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Principal Homogeneous Spaces under Flasque Tori: Applications

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INTRODUCTION

In this paper, we define flasque tori and flasque resolutions of tori over an arbitrary base scheme (Sect. 1) and we establish the basic cohomological properties of flasque tori over a regular scheme (Sect. 2). These properties are then used in a systematic and sometimes biased manner in the study of various problems, which will now be briefly listed. In Section 3, an alternative approach to R-equivalence upon tori [8] is given. Section 4 studies the behaviour of the first and second cohomology groups of arbitrary tori over a regular local ring, when going over to the fraction field. Applications to the representation of elements by norm forms and quadratic forms are described in Sections 5 and 6. In Sections 7 and 8, we study the behaviour of the group of sections of a torus, and of the first cohomology group of a group of multiplicative type when going over from a local ring to its residue class field, or when going over from a discretely valued field to its completion. We thus recover and generalize results of Saltman [30, 31] on the Grunwald–Wang theorem and its relation with the Noether problem [35]. In Section 9, Formanek's description [17] of the centre of the generic division ring as the function field of a certain torus provides a different route to two results of Saltman [31, 32].

Let us now give more details on the contents of this paper.

If U is an open set of an integral regular scheme X, the restriction map

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$\operatorname{Pic} X \to \operatorname{Pic} U$ is surjective, hence also, by Grothendieck's version of Hilbert's theorem 90, the map $H^1_{\text{et}}(X, \mathbb{G}_m) \to H^1_{\text{et}}(U, \mathbb{G}_m)$. In other words, any torseur (= principal homogeneous space) over U under $\mathbb{G}_m$ comes from a torseur over X under $\mathbb{G}_m$ (up to isomorphism). It is easy (cf. 3.2) to produce X-tori which do not share this property with $\mathbb{G}_{m,X}$. There exists however an important class of X-tori T, which we christened *flasque tori* precisely because for U and X as above the restriction map

$$H^1_{\text{et}}(X, T) \to H^1_{\text{et}}(U, T) \tag{1}$$

is *surjective* (Sect. 2, 2.2(i)). The general definition of flasque tori is given in Section 1. If X is a k-variety, for k a field, and T is a k-torus, the defining condition simply reads:

$$H^1(H, \operatorname{Hom}_{\mathbb{Z}}(\hat{T}, \mathbb{Z})) = 0 \qquad \text{for all subgroups } H \text{ of } G, \tag{2}$$

where $\hat{T}$ denotes the character group of T, and $G = \operatorname{Gal}(K/k)$ for K/k a finite Galois extension which splits T; in this case, the surjectivity of (1) was briefly sketched in [8, Proposition 9]. The general case was announced in [9] and developed in an unpublished part of our thesis [10], on which Sections 1, 2, 4, 5 of the present paper are built.

In the language of G-modules, tori as in (2) have already played an important role in the work of Endo–Miyata [15], Voskresenskiĭ [36, 37], and Lenstra [22]. In [8], we showed $H^1(k, T)$ finite for such tori over a field k which is finitely generated over the prime field. This property, which certainly does not hold for arbitrary tori, is shown in Section 2 (2.8) to be a consequence of the general surjectivity statement (1).

The main reason why flasque tori deserve attention is that for any torus T over a sensible scheme X, there exists an exact sequence of X-tori:

$$1 \to F \to E \to T \to 1, \tag{3}$$

where E is a quasitrivial torus (its character group, at least in the ground field case, is a permutation module) and F is a flasque torus; moreover, the F associated to T is almost unique: it is defined up to a product by a quasi-trivial torus (Sect. 1, 1.3; for the ground field case, see [8], or, in the language of G-modules, [15, 37]). We showed in [8] how sequences (3), called *flasque resolutions* (of T), compute R-equivalence on a torus T defined over a field k, and deduced the finiteness of $T(k)/R$ for k finitely generated over the prime field. In Section 3, we present a slightly different approach, based on the surjectivity of (1).

Flasque tori share another property with $\mathbb{G}_m$: if F is a flasque torus over an integral regular scheme X with field of fractions K, the restriction map

$$H^2_{\text{et}}(X, F) \to H^2_{\text{et}}(K, F) \tag{4}$$

is *injective* (Sect. 2, 2.2(ii)). This property, which is proved at the same stroke as the surjectivity of (1), extends the Auslander–Goldman–Grothendieck result on the Brauer group. This injectivity together with flasque resolutions is used in Section 4 to give a fancy proof of the following results: if T is an arbitrary torus over a regular local ring A, the restriction maps

$$H^1_{\text{et}}(A, T) \to H^1_{\text{et}}(K, T), \tag{5}$$

$$H^2_{\text{et}}(A, T) \to H^2_{\text{et}}(K, T) \tag{6}$$

are *injective*. This also holds for arbitrary A-groups of multiplicative type, flat cohomology replacing étale cohomology. The injectivity (6) extends the Auslander–Goldman–Grothendieck result. As for (5), it is a special case of a conjecture of Grothendieck for reductive A-group schemes. For a discrete valuation ring A, this conjecture has recently been proved by Nisnevich [23]. The injections (5) and (6) may be obtained in a more natural manner [9], but it seemed amusing to give a proof via flasque resolutions.

If $N_{B/A}$ denotes the norm form associated to a finite étale cover B of a regular local ring A, an element of A^* (a unit) is represented by $N_{B/A}$ over A as soon as it is represented by $N_{B/A}$ over the fraction field K of A: this special case of injection (5) is discussed at length in Section 5, together with extensions to localizations of polynomial rings over A. Various counterexamples show that the hypotheses A regular and B/A étale cannot be much relaxed. In Section 6, we give applications to the similar problem for representation of elements of A^* by a nondegenerate quadratic form over A, and we give an (independent) result of the same kind for representation of nonunits over an arbitrary valuation ring. This section complements a paper of Choi–Lam–Reznick–Rosenberg [6].

Much of the following two sections (Sect. 7 and 8) is motivated by Saltman's papers [30] and [31], and the beautiful relationship these papers established between lifting or approximation problems of the Grunwald–Wang type and the Noether problem. Given a local ring A with residue class field κ, T an A-torus and M an A-group of multiplicative type (e.g., an A-torus *or* $(\mathbb{Z}/n)_A$ for n invertible in A), one asks: *when are the natural maps*

$$T(A) \to T(\kappa) \tag{7}$$

$$H^1(A, M) \to H^1(\kappa, M) \tag{8}$$

surjective? Also, given a discretely valued field K and its completion $\hat{K}$, and given a K-torus T and a K-group of multiplicative type M, one asks:

$$\textit{is } T(K) \textit{ dense in } T(\hat{K})? \tag{9}$$

$$\textit{is } H^1(K, M) \to H^1(\hat{K}, M) \textit{ surjective}? \tag{10}$$

In a less general set-up, these are the questions studied in [30] and [31]: for $M = \mathbb{Z}/n$, (8) (resp. (10)) is the lifting (resp. approximation) problem for Galois extensions with group $\mathbb{Z}/n$, and for a suitable A-torus M, (8) turns out to be the lifting problem for crossed products with a given group.

Questions (7), (8), (9), (10) are very closely related. This relationship is made clear by means of resolutions of a type different from (3). Namely, if M is an X-group of multiplicative type over a sensible scheme X, there exist exact sequences of X-groups of multiplicative type (Sect. 1),

$$1 \to M \to F_1 \to P_1 \to 1, \tag{11}$$

$$1 \to M \to P_2 \to Q_2 \to 1, \tag{12}$$

where P_1 and P_2 are quasitrivial tori, F_1 is a flasque torus, and Q_2 is a coflasque torus (see Sect. 1 for the definition), and F_1 and Q_2 are well defined up to multiplication by a quasitrivial torus.

If T is a torus over a field k, the following conditions are equivalent: T is a k-birational direct factor of a k-rational variety, there exists a k-torus T_1 such that $T \times_k T_1$ is a k-rational variety, any F associated to T in a flasque resolution (3) is an invertible torus (direct factor, as a torus, of a quasi-trivial torus): see [8] or 7.4 (a k-rational variety is one which is k-birational to an affine space over k). It is a consequence of a theorem of Endo–Miyata (0.5) that these conditions are fulfilled by any k-torus which is split by a "metacyclic" Galois extension K/k (= K/k is finite and all Sylow subgroups of $\mathrm{Gal}(K/k)$ are cyclic). Since F invertible implies $H^1(k, F) = 0$, sequences (3), (11), (12) easily yield *positive answers to questions* (7), (8), (9), (10) *in the following cases*: T_κ is a direct factor of a κ-rational variety, T_κ is split by a metacyclic extension; $F_{1\kappa}$ or $Q_{2\kappa}$ in (11) or (12) are invertible κ-tori, M_κ is split by a metacyclic extension; $T_{\hat{K}}$ is a direct factor of a $\hat{K}$-rational variety, $T_{\hat{K}}$ is split by a metacyclic extension of $\hat{K}$; $F_{1\hat{K}}$ or $Q_{2\hat{K}}$ in (11) or (12) are invertible $\hat{K}$-tori, $M_{\hat{K}}$ is split by a metacyclic extension.

Let p be a prime, n be an integer, and let $M = (\mathbb{Z}/p^n)_A$ in (8), with p invertible in A. The κ-group of multiplicative type M_κ is split—*as a group of multiplicative type*—by the cyclotomic extension $\kappa(\mu_{p^n})/\kappa$. If p is odd, this extension is cyclic, and we recover the lifting property for abelian extensions of odd degree. For $p = 2$, we recover the lifting property under the usual condition: $\kappa(\mu_{2^n})/\kappa$ is cyclic. Using Theorem 5.3 of [30], one thus gets a simple proof of Theorem 2.1 in this same paper of Saltman.

Let us now start with a field k, a k-torus T, and a k-group of multiplicative type M. In [30] and especially [31], Saltman studies the question: if (7) or (9) hold universally (i.e., for any local ring A with $k \subset A$, or any K with $k \subset K$), what does it imply on T? and he also studies the similar question for M and (8), (10), at least for $M = \mathbb{Z}/n$ (see also Proposition 7.7 below). The answer is simple: T must be a k-birational direct factor of a k-rational variety, more precisely there must exist a k-torus T_1 such that $T \times_k T_1$ is a k-rational variety ([31, 3.8 and 3.14] for question (7); Proposition 7.4 below for (7) and 8.1 for (9)). As for (8), (10), and M, the answer is: any F_1 or Q_2 in sequences (11) or (12) must be an invertible k-torus (7.6; 8.4(iii)).

In simple terms, the answer for (9) is: the k-tori which universally satisfy weak approximation are the obvious ones.

Let k be a field, let C be a finite abelian group of order prime to char k. A special case of a problem of Emmy Noether, to which Swan and Voskresenskiĭ gave a negative answer, asked whether the field of invariants K^C of $K = k(x_g)_{g \in C}$, under the obvious action of C on the algebraically independent variables x_g, is a purely transcendental extension of k. Voskresenskiĭ's approach realizes K^C as the function field of a (coflasque) k-torus Q_C, which is part of a sequence of type (12):

$$1 \to C_k \to P_C \to Q_C \to 1. \tag{13}$$

The original example of Swan and Voskresenskiĭ was $k = \mathbb{Q}$ and $C = \mathbb{Z}/47$. In this case, the $\mathbb{Q}$-torus Q_C is split by the cyclic extension $\mathbb{Q}(\mu_{47})/\mathbb{Q}$, hence, as recalled above, is a direct factor of a $\mathbb{Q}$-rational variety. It was later noted by Endo–Miyata, Lenstra and Voskresenskiĭ that $k = \mathbb{Q}$ and $C = \mathbb{Z}/8$ provide a smaller negative answer to Noether's problem; in this case, Q_C is not even a direct factor of a $\mathbb{Q}$-rational variety (cf. the discussion after 7.10). Later, Saltman [30] obtained a very simple proof of this last fact. Up to the torus-theoretic language, he noted that Wang's counterexample to Grunwald's theorem implies that $Q_{\mathbb{Z}/8}(\mathbb{Q})$ is not dense in $Q_{\mathbb{Z}/8}(\mathbb{Q}_2)$ (for the $\mathbb{Q}_2$-adic topology). He then went on [30, 31] to analyse the relation between the failure of weak approximation, or of the lifting problem (7), for an arbitrary Q_C, and the failure of the weakened Noether problem: is Q_C a direct factor of a k-rational variety? We recover most of his results in Sections 7 and 8, and show: Q_C is a direct factor of a k-rational variety (i.e., the weakened Noether problem for C has a positive answer) if and only if abelian extensions with group C are universally liftable (i.e., (8) is surjective for any local ring A with $k \subset A$, and $M = C_k$), if and only if abelian extensions with group C are universally approximable (i.e., (10) is surjective for any discretely valued field K with $k \subset K$).

Our approach to these results is self-contained and in places more general than that of [30, 31], and we hope that this generality clarifies the results of [30] and [31], but we would like to stress our debt to Saltman's ideas. In particular, although we stubbornly refused to use the concept of retract rationality [31, Theorem 3.8] in any crucial fashion, the proof of Proposition 7.4 was inspired by [31].

We conclude Section 7 with some remarks to the effect that the failure of Noether's problem does not totally ruin this approach to the construction of Galois extensions of a given number field with a given Galois group.

Let k be a field, and let $Z_n(r)$ be the centre of the generic division algebra $UD(k, n, r)$ of r generic $n \times n$ matrices, as described in [17] and [29]. It is an open question whether $Z_n(r)$ is purely transcendental over k. This centre was described as the function field of a certain torus by Formanek [17]. In Section 9, we use this description to recover Saltman's result [31]: if $n = p$ is a prime, and Y is an integral k-variety with function field $k(Y) = Z_p(r)$, then Y is retract rational. Formanek's description together with a systematic use of tori and their flasque resolutions [8, 37] also enable us to recover—at least for $\operatorname{char} k = 0$—another result of Saltman [32] whose proof also depended on Formanek's description: for arbitrary n, if X is a smooth proper k-variety with function field equal to $Z_n(r)$, the Brauer group of X is trivial, i.e., coincides with the Brauer group of k.

0. Preliminaries

0.1. We denote by $\mathbb{G}_m$ (resp. $\mathbb{G}_a$), the multiplicative group Spec $\mathbb{Z}[t, t^{-1}]$ (resp. the additive group Spec $\mathbb{Z}[t]$), and by $\mathbb{G}_{m,X}$ (resp. $\mathbb{G}_{a,X}$), the X-group schemes $\mathbb{G}_m \times_{\operatorname{Spec} \mathbb{Z}} X$ (resp. $\mathbb{G}_a \times_{\operatorname{Spec} \mathbb{Z}} X$) over a scheme X. For simplicity, by an X-group scheme of multiplicative type we shall mean a finite type isotrivial X-group scheme of multiplicative type [SGA3, X]: if M is such an X-group scheme, there exists an étale cover (cover = finite locally free morphism) $X' \to X$ such that $M_{X'} = M \times_X X'$ is isomorphic to $\mathcal{H}om_{X'\text{-group}}(A_{X'}, \mathbb{G}_{m,X'})$ for a suitable finitely generated abelian group A. If A is torsion-free, M is called an X-torus (hence our X-tori are isotrivial). When X is connected, a splitting cover $X' \to X$ as above may be chosen connected and principal Galois over X. By a twisted constant X-group we shall mean an isotrivial twisted constant X-group with finite generation. Associating to an X-group scheme of multiplicative type its character group $\hat{M} = \mathcal{H}om_{X\text{-gr}}(M, \mathbb{G}_{m,X})$ defines an antiequivalence of categories between X-groups of multiplicative type and twisted constant X-groups: the natural map $M \to \mathcal{H}om_{X\text{-gr}}(\hat{M}, \mathbb{G}_{m,X})$ is an isomorphism.

0.2. Assume X is connected, and let $X' \to X$ be a connected principal Galois cover with group G. Associating $\hat{M}(X')$ to M induces an anti-equivalence of categories between X-groups of multiplicative type split by $X' \to X$ and G-modules of finite type. M is smooth over X if and only if the torsion of $\hat{M}(X')$ is of order prime to the residue characteristics of X, and

M is an X-torus if and only if $\hat{M}(X')$ is torsion-free. The reverse anti-equivalence transforms exact sequences of finitely generated G-modules into exact sequences of X-groups of multiplicative type, inducing exact sequences of *fppf* sheaves on X.

0.3. Let $p: X' \to X$ be a morphism of schemes, and let A be a twisted constant X-group. The natural map $p^*_{\text{et}} A \to A_{X'} = A \times_X X'$ is an isomorphism of étale sheaves on X', as may be seen by reducing to the constant case.

Let M be an X-group of multiplicative type. Reducing to M diagonalisable, one checks that the natural diagram of *étale sheaves*

$$\begin{array}{ccc} M & \longrightarrow & p_{*\text{et}} M_{X'} \\ \downarrow & & \downarrow \\ \mathcal{H}om_X(\hat{M}, \mathbb{G}_{m,X}) & \longrightarrow & \mathcal{H}om_X(\hat{M}, p_{*\text{et}} \mathbb{G}_{m,X'}) \end{array} \tag{0.3.1}$$

commutes, and that the vertical arrows are isomorphisms.

0.4. Let $p: X' \to X$ be a cover, and let G be an affine X'-group scheme. Restriction from X' to X defines an affine X-group scheme $R_{X'/X} G$. If H is an affine X-group scheme, we simply write $R_{X'/X} H = R_{X'/X}(H_{X'})$. Assume G commutative. Since p is finite, the étale sheaves $R^i_{p*\text{et}}(G)$ are zero for $i > 0$ [SGA4, VIII, 5.5 and 5.3]. One thus obtains [SGA3, XXIV, 8.5] canonical isomorphisms

$$H^i(X_{\text{et}}, R_{X'/X} G) \simeq H^i(X'_{\text{et}}, G) \qquad (i \geq 0). \tag{0.4.1}$$

If G is a commutative affine X-group scheme and X'/X is a cover, one denotes $N_{X'/X}: R_{X'/X} G \to G$ the "trace" morphism of X-group schemes defined in [SGA4, XVII, 6.3.13.2], and by $R^1_{X'/X} G$ the X-group scheme which is the kernel of $N_{X'/X}$. For $G = \mathbb{G}_{m,X}$ (resp. $G = \mathbb{G}_{a,X}$), $N_{X'/X}$ induces the usual norm $\mathbb{G}_m(X') \to \mathbb{G}_m(X)$ (resp. the usual trace $\mathbb{G}_a(X') \to \mathbb{G}_a(X)$) on X-points. If $p: X' \to X$ is of constant rank n, the composite morphism $G \to R_{X'/X} G \xrightarrow{N_{X'/X}} G$, where the first map is the natural one, is multiplication by n [SGA4, XVII, 6.13.15]. Finally, norm being compatible with arbitrary base change $Y \to X$ [SGA4, XVII, 6.13.15], so is the functor $R^1_{X'/X}$.

Assume $X' \to X$ is a connected étale cover, and let $X'' \to X$ be a majorizing connected étale cover, principal Galois with group G. Let H be the subgroup of G corresponding to the cover $X'' \to X'$. If M is an X-group of multiplicative type (resp. an X-torus), so are $R_{X'/X} M$ and $R^1_{X'/X} M$, and both are split by X'' if M is. Let $J_{G/H}$ (J_G if $H=\{1\}$) be the finitely generated torsion-free G-module defined by the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{N_{G/H}} \mathbb{Z}[G/H] \to J_{G/H} \to 0, \tag{0.4.2}$$

where $N_{G/H}=\sum_{x \in G/H} x$. Under the same assumptions for M, there is an exact sequence of X-groups of multiplicative type

$$1 \to R^1_{X'/X}M \to R_{X'/X}M \xrightarrow{N_{X'/X}} M \to 1 \tag{0.4.3}$$

which in the case $M=\mathbb{G}_{m,X}$ is dual (= antiequivalent) to (0.4.2) by means of $M \mapsto \hat{M}(X'')$.

