On the integral Tate conjecture for 1-cycles on the product of a curve and a surface over a finite field

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Let X be a smooth projective (geom. connected) variety over a finite field $\mathbb{F}$ of char. p. Unless otherwise mentioned, cohomology is étale cohomology.
We have $CH^1(X) = \mathrm{Pic}(X) = H^1_{Zar}(X, \mathbb{G}_m) = H^1(X, \mathbb{G}_m)$.
For r prime to p, the Kummer exact sequence of étale sheaves associated to $x \mapsto x^r$

$$1 \to \mu_r \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$

induces a map $\mathrm{Pic}(X)/r = H^1(X, \mathbb{G}_m)/r \to H^2(X, \mu_r)$.
Let $r = \ell^n$, with $\ell \neq p$. Passing over to the limit in n, we get the ℓ-adic cycle class map

$$\mathrm{Pic}(X) \otimes \mathbb{Z}_\ell \to H^2(X, \mathbb{Z}_\ell(1)).$$

Around 1960, Tate conjectured

(T^1) *For any smooth projective $X/\mathbb{F}$, the map*

$$\mathrm{Pic}(X) \otimes \mathbb{Z}_\ell \to H^2(X, \mathbb{Z}_\ell(1))$$

is surjective.

Via the Kummer sequence, one easily sees that this is equivalent to the finiteness of the ℓ-primary component $\mathrm{Br}(X)\{\ell\}$ of the Brauer group $\mathrm{Br}(X) := H^2_{et}(X, \mathbb{G}_m)$ (finiteness which itself is related to the conjectured finiteness of Tate-Shafarevich groups of abelian varieties over a global field $\mathbb{F}(C)$).
The conjecture is known for geometrically separably unirational varieties (easy), for abelian varieties (Tate) and for most $K3$-surfaces.

For any $i \geq 1$, there is an ℓ-adic **cycle class map**

$$CH^i(X) \otimes \mathbb{Z}_\ell \to H^{2i}(X, \mathbb{Z}_\ell(i))$$

from the Chow groups of codimension i cycles to the projective limit of the (finite) étale cohomology groups $H^{2i}(X, \mu_{\ell^n}^{\otimes i})$, which is a $\mathbb{Z}_\ell$-module of finite type.

For $i > 1$, Tate conjectured that the cycle class map

$$CH^i(X) \otimes \mathbb{Q}_\ell \to H^{2i}(X, \mathbb{Q}_\ell(i)) := H^{2i}(X, \mathbb{Z}_\ell(i)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is surjective. Very little is known.

For $i > 1$, ne may give examples where the statement with $\mathbb{Z}_\ell$ coefficients does not hold. However, for X of dimension d, it is unknown whether the *integral Tate conjecture* $T_1 = T^{d-1}$ *for 1-cycles* holds :

(T_1) *The map* $CH^{d-1}(X) \otimes \mathbb{Z}_\ell \to H^{2d-2}(X, \mathbb{Z}_\ell(d-1))$ *is onto.*

Under T^1 for X, the cokernel of the above map is finite.
Under T^1 for all surfaces, a limit version of T_1, over an algebraic closure of $\mathbb{F}$, holds for any X (C. Schoen 1998).

For $d = 2$, $T_1 = T^1$, original Tate conjecture.

For arbitrary d, the integral Tate conjecture for 1-cycles holds for X of any dimension $d \geq 3$ if it holds for any X of dimension 3. This follows from the Bertini theorem, the purity theorem, and the affine Lefschetz theorem in étale cohomology.

For X of dimension 3, some nontrivial cases have been established.
• X is a conic bundle over a geometrically ruled surface (Parimala and Suresh 2016).
• X is the product of a curve of arbitrary genus and a geometrically rational surface (Pirutka 2016).

For smooth projective varieties X over $\mathbb{C}$, there is a formally parallel surjectivity question for cycle maps

$$CH^i(X) \to Hdg^{2i}(X, \mathbb{Z})$$

where $Hdg^{2i}(X, \mathbb{Z}) \subset H^{2i}_{Betti}(X, \mathbb{Z})$ is the subgroup of rationally Hodge classes. The surjectivity with $\mathbb{Q}$-coefficients is the famous Hodge conjecture. With integral coefficients, several counterexamples were given, even with $dim(X) = 3$ and 1-cycles. A recent counterexample involves the product $X = E \times S$ of an elliptic curve E and an Enriques surface. For fixed S, provided E is "very general", the integral Hodge conjecture fails for X (Benoist-Ottem). The proof uses the fact that the torsion of the Picard group of an Enriques surface is nontrivial, it is $\mathbb{Z}/2$.

