General quartic threefolds are not stably rational

(Joint work with Alena Pirutka)

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§1. Rationality, stable rationality, unirationality

X a reduced irreducible variety over $\mathbf{C}$, $d = \dim(X)$.

rational : X is birational to $\mathbf{P}^d$
$\Longrightarrow$ **stably rational** : There exists $n \geq 0$ with $X \times \mathbf{P}^n$ rational
$\Longrightarrow$ **Retract Rational** : There exist open sets $\emptyset \neq U \subset X$, $V \subset \mathbf{P}^n$ and $U \to V \to U$ morphisms whose composite is identity on U (D. Saltman)
$\Longrightarrow$ **unirational** : There exists a dominant rational map from $\mathbf{P}^n$ to X – one can here assume $n = d$.
$\Longrightarrow$ **rationally connected** : Through any two points of X there is a curve of genus zero. Example : Fano varieties. (Campana; Kollár, Miyaoka, Mori).

For $d = 1$ (Lüroth) and $d = 2$ (Castelnuovo) these are all equivalent properties.

The situation changes as soon as $d \geq 3$.

$X/\mathbf{C}$ smooth and projective, unirational and not rational

1972 Clemens–Griffiths
X smooth cubic hypersurface in $\mathbf{P}^4$.
[X is birational to a conic bundle over $\mathbf{P}^2$]
Any such X is unirational
Method : Intermediate jacobian, Prym varieties (Mumford)
Shows X is not rational. Leaves open stable rationality.

1972 Iskovskikh–Manin
X quartic hypersurface in $\mathbf{P}^4$
Some of them are unirational (all of them ?)
Method: Rigidity : $BirAut(X) = Aut(X)$, finite, hence X not rational.
Shows X is not rational. Leaves open stable rationality.

1972 Artin–Mumford

X smooth projective over $\mathbf{C}$, birational to $z^2 = f_4(u, v, w)$, a double covering of $\mathbf{P}^3$ ramified along a certain (singular) quartic surface

X can also be viewed as a conic bundle over $\mathbf{P}^2$

X is unirational

Invariant detecting nonrationality : $H^3(X, \mathbf{Z})_{tors} \neq 0$, hence $Br(X) \neq 0$.

This implies that X is not retract rational.

Rational $\neq$ stably rational (Beauville, CT, Sansuc, Swinnerton-Dyer 1985) Method for nonrationality : Intermediate jacobian, Prym varieties (Clemens-Griffiths 1972, Mumford, Beauville 1977)

stably rational $\neq$ retract rational ? Unknown (over $\mathbf{C}$)

retract rational $\neq$ unirational : Brauer group, Artin–Mumford

unirational $\neq$ rationally connected ? Unknown.

$X \subset \mathbf{P}^n$ smooth cubic hypersurface, $n \geq 4$.
All unirational. Artin-Mumford Invariant $Br(X) = 0$.
$n = 4$. Never rational (Clemens–Griffiths). Are some, are all stably rational ? Open problem.
$n = 5$: some are rational (classical; Hassett). Is this an exception ?
n arbitrary. Are all stably rational ? Open problem.

$X \subset \mathbf{P}^4$ smooth quartic hypersurface.
Iskovskikh–Manin : X is never rational.
Artin-Mumford Invariant $Br(X) = 0$.

Is X stably rational ?

§2. Some stable birational invariants beyond the Brauer group

A basic stable birational invariant : the Chow group of zero-cycles over any field

k a field

X/k, smooth, projective, irreducible, retract rational

$\Longrightarrow$ For any field extension F/k, $Grad_F : CH_0(X_F) \to \mathbf{Z}$ is an isomorphism. We then say that X is universally CH_0-trivial.

(Merkurjev, Auel-CT-Parimala)
Even when $k = \mathbf{C}$, it is worth looking at fields F containing $\mathbf{C}$. The most interesting one is the function field $F = \mathbf{C}(X)$.