0.5. Let G be a finite group, and let $\mathscr{M}_G$ (resp. $\mathscr{L}_G$) denote the category of finitely generated (resp. finitely generated torsion-free (= $\mathbb{Z}$-free)), G-modules. The category $\mathscr{L}_G$ is antiequivalent to itself under the duality $A \mapsto A^0=\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$. A module in $\mathscr{L}_G$ is called a *permutation* module if it admits a $\mathbb{Z}$-basis stable under G. A module in $\mathscr{L}_G$ is called *invertible* if it is a G-direct summand of a permutation module. A module F in $\mathscr{L}_G$ is called *flasque* if any of the following equivalent conditions is satisfied:

(i) $\hat{H}^{-1}(H, F)=0$ for any subgroup H of G;

(ii) $\mathrm{Ext}^1_G(F, P)$ $(=H^1(G, \mathrm{Hom}_{\mathbb{Z}}(F, P)))=0$ for any permutation module P;

(iii) $\mathrm{Ext}^1_G(F, A)$ $(=H^1(G, \mathrm{Hom}_{\mathbb{Z}}(F, A)))=0$ for any invertible module A. A module Q in $\mathscr{L}_G$ is called *coflasque* if it satisfies:

(iv) $H^1(H, Q)=0$ for any subgroup H of G.

The duality $A \mapsto A^0$ transforms flasque modules into coflasque modules. It preserves permutation modules, hence also invertible modules. An invertible module is flasque (and coflasque). An important theorem of Endo and Miyata [15, Theorem 1.5; 8, Proposition 2] studies the converse: *any flasque G-module is invertible if and only if G is a "metacyclic" group (= all its Sylow subgroups are cyclic).*

We shall need the following extension to $\mathscr{M}_G$ of known facts about $\mathscr{L}_G$ (cf. [8, 15, 16]):

LEMMA 0.6. *For any A in $\mathscr{M}_G$, there exist exact sequences of G-modules*

$$0 \to Q_1 \to P_1 \to A \to 0, \tag{0.6.1}$$

$$0 \to P_2 \to F_2 \to A \to 0; \tag{0.6.2}$$

for any A in $\mathscr{L}_G$, there exist exact sequences of G-modules in $\mathscr{L}_G$:

$$0 \to A \to P_3 \to F_3 \to 0, \tag{0.6.3}$$

$$0 \to A \to Q_4 \to P_4 \to 0. \tag{0.6.4}$$

Here F_i (resp. Q_i) (resp. P_i) denotes a flasque (resp. coflasque) (resp. permutation module). Moreover Q_1, F_2, F_3, Q_4 in the above sequences are well defined up to addition of permutation modules.

Proof. Let A be in $\mathscr{M}_G$. For each subgroup H of G, choose a torsion-free G-module of finite type B_H together with a surjection $B_H \twoheadrightarrow A^H$ (A^H = invariants under H). Take $P_1 = \sum_{H \subset G} \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} B^H$ (with trivial action of G on B^H) and the natural map $P_1 \twoheadrightarrow A$. This gives a sequence (0.6.1). Using $A \mapsto A^0$, one then obtains (0.6.3). Now take any exact sequence

$$0 \to A_1 \to P_1 \to A \to 0,$$

with P_1 a permutation module, apply (0.6.3) to A_1, and form the push-out diagram:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & A \arrow[d, equal] \\
P_3 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & A \\
F_3 \arrow[r, equal] & F_3.
\end{tikzcd}$$

Since $\mathrm{Ext}^1_G(F_3, P_1) = 0$, we get a G-isomorphism $R \xrightarrow{\sim} P_1 \oplus F_3$, hence R is flasque, and the middle horizontal row if of type (0.6.2). Using $A \mapsto A^0$ once more, we get (0.6.4). That the Q_i and F_i in the lemma are well defined up to addition of permutation modules is proved as the standard Schanuel's lemma by forming suitable push-outs and pull-backs and using the defining properties of flasque and coflasque modules (cf. [8, Lemme 5]).

For more details, we refer to [8, 15, 16].

0.7. Finally, constant use is made of the fact: if X is a scheme and G/X is a flat finitely presented affine X-group scheme, the set $H^1(X_{\mathrm{fppf}}, G)$ classifies the torseurs (= principal homogeneous spaces) over X under G.

1. Flasque Tori

This notion was introduced in [8] in the case of a base field. It will now be extended to the case of an arbitrary base.

LEMMA 1.1. *Let X be a connected scheme, let X_0 be a closed subscheme with the same underlying space, let T be an X-torus, and let $X' \to X$ and $X'' \to X$ be two connected étale covers, principal Galois with respective groups G' and G'', which both split T. Set $X'_0 = X_0 \times_X X'$. The following conditions on T are equivalent:*

(i) *$\hat{T}(X')$ is a flasque (resp. coflasque), (resp. permutation) (resp. invertible) G'-module;*

(ii) *$\hat{T}(X'')$ is a flasque (resp. coflasque), (resp. permutation) (resp. invertible) G''-module;*

(iii) *$\hat{T}(X'_0)$ is a flasque (resp. coflasque), (resp. permutation) (resp. invertible) G'-module.*

Proof. There exists a connected étale cover of X which majorizes both coverings (use the induced scheme on a connected component of $X' \times_X X''$). To prove the equivalence of (i) and (ii) we may thus assume that $X'' \to X$ majorizes $X' \to X$, hence G' is the quotient of G'' by a normal subgroup H. The equivalence of (i) and (ii) then follows from the following proposition, applied to $A = \hat{T}(X'') = \hat{T}(X')$: a module A in $\mathscr{L}_{G''/H}$ is flasque (resp. coflasque) (resp. permutation) (resp. invertible) if and only if it is flasque (resp. coflasque) (resp. permutation) (resp. invertible) as a module in $\mathscr{L}_{G''}$ [8, Sect. 1, Lemme 2]. Since X'_0, as X', is connected, there is a G'-isomorphism $\hat{T}(X') \simeq \hat{T}(X'_0)$, hence (i) and (iii) are equivalent.

DEFINITION 1.2. Let X be a scheme and T be an X-torus. T is called flasque (resp. coflasque) (resp. quasitrivial) (resp. invertible) if for any connected component Z of X there exists a connected étale cover $Z' \to Z$, principal Galois with group G, which splits T_Z and such that $\hat{T}(Z')$ is a flasque (resp. coflasque) (resp. permutation) (resp. invertible) G-module.

Lemma 1.1 and [SGA3, X, Sect. 2] imply that the choice of the closed subscheme structure on Z is irrelevant. Also, if T is flasque, for any connected component Z of X and any connected étale Galois cover $Z'' \to Z$ with group G', splitting T, the G'-module $\hat{T}(Z'')$ is flasque (and similarly for coflasque, permutation, invertible).

When X is connected, one easily checks that a quasitrivial X-torus is none but a finite product of X-tori of the shape $R_{X'_i/X}\mathbb{G}_m$, for étale connected covers $X'_i \to X$; it may also be written as $R_{X'/X}\mathbb{G}_m$ for a not necessarily connected étale cover $X' \to X$. An invertible X-torus is a direct factor (as an

X-torus) of a quasitrivial X-torus; it is flasque (and coflasque) but in general there exist other flasque tori (cf. 0.5 and [8]).

PROPOSITION 1.3. *Let X be a scheme, and assume that the connected components of X are open, for instance, that X is locally noetherian or that X is the spectrum of a semilocal ring. Let M be an X-group of multiplicative type, and let T be an X-torus. There exist exact sequences of X-groups of multiplicative type*:

$$1 \to M \to P_1 \to Q_1 \to 1, \tag{1.3.1}$$

$$1 \to M \to F_2 \to P_2 \to 1, \tag{1.3.2}$$

$$1 \to F_3 \to P_3 \to T \to 1, \tag{1.3.3}$$

$$1 \to P_4 \to Q_4 \to T \to 1, \tag{1.3.4}$$

where P_i, F_i, Q_i denote, respectively, quasitrivial, flasque, coflasque, tori. Moreover the P_i and Q_i in the above sequences are well defined up to a product by a quasitrivial torus.

Proof. We may assume that X is connected, and choose a connected étale Galois cover $X' \to X$, with group G, which splits M, or T. The existence of sequences (1.3.1) to (1.3.4) now follows from 0.2 and Lemma 0.6. To show "uniqueness," one may again assume X connected, and choose a connected étale Galois cover of X with group G, splitting all X-groups of multiplicative type involved in two given exact sequences of the same type (1.3.i). By 0.2, we are then reduced to the "uniqueness" statement in Lemma 0.6.

As in [8], a sequence of type (1.3.3) will be called a *flasque resolution* of the X-torus T.

PROPOSITION 1.4. *Let $Y \to X$ be a morphism of schemes. If T is a flasque* (*resp. coflasque*) (*resp. quasitrivial*) (*resp. invertible*) *X-torus, so is T_Y. In particular the pull-back of a sequence of type* (1.3.i) *is of the same type.*

Proof. We may assume that X and Y are connected and ([SGA3, X, Sect. 2] and Lemma 1.1) reduced. By definition there exists a connected étale Galois cover $X' \to X$, with group G, which splits T and such that $\hat{T}(X')$ is a flasque (resp. coflasque) (resp. permutation) (resp. invertible) G-module. Now choose a connected component Z of $X' \times_X Y$. Since $X' \times_X Y \to Y$ is an étale cover, principal Galois with group G, the induced map $Z \to Y$ defines a connected étale cover, Galois with group H for H a subgroup of G, and T_Y is split by $Z \to Y$. Since $\hat{T}_Y(Z)$ is none but the G-module $\hat{T}(X')$ viewed as an H-module, it is a flasque (resp. coflasque) (resp. permutation) (resp. invertible) H-module, and T_Y is flasque (resp. coflasque) (resp. quasitrivial) (resp. invertible).

PROPOSITION 1.5. *Let X be an integral scheme, let K be its field of fractions, and let T be a flasque K-torus. There exists a Zariski open set $U \subset X$ and a flasque U-torus S such that $S \times_U K \simeq T$.*

Proof. We may assume X affine, say $X = \operatorname{Spec} A$. Let L/K be a (finite) Galois extension of fields, with group G, which splits T. Write $L = K[T]/P(T)$ for $P(T)$ a separable polynomial. Replacing X by an affine open set, we may assume that the coefficients of P are in A. Now $A = \varinjlim_{i \in I} A_i$, for A_i running through subrings A_i of A which are of finite type over $\mathbb{Z}$ (hence noetherian), regular, which contain all coefficients of P, and which satisfy: if K_i is the field of fractions of A_i, the extension $L_i = K_i[T]/P(T)$ is Galois with group G. Since L_i/K_i is separable and A_i normal noetherian, the integral closure B_i of A_i in L_i is finite over A_i [3, Chap. V, Sect. 1, No. 6]. By [EGA IV, 18.2.4] there exists $f_i \in A_i$ such that the restriction of B_i/A_i to A_{i,f_i} is étale. This restriction defines a connected étale Galois cover with group G. Define S_i as the $\operatorname{Spec} A_{i,f_i}$-torus split by $B_{i,f_i}/A_{i,f_i}$ with $\hat{S}_i(B_{i,f_i}) = \hat{T}(L)$. Pulling back S_i to $\operatorname{Spec} A_{f_i}$ produces a flasque torus of the required type. (The same proof goes for coflasque and quasitrivial tori.)

2. Cohomological Properties of Flasque Tori

In this section, the cohomology we use is *étale* cohomology. We therefore denote by $H^i(X, \cdot)$, $\operatorname{Ext}^i_X(\cdot, \cdot)$, and $\underline{\mathcal{E}xt}^i_X(\cdot, \cdot)$ étale cohomology groups, étale Ext groups and étale $\mathcal{E}xt$ sheaves of étale sheaves on a scheme X. Note, however, that for an X-torus, and more generally for a *smooth* commutative X-group M, there are canonical isomorphisms [GB III, Théorème 11.7],

$$H^i(X_{\mathrm{et}}, M) \simeq H^i(X_{\mathrm{fppf}}, M) \qquad (i \geqslant 0). \tag{2.0}$$

By $\mathbb{Z}_X$ (or by $\mathbb{Z}$ if there is no ambiguity) we denote the étale sheaf on X associated to the constant presheaf $\mathbb{Z}$.

KEY LEMMA 2.1. *Let X be a scheme and let T be a flasque X-torus.*

(i) *If X is connected and geometrically unibranch, then*

$$\operatorname{Ext}^1_X(\hat{T}, \mathbb{Z}_X) = 0.$$

(ii) *If X is quasicompact and quasiseparated, and if $\{i_j\}_{j\in J}$ is a family of morphisms $i_j\colon X_j \to X$ with each X_j connected and geometrically unibranch, then*

$$\operatorname{Ext}^1_X(\hat{T}, \bigoplus_{j\in J} i_{j*}\mathbb{Z}_{X_j}) = 0.$$

Proof. Let us first recall that a normal scheme is geometrically unibranch [EGA IV, 0_{IV}, 23.2.1], i.e., all its strictly local rings are irreducible [EGA IV, 18.8.15], that a connected geometrically unibranch scheme is irreducible and that a scheme is quasicompact and quasiseparated when it is a finite union of affine open sets and the intersection of two affine open sets is a finite union of affine open sets, which is certainly true if X is noetherian.

Let us prove (i). Let $X' \to X$ be a connected étale Galois cover, with group G, which splits T. The assumption on X implies that X' is irreducible and geometrically unibranch, hence [SGA4, IX, 3.6]

$$H^1(X', \mathbb{Z}) = 0.$$

Now $T_{X'}$ is trivial, i.e., isomorphic to $\mathbb{G}^n_{m,X'}$ for some integer n, hence $\hat{T}_{X'} \xrightarrow{\sim} \mathbb{Z}^n_{X'}$, and

$$\operatorname{Ext}^1_{X'}(\hat{T}_{X'}, \mathbb{Z}_{X'}) \simeq (\operatorname{Ext}^1_{X'}(\mathbb{Z}, \mathbb{Z}))^n \simeq (H^1(X', \mathbb{Z}))^n = 0.$$

The Hochschild–Serre spectral sequence

$$H^p(G, \operatorname{Ext}^q_{X'}(\hat{T}_{X'}, \mathbb{Z})) \Rightarrow \operatorname{Ext}^n_X(\hat{T}, \mathbb{Z}) \tag{2.1.1}$$

gives rise to the short exact sequence of terms of low degree,

$$0 \to H^1(G, \operatorname{Hom}_{\mathbb{Z}}(\hat{T}(X'), \mathbb{Z})) \to \operatorname{Ext}^1_X(\hat{T}, \mathbb{Z}) \to \operatorname{Ext}^1_{X'}(\hat{T}_{X'}, \mathbb{Z}) \quad (=0). \tag{2.1.2}$$

Since T is flasque, $\hat{T}(X')$ is a flasque G-module (cf. 1.2 and 1.1), and the left-hand-side group is zero (0.5(ii)). Hence (i).

(ii) Since $\hat{T}$ is of finite generation and X is quasicompact and quasiseparated, $\operatorname{Ext}^1_X(\hat{T}, \cdot)$ commutes with arbitrary direct sums (cf. SGA4, VII, 3.3, IX, 2.7.3). It is thus enough to prove the statement for one morphism $i\colon Y \to X$ with Y connected and geometrically unibranch. There is a straightforward injection

$$\operatorname{Ext}^1_X(\hat{T}, i_*\mathbb{Z}) \hookrightarrow \operatorname{Ext}^1_Y(i^*\hat{T}, \mathbb{Z})$$

which may be seen as the edge map $E_2^{1,0} \hookrightarrow E^1$ in the spectral sequence

$$\operatorname{Ext}^p_X(\hat{T}, R^q i_*\mathbb{Z}) \Rightarrow \operatorname{Ext}^n_Y(i^*\hat{T}, \mathbb{Z}). \tag{2.1.3}$$

Since T_Y is a flasque Y-torus (Proposition 1.4) the étale sheaves isomorphism $i^*\hat{T} \xrightarrow{\sim} \hat{T}_Y$ (0.3) together with (i) imply $\mathrm{Ext}^1_Y(i^*\hat{T}, \mathbb{Z}) = 0$, and the above injection now implies $\mathrm{Ext}^1_X(\hat{T}, i_*\mathbb{Z}) = 0$, hence (ii).

THEOREM 2.2. *Let X be an integral noetherian scheme. Assume that its strictly local rings are factorial, e.g., X is regular. Let K be the field of fractions of X, and let $U \subset X$ be a nonempty open set. Let T be a flasque X-torus.*

(i) *The restriction maps*

$$H^1(X, T) \to H^1(U, T_U) \qquad \text{and} \qquad H^1(X, T) \to H^1(K, T_K)$$

are onto.

(ii) *The restriction maps*

$$H^2(X, T) \to H^2(U, T_U) \qquad \text{and} \qquad H^2(X, T) \to H^2(K, T_K)$$

are injective.

Note that $(\mathrm{ii})_U$ trivially follows from $(\mathrm{ii})_K$ and that by [SGA4, VII, 5.9], $(\mathrm{i})_K$ follows from $(\mathrm{i})_U$ for all U, and $(\mathrm{ii})_K$ follows from $(\mathrm{ii})_U$ for all U. Statement $(\mathrm{i})_U$ lead us to call such tori "flasque": they should really be called torseur-flasque. Statement $(\mathrm{ii})_K$, as indeed the following proof, is an extension of Grothendieck's result [GB II, Corollary 1.8] for the cohomological Brauer group $H^2(X, \mathbb{G}_m)$, itself an extension [GB II, Corollary 1.10] of the Auslander–Goldman result that the "usual" Brauer group of a regular integral ring injects into the Brauer group of its field of fractions.

Let us recall: the strict local rings of X are the strict henselizations of its local rings; if ${}^{\mathrm{hs}}A$ is the strict henselization of a noetherian local ring A, A is regular if and only if ${}^{\mathrm{hs}}A$ is regular [EGA IV, 18.8.13]; if this is the case, then A and ${}^{\mathrm{hs}}A$ are factorial (Auslander–Buchsbaum [EGA IV, 21.11.1]); ${}^{\mathrm{hs}}A$ factorial implies A factorial [EGA IV, 21.13.12].

Proof of Theorem 2.2. Let first X be an arbitrary scheme and let $\mathscr{F}$ and $\mathscr{G}$ be two abelian étale sheaves on X. There is a local-to-global spectral sequence [SGA4, V, 6.1],

$$H^p(X, \mathscr{E}xt^q_X(\mathscr{F}, \mathscr{G})) \Rightarrow \mathrm{Ext}^n_X(\mathscr{F}, \mathscr{G}). \tag{2.2.1}$$

Let $\theta_n = \theta_n(X; \mathscr{F}, \mathscr{G})$ be its edge map $E_2^{n,0} \to E^n$, i.e.,

$$\theta_n\colon\ H^n(X, \mathscr{H}om_X(\mathscr{F}, \mathscr{G})) \to \mathrm{Ext}^n_X(\mathscr{F}, \mathscr{G}). \tag{2.2.2}$$

Let T be an arbitrary X-torus. We have already mentioned (0.3.1) that the natural map

$$T \to \mathscr{H}om_X(\hat{T}, \mathbb{G}_{m,X}) \tag{2.2.3}$$

is an isomorphism of étale sheaves. On the other hand,

$$\mathscr{E}xt^q_X(\hat{T}, \mathbb{G}_{m,X}) = 0 \qquad \text{for} \quad q > 0, \tag{2.2.4}$$

a statement which may be checked locally (for the étale topology) hence boils down to the obvious case where $\hat{T} = \mathbb{Z}_X$.