It is reasonable to investigate the Tate conjecture for cycles of codimension $i \geq 2$ assuming the original Tate conjecture :

T^1_{all} : The surjectivity conjecture T^1 is true for cycles of codimension 1 over any smooth projective variety.

Theorem (CT-Scavia 2020). *Let $\mathbb{F}$ be a finite field, $\overline{\mathbb{F}}$ a Galois closure, $G = Gal(\overline{\mathbb{F}}/\mathbb{F})$. Let $E/\mathbb{F}$ be an elliptic curve and $S/\mathbb{F}$ be an Enriques surface. Let $X = E \times_{\mathbb{F}} S$. Let ℓ be a prime different from $p = \mathrm{char.}(\mathbb{F})$. Assume T^1_{all}.*
If $\ell \neq 2$, or if $\ell = 2$ but $E(\mathbb{F})$ has no nontrivial 2-torsion, then the map $CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$ is onto.

We actually prove a general theorem, for the product $X = C \times S$ of a curve C and a surface S which is *geometrically CH_0-trivial*, which here means :

Over **any** *algebraically closed field extension Ω of $\mathbb{F}$, the degree map $CH_0(S_\Omega) \to \mathbb{Z}$ is an isomorphism.*

In that case $\mathrm{Pic}(S_\Omega)$ is a finitely generated abelian group.

For $\overline{\mathbb{F}}$ a Galois closure of $\mathbb{F}$, $G = \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$, and J the jacobian of C, still assuming T^1_{all}, we prove that $CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$ is onto under the condition $\mathrm{Hom}_G(\mathrm{Pic}(S_{\overline{\mathbb{F}}})\{\ell\}, J(\overline{\mathbb{F}})) = 0$.

We do not know whether this condition is necessary.

The case $\mathrm{Pic}(S_{\overline{\mathbb{F}}})\{\ell\} = 0$ is a theorem of A. Pirutka (2016).

In the rest of the talk, I shall sketch some ingredients of the proof.

Let M be a finite Galois-module over a field k. Given a smooth, projective, integral variety X/k with function field $k(X)$, and $i \geq 1$ an integer, one lets

$$H^i_{nr}(k(X), M) := \mathrm{Ker}[H^i(k(X), M) \to \oplus_{x \in X^{(1)}} H^{i-1}(k(x), M(-1))]$$

Here $k(x)$ is the residue field at a codimension 1 point $x \in X$, the cohomology is Galois cohomology of fields, and the maps on the right hand side are "residue maps".
One is interested in $M = \mu_{\ell^n}^{\otimes j} = \mathbb{Z}/\ell^n(j)$, hence $M(-1) = \mu_{\ell^n}^{\otimes (j-1)}$, and in the direct limit $\mathbb{Q}_\ell/\mathbb{Z}_\ell(j) = \lim_j \mu_{\ell^n}^{\otimes j}$, for which the cohomology groups are the limit of the cohomology groups.

The group $H^1_{nr}(k(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell) = H^1_{et}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ classifies ℓ-primary cyclic étale covers of X.

The group

$$H^2_{nr}(k(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) = \mathrm{Br}(X)\{\ell\}$$

turns up in investigations on the original Tate conjecture for divisors.
As already mentioned, its finiteness for a given X is equivalent to the ℓ-adic Tate conjecture for codimension 1 cycles on X.

The group $H^3_{nr}(k(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ is mysterious. It turns up when investigating cycles of codimension 2.

For $k = \mathbb{F}$ a finite field, examples of X with $dim(X) \geq 5$ and $H^3_{nr}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \neq 0$ are known (Pirutka 2011).

Open questions :

Is $H^3_{nr}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ of cofinite type ?

Is $H^3_{nr}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ finite ?

Is $H^3_{nr}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$ if $dim(X) = 3$?
[Known for a conic bundle over a surface, Parimala–Suresh 2016]

Theorem (Kahn 2012, CT-Kahn 2013) *For $X/\mathbb{F}$ smooth, projective of arbitrary dimension, the torsion subgroup of the (conjecturally finite) group*

$$\mathrm{Coker}[CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))]$$

is isomorphic to the quotient of $H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ by its maximal divisible subgroup.

