Unramified third cohomology and integral Hodge conjecture

(Joint work with Claire Voisin)

Jean-Louis Colliot-Thélène (CNRS et Université Paris-Sud)

KIAS

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Let $X/\mathbb{C}$ be a smooth, projective variety and $d = \dim(X)$. Let $H^i_B(X, R(j)) := H^{2i}_B(X(\mathbb{C}), R(j))$, where $R = \mathbb{Z}, \mathbb{Q}, \mathbb{C}$ or $\mathbb{Q}/\mathbb{Z}$, and $R(j) = R \otimes (\mathbb{Z}.(2\pi\sqrt{-1})^{\otimes i})$.
For any $i \geq 0$, there is a cycle map with values in Betti cohomology

$$\mathrm{c}^i : CH^i(X) \to H^{2i}_B(X, \mathbb{Z}(i)).$$

Let $H^{2i}_{alg}(X, \mathbb{Z}) \subset H^{2i}_B(X, \mathbb{Z}(i))$ denote the image of this map. Using the embedding $H^{2i}_B(X, \mathbb{Q}) \subset H^{2i}_B(X, \mathbb{C}(i))$ one defines the subgroup $H^{2i}_{Hdg}(X, \mathbb{Q})$ of classes of type (i, i).

One defines the group $H^{2i}_{Hdg}(X,\mathbb{Z}) \subset H^{2i}_{B}(X,\mathbb{Z}(i))$ as the inverse image of $H^{2i}_{Hdg}(X,\mathbb{Q})$ in $H^{2i}_{B}(X,\mathbb{Z}(i))$.
One then has $H^{2i}_{alg}(X,\mathbb{Z}) \subset H^{2i}_{Hdg}(X,\mathbb{Z}) \subset H^{2i}_{B}(X,\mathbb{Z}(i))$
The Hodge conjecture predicts that the quotient $H^{2i}_{Hdg}(X,\mathbb{Z})/H^{2i}_{alg}(X,\mathbb{Z})$ is finite.

Trivial remark : the embedding

$$Z^{2i}(X) := H^{2i}_{Hdg}(X,\mathbb{Z})/H^{2i}_{alg}(X,\mathbb{Z}) \subset H^{2i}_{B}(X,\mathbb{Z}(i))/H^{2i}_{alg}(X,\mathbb{Z})$$

induces an isomorphism on torsion subgroups.

We know :

For $i = 0, 1, d$, we have $Z^{2i}(X) = 0$.
For $i = 1$: Lefschetz's theorem on class of type $(1, 1)$.
In this case one has an embedding $NS(X) \subset H^2_B(X, \mathbb{Z}(1))$, and it induces an isomorphism $NS(X)\{\mathrm{tor}\} \xrightarrow{\simeq} H^2_B(X, \mathbb{Z}(1))\{\mathrm{tor}\}$.

For $i = d - 1$, the group $Z^{2d-2}(X)$ is finite (follows from the hard Lefschetz theorem and the case $d = 1$).

For $i = 2$, if there exists a proper map $f : V \to X$, from a 3-dimensional variety V such that the induced homomorphism $f_* : \mathrm{CH}_0(V) \to \mathrm{CH}_0(X)$ is onto, then
$Z^4(X) = H^4_{Hdg}(X, \mathbb{Z})/H^4_{alg}(X, \mathbb{Z})$ is finite (Bloch-Srinivas).

One knows that the integral Hodge conjecture does not hold in general. There are examples with
$Z^4(X) = H^4_{Hdg}(X, \mathbb{Z})/H^4_{alg}(X, \mathbb{Z}) \neq 0$.

More precisely : there are examples (Atiyah-Hirzebruch) for which the finite group $Z^4(X)\{\mathrm{tors}\} \neq 0$.

Questions which we want to ask :

Is there a systematic method to compute the finite group $Z^4(X)\{\mathrm{tors}\}$?