Let $k = \mathbf{C}$.
Bloch-Srinivas (1983) and others have studied the consequences of $CH_0(X_\Omega) = \mathbf{Z}$, where Ω is an arbitrary algebraically closed field containing $\mathbf{C}$.
Let $\Delta \subset X \times X$ be the diagonal.
The above hypothesis is equivalent to : There exists an integer $N > 0$ and a point $x \in X$, such that
$N\Delta = Z_1 + Z_2 \in CH^d(X \times X)$, with the support of Z_1 in $x \times X$ the support of Z_2 in $X \times Y$, $Y \subset X$, $Y \neq X$ closed.
Under this hypothesis $H^i(X, O_X) = 0$ for all $i \geq 1$.

$X/\mathbf{C}$ is universally CH_0-trivial if and only if there exists such a decomposition of the diagonal with $N = 1$.

Warning

There exist surfaces $X/\mathbf{C}$ which are universally CH_0-trivial while being of general type, hence in particular not stably rational.
For these surfaces $H^0(X, K) = 0$, but some have $H^0(X, 2K) \neq 0$.

Unramified cohomology

X/k, smooth, projective, irreducible, function field $k(X)$
$H^i_{nr}(k(X)/k, \mu_n^{\otimes j}) :=$
$\cap_{x \in X^{(1)}} Ker[\partial_x : H^i(k(X), \mu_n^{\otimes j}) \to H^{i-1}(\kappa(x), \mu_n^{\otimes j-1})]$

$=$ (Bloch-Ogus)
$\cap_v Ker[\partial_v : H^i(k(X), \mu_n^{\otimes j}) \to H^{i-1}(\kappa_v, \mu_n^{\otimes j-1})]$
where v runs through all discrete rank one valuations of $(k(X)$ trivial on k.

These are k-birational invariants. If X/k is retract rational, then $H^i(k, \bullet) = H^i_{nr}(k(X)/k, \bullet)$ (CT-Ojanguren 1989)

For algebraically closed fields $k \subset K$, rigidity :
$H^i_{nr}(k(X)/k, \bullet) = H^i_{nr}(K(X)/K, \bullet)$ (CT, Jannsen, method due to Suslin)

$X/\mathbf{C}$ smooth, projective

X retract rational
$\Longrightarrow X$ universally CH_0-trivial
$\Longrightarrow$ For any overfield F/k, any $i, n \in \mathbf{N}_{>0}$, any $j \in \mathbf{Z}$,

$$H^i(F, \mu_n^{\otimes j}) \to H^i_{nr}(F(X)/F, \mu_n^{\otimes j})$$

is an isomorphism

$$H^1_{nr}(k(X)/k, \mathbf{Q}/\mathbf{Z}) = H^1_{ét}(X, \mathbf{Q}/\mathbf{Z}) = Hom(\pi_1(X), \mathbf{Q}/\mathbf{Z}).$$

For $k = \mathbf{C}$, this group is an extension of the finite group $NS(X)_{tors} = H^2_{Betti}(X, \mathbf{Z})_{tors}$ by $(\mathbf{Q}/\mathbf{Z})^{b_1}$.
If X is rationally connected, then $\pi_1(X) = 0$

$$H^2_{nr}(k(X)/k, \mathbf{Q}/\mathbf{Z}(1)) = Br(X), \text{ Brauer group of } X.$$

For $k = \mathbf{C}$, this group is an extension of the finite group $H^3_{Betti}(X, \mathbf{Z})_{tors}$ by the group $(\mathbf{Q}/\mathbf{Z})^{b_2-\rho}$.
If $X$is rationally connected, then $b_2 - \rho = 0$.

$H^3_{nr}(k(X)/k, \mathbf{Q}/\mathbf{Z}(2))$
$k = \mathbf{C}$
Let $Z^4(X) := Hdg^4(X, \mathbf{Z})/Im(CH^2(X))$ (conjecturally finite).