There is an analogue of this for the integral Hodge conjecture (CT-Voisin 2012).

A **basic exact sequence** (CT-Kahn 2013). Let $\overline{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$, let $\overline{X} = X \times_{\mathbb{F}} \overline{\mathbb{F}}$ and $G = \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$.

For $X/\mathbb{F}$ a smooth, projective, geometrically connected variety over a finite field, long exact sequence

$$0 \to \mathrm{Ker}[CH^2(X)\{\ell\} \to CH^2(\overline{X})\{\ell\}] \to H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)))$$

$$\to \mathrm{Ker}[H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \to H^3_{\mathrm{nr}}(\overline{\mathbb{F}}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))]$$

$$\to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\} \to 0.$$

The proof relies on early work of Bloch and on the Merkurjev-Suslin theorem (1983). Via Deligne's theorem on the Weil conjectures, one has

$$H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) = H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tors}})$$

and this is finite.

For X a curve, all groups in the sequence are zero.
For X a surface, $H^3(\overline{\mathbb{F}}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.
For $X/\mathbb{F}$ a surface, one also has

$$H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0.$$

This vanishing was remarked in the early stages of higher class field theory (CT-Sansuc-Soulé, K. Kato, in the 80s). It uses a theorem of S. Lang, which relies on Tchebotarev's theorem.

For our 3-folds $X = C \times S$, S as above, we have an isomorphism of finite groups

$$\mathrm{Coker}[CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))] \simeq H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)),$$

and, under the assumption T^1 for all surfaces over a finite field, a theorem of Chad Schoen implies $H^3_{\mathrm{nr}}(\overline{\mathbb{F}}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.

Under T^1 for all surfaces, for our threefolds $X = C \times S$ with S geometrically CH_0-trivial, we thus have an exact sequence of finite groups

$$0 \to \mathrm{Ker}[CH^2(X)\{\ell\} \to CH^2(\overline{X})\{\ell\}] \to H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tors}})$$

$$\xrightarrow{\theta_X} H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\} \to 0.$$

Under T^1 for all surfaces, for our threefolds $X = C \times S$ with S geometrically CH_0-trivial, the surjectivity of

$$CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$$

(integral Tate conjecture) is therefore equivalent to the combination of two hypotheses :

Hypothesis 1
The composite map

$$\rho_X : H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \to H^3(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$$

of θ_X and $H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \subset H^3(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ vanishes.

Hypothesis 2 $\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\} = 0$.

Hypothesis 1 is equivalent to each of the following hypotheses :

Hypothesis 1a. The (injective) map from

$$\mathrm{Ker}[CH^2(X)\{\ell\} \to CH^2(\overline{X})\{\ell\}]$$

to the (finite) group

$$H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \simeq H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{tors})$$

is onto.

Hypothesis 1b. For any $n \geq 1$, if a class $\xi \in H^3(X, \mu_{\ell^n}^{\otimes 2})$ vanishes in $H^3(\overline{X}, \mu_{\ell^n}^{\otimes 2})$, then it vanishes after restriction to a suitable Zariski open set $U \subset X$.

For all we know, these hypotheses 1,1a,1b could hold for any smooth projective variety X over a finite field.

For X of dimension > 2 , we do not see how to establish them directly – unless of course when the finite group $H^3(\overline{X}, \mathbb{Z}_\ell(2))_{tors}$ vanishes.
The group $H^3(\overline{X}, \mathbb{Z}_\ell(1))_{tors}$ is the nondivisible part of the ℓ-primary Brauer group of $\overline{X}$.
The finite group $H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(1))_{tors})$ is thus the most easily computable group in the 4 terms exact sequence.
For $\mathrm{char}(\mathbb{F}) \neq 2$, $\ell = 2$, $X = E \times_F S$ product of an elliptic curve E and an Enriques surface S, one finds that this group is $E(\mathbb{F})[2] \oplus \mathbb{Z}/2$.

Discussion of Hypothesis 1 : *The map* $\rho_X : H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \to H^3(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ *vanishes.*

This map is the composite of the Hochschild-Serre map

$$H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \to H^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)))$$

with the restriction to the generic point of X.