For $\mathscr{F} = \hat{T}$ and $\mathscr{G} = \mathbb{G}_{m,X}$, the spectral sequence (2.2.1) completely degenerates by (2.2.4), and yields canonical isomorphisms

$$\theta_n\colon\ H^n(X, \mathscr{H}om_X(\hat{T}, \mathbb{G}_{m,X})) \xrightarrow{\sim} \operatorname{Ext}^n_X(\hat{T}, \mathbb{G}_{m,X}). \tag{2.2.5}$$

Let $i\colon Y \to X$ be a morphism of schemes. For each $n \geqslant 0$, there is a commutative diagram

$$\begin{tikzcd}
H^n(X, \mathscr{H}om_X(\hat{T}, \mathbb{G}_{m,X})) \arrow[r] \arrow[d, "\theta_n"] & H^n(Y, \mathscr{H}om_Y(i^*\hat{T}, i^*\mathbb{G}_{m,X})) \arrow[r] \arrow[d, "\theta_n"] & H^n(Y, \mathscr{H}om_Y(i^*\hat{T}, \mathbb{G}_{m,Y})) \arrow[d, "\theta_n"] \\
\operatorname{Ext}^n_X(\hat{T}, \mathbb{G}_{m,X}) \arrow[r] & \operatorname{Ext}^n_Y(i^*\hat{T}, i^*\mathbb{G}_{m,X}) \arrow[r] & \operatorname{Ext}^n_Y(i^*\hat{T}, \mathbb{G}_{m,Y}).
\end{tikzcd} \tag{2.2.6}$$

Here the horizontal maps in the left-hand square are the natural maps induced by i and those in the right-hand square are induced by the canonical morphism $i^*\mathbb{G}_{m,X} \to \mathbb{G}_{m,Y}$. Functoriality of (2.2.1) in X and $\mathscr{G}$ implies that the diagram is commutative.

The adjunction formula and the spectral sequence of composite functors give rise to the spectral sequence

$$\operatorname{Ext}^p_X(\hat{T}, R^q i_* \mathbb{G}_{m,Y}) \Rightarrow \operatorname{Ext}^n_Y(i^*\hat{T}, \mathbb{G}_{m,Y}) \tag{2.2.7}$$

whose edge maps $E_2^{n,0} \to E^n$ define maps

$$\varphi_n\colon\ \operatorname{Ext}^n_X(\hat{T}, i_*\mathbb{G}_{m,Y}) \to \operatorname{Ext}^n_Y(i^*\hat{T}, \mathbb{G}_{m,Y}). \tag{2.2.8}$$

Using the identification of étale sheaves $i^*\hat{T} \xrightarrow{\sim} \hat{T}_Y$ (0.3) and putting (2.2.6) and (2.2.8) together, we get the diagram

$$\begin{tikzcd}
H^n(X, T) \arrow[r, "\rho_n"] \arrow[d, "\theta_n", "\approx"'] & H^n(Y, T_Y) \arrow[d, "\theta_n", "\approx"'] \\
\operatorname{Ext}^n_X(\hat{T}, \mathbb{G}_{m,X}) \arrow[r, "\sigma_n"] \arrow[dr, "\lambda_n"'] & \operatorname{Ext}^n_Y(\hat{T}_Y, \mathbb{G}_{m,Y}) \\
& \operatorname{Ext}^n_X(\hat{T}, i_*\mathbb{G}_{m,Y}) \arrow[u, "\varphi_n"']
\end{tikzcd} \tag{2.2.9}$$

where λ_n is induced by the canonical map $\mathbb{G}_{m,X} \to i_* \mathbb{G}_{m,Y}$, and where ρ_n and σ_n are the natural maps induced by i. We have seen that the square is commutative. That the triangle commutes up to a sign may be checked by tedious but standard homological algebra [10].

Assume i satisfies

$$R^1 i_* \mathbb{G}_{m,Y} = 0. \tag{2.2.10}$$

Then $E_2^{0,1} = 0$ in spectral sequence (2.2.7), so that φ_1 is an isomorphism and φ_2 is an injection. Diagram (2.2.9) for $n = 1$ and $n = 2$ now gives:

$$\rho_1 \text{ onto} \Leftrightarrow \lambda_1 \text{ onto}; \qquad \rho_2 \text{ injective} \Leftrightarrow \lambda_2 \text{ injective}. \tag{2.2.11}$$

For X as in Theorem 2.2, and $i = i_K$: Spec $K \to X$ or $i = i_U$: $U \to X$, let us check (2.2.10). The étale sheaf $R^1 i_* \mathbb{G}_{m,Y}$ is the sheaf associated to the presheaf $X' \mapsto \operatorname{Pic}(Y \times_X X')$ [SGA4, V, 5.1]. When $Y = \operatorname{Spec} K$ and $i = i_K$, this presheaf is already zero by Hilbert's Theorem 90. When $Y = U$ and $i = i_U$, the assumption on X implies that the local rings of X' are factorial, hence the restriction maps $\operatorname{Pic} X' \to \operatorname{Pic} U'$ for $U' = U \times_X X'$ are onto [EGA IV, 21.6.11]. Since the sheaf associated to $X' \mapsto \operatorname{Pic} X'$ is zero so is the sheaf associated to $X' \mapsto \operatorname{Pic} U'$.

We are thus reduced to study λ_1 and λ_2. Since X is noetherian and its strictly local rings are factorial, there is a canonical exact sequence of étale sheaves on X

$$0 \to \mathbb{G}_{m,X} \to i_* \mathbb{G}_{m,Y} \to \bigoplus_x i_{x*} \mathbb{Z}_x \to 0 \tag{2.2.12}$$

for $Y = \operatorname{Spec} K$ and $i = i_K$ or $Y = U$ and $i = i_U$ [GB II, Sect. 1 (2) and (3)]. Here x runs through the codimension 1 points of X outside Y, the sheaf $\mathbb{Z}_x$ is the "constant" sheaf $\mathbb{Z}$ on the spectrum of the residue class field $\kappa(x)$ of X at x, and i_x: Spec $\kappa(x) \to X$ is the canonical morphism. From (2.2.12) we deduce the exact sequence

$$\begin{aligned} \operatorname{Ext}^1_X(\hat{T}, \mathbb{G}_{m,X}) &\xrightarrow{\lambda_1} \operatorname{Ext}^1_X(\hat{T}, i_* \mathbb{G}_{m,Y}) \to \operatorname{Ext}^1_X\left(\hat{T}, \bigoplus_x i_{x*} \mathbb{Z}_x\right) \to \operatorname{Ext}^2_X(\hat{T}, \mathbb{G}_{m,X}) \\ &\xrightarrow{\lambda_2} \operatorname{Ext}^2_X(\hat{T}, i_* \mathbb{G}_{m,Y}), \end{aligned}$$

where the extreme maps are precisely λ_1 and λ_2. Now, since T is a *flasque* torus, Lemma 2.1(ii) (with $i_j = i_x$, and $X_j = \operatorname{Spec} \kappa(x)$, the spectrum of a field!) implies that the middle term $\operatorname{Ext}^1_X(\hat{T}, \bigoplus_x i_{x*} \mathbb{Z}_x)$ vanishes. Hence λ_1 is onto and λ_2 is injective, which by (2.2.11) completes the proof.

Remark 2.3. In view of Proposition 1.4, statement $(\mathrm{i})_U$ appears as a generalization of Proposition 9 of [8], which handled the case of a smooth variety X over a field k and an X-torus coming from a flasque k-torus.

LEMMA 2.4. *Let A be a normal integral noetherian ring. Let S be an A-group of multiplicative type. The natural map*

$$H^1(A_{\mathrm{fppf}}, S) \to H^1(A[t]_{\mathrm{fppf}}, S_{A[t]}),$$

where t denotes an indeterminate, is an isomorphism.

Proof. This is well known for $S = \mathbb{G}_{m,A}$. Since the units A^* of A coincide with the units $A[t]^*$ of $A[t]$, the analogous result for the group $\mu_{n,A}$ of nth roots of unity (n positive integer) follows from taking the cohomology of the Kummer sequence. Hence the result holds for a split A-group of multiplicative type. For a general S, let B/A be a finite étale integral Galois extension, with group G, which splits S. Writing the first four terms of the exact sequence of terms of low degree for the Hochschild–Serre spectral sequence

$$H^p(G, H^q(B_{\mathrm{fppf}}, S_B)) \Rightarrow H^n(A_{\mathrm{fppf}}, S) \tag{2.4.1}$$

and comparing it with the analogous sequence for $B[t]/A[t]$ now yields the result.

Remark 2.5. When A is regular and the characteristic of the fraction field of A is zero, the same result holds with H^2 in place of H^1 [9].

COROLLARY 2.6. *Let A be a regular integral domain, and let F be a flasque A-torus. Let U be a nonempty open set of the n-dimensional affine space $\mathbb{A}^n_A$. The composite map*

$$H^1(A, F) \to H^1(\mathbb{A}^n_A, F) \to H^1(U, F)$$

is a surjective map, and it is a bijection if A is a field.

Proof. The surjectivity assertion follows from $2.2(\mathrm{i})_U$ and 2.4. If $A = k$ is a field, either k is infinite and $U(k)$ is not empty, hence the composite map has a section, or k is finite and $H^1(k, T) = 0$ for any k-torus T by Lang's theorem.

LEMMA 2.7. *Let A be a regular ring of finite type over $\mathbb{Z}$, and let S be an A-group of multiplicative type. The group $H^1(A_{\mathrm{fppf}}, S)$ is an abelian group of finite type.*

Proof. We may assume that A is connected, i.e., integral. Let B/A be a finite étale integral Galois extension with group G, which splits S. The ring

B is regular and of finite type over $\mathbb{Z}$, hence B^* and $\operatorname{Pic} B$ are abelian groups of finite type by the Dirichlet unit theorem and the Mordell–Weil–Néron theorem (Roquette, cf. [8, Sect. 3]). The same therefore holds for $S_B(B)$ and $H^1(B_{\text{fppf}}, S_B)$. The result now follows from writing down the first three terms of the exact sequence of terms of low degrees of the spectral sequence (2.4.1) (i.e., the restriction-inflation sequence).

COROLLARY 2.8 [8, Sect. 3, théorème 1]. *Let K be a field of finite type over the prime field. If F is a flasque K-torus, the group $H^1(K, F)$ is a finite group.*

Proof. Since K is of finite type over its prime field, K is the field of fractions of a subring A which is regular and of finite type (as algebra) over $\mathbb{Z}$. This follows from a result of Nagata [EGA IV, 6.12.6]. Here is a simple proof: write K as a finite separable extension of a field K_0 which is purely transcendental over the prime field, $K_0 = k(x_1, ..., x_n)$, with $k = \mathbb{Q}$ or $k = \mathbb{F}_p$. Write $A_0 = \mathbb{Z}[x_1, ..., x_n]$ or $A_0 = \mathbb{F}_p[x_1, ..., x_n]$, and let A_1 be the integral closure of A_0 in K. As in 1.5, inverting some element of A_0 in A_1 produces a suitable A. Now by 1.5 we may assume, after inverting another element in A, that F comes from a flasque A-torus F_1. The restriction map

$$H^1(A, F_1) \to H^1(K, F)$$

is onto by Theorem 2.2 $(\mathrm{i})_K$, the left-hand-side group is of finite type by Lemma 2.7, and the right-hand-side group is torsion, hence the result.

3. R-EQUIVALENCE ON TORI

Let k be a field. R-equivalence on the set $X(k)$ of k-rational points of a k-variety X is the coarsest equivalence relation for which two points A and B of $X(k)$ are in the same class if there exists a k-morphism $f\colon U \to X$ with U open in $\mathbb{A}^1_k$ such that A and B belong to $f(U(k))$.

THEOREM 3.1 [8, Theorem 2, p. 199]. *Let k be a field and T be a k-torus. If $1 \to F \to P \to T \to 1$ is a flasque resolution of T, it induces an isomorphism of abelian groups*

$$T(k)/R \xrightarrow{\sim} H^1(k, F);$$

if k is of finite type over the prime field, $T(k)/R$ is finite.

Proof. Let $T \subset X$ be a smooth k-compactification of the torus T. Such compactifications may be constructed by the method of toroidal embeddings (Brylinski [4]). Since F is flasque and X is smooth, the torseur over T under F defined by $P \to T$ extends (up to isomorphism) to a torseur $\mathscr{T}$ over X, by Theorem 2.2(i). Taking fibres of $\mathscr{T}$ defines a map $X(k) \to H^1(k, F)$ whose composite with $T(k) \to X(k)$ coincides (up to a sign) with the coboundary map ∂ in the étale (or Galois) cohomology sequence of the flasque resolution. Now since X/k is proper, the map $X(k) \to H^1(k, F)$ factorizes through $X(k)/R$: any $f: U \to X$ as in the preliminaries of the theorem extends to a k-morphism $f: \mathbb{A}^1_k \to X$, and $H^1(k, F) \xrightarrow{\sim} H^1(\mathbb{A}^1_k, F)$ for F as for any k-torus by Lemma 2.4. Since P is quasitrivial, $P(k)/R = 1$ because P is an open set in an affine space over k, and $H^1(k, P) = 0$ by Hilbert's Theorem 90 (cf. 0.4.1). The cohomology sequence thus yields an isomorphism $T(k)/\mathrm{im}(P(k)) \xrightarrow{\sim} H^1(k, F)$ which *factorizes* through $T(k)/R$. Hence the first statement of the theorem. The second now follows from Corollary 2.8.

Remark 3.2. The proof of Theorem 3.1 given in [8] did not use the extension theorem for torseurs, but it used a special case of Corollary 2.6. For yet another proof, using a natural construction of $\mathscr{T}/X$, we refer to [12].

As an example that the extension property is by no means a general fact, let us consider the exact sequence

$$1 \longrightarrow R^1_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \longrightarrow R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \xrightarrow{N_{\mathbb{C}/\mathbb{R}}} \mathbb{G}_{m,\mathbb{R}} \longrightarrow 1.$$

The induced surjective coboundary map

$$\mathbb{R}^* \twoheadrightarrow H^1(\mathbb{R}, R^1_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m) = \mathbb{Z}/2$$

does not factorize through R-equivalence (clearly trivial on $\mathbb{G}_{m,\mathbb{R}}(\mathbb{R})$). Hence the torseur over $\mathbb{G}_{m,\mathbb{R}}$ which this sequence defines is not isomorphic to the restriction of a torseur over $\mathbb{P}^1_{\mathbb{R}}$ (for a similar situation of a torseur which is "ramified at infinity," think of the Kummer sequence).

4. Going Over from a Regular Local Ring to Its Field of Fractions

In contrast with Section 2, the cohomology used in this section is the *fppf* topology, and we write $H^i(X, M) = H^i(X_{\mathrm{fppf}}, M)$. For M/X smooth, e.g., M a torus (see 0.2), these groups coincide with the étale cohomology groups (2.0).

Theorem 4.1. *Let X be an integral noetherian scheme and let K be its field of fractions. Assume that the strictly local rings of X are factorial, for* *instance that X is regular. Let M be an X-group of multiplicative type, and let $X' \to X$ be a connected étale Galois cover which splits M.*

The restriction map

$$\rho_1\colon H^1(X, M) \to H^1(K, M_K)$$

is injective in any of the following cases:

(i) *$X = \operatorname{Spec} A$ and A is a semilocal ring;*

(ii) *X is an open set of an affine space $\mathbb{A}^n_A$ over an integral noetherian semilocal ring A, the strictly local rings of A are factorial, and $M = N \times_A X$ for N an A-group of multiplicative type;*

(iii) *any subcover $X'' \to X$ of $X' \to X$ satisfies* $\operatorname{Pic} X'' = 0$.

Proof. (ii) clearly generalizes (i). Let us show that (iii) generalizes (ii). Let B/A be an integral étale cover, Galois with group G, which splits N. As $X' \to X$ we may take the connected étale cover $X \times_A B \to X$, which is Galois with group G. Any subcover $X'' \to X$ is then of the type $X \times_C A \to X$ for C/A an étale subcover of B/A. Since A is noetherian semilocal, so is C, hence $\operatorname{Pic} C = 0$ [3, Chap. 2, Sect. 5, No. 3, Proposition 5]. Since C is normal noetherian, $\operatorname{Pic} C = \operatorname{Pic} \mathbb{A}^n_C$ (this was used in Lemma 2.4; cf. [3, Chap. 7, Sect. 1, Nos. 9, 10, Proposition 18; EGA IV, $\mathrm{Err}_{\mathrm{IV}}$, 21.4.13]). Thus $\operatorname{Pic} \mathbb{A}^n_C = 0$. Since the strictly local rings of A are factorial, so are the local rings of the noetherian scheme $\mathbb{A}^n_C$, hence the restriction map $\operatorname{Pic} \mathbb{A}^n_C \to \operatorname{Pic}(X \times_A C)$ is surjective [EGA IV, 21.6.11] and $\operatorname{Pic}(X \times_A C) = 0$.

It is thus enough to prove (iii). Assume first that M is a torus T, and let

$$1 \to F \to P \to T \to 1$$

be a flasque resolution (1.3.3) of T, built as in Proposition 1.3, i.e., with P and F split by $X' \to X$. This resolution yields a commutative diagram

$$\begin{array}{ccccc}
H^1(X, P) & \longrightarrow & H^1(X, T) & \longrightarrow & H^2(X, F) \\
 & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} \\
 & & H^1(K, T_K) & \longrightarrow & H^2(K, F_K)
\end{array}$$

with exact first row. Now the quasitrivial torus P is isomorphic to a finite product of tori $R_{X''/X}\mathbb{G}_m$, for $X'' \to X$ étale subcovers of $X' \to X$. Assumption (iii) together with (2.0) and (0.4.1) then implies $H^1(X, P) = 0$. The general assumptions of the theorem together with (2.0) and Theorem 2.2(ii) imply that ρ_2 is injective. That ρ_1 is injective now follows from the diagram.

If M is an arbitrary X-group of multiplicative type, there is an exact sequence of X-groups of multiplicative type,

$$1 \to T \to M \to \mu \to 1,$$

where T is a torus and μ is a finite X-group scheme, and T and μ are split by $X' \to X$. Indeed it is enough to define μ by taking for $\hat{\mu}(X')$ the torsion subgroup of $\hat{M}(X')$ (cf. [SGA3, VIII, 3.1] in the diagonalizable case). This exact sequence induces an exact sequence of *fppf* sheaves on X, hence the following commutative diagram of exact sequences

$$\begin{array}{ccccccc}
\mu(X) & \longrightarrow & H^1(X, T) & \longrightarrow & H^1(X, M) & \longrightarrow & H^1(X, \mu) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\rho_1^T} & & \downarrow{\scriptstyle\rho_1^M} & & \downarrow{\scriptstyle\beta} \\
\mu(K) & \longrightarrow & H^1(K, T_K) & \longrightarrow & H^1(K, M_K) & \longrightarrow & H^1(K, \mu_K).
\end{array}$$

Since X is locally factorial hence normal and since μ is finite over X, α is a surjective map [EGA II, 6.1.14], and is thus an isomorphism. Similarly, since any X-torseur under μ is representable by a finite X-scheme, and X is normal, any rational section of such a torseur extends to a section: β is injective. Since T is split by X', we already know that ρ_1^T is injective. Chasing through the diagram yields ρ_1^M injective.

COROLLARY 4.2. *With the same assumptions and notations as in Theorem* 4.1, *if M is a flasque X-torus, the restriction map*

$$\rho_1\colon\ H^1(X, M) \to H^1(K, M_K)$$

is an isomorphism in cases (i), (ii), (iii).

This follows immediately from the above theorem and Theorem $2.2(\mathrm{i})_K$. The same result also holds for $H^1(X, M) \to H^1(U, M)$ for U nonempty open in X.

THEOREM 4.3. *With the same assumptions and notations as in Theorem* 4.1, *the restriction map*

$$\rho_2\colon\ H^2(X, M) \to H^2(K, M_K)$$

is injective in all cases (i), (ii), (iii).

