1_k$.

In the geometric set-up, I could prove :

Let $\mathrm{char}(\mathbf{F}) \neq 2$. Let $X \to S$ be a conic fibration, where X, resp. S, is a smooth projective threefold, resp. surface, over $\mathbf{F}$. Assume that the ramification curve of the conic fibration is a smooth, geometrically connected curve. Then $H^3_{nr}(\mathbf{F}(X)/\mathbf{F}, \mathbf{Q}_2/\mathbf{Z}_2(2)) = 0$.

However, Parimala and Suresh have just announced a proof of the same vanishing result without any hypothesis on the ramification curve. Their proof actually handles conic bundles over an arithmetic surface (for the time being up to the real phenomena), hence will very likely produce an alternate proof for Salberger's 1988 results and some of its followers.

Let us finish the talk by a discussion of the "trivial case" of **F**-rational threefolds. This is recent joint work with Sir Peter Swinnerton-Dyer.

Theorem *Let C be a smooth, projective, g. i. curve over a finite field* $\mathbf{F}$*. Let $f : X \to C$ be a dominant* $\mathbf{F}$*-morphism of smooth, projective, g. i.* $\mathbf{F}$*-varieties. Assume the generic fibre X_η of f is a smooth, geometrically integral surface over* $\mathbf{F}(C)$*. Assume that* $X/\mathbf{F}$ *is an* $\mathbf{F}$*-rational variety. Let ℓ be a prime, $\ell \neq \mathrm{char}(\mathbf{F})$. Then on the* $\mathbf{F}(C)$*-variety X_η, the Brauer–Manin obstruction to the existence of a zero-cycle of degree prime to ℓ is the only obstruction.*

Proof. By S. Saito's 1989 result, the conclusion holds if the map $CH^2(X) \otimes \mathbf{Z}_\ell \to H^4(X, \mathbf{Z}_\ell(2))$ is onto. As recalled above, enough is known on the resolution of indeterminacies of rational maps in positive characteristic in this special case (Abhyankar, Cossart) to reduce to the case $X = \mathbf{P}^3_{\mathbf{F}}$.

Corollary *Let* $\mathbf{F}$ *be a finite field of characteristic* p*, let* f *and* g *be two homogeneous forms of degree* d *in 4 variables. Assume* f *and* g *have no common divisor, and assume* d *prime to* p*. Assume that the homogeneous form* $f + tg$ *defines a smooth surface* $Y \subset \mathbf{P}^3_{\mathbf{F}(t)}$*. Then the Brauer–Manin obstruction for the* $\mathbf{F}(t)$*-variety* Y *is the only obstruction to the existence of a zero-cycle of degree 1 on* Y*.*

The existence of a smooth projective threefold $\mathcal{X}$ fibred over $\mathbf{P}^1_{\mathbf{F}}$, with generic fibre Y, follows from a result of Cossart on the resolution of singularities of a threefold over an arbitrary field, when it is already birational to a smooth projective threefold.

This corollary is trivial if the (geometrically) connected curve $Z \subset \mathbf{P}^3_{\mathbf{F}}$ defined by $f = g = 0$ is smooth. More generally, if the curve Z contains a geometrically integral component over $\mathbf{F}$, then by the Weil estimates for curves over a finite field, this component contains a (smooth) zero-cycle of degree 1 over $\mathbf{F}$, hence $Y/\mathbf{F}(t)$ also contains such a zero-cycle of degree 1. Better, for any prime q, the curve Z then contains a point in a field extension of $\mathbf{F}$ of degree a power of q, hence the same holds for Y.

The corollary acquires substance only when the curve Z breaks up into pieces over $\overline{\mathbf{F}}$.

For $d = 3$, we obtain the following very special case of the conjecture (CT/Sansuc 1979) that the Brauer-Manin obstruction to the existence of a *rational* point on a geometrically rational surface is the only obstruction.

Theorem *Let* $\mathbf{F}$ *be a finite field of characteristic* $p \neq 3$*, let* f *and* g *be two homogeneous forms of degree* 3 *in 4 variables. Assume that the homogeneous form* $f + tg$ *defines a smooth surface* $Y \subset \mathbf{P}^3_{\mathbf{F}(t)}$*. If there is no Brauer–Manin obstruction to the existence of a rational point on* Y*, then* Y *has a rational point.*

Proof : Under the assumptions of the theorem, there exists a zero-cycle of degree 1 on Y. A general result (CT/Levine 1990/2009) implies that if $Y/\mathbf{F}(t)$ as above contains a zero-cycle of degree 1 then the curve $Z/\mathbf{F}$ defined by $f = g = 0$ contains a zero-cycle of degree 1. By a case by case discussion of the possible decompositions of the curve Z over $\overline{\mathbf{F}}$ one then shows that Z contains a point in an extension of $\mathbf{F}$ of degree a power of 2. Thus the cubic surface $Y/\mathbf{F}(t)$ has a point in a tower of quadratic extensions of $\mathbf{F}(t)$, hence in $\mathbf{F}(t)$.

Here are possible decompositions for $Z \times_F \overline{\mathbf{F}}$:

$9 = 3(1 + 1 + 1)$
$9 = 2(1 + 1 + 1) + (1 + 1 + 1)$
$9 = (3 + 3 + 3)$
$9 = (1 + \cdots + 1)$ (9 times)
$9 = (2 + 2 + 2) + (1 + 1 + 1)$
$9 = (1 + 1 + 1) + (1 + 1 + 1) + (1 + 1 + 1)$

$(a + a + a)$ means sum of three conjugate curves, each of degree a.

In the cases $9 = (3 + 3 + 3)$, sum of three conjugate twisted cubics, and $9 = (2 + 2 + 2) + (1 + 1 + 1)$, three conjugate conics and three conjugate lines, we may have $\mathrm{Br}(Y)/\mathrm{Br}(\mathbf{F}(t)) \neq 0$.

Theorem

Let $k = \mathbf{F}(C)$ with $\mathbf{F} = \mathbf{F}_q$ and q odd, $q \equiv 2$ mod. 3.
Let $a, b, c, d \in k^*$. Let us say that v appears in $z \in k^*$ if $v(z) \neq 0 \bmod 3$.
If there exist a place v, resp. a place w, which appears in a and only in a, resp. in b and only in b, then the Hasse principle holds for the diagonal cubic surface

$$ax^3 + by^3 + cz^3 + dt^3 = 0.$$

In the number field case, proven by Swinnerton-Dyer (2001) assuming finiteness of Tate–Shafarevich groups
In the function field case (CT 2003) : no assumption needed, the relevant cases of the Tate conjecture are known