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ON THE CHOW GROUPS OF CERTAIN RATIONAL SURFACES: A SEQUEL TO A PAPER OF S. BLOCH

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S. Bloch has recently applied the methods of algebraic K-theory to the study of 0-dimensional cycles on rational surfaces, modulo rational equivalence. The best results are obtained for conic bundles over the projective line [1]. In this paper, building upon Bloch's very original ideas and upon some more or less classical facts pertaining to quadratic forms, we shall refine the results of [1], thereby answering some of the questions raised there.

Let k be a perfect field, $\bar{k}$ an algebraic closure of k, and $\mathfrak{g} = \mathrm{Gal}(\bar{k}/k)$. Let X be a *rational*, proper, smooth, geometrically integral variety over k. We denote the function field of X, resp. $\bar{X} = X \times_k \bar{k}$, by $F = k(X)$, resp. $\bar{F} = \bar{k}(X)$. By the very definition of a rational variety, the latter field is purely transcendental over $\bar{k}$. Moreover, for such an X, the $\mathfrak{g}$-module $\mathrm{Pic}\,\bar{X}$ is a free $\mathbf{Z}$-module of finite type: we can regard it as the character group $\hat{S}$ of a k-torus S. Following [1] (as opposed to [2] or [4]) we denote by $A_0(X)$ the group of classes of degree nought 0-dimensional cycles on X with respect to rational equivalence.

In section 1 of this paper, we define a "characteristic" homomorphism

$$\Phi : A_0(X) \to H^1(k, S)$$

and we show that its image is finite when k is any finitely generated extension of $\mathbf{Q}$. This raises the question: what about the kernel of Φ? Examples with $\dim X \geqslant 3$ suggest one should not expect a general answer, except in the case of *surfaces*.

In this last case, Bloch [1] has produced a K-theoretical interpretation of the kernel and the cokernel of Φ: starting from another definition of Φ, special to dimension 2, he constructs the basic exact sequence:

$$S(k) \to H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{k}) \to A_0(X) \xrightarrow{\Phi} H^1(k, S) \to H^2(\mathfrak{g}, K_2\bar{F}/K_2\bar{k}). \quad (*)$$

He uses this sequence to show that the image of Φ is finite if k is global, and that the kernel of Φ is finite when X is a conic bundle over $\mathbf{P}^1_k$ and k is local or global. This gives the finiteness of $A_0(X)$ for $X/\mathbf{P}^1_k$ a conic bundle over a local or a global field. He also gets $A_0(X) = 0$ for $X/\mathbf{P}^1_k$ a conic bundle over a C_1-field.

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The main "technical" result of the present paper is that for X a conic bundle over $\mathbf{P}_k^1$, with k nearly *arbitrary*, and $X(k) \neq \emptyset$, the map Φ is *injective*. This is certainly more than a technical point, but the geometric interpretation of this fact has so far eluded our efforts. The main "concrete" result is that for X as above and k any *finitely generated extension of* $\mathbf{Q}$, the group $A_0(X)$ is *finite*. This last result relies on the technical one and on the general finiteness statement for the image of Φ. Another important feature of the technical result is that it prompts us to put forward a *conjecture* on the precise value of $A_0(X)$, when k is a number field. The main part of this conjecture claims the existence of an exact sequence of finite abelian groups

$$A_0(X) \to \coprod_v A_0(X_{k_v}) \to \operatorname{Hom}\big(H^1(k, \hat{S}), \mathbf{Q}/\mathbf{Z}\big).$$

Here $\hat{S} = \operatorname{Pic} \overline{X}$ as above, and k_v runs through all completions of k. In loose terms: for a given (algebraic) type of rational surface, the fact that many $A_0(X_{k_v})$ are non-zero should force $A_0(X)$ to be big. Some evidence is provided in the case of Châtelet surfaces.

Here are the precise statements of our main results.