Proof. Let

$$1 \to M \to F \to P \to 1$$

be an exact sequence of X-groups of multiplicative type of type (1.3.2), i.e., with F a flasque X-torus and P a quasitrivial X-torus, both split by $X' \to X$. This gives the commutative diagram of exact sequences

$$\begin{tikzcd} H^1(X,P) \arrow[r] & H^2(X,M) \arrow[r] \arrow[d, "\rho_2^M"] & H^2(X,F) \arrow[d, "\rho_2^F"] \\ & H^2(K,M_K) \arrow[r] & H^2(K,F_K). \end{tikzcd}$$

As already seen in the proof of Theorem 4.1, we have $H^1(X,P)=0$, and ρ_2^F is injective by Theorem 2.2(ii). Hence ρ_2^M is injective.

Remark 4.4. For less farfetched proofs of 4.1 and 4.3, see [9]: one can first prove 4.1 and 4.3 and then deduce 2.2.

5. Norm Forms

Letting B/A be a cover of rings, we shall use the A-group scheme $R^1_{B/A}\mathbb{G}_m$ (cf. 0.4) to illustrate Theorem 4.1 when B/A is étale and A is for instance regular, and to give counterexamples when these assumptions are weakened.

LEMMA 5.1. *Let B/A be a cover of rings. There is a natural injection*

$$A^*/N(B^*) \hookrightarrow H^1(A_{\mathrm{fppf}}, R^1_{B/A}\mathbb{G}_m) \tag{5.1.1}$$

where $N=N_{B/A}$, which is functorial in A. In particular, when A is integral with field of fractions K, there is a commutative diagram with injective horizontal maps (here $L=B\otimes_A K$):

$$\begin{tikzcd} A^*/N(B^*) \arrow[r, hook] \arrow[d, "\rho"] & H^1(A_{\mathrm{fppf}}, R^1_{B/A}\mathbb{G}_m) \arrow[d, "\rho_1"] \\ K^*/N(L^*) \arrow[r, hook] & H^1(K_{\mathrm{fppf}}, R^1_{B_K/K}\mathbb{G}_m). \end{tikzcd} \tag{5.1.2}$$

Proof. Let us show that $N_{B/A}: R_{B/A}\mathbb{G}_m \to \mathbb{G}_{m,A}$ is a faithfully flat morphism of finite presentation. We may assume that B/A is of constant rank n, in which case the composite map $\mathbb{G}_{m,A} \to R_{B/A}\mathbb{G}_m \xrightarrow{N_{B/A}} \mathbb{G}_{m,A}$ is multiplication by n, hence finite, hence surjective. Flatness may be checked fibrewise [EGA IV, 11.3.11], hence is reduced to the case A is a field, where it is easy to prove since $\mathbb{G}_{m,A}$ is then of dimension 1. We thus get an "exact sequence"

$$1 \longrightarrow R^1_{B/A}\mathbb{G}_m \longrightarrow R_{B/A}\mathbb{G}_m \xrightarrow{N_{B/A}} \mathbb{G}_{m,A} \longrightarrow 1 \tag{5.1.3}$$

which turns $R_{B/A}\mathbb{G}_m$ into an *fppf* torseur over $\mathbb{G}_{m,A}$ under $R^1_{B/A}\mathbb{G}_m$ and defines an exact sequence of abelian *fppf* sheaves on X. Taking the cohomology sequence yields the injection (5.1.1). Functoriality follows from the fact that norm is compatible with base change.

LEMMA 5.2. *Let B/A be a cover. The A-group scheme $R^1_{B/A}\mathbb{G}_m$ is flat of finite presentation. It is smooth if and only if B/A is tamely ramified in at least one point of each fibre of* $\operatorname{Spec} B \to \operatorname{Spec} A$. *It is a torus if and only if B/A is étale.*

Proof. We have already proved the first statement. Let us recall that $Y/X = \operatorname{Spec} B/\operatorname{Spec} A$ is tamely ramified at $y \in Y$ with image $x \in X$ when the residue extension $\kappa(y)/\kappa(x)$ is separable and the length of the artinian ring $\mathcal{O}_{Y,y} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is prime to the characteristic of the residue field $\kappa(x)$.

Smoothness may be checked fibrewise [EGA IV, 17.5.1*b*], so we may assume A is a field. Since the tangent linear map, at the origin, of $N\colon R_{B/A}\mathbb{G}_m \to \mathbb{G}_{m,A}$ is the trace map $R_{B/A}\mathbb{G}_a \to \mathbb{G}_{a,A}$, smoothness of the A-algebraic group $R^1_{B/A}\mathbb{G}_m$ amounts to the non-vanishing of this trace map (i.e., cf. [3, Sect. 12, Proposition 6]) to the existence of $y \in Y$ where $Y \to \operatorname{Spec} A$ is tamely ramified.

We already know (0.4.2 and 0.4.3) that $R^1_{B/A}\mathbb{G}_m$ is a torus if B/A is étale. Conversely, simply note that it is not a torus when A is a field and B is a nonseparable artinian algebra, finite over A.

PROPOSITION 5.3. *Let A be an integral semilocal ring whose strictly local rings are factorial, for instance regular. Let K be the field of fractions of A, let B/A be an étale cover, let $L = B \otimes_A K$, i.e., the field of fractions of B if B is integral. If an element of A^* is a norm of an element of L^*, it is a norm of an element of B^*.*

The proof immediately follows from Theorem 4.1 and Lemma 5.1.

PROPOSITION 5.4. *Let A and K be as in 5.3. Let B_i/A $(i = 1,..., n)$ be integral étale covers with fields of fractions L_i, let a_i $(i = 1,..., n)$ be integers, and set $N_i = N_{B_i/A}$. If an element of A^* may be written as $\prod_{i=1}^n N_i(x_i^{a_i})$ with $x_i \in L_i^*$, it can also be written as such a product with $x_i \in B_i^*$.*

Proof. One may assume that none of the a_i is zero. The proof is then similar to the proof of 5.3. One applies Theorem 4.1 and Lemma 5.1 to the A-group of multiplicative type defined as the kernel of the map

$$\prod_{i=1}^{n} R_{B_i/A}\mathbb{G}_m \to \mathbb{G}_{m,A},$$

$$\{x_i\}_{i=1,\dots,n} \mapsto \prod_{i=1}^{n} N_i(x_i)^{a_i}.$$

Using Theorem 4.1(ii) rather than 4.1(i) it is easy to extend the above two propositions. Let us simply state:

PROPOSITION 5.5. *Let B/A be an étale cover of regular integral semilocal rings and let L/K be the induced extension of fraction fields. Let n be a non-negative integer, and let g be a nonzero element of the polynomial ring $A[t_1,\dots,t_n]$. If an element $f \in A[t_1,\dots,t_n]_g^*$ (g is inverted) is a norm for the extension of fields $L(t_1,\dots,t_n)/K(t_1,\dots,t_n)$, then it is a norm of an element of $B[t_1,\dots,t_n]_g^*$.*

(Recall that if A is regular, so is $A[t_1,\dots,t_n]$.)

When B/A is Galois, and $f \in A[t_1,\dots,t_n]_g$ is not necessarily a unit, a similar result holds; the following proposition extends results of Choi–Lam–Reznick–Rosenberg [6].

PROPOSITION 5.6. *Let B/A and L/K be as in* 5.5, *and assume that B/A is a Galois cover with group G. Let g be as in* 5.5. *If $f \in A[t_1,\dots,t_n]_g$ is a norm of an element of $L(t_1,\dots,t_n)$, it is a norm of an element of $B[t_1,\dots,t_n]_g$.*

Proof. Let us denote $U = \operatorname{Spec} A[t_1,\dots,t_n]_g$ and $V = \operatorname{Spec} B[t_1,\dots,t_n]_g$. As usual, X^1 is the set of codimension 1 points of a scheme X. We shall use compatibilities between the norm at the function and divisor levels which may all be found in [EGA IV, 21]. We have

$$\operatorname{div}_U(f) = \operatorname{div}_U(N(h)) = N(\operatorname{div}_V(h)) = N\left(\sum_{y \in V^1} m_y\, y\right)$$

with $m_y \in \mathbb{Z}$ zero for almost all y. Now, since V/U is *Galois* with group G, the various $y \in V^1$ above a given $x \in U^1$ are transitively permuted by G; in particular for such y and x, there exists a positive integer r_x depending only on x such that $N(y) = r_x x$. Now,

$$\sum_{x \in U^1} n_x x = \operatorname{div}_U(f) = \sum_{x \in U^1} N\left(\sum_{y \mapsto x} m_y\, y\right) = \sum_{x \in U^1}\left(\sum_{y \mapsto x} m_y\right) r_x x,$$

hence for each $x \in U^1$,

$$n_x = \left(\sum_{y \mapsto x} m_y\right) r_x.$$

From $n_x \geqslant 0$, we conclude $p_x = \sum_{y \mapsto x} m_y \geqslant 0$, and

$$n_x x = N(p_x y_x) \qquad \text{for a fixed but arbitrary } y_x \in V^1 \text{ above } x.$$

The assumptions of the proposition imply that V is regular and Pic $V = 0$. Thus there exists $h_x \in B[t_1,..., t_n]_g$ with

$$\operatorname{div}_V(h_x) = p_x y_x.$$

Upon (finite) summation, we find $h_1 = \prod_{x \in U^1, p_x \neq 0} h_x \in B[t_1,..., t_n]_g$ with

$$\operatorname{div}_U(f) = N(\operatorname{div}_V(h_1)) = \operatorname{div}_U(N(h_1)).$$

Hence $f/N(h_1)$ is in $A[t_1,..., t_n]_g^*$, and by assumption it is a norm at the function field level. By 5.5, there exists $h_2 \in B[t_1,..., t_n]_g^*$ such that $f/N(h_1) = N(h_2)$. Hence f is the norm of $h_1 h_2 \in B[t_1,..., t_n]_g$.

Remark 5.7. Let A be an integral noetherian semilocal ring, let K be its field of fractions, let B/A be a cover, and let $L = B \otimes_A K$. Let us first list *cases when the natural map*

$$\rho\colon\ A^*/N(B^*) \to K^*/N(L^*)$$

is an injective map. By 5.3, this is true if

(i) *A is regular and B/A is étale.*

The map ρ is also injective under the sole assumption

(ii) *B is a discrete valuation ring.*

Under this assumption, A is automatically a discrete valuation ring, and the statement follows from considering the commutative diagram of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & B^* \arrow[r] \arrow[d, "N"] & L^* \arrow[r] \arrow[d, "N"] & \operatorname{Div} B \arrow[r] \arrow[d, "N"] & 0 \\
1 \arrow[r] & A^* \arrow[r] & K^* \arrow[r] & \operatorname{Div} A \arrow[r] & 0
\end{tikzcd}$$

since $N\colon \operatorname{Div} B \to \operatorname{Div} A$ simply reads as multiplication by the integer $[B\colon A]$ on $\mathbb{Z}$. Finally, ρ is also injective under the sole assumption

(iii) *B is regular integral, and $A = B^G$ for G a finite group of automorphisms of B.*

In this case, A is regular [EGA IV, 15.4.2]. Note that this is a quite natural case: it occurs for instance when B/A is a cover of semilocal regular integral domains such that L/K is a Galois extension of fields with group G. The proof in this case is as follows. Consider the exact sequence of G-modules

$$1 \to B^* \to L^* \to \operatorname{Div} B \to 0$$

(this uses $\operatorname{Pic} B = 0$). The G-module $\operatorname{Div} B$ is a permutation module, since B is regular. By Shapiro's lemma, the Tate cohomology group $\hat{H}^{-1}(G, \operatorname{Div} B)$ therefore vanishes. The cohomology sequence associated to the above sequence of G-modules now yields the injection

$$\hat{H}^0(G, B^*) \hookrightarrow \hat{H}^0(G, L^*),$$

hence the result by identifying $N_{B/A}$ with $\prod_{\sigma \in G} \sigma$, which may be checked at the level of L/K.

EXAMPLE 5.8. Even for A and B local integral domains, the map ρ, and a fortiori the map ρ_1 (5.1.2), *need not be injective* under the mere assumption

(iv) *A is normal and B/A is étale.*

Indeed, let A be the local ring at $(0, 0, 0)$ of the normal affine $\mathbb{R}$-surface defined by $x^2 + y^2 + z^2 = x^3$ in the affine space $\mathbb{A}^3_{\mathbb{R}}$. Take $B = A[\sqrt{-1}]$. The element $f = x - 1 \in A^*$ coincides with $(y^2 + z^2)/x^2 \in K^*$, hence is a norm for L/K. But it is not a norm for B/A, since $f(0, 0, 0) = -1$ is not a norm for $\mathbb{C}/\mathbb{R}$! Hence neither 4.1 nor, as is well known, 2.2(ii) holds under the mere assumption X normal (local).

EXAMPLE 5.9. For A and B local integral domains, ρ and a fortiori ρ_1 (cf. 5.1.2) *need not be injective* under the mere assumption

(v) *A is regular.*

Let indeed A be the local ring at the origin of the affine line $\operatorname{Spec} \mathbb{R}[x]$, and let B be the local ring at the double point of the cubic curve defined by $y^2 = x^2(x + 1)$ in the affine plane $\operatorname{Spec} \mathbb{R}[x, y]$. Let $\operatorname{Spec} B \to \operatorname{Spec} A$ be the projection map $(x, y) \mapsto x$. The function $f = -x - 1$ is in A^*. It is also in $N(L^*)$, since it is the norm of $(y/x) \in L^*$. But it cannot be the norm of $g \in B^*$, which would imply $-1 = f(0) = (g(0, 0))^2$.

Hence the assumption in (ii) cannot be replaced by the same assumption on A. This example also shows that even for X regular local, Theorem 4.1 does not hold for an arbitrary *smooth* X-group scheme: in Example 5.9, $R^1_{B/A}\mathbb{G}_m$ is smooth by Lemma 5.2, and of relative dimension 1. Note, however, that the generic fibre (over A) of this group scheme is a torus, hence connected, but that the special fibre is $\mu_{2,\mathbb{R}} \times_{\mathbb{R}} \mathbb{G}_{a,\mathbb{R}}$, hence is not connected.

6. Quadratic Forms

This section elaborates on [11] and [6]. The results, as well as Proposition 5.6, were announced in [6].

Proposition 6.1 [11, Proposition 4.3.2]. *Let A be an integral semilocal noetherian domain whose strictly local rings are factorial, e.g., A is regular. Let K be the field of fractions of A, assume $2 \in A^*$, and let f be a non-degenerate quadratic form over A, with rank $\leqslant 4$. If f is isotropic over K, it is also isotropic over A, i.e., it has a primitive zero in A.*

Proof. A primitive zero is a zero whose coordinates span A. Since A is semilocal and $2 \in A^*$, a nondegenerate quadratic form over A, of rank n, is equivalent over A to a form $\sum_{i=1}^{n} a_i x_i^2$ with $a_i \in A^*$ for all i. Also, for $a \in A^*$, isotropy of f amounts to isotropy of af. If $n=1$, the statement is obvious. For $n=2$, we may assume $f = x^2 - ay^2$ with $a \in A^*$, and the statement follows from the fact that A is a normal ring. For $n=3$, we may assume $f = x^2 - ay^2 - bz^2$ and exclude the trivial case when a is a square. Let B/A be the étale quadratic cover $A[\sqrt{a}]/A$ and let $L = B \otimes_A K$ be the field of fractions of B. If f is isotropic over K, the element $b \in A^*$ is a norm of an element of L^*, hence of an element of B^* by 5.3, and f has a primitive zero of the shape $(x_0, y_0, 1)$ in A. For $n=4$, we may assume $f = (x^2 - ay^2) - c(z^2 - bt^2)$ and exclude the trivial case when a or b (or c) is a square. The result then follows on applying Proposition 5.4 to the étale quadratic covers $B_1/A = A[\sqrt{a}]/A$ and $B_2/A = A[\sqrt{b}]/A$, with $a_1 = -a_2 = 1$: if f is isotropic over K, the element $c \in A^*$ may be written as $N_1(x_1)/N_2(x_2)$ with $x_i \in L_i^*$, hence also, according to 5.4, with $x_i \in B_i^*$; from $N_1(x_1) \in A^*$ follows readily that the zero of f we obtain is primitive.

Remark 6.2. Let us detail the case $n=4$ from the point of view of tori. Take f as above, and let us exclude the trivial cases when a or b or ab is a square (the last case reduces to the three variables case). Consider the étale Galois (connected) covers $B_3/A = A[\sqrt{ab}]/A$ and $B/A = A[\sqrt{a}, \sqrt{b}]/A$. This last cover is Galois with group $(\mathbb{Z}/2)^2 = \langle \sigma, \tau \rangle$, with $\sigma(\sqrt{a}) = \sqrt{a}$ and $\tau(\sqrt{b}) = \sqrt{b}$. Let T be the A-torus defined by the exact sequence

$$1 \longrightarrow T \longrightarrow R_{B_1/A}\mathbb{G}_m \times_A R_{B_2/A}\mathbb{G}_m \xrightarrow{\nu} \mathbb{G}_{m,A} \longrightarrow 1 \qquad (6.2.1)$$

with $\nu(x_1, x_2) = N_1(x_1)/N_2(x_2)$. Here is an explicit flasque resolution (1.3.3) for T,

$$1 \longrightarrow R_{B_3/A}\mathbb{G}_m \xrightarrow{\ \iota\ } R_{B/A}\mathbb{G}_m \times_A \mathbb{G}_{m,A} \xrightarrow{\ \pi\ } T \longrightarrow 1, \qquad (6.2.2)$$

with $\iota(x) = (x, N_3(x))$ and $\pi(x, y) = (x\sigma(x)/y, x\tau(x)/y)$. Via (6.2.1), $c \in A^*$ defines an element in $H^1(A, T)$ which goes over to $0 \in H^1(K, T)$. The conclusion follows from the fact that $\rho_1 : H^1(A, T) \to H^1(K, T)$ is an injection, as may be read on (6.2.2): one uses the vanishing of $\operatorname{Pic} A$ and $\operatorname{Pic} B$, together with the injectivity of Brauer groups $H^2(B_3, \mathbb{G}_m) \hookrightarrow H^2(L_3, \mathbb{G}_m)$ (use is made of (0.4.1)). Note that $B_3 = A[\sqrt{d}]$, for d the discriminant of f. The approach in [11, 4.3.2] is none but a computational transcription of the above torus-theoretic approach.

Remark 6.3. As pointed out in [11, 2.3.3 and 4.3.3], taking A to be the local ring at the origin of $\mathbb{R}[t_1,..., t_n]/(t_1^2 + \cdots + t_n^2)$ and $f = x_1^2 + \cdots + x_n^2$ for $n \geq 5$ (an example due to Craven–Rosenberg–Ware) shows that 6.1 does not extend verbatim to rank $f \geq 5$. For *A regular*, the question is open (cf. [11]); results for A regular of dimension 2 have been obtained by Ojanguren [26], who also handled the weaker problem of isomorphy of forms (as opposed to isotropy) in a very general manner ([25, 24]; see also Pardon [27, 28]).