Theorem (CT-Voisin 2012)
For any smooth projective variety X over $k = \mathbf{C}$, the group $H^3_{nr}(\mathbf{C}(X)/\mathbf{C}, \mathbf{Q}/\mathbf{Z}(2))$ is an extension of the finite group $Z_4(X)_{tors}$ by a divisible group.
If $CH_0(X) = \mathbf{Z}$ (e.g. if X is rationally connected), then $Z_4(X)$ is finite and
$$H^3_{nr}(\mathbf{C}(X)/\mathbf{C}, \mathbf{Q}/\mathbf{Z}(2)) = Z^4(X).$$

Theorem (Voisin 2006). *For X rationally connected of dimension 3, $Z^4(X) = 0$.*

Hence also $H^3_{nr}(\mathbf{C}(X)/\mathbf{C}, \mathbf{Q}/\mathbf{Z}(2)) = 0$.
There exist examples of smooth, projective, unirational varieties X of dimension ≥ 6 with $Br(X) = 0$, $H^3_{nr}(\mathbf{C}(X)/\mathbf{C}, \mathbf{Q}/\mathbf{Z}(2)) \neq 0$
CT-Ojanguren 1989, via Arason 1975 (pre-Merkurjev-Suslin 1983).
Hence there are rationally connected varieties of any dimension $d \geq 6$ for which the integral Hodge conjectures fails for codimension 2 cycles.

What about dimensions 4 and 5 ? Open.

A "new" stable birational invariant

$X/\mathbf{C}$ smooth and projective, $F/\mathbf{C}$ an overfield, $\overline{F}$ algebraic closure of F, $G = \mathrm{Gal}(\overline{F}/F)$.

Theorem : Coker $[CH^2(X_F) \to CH^2(X_{\overline{F}})^G]$ *is a birational invariant of* $X/\mathbf{C}$, *trivial on stably rational varieties*.

Proof : Behaviour of Chow groups under blow up of smooth $\mathbf{C}$-subvariety. For $Y/\mathbf{C}$ smooth, projective, connected and $F/\mathbf{C}$ we have $Pic(Y_F) = Pic(Y_{\overline{F}})^G$ since $Y(\mathbf{C}) \neq \emptyset$ hence $Y_F(F) \neq \emptyset$.

In some cases, related to the previous invariants.

Theorem. *Assume* $CH_0(X) = \mathbf{Z}$, *and* $H^i(X, \mathbf{Z})_{tors} = 0$ *for* $i = 2, 3$, *and* $H^3_{nr}(\mathbf{C}(X)/\mathbf{C}, \mathbf{Q}/\mathbf{Z}(2)) = 0$. *Then for any field* $F/\mathbf{C}$ *there is an exact sequence*

$$
\begin{aligned}
0 \to H^3_{nr}(F(X)/F, \mathbf{Q}/\mathbf{Z}(2))/H^3(F, \mathbf{Q}/\mathbf{Z}(2)) &\to \\
\to Coker[CH^2(X_F) \to CH^2(X_{\overline{F}})^G] & \\
\to H^2(G, \mathrm{Pic}(X_{\overline{F}}) \otimes \overline{F}^\times)&.
\end{aligned}
$$

(Bloch, CT-Raskind 1985, Kahn 1996, CT 2013)

My recent involvement in the topic began with an AIM (Palo Alto, March 2013) “project” which led to the following result on cubic fourfolds (Auel-CT-Parimala 2013) :
Theorem. *For a very general smooth projective cubic hypersurface $X \subset \mathbf{P}^5_{\mathbf{C}}$ containing a plane, for any field F containing $\mathbf{C}$, the map $H^3(F, \mathbf{Q}/\mathbf{Z}(2)) \to H^3_{nr}(F(X)/F, \mathbf{Q}/\mathbf{Z}(2))$ is an isomorphism.*
Such a fourfold is fibred into quadrics over $\mathbf{P}^2$. We used quadratic forms and K-theoretic techniques, which give information on the unramified cohomology of quadrics (Kahn, Rost, Sujatha).

