Brauer–Manin obstruction and integral points

Jean-Louis Colliot-Thélène (CNRS et Université Paris-Sud)

Simons Symposium
Caneel Bay, St. John, US Virgin Islands
February 27th, 2012

Classical statements.

p prime, $p = x^2 + y^2$ in $\mathbb{Z}$ iff $p \equiv 1 \bmod 4$.

$n \in \mathbb{Z}, n = x^2 + y^2 + z^2$ in $\mathbb{Z}$ iff $n > 0$ and $n \neq 4^r(8m+7)$.

$n \in \mathbb{Z}$, $n = x^2 + y^2 + z^2 + t^2$ in $\mathbb{Z}$ iff $n > 0$

$q(x, y, z, t)$ quaternary quadratic form over $\mathbb{Z}$, indefinite over $\mathbb{R}$, then $n \in \mathbb{Z}$ is represented by q over $\mathbb{Z}$ if and only if any congruence $n \equiv q(x, y, z, t) \bmod m$ has a solution.

$\mathbb{Z} \subset \mathbb{Z}_p \subset \mathbb{Q}_p$

For each prime p, there is a ring $\mathbb{Z}_p$, projective limit of $\mathbb{Z}/p^m$. This is an integral domain. Its field of fractions $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value sending $p^n.(a/b)$ with $a, b \in \mathbb{Z}$ prime to p to $1/p^r$.

A polynomial equation $f(x_1, \ldots, x_r)$ with coefficients in $\mathbb{Z}$ has solutions in $\mathbb{Z}_p$ if and only if any congruence $f(x_1, \ldots, x_r) \equiv 0 \mod p^t$ has a solution.

Let us denote by $\mathcal{X}$ the scheme over $\mathbb{Z}$ defined by $f(x_1, \ldots, x_r) = 0$. For any commutative ring A, $\mathcal{X}(A)$ is the set of solutions of $f(x_1, \ldots, x_r) = 0$ with coordinates in A.

By convention, $\mathbb{Z} \subset \mathbb{Z}_\infty = \mathbb{R}$.

We have a diagonal embedding

$$\mathcal{X}(\mathbb{Z}) \hookrightarrow \prod_{p \cup \infty} \mathcal{X}(\mathbb{Z}_p).$$

The classical results mentioned above may each be rewritten as :

LHS not empty iff RHS not empty.
$\mathcal{X}(\mathbb{Z}) \neq \emptyset$ iff for each prime p (also $p = \infty$) $\mathcal{X}(\mathbb{Z}_p) \neq \emptyset$.
In other words, a local-global principle for existence of integral points.

Very quickly, one realizes that such a local-global principle often fails. There are solutions in all $\mathbb{Z}_p$ and $\mathbb{R}$ but no solutions in $\mathbb{Z}$ for :
The equation $23 = x(x + 7y)$
The system $\{2x - 5y = 1, xt = 1\}$
The equation $1 = 4x^2 + 25y^2$
The equation $1 = 4x^2 - 475y^2$ (harder)
(Over $\mathbb{Q}$, these are projective conics minus 2 rational points or minus 2 congugate points.)
The literature contains such examples as :

For q prime, the equation $q = x^2 + 27y^2$
has solutions in all $\mathbb{Z}_p$ iff $q \equiv 1 \mod 3$
and if so
it has a solution in $\mathbb{Z}$ iff 2 is a cube in $\mathbb{F}_q$.

(Euler, Gauß, see book by D. Cox)

Let $\mathcal{X}_{n,m}$ with $n, m \in \mathbb{N}$, $(n, m) = 1$ be given by

$$m^2x^2 + n^{2k}y^2 - nz^2 = 1.$$

Then $\mathcal{X}_{n,m}(\mathbb{Z}_p) \neq \emptyset$ for all prime p.
$\mathcal{X}_{n,m}(\mathbb{Z}) = \emptyset$ iff
either 2 divides exactly m and $n \equiv 5 \bmod 8$
or 4 divides m and $n \equiv \pm 3 \bmod 8$.

(Schulze-Pillot and Xu)

(Borovoi–Rudnick)

$$(y-x)(9x+7y) = 1-2z^2$$

Solutions (x,y,z) over $\mathbb{Q}$: $(-1/2, 1/2, 1)$ and $(1/3, 0, 1)$ hence solution over each $\mathbb{Z}_p$.

For any solution over $\mathbb{Z}_2$, one has $y - x \equiv \pm 3 \bmod 8$.