Are there classes of varieties for which $Z^4(X)\{\mathrm{tors}\} = 0$?
[C. Voisin for instance proves this for rational varieties.]

If X is rationally simply connected (in the sens of Kollár, Miyaoka, Mori and Campana), is the finite group $Z^4(X) = 0$?
(question raised by C. Voisin, 2004)

Using methods and results from algebraic K-theory, we shall partially answer these questions.

Bloch–Ogus-Theory and Betti-Cohomology (1974)

Let X be a complex variety. Let X_{cl} denote the classical topology on $X(\mathbb{C})$. There is a morphism of sites $h : X_{cl} \to X_{Zar}$. An abelian group A defines a constant sheaf A on $X(\mathbb{C})$. For $i \in \mathbb{N}$, the sheaf

$$\mathcal{H}^i(A) := R^i h_* A$$

on X_{Zar} is the sheaf associated to the presheaf $U \mapsto H^i_B(U, A)$.

We have the spectral sequence

$$E_2^{pq} = H^p(X_{Zar}, \mathcal{H}^q(A)) \Longrightarrow H^n_B(X, A).$$

Let $i_D : D \hookrightarrow X$ be a closed integral subvariety, let $\mathbb{C}(D)$ be its function field.
Let

$$H^i(\mathbb{C}(D), A) := \varinjlim_{U \subset D, U \neq \emptyset} H^i(U(\mathbb{C}), A).$$

This defines a constant sheaf on D, which itself defines the sheaf $i_{D_*} H^i(\mathbb{C}(D), A)$ on X_{Zar}.
For $E \subset D$ of codimension 1, there is a residue map

$$H^i(\mathbb{C}(D), A) \to H^{i-1}(\mathbb{C}(E), A(-1)).$$

Main theorem of the Bloch–Ogus Theory (Gersten conjecture for étale cohomology)

Let X be a smooth irreducible variety over $\mathbb{C}$. Then for all $i \in \mathbb{N}$ there is an exact sequence of sheaves

$$0 \to \mathcal{H}^i_X(A) \to i_{X*}H^i(\mathbb{C}(X), A) \xrightarrow{\partial} \bigoplus_{\substack{D \text{ integral} \\ codim\, D=1}} i_{D*}H^{i-1}(\mathbb{C}(D), A(-1))$$

$$\xrightarrow{\partial} \dots \xrightarrow{\partial} \bigoplus_{\substack{D \text{ integral} \\ codim\, D=i}} i_{D*}A_D(-i) \to 0.$$

The maps ∂ are induced by the above mentioned residue maps.

Definition
Let X be complex variety, and A an abelian group. The i-th unramified cohomology group of X with values in A is the group

$$H^i_{nr}(X, A) := H^0(X, \mathcal{H}^i(X, A)).$$

Let X be a smooth, connected, projective variety over $\mathbb{C}$. The groups H^1_{nr} and H^2_{nr} were understood (Grothendieck) without Bloch-Ogus theory.

$$H^2_{nr}(X,\mu_n) \simeq \mathrm{Br}(X)[n] \qquad \text{birational} \quad \text{invariant}$$

Exact sequence

$$0 \to NS(X) \to H^2_B(X,\mathbb{Z}(1)) \to H^2_{nr}(X,\mathbb{Z}(1)) \to 0.$$

$$H^2_{nr}(X,\mathbb{Z}) \simeq \mathbb{Z}^{(b_2-\rho)}.$$

Exact sequence

$$0 \to (\mathbb{Q}/\mathbb{Z})^{(b_2-\rho)} \to H^2_{nr}(X,\mathbb{Q}/\mathbb{Z}) \to H^3_B(X,\mathbb{Z})\{\mathrm{tor}\} \to 0.$$

and $b_2 - \rho = 0$ if and only if $H^2(X, O_X) = 0$.