This result was superseded by C. Voisin, who proved
Theorem. *For any smooth projective cubic hypersurface $X \subset \mathbf{P}^5_{\mathbf{C}}$, for any field F containing $\mathbf{C}$, the map $H^3(F, \mathbf{Q}/\mathbf{Z}(2)) \to H^3_{nr}(F(X)/F, \mathbf{Q}/\mathbf{Z}(2))$ is an isomorphism.*

Here is a partly alternative proof for her result.
One uses $H^3_{nr}(\mathbf{C}(X)/\mathbf{C}, \mathbf{Q}/\mathbf{Z}(2)) = 0$, which is proved as a consequence of the validity of the integral Hodge conjecture for codimension 2 cycles on X (Voisin).
This implies $H^3_{nr}(\overline{F}(X)/\overline{F}, \mathbf{Q}/\mathbf{Z}(2)) = 0$.
From two slides above we then have the inclusion

$$H^3_{nr}(F(X)/F, \mathbf{Q}/\mathbf{Z}(2))/H^3(F, \mathbf{Q}/\mathbf{Z}(2)) \subset Coker[CH^2(X_F) \to CH^2(X_{\overline{F}})^G]$$

For cycles of codimension 2 on X as above on an algebraically closed field, rational equivalence, algebraic equivalence, homological equivalence all coincide. Thus $CH^2(X) \to CH^2(X_{\overline{F}})$ is an isomorphism. Hence $CH^2(X_F) \to CH^2(X_{\overline{F}})^G$ is onto.
Hence $H^3_{nr}(F(X)/F, \mathbf{Q}/\mathbf{Z}(2))/H^3(F, \mathbf{Q}/\mathbf{Z}(2)) = 0$.

Smooth cubic hypersurfaces $X \subset \mathbf{P}^4_{\mathbf{C}}$.
For any such X and any overfield $F/\mathbf{C}$, one may prove

$$H^3_{nr}(F(X)/F, \mathbf{Q}/\mathbf{Z}(2))/H^3(F, \mathbf{Q}/\mathbf{Z}(2)) \simeq Coker[CH^2(X_F) \to CH^2(X_{\overline{F}})^G]$$

Can this this group be non-zero (hence X not stably rational) ?
The key case is when F is the function field of the intermediate jacobian $J^3(X)$, which is an abelian variety parametrizing cycles of codimension 2 which are homologous to zero, one considers the obvious class in $CH^2(X_{\overline{F}})^G$, and one asks whether it comes from a class in $CH^2(X_F)$, defining a "universal codimension 2 cycle".

This was investigated by C. Voisin (2014) from a different point of view. She proved that the existence of a universal codimension 2 cycle on X is equivalent to
$H^3_{nr}(F(X)/F, \mathbf{Q}/\mathbf{Z}(2))/H^3(F, \mathbf{Q}/\mathbf{Z}(2)) = 0$ for any field F.
In recent work, C. Voisin has produced families of cubic threefolds for which $CH_0(X) \simeq \mathbf{Z}$ universally.
She also found examples of rationally connected threefolds with $H^3_{nr}(F(X)/F, \mathbf{Q}/\mathbf{Z}(2))/H^3(F, \mathbf{Q}/\mathbf{Z}(2)) \neq 0$ for some F, the proof relying on the specialisation arguments now to be discussed.

§3. Proving non-rationality by specialisation arguments

In unequal characteristic, a specialisation method to prove non-ruledness (based on a specialisation result of Matsusaka) was used by J. Kollár (1995).

In equal characteristic zero, a different method was used by C. Voisin (Dec. 2013) : action of correspondances on Chow groups and on Betti cohomology. This will be described in her talk, with a bigger emphasis on the decomposition of the diagonal aspect.

A variant of her method was produced and applied by CT-Pirutka (Feb. 2014) : use of the specialisation map (Fulton) on the Chow group of zero-cycles over the field of fractions of a DVR.