Let us consider the question of representation of elements of a ring A, which we assume semilocal, integral and normal, with fraction field K, by a nondegenerate quadratic form f:

If $a \in A^*$ is represented by f over K, is it represented by f over A? (6.3.1)

If $a \in A$ is represented by f over K, is it represented by f over A? (6.3.2)

Standard considerations [11, Proposition 1.2], reduce (6.3.1) to the isotropy question in dimension one more than that of f. Hence 6.1 gives a positive answer to (6.3.1) for A as in 6.1 and rank $f \leq 3$. An analysis of the proof of 5.6 shows that at least in the case $n = 0$ of 5.6, the statement still holds for A as in 6.1. Now 5.6 yields a positive answer to (6.3.2) for such A when rank $f \leq 2$ (cf. [6, Theorem B]). (Even for A regular, (6.3.2) does not hold when rank $f \geq 4$ [6, 3.5]; the rank 3 case is open.) When A is a semilocal Dedekind domain (hence a principal ideal domain), for instance a discrete valuation ring, and also when A is an arbitrary valuation ring, the isotropy problem for nondegenerate quadratic forms of arbitrary rank has an easy positive answer; hence also (6.3.1); we shall now see that even (6.3.2), where a is not assumed to be a unit, has a positive answer. The following proposition was given an alternative proof by M. Kneser [6, Theorem 4.5]:

PROPOSITION 6.4. *Let A be an arbitrary valuation ring, with $2 \in A^*$, and let f be a nondegenerate quadratic form over A. If $a \in A$ (not necessarily a unit) is represented by f over the fraction field K of A, it is represented by f over A.*

Proof. We may assume $f = \sum_{i=1}^n a_i x_i^2$ with $a_i \in A^*$ for all i. The case $a \in A^*$ is known (see above) hence we may assume

$$a \notin A^*. \tag{6.4.1}$$

By hypothesis, there is an equality:

$$at^2 = \sum_{i=1}^{n} a_i x_i^2 \tag{6.4.2}$$

with t and all x_i in A. Since A is a valuation ring, we may assume that at least one of the elements $t, x_1, \ldots, x_n$ (not all zero!) is in A^*. If t is in A^*, we are done. Upon renumbering the variables, we may therefore assume:

$$t \notin A^* \quad \text{and} \quad x_1 \in A^*. \tag{6.4.3}$$

Now let

$$X_1 = \frac{x_1(1 + a_1^{-1}a) - 2a_1^{-1}t}{1 - a_1^{-1}a}, \qquad T = \frac{t(1 + a_1^{-1}a) - 2x_1}{1 - a_1^{-1}a}.$$

By (6.4.1) and (6.4.3), X_1 is in A (in fact in A^*) and T is in A^* (this is the important point). Now the equality

$$at^2 - a_1 x_1^2 = aT^2 - a_1 X_1^2 \tag{6.4.4}$$

implies, since T is in A^*, that a is represented by f over A. This equality actually comes from the formal identity

$$X_1 + \sqrt{a_1^{-1}a}\, T = (x_1 + \sqrt{a_1^{-1}a}\, t)[(1 - \sqrt{a_1^{-1}a})/(1 + \sqrt{a_1^{-1}a})]$$

which was used to produce X_1 and T (in a classical set-up, i.e., over a field, the same idea of letting elements of norm 1, i.e., rational points of a suitable torus $R^1 \mathbb{G}_m$, act is used to produce zeros of quadratic forms none of which components vanishes).

Proposition 6.4 admits the following extension, which applies for instance to a semilocal Dedekind domain with $2 \in A^*$, thus improving on [6, Theorems C and 4.1].

PROPOSITION 6.5. *Let K be a field, and let A be the intersection in K of finitely many proper independent valuation rings A_j ($j \in J$). Assume $2 \in A^*$. Let f be a nondegenerate quadratic form over A. If $a \in A$ is represented by f over K, it is represented by f over A.*

Proof. For the basic properties of such rings, see [3, Chap. 6, Sect. 7].

Step 1. Let v be a proper valuation on a field K. Let q be a regular quadratic form over K of rank $\geqslant 3$, and let h be a nonzero linear form in the same variables. Let $X \in K^n$ be a nontrivial zero of q. There exists a solution of $q(Y) = 0$ with $h(Y) \neq 0$ in every neighbourhood of X—for the product topology on K^n and the topology defined by v on K [3, Chap. 6, Sect. 5). This is proved by mimicking [5, p. 62, Lemma 2.8].

Step 2. Let v_j be the valuation associated to A_j. Let $q(x_1, ..., x_n)$ be a nondegenerate isotropic quadratic form over K in $n \geqslant 3$ variables. For each j, let $X_j = \{x_{j,i}\} \in K^n$ be a nontrivial zero of q. There exists a nontrivial zero $X = \{x_i\}$ of q in K^n such that

$$\forall i, \quad \forall j, \qquad v_j(x_i - x_{j,i}) > 0.$$

This is proved by mimicking [5, p. 89, Lemma 9.1]. We use Step 1 together with the weak approximation theorem [3, Chap. 6, Sect. 7].

Step 3. For f of rank one, the statement of Proposition 6 is trivial. Let rank $f \geqslant 2$. We may assume f in diagonal form $\sum_{i=1}^n a_i x_i^2$. Set

$$q(t, x_1, ..., x_n) = at^2 - \sum_{i=1}^n a_i x_i^2.$$

According to Proposition 6.4, for each j, there is a zero $(t, X_1, ..., X_n)$ of q with coefficients in A_j and $t \in A_j^*$. It then remains to apply Step 2.

7. GOING OVER TO THE RESIDUE CLASS FIELD OF A LOCAL RING

The aim of this and the following section is to unravel the torus-theoretic mechanism underlying Saltman's papers [30] and [31]. Extensive use is made of flasque tori and of their cohomological properties. Following Saltman and Swan [35], we then discuss the relation between the Grunwald–Wang theorem and the Noether problem.

LEMMA 7.1. *Let A be a semilocal ring, let κ_i be the residue class fields at the maximal ideals of A, and let P be an invertible A-torus. The natural map*

$$P(A) \to \prod_i P(\kappa_i)$$

is a surjective map.

Proof. It is enough to prove the result for a quasitrivial torus, and even for a torus of the shape $R_{B/A}\mathbb{G}_m$ for B/A an *étale* cover. Since B is semilocal and B/A is finite étale, the statement follows from the Chinese remainder theorem, as applied to B and its maximal ideals.

LEMMA 7.2. *Let A be a semilocal ring, and let κ_i be the residue class fields at the maximal ideals of A. Let T be an A-torus, and let*

$$1 \to F \to P \to T \to 1 \tag{7.2.1}$$

be an exact sequence of A-tori, with P a quasitrivial A-torus. The following statements are equivalent:

(i) *$T(A)$ maps onto the product $\prod_i T(\kappa_i)$;*

(ii) *$U(A)$ maps onto the product $\prod_i U(\kappa_i)$ for any open set U of T with $U_{\kappa_i} \neq \varnothing$ for each i;*

(iii) *there exists an open set U of T with $U_{\kappa_i} \neq \varnothing$ for each i such that $U(A)$ maps onto the product $\prod_i U(\kappa_i)$;*

(iv) *$H^1(A, F)$ maps onto the product $\prod_i H^1(\kappa_i, F_{\kappa_i})$.*

(for X an A-scheme, we denote $X_{\kappa_i} = X \times_{\operatorname{Spec} A} \operatorname{Spec} \kappa_i$, and $X(\kappa_i) = X_{\kappa_i}(\kappa_i)$.)

Proof. We have a commutative diagram of (*étale* or *fppf*, cf. 0.2) cohomology sequences

$$\begin{array}{ccccccc}
P(A) & \longrightarrow & T(A) & \longrightarrow & H^1(A, F) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\prod_i P(\kappa_i) & \longrightarrow & \prod_i T(\kappa_i) & \longrightarrow & \prod_i H^1(\kappa_i, F_{\kappa_i}) & \longrightarrow & 0,
\end{array} \tag{7.2.2}$$

since $H^1(A, P)$ and $H^1(\kappa_i, P_{\kappa_i})$ vanish by (0.4.1), Proposition 1.4 and Grothendieck's version of Hilbert's Theorem 90. Lemma 7.1 and this diagram show that (i) and (iv) are equivalent. That (i) implies (ii) follows from the fact that any element of $T(A)$ whose image in $T(\kappa_i)$ (for each i) belongs to $U(\kappa_i)$ necessarily belongs to $U(A)$. As for (ii) implies (iii), this is trivial. To show that (iii) implies (iv), it is enough to prove: for each i, the composite map

$$U(\kappa_i) \to T(\kappa_i) \to H^1(\kappa_i, F_{\kappa_i})$$

is surjective for any U with $U_{\kappa_i} \neq \emptyset$. Let us drop the index i. Either κ is finite, in which case $H^1(\kappa, F_\kappa) = 0$ since F_κ is a torus, or κ is infinite, which we now assume. The surjectivity assertion then follows from the exact sequence

$$P(\kappa) \to T(\kappa) \to H^1(\kappa, F_\kappa) \to 0$$

and the remark: for $\alpha \in T(k)$, the inverse image of $\alpha^{-1} \cdot U$ under $P \to T$ is a nonempty open set of P, itself κ-isomorphic to an open set of an affine space over κ, hence this open set of P contains a κ-rational point.

EXAMPLE 7.2.1. Although Proposition 7.4 will yield many theoretical examples of tori T over a discrete valuation ring (A, κ) such that $T(A)$ does not surject onto $T(\kappa)$, we want to give a concrete example. Let $A = \mathbb{Q}_3[t]_{(t)}$ be the localization at the prime ideal (t) of the polynomial ring in one variable t over the 3-adic field $\mathbb{Q}_3$. Let $B = A[\sqrt{3}, u]$, where $u = \sqrt{t+2}$. Then B is a discrete valuation ring with uniformizing parameter t, and B/A is étale Galois with group $G = \mathbb{Z}/2 \times \mathbb{Z}/2$. The residue extension is the Galois field extension $\kappa_B/\kappa_A = \mathbb{Q}_3(\sqrt{2}, \sqrt{3})/\mathbb{Q}_3$. Let K (resp. L) denote the fraction field of A (resp. B). Let T be the A-torus $R^1_{B/A}\mathbb{G}_m$ (cf. Lemma 5.2). According to [8, Propositions 1 and 15], we can find a sequence (7.2.1) such that the map $H^1(A, F) \to H^1(\kappa_A, F_{\kappa_A})$ simply reads $\hat{H}^{-1}(G, B^*) \to \hat{H}^{-1}(G, \kappa_B^*)$. That T is the announced example now follows from the following (a) and (b) together with Lemma 7.2.

(a) $\hat{H}^{-1}(G, \kappa_B^*) = \mathbb{Z}/2$. Indeed, local class field theory shows that $\hat{H}^{-1}(G, \kappa_B^*)$ is dual to $H^3(G, \mathbb{Z})$, and this last group is $\mathbb{Z}/2$ by the Künneth formula.

(b) $\hat{H}^{-1}(G, B^*) = 0$. First, the uniformizing parameter t yields a G-isomorphism of L^* with $B^* \oplus \mathbb{Z}$, hence $\hat{H}^{-1}(G, B^*) = \hat{H}^{-1}(G, L^*)$. Now let $F = \mathbb{Q}_3(\sqrt{3})$, so that $L = F(u)$. One may write any element $f \in L^*$ in a unique way as a product

$$f = cu^n \prod_p p^{v_p(f)},$$

where $c \in F^*$, $n \in \mathbb{Z}$, and p runs through the irreducible monic polynomials in $F[u]$, and $p \neq u$. This decomposition identifies the G-module L^* with the direct sum of an infinite permutation G-module (the free abelian group on the above p's) and of the abelian semidirect product $M = F^* \rtimes \mathbb{Z}$, where G acts on F^* and $\mathbb{Z}$ in the obvious way. Hence $\hat{H}^{-1}(G, L^*) = \hat{H}^{-1}(G, M)$ by Shapiro's lemma. The exact sequence

$$1 \to F^* \to M \to \mathbb{Z} \to 1$$

shows that $\hat{H}^{-1}(G, M)$ is a quotient of $\hat{H}^{-1}(G, F^*)$. But this last group is easily identified with $\{\pm 1\} \cap N_{F/\mathbb{Q}_3}(F^*)$, hence with $\{1\}$, since (-1) cannot be written as $x^2 - 3y^2$ with x and y in $\mathbb{Q}_3$.

PROPOSITION 7.3. *Let A be a semilocal ring, and let κ_i be the residue class fields at the maximal ideals. Let T be an A-torus, and assume that for each i, either* cd $\kappa_i \leqslant 1$, *or T_{κ_i} is κ_i-birationally a direct factor of a κ_i-rational variety, or T_{κ_i} is split by a metacyclic extension of κ_i. Then the natural map*

$$T(A) \to \prod_i T(\kappa_i)$$

is a surjective map, and the same statement holds for any open set U of T with $U_{\kappa_i} \neq \varnothing$ for each i.

Proof. After Proposition 7.2, it is enough to show: if

$$1 \to F \to P \to T \to 1 \tag{7.3.1}$$

is a *flasque resolution* (1.3.3) of the torus T, then $H^1(\kappa_i, F_{\kappa_i}) = 0$ for each i (note this is certainly true if the cohomological dimension cd κ_i is at most one). Let us drop the index i. By Proposition 1.4, the sequence

$$1 \to F_\kappa \to P_\kappa \to T_\kappa \to 1$$

is a flasque resolution of T_κ. As can be read from [8] (see the following Proposition 7.4), T_κ is a direct factor of a κ-rational variety if and only if F_κ is an invertible torus. If T_κ is split by a metacyclic extension κ' of κ, there exists a flasque resolution of T_κ:

$$1 \to F_1 \to P_1 \to T_\kappa \to 1$$

with F_1 (and P_1) split by κ' (cf. proof of Proposition 1.3). Now the Endo–Miyata theorem quoted in 0.5 shows that F_1 is an invertible κ-torus. By the same Proposition 1.3 (or by the above assertion), there exist quasitrivial κ-tori P_2 and P_3 and an isomorphism of κ-tori:

$$F_\kappa \times_\kappa P_2 \xrightarrow{\sim} F_1 \times_\kappa P_3,$$

hence F_κ is also an invertible κ-torus, and the third possibility in the proposition is none but a special case of the second possibility. In all cases, we conclude $H^1(\kappa, F_\kappa)=0$.

PROPOSITION 7.4. *Let k be a field, let T be a k-torus, and let*

$$1 \to F \to P \to T \to 1 \tag{7.4.1}$$

be a flasque resolution of the torus T. The following statements are equivalent:

(i) *F is an invertible k-torus;*

(ii) *for any field K with $k \subset K$, $H^1(K, F_K)=0$;*

(iii) *the equivalent statements in Lemma 7.2 hold for any local ring (A, κ) with $k \subset A$;*

(iv) *they hold for any regular local domain (A, κ) with $k \subset A$;*

(v) *they hold for any discrete valuation ring (A, κ) with $k \subset A$;*

(vi) *there exists a k-torus T_1 such that $T \times_k T_1$ is a k-rational variety, i.e., k-birational to an affine space over k;*

(vii) *T is a (k-birational) direct factor of a k-rational variety, i.e., there exists an integral k-variety Y such that $T \times_k Y$ is a k-rational variety;*

(viii) *T is "retract rational", i.e.,* [31, Sect. 3] *there exists a nonempty open set U of T such that the identity morphism $U \rightrightarrows U$ factorizes (as a k-morphism) through an open set of an affine space over k.*

(In (iii), (iv), (v), one uses the flasque resolution of T_A gotten by pulling (7.4.1) back to A, via Spec $A \to$ Spec k.)

Note that since any k-torus F is easily seen to be part of an exact sequence (7.4.1), the equivalence of (i)–(v) is a statement about flasque tori.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) and (vi) $\Rightarrow$ (vii) are obvious. Also (viii) implies (iii): for U as in (viii), $U(A)$ maps onto $U(\kappa)$ for any local ring (A, κ) since this is certainly true for an open set of an affine space. For an infinite field k, it is easy to show (vii) implies (viii) for an arbitrary k-variety T (using, e.g., [31, Sect. 3]) but we shall not use this implication.

Let us assume (iv). Let $T \hookrightarrow \mathbb{A}^n_k$ be a closed k-immersion of T into an affine space, i.e., write the affine ring $k[T]$ of T as a quotient $k[x_1,..., x_n]/\mathfrak{p}$ for some polynomial ring and some prime ideal $\mathfrak{p}$. Let A be the regular local ring of $\mathbb{A}^n_k$ at the generic point of T, i.e., $A=k[x_1,..., x_n]_\mathfrak{p}$. The residue class field of A coincides with the field of rational functions on T.

Let $e \in T(k)$ be the neutral element, and let $\mathcal{O}_{T,e}$ be the local ring of T at e. Let us consider—and construct—the diagram:

$$\begin{tikzcd}
\mathscr{T}_2 \arrow[rr] \arrow[drrr] \arrow[dd] \arrow[dddr] & & \mathscr{T}_1 \arrow[rr] \arrow[dd] & & P = \mathscr{T}_0 \arrow[dd] \arrow[dddrr] & & \\
& & & \mathscr{T}_6 \arrow[rr] \arrow[dl] & & \mathscr{T}_5 \arrow[dl] \arrow[ddr] & \\
\operatorname{Spec} \kappa \arrow[rr] \arrow[ddr] & & \operatorname{Spec} \mathcal{O}_{T,e} \arrow[rr] & & T \arrow[ddrr] & & \\
& \mathscr{T}_3 \arrow[d] & & & & & \mathscr{T}_4 \arrow[d] \\
& \operatorname{Spec} A \arrow[rrrrr] & & & & & \mathbb{A}^n_k .
\end{tikzcd}$$

The lower level is clear. Each $\mathscr{T}_i$ is a torseur under F, and the vertical squares are cartesian squares, i.e., pull-backs of torseurs. The torseurs $\mathscr{T}_i$ are constructed as follows. First, $\mathscr{T}_0 \to T$ is none but the torseur given by $P \to T$ in 7.4.1, and $\mathscr{T}_1$ and $\mathscr{T}_2$ are the mere pull-backs of $\mathscr{T}_0$. Since the statements in Lemma 7.2 are supposed to hold for the regular local domain A, the map $H^1(A, F) \to H^1(\kappa, F)$ is *surjective*, and there exists a torseur $\mathscr{T}_3$ over $\operatorname{Spec} A$ whose pull-back over $\operatorname{Spec} \kappa$ is $\mathscr{T}_2$ (we should really argue up to isomorphisms of torseurs, but this does not matter here). Since F is *flasque*, the torseur $\mathscr{T}_3$ extends to a torseur $\mathscr{T}_4$ over $\mathbb{A}^n_k$ by Theorem 2.2(i). Finally, $\mathscr{T}_5$ and $\mathscr{T}_6$ are the pull-backs of $\mathscr{T}_4$.

By Lemma 2.4, there exists a torseur $\mathscr{F}_0$ over $\operatorname{Spec} k$ under F (i.e., a principal homogeneous space) such that $\mathscr{T}_4$ is the pull-back $\mathscr{F}_0 \times_k \mathbb{A}^n_k$ of $\mathscr{F}_0$ under the projection $\mathbb{A}^n_k \to \operatorname{Spec} k$. Hence $\mathscr{T}_5 = \mathscr{F}_0 \times_k T$, and $\mathscr{T}_6 = \mathscr{F}_0 \times_k \operatorname{Spec} \mathcal{O}_{T,e}$. But the restrictions of $\mathscr{T}_1$ and $\mathscr{T}_6$ to the field of fractions κ of the regular local ring $\mathcal{O}_{T,e}$ are the same (namely $\mathscr{T}_2$). By Theorem 4.1, the torseurs $\mathscr{T}_1$ and $\mathscr{T}_6$ are isomorphic. The fibre of $\mathscr{T}_1$ at e is trivial, since this is clearly so for $\mathscr{T}_0 = P$. Hence so is the fibre of $\mathscr{T}_6$, which as seen above is none but $\mathscr{F}_0$: we conclude that $\mathscr{F}_0$ is the trivial torseur F under F over k. Now $\mathscr{T}_5 = F \times_k T$, and the torseurs $P \to T$ and $F \times_k T$ over T coincide at the generic point $\operatorname{Spec} \kappa$ of T, hence also [SGA4, VII, 5.9] over a nonempty open set U of T. That is, there is an isomorphism of torseurs over U under F (hence also of k-varieties):

$$P \times_T U \simeq F \times_k U. \tag{7.4.1}$$

Since $P \times_T U$ is an open set of P, itself open in some affine space, this shows (iv) $\Rightarrow$ (vi), (iv) $\Rightarrow$ (vii), and (iv) $\Rightarrow$ (viii), since the identity morphism of U clearly factors through $F \times_k U$.

