Local-global principle for constant reductive groups over arithmetic curves

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After joint works with

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Semi-global field
R a complete dvr, K its field of fractions, κ its residue field.
Example of original interest : K a p-adic field, κ finite residue field.
X/K a smooth, projective, geometrically integral curve.
$F = K(X)$ is called a semi-global field.
Such a curve admits regular, projective integral models $\mathcal{X}/R$.
A normal crossings model (NC model) of such a curve X/K is a 2-dimensional, regular, projective scheme $\mathcal{X}/R$ with generic fibre X/K and with special fibre Y/κ such that Y_{red} is a union of regular connected curves over κ which intersect transversally. Such models exist.

Valuations and completions of a semi-global field.

$\Omega_{\mathcal{X}}$ denotes the set of codimension one points of $\mathcal{X}$.
Two types of such points : closed points of the generic fibre X/K (residue field finite extension of K) and generic points of the components of the special fibre (residue field function field of a curve over κ). To any such a point is associated a discrete valuation v. We let F_v denote the completion of F wrt to v.

$\Omega_F = \bigcup_{\mathcal{X}} \Omega_{\mathcal{X}}$ for all NC models $\mathcal{X}/R$ of X/K.

[In this talk we do not consider other valuations of F.]

Let G/F be a reductive group.

We are interested in

$$Ш_{\mathcal{X}}(F, G) := \mathrm{Ker}[H^1(F, G) \to \prod_{v \in \Omega_{\mathcal{X}}} H^1(F_v, G)]$$

and

$$Ш(F, G) = \mathrm{Ker}[H^1(F, G) \to \prod_{v \in \Omega_F} H^1(F_v, G)]$$

which express the possible lack of a local-global principle for principal homogeneous spaces under G over F

Motivation

Analogy with local-global problems over number fields

Natural intermediate problem on the way to local-global problems for function fields of varieties over number fields

Already presents challenges :
• Find classes of groups G/F such that $\text{Ш}(F, G) = 1$
• Produce counterexamples to the local-global counterexamples in this context.
Many results for F/K with K a p-adic field, cf. Parimala's talk.

The HHK (Harbater, Hartmann, Krashen) method (2009) :
The patching set-up

R a complete dvr, κ its residue field, K its fraction field, $t \in R$ a uniformizing parameter.
$\mathcal{X}/R$ a regular, proper, integral curve over R.
$F = K(\mathcal{X})$ function field, referred to as a *semi-global field*
Y/κ the special fibre
$Y^{red} = \cup_{i \in I} Y_i$ with each Y_i/κ smooth. One assumes normal crossings.
$\mathcal{P}$ a finite set of closed points of Y containing all the singular points of Y and at least one point of each component Y_i.

For each connected component U of $Y^{red} \setminus \mathcal{P}$, one denotes $R_U \subset F$ the ring of functions regular on U, then $\hat{R}_U$ its t-completion, and F_U the field of fractions of $\hat{R}_U$. There is a surjective map $\hat{R}_U \to \kappa[U]$.
For $P \in Y \subset \mathcal{X}$ one lets F_P denote the quotient of the complete local ring $\hat{R}_P = O_{\mathcal{X},P}$.
For a closed point $P \in X$ in the closure of U, one considers the local ring of $\hat{R}_P = O_{\mathcal{X},P}$ at the codimension 1 point defined by U, completes it, and denotes $F_{U,P}$ the field of fractions of that dvr. One calls such a pair (U,P) a branch. There are inclusions $F_U \subset F_{U,P}$ and $F_P \subset F_{U,P}$. The field $F_{U,P}$ is in a sense built out of the fields F_U and F_P.
The original field $F = K(\mathcal{X})$ is the inverse limit of the entire system $\{F_U, F_P, F_{U,P}\}_{P \in \mathcal{P}}$.

Simplest case $\mathcal{X} = \mathbb{P}^1_{\kappa[[t]]}$.
$F = \kappa((t))(x)$
$U = \mathrm{Spec}\,(\kappa[x^{-1}]) = \mathbb{A}^1_\kappa \subset \mathbb{P}^1_\kappa$
$P = \mathbb{P}^1_\kappa \setminus \mathbb{A}^1_\kappa$

$F_P = \kappa((t,x))$
F_U is the field of fractions of $\kappa[x^{-1}][[t]]$; this is a subfield of $\kappa(x)((t))$.
$F_{U,P} = \kappa((x))((t))$.

$F_U \subset F_{U,P}$
$F_P \subset F_{U,P}$.