The main theorem of Bloch–Ogus theory implies :
Let X be smooth and irreducible. Then

$$H^i_{nr}(X,A) = Ker[H^i(\mathbb{C}(X),A) \xrightarrow{\partial} \bigoplus_{\substack{D \text{ integral} \\ codim\, D=1}} H^{i-1}(\mathbb{C}(D),A(-1))].$$

In particular : If X is smooth and $U \subset X$ open with complement of codimension at least 2, then $H^i_{nr}(X,A) \stackrel{\simeq}{\to} H^i_{nr}(U,A)$.

Hence : if X and Y are smooth, projective, irreducible and birational to each other, then $H^i_{nr}(X,A) \simeq H^i_{nr}(Y,A)$. For $i \geq 1$ and $X = \mathbf{P}^n_{\mathbb{C}}$, these groups vanish.

Moreover :

If X is smooth, then $H^r(X_{Zar}, \mathcal{H}^i(A)) = 0$ for $r > i$.

Let X over $\mathbb{C}$ be smooth, connected, projective. Then

$$\mathrm{CH}^i(X)/n \xrightarrow{\simeq} H^i_{Zar}(X, \mathcal{H}^i(\mathbb{Z}/n(i)))$$

and

$$\mathrm{CH}^i(X)/\mathrm{alg} \xrightarrow{\simeq} H^i_{Zar}(X, \mathcal{H}^i(\mathbb{Z}(i))).$$

Let X over $\mathbb{C}$ be smooth and connected.
Let A be an abelian group.
Exact sequence :

$$H^3_B(X(\mathbb{C}),A) \to H^3_{nr}(X,A) \xrightarrow{d_2} H^2(X_{Zar},\mathcal{H}^2_X(A)) \to H^4_B(X(\mathbb{C}),A)$$

If X moreover is projective, then exact sequence :

$$H^3_B(X,\mathbb{Z}(2)) \to H^3_{nr}(X,\mathbb{Z}(2)) \to CH^2(X)/alg \xrightarrow{c^2} H^4_B(X,\mathbb{Z}(2))$$

Hence (Definition of $\mathrm{Griff}^2(X)$)

$$0 \to \mathrm{Griff}^2(X) \to CH^2(X)/alg \xrightarrow{c^2} H_B^4(X, \mathbb{Z}(2))$$

and

$$H_B^3(X, \mathbb{Z}(2)) \to H_{nr}^3(X, \mathbb{Z}(2)) \to \mathrm{Griff}^2(X) \to 0$$

The Bloch–Kato conjecture

F field, $\mathrm{char}(F) = 0$.
Map (Tate) from Milnor K-theory to Galois cohomology :

$$K_i^M(F)/n \to H^i(F, \mu_n^{\otimes i})$$

Conjecture $BK_{i,n}$: This is an isomorphism.

If this holds for all i and n, then

$$H^{i+1}(F, \mu_n^{\otimes i}) \hookrightarrow H^{i+1}(F, \mu_{nm}^{\otimes i})$$

hence

$$H^{i+1}(F, \mathbb{Q}/\mathbb{Z}(i)) = \bigcup_n H^{i+1}(F, \mu_n^{\otimes i}).$$

$$K_i^M(F)/n \xrightarrow{\simeq} H^i(F, \mu_n^{\otimes i}) \quad ?$$

Long history

$i = 1$: Hilbert's theorem 90 (Kummer theory)
$i = 2$: Merkurjev and Suslin (1982)
$i = 3, n = 2^m$: Merkurjev and Suslin, Rost (1990)
$i > 4, n = 2^m$: Voevodsky (2003)
$i > 4$: Voevodsky, Rost 2003-2010
(Voevodsky, arXiv 0805.4430v2, 10.2.2010)

Theorem

Let X be a complex variety. Multiplication by $n > 0$ induces short exact sequences of Zariski sheaves

$$0 \to \mathcal{H}^i(X, \mathbb{Z}(j)) \stackrel{\times n}{\to} \mathcal{H}^i(X, \mathbb{Z}(j)) \to \mathcal{H}^i(X, \mathbb{Z}/n(j))) \to 0.$$

In particular the groups $H^i_{nr}(X, \mathbb{Z}(j)) = H^0(X, \mathcal{H}^i(X, \mathbb{Z}(j)))$ are torsion free.

