On the integral Tate conjecture and on the integral Hodge conjecture for 1-cycles on the product of a curve and a surface

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Shafarevich Seminar
(online)
Москва, Tuesday, January 19th, 2021

This talk will have two parts.

I. On the integral Tate conjecture over a finite field (work with Federico Scavia)
II. On the integral Hodge conjecture

This file (January 25th) is slightly edited.

Let X be a smooth projective (geom. connected) variety over a finite field $\mathbb{F}$ of char. p. Unless otherwise mentioned, cohomology is étale cohomology (Galois cohomology over a field).
We have $CH^1(X) = \mathrm{Pic}(X) = H^1_{Zar}(X, \mathbb{G}_m) = H^1(X, \mathbb{G}_m)$. Also $\mathrm{Br}(X) := H^2(X, \mathbb{G}_m)$.
For r prime to p, the Kummer exact sequence of étale sheaves associated to $x \mapsto x^r$

$$1 \to \mu_r \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$

induces an exact sequence

$$0 \to \mathrm{Pic}(X)/r \to H^2(X, \mu_r) \to \mathrm{Br}(X)[r] \to 0.$$

Let $r = \ell^n$, with $\ell \neq p$. Passing over to the limit in n, we get the ℓ-adic cycle class map

$$\mathrm{Pic}(X) \otimes \mathbb{Z}_\ell \to H^2(X, \mathbb{Z}_\ell(1)).$$

Around 1960, Tate conjectured

(T^1) *For any smooth projective* $X/\mathbb{F}$, *the map*

$$\mathrm{Pic}(X) \otimes \mathbb{Z}_\ell \to H^2(X, \mathbb{Z}_\ell(1))$$

is surjective.

Via the Kummer sequence, one easily sees that this is equivalent to the finiteness of the ℓ-primary component $\mathrm{Br}(X)\{\ell\}$ of the Brauer group $\mathrm{Br}(X) := H^2_{et}(X, \mathbb{G}_m)$. [Known : If true for one $\ell \neq p$ then true for all $\ell \neq p$ and $\mathrm{Br}(X)\{\ell\} = 0$ for almost all p.]
This finiteness is closely related to the conjectured finiteness of Tate-Shafarevich groups of abelian varieties over a global field $\mathbb{F}(C)$.
The conjecture is known for geometrically separably unirational varieties (easy), for abelian varieties (Tate) and for all $K3$-surfaces.

For any $i \geq 0$, let $CH^i(X)$ denote the Chow group of codimension i cycles modulo rational equivalence. For any $i \geq 1$, there is an ℓ-adic **cycle class map**

$$CH^i(X) \otimes \mathbb{Z}_\ell \to H^{2i}(X, \mathbb{Z}_\ell(i)),$$

with values in the projective limit of the (finite) étale cohomology groups $H^{2i}(X, \mu_{\ell^n}^{\otimes i})$, which is a $\mathbb{Z}_\ell$-module of finite type.

For $i > 1$, Tate conjectured that the cycle class map

$$CH^i(X) \otimes \mathbb{Q}_\ell \to H^{2i}(X, \mathbb{Q}_\ell(i)) := H^{2i}(X, \mathbb{Z}_\ell(i)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is surjective. Very little is known.

For $i = 1$, the surjectivity conjecture with $\mathbb{Z}_\ell$ coefficients is equivalent to the conjecture with $\mathbb{Q}_\ell$ coefficients.
For $i > 1$, one may give examples where the surjectivity statement with $\mathbb{Z}_\ell$ coefficients does not hold. However, for X of dimension d, it is unknown whether the (strong) *integral Tate conjecture* $T_1 = T^{d-1}$ *for 1-cycles* holds :

(T_1) *The map* $CH^{d-1}(X) \otimes \mathbb{Z}_\ell \to H^{2d-2}(X, \mathbb{Z}_\ell(d-1))$ *is onto.*

Under T^1 for X, the cokernel of the above map is finite (proof using Deligne's theorem on the Weil conjectures, including the hard Lefschetz theorem).

For $d = 2$, $T_1 = T^1$, original Tate conjecture.