$X \subset \mathbf{P}(2,1,1,1,1)$ double cover of $\mathbf{P}^3$ ramified along a quartic surface $S \subset \mathbf{P}^3$.

Theorem (Voisin) *If S has $n \leq 7$ ordinary singularities, and is otherwise very general, then X is not stably rational, but it satisfies $Br(X) = 0$ and $H^3_{nr}(\mathbf{C}(X)/\mathbf{C}, \mathbf{Q}/\mathbf{Z}(2)) = 0$.*
Very general means : outside a countable union of proper subvarieties of the projective space parametrizing the quartic surfaces.
Method : Specialisation to an Artin-Mumford threefold Y. This is a double cover of $\mathbf{P}^3$ ramified along a suitable quartic surface $S \subset \mathbf{P}^3$ with 10 ordinary singularities.
Artin and Mumford (1972) produced such a Y with a desingularisation $Z \to Y$ satisfying $Br(Z) \neq 0$.

Specialisation theorem (CT-Pirutka 2014)

A = discrete valuation ring, K = field of fractions, k = residue class field, algebraically closed, Let $\mathcal{X}/A$ be flat and projective over A. Assume :
1) The generic geometric fibre $\mathcal{X} \times_A \overline{K}$ is smooth, integral and universally CH_0-trivial.
2) The special fibre $Y := \mathcal{X} \times_A k$ is integral, and there exists a desingularisation morphism $p : Z \to Y$ which is universally CH_0-trivial, that is, for any overfield F/k the projection map $p_{F,*} : CH_0(Z_F) \to CH_0(Y_F)$ is an isomorphism.

Then Z/k is universally CH_0-trivial. In particular $Br(Z) = 0$.

Proof. One reduces to the case A is a complete DVR with residue field F and $X := \mathcal{X} \times_A K$ is stably rational over K. Then $deg_K : CH_0(X) \to \mathbf{Z}$ is an isomorphism.
Let $U \subset Y$ be a nonempty Zariski open set such that $p : p^{-1}(U) \to U$ is an isomorphism. Let $V = p^{-1}(U)$. Let z be a zero-cycle of degree 0 on Z. Since Z/F is smooth, z is rationally equivalent on Z to a degree 0 zero-cycle z_1 with support in V. The degree 0 zero-cycle $p_*(z_1)$ then has its support in U. Since U is smooth and A is complete, one may lift the degree 0 zero-cycle $p_*(z_1)$ to a 1-cycle on $\mathcal{X}$ of relative degree 0 over A.
Fact (Fulton) : *The homomorphism $CH_1(\mathcal{X}) \to CH_0(Y)$ induces a homomorphism $CH_0(X) \to CH_0(Y)$.*
From $deg_K : CH_0(X) \simeq \mathbf{Z}$ we then deduce $p_*(z) = p_*(z_1) = 0 \in CH_0(Y)$. Since $p_* : CH_0(Z) \to CH_0(Y)$ is an isomorphism, we conclude $z = 0 \in CH_0(Z)$. QED

Note :

- We do not impose any regularity condition on $\mathcal{X}$.
- One need not even assume that $\mathcal{X} \times_A \overline{K}$ is smooth.

Hypotheses on the special fibre

• The requirement *the morphism* $p : Z \to Y$ *universally* CH_0*-trivial* depends only on Y.

• If for each schematic point $M \in Y$, with residue field $\kappa(M)$, the fibre $Z_M/\kappa(M)$, is universally CH_0-trivial, then the morphism $p : Z \to Y$ is universally CH_0-trivial.

• Simple example for a universally CH_0-trivial morphism $p : Z \to Y$:
k algebraically closed, $dim(Y) \geq 2$, and all singularities of Y are ordinary quadratic singularities.