Follows from the Bloch-Kato conjecture and further work.

Corollary

Let X be smooth, connected, projective variety. If there exists Y of dimension r and a morphism $Y \to X$ such that $CH_0(Y) \to CH_0(X)$ is onto, then $H^i_{nr}(X, \mathbb{Z}(j)) = 0$ for $i > r$.

Proof : The correspondance method of Bloch-Srinivas shows that these groups are torsion groups.

For X smooth, injectivity of $\mathcal{H}^3(X, \mathbb{Z}(j)) \stackrel{\times n}{\to} \mathcal{H}^3(X, \mathbb{Z}(j))$ follows from Merkurjev-Suslin: remark of Bloch and Srinivas (1983). If moreover $\dim(X) = 3$, then $\mathcal{H}^4(X, \mathbb{Z}(j)) = 0$ (Lefschetz), hence one already has the exact sequence.

$$0 \to \mathcal{H}^3(X, \mathbb{Z}(j)) \stackrel{\times n}{\to} \mathcal{H}^3(X, \mathbb{Z}(j)) \to \mathcal{H}^3(X, \mathbb{Z}/n(j))) \to 0.$$

This was noticed by Barbieri-Viale (1992).

Main theorem

Let X be a smooth variety over $\mathbb{C}$.

(i) For $n > 1$, exact sequence

$$0 \to H^3_{nr}(X, \mathbb{Z}(2)))/n \to H^3_{nr}(X, \mu_n^{\otimes 2}) \to Z^4(X)[n] \to 0$$

(ii) Exact sequence

$$0 \to H^3_{nr}(X, \mathbb{Z}(2))) \otimes \mathbb{Q}/\mathbb{Z} \to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))) \to Z^4(X)\{\mathrm{tor}\} \to 0.$$

Proof

By the Bloch-Ogus theorem, the spectal sequence

$$E_2^{pq} = H^p(X_{Zar}, \mathcal{H}^q(\mathbb{Z}(2))) \Longrightarrow H_B^n(X, \mathbb{Z}(2))$$

is concentrated in the second octant. When one analyses the filtration on $H_B^4(X, \mathbb{Z}(2))$ given by the spectral sequence, one gets the exact sequence

$$0 \to H^1(X, \mathcal{H}_X^3(\mathbb{Z}(2))) \to [H_B^4(X, \mathbb{Z}(2))/H_{alg}^4(X, \mathbb{Z}(2))] \to H^0(X, \mathcal{H}_X^4(\mathbb{Z}(2))).$$

As the group $H_{nr}^4(X, \mathbb{Z}(2)) = H^0(X, \mathcal{H}_X^4(\mathbb{Z}(2)))$ has no torsion, this yields

$$H^1(X, \mathcal{H}_X^3(\mathbb{Z}(2)))\{\mathrm{tor}\} \xrightarrow{\simeq} Z^4(X)\{\mathrm{tor}\}.$$

The exact sequence of sheaves

$$0 \to \mathcal{H}_X^3(\mathbb{Z}(2)) \xrightarrow{\times n} \mathcal{H}_X^3(\mathbb{Z}(2)) \to \mathcal{H}_X^3(\mu_n^{\otimes 2}) \to 0$$

gives rise to the exact sequence of groups

$$0 \to H^0(X, \mathcal{H}_X^3(\mathbb{Z}(2)))/n \to H^0(X, \mathcal{H}_X^3(\mu_n^{\otimes 2}))$$

$$\to H^1(X, \mathcal{H}_X^3(\mathbb{Z}(2)))[n] \to 0,$$

from which we get the announced exact sequences

$$0 \to H^3_{nr}(X, \mathbb{Z}(2)))/n \to H^3_{nr}(X, \mu_n^{\otimes 2}) \to Z^4(X)[n] \to 0$$

$$0 \to H^3_{nr}(X, \mathbb{Z}(2))) \otimes \mathbb{Q}/\mathbb{Z} \to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))) \to Z^4(X)\{\mathrm{tor}\} \to 0.$$

1) For varieties X of dimension 3 with $H^3_{nr}(X, \mathbb{Z}(2))) = 0$, for instance unitational varieties of dimension 3, the argument goes back to a 1992 paper by Barbieri-Viale.