That (vi) implies (i) is proved in [8, Proposition 6, p. 189]. The proof of

(vii) $\Rightarrow$ (i) is analogous and will be left to the reader (use [8, Lemmas 10 and 11 and Proposition 5]).

It only remains to show that (v) implies (iv), i.e., that surjectivity $H^1(A, F) \to H^1(\kappa, F)$ for all discrete valuation rings (A, κ) with $k \subset A$ implies surjectivity for regular local domains (A, κ) with $k \subset A$. The proof is by induction on the dimension of the regular local domain A. Let $\mathfrak{p}$ be a height one prime ideal of A. The dimensions of the regular local rings $A_{\mathfrak{p}}$ and $A/\mathfrak{p}$ are strictly less than the dimension of A, and the residue class field κ_1 of the first is the fraction field of the second. The residue class field of $A/\mathfrak{p}$ coincides with the residue class field κ of A. Consider the diagram

$$\begin{tikzcd} H^1(A, F) \arrow[r] \arrow[d, "\alpha"'] & H^1(A/\mathfrak{p}, F) \arrow[r, "\lambda"] \arrow[d, "\beta"'] & H^1(\kappa, F) \\ H^1(A_{\mathfrak{p}}, F) \arrow[r, "\mu"] & H^1(\kappa_1, F). & \end{tikzcd}$$

The maps λ and μ are surjective by the induction hypothesis, and α and β are isomorphisms, since F is flasque (4.2). Hence the result.

PROPOSITION 7.5. *Let A be a semilocal ring, and let κ_i be the residue class fields at the maximal ideals of A. Let M be an A-group of multiplicative type, and assume that for each i, either M_{κ_i} is split by a metacyclic extension of κ_i, or* $\operatorname{cd} \kappa_i \leqslant 1$. *Then the reduction map in fppf cohomology*

$$H^1(A, M) \to \prod_i H^1(\kappa_i, M_{\kappa_i})$$

is surjective.

Proof. Let

$$1 \to M \to F \to P \to 1$$

be an exact sequence of A-groups of multiplicative type of type (1.3.2), i.e., F is flasque and P is quasitrivial. This sequence induces a commutative diagram of exact sequences of *fppf* cohomology groups

$$\begin{tikzcd} P(A) \arrow[r] \arrow[d] & H^1(A, M) \arrow[r] \arrow[d] & H^1(A, F) \arrow[r] \arrow[d] & 0 \\ \prod_i P(\kappa_i) \arrow[r] & \prod_i H^1(\kappa_i, M_{\kappa_i}) \arrow[r] & \prod_i H^1(\kappa_i, F_{\kappa_i}) \arrow[r] & 0 \end{tikzcd} \tag{7.5.1}$$

since $H^1(A, P)$ and $H^1(\kappa_i, P_{\kappa_i})$ vanish, as seen in the proof of 7.2, whose arguments immediately adapt to the situation under consideration.

PROPOSITION 7.6. *Let k be a field, and let M be a k-group of multiplicative type. Let*

$$1 \to M \to F \to P \to 1$$

be an exact sequence of type (1.3.2) *over k. The following conditions are equivalent*:

(i) *$H^1(A, M) \to H^1(\kappa, M)$ is surjective for any local k-algebra (A, κ)*;

(ii) *$H^1(A, M) \to H^1(\kappa, M)$ is surjective for any regular local k-algebra (A, κ)*;

(iii) *$H^1(A, M) \to H^1(\kappa, M)$ is surjective for any discrete valuation ring (A, κ) with $k \subset A$*;

(iv) *F is an invertible k-torus*;

(v) *there exists an exact sequence of k-groups of multiplicative type*

$$1 \to M \to P_1 \to F_1 \to 1$$

with P_1 quasitrivial and F_1 invertible.

Proof. For any local k-algebra (A, κ), Lemma 7.1 and diagram (7.5.1) show that $H^1(A, M) \to H^1(\kappa, M)$ is onto if and only if $H^1(A, F) \to H^1(\kappa, F)$ is. The equivalence of (i), (ii), (iii), (iv) now follows from Proposition 7.4. Let F_2 be a k-torus such that $F \times_k F_2 = P_1$ is a quasitrivial torus. To produce a sequence as in (v), it is enough to multiply F and P in the sequence of type (1.3.2) by F_2, hence (iv) $\Rightarrow$ (v), and the proof of the converse is similar.

As a first application of the general results, let us show

PROPOSITION 7.7 [31, Theorem 3.18]. *Let L/k be a finite Galois extension of fields, with Galois group G. The following conditions are equivalent*:

(i) *G is a metacyclic group*;

(ii) *for any local k-algebra (A, κ) and any étale cover B/A, principal Galois with group G, the restriction map*

$$\mathrm{Br}(B/A) \to \mathrm{Br}(B \otimes_A \kappa/\kappa)$$

is surjective;

(iii) *for any local k-algebra (A, κ), the restriction map*

$$\mathrm{Br}(L \otimes_k A/A) \to \mathrm{Br}(L \otimes_k \kappa/\kappa)$$

is surjective.

Proof. Let B/A be as in (ii). Let us first define $\mathrm{Br}(B/A)$:

$$\mathrm{Br}(B/A) = \mathrm{Ker}(H^2(A, \mathbb{G}_m) \to H^2(B, \mathbb{G}_m)).$$

Since $\mathrm{Pic}\, B = 0$ (because B is semilocal), the spectral sequence (2.4.1) shows that this group is none but $H^2(G, B^*)$, hence coincides with the similar group defined via the Azumaya–Brauer group. Let us consider the obvious exact sequence of *A-tori*:

$$1 \to \mathbb{G}_{m,A} \to R_{B/A}\mathbb{G}_m \to T_{B/A} \to 1, \tag{7.7.1}$$

where $T_{B/A}$ is the quotient of the "diagonal" map. The étale cohomology sequence together with $\mathrm{Pic}\, B = 0$ and (0.4.1) yields the basic isomorphism

$$H^1(A, T_{B/A}) \xrightarrow{\sim} \mathrm{Br}(B/A), \tag{7.7.2}$$

and this isomorphism, just as (7.7.1), is functorial in the local ring A (and holds more generally for A semilocal). Now (i) $\Rightarrow$ (ii) appears as a special case of Proposition 7.5: since $T_{B/A}$ is split by the étale Galois cover B/A with metacyclic group G, and since any subgroup of a metacyclic group also is metacyclic. Statement (iii) is a special case of (ii). Assume (iii), hence (7.7.2) assume that $H^1(A, T_{L/k}) \to H^1(\kappa, T_{L/k})$ is a surjective map for all local k-algebras (A, κ). We may now apply Proposition 7.6, or rather its proof, which shows that there exists an exact sequence of k-tori *split by L/k*

$$1 \to T_{L/k} \to P_1 \to F_1 \to 1 \tag{7.7.3}$$

with P_1 quasitrivial and F_1 invertible. Let us now go over to character groups, i.e., torsion-free G-modules (0.2). Dualizing (7.7.1), resp. (7.7.3), yields:

$$0 \longrightarrow I_G \longrightarrow \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \tag{7.7.4}$$

$$0 \longrightarrow F_1 \longrightarrow P_1 \longrightarrow I_G \longrightarrow 0, \tag{7.7.5}$$

where ε is the augmentation map, P_1 is a permutation module and F_1 an invertible module. The dual under $A \mapsto A^0 = \mathrm{Hom}(A, \mathbb{Z})$ (0.5) of I_G is J_G, as defined in (0.4.2). But the dual under $A \mapsto A^0$ of (7.7.5) is a flasque resolution of J_G

$$0 \to J_G \to P_1^0 \to F_1^0 \to 0$$

with F_1^0 invertible, and Endo and Miyata have shown [15, Theorem 1.5; 8, Proposition 2] that such a resolution exists if and only if G is a metacyclic group.

Remark 7.8. (a) The main point in the above proof is the isomorphism (7.7.2), which reduces a problem which looked like a problem on Brauer groups to a problem on the H^1-cohomology of a torus. For another analogous example, cf. [7, Sect. 2, Example a].

(b) There are various ways in which to conduct the rest of the proof: one may indeed give other equivalent conditions in 7.6 and 7.4, in terms of the other exact sequences in Proposition 1.3 (those which involve coflasque tori). This is left to the reader.

Proposition 7.7 was devoted to the lifting of crossed products with a given group. The second application of the general results will be to the lifting of abelian coverings, as studied in [30] and [31]. Let C be a finite abelian group. It defines a constant group scheme C_X over any scheme X, and $H^1(X, C) = H^1(X_{\mathrm{fppf}}, C_X) = H^1(X_{\mathrm{et}}, C_X)$ (recall 2.0: C_X is clearly smooth) classifies the étale covers of X which are principal Galois under C_X, in short the abelian étale extensions of X with group C. If the exponent of C is prime to the residue characteristics of X, the group C_X is of multiplicative type (and smooth), but it need not be split as an X-group of multiplicative type: this is where the roots of unity come in.

The following result, which was also pointed out to us by A. S. Merkur'ev, was proved in [30] for k-algebras (k a field):

PROPOSITION 7.9. *Let A be a semilocal ring, and let C be a finite abelian group whose exponent is prime to the characteristics of the residue class fields κ_i of A at its maximal ideals. Let $2^r = e_2(C)$ be the highest power of* 2 *which divides the exponent of C. For each i, assume that either* $\operatorname{cd} \kappa_i \leqslant 1$, *or that the field $\kappa_i(\mu_{2^r})$ obtained by adjoining the* 2^rth *roots of unity to κ_i is a cyclic extension of κ_i. Then the restriction map*

$$H^1(A, C) \to \prod_i H^1(\kappa_i, C)$$

is surjective.

Proof. Since $H^1(A, \cdot)$ is additive, we may assume $C = \mathbb{Z}/p^n$ for p prime. For arbitrary p and i, the extension $\kappa_i(\mu_{p^n})/\kappa_i$ splits the κ_i-group of multiplicative type $(\mathbb{Z}/p^n)_{\kappa_i}$. Either $\operatorname{cd} \kappa_i \leqslant 1$ or this extension is *cyclic*: this is clear for p odd, and has been assumed for $p = 2$. The statement now appears as a special case of Proposition 7.5.

Let k be a field, $\bar{k}$ a separable closure of k, and $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$. Let C be a finite abelian group of order prime to char k, let

$$X_C = C^0(\bar{k}) = \mathrm{Hom}_{\mathbb{Z}}(C, \bar{k}^*) = \mathrm{Hom}_{\mathbb{Z}}(C, \mu_\infty(\bar{k}))$$

be the character group of C, viewed as a $\mathfrak{g}$-module. Let k_C/k be the smallest (finite) cyclotomic subextension of $\bar{k}/k$ such that $\mathrm{Gal}(\bar{k}/k_C)$ acts trivially on X_C, and let $G = \mathrm{Gal}(k_C/k)$. View X_C as a G-module. There is a standard resolution of X_C of type (0.6.1):

$$0 \to \hat{Q}_C \to \hat{P}_C \to X_C \to 0. \tag{7.10.1}$$

One simply takes $\hat{P}_C$ to be $\mathbb{Z}[X_C]$, the free abelian group on the elements of X_C, and the map $\mathbb{Z}[X_C] \to X_C$ sends $\sum_\chi a_\chi \chi$ to $\prod_\chi \chi^{a_\chi}$. The G-module $\hat{P}_C$ is a permutation module, and for any subgroup $H \subset G$, the map $\hat{P}_C^H \to X_C^H$ is clearly surjective, hence the kernel $\hat{Q}_C$ is a coflasque G-module. As explained by Voskresenskiĭ ([36, Theorem 1], see also Kervaire [21, Sect. 1]), the dual exact sequence of k-groups of multiplicative type

$$1 \to C \to P_C \to Q_C \to 1 \tag{7.10.2}$$

defines a k-torus Q_C whose function field is none but the field of invariants of the obvious action of C on $k(x_g)_{g \in C}$: this is Voskresenskiĭ's torus-theoretic approach to the Noether problem for abelian groups. Applying the exact sequence of $\mathrm{Ext}_{\mathbb{Z}}(\cdot, \mathbb{Z})$ to (7.10.1) yields an exact sequence of G-modules (use $\mathrm{Ext}^1_{\mathbb{Z}}(X_C, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(X_C, \mathbb{Q}/\mathbb{Z}) \cong X_C$ (noncanonically) as G-modules),

$$0 \to \hat{R}_C \to \hat{F}_C \to X_C \to 0 \tag{7.10.3}$$

with $\hat{R}_C = \hat{P}_C^0$ permutation and $\hat{F}_C = \hat{Q}_C^0$ flasque, i.e., a sequence of type (0.6.2).

PROPOSITION 7.10. *With notations as above, the following conditions are equivalent*:

(i) *the restriction map* $H^1(A, C) \to H^1(\kappa, C)$ *is surjective for any local* k*-algebra* (A, κ);

(ii) *it is surjective for any discrete valuation ring* (A, κ) *with* $k \subset A$;

(iii) *the* k*-torus* Q_C *is invertible*;

(iv) *the* k*-torus* Q_C *is a direct factor of a* k*-rational variety* (*i.e.*, $Q_C \times_k Y$ *is* k*-birational to* $\mathbb{A}_k^m$ *for some* m *and some* k*-variety* Y).

In particular, if the universal lifting property (i) fails, the Noether problem has a negative answer for k and C.

(This last fact is Saltman's basic remark in [30].)

Proof. The exact sequence of k-groups of multiplicative type dual to (7.10.3) is of type (1.3.2). It now follows from Proposition 7.6 that (i) or (ii) are equivalent to $\hat{F}$ invertible, hence $\hat{Q}_C$ invertible (since $\hat{F}_C = \hat{Q}_C^0$), i.e., (iii). If Q_C satisfies (iv), Proposition 7.4 shows that there exists an exact sequence of G-modules

$$0 \to \hat{Q}_C \to \hat{P} \to \hat{F} \to 0$$

with $\hat{P}$ permutation and $\hat{F}$ invertible. Since $\hat{Q}_C$ is coflasque, this implies that the sequence splits, hence $\hat{Q}_C \oplus \hat{F} \simeq \hat{P}$ and Q_C is invertible.

Swan [34] and Voskresenskiĭ (see [37, Chap. VII] found the first negative answer to the Noether problem:

For $k = \mathbb{Q}$ and $C = \mathbb{Z}/n$ with $n = 47$, the k-torus $Q_n = Q_{\mathbb{Z}/n}$ is not a k-rational variety. It was later shown by Endo–Miyata [14], Voskresenskiĭ [36], and Lenstra [22] that $k = \mathbb{Q}$ and $n = 8$ also yield a negative answer. For k arbitrary and $n = 2^m$, it is known [36, Sect. 6; 37, Chap. VII, Sect. 4] that Q_n is k-birational to an affine space if and only if $k(\mu_{2^m})/k$ is cyclic, and that if this is not the case, then Q_n is not even a direct factor of a k-rational variety. Another approach to this last fact is given in [31]. It can also be proved as follows. For $i \geqslant 0$ an integer, and M a G-module, let

$$Ш^i_\omega(G, M) = \operatorname{Ker}\left(H^i(G, M) \to \prod_{g \in G} H^i(\langle g \rangle, M)\right).$$

One easily checks $Ш^2_\omega(G, P) = 0$ for an arbitrary permutation G-module P [33, Sect. 1]. If Q_n is a direct factor of a k-rational variety, then as seen above, $\hat{Q}_n$ and $\hat{F}_n$ are invertible G-modules. From (7.10.3) we conclude

$$Ш^1_\omega(G, \mu_{2^m}) = 0, \tag{7.10.4}$$

with $G = \operatorname{Gal}(k(\mu_{2^m})/k)$. Assume that G is not cyclic. Then $G \overset{\varphi}{\hookrightarrow} \operatorname{Aut}(\mu_{2^m}) = (\mathbb{Z}/2^m)^* \cong \mathbb{Z}/2 \times \mathbb{Z}/2^{m-2}$ is generated by two elements s and t, with $\varphi(s) = -1$ and $\varphi(t) = 5^j$. Any cyclic subgroup of G which contains s coincides with $\langle s \rangle$. If $H \subset G$ is a cyclic subgroup different from $\langle s \rangle$, Voskresenskiĭ shows by direct computation [36, Sect. 6; 37, Chap. VII, Sect. 4] $H^1(H, \mu_{2^m}) = 0$. Hence $Ш^1_\omega(G, \mu_{2^m}) = \operatorname{Ker}[H^1(G, \mu_{2^m}) \to H^1(\langle s \rangle, \mu_{2^m})] = H^1(G/\langle s \rangle, \mu^s_{2^m})$, and $\mu^s_{2^m} = \{\pm 1\}$, hence $Ш^1_\omega(G, \mu_{2^m}) = \operatorname{Hom}(G/\langle s \rangle, \pm 1) = \mathbb{Z}/2 \neq 0$, contradicting (7.10.4).

Let k be a number field. As Hilbert first showed, if $K = k(t_1, \ldots, t_n)$ is a purely transcendental extension of k, and if there exists a finite Galois extension of fields L/K with group G, then there exists a Galois extension of k with group G—and there are many ways in which one may specialize the variables t_i in k so that the specialized extension still makes sense, and is Galois with group G (such values of the variables are Zariski-dense in $\mathbb{A}^n_k$). A positive answer to Emmy Noether's question: If G is a finite group, is the field of invariants of $k(x_g)_{g \in G}$ under the obvious action of G purely transcendental over k? would have implied the existence of a field extension of k, Galois with group G.

On the occasion of Emmy Noether's centenary, there has been a number of papers devoted to her work and problems (e.g., [35]). The following remarks seem to us relevant.

Remark 7.11. Although the Noether problem has a negative answer for abelian $G = C$, it is well known that there are field extensions of a number field k which are Galois with group C, and this may be proved with the Hilbert approach. Let

$$1 \to C \to F_C \to R_C \to 1 \tag{7.11.1}$$

be the exact sequence of k-groups of multiplicative type dual to (7.10.3) (or to any resolution of X_C of type (0.6.2)). The map $F_C \to R_C$ is an étale cover of k-tori, Galois with group C, hence the generic fibre of this morphism defines a Galois extension of function fields $k(F_C)/k(R_C)$, Galois with group C. Since R_C is a quasitrivial torus, the function field $k(R_C)$ is purely transcendental over k, and Hilbert's approach applies (this proof works for any Hilbertian field k).

Remark 7.12. Had we tried to apply a similar argument to

$$1 \to C \to P_C \to Q_C \to 1, \tag{7.10.2}$$

we would have run into the problem that Q_C need not be a k-rational variety ($= k$-birational to some $\mathbb{A}_k^m$). When Q_C is a direct factor of a k-rational variety (say $C = \mathbb{Z}/47$), this can easily be obviated since Hilbert's argument extends in a straightforward manner for such k-varieties, but as seen above Q_C need not be a direct factor of a k-rational variety. It is all the more striking that Hilbert's approach can be extended to this case: simply apply the following Corollary 7.14 to the k-torus Q_C.

Let V/k be a geometrically integral variety over a Hilbertian field k. We shall say that V/k is *of Hilbert type* if the following statement holds:

(H) If $L/k(V)$ is a finite extension field of the field $k(V)$ of rational functions of V, Galois with group G, the set of points of $V(k)$ at which the specialized extension makes sense and is a Galois extension of k with group G is Zariski-dense in V.