THEOREM 1. *Let X be a conic bundle over $\mathbf{P}_k^1$. If k is a finitely generated extension of $\mathbf{Q}$, and if there is a 0-cycle of degree one on X, the group $A_0(X)$ is* finite. *If k is a perfect field of characteristic $\neq 2$ and of cohomological dimension* 1, *the group $A_0(X)$ is zero.*

THEOREM 2. *Let k be a perfect field of characteristic $\neq 2$, and let X be a conic bundle over $\mathbf{P}_k^1$. The homomorphism $\Phi : A_0(X) \to H^1(k, S)$ is* injective, *provided at least one of the following assumptions holds*:

(i) *there is a 0-cycle of degree one on X*;

(ii) *k is a local field*;

(iii) *k is a number field.*

For any conic bundle $X/\mathbf{P}_k^1$, the group $H^1(k, S)$ is 2-torsion (§2 proposition 1). As for $A_0(X)$, it was already known to be 2-torsion ([2] §6).

Note that, for k local or global, Bloch [1] gives the bound 2^s for the order of the kernel of Φ, where s is the number of real imbeddings of k.

The paper is organized as follows. The homomorphism Φ is defined in quite general circumstances in §1, where it is shown to agree with several previously defined homomorphisms, including Bloch's for rational surfaces. We prove the finiteness of the image of Φ when k is a finitely generated extension of $\mathbf{Q}$—for this, no K-theory is needed. Section 2 contains basic facts about conic bundles over $\mathbf{P}_k^1$: although they are essentially well known, no convenient reference is on hand. These facts are used in §3, where we prove theorem 2 by a refinement of Bloch's fundamental argument ([1] §3). Theorem 1 is then an immediate corollary of the results of §1. The precise computations made in §3 also allow us to answer a question of Bloch on the value of $H^1(\mathfrak{g}, K_2\overline{F}/K_2\bar{k})$ (theorem 5). In

§4, we discuss the above mentioned conjecture on the precise value of $A_0(X)$ when k is a number field.

A word of warning: a number of technical difficulties may be avoided if one assumes that there is a 0-cycle of degree one on X. First, we can study the image of Φ as in [4]. Second, only a part of §§2 and 3 is needed to prove theorem 2(i): one need not study the upper half of diagram (3.4).

We are grateful to S. Bloch for some very useful conversations: it was he who pointed out the relevance of Weil's reciprocity law in the general construction of Φ given in §1.

Contents

§1. The characteristic homomorphism. Let X be a smooth, projective, geometrically integral variety over the perfect field k. Following a procedure inspired by classical analogues (Severi, Weil, Serre, Tate, cf. a very similar construction in [9] 3.8), we shall define a "characteristic" homomorphism

$$\Phi : A_0(X) \to \mathrm{Ext}^1_{\mathfrak{g}}(\mathrm{Pic}\,\bar{X}, \bar{k}^*).$$

Let $Z_0(X)$ be the group of 0-cycles on X, and let $\tilde{Z}_0(X)$ be the subgroup of those of degree zero. Let ξ be a 0-cycle of X, let $\mathrm{supp}(\xi)$ be its support, $Y = X - \mathrm{supp}(\xi)$ and $\bar{\xi}$ the extension of ξ to $\bar{k}$. In the natural exact sequence of $\mathfrak{g}$-modules

$$0 \to \mathcal{O}^*_{\bar{X},\bar{\xi}}/\bar{k}^* \to \mathrm{Div}_{\bar{Y}}\bar{X} \to \mathrm{Pic}\,\bar{X} \to 0 \tag{1.1}$$

$\mathcal{O}^*_{\bar{X},\bar{\xi}}$ denotes the group of units of the semi-local ring of $\bar{X}$ at the points of $\mathrm{supp}(\bar{\xi})$, and $\mathrm{Div}_{\bar{Y}}\bar{X}$ is the group of divisors on $\bar{X}$, with support in $\bar{Y}$. Let $\bar{\xi} = \sum_i n_i x_i$, with $x_i \in X(\bar{k})$ and $n_i \in \mathbf{Z}$, and let $g \in \bar{k}(\bar{X})^*$ be a unit at all x_i. When ξ is a degree zero 0-cycle, the evaluation formula

$$[\xi]_{\bar{X}}(g) = g(\xi) = \prod_i g(x_i)^{n_i}$$

defines a $\mathfrak{g}$-homomorphism $[\xi]_{\bar{X}} : \mathcal{O}^*_{\bar{X},\bar{\xi}}/\bar{k}^* \to \bar{k}^*$. Using it to push out the extension (1.1), we get an extension of $\mathfrak{g}$-modules

$$\mathcal{E}_\xi : 0 \to \bar{k}^* \to E_\xi \to \mathrm{Pic}\,\bar{X} \to 0. \tag{1.2}$$