If solution over $\mathbb{Z}$, if p prime divides $y-x$ then $1-2z^2 \equiv 0 \bmod p$ so p odd and 2 square mod p. So (complementary law of quadratic reciprocity) $p \equiv \pm 1 \bmod 8$. So $y - x \equiv \pm 1 \bmod 8$. Contradiction.

All the counterexamples to the local-global principle given above may be explained by means of

the integral Brauer–Manin obstruction

This is a variant, formulated by XU Fei and the speaker (2009), of the Brauer–Manin obstruction to the local-global principle for rational points.

The proof of local-global principles, when they hold, often go hand in hand with density properties.

Let us explain this over an arbitrary number field k. Given a finite set S of places of k, one says that *strong approximation off S holds for a k-variety X* if given any finite set T of places containing S, any integral model $\mathcal{X}$ over O_S of X, for each $v \in T \setminus S$ an open set $U_v \subset X(k_v)$ such that the product

$$\prod_{v \in S} X(k_v) \times \prod_{v \in T \setminus S} U_v \times \prod_{v \notin T} \mathcal{X}(O_v)$$

is not empty, then there exists a point of $X(k)$, hence of $\mathcal{X}(O_T)$, in this product.

Such a statement contains a local-global statement for the existence of O_S-integral points for O_S-models of X.
The archetypical example for strong approximation is the additive group $\mathbb{G}_a$ over a number field.
The first case is $\mathbb{Q}$ with $S = \{\infty\}$. The map $\mathbb{Z} \to \prod_{p \neq \infty} \mathbb{Z}_p$ has dense image.
We may replace $\{\infty\}$ by any finite place ℓ, the map $\mathbb{Z}[1/\ell] \to \mathbb{R} \times \prod_{p \neq \ell} \mathbb{Z}_p$ has dense image.

Theorem (Eichler, Kneser, Platonov) Let G be a semisimple simply connected group over a number field. Let v be a place. If for each simple factor H of G, the group $H(k_v)$ is noncompact, then strong approximation off v holds for the group G.

Theorem (Eichler, Kneser) Let q be a nondegenerate quadratic form in $n \geq 4$ variables over a number field, and assume q isotropic over k_v. Then for any $a \in k^\times$ strong approximation off v holds for

$$q(x_1, \ldots, x_n) = a.$$

Let k be a number field, $\mathbb{A}_k$ the ring of adèles. Let X be a k-variety. Let $X(\mathbb{A}_k)$ denote the adèles of X and $\mathrm{Br}(X)$ sthe Brauer group of X.
There is a natural pairing

$$X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$$

$$(\{M_v\}, A) \mapsto \sum_v \mathrm{inv}_v A(M_v),$$

which vanishes on $X(k) \times \mathrm{Br}(X)$ (reciprocity law in class field theory).
One lets

$$X(\mathbb{A}_k)^{\mathrm{Br}(X)}$$

denote the kernel on the left. We thus have $X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}(X)}$.

Assume $X(\mathbb{A}_k) \neq \emptyset$. Let S be a finite set of places. Let $S \subset T$ with T a finite set of places containing all the archimedean places and let $\mathcal{X}/O_T$ be a model of X/k, then for each $v \in T \setminus S$, let $U_v \subset X(k_v)$ be an open set (for the k_v-topology). Assume that in any such situation, if the set

$$[\prod_{v \in S} X(k_v) \times \prod_{v \in T \setminus S} U_v \times \prod_{v \notin T} \mathcal{X}(O_v)]^{\mathrm{Br}(X)}$$

is not empty, then it contains the diagonal image of a point in $X(k)$ (hence in $\mathcal{X}(O_T)$).
One then says that *strong approximation off S with Brauer–Manin condition holds for X*.

In the counterexamples to the local-global principle given above, one has $k = \mathbb{Q}$, $S = \infty$, $\mathcal{X}$ over $\mathbb{Z}$, and one may show :

$$[X(\mathbb{R}) \times \prod_{p \neq \infty} \mathcal{X}(\mathbb{Z}_p)]^{\mathrm{Br} X} = \emptyset,$$

hence (by reciprocity) $\mathcal{X}(\mathbb{Z}) = \emptyset$.

In the Borovoi–Rudnick example $X/\mathbb{Q}$

$$(y - x)(9x + 7y) = 1 - 2z^2$$

on uses the quaternion class $(y - x, 2) = (9x + 7y, 2) \in \mathrm{Br}(X)$.