For arbitrary d, the integral Tate conjecture for 1-cycles holds for X of any dimension $d \geq 3$ if it holds for any X of dimension 3. This follows from the Bertini theorem, the purity theorem, and the affine Lefschetz theorem in étale cohomology.
We shall write T^1_{surf} for the conjecture T^1 restricted to surfaces.

For X of dimension 3, some nontrivial cases of T_1 have been established.

• X is a conic bundle over a geometrically ruled surface (Parimala and Suresh 2016).

• X is the product of a curve of arbitrary genus and a geometrically rational surface (Pirutka 2016).

Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$ and $G = \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$.

Theorem (C. Schoen, 1998)

Let $X/\mathbb{F}$ be smooth, proj., geom. connected. Let $\ell \neq \mathrm{char}(\mathbb{F})$. Let $\overline{X} = X \times_{\mathbb{F}} \overline{\mathbb{F}}$. If T^1_{surf} holds, then the map

$$CH^{d-1}(\overline{X}) \otimes \mathbb{Z}_\ell \to \bigcup_{U \subset G} H^{2d-2}(\overline{X}, \mathbb{Z}_\ell(d-1))^U,$$

where $U \subset G$ run through the open subgroups of G, is onto.

Corollary. *Let $X/\mathbb{F}$ be smooth, proj., geom. connected of dimension d. Let $\overline{X} = X \times_{\mathbb{F}} \overline{\mathbb{F}}$. Suppose $\mathrm{Br}(\overline{X})\{\ell\}$ is finite. If T^1_{surf} holds, then the cycle class map*

$$CH^{d-1}(\overline{X}) \otimes \mathbb{Z}_\ell \to H^{2d-2}(\overline{X}, \mathbb{Z}_\ell(d-1))$$

is onto.

Remark. The condition $\mathrm{Br}(\overline{X})\{\ell\}$ finite is a positive characteristic version of $H^2(X, \mathcal{O}_X) = 0$.

What about the situation over a finite field $\mathbb{F}$ itself?

Definition. A smooth, projective, connected variety S over a field k is called *geometrically CH_0-trivial* if for **any** algebraically closed field extension Ω of k, the degree map $CH_0(S_\Omega) \to \mathbb{Z}$ is an isomorphism.

Examples : Rationally connected varieties. Enriques surfaces. Some surfaces of general type.

Theorem A (main theorem of the talk) (CT/Scavia)
Let $\mathbb{F}$ *be a finite field,* $G = \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$*. Let* ℓ *be a prime,* $\ell \neq \mathrm{char.}(\mathbb{F})$*. Let* C *be a smooth projective curve over* $\mathbb{F}$*, let* $J/\mathbb{F}$ *be its jacobian, and let* $S/\mathbb{F}$ *be a smooth, projective, geometrically* CH_0*-trivial surface.*
Let $X = C \times_{\mathbb{F}} S$*.*
Assume T^1_{surf}*. Under the assumption*

$$(**) \qquad \mathrm{Hom}_G(\mathrm{Pic}(S_{\overline{\mathbb{F}}})\{\ell\}, J(\overline{\mathbb{F}})) = 0,$$

the cycle class map $CH^2(X) \otimes \mathbb{Z}_\ell \to H^4_{et}(X, \mathbb{Z}_\ell(2))$ *is onto.*

Concrete case

Let $p = \mathrm{char}(\mathbb{F}) \neq 2$ and let $E/\mathbb{F}$ be an elliptic curve defined by the affine equation $y^2 = P(x)$ with $P \in \mathbb{F}[x]$ a separable polynomial of degree 3.
Let $S/\mathbb{F}$ be an Enriques surface. This is a geometrically CH_0-trivial variety.
One has $\mathrm{Pic}(S_{\overline{\mathbb{F}}})_{tors} = \mathbb{Z}/2$, automatically with trivial Galois action.
The assumption $(**)$ reads : $E(\mathbb{F})[2] = 0$, which translates as : $P \in \mathbb{F}[x]$ is an *irreducible* polynomial.