On letting the support of ξ grow, one checks that this construction defines a homomorphism $\tilde{Z}_0(X) \to \mathrm{Ext}^1_{\mathfrak{g}}(\mathrm{Pic}\,\bar{X}, \bar{k}^*)$.

Let us show that it depends only on rational equivalence classes. This amounts to proving that, for any non-constant k-morphism $C \xrightarrow{\pi} X$ from a proper, smooth, integral k-curve C to X, and any non-constant $f \in k(C)^*$, the extension $\mathcal{E}_{\pi_*(\mathrm{div}(f))}$ is trivial. Now, for $Z = \mathrm{supp}(\mathrm{div}(f))$, there is a commutative diagram of $\mathfrak{g}$-modules

$$\begin{array}{ccc}
\mathcal{O}^*_{\bar{X},\pi(\bar{Z})}/\bar{k}^* & \longrightarrow & \mathrm{Div}_{\bar{X}-\pi(\bar{Z})}\bar{X} \\
\pi^* \big\downarrow & & \big\downarrow \pi^* \\
\mathcal{O}^*_{\bar{C},\bar{Z}}/\bar{k}^* & \longrightarrow & \mathrm{Div}_{\bar{C}-\bar{Z}}\bar{C} \\
[\mathrm{div}(f)]_{\bar{C}} \big\downarrow & \swarrow f & \\
\bar{k}^* & &
\end{array}$$

where the skew arrow is given by evaluation. That the triangle commutes is a consequence of Weil's reciprocity law (see [1] A.8)—which one must apply to all components of $\bar{C}$ (one cannot a priori restrict oneself to geometrically integral curves in the definition of rational equivalence). The $\mathfrak{g}$-homomorphism $[\pi_*(\mathrm{div}(f))]_{\bar{X}} = [\mathrm{div}(f)]_{\bar{C}} \circ \pi^*$ therefore extends to a $\mathfrak{g}$-homomorphism $\mathrm{Div}_{\bar{X}-\pi(\bar{Z})}\bar{X} \to \bar{k}^*$, which shows $\mathcal{E}_{\pi_*(\mathrm{div}(f))}$ to be trivial.

One thus gets a canonical homomorphism

$$A_0(X) \xrightarrow{\ \Phi\ } \mathrm{Ext}^1_{\mathfrak{g}}(\mathrm{Pic}\,\bar{X}, \bar{k}^*). \tag{1.3}$$

When X is rational, $\mathrm{Pic}\,\bar{X}$ is a free finitely generated $\mathbf{Z}$-module; for the dual k-torus S, there is an identification of $\mathfrak{g}$-modules $S(\bar{k}) = \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Pic}\,\bar{X}, \bar{k}^*)$, hence

$$\mathrm{Ext}^1_{\mathfrak{g}}(\mathrm{Pic}\,\bar{X}, \bar{k}^*) = H^1\big(\mathfrak{g}, S(\bar{k})\big) = H^1(k, S).$$

THEOREM 3. *Let k be a perfect field and let X be a smooth projective geometrically integral rational variety over k. Let S be the k-torus dual to* $\mathrm{Pic}\,\bar{X}$. *The homomorphism*

$$\Phi : A_0(X) \to H^1(k, S) \tag{1.4}$$

which has just been defined enjoys the following properties:

(i) *when there is a universal torseur on X, e.g. $X(k) \neq \emptyset$, it coincides with the homomorphism attached in* [4] *to such a torseur*;

(ii) *when X is a surface, it coincides with the map defined by Bloch in* [1];

(iii) *if* $\dim X = 2$ *or* $\mathrm{char}.\,k = 0$, *then Φ is a k-birational invariant*;

(iv) *its image is finite if k is finitely generated over* $\mathbf{Q}$.