In the trivial example $\{2x - 5y = 1, xt = 1\}$ over $\mathbb{Q}$, one may use the quaternion algebra $(x, 5)$. The argument shows that the system has no integral point in a field extension $K/\mathbb{Q}$ of odd degree, unramified and totally split at 2 and 5.

Strong approximation off S with Brauer–Manin condition has been established for

X/k homogeneous space of a connected linear algebraic group G/k, with connected geometric stabilizers $\overline{H}$, under a suitable noncompactness hypothesis for G at the places of S :

• CT/Xu 2005-2009, homogeneous spaces of G semisimple simply connected), application to the representation of an integer by a quadratic form of rank at least 3. Here $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite.
• Harari 2008, G a torus, $\overline{H} = 1$, e.g. $x^2 - ay^2 = b$. Here $\mathrm{Br}(X)/\mathrm{Br}(k)$ is infinite.
• Demarche 2011 (G any connected group, $\overline{H} = 1$);
• Borovoi and Demarche 2011 (homogeneous spaces of arbitrary G).

The proofs use :

Hasse principle for semisimple simply connected groups (Kneser, Harder, Chernousov)

Class field theory : Tate-Nakayama duality theorems for tori, non commutative generalization (Kottwiz), extension to duality theorems for complexes of tori (Demarche)

Strong approximation off S for semisimple simply connected groups with noncompacity condition.

An interesting special case.

Let $q(x, y, z)$ be a nondegenerate ternary quadratic form, let $c \in k^\times$. Let Y/k be the k-variety defined by $q(x, y, z) = c$. If it has a k-point it is the quotient of a semisimply connected group by a subtorus.

Let $d = -c.\mathrm{det}(q)$.

If $d \in k^{\times 2}$ then $\mathrm{Br}(Y)/\mathrm{Br}(k) = 0$.

If $d \notin k^{\times 2}$ then $\mathrm{Br}(Y)/\mathrm{Br}(k) = \mathbb{Z}/2$. The group is spanned by one element $\xi \in \mathrm{Br}(Y)$ of order 2, of the shape $(l(x, y, z), d)$ with $l(x, y, z)$ a computable linear affine function.

Over a number field k, if S is a finite set of places containing one place v with q isotropic at v, strong approximation off S with Brauer–Manin condition holds. The condition is given by ξ.

Let Y/k be the k-variety defined by $q(x, y, z) = c$.
We have $\xi \in \mathrm{Br}(Y)[2]$ and $\mathrm{Br}(Y)/\mathrm{Br}(k) = \mathbb{Z}/2.\xi$.

A computation

Over a nondyadic p-adic field, if q is a nondegerate quadratic form over O_v and $c \in O_v$, if $d = -c.\det(q)$ is not a square, then ξ takes two distinct values on the points $(x, y, z) \in Y(O_v)$ with $(x, y, z) = 1$ (primitive points) iff $v(c)$ is odd.

Application. “Spinor Exceptions”.
(Endliche Anzahl von Spinorausnahmen, M. Kneser, A. Weil)

Let $q(x, y, z) \in \mathbb{Z}[x, y, z]$ be indefinite. For each $c \in \mathbb{Z}$ outside a finite set of square classes $E = E(q) \subset \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ the local-global principle holds for integral solutions of $\mathcal{Y}/\mathbb{Z}$ given by the equation

$$q(x, y, z) = c.$$

Indeed, a necessary condition for $[\mathcal{Y}(\mathbb{R}) \times \prod_p \mathcal{Y}(\mathbb{Z}_p)]^{\mathrm{Br} X}$ to be empty is that ξ takes just one value on each $\mathcal{Y}(\mathbb{Z}_p)$. In particular for any prime p which does not divide $2.det(q)$, we must have $v_p(c)$ even.

What about integral points when one leaves the world of homogeneous spaces under linear algebraic groups ?
We shall consider three types of such problems.

Pencils of quadrics.
Some affine cubic surfaces.
Curves.

Pencils of quadrics

Let k be a field, $q(x, y, z)$ a nondegenerate ternary quadratic form over k and $p(t) \in k[t]$ nonzero. Let X/k be the affine variety

$$q(x, y, z) = p(t).$$

If $p(t)$ is separable, X is a smooth variety. Let $U \subset X$ be the complement of $x = y = z = 0$. This is a smooth variety. Let $\tilde{X} \to X$ be a resolution of singularities for X, with $U \subset \tilde{X}$.