For $\ell = 2$, $p = \mathrm{char}(\mathbb{F})) \neq 2$ and $P(x) \in \mathbb{F}[x]$ *reducible*, the integral Tate conjecture $T_1(X)$ with $\mathbb{Z}_2$ coefficients for $X = E \times_{\mathbb{F}} S$ remains open.

Unramified cohomology, cycles of codimension 2

I first recall various results, in particular from a paper with Bruno Kahn (2013).

Let M be a finite Galois-module over a field k. Given a smooth, projective, integral variety X/k with function field $k(X)$, and $i \geq 1$ an integer, one lets

$$H^i_{nr}(k(X), M) := \mathrm{Ker}[H^i(k(X), M) \to \oplus_{x \in X^{(1)}} H^{i-1}(k(x), M(-1))]$$

Here $k(x)$ is the residue field at a codimension 1 point $x \in X$, the cohomology is Galois cohomology of fields, and the maps on the right hand side are "residue maps".
For $\ell \neq \mathrm{char.}(k)$, one is interested in $M = \mu_{\ell^n}^{\otimes j} = \mathbb{Z}/\ell^n(j)$, hence $M(-1) = \mu_{\ell^n}^{\otimes (j-1)}$, and in the direct limit $\mathbb{Q}_\ell/\mathbb{Z}_\ell(j) = \lim_j \mu_{\ell^n}^{\otimes j}$, for which the cohomology groups are the limit of the cohomology groups. By Voevodsky, for $j \geq 1$ and any field F of char. $\neq \ell$,

$$H^j(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j-1)) = \bigcup_n H^j(F, \mu_{\ell^n}^{\otimes j-1}).$$

The group $H^1_{nr}(k(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell) = H^1_{et}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ classifies ℓ-primary cyclic étale covers of X.

One has

$$H^2_{nr}(k(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) = \mathrm{Br}(X)\{\ell\}.$$

For $k = \mathbb{F}$ a finite field, this turns up in investigations on the Tate conjecture for divisors. As already mentioned, its finiteness for a given X is equivalent to the ℓ-adic Tate conjecture for codimension 1 cycles on X.

The group $H^3_{nr}(k(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ turns up when investigating cycles of codimension 2. It vanishes for $\dim(X) \leq 2$ (higher class field theory, 80s).

Let $k = \mathbb{F}$. Open questions :

Is $H^3_{nr}(\mathbb{F}(X), \mu_\ell^{\otimes 2})$ finite ?
(Equivalent question : is $CH^2(X)/\ell$ finite ?)

Is $H^3_{nr}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ of cofinite type ?

Is $H^3_{nr}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ finite ?

Do we have $H^3_{nr}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$ for any threefold X ?
[Known for a conic bundle over a surface, Parimala–Suresh 2016]

Examples of $X/\mathbb{F}$ with $dim(X) \geq 5$ and $H^3_{nr}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \neq 0$ are known (Pirutka 2011).

Theorem B (Kahn 2012, CT-Kahn 2013) *For $X/\mathbb{F}$ smooth, projective of arbitrary dimension, the torsion subgroup of the (conjecturally finite) group*

$$\mathrm{Coker}[CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))]$$

is isomorphic to the quotient of $H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ by its maximal divisible subgroup.

There is an analogue of this for the integral Hodge conjecture (CT-Voisin 2012).

A **basic exact sequence** (CT-Kahn 2013). Let $\overline{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$, let $\overline{X} = X \times_{\mathbb{F}} \overline{\mathbb{F}}$ and $G = \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$.

Theorem C. *For $X/\mathbb{F}$ a smooth, projective, geometrically connected variety over a finite field, there is a long exact sequence*

$$0 \to \mathrm{Ker}[CH^2(X)\{\ell\} \to CH^2(\overline{X})\{\ell\}] \to H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)))$$

$$\to \mathrm{Ker}[H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \to H^3_{\mathrm{nr}}(\overline{\mathbb{F}}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))]$$

$$\to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\} \to 0.$$

Moreover $H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) = H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tors}})$ *and this is a finite group.*

The proof relies on early work of Bloch and on the Merkurjev-Suslin theorem (1983).

The last statement follows from Deligne's theorem on the Weil conjectures.

For $X/\mathbb{F}$ a curve, all groups in the sequence are zero.