The proof will occupy the rest of this section.

Proof of (i). For facts concerning universal torseurs, we refer to [4] and the references therein. Given a (universal) torseur $\mathfrak{T}$ over X, under the torus S, there is a well defined homomorphism $\varphi : A_0(X) \to H^1(k, S)$ which comes from the linear map $\theta_{\mathfrak{T}} : Z_0(X) \to H^1(k, S)$ sending the closed point x, with residue class field $k(x)$, to $\theta_{\mathfrak{T}}(x) = \mathrm{cor}_{k(x)/k}(\mathfrak{T}_x)$, the corestriction of the fibre $\mathfrak{T}_x$ of $\mathfrak{T}$ at x. The identification of φ with Φ will be deduced from the local description of universal torseurs ([4] II.C). Let x be a closed point on X, and let U be an open neighbourhood of x with $\mathrm{Pic}\,\bar{U} = 0$. Upon dualizing the exact sequence of $\mathfrak{g}$-modules

$$0 \to \bar{k}[U]^*/\bar{k}^* \to \mathrm{Div}_{\bar{Y}}\bar{X} \to \mathrm{Pic}\,\bar{X} \to 0 \tag{1.5}$$

where $Y = X - U$ and where $\bar{k}[U]^* = H^0(\bar{U}, \mathbf{G}_m)$, we get the exact sequence of k-tori

$$1 \to S \to M \to T \to 1. \tag{1.6}$$

The existence of a universal torseur $\mathfrak{T}$ implies that of a $\mathfrak{g}$-section σ of the projection $\bar{k}[U]^* \to \bar{k}[U]^*/\bar{k}^*$. This defines a k-morphism $\varphi_\sigma : U \to T$ from which one regains the restriction of $\mathfrak{T} = \mathfrak{T}^\sigma$ to U: it is the pull-back through φ_σ of the torseur M over T defined by (1.6). Since $\mathrm{Pic}\,\bar{T} = 0$ and $\mathcal{E}xt^1_{T_{\acute{e}t}}(\hat{S}, \mathbf{G}_m) = 0$, easy spectral sequence arguments give canonical isomorphisms (where all Ext^1 and H^1 are relative to the étale topology): $\mathrm{Ext}^1_k(\hat{S}, \bar{k}[T]^*) \xrightarrow{\sim} \mathrm{Ext}^1_T(\hat{S}, \mathbf{G}_m) \xleftarrow{\sim} H^1(T, S)$. We get the commutative diagram

$$\begin{array}{ccccc}
H^1(T,S) & \xrightarrow{\sim} & \mathrm{Ext}^1_k(\hat{S}, \bar{k}[T]^*) & \longleftarrow & \mathrm{Ext}^1_k(\hat{S}, \bar{k}[U]^*/\bar{k}^*) \\
\downarrow & & \downarrow & & \\
H^1(k(x),S) & \xrightarrow{\sim} & \mathrm{Ext}^1_k\big(\hat{S}, (k(x) \otimes_k \bar{k})^*\big) & & \swarrow \lambda \\
\downarrow \mathrm{cor}_{k(x)/k} & & \downarrow N_{k(x)/k} & & \\
H^1(k,S) & \xrightarrow{\sim} & \mathrm{Ext}^1_k(\hat{S}, \bar{k}^*). & &
\end{array}$$

It is now clear that the extension of $\hat{S}$ by $\bar{k}^*$ corresponding to $\mathrm{cor}_{k(x)/k}(\mathfrak{T}^\sigma_x) \in H^1(k, S)$ is obtained by pushing out the extension (1.5) through the homomorphism $\bar{k}[U]^*/\bar{k}^* \xrightarrow{\lambda} \bar{k}^*$ defined by $g \mapsto (\sigma(g))(x)$. Extending this to 0-cycles and comparing with the above definition of Φ, we get $\varphi = \Phi$.