Theorem (CT and Fei XU, 2011)

Let k be a number field and X/k as above. Let v_0 be a place of k such that q is isotropic over k_{v_0}.
Then strong approximation off v_0 with Brauer–Manin condition holds for any Zariski open set V of $\tilde{X}$ with $U \subset V \subset \tilde{X}$.

If $p(t) \neq c.r(t)^2$, strong approximation off v_0 holds for V.

[If $p(t) \neq c.r(t)^2$, then $\mathrm{Br}(V)/\mathrm{Br}(k) = 0$, hence there is no Brauer–Manin condition.]

$k = \mathbb{Q}$, $S = \infty$

$$(y - x)(9x + 7y) + 2z^2 = (2t^2 - 1)^2$$

is a counterexample to the local-global principle for integral solutions, hence to strong approximation off ∞ for $\tilde{X}/\mathbb{Q}$.

$$x^2 - 2y^2 + 64z^2 = (2t^2 + 3)^2.$$

is a counterexample to the local-global principle for primitive integral solutions ($(x, y, z) = 1$), hence to strong approximation off ∞ for $U/\mathbb{Q}$.

Proof of a special case of the theorem.

Theorem. *Let* $q(x, y, z) \in \mathbb{Z}[x, y, z]$ *be an indefinite integral nondegenerate ternary quadratic form over* $\mathbb{Z}$*. If* $p(t) \in \mathbb{Z}[t]$ *is not a constant times a square, the local-global principle holds for the integral solutions of the diophantine equation* $q(x, y, z) = p(t)$*.*

[G. L. Watson (1967) obtained some results in this direction.]

Proof. There exists a finite set S of primes such that over $\mathbb{Z}_p$ with $p \notin S$, the form $q(x, y, z)$ represents all of $\mathbb{Z}_p$.
We are given local solutions $(x_p, y_p, z_p, t_p) \in \mathbb{Z}_p^4$. Let $t_0 \in \mathbb{Z}$ be very close to t_p for $p \in S$. There then exists $t_0 \in \mathbb{Z}$ very close to t_p for $p \in S$ and an integer $r > 0$ such that for any integer $m > 0$

$$P(t_0 + (\prod_{p \in S} p)^r . m)$$

is represented by $q(x, y, z)$ over each $\mathbb{Z}_p$.

Lemma. *Let $P(t) \in \mathbb{Q}[t]$. If P is not a constant times a square, then the set of $P(m)$ for $m \in \mathbb{N}$ run through an infinite number of classes in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$.* $\square$

One may thus choose $m = m_0$ so that $P(t_0 + (\prod_{p \in S} p)^r.m_0)$ is not a spinor exception for q (we saw these exceptions fall in finitely many classes in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$). The equation

$$q(x, y, z) = P(t_0 + (\prod_{p \in S} p)^r.m_0)$$

has solutions in each $\mathbb{Z}_p$ and $\mathbb{R}$, it then has a solution $(x, y, z) \in \mathbb{Z}$. $\square$

We have a theorem for

$$\sum_{i=1}^{3} a_i x_i^2 = p(t)$$

with $a_i \in k^\times$ and $p(t) \neq 0$.
What about

$$\sum_{i=1}^{3} a_i(t) x_i^2 = p(t)$$

with $a_i(t) \in k[t]$ and $p(t).\prod_i a_i(t) \neq 0$?

Theorem (CT-Harari, being written).
Let k be a number field with a complex place v_0. Let X be smooth geometrically connected k-variety and $f : X \to \mathbb{A}^1_k$ a k-morphism. Assume $X(\mathbb{A}_k) \neq \emptyset$. Let $K = k(\mathbb{A}^1)$ and let G be an absolutely almost simple simply connected semisimple group.
Assume :
(i) The generic fibre X_η/K of f is a homogeneous space of G with toric stabilizers.
(ii) All geometric fibres of f are nonempty and integral.
Then strong approximation off v_0 with Brauer–Manin condition holds for X.

[We have serious difficulties giving an elegant statement for $k = \mathbb{Q}$ and $v_0 = \infty$. Asking that $\sum_{i=1}^{3} a_i(t)x_i^2$ is isotropic over $k_{v_0}(t)$ does not seem enough.]