For $X/\mathbb{F}$ a surface, trivially $H^3(\overline{\mathbb{F}}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$. One actually has $H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$. This vanishing was remarked in the early stages of higher class field theory (CT-Sansuc-Soulé, K. Kato, in the 80s). It uses a theorem of S. Lang, which relies on Tchebotarev's theorem. The above exact sequence then gives the prime-to-p part of the main theorem of unramified class field theory for surfaces over a finite field (studied by Parshin, Bloch, Kato, Saito).

For a 3-fold $X = C \times_{\mathbb{F}} S$ as in Theorem A, Theorem B gives an isomorphism of finite groups

$$\mathrm{Coker}[CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))] \simeq H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)),$$

and, under the assumption T^1_{surf}, Chad Schoen's theorem implies $H^3_{\mathrm{nr}}(\overline{\mathbb{F}}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$. Thus :

Under T^1_{surf}, for our threefolds $X = C \times_{\mathbb{F}} S$ with S geometrically CH_0-trivial, there is an exact sequence of finite groups

$$0 \to \mathrm{Ker}[CH^2(X)\{\ell\} \to CH^2(\overline{X})\{\ell\}] \to H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)))$$

$$\xrightarrow{\theta_X} H^3_{\mathrm{nr}}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\} \to 0.$$

Under T^1_{surf}, for our threefolds $X = C \times_{\mathbb{F}} S$ with S geometrically CH_0-trivial, the surjectivity of $CH^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$ (integral Tate conjecture for 1-cycles) is therefore equivalent to the combination of two hypotheses :

Hypothesis 1 *The composite map*

$$\rho_X : H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \to H^3(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$$

of $H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \to H^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ *and* $H^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \to H^3(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ *vanishes.*

Hypothesis 2 $\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\} = 0.$

Results with Federico Scavia

On Hypothesis 1

Hypothesis 1. *Let $X/\mathbb{F}$ be a smooth projective, geometrically connected variety. The map*

$$\rho_X : H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \to H^3(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$$

vanishes.

This map is the composite of the Hochschild-Serre map

$$H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \to H^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$$

with the restriction map to the generic point of X.

Hypothesis 1 is equivalent to each of the following hypotheses :

Hypothesis 1a. The (injective) map from

$$\mathrm{Ker}[CH^2(X)\{\ell\} \to CH^2(\overline{X})\{\ell\}]$$

to the (finite) group

$$H^1(\mathbb{F}, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \simeq H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{tors})$$

is onto.

Hypothesis 1b. For any $n \geq 1$, if a class $\xi \in H^3(X, \mu_{\ell^n}^{\otimes 2})$ vanishes in $H^3(\overline{X}, \mu_{\ell^n}^{\otimes 2})$, then it vanishes after restriction to a suitable Zariski open set $U \subset X$.

For all we know, these hypotheses 1,1a,1b could hold for any smooth projective variety X over a finite field.

For X of dimension > 2 , we do not see how to establish them directly – unless of course when the finite group $H^3(\overline{X}, \mathbb{Z}_\ell(2))_{tors}$ vanishes.
The group $H^3(\overline{X}, \mathbb{Z}_\ell(1))_{tors}$ is the nondivisible part of the ℓ-primary Brauer group of $\overline{X}$.
The finite group $H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(1))_{tors})$ is thus the most easily computable group in the 4 terms exact sequence.
For $\mathrm{char}(\mathbb{F}) \neq 2$, $\ell = 2$, $X = E \times_F S$ product of an elliptic curve E and an Enriques surface S, one finds that this group is isomorphic to $E(\mathbb{F})[2] \oplus \mathbb{Z}/2$. How can one lift elements of this group to cycles in $\mathrm{Ker}[CH^2(X)\{\ell\} \to CH^2(\overline{X})\{\ell\}]$?

We prove :

Theorem. *Let Y be a smooth, projective geometrically connected varieties over a finite field $\mathbb{F}$. Let C be a smooth projective curve over $\mathbb{F}$. Let $X = C \times_{\mathbb{F}} Y$. If the maps ρ_Y vanishes, then so does the map ρ_X.*

One must study $H^1(\mathbb{F}, H^2(\overline{X}, \mu_{\ell^n}^{\otimes 2}))$ under restriction from X to its generic point.
As may be expected, the proof uses a specific Künneth formula, along with standard properties of Galois cohomology of a finite field.