Proof of (ii). When X is a surface, the map defined by Bloch in [1] is actually a homomorphism $\Phi' : A_0(X) \to H^1(k, S')$, where S' is the k-torus with $S'(\bar{k}) = \mathrm{Pic}\,\bar{X} \otimes_{\mathbf{Z}} \bar{k}^*$. But the intersection form induces a self-duality of $\mathrm{Pic}\,\bar{X}$ which defines a canonical isomorphism of S and S'. From now on they will be identified. The fact that up to this identification Φ and Φ' coincide is a consequence of the Appendix of [1], as we shall now see. First recall from [1] that for U open in X with $\mathrm{Pic}\,\bar{U} = 0$ and $Y = X - U$, there is a commutative diagram of exact sequences of $\mathfrak{g}$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{T}_{\bar Y} & \longrightarrow & \coprod_{D \not\subset \bar Y} \bar k(D)^*_{\bar Y} & \longrightarrow & \left(\bigoplus^0_{\bar X} \mathbf{Z}\right)_{\bar Y} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(\mathrm{Pic}\,\bar X, \bar k^*) & \longrightarrow & \mathrm{Hom}(\mathrm{Div}_{\bar Y} \bar X, \bar k^*) & \longrightarrow & \mathrm{Hom}(\bar k[U]^*/\bar k^*, \bar k^*) & \longrightarrow & 0
\end{array}$$

where D runs through all integral curves on $\bar X$ not in $\bar Y$, and where $\bar k(D)^*_{\bar Y}$ denotes the group of rational functions on D which are invertible at the points of $\pi^{-1}(\bar Y)$ in the normalisation $\tilde D \xrightarrow{\pi} D$. The top right arrow is deduced from the natural map $\coprod_{D \not\subset \bar Y} \bar k(D)^*_{\bar Y} \xrightarrow{\mathrm{div}} \bigoplus_{\bar X} \mathbf{Z}$, whose kernel is $\mathfrak{T}_{\bar Y}$ and whose image is $(\bigoplus^0_{\bar X} \mathbf{Z})_{\bar Y} \hookrightarrow \bigoplus^0_{\bar U} \mathbf{Z} = \tilde Z_0(\bar U)$. The two right vertical arrows are given by evaluation; that the right square commutes is proven in [1] (A.8). This defines the left vertical arrow. Going over to cohomology, we obtain the commutativity of the square in the following diagram:

$$\begin{array}{ccccccc}
\tilde Z_0(U) = \left(\bigoplus^0_{\bar U} \mathbf{Z}\right)^{\mathfrak{g}} & \hookleftarrow & H^0\left(\mathfrak{g}, \left(\bigoplus^0_{\bar X} \mathbf{Z}\right)_{\bar Y}\right) & \xrightarrow{\partial} & H^1(\mathfrak{g}, \mathfrak{T}_{\bar Y}) & \xrightarrow{\beta} & H^1\left(\mathfrak{g}, \mathrm{Pic}\,\bar X \otimes_{\mathbf{Z}} \bar k^*\right) \\
 & \searrow & \downarrow \epsilon & & \downarrow & \swarrow \wr\, \omega & \\
 & & \mathrm{Hom}_{\mathfrak{g}}\left(\bar k[U]^*/\bar k^*, \bar k^*\right) & \xrightarrow{\partial} & H^1\left(\mathfrak{g}, \mathrm{Hom}(\mathrm{Pic}\,\bar X, \bar k^*)\right). & &
\end{array}$$