We prove :

Theorem. *Let Y and Z be two smooth, projective geometrically connected varieties over a finite field $\mathbb{F}$. Let $X = Y \times_{\mathbb{F}} Z$. Assume that the Néron-Severi group of $\overline{Z}$ is free with trivial Galois action. If the maps ρ_Y and ρ_Z vanish, then so does the map ρ_X.*

On must study $H^1(\mathbb{F}, H^2(\overline{X}, \mu_{\ell^n}^{\otimes 2}))$ under restriction from X to its generic point.
As may be expected, the proof uses a Künneth formula, along with standard properties of Galois cohomology of a finite field.
As a matter of fact, it is an unusual Künneth formula, with coefficients $\mathbb{Z}/\ell^n$, $n > 1$. That it holds for H^2 of the product of two smooth, projective varieties over an algebraically closed field, is a result of Skorobogatov and Zarhin (2014), who used it in an other context (the Brauer-Manin set of a product).

Corollary. *For the product X of a surface and arbitrary many curves, the map ρ_X vanishes.*
This establishes Hypothesis 1 for the 3-folds $X = C \times_F S$ under study.

Discussion of Hypothesis 2 :

$\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\} = 0.$

For X of dimension at least 5, A. Pirutka gave counterexamples.

Here we restrict to the special situation : C is a curve, S is geometrically CH_0-trivial surface, and $X = C \times_{\mathbb{F}} S$.
One lets $K = \mathbb{F}(C)$ and $L = \overline{\mathbb{F}}(C)$.
On considers the projection $X = C \times S \to C$, with generic fibre the K-surface S_K. Restriction to the generic fibre gives a natural map from

$$\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\}$$

to

$$\mathrm{Coker}[CH^2(S_K) \to CH^2(S_L)^G]\{\ell\}.$$

Using the hypothesis that S is geometrically CH_0-trivial, which implies $b_1 = 0$ and $b_2 - \rho = 0$ (Betti number b_i, rank ρ of Néron-Severi group), one proves :

Theorem. *The natural, exact localisation sequence*

$$\mathrm{Pic}(\overline{C}) \otimes \mathrm{Pic}(\overline{S}) \to CH^2(\overline{X}) \to CH^2(S_L) \to 0.$$

may be extended on the left with a finite p-group.

To prove this, we use correspondences on the product $X = C \times S$, over $\overline{\mathbb{F}}$.
We use various pull-back maps, push-forward maps, intersection maps of cycle classes :

$$\mathrm{Pic}(C) \otimes \mathrm{Pic}(S) \to \mathrm{Pic}(X) \otimes \mathrm{Pic}(X) \to CH^2(X)$$

$$CH^2(X) \otimes \mathrm{Pic}(S) \to CH^2(X) \otimes \mathrm{Pic}(X) \to CH^3(X) = CH_0(X) \to CH_0(C)$$

$$\mathrm{Pic}(C) \otimes \mathrm{Pic}(S) \to CH^2(X) = CH_1(X) \to CH_1(S) = \mathrm{Pic}(S)$$

Not completely standard properties of G-lattices for $G = \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ applied to the (up to p-torsion) exact sequence of G-modules

$$0 \to \mathrm{Pic}(\overline{C}) \otimes \mathrm{Pic}(\overline{S}) \to CH^2(\overline{X}) \to CH^2(S_L) \to 0$$

then lead to :

Theorem. *The natural map from* $\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\}$ *to* $\mathrm{Coker}[CH^2(S_K) \to CH^2(S_L)^G]\{\ell\}$ *is an isomorphism.*
(Recall $K = \mathbb{F}(C)$ and $L = \overline{\mathbb{F}}(C)$.)

One is thus left with controlling this group. Under the CH_0-triviality hypothesis for S, it coincides with

$$\mathrm{Coker}[CH^2(S_K)\{\ell\} \to CH^2(S_L)\{\ell\}^G].$$

At this point, for a geometrically CH_0-trivial surface over $L = \overline{\mathbb{F}}(C)$, which is a field of cohomological dimension 1, like $\mathbb{F}$, using the K-theoretic mechanism, one may produce an exact sequence parallel to the basic four-term exact sequence over $\mathbb{F}$ which we saw at the beginning. In the particular case of the constant surface $S_L = S \times_{\mathbb{F}} L$, the left hand side of this sequence gives an injection

$$0 \to A_0(S_L)\{\ell\} \to H^1_{Galois}(L, H^3(\overline{S}, \mathbb{Z}_\ell(2)\{\ell\})$$

where $A_0(S_L) \subset CH^2(S_L)$ is the subgroup of classes of zero-cycles of degree zero on the L-surface S_L.