Corollary. *For the product X of a surface and arbitrary many curves, the map ρ_X vanishes.*
This establishes Hypothesis 1 for the 3-folds $X = C \times_F S$ under study.

On Hypothesis 2

Hypothesis 2. Let $X/\mathbb{F}$ be a smooth projective, geometrically connected variety. If $\dim(X) = 3$, then

$$\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\} = 0.$$

[For X of dimension at least 5, A. Pirutka gave counterexamples.]

Here we restrict to the special situation : C is a curve, S is geometrically CH_0-trivial surface, and $X = C \times_{\mathbb{F}} S$.
One lets $K = \mathbb{F}(C)$ and $L = \overline{\mathbb{F}}(C)$.
On considers the projection $X = C \times S \to C$, with generic fibre the K-surface S_K. Restriction to the generic fibre gives a natural map from

$$\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\}$$

to

$$\mathrm{Coker}[CH^2(S_K) \to CH^2(S_L)^G]\{\ell\}.$$

Using the hypothesis that S is geometrically CH_0-trivial, which implies $b_1 = 0$ and $b_2 - \rho = 0$ (Betti number b_i, rank ρ of Néron-Severi group), one proves :

Theorem. *The natural, exact localisation sequence*

$$\mathrm{Pic}(\overline{C}) \otimes \mathrm{Pic}(\overline{S}) \to CH^2(\overline{X}) \to CH^2(S_L) \to 0.$$

may be extended on the left with a finite p-group.

To prove this, we use correspondences on the product $X = C \times S$, over $\overline{\mathbb{F}}$.

We use various pull-back maps, push-forward maps, intersection maps of cycle classes :

$$\mathrm{Pic}(C) \otimes \mathrm{Pic}(S) \to \mathrm{Pic}(X) \otimes \mathrm{Pic}(X) \to CH^2(X)$$

$$CH^2(X) \otimes \mathrm{Pic}(S) \to CH^2(X) \otimes \mathrm{Pic}(X) \to CH^3(X) = CH_0(X) \to CH_0(C)$$

$$\mathrm{Pic}(C) \otimes \mathrm{Pic}(S) \to CH^2(X) = CH_1(X) \to CH_1(S) = \mathrm{Pic}(S)$$

Not completely standard properties of G-lattices for $G = \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ applied to the (up to p-torsion) exact sequence of G-modules

$$0 \to \mathrm{Pic}(\overline{C}) \otimes \mathrm{Pic}(\overline{S}) \to CH^2(\overline{X}) \to CH^2(S_L) \to 0$$

then lead to :

Theorem. *The natural map from* $\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\}$ *to* $\mathrm{Coker}[CH^2(S_K) \to CH^2(S_L)^G]\{\ell\}$ *is an isomorphism.*
(Recall $K = \mathbb{F}(C)$ and $L = \overline{\mathbb{F}}(C)$.)

One is thus left with controlling this group. Under the CH_0-triviality hypothesis for S, it coincides with

$$\mathrm{Coker}[CH^2(S_K)\{\ell\} \to CH^2(S_L)\{\ell\}^G].$$

At this point, for a geometrically CH_0-trivial surface over $L = \overline{\mathbb{F}}(C)$, which is a field of cohomological dimension 1, like $\mathbb{F}$, using the K-theoretic mechanism, one may produce an exact sequence parallel to the basic four-term exact sequence over $\mathbb{F}$ which we saw at the beginning. In the particular case of the constant surface $S_L = S \times_{\mathbb{F}} L$, the left hand side of this sequence gives an injection

$$0 \to A_0(S_L)\{\ell\} \to H^1_{Galois}(L, H^3(\overline{S}, \mathbb{Z}_\ell(2)\{\ell\})$$

where $A_0(S_L) \subset CH^2(S_L)$ is the subgroup of classes of zero-cycles of degree zero on the L-surface S_L.