2) From the sequence and the Bloch–Ogus Theory one concludes that the group $Z^4(X)\{\mathrm{tor}\}$ is a birational invariant. C. Voisin had already remarked that the groups $Z^4(X)$ and $Z^{2d-2}(X)$ are birational invariant. Her proof was by reduction to the case of blowing up of a smooth closed subvariety.

On the group $H^3_{nr}(X,\mathbb{Z})$

We have the exact sequence

$$H^3_B(X,\mathbb{Z}(2)) \to H^3_{nr}(X,\mathbb{Z}(2)) \to \mathrm{Griff}^2(X) \to 0.$$

As already mentioned, if $CH_0(X)$ is covered a surface, then $H^3_{nr}(X,\mathbb{Z}(2)) = 0$. One then has an isomorphism of finite abelian groups

$$H^3_{nr}(X,\mathbb{Q}/\mathbb{Z}(2))) \overset{\simeq}{\to} Z^4(X).$$

Basic diagram

$$\begin{array}{ccccc}
H^3_B(X(\mathbb{C}),\mathbb{Z}(2))/n & \hookrightarrow & H^3_{\mathrm{et}}(X,\mu_n^{\otimes 2}) & \to & H^4_B(X(\mathbb{C}),\mathbb{Z}(2))[n] \\
\downarrow & & \downarrow & & \downarrow \\
H^3_{nr}(X,\mathbb{Z}(2)))/n & \hookrightarrow & H^3_{nr}(X,\mu_n^{\otimes 2}) & \to & Z^4(X)[n] \\
\downarrow & & \downarrow & & \\
\mathrm{Griff}^2(X)/n & \to & Ker & & \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

where $Ker = \mathrm{Ker}[CH^2(X)/n \to H^4_{\mathrm{et}}(X,\mu_n^{\otimes 2})]$ and the top two sequences are short exact sequences.

Varieties with $H^3_{nr}(X, \mu_n^{\otimes 2}) \neq 0$ or $Z^4(X) \neq 0$

Atiyah and Hirzebruch (1962); Totaro (1997) : Topological methods. Torsion in $H^4_B(X,\mathbb{Z})$. Does not come from $H^4_{alg}(X,\mathbb{Z})$. Examples with $\dim(X) \geq 7$.

Kollár (1990). Specialization argument. For $X \subset \mathbf{P}^4_{\mathbb{C}}$ a very general hypersurface of degree p^3 with a prime $p \neq 2,3$, $H^4_B(X,\mathbb{Z}) = \mathbb{Z}$ and $H^4_{alg}(X,\mathbb{Z}) \subset p\mathbb{Z}$. Thus $Z^4(X) \neq 0$ and $H^3_{nr}(X,\mathbb{Z}/p) \neq 0$.

Bloch and Esnault (1996); Schoen (2002). Arithmetical method (p-adic cohomology, Hilbert's irreducibility theorem). Examples with $\dim(X) = 3$. $Griff^2(X)/n \neq 0$, resp. $Griff^2(X)/n$ infinite. The same therefore holds for $H^3_{nr}(X,\mathbb{Z})/n$ and $H^3_{nr}(X,\mathbb{Z}/n)$. Open question : is $Z^4(X) \neq 0$?

For rationally connected varieties, could the situation be better ? (Question by C. Voisin 2004) Answer (CT/Voisin 2009) : No.

