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Division algebras over p-adic curves. (English)

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This interesting paper has already fostered further work [*D. W. Hoffmann* and *J. Van Geel*, Zeroes and norm groups of quadratic forms over function fields in one variable over a local non-dyadic field, Preprint (1997); *R. Parimala* and *V. Suresh*, Isotropy of quadratic forms over function fields of curves over p-adic fields, Isaac Newton preprint NI98006-AMG (1998)]. The paper itself contains a gap, which was pointed out and simultaneously fixed by O. Gabber in October 1997, during a lecture given by the reviewer. I will give the main statement, present the author's arguments in a slightly different guise, then report on Gabber's contribution.

Theorem *Let l be a prime number, and let p be a prime different from l. Let k be a p-adic field containing an l-th root of unity, and let K be a function field in one variable over k. Given any finite set of central simple algebras $\mathcal{A}_i$, $i = 1, \cdots, m$, over K, each of exponent l in the Brauer group of K, there exist rational functions f and g in K such that the field extension $K(f^{1/l}, g^{1/l})$ splits each of the $\mathcal{A}_i$'s.* (Standard reductions then enable one to show that for K a function field in one variable over an arbitrary p-adic field k, and $\mathcal{A}$ a central simple algebra over K of exponent n prime to p, the index of $\mathcal{A}$ divides n^2. Examples of algebras of exponent n and degree exactly n^2 are given by W. Jacob and J.-P. Tignol in an appendix to the paper.)

Saltman's idea is to use resolution of singularities of two-dimensional excellent schemes (Lipman) as well as embedded resolution of singularities of curves on two-dimensional excellent schemes (Lipman) to reduce to the following situation: There exists a regular, noetherian, integral two-dimensional scheme X, which is projective over the spectrum $S = \mathrm{Spec}(O_k)$ of the ring of integers of k, and there exist two closed regular curves C and E (not necessarily connected) on X, which meet transversally, such that the algebras $\mathcal{A}_i$ have non-trivial residues only at generic points of components of C or E. A first lemma, which may be seen as an easy application of the general results in *K. Kato*'s paper [J. Reine Angew. Math. 366, 142-183 (1986; Zbl 0576.12012)], ensures that at any point $x \in X$, each $\mathcal{A}_i$ may be written as a sum of an unramified element and of elements of one of the following shapes: (u, s), (v, t), (s, t), where u, v denote units in the local ring $\mathcal{O}_{X,x}$, and s, resp. t, denote local equations (possibly units) for C, resp. E, at x. Here and further below, a primitive l-th root of unity $\zeta \in O_k^* \subset k^*$ has been fixed, and for $\alpha, \beta \in K^*$, one sets $(\alpha, \beta) := (\alpha, \beta)_\zeta \in \mathrm{Br}(K)$. Let T be the non-empty finite set of points of X consisting of the generic point of X, the generic points of each component of C and E, and the points of $\mathrm{supp}(C) \cap \mathrm{supp}(E)$. Since X is projective over S, one may find an affine open set U of X such that U contains all points of T. Let A be the semi-local ring associated to this finite set T of points. This is a regular semi-local ring, hence a unique factorization domain. Hence one may find rational functions f, resp. g, on X such that their divisors on $\mathrm{Spec}(A)$ coincide with the restriction of C, resp. E, to Spec(A).

Saltman then considers the extension $L = K(f^{1/l}, g^{1/l})$ and *claims* that each $\mathcal{A}_i$ becomes unramified in L with respect to any rank one discrete valuation ring R centered on X (see the discussion on this *claim* further below). Now there is a classical result: *Let Y/O be a regular, flat, proper (relative) curve over the ring of integers of a p-adic field. Then the Brauer group of Y is trivial.* For a proof of this, see *J. Tate* [*WC-groups over p-adic fields*, Séminaire Bourbaki, Exp. 156 (1958; Zbl 0091.33701)], complemented by *S. Lichtenbaum* [Invent. Math. 7, 120-136 (1969; Zbl 0186.26402)]; for another proof, see *A. Grothendieck* [Le groupe de Brauer III, (2.15) et (3.1), in Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math. 3, 88-188 (1968; Zbl 0198.25901)]. Grothendieck's proof is rather terse; for details on the Picard functor of a singular curve, see *S. Bosch, W. Lütkebohmert, M. Raynaud*, [Néron models, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd. 21, Springer-Verlag (1990; Zbl 0705.14001), section 9.2]. Using this result applied to a regular model Y (proper over O) of the field L, one may then conclude that each $\mathcal{A}_i$ actually vanishes in $L = K(f^{1/l}, g^{1/l})$.

Let us detail the *claim*. Let $\mathrm{div}_X(f) = C + F$ and $\mathrm{div}_X(g) = E + G$. Let $x \in X$ be the point where the discrete valuation ring $R \subset L$ is centered. Let notation be as above, namely let u, v denote units in $\mathcal{O}_{X,x}$, let $s \in \mathcal{O}_{X,x}$ be a local equation for C at x and $t \in \mathcal{O}_{X,x}$ a local equation for E. In order to show that each $\mathcal{A}_i \otimes_K L$ is unramified at R, it is enough to show:

(C) Each of the algebras $(u, s) \otimes_K L$, $(v, t) \otimes_K L$, $(s, t) \otimes_K L$ is unramified at R.

For this, it is enough (but not necessary) to show

($\mathbf{C_1}$) Both s and t belong to the subgroup of the multiplicative group of K^* spanned by $\mathcal{O}^*_{X,x}$, the elements f and g, and the subgroup K^{*l} of l-th powers.