Study of this situation over completions of $\overline{\mathbb{F}}(C)$ (Raskind 1989) and a good reduction argument in the weak Mordell-Weil style, plus a further identification of torsion groups in cohomology of surfaces over an algebraically closed field then yield a Galois embedding

$$A_0(S_L)\{\ell\} \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(\overline{S})\{\ell\}, J(C)(\overline{\mathbb{F}})),$$

hence an embedding

$$A_0(S_L)\{\ell\}^G \hookrightarrow \mathrm{Hom}_G(\mathrm{Pic}(\overline{S})\{\ell\}, J(C)(\overline{\mathbb{F}})).$$

If the group $\mathrm{Hom}_G(\mathrm{Pic}(\overline{S})\{\ell\}, J(C)(\overline{\mathbb{F}}))$ vanishes, then $A_0(S_L)\{\ell\}^G = CH^2(S_L)\{\ell\}^G$ vanishes, and since S_K obviously has a K-rational point, then trivially

$$\mathrm{Coker}[CH^2(S_K)\{\ell\} \to CH^2(S_L)\{\ell\}^G] = 0,$$

from which one then deduces

$$\mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^G]\{\ell\} = 0,$$

which is Hypothesis 2, and this completes the proof of Theorem A.

One has actually proved :

Theorem *Let* $\mathbb{F}$ *be a finite field,* $G = \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$*. Let* ℓ *be a prime,* $\ell \neq \mathrm{char.}(\mathbb{F})$*. Let* C *be a smooth projective curve over* $\mathbb{F}$*, let* $J/\mathbb{F}$ *be its jacobian, and let* $S/\mathbb{F}$ *be a smooth, projective, geometrically* CH_0*-trivial surface. Let* $X = C \times_{\mathbb{F}} S$*.*
Assume

$$(**) \qquad \mathrm{Hom}_G(\mathrm{Pic}(S_{\overline{\mathbb{F}}})\{\ell\}, J(\overline{\mathbb{F}})) = 0,$$

Then $\mathrm{Ker}[H^3_{nr}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2) \to H^3_{nr}(\overline{\mathbb{F}}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))] = 0$.
If moreover T^1_{surf} *holds, then* $H^3_{nr}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$ *and the cycle class map* $CH^2(X) \otimes \mathbb{Z}_\ell \to H^4_{et}(X, \mathbb{Z}_\ell(2))$ *is onto.*

Basic question : Is the assumption $(**)$ necessary ?

Over the complex field

The situation over finite fields should be confronted with the situation over the complex field, which actually stimulated the work with F. Scavia.

For smooth projective varieties X over $\mathbb{C}$, the integral Tate conjecture admits an earlier, formally parallel surjectivity question, the integral Hodge conjecture (known to fail in general) for the Betti cycle maps

$$CH^i(X) \to Hdg^{2i}(X, \mathbb{Z}),$$

where $Hdg^{2i}(X, \mathbb{Z}) \subset H^{2i}_{Betti}(X, \mathbb{Z})$ is the subgroup of rationally Hodge classes. The surjectivity with $\mathbb{Q}$-coefficients is the classical (rational) Hodge conjecture.

For $i = 1$, the integral Hodge conjecture is known (Lefschetz (1,1)-theorem).

By the Lefschetz hyperplane theorem it implies the rational Hodge conjecture for for $i = d - 1$.

With integral coefficients, counterexamples to the integral Hodge conjecture for 1-cycles on threefolds have been constructed.

Kollár : "very general" hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ of degree $d = p^3.n$ with p prime, $p \neq 2, 3$).

A recent counterexample (Benoist-Ottem 2018) involves the product $X = E \times S$ of an elliptic curve E and an Enriques surface. For fixed S, provided E is "very general", the integral Hodge conjecture fails for X.

In both cases, we have $H^2(X, O_X) = 0$ and $\mathrm{Br}(X)$ finite. (Compare with the Schoen result on $\overline{\mathbb{F}}$, depending on T_{surf}.)