A unirational variety with $H^3_{nr}(X,\mathbb{Z}/2) = Z^4(X)[2] \neq 0$

Let F be a field, $Char.(F) = 0$, $f,g,h \in F^*$, let Q/F be the 3-dimensional quadric in $\mathbf{P}^5_F$ defined by the equation $X^2 - fY^2 - gZ^2 + fgT^2 - hW^2 = 0$. Let $F(Q)$ denote its function field.

Theorem (Arason, 1974) *The kernel of the map* $H^3(F,\mathbb{Z}/2) \to H^3(F(Q),\mathbb{Z}/2)$ *is* 0 *or* $\mathbb{Z}/2$*, and it is spanned by the cup-product* $(f) \cup (g) \cup (h)$*.*

This result is a forerunner of the big theorems in algebraic K-theory : Merkur'ev-Suslin, Rost, Voevodsky.
[Similar result : For H^1, Hilbert. For H^2, Witt. For H^n, $n \geq 4$, Jacob and Rost; Orlov, Vishik and Voevodsky.]

Theorem (CT/Ojanguren, 1988)
Let $F = \mathbb{C}(x, y, z)$. There exist $f, g, h = h_1 h_2$ such that the class $(f) \cup (g) \cup (h_1) \in H^3(F, \mathbb{Z}/2)$ does not vanish in $H^3(F(Q), \mathbb{Z}/2)$, but becomes unramified on any smooth projective variety $X/\mathbb{C}$ with $\mathbb{C}(X) = F(Q)$. On may choose f, g, h such that the 6-dimension variety X is unirational.

For the proof of this result, one uses residues with respect to rank one discrete valuation on $\mathbb{C}(X)$.

[E. Peyre later produced many examples of unirational varities with $H^i_{nr}(X, \mathbb{Z}/n) \neq 0$ for suitable i. Further recent results by A. Asok.]

1-dimensional version of the argument, with H^1_{nr}
(Abhyankar's Lemma – Ramification eats up ramification)

Let Γ be the curve $y^2 = x(x-1)(x+1)$. The function field $L = \mathbb{C}(\Gamma)$ is a zero-dimensional quadric over $F = \mathbb{C}(x)$. The class $(x) \in F^*/F^{*2} = H^1(F, \mathbb{Z}/2)$ does not vanish in L^*/L^{*2}, because the kernel is $\mathbb{Z}/2.(x(x-1)(x+1))$, and the classes x and $(x(x-1)(x+1))$ have different valuation mod. 2 at the place $x-1$ of $\mathbb{C}(x)$. The class x ist unramified in $H^1(L, \mathbb{Z}/2)$, because its ramification in $\mathbb{C}(x)$ at every place is eaten up by the ramfication of $x(x-1)(x+1)$.

Examples of Artin-Mumford (1970) : here one uses the group $H^2_{nr}(X, \mathbb{Z}/2)$. As ramification locus one may take a suitable configuration of 10 lines in $\mathbf{P}^2_{\mathbb{C}}$. In CT/Ojanguren, for H^3_{nr}, we use a suitable configuration of 36 planes in $\mathbf{P}^3_{\mathbb{C}}$.

Theorem (2010, CT/Voisin)
There exists a smooth, projective variety X *of dimension 3 with* $H^i(X, O_X) = 0$ *for any* $i > 0$*, but with* $Z^4(X)\{\mathrm{tors}\} \neq 0$.

These varieties X admit a fibration $X \to \mathbf{P}^1$ whose general fibre is a $K3$-surface.
The index $I(X_{eta}/\mathbb{C}(\mathbf{P}^1)) \neq 1$.

Proof rather elaborate. In principle the argument is in the same spirit as Kollár's specialization argument.

Varieties for which $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$ and $Z^4(X) = 0$.

Two theorems proven using methods of algebraic K-theory.

Theorem 1 (1988) *Let* $X \to S$ *be a dominant morphism of smooth, projective, complex varieties,* $dim(S) = 2$ *whose generic fibre is a conic. Then* $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) = 0$.
Hence also $Z^4(X) = 0$.