Here the left diagonal arrow is evaluation, hence the left triangle is commutative. The right diagonal arrow is deduced from the intersection form on $\mathrm{Pic}\,\bar X$, and β is built up from the maps $\mathfrak{T}_{\bar Y} \to H^1(\bar X, \mathcal{K}_2) \xleftarrow{\sim} \mathrm{Pic}\,\bar X \otimes_{\mathbf{Z}} \bar k^*$. Bloch's map Φ', restricted to $H^0(\mathfrak{g}, (\bigoplus^0_{\bar X} \mathbf{Z})_{\bar Y}) \hookrightarrow \tilde Z_0(U)$, is, by definition, $\beta \circ \partial$. On the other hand, $\partial \circ \epsilon$ is induced by the restriction of Φ to $\tilde Z_0(U)$, because ϵ is the evaluation map $(\epsilon(\xi))(g) = g(\xi)$ for $g \in \bar k[U]^*$. Since the right triangle commutes ([1] A.11), we conclude that, for any ξ in $((\bigoplus^0_{\bar X} \mathbf{Z})_{\bar Y})^{\mathfrak{g}}$, we have $\Phi(\xi) = \omega(\Phi'(\xi))$. We are thus reduced to proving the following *assertion*: for *any* ξ in $\tilde Z_0(X)$, there is an open set U as above, with $Y = X - U$ and $\bar\xi \in (\bigoplus^0_{\bar X} \mathbf{Z})_{\bar Y}$. This fact is also used implicitly at the end of [1]. Since $A_0(\bar X) = 0$, there is a finite family $\{(D_i, f_i)\}$, where D_i is an integral curve on $\bar X$, and $f_i \in \bar k(D_i)^*$, such that $\bar\xi = \mathrm{div}(\sum_i f_i|_{D_i})$. Letting $\pi_i : \tilde D_i \to D_i$ be the normalizations, we can find an open subset U of X containing all $\pi_i(\mathrm{supp}(\mathrm{div}_{\tilde D_i}(f_i)))$ and such that $\mathrm{Pic}\,\bar U = 0$. For this U and $Y = X - U$, we have $f_i \in \bar k(D_i)^*_{\bar Y}$, hence $\bar\xi \in (\bigoplus^0_{\bar X} \mathbf{Z})_{\bar Y}$.

Proof of (iii). As in [2] (proposition 6.3, birational invariance of $A_0(X)$), by known theorems on resolution of singularities, it is enough to study the case of a proper birational k-morphism $p: X' \to X$, where X and X' are both geometrically integral, projective and smooth. There are non-empty open sets $V \subset U \subset X$, satisfying the following properties: U contains all points of X with codimension 1, the restriction of p to $p^{-1}(U)$ has a section—we shall call it r ($p \circ r = id_U$)—and the restriction of p to $p^{-1}(V)$ is an isomorphism. According to Chow's moving lemma, any element in $A_0(X)$ can be represented by some 0-cycle ξ of degree zero with support in U. Letting $\xi' = p^{-1}(\xi)$, taking $Y = X - \mathrm{supp}(\xi)$ and $Y' = X' - \mathrm{supp}(\xi')$, we have the commutative diagram of exact sequences of $\mathfrak{g}$-modules:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^*_{\bar{X},\xi}/\bar{k}^* & \longrightarrow & \mathrm{Div}_{\bar{Y}}\bar{X} & \longrightarrow & \mathrm{Pic}\,\bar{X} & \longrightarrow & 0 \\
 & & \wr\downarrow p^* & & \downarrow p^* & & \downarrow p^* & & \\
0 & \longrightarrow & \mathcal{O}^*_{\bar{X}',\xi'}/\bar{k}^* & \longrightarrow & \mathrm{Div}_{\bar{Y}'}\bar{X}' & \longrightarrow & \mathrm{Pic}\,\bar{X}' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M & = & M & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Indeed the left vertical arrow is obviously an isomorphism, and the other two p^*'s are split by r^*. From this diagram, we deduce $p^*(\Phi(\xi')) = \Phi(\xi)$, and we also deduce that $p^*: \mathrm{Ext}^1_{\mathfrak{g}}(\mathrm{Pic}\,\bar{X}', \bar{k}^*) \to \mathrm{Ext}^1_{\mathfrak{g}}(\mathrm{Pic}\,\bar{X}, \bar{k}^*)$ is an isomorphism, because the two vertical sequences are split and $\mathrm{Div}_{\bar{Y}'}\bar{X}'$ is a permutation $\mathfrak{g}$-module, which implies (Hilbert 90) $\mathrm{Ext}^1_{\mathfrak{g}}(M, \bar{k}^*) = 0$.