Theorem (CT-Voisin 2012).
Let $X/\mathbb{C}$ be a smooth, projective, connected variety.
The following finite groups are isomorphic :
(i) The torsion subgroup of $Coker[CH^2(X) \to H^4(X,\mathbb{Z}(2))]$
(ii) The torsion subgroup of the conjecturally finite group $Coker[CH^2(X) \to Hdg^4(X,\mathbb{Z}(2))]$
(iii) The quotient of $H^3_{nr}(\mathbb{C}(X),\mathbb{Q}/\mathbb{Z}(2))$ *by its maximal divisible subgroup.*

These groups are birational invariants.

Corollary. *Let $X/\mathbb{C}$ be a smooth, projective, connected variety. Suppose that the Chow group of zero-cycles is representable by a surface, that is to say, there exists a morphism $f : S \to X$ from a smooth, projective, connected surface S such that the induced map $f_* : CH_0(S) \to CH_0(X)$ is surjective.*
Then the following groups are finite and isomorphic :
(i) The torsion subgroup of $Coker[CH^2(X) \to H^4(X, \mathbb{Z}(2))]$.
(ii) The group $Coker[CH^2(X) \to Hdg^4(X, \mathbb{Z}(2))]$.
(iii) The group $H^3_{nr}(\mathbb{C}(X), \mathbb{Q}/\mathbb{Z}(2))$.

There is in general no "simple formula" for the value of $H^3_{nr}(\mathbb{C}(X), \mathbb{Q}/\mathbb{Z}(2))$, in contrast with $H^1_{nr}(\mathbb{C}(X), \mathbb{Q}/\mathbb{Z})$ and $H^2_{nr}(\mathbb{C}(X), \mathbb{Q}/\mathbb{Z}(1))$.

In some cases, one may compute these groups by using complex algebraic geometry, in some other cases by using algebraic K-theory.

Voisin 2006 proved that these groups are zero for any uniruled threefold, and also for Calabi-Yau threefolds.

Examples of unirational varieties X with $dim(X) \geq 6$ and $H^3_{nr}(\mathbb{C}(X), \mathbb{Q}/\mathbb{Z}(2)) \neq 0$ were given by CT-Ojanguren 1989 (talk at the Shafarevich seminar, Moscow 1988).

Examples with $dim(X) \geq 4$ were recently given by Schreieder.

In 2002 I asked the following question. Let E_1, E_2, E_3 be elliptic curves. Let $X = E_1 \times E_2 \times E_3$. Let $\xi_i \in H^1(E_i, \mathbb{Z}/2)$ be nonzero elements. Consider the image of the cup-product $\xi_1 \cup \xi_2 \cup \xi_3 \in H^3(X, \mathbb{Z}/2)$. Can its image in $H^3(\mathbb{C}(X), \mathbb{Z}/2)$ be nonzero ?

Gabber immediately showed how, in the "very general" case, the answer is yes. I recently used his technique to prove a result which, via the above theorem with Voisin, extends the Benoist–Ottem result.

Proposition (Gabber 2002). *Let* $\pi : W \to U$ *be a smooth morphism of integral noetherian schemes with geometrically connected fibres. Let* $\alpha \in H^i(W, \mathbb{Z}/\ell))$. *The set of (scheme-theoretic) points* $s \in U$ *such that the restriction of* α *to the generic point of the geometric fibre of* π *at* s *vanishes is a countable union of closed subsets of* U.

(The property is stable under specialisation.)

As a consequence, if U is a variety over $\mathbb{C}$; if there exists one point $s \in U(\mathbb{C})$ such that $\alpha_s \in H^i(\mathbb{C}(W_s), \mathbb{Z}/\ell)$ does not vanish, then the set of such points $s \in U(\mathbb{C})$ is Zariski dense in $U(\mathbb{C})$.

The following theorem (CT 2018) is a variant of a result of Gabber (2002).