One might ask : what about integral solutions of

$$q(x, y) = p(t)$$

Already for $p(t)$ of degree at three this problem on integral points of a cubic surface looks very hard. Even computing $\mathrm{Br}(X)$ is not easy. It is not clear a priori whether $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite.

Affine cubic surfaces

The equation $a = x^3 + y^3 + z^3$, with $a \in \mathbb{Z}$ nonzero.

There are solutions with $x, y, z \in \mathbb{Q}$.

For $a = 9n \pm 4$ with $n \in \mathbb{Z}$, there are no solutions with $x, y, z \in \mathbb{Z}$.

Famous open question : if a is not of the shape $9n \pm 4$, is there a solution with $x, y, z \in \mathbb{Z}$?

Open already for $a = 33$.

Theorem (CT/Wittenberg 2009) *Let* $\mathcal{X}_a$ *be the* $\mathbb{Z}$*-scheme defined by* $x^3 + y^3 + z^3 = a$, *with* $a \neq 0$. *Let* $X_a = \mathcal{X}_{a,\mathbb{Q}}$. *If* $a \neq 9n \pm 4$, *then*

$$\Big(\prod_p \mathcal{X}_a(\mathbb{Z}_p)\Big)^{\mathrm{Br}(X_{a,\mathbb{Q}})} \neq \emptyset.$$

This makes it unlikely that use of a reciprocity law will prevent this equation from having an integral solution.

To prove such a result, one must compute $\mathrm{Br}(X_a)/\mathrm{Br}(\mathbb{Q})$.
Let $X_a^c \subset \mathbb{P}^3_{\mathbb{Q}}$ be the cubic surface with homogeneous equation $x^3 + y^3 + z^3 = at^3$. Let E be the elliptic curve over $\mathbb{Q}$ with equation $x^3 + y^3 + z^3 = 0$. This is the complement of X_a in X_a^c.
There is a localisation exact sequence

$$0 \to \mathrm{Br}(X_a^c) \to \mathrm{Br}(X_a) \to H^1(E, \mathbb{Q}/\mathbb{Z}).$$

The last group classifies abelian unramified covers of E.
We may assume that a is not a cube. An algebraic computation yields $\mathrm{Br}(X_a^c)/\mathrm{Br}(\mathbb{Q}) = \mathbb{Z}/3$, with an explicit generator $\beta \in \mathrm{Br}(X_a^c)$, of order 3.

An algebraic argument shows that the image of $\mathrm{Br}(X_a) \to H^1(E, \mathbb{Q}/\mathbb{Z})$ consist of classes which vanish at each of the points $(1, -1, 0)$, $(0, 1, -1)$, $(1, 0, -1)$.

One then uses arithmetic for the elliptic curve E over $\mathbb{Q}$ (knowledge of all isogeneous curves) to show that such a class in $H^1(E, \mathbb{Q}/\mathbb{Z})$ is zero. Thus $\mathrm{Br}(X_a^c) = \mathrm{Br}(X_a)$.

One then shows that for any $a \in \mathbb{Z}$ not a cube and not of the shape $9n \pm 4$, there exists a prime p such that β takes three distinct values on $\mathcal{X}_a(\mathbb{Z}_p)$.
Thus

$$(\prod_p \mathcal{X}_a(\mathbb{Z}_p))^{\mathrm{Br}(X_a)} = (\prod_p \mathcal{X}_a(\mathbb{Z}_p))^{\beta} \neq \emptyset$$

It is an open question whether any integer a may be written as $x^3 + y^3 + 2z^3$, with $x, y, z \in \mathbb{Z}$.

Theorem (CT/Wittenberg 2009)
Let $\mathcal{Y}_a$ be the $\mathbb{Z}$-scheme defined by $x^3 + y^3 + 2z^3 = a$, with $a \neq 0$.
Let $Y_a = \mathcal{Y}_{a,\mathbb{Q}}$.
Then

$$\Big(\prod_p \mathcal{Y}_a(\mathbb{Z}_p)\Big)^{\mathrm{Br}(Y_a)} \neq \emptyset.$$

This makes it unlikely that use of a reciprocity law will prevent this equation from having an integral solution.

The proof here is more delicate : the restriction map $\mathrm{Br}(Y_a^c) \to \mathrm{Br}(Y_a)$ is not onto. We have $\mathrm{Br}(Y_a)/\mathrm{Br}(\mathbb{Q}) \simeq \mathbb{Z}/3 \oplus \mathbb{Z}/2$.