Let G/F be a linear algebraic group, $\mathcal{X}/R$ and $\mathcal{P}$ as above, let $Ш_{\mathcal{P}}(F, G)$ be the kernel of the finite product of maps :

$$H^1(F, G) \to \prod_P H^1(F_P, G) \times \prod_U H^1(F_U, G).$$

Theorem (HHK 2015). *There is a bijection of pointed sets between* $Ш_{\mathcal{P}}(F, G)$ *and the double coset*

$$\prod_P G(F_P) \Big\backslash \prod_{U,P} G(F_{U,P}) \Big/ \prod_U G(F_U)$$

This comes from a closer analysis of the proof of :

Theorem (HHK 2009) *Let F be an arbitrary semi-global field. If G is a connected reductive group and its underlying F-variety is F-***rational***, then this double quotient is reduced to one point.*

This gives local-global statements with respect to the finite set of overfields $\{F_U, F_P\}_{P \in \mathcal{P}}$.
This was used by HHK to reprove and extend the Parimala–Suresh theorem that quadratic forms in 9 variables over F with κ finite are isotropic.

For G/F reductive and $\mathcal{P} \subset \mathcal{X}$ as above, one proves :

$$Ш_{\mathcal{P}}(F, G) \subset Ш(F, G) \subset Ш_{\mathcal{X}}(F, G)$$

If G/F comes from a reductive group over a given regular projective model $\mathcal{X}$ ("G/F has good reduction over $\mathcal{X}$"), one also knows

$$\bigcup_{\mathcal{P}} Ш_{\mathcal{P}}(F, G) = Ш(F, G) = Ш_{\mathcal{X}}(F, G)$$

If G comes from a reductive group over R, we have

$$Ш_{\mathcal{P}}(F, G) = Ш(F, G) = Ш_{\mathcal{X}}(F, G).$$

In the classical case of a number field k, the basic theorem is $Ш(k, G) = 1$ for G a *semisimple simply connected group* over k. And the triviality of $Ш(k, G)$ for G k-rational is ultimately a consequence of that fact.

Question (cf. CPS12). *Let $F = K(X)$ be a an arbitrary semi-global field (no restriction on the residue field κ). If G/F is a semisimple simply connected group, is $Ш(F, G) = 1$?*

For curves X over a p-adic field, i.e. residue field κ finite, this has now been proved in many cases, see Parimala's talk.

In CHHKPS20 (case of tori) and CHHKPS21, we have obtained results on $\text{Ш}(F, G)$ over arbitrary semi-global fields F/K (i.e. arbitrary residue field κ) in the case where the reductive group G/F is obtained by base change $R \to F$ from a reductive group over the complete dvr $R \subset K$.

Theorem A. *Let K be a complete discretely valued field, R its ring of integers. Let $F = K(X)$ be a semi-global field over K and $\mathcal{X}$ a regular projective NC model of F over R. Assume that the residue field κ of R is of characteristic zero ; that the closed fiber Y/κ of $\mathcal{X}$ is reduced ; and that the reduction graph associated to Y is a tree and remains a tree under all finite extensions κ'/κ. Then for any reductive group G over R we have $Ш(F, G) = 1$.*

The proof is rather elaborate. One first proves that a torsor over F with class in $Ш(F, G)$ may be represented by a torsor over $\mathcal{X}$ under G *which is trivial when restricted to the (reduced) closed fibre*. One then invokes a recent result of P. Gille, Parimala and Suresh to conclude that it is trivial over F.

Recall the notion of R-equivalence.
Given a connected algebraic group G over a field k, the set of points $P \in G(k)$ such that there exists an F-morphism $\phi : U \to G$ with U open in $\mathbb{P}^1_k$ and both e_G and P in $\phi(U(k))$ build up a normal subgroup of $G(k)$. The quotient by this subgroup is denoted $G(k)/\mathrm{R}$.

For a connected reductive group G/k and $cd(k) \leq 1$ we have $G(k)/\mathrm{R} = 1$.

There are *tori* T over a field k with $cd(k) = 2$ and $T(k)/\mathrm{R} \neq 1$.

For G/k semisimple simply connected, $G(k)/\mathrm{R} = 1$ if $cd(k) = 2$ is known in many cases. Whether $G(k)/\mathrm{R} = 1$ if $cd(k) = 3$ is an open question. There exist G/k with $cd(k) = 4$ and $G(k)/\mathrm{R} \neq 1$ (see below).