Mike Fried tells us that the following proposition is implicit in the work of Uchida:

PROPOSITION 7.13. *Let k be a Hilbertian field, and let $f\colon W \to V$ be a dominant k-morphism of geometrically integral k-varieties. Assume that the generic fibre of f is geometrically integral, i.e., that the extension of function* *fields* $k(W)/k(V)$ *is separable* (*e.g.*, char $k=0$) *and that* $k(V)$ *is algebraically closed in* $k(W)$. *If* W *is of Hilbert type, so is* V.

Proof. If $L/k(V)$ is a Galois extension of *fields* with group G, so is $L\otimes_{k(V)}k(W)/k(W)$. Shrinking V, we may assume that $L/k(V)$ is the generic fibre of an étale Galois cover V_1/V with group G with V_1 integral and $k(V_1)=L$. Since shrinking W clearly does not modify (**H**) for W, the set $U\subset W(k)$ where the specialized extension of $V_1\times_k W/W$ is a field extension with group G is Zariski-dense in W, and $f(U)\subset V(k)$ is Zariski-dense in V and has the same property for V_1/V.

COROLLARY 7.14. *Any k-torus T over a Hilbertian field k is of Hilbert type.*

Proof. Any k-torus T may be included in an exact sequence of *k-tori*:

$$1\to T_1\to P\to T\to 1 \tag{7.14.1}$$

(use a surjection of a permutation module onto $\hat{T}^0$, cf. also (1.3.3)) with P a quasitrivial torus. Since P is quasitrivial, it is a k-rational variety, hence satisfies (**H**) by Hilbert's theorem, and since T_1 is connected and smooth over k, the extension $k(P)/k(T)$ satisfies the assumptions of the previous proposition. Hence T is of Hilbert type.

COROLLARY 7.15. *Any connected reductive k-algebraic group G over a Hilbertian field k is of Hilbert type.*

Proof. For G/k as above, it is known [SGA3, XIV, 6.5] that the function field $k(G)$ coincides with the function field of a K-torus T, for K a purely transcendental extension of k (K is the function field of the variety of maximal tori of G). Let us write a sequence (7.14.1) for T over K. The field $K(P)$ is purely transcendental over K, hence over k, and the extension $K(P)/K(T)=K(P)/k(G)$ satisfies the assumptions of Proposition 7.13. Hence G is of Hilbert type.

It would certainly be of interest to discuss the existence of k-unirational varieties which are not of Hilbert type.

8. GOING OVER TO THE COMPLETION OF A DISCRETELY VALUED FIELD

This section closely parallels Section 7. Many of its results could be extended to arbitrary real valuations. If (A,κ) is a discrete valuation ring with field of fractions K and residue class field κ, we denote by $\hat{A}$ (resp. $\hat{K}$)

the completion of A (resp. K). Recall [GB III, 11.7] that for any smooth $\hat{A}$-group scheme M there is a natural Hensel isomorphism:

$$H^1(\hat{A}, M) \xrightarrow{\sim} H^1(\kappa, M_\kappa). \tag{8.0}$$

PROPOSITION 8.1. (i) *Let T/A be a torus over a discrete valuation ring A. Then*

$$T(\hat{A})\, T(K) = T(\hat{K}).$$

(ii) *For T/A as in* (i), *$T(K)$ is dense in $T(\hat{K})$ if any of the following conditions hold: T_κ is split by a metacyclic extension of κ, or* cd $\kappa \leqslant 1$, *or T_κ is κ-birationally a direct factor of a κ-rational variety.*

(iii) *Let T/K be a torus over a discretely valued field K. If $T_{\hat{K}}$ is split by a metacyclic extension of $\hat{K}$, or if $T_{\hat{K}}$ is $\hat{K}$-birationally a direct factor of a $\hat{K}$-rational variety, $T(K)$ is dense in $T(\hat{K})$.*

(iv) *Let T/k be a torus over a field k. If for any discrete valuation ring A with $k \subset A$, the group $T(K)$ is dense in $T(\hat{K})$, then there exists a k-torus T_1 such that $T \times_k T_1$ is a k-rational variety.*

Thus the k-tori which "universally" satisfy weak approximation are the obvious ones.

Proof. Consider a flasque resolution of the A-torus T:

$$1 \to F \to P \to T \to 1. \tag{8.1.1}$$

It induces a commutative diagram of exact sequences:

$$\begin{array}{ccccccc} P(\hat{A}) & \longrightarrow & T(\hat{A}) & \longrightarrow & H^1(\hat{A}, F) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ P(\hat{K}) & \xrightarrow{\pi} & T(\hat{K}) & \longrightarrow & H^1(\hat{K}, F) & \longrightarrow & 0. \end{array} \tag{8.1.2}$$

Since $F_{\hat{A}}$ is flasque over the discrete valuation ring $\hat{A}$, the map $H^1(\hat{A}, F) \to H^1(\hat{K}, F)$ is surjective by Theorem 2.2(i) and the above diagram then gives $T(\hat{K}) = T(\hat{A}) \cdot \pi(P(\hat{K}))$. But P_K is a K-rational variety, hence $P(K)$ is dense in $P(\hat{K})$. Since $P(\hat{A})$ is an open subgroup of $P(\hat{K})$, this gives $P(\hat{K}) = P(\hat{A}) \cdot P(K)$, hence (i) on combining the last two equalities.

Let us consider (ii). The flasque resolution (8.1.1) induces a similar resolution of T_κ over κ (1.4). In all the cases considered in (ii), we have $H^1(\kappa, F_\kappa) = 0$. This is clear if cd $\kappa \leqslant 1$ and was proved in 7.3 (or rather its proof) and 7.4 for the other cases. By (8.0) we conclude $H^1(\hat{A}, F) = 0$, and the first line of (8.1.2) now implies that $P(\hat{A})$ surjects onto $T(\hat{A})$. The result now follows from (i) and the density of $P(K)$ in $P(\hat{K})$.

As for (iii), if we take a flasque resolution (8.1.1) of T over K, it induces a flasque resolution of $T_{\hat{K}}$ over $\hat{K}$. Just as above, the assumptions imply $H^1(\hat{K}, F)=0$, hence $P(\hat{K})$ surjects onto $T(\hat{K})$ and we conclude using the density of $P(K)$ in $P(\hat{K})$.

Let T/k and (A, κ) be as in (iv). The residue class field of A is κ. The natural map $T(\hat{A}) \to T(\kappa)$ is surjective by Hensel's lemma, and the inverse image of any element of $T(\kappa)$ is *open* in $T(\hat{A})$, hence in $T(\hat{K})$. (This is clear for a split torus, cf. [3, Chap. 6, Sect. 5], can then be proved for a quasi-trivial torus, and then for an arbitrary torus by using a sequence as (8.1.1).) Since $T(K)$ is dense in $T(\hat{K})$, we conclude that the map $T(A) \to T(\kappa)$ is surjective, whence (iv) by Proposition 7.4.

EXAMPLE 8.1.1. The A-torus of Example 7.2.1 had a flasque resolution with $H^1(K, F)=0$ and $H^1(\kappa, F)\neq 0$, hence $H^1(\hat{K}, F)\neq 0$ by arguments used above. Since $\pi(P(\hat{K}))$ is open in $T(\hat{K})$ by the implicit function theorem, the exact cohomology sequences associated to (8.1.1) over K and $\hat{K}$ show that $T(K)$ is not dense in $T(\hat{K})$.

Remark 8.2. Result 8.1(i) is given a different proof by Nisnevich in [23], where he also extends the result to an arbitrary reductive A-group scheme G. In the same Note, this extended result is used in a crucial way to prove the analogue of 4.1 for G/A when A is a discrete valuation ring.

Remark 8.3. Let A be a discrete valuation ring with finite residue class field κ, and with fraction field K. Let T be a K-torus (not necessarily an A-torus). Abuse notations and write $T(\hat{A})$ for the maximal compact subgroup of $T(\hat{K})$. Bruhat and Tits have asked whether statement 8.1(i) still holds. Using a flasque resolution of T as in (8.1.1), one checks that $A(T)$, the quotient of $T(\hat{K})$ by the closure of $T(K)$, coincides with the cokernel of the map $H^1(K, F) \to H^1(\hat{K}, F)$ (this last group is finite): see [8, Proposition 18, p. 219]. Hence the above question has a positive answer if the composite map $T(\hat{A}) \to T(\hat{K}) \to H^1(\hat{K}, F)$ is surjective. Let L/K be a finite Galois extension which splits T (which may be assumed ramified, since otherwise T/K extends to an A-torus), let B be the integral closure of A in L, let $\hat{B}=B\otimes_A \hat{A}$ and $\hat{L}=L\otimes_K \hat{K}$, and let $G=\mathrm{Gal}(L/K)$. Then $T(\hat{A})=\mathrm{Hom}_G(\hat{T}, \hat{B}^*)$ and $T(\hat{K})=\mathrm{Hom}_G(\hat{T}, \hat{L}^*)$. Using a flasque resolution of T which splits over L, we get the commutative diagram of exact sequences:

$$\begin{array}{ccccc} T(\hat{A}) & \longrightarrow & \mathrm{Ext}^1_G(\hat{F}, \hat{B}^*) & \longrightarrow & \mathrm{Ext}^1_G(\hat{P}, \hat{B}^*) \\ \downarrow & & \downarrow & & \\ T(\hat{K}) & \longrightarrow & \mathrm{Ext}^1_G(\hat{F}, \hat{L}^*) & (=H^1(\hat{K}, F)). & \end{array}$$

The right vertical map is surjective, because $\hat{L}^*/\hat{B}^*$ is a permutation module, hence $\operatorname{Ext}^1_G(\hat{F}, \hat{L}^*/\hat{B}^*) = 0$. We conclude that the above question has a positive answer if P may be taken of the shape $\mathbb{Z}[G]^i$ for some i. This does not always happen, but in the usual counterexample for weak approximation, namely $T = R^1_{L/K}\mathbb{G}_m$ for L/K a *Galois* extension, this is the case, cf. [8, Proposition 1]. Hence a positive answer to the Bruhat–Tits question for such T's.

PROPOSITION 8.4. (i) *Let (A, κ) be a discrete valuation ring, and let M be an A-group of multiplicative type. If M_κ is split by a metacyclic extension of κ, or if* $\operatorname{cd} \kappa \leqslant 1$, *the natural map $H^1(K, M) \to H^1(\hat{K}, M)$ is surjective.*

(ii) *Let K be a discretely valued field, and let M be a K-group of multiplicative type. If $M_{\hat{K}}$ is split by a metacyclic extension of $\hat{K}$, the map $H^1(K, M) \to H^1(\hat{K}, M)$ is surjective.*

(iii) *Let k be field and let M be a k-group of multiplicative type. Assume that for each discrete valuation ring A with $k \subset A$, the natural map $H^1(K, M) \to H^1(\hat{K}, M)$ is surjective. Then for any exact sequence of k-groups of multiplicative type of one of the following types*

$$1 \to M \to P \to Q \to 1 \tag{1.3.1}$$

$$1 \to M \to F \to P \to 1 \tag{1.3.2}$$

the k-tori Q and F are invertible. In particular there exist such sequences with Q and F invertible.

(iv) *Conversely, if there exist such sequences with either Q or F invertible, then $H^1(K, M) \to H^1(\hat{K}, M)$ is surjective for any discretely valued field K with $k \subset K$.*

Proof. Let us consider an exact sequence of type (1.3.2):

$$1 \to M \to F \to P \to 1$$

over A in case (i) and over K in case (ii). In both cases, this sequence induces a commutative diagram of exact sequences

$$\begin{array}{ccccccc} & & P(K) & \longrightarrow & H^1(K, M) & & \\ & & \downarrow & & \downarrow & & \\ F(\hat{K}) & \longrightarrow & P(\hat{K}) & \longrightarrow & H^1(\hat{K}, M) & \longrightarrow & H^1(\hat{K}, F). \end{array}$$

As in the proof of (ii) and (iii) in Proposition 8.1, the assumptions imply $H^1(\hat{K}, F) = 0$. Since the image of $F(\hat{K})$ in $P(\hat{K})$ is open by the implicit function theorem, and since $P(K)$ is dense in $P(\hat{K})$ because P is quasitrivial, the composite map $P(K) \to P(\hat{K}) \to H^1(\hat{K}, M)$ is surjective, hence also $H^1(K, M) \to H^1(\hat{K}, M)$.

As for (iii), let us consider an exact sequence of type (1.3.1) for a change (cf. Remark 7.8(b)). For A as in (iii), it induces a commutative diagram of exact sequences

$$\begin{tikzcd} P(K) \arrow[r] \arrow[d] & Q(K) \arrow[r] \arrow[d] & H^1(K, M) \arrow[r] \arrow[d] & 0 \\ P(\hat{K}) \arrow[r] & Q(\hat{K}) \arrow[r] & H^1(\hat{K}, M). \end{tikzcd}$$

From the surjection $H^1(K, M) \to H^1(\hat{K}, M)$ and the density of $P(K)$ in $P(\hat{K})$, we conclude with the above diagram that $Q(K)$ is dense in $Q(\hat{K})$. We may now apply 8.1(iv) to Q: there exists a k-torus Q_1 such that $Q \times_k Q_1$ is k-birational to an affine space. By 7.4, (vi) $\Rightarrow$ (i), this implies the existence of an exact sequence of type (0.6.3):

$$0 \to \hat{Q} \to \hat{P} \to \hat{F} \to 0$$

with $\hat{P}$ permutation and $\hat{F}$ invertible. Since $\hat{Q}$ is coflasque, this sequence splits, hence Q is invertible. As in 7.6(v), this implies the existence of an exact sequence of type (1.3.2) with F invertible, and (iii) now follows from the "uniqueness" of such sequences (cf. 1.3).

As we have just seen, the existence of (1.3.1) with Q invertible amounts to the existence of (1.3.2) with F invertible, and any such sequences then have Q and F invertible, hence in particular $H^1(\hat{K}, F) = 0$. The proof of (iv) is now identical with the proof of (ii).

We now recover Saltman's approach [30, 31] to the Grunwald–Wang theorem. Let C be a finite abelian group, let e be its exponent, and let $e_2(C) = 2^r$ be the highest power of 2 which divides e.

COROLLARY 8.5. (i) *Let (A, κ) be a discrete valuation ring, let K be its fraction field and $\hat{K}$ be the completion of K. The natural map*

$$H^1(K, C) \to H^1(\hat{K}, C)$$

is surjective, and abelian extensions of $\hat{K}$ with group C are approximable, in any of the following cases:

(a) *the characteristic of K is prime to e, and $\hat{K}(\mu_{2^r})/\hat{K}$ is cyclic*;

(b) *the characteristic of κ is prime to e, and either* $\operatorname{cd} \kappa \leqslant 1$, *or $\kappa(\mu_{2^r})/\kappa$ is cyclic.*

(ii) *Let k be a field of characteristic prime to e. Assume that for any discrete valuation ring A with $k \subset A$, the map $H^1(K, C) \to H^1(\hat{K}, C)$ is surjective. Then the k-torus Q_C in* 7.10 *is invertible, in particular it is a direct factor of a k-rational variety. Conversely, if this last property of Q_C holds, then $H^1(K, C) \to H^1(\hat{K}, C)$ is surjective for any discretely valued field K with $k \subset K$.*

Proof. The conditions under which C_A or C_K define a group of multiplicative type have been discussed in the preliminaries to 7.9. Part (i) immediately follows from 8.4(i) and (ii) (cf. the proof of 7.9). The direct assertion in (ii) follows from 8.4(iii) and (7.10.2). The converse property is a special case of 8.4(iv).

As pointed out by Saltman [30], taking $k = \mathbb{Q}$ and $C = \mathbb{Z}/8$ then transforms Wang's counterexample to Grunwald's theorem into a negative answer to Noether's problem.

9. On the Centre of the Ring of Generic Matrices

When studying the centre of the ring of $p \times p$ generic matrices for p prime, we shall use the following proposition.

PROPOSITION 9.1. *Let K/k be a separable extension of fields of prime degree p, and let $T = R^1_{K/k}\mathbb{G}_m$ be the k-torus of norm 1 elements for K/k. There exists a k-torus T_1 such that $T \times_k T_1$ is a k-rational variety.*

Proof. Step 1. If K/k is Galois, hence cyclic, the k-torus $R^1_{K/k}\mathbb{G}_m$ is k-isomorphic to $R_{K/k}\mathbb{G}_m/\mathbb{G}_{m,k}$, which is clearly k-rational [37, 4.8].

Step 2. In the general case, there is a diagram of field extensions

$$\begin{tikzcd} & L \arrow[dl, dash, "m"'] \arrow[dr, dash, "p"] & \\ K \arrow[dr, dash] & & M \arrow[dl, dash] \\ & k & \end{tikzcd}$$

where L/k is the Galois closure of K/k, and M is the fixed field of a chosen Sylow p-subgroup G_p of $G = \mathrm{Gal}(L/k)$. Since G is a subgroup of the symmetric group $\mathfrak{S}_p$, we have $G_p \simeq \mathbb{Z}/p$ and the degree $m = [L:K] = [M:k]$ is prime to p.

Step 3. If K/k and M/k are two extensions of coprime degrees p, m, and $a, b \in \mathbb{Z}$ are such that $ap + bm = 1$, and T is a k-torus, composing the morphisms of k-tori:

$$T \xrightarrow{(i_{K/k}, i_{M/k})} R_{K/k} T \times_k R_{M/k} T \xrightarrow{(N^a_{K/k} \cdot N^b_{M/k})} T,$$

where $i_{K/k}$ and $i_{M/k}$ are the diagonal maps and the second map sends (x, y) to $(N_{K/k}(x))^a \cdot (N_{M/k}(y))^b$, yields the identity map on T; cf. 0.4 (it is easy to give a direct proof by going over to G-modules). In particular, the k-torus T is a direct factor, as a k-torus, of the middle torus.

Step 4. Let k'/k be a finite separable extension of fields, and let X be a k'-rational k'-variety. Then $R_{k'/k} X$ is a k-rational variety: indeed $R_{k'/k}$ commutes with open immersions, and $R_{k'/k}(\mathbb{A}^n_{k'}) \simeq \mathbb{A}^{nm}_k$ as k-varieties, for $m = [k' : k]$.

Step 5. For T as in 9.1 and L, M as in Step 2, the M-torus T_M is none but $R^1_{L/M} \mathbb{G}_m$ (0.4 or direct check) hence is an M-rational variety according to Step 1.

Step 6. Recall (0.4.2) that the G-module $\hat{T}$ may be defined by the exact sequence of G-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{N_{G/H}} \mathbb{Z}[G/H] \longrightarrow \hat{T} \longrightarrow 0,$$

where $H = \mathrm{Gal}(L/K)$. Projecting $\mathbb{Z}[G/H]$ onto $\mathbb{Z} \cdot 1_G \cdot H$ defines an H-retraction of $N_{G/H}$, hence the sequence splits as a sequence of H-modules. This gives a K-isomorphism of K-tori:

$$T_K \times_K \mathbb{G}_{m,K} \simeq (R_{K/k} \mathbb{G}_m)_K$$

and this last K-variety is clearly K-rational.

Step 7. From Step 3 there exists a k-torus T_0 and a k-isomorphism of k-tori

$$T \times_k T_0 \simeq R_{K/k} T \times_k R_{M/k} T,$$

hence also

$$T \times_k T_0 \times_k R_{K/k} \mathbb{G}_m \simeq R_{K/k}(T \times_k \mathbb{G}_m) \times_k R_{M/k} T$$

and Steps 4, 5, and 6 show that the right-hand-side k-torus is k-rational.

Remark 9.2. Here is some help for the algebraically inclined reader. If L/k is a finite Galois splitting field of a k-torus T, and if E/k is a subextension, and $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/E)$, the character group of the k-torus $R_{E/k}(T_E)$ is the G-module $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \hat{T}$, where $\hat{T}$ is viewed as an

H-module, and the action of G is by left multiplication on $\mathbb{Z}[G]$. There is a G-isomorphism

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \hat{T} \xrightarrow{\sim} \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} \hat{T}$$
$$g \otimes m \mapsto g \otimes gm,$$

the left G-module structure on $\mathbb{Z}[G/H) \otimes_{\mathbb{Z}} \hat{T}$ being given by $g(x \otimes m) = gx \otimes gm$.