Two examples of quartic hypersurfaces $Y \subset \mathbf{P}^4_{\mathbf{C}}$ whose singular locus is of dimension 1 and whose desingularisations $p : Z \to Y$ have the two properties :
(i) the morphism $p : Z \to Y$ is universally CH_0-trivial :
(ii) $Br(Z) \neq 0$, i.e. $H^3(Z, \mathbf{Z})_{tors} \neq 0$, i.e. $H^4(X, \mathbf{Z})_{tors} \neq 0$.

One starts with an Artin-Mumford surface in $\mathbf{P}^3_{\mathbf{C}}$:

$$\alpha_2(z_0, z_1, z_2)z_3^2 + \beta_3(z_0, z_1, z_2)z_3 + \gamma_4(z_0, z_1, z_2) = 0$$

$\beta^2 - \alpha\gamma = \varepsilon_1.\varepsilon_2$, with each $\varepsilon_i = 0$ an elliptic curve in $\mathbf{P}^2_{\mathbf{C}}$, and with the smooth conic $\alpha = 0$ tangent to each of them.

• June Huh (A counterexample to the geometric Chevalley-Lang conjecture, 2013)
$\alpha_2(z_0,z_1,z_2)z_3^2+\beta_3(z_0,z_1,z_2)z_3+\gamma_4(z_0,z_1,z_2)+\delta_2(z_0,z_1,z_2)z_4^2=0$
Here $\delta_2(z_0,z_1,z_2)=0$ is smooth and general enough.
J. Huh constructs a desingularisation $p : Z \to Y$, shows :
$\mathbb{L}$-desingularisation (akin to CH_0-trivial desingularisaton).
Computes $H^4(Z,\mathbf{Z})_{tors}$, finds it is not zero (delicate computation, as in Artin-Mumford).

• CT-Pirutka (2014)
$\alpha_2(z_0,z_1,z_2)z_3^2+\beta_3(z_0,z_1,z_2)z_3+\gamma_4(z_0,z_1,z_2)+z_0^2z_4^2=0$
We compute a desingularisation $p : Z \to Y$ and show p is universally CH_0-trivial. We need not compute $H^4(Z,\mathbf{Z})_{tors}$, because our variety is birational to the 3-dimensional Artin-Mumford variety, hence one already knows $Br(Z) \neq 0$.

One may take these singular quartics with coefficients in $\overline{\mathbf{Q}}$.
In the projective space P parametrizing quartics in $\mathbf{P}^4$, take a line L defined over $\overline{\mathbf{Q}}$, not contained in the locus parametrizing singular quartics, and take a point R in $L(\mathbf{C})$ which does not belong to $L(\overline{\mathbf{Q}})$.
By application of the specialisation theorem we get (CT et Pirutka, February 2014) :

The quartic threefold with parameter R is not retract rational.

The method produces smooth quartic hypersurfaces defined on an algebraic closure of $\mathbf{Q}(t)$ which are not retract rational.

To be compared with Iskovskikh-Manin (1972) : no smooth quartic hypersurface in $\mathbf{P}^4_{\mathbf{C}}$ is rational.

One can do better (following a suggestion of O. Wittenberg) :
There exist smooth quartic hypersurfaces defined over $\overline{\mathbf{Q}}$ which are not retract rational.
One starts with a singular quartic $Y \subset \mathbf{P}^4_{\overline{\mathbf{Q}}}$ as considered above, with its resolution of singularities $Z \to Y$. This exists over a finite extension k of $\mathbf{Q}$ and spreads out over a suitable open set T of the ring of integers of k, the fibres over closed points v of T giving rise to resolutions $Z_v \to Y_v$ as nice as $Z \to Y$. By proper base change for étale cohomology, the 2-torsion of the Brauer group of the geometric fibres Z_v is nontrivial.
Pick $v \in T$. Consider the $\mathbf{P}^N$ parametrizing quartics. Away from a hypersurface $f = 0$ the points correspond to smooth quartics. One picks up a point $m \in \mathbf{P}^N(k)$ which is not in $f = 0$ but reduces to a point m_v associated to a Y_v a above. The specialisation theorem shows that the quartic $X_m \times_k \overline{k}$ is not retract rational.