Theorem B. *Let K be a complete discretely valued field, R its ring of integers. Let $F = K(X)$ be a semi-global field over K and $\mathcal{X}$ a regular projective NC model of F over R. Let G be a reductive group over R. If the closed fiber Y/κ of $\mathcal{X}/R$ is reduced and consists of copies of $\mathbb{P}^1_\kappa$ meeting at κ-points and forming m cycles, and if char(κ) is not one of the bad primes for the reductive group G_κ then $Ш(F, G)$ is in bijection with the quotient of $(G(\kappa)/\mathrm{R})^m$ by simultaneous conjugation by $G(\kappa)$:*

$$^g(g_1, \ldots, g_m) := (gg_1g^{-1}, \ldots, gg_mg^{-1}).$$

If $G(\kappa)/\mathrm{R}$ is commutative, this quotient is nothing but $(G(\kappa)/\mathrm{R})^m$.

Using Theorem B, one gets a negative answer to the above question on semisimple simply connected groups :

Theorem (CHHKPS21) *There exists a field* κ *of cohomological dimension 4, a semisimple simply connected group* G *over* κ *and a geometrically connected curve* X *over* $K = \kappa((t))$*, with function field* F*, such that* $Ш(F, G) \neq 1$.

More precisely, we may take
$\kappa = \mathbb{C}((a))((b))((c))((d))$
$R = \kappa[[t]]$, $K = \kappa((t))$
$D = (a, b) \otimes_{\kappa} (c, d)$ (tensor product of quaternion algebras) and
$G = SL(D)/\kappa$ defined by equation $\mathrm{Nrd}_D(\xi) = 1$
X/K a curve with $\mathcal{X}/R$ a regular model with special fibre Y/κ consisting of a triangle of $\mathbb{P}^1_{\kappa}$'s intersecting transversally in rational points.

To produce the required counterexamples to the local-global principle for a simply connected group over the field $F = \mathbb{C}((a))((b))((c))((d))((t))(X)$, it remains to recall the following results from the 70s, which gives a group $G = SL(D)$ over κ with $G(\kappa)/\mathrm{R}$ commutative and $G(\kappa)/\mathrm{R} \neq 1$.

(Platonov) Let $\kappa = \mathbb{C}((a))((b))((c))((d))$. Let $D = (a, b) \otimes (c, d)$ (tensor product of two quaternion algebras). Then the quotient $SK_1(D)$ of D^{*1} (elements of reduced norm 1) by the commutator subgroup of D^* is non-zero.

(Voskresenskiĭ, using a result of Platonov) Let D/k be a central simple algebra over a field k. Let $G = SL(D)$. Then $SK_1(D) = G(k)/\mathrm{R}$.

On the proof of Theorem B

For an arbitrary connected reductive group G over F we have

$$\text{Ш}_{\mathcal{P}}(F,G) \simeq \prod_{P} G(F_P) \Big\backslash \prod_{U,P} G(F_{U,P}) \Big/ \prod_{U} G(F_U)$$

The pointed set $\text{Ш}_{\mathcal{P}}(F,G)$ admits the pointed double coset

$$\prod_{P} G(F_P)/\mathrm{R} \Big\backslash \prod_{U,P} G(F_{U,P})/\mathrm{R} \Big/ \prod_{U} G(F_U)/\mathrm{R}$$

as a quotient.

Now assume that G *is induced by a reductive group over* R, *also denoted* G, *and that the special fibre* Y/κ *is a (reduced) union of* $\mathbb{P}^1_\kappa$*'s meeting transversally at* κ*-rational points*.
One then proves that there are **specialisation maps** for
R-equivalence classes
$sp_P : G(F_P)/\mathrm{R} \to G(\kappa)/\mathrm{R}$
$sp_U : G(F_U)/\mathrm{R} \to G(\kappa)/\mathrm{R}$
$sp_{U,P} : G(F_{U,P})/\mathrm{R} \to G(\kappa)/\mathrm{R}$
which are compatible with the maps $G(F_P) \to G(F_{U,P})$ and
$G(F_U) \to G(F_{U,P})$,

and one proves that the induced map from

$$\prod_P G(F_P) \Big\backslash \prod_{U,P} G(F_{U,P}) \Big/ \prod_U G(F_U)$$

to

$$\prod_P G(\kappa)/\mathrm{R} \Big\backslash \prod_{U,P} G(\kappa)/\mathrm{R} \Big/ \prod_U G(\kappa)/\mathrm{R}$$

is a bijection.