Remark 9.3. The conclusion of 9.1 is equivalent to: for any flasque resolution (0.6.3) of $\hat{T}$, i.e., $0 \to \hat{T} \to \hat{P} \to \hat{F} \to 0$, the module $\hat{F}$ is invertible. As a first consequence, $T(k')/R = H^1(k', F) = 0$ for any overfield k' of k (cf. Sect. 3). As a second consequence, if k is a number field, and k'/k a finite extension, $T_{k'}$ satisfies weak approximation and principal homogeneous spaces under $T_{k'}$ satisfy the Hasse principle. This follows from the exact sequence

$$0 \to A(T_{k'}) \to \mathrm{Hom}(H^1(k', \hat{F}), \mathbb{Q}/\mathbb{Z}) \to Ш^1(T_{k'}) \to 0$$

(Voskresenskiĭ [37, 6.38]; [8, Proposition 19]). In particular, if an element of k is everywhere locally a norm for K/k, it is a norm for K/k; this last fact has already been noticed by various authors, and lead us to Proposition 9.1. Note that T in 9.1 need not be stably k-rational [8, R4, p. 186 and bottom of p. 224].

Let $n \geqslant 2$ and $r \geqslant 2$ be natural integers, and let k be an arbitrary field. We refer to [29] and [17] for the definition of the generic division ring $UD(k, n, r)$ of r generic $n \times n$ matrices over k. Let $Z_n(k, r) = Z_n(r)$ denote its centre. A basic question is whether the field $Z_n(r)$ is purely transcendental over k. Procesi has shown that $Z_n(r+1)$ is purely transcendental over $Z_n(r)$. Hence, to a great extent, the crucial case is $Z_n(2)$, on which we shall now concentrate. Extending work of Procesi, Formanek [17] gave a description of $Z_n(2)$ as the function field of a torus over a purely transcendental extension of k. Let us recall this description.

Denote $G = \mathfrak{S}_n$ and $H = \mathfrak{S}_{n-1}$. Let $\varepsilon: \mathbb{Z}[G/H] \to \mathbb{Z}$ be the augmentation map $gH \mapsto 1$. Let $I_{G/H}$ be its kernel. The G-homomorphism

$$\eta: \mathbb{Z}[G/H \times G/H] \to \mathbb{Z}[G/H]$$

defined on generators by

$$(gH, g'H) \mapsto gH - g'H$$

has image equal to $I_{G/H}$. Let A be the kernel of η: we have the exact sequence of finitely generated $\mathbb{Z}$-free G-modules:

$$0 \longrightarrow A \longrightarrow \mathbb{Z}[G/H \times G/H] \xrightarrow{\eta} I_{G/H} \longrightarrow 0. \qquad (9.2.1)$$

Let us denote by $k(R)$ the function field of the group algebra (over k) of a finitely generated torsion-free abelian group R. Consider the diagram of field extensions:

$$\begin{tikzcd}[column sep=small]
& & L = K(A) = k(\mathbb{Z}[G/H] \oplus A) \arrow[d, dash] \\
L^G \arrow[urr, dash] \arrow[dd, dash] & & K = k(\mathbb{Z}[G/H]). \\
& F = K^H \arrow[ur, dash] & \\
K^G = E \arrow[ur, dash] \arrow[d, dash] & & \\
k & &
\end{tikzcd} \qquad (9.2.2)$$

Let us identify G/H with $\{1, ..., n\}$ equipped with the natural action of G (i.e., identify $H = \mathfrak{S}_{n-1}$ with the isotropy subgroup of $G = \mathfrak{S}_n$ at 1). Then $K = k(t_1, ..., t_n)$ for independent variables t_i, and E is the purely transcendental field $k(\sigma_1, ..., \sigma_n)$ for σ_i the elementary symmetric functions on the t_i's. Now $G = \mathrm{Gal}(K/E)$, and L^G is the function field of the E-torus T, split by K, whose G-module of characters is A. Note $\dim_E T = \mathrm{rank}_{\mathbb{Z}} A = n^2 - n + 1$. We have the important:

PROPOSITION 9.3 (Formanek [17]). *The centre $Z_n(2)$ of the generic division algebra $UD(k, n, 2)$ coincides with the field L^G, i.e., with the function field of the E-torus $T_n = T$ whose G-module of characters is A.*

We shall need the following notations and easy lemma from [8, Sect. 1]. For $A \in \mathscr{L}_G$, we denote by $\rho(A)$ (resp. $\varsigma(A)$) the class, up to permutation modules of F_3 (resp. Q_1) in a resolution (0.6.3) (resp. (0.6.1)).

LEMMA 9.4. (i) *For $A \in \mathscr{L}_G$, $\varsigma(A^0) = [\rho(A)]^0$;*

(ii) *if $0 \to P \to A \to B \to 0$ is an exact sequence in $\mathscr{L}_G$, and P is a permutation module, then $\varsigma(A) = \varsigma(B)$;*

(iii) *for a flasque G-module, F invertible $\Leftrightarrow \varsigma(F)$ invertible ($\Leftrightarrow F = -\varsigma(F)$).*

As for tori, we shall use

PROPOSITION 9.5. *Let K/E be a finite extension of fields, Galois with group G. Let T be an E-torus which is split by K, and let $\hat{T} \in \mathscr{L}_G$ be its character group. Then:*

(i) *The following conditions are equivalent*:

(a) $\rho(\hat{T})$ *is invertible*;

(b) *there exists an E-torus* T' *such that* $T \times_E T'$ *is an E-rational variety*;

(c) *there exists an E-variety* Y *with* $T \times_E Y$ *an E-rational variety.*

(ii) *Assume* char $E = 0$. *Let* X *be a smooth E-compactification of* T. *Then*:

$$\begin{aligned}\operatorname{Br} X/\operatorname{Br} E &= H^1(G, \rho(\hat{T})) \\ &= Ш^2_\omega(G, \hat{T}) \quad \left(:= \operatorname{Ker}\left(H^2(G, \hat{T}) \to \prod_{g \in G} H^2(\langle g \rangle, \hat{T})\right)\right).\end{aligned}$$

Proof. For (i), see 7.4. In (ii), $\operatorname{Br} X$ denotes the étale cohomological Brauer group of X, but one easily checks that for such an X it is the same as the Azumaya–Brauer group (use $\operatorname{Br}(X \times_E \bar{E}) = 0$). It was Voskresenskiĭ's insight (cf. [37] or [8, Proposition 6]) that $\rho(\hat{T}) = [\operatorname{Pic} X_K]$. Standard arguments (see [8, Sect. 7]) then yield the first equality in (ii). As for the second, which seems to appear for the first time here (but see [33, Proposition 9.8]), simply use a flasque resolution

$$0 \to \hat{T} \to P \to F \to 0$$

of $\hat{T}$, and use $H^1(G, P) = 0$ and $Ш^2_\omega(G, P) = 0$ for any permutation module P together with $H^1(\langle g \rangle, F) \simeq \hat{H}^{-1}(\langle g \rangle, F) = 0$ (since F is flasque).

COROLLARY 9.6. *Let* T *be the E-torus of Proposition* 9.3, *assume* char $E = 0$, *and let* X *be a smooth E-compactification of* T. *Then*

$$\operatorname{Br} X = \operatorname{Br} E.$$

Proof. Let $g \in G = \mathfrak{S}_n$ be the cyclic permutation $(1, \ldots, n)$. Taking the G-cohomology and the $\langle g \rangle$-cohomology of $0 \to I_{G/H} \to \mathbb{Z}[G/H] \to \mathbb{Z} \to 0$ gives the commutative diagram of exact sequences

$$\begin{tikzcd}
\mathbb{Z} \cdot N_{G/H} \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Z}/n = H^1(G, I_{G/H}) \arrow[r] \arrow[d] & 0 \\
\mathbb{Z} \cdot N_{G/H} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/n = H^1(\langle g \rangle, I_{G/H}) \arrow[r] & 0
\end{tikzcd}$$

which shows $Ш^1_\omega(G, I_{G/H}) = 0$. Using $H^1(G, P) = 0$ and $Ш^2_\omega(G, P) = 0$ for the permutation module $P = \mathbb{Z}[G/H \times G/H]$, we conclude $Ш^2_\omega(G, A) = 0$ from (9.2.1), hence the result by 9.5(ii) (since $T(E)$, hence $X(E)$, is not empty, $\operatorname{Br} E$ injects into $\operatorname{Br} X$).

In characteristic zero, the previous results now enable us to recover the main result of Saltman's paper [32]:

THEOREM 9.7. *Let Y/k be a smooth proper model of the function field $Z_n(r)$, where $Z_n(r)$ is the centre of the generic division ring $UD(k, n, r)$ over a field k of characteristic zero. Then the Brauer group of Y is trivial, i.e.,* $\mathrm{Br}\, Y = \mathrm{Br}\, k$.

Proof. Since $Z_n(r+1)$ is purely transcendental over $Z_n(r)$ by Procesi's result, the birational invariance of the Brauer group [GB III, 7.4] and its triviality over a projective space (or even, since we are in characteristic zero, over an affine space) reduce the problem to $r=2$. Let T/E be as in 9.3, and let X/E be a smooth E-compactification of T. Recall that E/k is purely transcendental, i.e., is the function field of $\mathbb{P}^n_k$. By Hironaka's theorem, there exists a smooth proper model Y/k of $Z_n(2) = E(T)$, equipped with a proper k-morphism $p_2 \colon Y \to \mathbb{P}^n_k$, and such that X/E is the generic fibre of p_2:

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "p_1"'] & Y \arrow[d, "p_2"] \\ \mathrm{Spec}\, E \arrow[r, "i_2"] & \mathbb{P}^n_k . \end{tikzcd} \tag{9.7.1}$$

Let $s_1 \colon \mathrm{Spec}\, E \to X$ be a section of p_1 (e.g., the section given by the neutral element in $T(E) \subset X(E)$). Since $\mathbb{P}^n_k$ is smooth, hence regular in codimension 1, and since p_2 is a proper morphism, s_1 extends to a k-morphism $s_2 \colon U \to Y$ with U an open set of Y which contains all codimension 1 points of $\mathbb{P}^n_k$: there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i_1"] & Y \arrow[dr, "p_2"] & \\ \mathrm{Spec}\, E \arrow[u, "s_1"] \arrow[r, "i_3"] \arrow[rr, bend right, "i_2"'] & U \arrow[u, "s_2"] \arrow[r, hook, "i_4"] & \mathbb{P}^n_k \end{tikzcd} \tag{9.7.2}$$

Since U contains all codimension 1 points of the smooth scheme $\mathbb{P}^n_k$, the purity theorem [GB III, 6.1] asserts that $i_4^* \colon \mathrm{Br}\, \mathbb{P}^n_k \to \mathrm{Br}\, U$ is an isomorphism. Let α be in $\mathrm{Br}\, Y$. We have the chain of equalities:

$$\begin{aligned} i_1^*(\alpha) &= p_1^* s_1^* i_1^*(\alpha) && \text{because } \mathrm{Br}\, X = \mathrm{Br}\, E \quad (9.6) \\ &= p_1^* i_3^* s_2^*(\alpha) && \text{by } (9.7.2) \\ &= p_1^* i_3^* i_4^*(\beta) && \text{for some } \beta \in \mathrm{Br}\, \mathbb{P}^n_k \text{ by purity} \\ &= p_1^* i_2^*(\beta) && \text{by } (9.7.2) \\ &= i_1^* p_2^*(\beta) && \text{by } (9.7.1). \end{aligned}$$

Now since Y/k is smooth, hence Y regular, i_1^* is injective (indeed Br Y injects into $\operatorname{Br} k(Y) = \operatorname{Br} E(X)$, see [GB II, Sect. 1] or 2.2(ii)). Hence $\alpha = p_2^*(\beta)$ for $\beta \in \operatorname{Br} \mathbb{P}_k^n$ and this last group is none but $\operatorname{Br} k$. Since $s_2(U(k)) \subset Y(k)$ is not empty, we conclude $\operatorname{Br} Y = \operatorname{Br} k$ for the specific Y used in the proof. The conclusion for an arbitrary smooth proper model of $Z_n(2)$ now follows from the birational invariance of the Brauer group, already mentioned above.

Remark 9.8. The characteristic zero hypothesis is used in many places in the proof of 9.7. There is nevertheless some scope for extending this proof in arbitrary characteristic, and showing, as Saltman does [32], that the "unramified" Brauer group of $Z_n(r)$ is trivial, or at least that the theorem holds up to p-torsion, for $p = \operatorname{char} k$. Indeed, smooth compactifications of tori exist in arbitrary characteristic [4] and $\operatorname{Br} \mathbb{P}_k^n = \operatorname{Br} k$ over an arbitrary field k (cf. [32]; the crucial case is $n = 1$, and follows from [GB III, 5.8]).

Remark 9.9. Theorem 9.7 shows that one of the suggested approaches [29] to the would-be non-k-rationality of Y, i.e., the would-be nonpurity of $Z_n(r)$, completely fails. Subtler k-birational invariants than the Brauer group are needed. For $n = 2$, Procesi showed that $Z_2(2)$ (hence $Z_2(r)$) is purely transcendental over k. This easily follows from 9.3, since the E-torus $T = T_2$ is then split by the quadratic extension K/E, and the structure of torsion-free finitely generated $\mathbb{Z}/2$-modules shows that any such E-torus is an E-rational variety. For $n = 3$, Formanek [17] showed that T_3 is an E-rational variety. Since $\mathfrak{S}_3 = D_3$ is a metacyclic group, results of Endo and Miyata (cf. [8, Proposition 2 and Remark R5]) already show that T_3 is a stably rational E-variety, i.e., $T_3 \times_E \mathbb{A}_E^n$ is E-birational to $\mathbb{A}_E^m$ for some n, m. Thus the function field of $T_3 \times_k \mathbb{A}_k^n$ is purely transcendental over k. In other words, $Z_3(2)$, which is the function field of T_3, becomes purely transcendental over k provided one adds a few transcendental variables; the same therefore holds for $Z_3(r)$ for any $r \geqslant 2$. Formanek's result [17] is that $Z_3(r)$ itself is purely transcendental over k.

As noted by Snider (mentioned in [18, p. 319]), this approach to the k-rationality of $Z_n(2)$ already fails for $n = 4$, namely T_4 is not E-rational (Formanek however shows [18] that $Z_4(r)$ is purely transcendental over k). Here is a proof of Snider's result, which shows that T_4 is not even (E-birationally) a direct factor of an E-rational variety.

Let $G' \subset G = \mathfrak{S}_4$ be the subgroup isomorphic to $V_4 = \mathbb{Z}/2 \times \mathbb{Z}/2$. Arguing as in 9.6, we find $H^1(G', \rho(A)) = Ш_\omega^2(G', A) = Ш_\omega^1(G', I_{G/H})$. Let $g = (1, 2)(3, 4) \in \mathfrak{S}_4$ be a typical nontrivial element of G'. As in 9.6, we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
\mathbb{Z}\cdot N_{G/H} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/4 = H^1(G', I_{G/H}) & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow \qquad\quad \downarrow & & \\
\mathbb{Z}\cdot(t_1+t_2)\oplus\mathbb{Z}\cdot(t_3+t_4) & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 = H^1(\langle g\rangle, I_{G/H}) & \longrightarrow & 0.
\end{array}$$

Writing similar diagrams for the other nontrivial elements $g \in G'$, we conclude:

$$H^1(G', \rho(A)) = Ш^1_\omega(G', I_{G/H}) = \mathbb{Z}/2.$$

Hence $\rho(A)$ is not an invertible G-module, T_4 is not a direct factor of an E-rational variety, and if $\operatorname{char} k = 0$ and E_1 denotes the fixed field of K under G', $\operatorname{Br} E_1 \neq \operatorname{Br} X_{E_1}$ for any smooth E-compactification of T_4 (use 9.5(ii)). Thus 9.6 may fail to hold after a finite extension of the ground field E!

PROPOSITION 9.10. *The torus $T = T_n$ in 9.3 is E-birationally a direct factor of an E-rational variety (resp. is stably E-rational), if and only if the E-torus $R^1_{F/E}\mathbb{G}_m$ enjoys the same property.*

(See 7.4 and [8, Proposition 6] for equivalent properties.)

Proof. Let $0 \to J_{G/H} \to P \to F \to 0$ be a flasque resolution of the G-module $J_{G/H} = I^0_{G/H}$ (cf. (0.4.2)). Dualizing ($R \mapsto R^0$) the sequence (9.2.1), we form the push-out diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & J_{G/H} & \longrightarrow & \mathbb{Z}[G/H\oplus G/H] & \longrightarrow & A^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & P & \longrightarrow & M & \longrightarrow & A^0 & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & F & = & F & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The following equality now follows from Lemma 9.4:

$$(\rho(A))^0 = \varsigma(A^0) = \varsigma(M) = \varsigma(F)$$

hence

$$\rho(A) = (\varsigma(F))^0 \ (= \rho(F^0)).$$

Now T_n is a direct factor of an E-rational variety if and only if $\rho(A)$ is invertible (7.4), hence if and only if $\varsigma(F)$ is invertible, hence by 9.4(iii) if and only if F is invertible, hence by 7.4 if and only if $R^1_{F/E}\mathbb{G}_m$ (whose character group is $J_{G/H}$) is a direct factor of an E-rational variety. The same proof goes for stable rationality.

Remark 9.11. Although the middle vertical sequence in the above diagram splits, we cannot produce a kind of E-birational equivalence between T and $R^1_{F/E}\mathbb{G}_m$; indeed the last equality connects $\rho(A)$ and $\rho(F^0)$, not $\rho(F)$.

COROLLARY 9.12. *If $n = p$ is a prime, the E-torus $T = T_p$ in 9.3 is a direct factor of an E-rational variety; in fact there exists an E-torus T'_p such that $T_p \times_E T'_p$ is E-birational to some affine space over E.*

Proof. This follows from 9.1, 9.10, and 7.4.

We thus get a more explicit proof of Saltman's result [31, Corollary 5.3]:

COROLLARY 9.13. *If p is prime, the centre $Z_p(r)$ of $UD(k, p, r)$ is retract rational over k.*

Proof. According to Procesi's result, $Z_p(r+1)$ is purely transcendental over $Z_p(r)$ for $r \geqslant 2$. So it is enough to prove the result for $r = 2$, i.e. (Proposition 9.3) for the function field $E(T_p)$. Let T'_p be the E-torus in Corollary 9.12. Since E is purely transcendental over k, the function field of $T_p \times_E T'_p$ is purely transcendental over k. We may find an integral affine k-variety X with function field E such that T_p and T'_p are the restrictions over E of X-tori $\tilde{T}_p$ and $\tilde{T}'_p$ (see the proof of Proposition 1.5). The function field of the integral affine k-variety $\tilde{T}_p \times_X \tilde{T}'_p$ coincides with the function field of $T_p \times_E T'_p$, hence is purely transcendental over k, and the identity map of the integral affine k-variety $\tilde{T}_p$ (whose function field is $Z_p(2)$) factorizes through $\tilde{T}_p \times_X \tilde{T}'_p$:

$$\tilde{T}_p \xrightarrow{\ i\ } \tilde{T}_p \times_X \tilde{T}'_p \xrightarrow{\ p_1\ } \tilde{T}_p,$$

where i sends α to (α, e') for e' the zero section of the X-torus $\tilde{T}'_p$, and p_1 denotes the projection onto the first factor of the fibre product.

Remark 9.14. Saltman's proof uses his lifting criterion for retract rationality [31, Theorem 3.8].

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