Proof of (iv). When there is a universal torseur over X, for instance when there is a 0-cycle of degree one on X, it suffices to combine (i) with [4] (proposition 2). We shall sketch a proof in the general case, referring the reader to [4] for some details. Given X/k, one can find an integral domain A and a projective morphism $\tilde{p}: \tilde{X} \to \mathrm{Spec}\,A$, so that the following holds: A is of finite type over $\mathbf{Z}$, it is regular and k is its quotient field (this last and obvious assumption was forgotten in [4] p. 225, démonstration); the morphism $\tilde{p}$ is smooth with geometrically integral fibres (hence $\mathbf{G}_{m,A} \xrightarrow{\sim} \tilde{p}_*\mathbf{G}_{m,\tilde{X}}$ as sheaves for the étale topology over A), and X/k is its generic fibre. Since X is rational, the étale sheaf $R^1p_*\mathbf{G}_{m,X} = \mathcal{P}ic_{X/k}$ over k is representable by a finite type torsion free twisted constant group $\hat{S}$ over k, the dual of which is the k-torus S. Upon inverting some element in A, we can assume that the isomorphism of étale sheaves over $k: \hat{S} \xrightarrow{\sim} R^1p_*\mathbf{G}_{m,X}$ comes from a homomorphism of étale sheaves over $A: \hat{\tilde{S}} \xrightarrow{\lambda} R^1\tilde{p}_*\mathbf{G}_{m,\tilde{X}}$, where $\hat{\tilde{S}}$ is a finite type torsion free twisted constant group over A, the dual of which is an A-torus $\tilde{S}$ with fibre S over k.

In order to show that $\Phi(A_0(X)) \subset H^1(k, S)$ is finite, it is enough to show that for every height 1 prime ideal $\mathfrak{p}$ in A, the image of Φ belongs to the image of

$H^1(A_{\mathfrak p}, \tilde S) \to H^1(k, S)$. Indeed, $H^1(k,S)$ is a torsion group, and an element of $H^1(k,S)$ which is in the image of each $H^1(A_{\mathfrak p}, \tilde S)$ (for all such $\mathfrak p$) is in the image of $H^1(A, \tilde S)$ (because A is regular and $\tilde S$ is an A-torus). Now this last group is of finite type, because A is of finite type over $\mathbf Z$ (this uses the generalized unit theorem and the Mordell-Weil-Néron theorem, cf. [4] proposition 2, démonstration, p. 226).

We now fix $\mathfrak p$ as above, denote $A_{\mathfrak p}$ by A, similarly $\tilde X \times_A A_{\mathfrak p}$ by $\tilde X$, etc. . . . . Let $\sum_j n_j P_j$ be a degree nought 0-cycle on X, and let $K_j = k(P_j)$. The integral closure B_j of A in K_j is a semi-local Dedekind ring, which is a (finite) free A-module. Since $\tilde p : \tilde X \to \operatorname{Spec} A$ is proper, the k-point P_j comes from an A-morphism i_j:

$$\begin{array}{ccc} \operatorname{Spec} B_j & \xrightarrow{\quad i_j \quad} & \tilde X \\ & {}_{r_j}\searrow \qquad \swarrow_{\tilde p} & \\ & \operatorname{Spec} A. & \end{array}$$

Since $\tilde p$ is projective and A is a discrete valuation ring, we can talk of the semi-local ring $\mathcal{O}_{\tilde X, V}$ which is the intersection of the local rings of $\tilde X$ at the (finitely many) points of the union V of the images of the i_j. We shall denote this ring $\mathcal{O}_{\tilde X, V}$ by R. Each i_j factorizes through $\operatorname{Spec} R$:

$$\begin{array}{ccccc} \operatorname{Spec} B_j & \xrightarrow{\ i'_j\ } & \operatorname{Spec} R & \xrightarrow{\ h\ } & \tilde X \\ & {}_{r_j}\searrow & \downarrow q & \swarrow_{\tilde p} & \\ & & \operatorname{Spec} A. & & \end{array}$$