If x does not belong to the support of $C \cup E$, then these assertions are clear. Assume $x \in T$. Then f/s and g/t are units in $\mathcal{O}_{X,x}$, hence **($\mathbf{C_1}$)** holds. Assume $x \in C$, $x \notin T \cup \mathrm{supp}(F)$. Then t and f/s are units at x, hence **($\mathbf{C_1}$)** holds. Similarly, for $x \in E$, $x \notin T \cup \mathrm{supp}(G)$, **($\mathbf{C_1}$)** holds, since s and g/t are units at x. However for $x \in C \cap \mathrm{supp}(F)$ or $x \in E \cap \mathrm{supp}(G)$, the situation is unclear (this is the situation ignored in Saltman's paper, see end of his section 2, which refers to Proposition 1.5).

Gabber suggests two ways out of this difficulty. Both ways involve a non-symmetrical choice of some other rational functions f and g.

First method, assume $l \neq 2$. Take a function f whose divisor on the whole surface X is $C + 2E + F$, where F is a divisor whose support does not contain any of the points in T as above. Let T_1 be the intersection of the support of F with the support of $C + E$. Then choose g a rational function whose divisor is $C + E + G$, where G is a divisor whose support does not contain any of the points in $T \cup T_1$. We may assume $x \in C \cup E$. If x lies in T, then s and g^2/f differ by a unit, and t and f/g differ by a unit, hence **($\mathbf{C_1}$)** holds. If x lies on E but not in $T \cup T_1$, then s is a unit, and f/t^2 is a unit, hence, since l is odd, t is a product of a unit, a power of f and an l-th power: **($\mathbf{C_1}$)** holds. If x lies on C but not in $T \cup T_1$, then t is a unit and f/s is a unit: **($\mathbf{C_1}$)** holds. Assume x belongs to T_1. Then x lies either on C or on E but not on both. At such a point, the divisor of g is equal to $C + E$, and g defines the relevant component of C or E. If x lies on C, then t and g/s are units: if x lies on E, then s and g/t are units. In both cases, **($\mathbf{C_1}$)** holds.

Second method (valid also for $l = 2$). Start with a function f whose divisor is $C+E+F$, where F is a divisor whose support does not contain any of the closed points in T as above. Let T_1 be the intersection of the support of F with the support of $C+E$. The semi-local ring of X at the points of $T \cup T_1$ is a regular unique factorization domain. One may thus find a rational function $g \in K^*$ whose divisor on X is $C+G$, such that the support of G does not contain any point of $T \cup T_1$, and such that moreover g, at any of the finitely many closed points x where the support of F and E meet, is not only a unit but also is a non-l-th power in the (finite) residue field κ_x (note that since O_k contains the l-th roots of unity, so does κ_x). For $x \in T$, g/s and f/gt are units. For $x \in C \cup E$ not in $T \cup T_1$, the function f is a local equation for the relevant component of $C+E$. If x lies on C, then t and f/s are units: if x lies on E, then s and f/t are units. In both cases, $(\mathbf{C_1})$ holds. If x lies in T_1 and on C, then t and g/s are units. Thus in all these cases, $(\mathbf{C_1})$ holds. If x lies in T_1 and on E, then at such a point, s is a unit. To prove **(C)**, it is thus enough to show that for any unit $u \in \mathcal{O}^*_{X,x}$, the algebra $(u,t) \otimes_K L$ is unramified at R. Let v_R be the discrete rank one valuation associated to R and let $\delta_R{:}\,{}_l\mathrm{Br}(L) \to H^1(\kappa_R, \mathbf{Z}/l) = \kappa_R^*/{\kappa_R}^{*l}$ be the residue map associated to R (a primitive l-th root of unity ζ has been fixed in the ring of integers of k, hence the last identification). One has the classical formula $\delta_R((u,t)_\zeta) = \overline{u}^{v_R(t)}$ where $\overline{u}$ is the class in $\kappa_R^*/{\kappa_R}^{*l}$ of the unit u, i.e. the image of the class of u in the residue field κ_x under the natural map $\kappa_x^*/{\kappa_x}^{*l} \to \kappa_R^*/{\kappa_R}^{*l}$. Since g is a unit at x and an l-th power in L, it is an l-th power in R. The natural map $\mathcal{O}_{X,x} \to R$ factorizes through $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}[T]/(T^l - g) \to R$, hence the map $\kappa_x \to \kappa_R$ factorizes through $\kappa_x \to \kappa_x[T]/(T^l - g(x)) \to \kappa_R$. The extension $\kappa_x[T]/(T^l - g(x))$ over κ_x is a *field* extension. Since κ_x is a finite field, any element in κ_x becomes an l-th power in the field $\kappa_x[T]/(T^l - g(x))$. Hence a fortiori $\overline{u} = 1 \in \kappa_R^*/{\kappa_R}^{*l}$. The above formula now implies $\delta_R((u,t)) = 0$, hence **(C)** holds.

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Classification :

$*$16K20 Finite-dimensional division rings
11G20 Curves over finite and local fields
14H25 Arithmetic ground fields (curves)
14F22 Brauer groups of schemes
12E15 Skew fields over special fields
11S45 Algebras and orders, and their zeta functions
14H20 Singularities, local rings
14H05 Algebraic functions
14E15 Global theory of singularities
14G20 p-adic ground fields