Theorem. *Let $X/\mathbb{C}$ be smooth, projective, connected variety. Let ℓ be a prime number. Let $\alpha \in H^i(X, \mathbb{Z}/\ell)$ have a nonzero image in $H^i(\mathbb{C}(X), \mathbb{Z}/\ell)$.*
There exist an elliptic curve $E/\mathbb{C}$ and $\beta \in H^1(E, \mathbb{Z}/\ell)$ such that the image of $\alpha \cup \beta \in H^{i+1}(X \times E, \mathbb{Z}/\ell)$ in $H^{i+1}(\mathbb{C}(X \times E), \mathbb{Z}/\ell)$ is nonzero. In particular the groups $H^{i+1}_{nr}(\mathbb{C}(X \times E), \mathbb{Z}/\ell)$ and $H^{i+1}_{nr}(\mathbb{C}(X \times E), \mathbb{Q}/\mathbb{Z})$ are nonzero.
(Passing from $\mathbb{Z}/\ell$ to $\mathbb{Q}/\mathbb{Z}$ uses Voevodsky.)

The idea is to use a family of elliptic curves over an open set $U \subset \mathbb{P}^1$ which degenerates to a nodal curve over a point $P \in U(\mathbb{C})$. The same idea is used in the paper by Benoist–Ottem.

Proof. One produces an exact sequence of abelian U-group schemes

$$1 \to (\mathbb{Z}/\ell)_U \to \mathcal{E}' \to \mathcal{E} \to 1$$

which on $U \setminus P$ is an isogeny of elliptic curves over U and whose fibre above the point P is $1 \to \mathbb{Z}/\ell \to \mathbb{G}_m \to \mathbb{G}_m \to 1$, where $x \mapsto x^\ell$. Let $\mathcal{E}_P = \mathbb{G}_m \subset \mathbb{P}^1$. Let $\beta \in H^1(\mathcal{E}, \mathbb{Z}/\ell)$ be the class associated to the first sequence. It induces a class in $H^1(\mathcal{E}_P, \mathbb{Z}/\ell) = H^1(\mathbb{G}_m, \mathbb{Z}/\ell)$ which is ramified at $\infty \in \mathbb{P}^1$. Consider the cup-product $\alpha \cup \beta \in H^{i+1}(X \times \mathcal{E}, \mathbb{Z}/\ell)$. On the subvariety $X \times \mathbb{G}_m = X \times \mathcal{E}_P \subset X \times \mathcal{E}$, it induces a class whose residue at the generic point of $X \times \infty$ is the image of α in $H^i(\mathbb{C}(X), \mathbb{Z}/\ell)$. Thus the image of $\alpha \cup \beta$ in $H^{i+1}(\mathbb{C}(X \times \mathcal{E}_P), \mathbb{Z}/\ell)$ is nonzero. The previous proposition implies that the same holds for the image of $\alpha \cup \beta$ in $H^{i+1}(\mathbb{C}(X \times \mathcal{E}_s), \mathbb{Z}/\ell)$ for s in a Zariski dense subset of $U(\mathbb{C})$.

Corollary. *Let $X/\mathbb{C}$ be smooth, projective, connected variety with nontrivial Brauer group. Then there exists an elliptic curve $E/\mathbb{C}$ and a nonzero class in $H^3_{nr}(\mathbb{C}(X \times E), \mathbb{Q}/\mathbb{Z})$.*
If the Chow group of zero-cycles on X is supported on a curve, then the integral Hodge conjecture for codimension 2 cycles fails on $X \times E$.

Indeed, since $\mathrm{Br}(X) \to \mathrm{Br}(\mathbb{C}(X))$ is injective, one produces a class $\alpha \in H^2(X, \mathbb{Z}/\ell)$ with nontrivial image in $H^2(\mathbb{C}(X), \mathbb{Z}/\ell)$. The previous proposition then gives an elliptic curve E with $H^3_{nr}(\mathbb{C}(X \times E), \mathbb{Q}/\mathbb{Z}) \neq 0$. The additional hypothesis implies that the Chow group of zero-cycles on $X \times E$ is supported on a surface. The corollary of the CT–Voisin result then gives the failure of the integral Hodge conjecture.

If $X = S$ is an Enriques surface, then $\mathrm{Br}(S) = \mathbb{Z}/2$ and $CH_0(S) = \mathbb{Z}$. There thus exist elliptic curves E such that the integral Hodge conjecture fails for the 3-fold $S \times E$. One recovers the Benoist-Ottem examples.

Конец