Warning

For $X \subset X^c$ the complement of a smooth curve C in say a geometrically rational smooth projective surface X^c, the quotient $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q})$ need not be finite.

Example : complement X of a smooth conic $q(x, y, t) = 0$ in $\mathbb{P}^2_{\mathbb{Q}}$ (a log del Pezzo surface). In this case $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) = \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$.

Given by the quaternion classes $(q(x, y, t)/t^2, a)$, with $a \in \mathbb{Q}^\times$.

An example for which failure of the local-global principle is not accounted for by the Brauer-Manin obstruction on the variety.

Let $q(x, y, z) = 16x^2 + 9y^2 - 3z^2$. Consider the $\mathbb{Z}$-scheme $\mathcal{X} \subset \mathbb{P}^2_{\mathbb{Z}}$ defined by $q(x, y, z) \neq 0$. Let $\mathcal{Q}$ be the affine quadric over $\mathbb{Z}$ defined by $q(x, y, z) = 1$.

Using the obvious μ_2-covering $\mathcal{Q}_{\mathbb{Q}} \to \mathcal{X}_{\mathbb{Q}}$, one shows :

$$[\prod_p \mathcal{X}(\mathbb{Z}_p)]^{\mathrm{Br}\mathcal{X}_{\mathbb{Q}}} \neq \emptyset \text{ but } \mathcal{X}(\mathbb{Z}) = \emptyset.$$

This provides a rather simple "Skorobogatov" type of example in the affine context. Other examples were found by Kresch and Tschinkel.

Hyperbolic curves

$\mathbb{P}^1$ *minus three points*

F. Voloch pointed out the following conjecture of Skolem (1937). Let S be a finite set of prime numbers $p_i, i = 1, \cdots, n$. Let $R \subset \mathbb{Q}^\times$ be the subgroup generated by the p_i. Let a_1, a_2, a_3 be elements in R.
Skolem's conjecture :
The equation $\sum_{i=1}^{3} a_i x_i = 0$ has solutions with $x_i \in R$ if and only if for all integer m prime to S, the equation $\sum_{i=1}^{3} a_i x_i = 0 \ \ mod \ \ m$ has a solution with all $x_i \in R$.

This may be viewed as a special case of a general conjecture.

Let $X \subset \mathbb{P}^1_{\mathbb{Q}}$ be an open set whose geometric complement consist of at least 3 points. One may view X as a closed curve in a $\mathbb{Q}$-torus T. The whole situation may be realized over the ring O_S of S-integers, for some finite set S of places. We thus have $\mathcal{X} \subset \mathcal{T}$.

Conjecture (Harari and Voloch 2009)

$$\mathcal{X}(O_S) = [\prod_{v \notin S} \mathcal{X}(O_v)] \cap \mathcal{T}(O_S)^{closure} \subset \prod_{v \notin S} \mathcal{T}(O_v).$$

They show that $[\prod_{v \notin S} \mathcal{X}(O_v)] \cap \mathcal{T}(O_S)^{closure}$ may be interpreted as a Brauer-Manin set of $\mathcal{X}$ (analogue of a statement by Scharashkin and Skorobogatov for rational points on projective curves).

Arbitrary hyperbolic curves

One might be tempted to produce further conjectures of the kind for integral points of arbitrary hyperbolic curves. A local-global principle for integral points (of which there are finitely many !) in the Brauer–Manin style is not excluded, but as regards strong approximation, Harari and Voloch have a strikingly bad example.

They take the affine curve $\mathcal{X}/\mathbb{Z}$ given by $y^2 = x^3 + 3$. Over $\mathbb{Q}$, this is the complement of one rational point in an elliptic curve E. Let P be the point $(x, y) = (1, 2)$. One has $E(\mathbb{Q}) = \mathbb{Z}.P$ and $\mathcal{X}(\mathbb{Z}) = \{\pm P\}$. Let p run through primes $p \equiv 3 \bmod 8$. Take a subsequence of the pP in $E(\mathbb{Q})$ so that its image under the embedding $E(\mathbb{Q}) \hookrightarrow E(A_{\mathbb{Q}})$ converges.
Claim :
(i) The limit is in $\prod_p \mathcal{X}(\mathbb{Z}_p)^{\mathrm{Br}\mathcal{X}_{\mathbb{Q}}}$.
(ii) This limit is neither P nor $-P$.

They write that a "Skorobogatov" type of argument (mixture of descent and Brauer–Manin obstruction) accounts for this example.