Further work

Beauville (Nov., Dec. 2014) used the specialisation argument to show that double covers of $\mathbf{P}^4$ and of $\mathbf{P}^5$ ramified along a very general quartic hypersurface are not stably rational, and also that double covers of $\mathbf{P}^3$ ramified along a very general sextic surface are not retract rational. The specialisation is to characteristic zero.

Burt Totaro (Feb. 2015) very recently pushed the method to get impressive results. He shows that for any integers d and $n \geq 3$ for which there exists an integer m with $d/2 \geq m \geq (n+2)/3$, a very general hypersurface of degree d in $\mathbf{P}^n$ is not retract rational. This includes quartic 4-folds, for which nonrationality was not known.

Totaro uses the specialisation method in unequal characteristic, more precisely to characteristic 2. For hypersurfaces of even degree, he reduces to varieties Y with a (simple) resolution of singularities $Z \to Y$, in particular for which $H^0(Z, \Omega^i) \neq 0$ for some i by results of Kollár. Such a specialisation had been used by Kollár (1995) to show that many Fano hypersurfaces are not ruled. Totaro notes that $Z \to Y$ is a universal CH_0-isomorphism and uses a cycle map defined by M. Gros combined with an argument in the Bloch-Srinivas style to prove that Z is not universally CH_0-trivial.

Appendix : Smooth cubic hypersurfaces $X \subset \mathbf{P}^n_k$, $n \geq 3$, over a non-algebraically-closed field k

There exist cubic hypersurfaces

$X \subset \mathbf{P}^3_k$ with $Br(X) \neq Br(k)$ hence X not retract rational over $k = \mathbf{C}((t))$ and over $k = \mathbf{F}$ a finite field (Shafarevich, Manin).

$X \subset \mathbf{P}^3_{\mathbf{Q}}$ which are stably rational over $\mathbf{Q}$, but not rational over $\mathbf{Q}$. Also over $\mathbf{Q}_p$ (Beauville, CT, Sansuc, Swinnerton-Dyer)

$X \subset \mathbf{P}^n_{\mathbf{R}}$ with $X(\mathbf{R})$ not connected, hence X not stably rational, and $A_0(X) = \mathbf{Z}/2$, any $n \geq 3$. Deformation of

$$x(x-z)(x+z) + (\sum_{i=1}^{i=m} y_i^2)z = 0.$$

$X \subset \mathbf{P}^4_k$ with $k = \mathbf{C}((x))((y))$ and $A_0(X) \neq 0$, hence X not stably rational

$$T_0^3 + T_1^3 + xT_2^3 + yT_3^3 + xyXT_4^3 = 0.$$

(Specialisation on Chow groups, D. Madore 2008).

$X \subset \mathbf{P}^4_k$ not stably rational, with $k = \mathbf{Q}_p$, $K = k(\omega)$ cubic, unramified.

$$Norm_{K/k}(u + \omega.y + \omega^2.z) + x.y.(x + y) + p.\Phi(u, v, w, x, y) = 0$$

with suitable $\mathbf{Z}_p$-smooth $\Phi \in \mathbf{Z}_p[u, v, w, x, y] = 0$. (CT-Pirutka, Same technique as in the present talk, reduction to a finite field and computation of Brauer group of a nonsingular model of the special fibre)

Open problems

In any of the following cases does there exist a smooth cubic hypersurface $X \subset \mathbf{P}^n_k$ which is not stably rational ?

Over $\mathbf{C}$ (any $n \geq 4$)

Over $\mathbf{C}((t))$ or $\mathbf{C}(t)$ (any $n \geq 4$)

Over a finite field (any $n \geq 4$)

Over $\mathbf{C}((x))((y))$ (any $n \geq 5$)

Over $\mathbf{Q}_p$ (any $n \geq 5$)