Theorem 2 *Let* $X \to \Gamma$ *be be a dominant morphism of smooth, projective, complex varieties,* $dim(\Gamma) = 1$, *geometric generic fibre a rational surface. Then* $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z})$ *and* $Z^4(X) = 0$.

Proof of theorem 1. The generic fibre X_η is a conic C over $F = \mathbb{C}(S)$. Let D/F be the associated quaternion algebra. One may restrict attention to 2-torsion. One has $H^3_{nr}(X/\mathbb{C}, \mathbb{Z}/2)) \subset H^3_{nr}(C/F, \mathbb{Z}/2)$. One considers the localisation sequence for étale cohomology and the Leray spectral sequence for $C \to Spec(F)$. From this follows $F^*/Nrd(D^*) \simeq H^3_{nr}(C, \mathbb{Z}/2)$. But $F^*/Nrd(D^*) \hookrightarrow H^3(F, \mathbb{Z}/2)$ (Merkurjev-Suslin) and $H^3(F, \mathbb{Z}/2) = 0$, since $\mathrm{coh.dim}(\mathbb{C}(S)) \leq 2$.

Proof of Theorem 2. Using methods of algebraic K-Theory one shows :

Theorem (B. Kahn (1996) $+ \varepsilon$) *Let F be a field of char. zero and cohomological dimension ≤ 1. Let V/F be a smooth projective surface. Let $\overline{F}$ be an algebraic closure of F, G the absolute Galois group of F and $\overline{V} = V \times_F \overline{F}$. If $H^2(V, O_V) = 0$ and the third integral cohomology of $\overline{V}$ has no torsion, then one has an exact sequence*

$$0 \to CH^2(V) \to CH^2(\overline{V})^G \to H^3_{nr}(V, \mathbb{Q}/\mathbb{Z}(2)) \to 0.$$

If the surface $\overline{V}$ is rational, then $deg : CH^2(\overline{V}) \stackrel{\simeq}{\to} \mathbb{Z}$, hence

$$\mathbb{Z}/I(V/F) \simeq H^3_{nr}(V, \mathbb{Q}/\mathbb{Z}(2)),$$

where $I(V/F)$ is the index of V.
The F-birational classification of geometrically F-rational surfaces (Iskovskikh) (or the Graber-Harris-Starr theorem) implies : over $F = \mathbb{C}(\Gamma)$, we have $V(F) \neq \emptyset$, hence $I(V/F) = 1$. From this, one deduces $H^3_{nr}(V, \mathbb{Q}/\mathbb{Z}(2) = 0$.

If V/F is the generic fibre of $X \to \Gamma$, then $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) \subset H^3_{nr}(V, \mathbb{Q}/\mathbb{Z}(2) = 0$. This proves Theorem 2.

Theorems 1 and 2 are but special cases of a theorem which is proven using Hodge theoretic methods (infinitesimal variations of Hodge structures).

Theorem (Voisin, 2004) *Let X be a smooth, projective uniruled threefold. Then $Z^4(X) = 0$.*

Hence, by the main theorem in this lecture, $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$.

[C. Voisin also proves $Z^4(X) = 0$ for Calabi-Yau threefolds.]

Open problems

Let X be a smooth projective variety, $d = \dim(X)$.

(Voisin, 2004) If X is rationally connected, is $Z^{2d-2}(X) = 0$?

If X is rationally connected, $d = 4$ or $d = 5$, is the finite group $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = Z^4(X)$ zero ?

There are parallel problems for varieties over a finite field. The analogue of the Hodge conjecture is the Tate conjecture.

Let me mention just one specific problem.

Let $X/\mathbb{F}$ be a smooth projective variety of dimension 3 over a finite field $\mathbb{F}$ of characteristic p. Let $\ell \neq p$ be a prime. If X is geometrically covered by the product of a surface and $\mathbf{P}^1$, is the group $H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$?

For threefolds $X/\mathbb{F}$ which admit a conic bundle structure over a surface, this was recently proved by Parimala and Suresh.