Study of this situation over completions of $\overline{\mathbb{F}}(C)$ (Raskind 1989) and a good reduction argument in the weak Mordell-Weil style, plus a further identification of torsion groups in cohomology of surfaces over an algebraically closed field then yield a Galois embedding

$$A_0(S_L)\{\ell\} \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(\overline{S})\{\ell\}, J(C)(\overline{\mathbb{F}})),$$

hence an embedding

$$A_0(S_L)\{\ell\}^G \hookrightarrow \mathrm{Hom}_G(\mathrm{Pic}(\overline{S})\{\ell\}, J(C)(\overline{\mathbb{F}})).$$

If this group $\mathrm{Hom}_G(\mathrm{Pic}(\overline{S})\{\ell\}, J(C)(\overline{\mathbb{F}}))$ vanishes, then

$$\mathrm{Coker}[CH^2(S_K)\{\ell\} \to CH^2(S_L)\{\ell\}^G] = 0$$

hence

$$\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\} = 0,$$

which is Hypothesis 2, and completes the proof of the theorem :

Theorem (CT/Scavia) *Let $\mathbb{F}$ be a finite field, $G = \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$. Let ℓ be a prime, $\ell \neq \mathrm{char.}(\mathbb{F})$. Let C be a smooth projective curve over $\mathbb{F}$, let $J/\mathbb{F}$ be its jacobian, and let $S/\mathbb{F}$ be a smooth, projective, geometrically CH_0-trivial surface. Let $X = C \times_{\mathbb{F}} S$.*
Assume the usual Tate conjecture for codimension 1 cycles on varieties over a finite field.
Under the assumption

$$(**) \qquad \mathrm{Hom}_G(\mathrm{Pic}(S_{\overline{\mathbb{F}}})\{\ell\}, J(\overline{\mathbb{F}})) = 0,$$

the cycle class map $CH^2(X) \otimes \mathbb{Z}_\ell \to H^4_{et}(X, \mathbb{Z}_\ell(2))$ is onto.

Basic question : Is the assumption $(**)$ necessary ?

Concrete case

Let $p \neq 2$ and let E be an elliptic curve defined by the affine equation $y^2 = P(x)$ with $P \in \mathbb{F}[x]$ a separable polynomial of degree 3.
Let $S/\mathbb{F}$ be an Enriques surface. Then $\mathrm{Pic}(S_{\overline{\mathbb{F}}})_{tors} = \mathbb{Z}/2$, automatically with trivial Galois action.
The assumption $(**)$ reads : $E(\mathbb{F})[2] = 0$, which translates as : $P \in \mathbb{F}[x]$ is an *irreducible* polynomial. .

Thus, for $p \neq 2$ and $P(x) \in \mathbb{F}[x]$ *reducible*, the integral Tate conjecture $T_1(X, \mathbb{Z}_2)$ for $X = E \times_{\mathbb{F}} S$ remains open.

What we have done should be confronted with the situation over the complex field, which actually stimulated our work.
For X a smooth projective variety over $\mathbb{C}$ there is a cycle map

$$CH^i(X) \to Hdg^{2i}(X, \mathbb{Z}) \subset H^4_{Betti}(X, \mathbb{Z}).$$

whose cokernel is conjecturally finite (Hodge conjecture). If the map is onto, one says the integral Hodge conjecture holds.

Theorem (Benoist-Ottem 2018). *Let S be an Enriques surface over $\mathbb{C}$. Then the integral Hodge conjecture for codimension 2 cycles fails for the product $X = E \times S$ of S and a "very general" elliptic curve.*
The proof uses a degeneration technique of E to $\mathbb{G}_m$ which one may already find in a paper of Gabber (2002).

Using earlier joint work with C. Voisin on the connexion between the integral Hodge conjecture for codimension 2 cycles and unramified H^3, I could extend the result of Benoist-Ottem. In particular :

Theorem (CT, 2018). *Let Y be a smooth projective variety over $\mathbb{C}$ with $\mathrm{Br}(Y) \neq 0$. Assume that Y is a geometrically CH_0-trivial variety. Then there exists an elliptic curve E such that the integral Hodge conjecture for codimension 2 cycles fails on $X = E \times Y$. If Y is defined over a number field, it fails for X if the j-invariant of E is transcendental.*