The following exact sequence of étale sheaves over $\tilde X$ defines $\mathcal{D}$:

$$1 \to \mathbf{G}_{m, \tilde X} \to h_* \mathbf{G}_{m, R} \to \mathcal{D} \to 1.$$

Applying $\tilde p_*$ and taking cohomology, we get the exact sequence of étale sheaves over A:

$$0 \to \mathbf{G}_{m, A} \to q_* \mathbf{G}_{m, R} \to \tilde p_* \mathcal{D} \to R^1 \tilde p_* \mathbf{G}_{m, \tilde X} \to 0 \tag{1.7}$$

using $\mathbf{G}_{m,A} \xrightarrow{\sim} \tilde p_* \mathbf{G}_{m, \tilde X}$ and $R^1 \tilde p_* (h_* \mathbf{G}_{m,R}) = 0$. This last equality comes from the injection $R^1 \tilde p_* (h_* \mathbf{G}_{m,R}) \hookrightarrow R^1 q_* \mathbf{G}_{m,R}$ (spectral sequence attached to $\tilde p \circ h = q$) and from the vanishing of this last sheaf, which one checks by computing fibres: since A is normal, it suffices to prove that $\operatorname{Pic} U_R = 0$ for any open subset U of $\operatorname{Spec} A'$ where A' denotes the integral closure of A in some finite extension k'/k, and where $U_R = U \times_{\operatorname{Spec} A} \operatorname{Spec} R$; since A'/A is finite (char. $k = 0$), the ring

$R' = R\otimes_A A'$ is finite over the semi-local ring R, hence itself semi-local, which implies $\operatorname{Pic} R' = 0$; it is also regular, since it is formally smooth over the Dedekind ring A'; therefore the restriction map $0 = \operatorname{Pic} R' \to \operatorname{Pic} U_R$ is surjective.

From i'_j we get $\mathbf{G}_{m,R} \to i'_{j*}\mathbf{G}_{m,B_j}$, which yields $q_*\mathbf{G}_{m,R} \xrightarrow{\mu_j} r_{j*}\mathbf{G}_{m,B_j}$ upon applying q_*. Since B_j/A is finite and free, there is a trace map of étale sheaves $T_j : r_{j*}\mathbf{G}_{m,B_j} \to \mathbf{G}_{m,A}$, which composed with the natural map $\mathbf{G}_{m,A} \to r_{j*}\mathbf{G}_{m,B_j}$ yields exponentiation to the power $\dim_A B_j = \dim_k K_j$ on $\mathbf{G}_{m,A}$. On taking $\sum_j n_j T_j \circ \mu_j$, one gets a map of sheaves $q_*\mathbf{G}_{m,R} \to \mathbf{G}_{m,A}$ which composed with $\mathbf{G}_{m,A} \to q_*\mathbf{G}_{m,R}$ in (1.7) gives 1, that is we get a map $\rho : q_*\mathbf{G}_{m,R}/\mathbf{G}_{m,A} \to \mathbf{G}_{m,A}$. From (1.7) one gets the exact sequence

$$0 \to q_*\mathbf{G}_{m,R}/\mathbf{G}_{m,A} \to \tilde{p}_*\mathfrak{D} \to R^1\tilde{p}_*\mathbf{G}_{m,\tilde{X}} \to 0.$$

On first pulling back this sequence via $\lambda : \hat{\tilde{S}} \to R^1\tilde{p}_*\mathbf{G}_{m,\tilde{X}}$ and then pushing out the resulting sequence via ρ, one gets an extension of sheaves over A, of $\hat{\tilde{S}}$ by $\mathbf{G}_{m,A}$, whose restriction to $\operatorname{Spec} k$ is isomorphic to the extension (1.2) associated with $\xi = \sum_j n_j P_j$. This last element of $H^1(k,S) \tilde{\to} \operatorname{Ext}^1_k(\hat{S}, \mathbf{G}_{m,k})$ therefore comes from an element of $H^1(A,\tilde{S}) \tilde{\to} \operatorname{Ext}^1_A(\hat{\tilde{S}}, \mathbf{G}_{m,A})$. □

§2. Conic bundles over the projective line. Let k be a perfect field of characteristic $\neq 2$, and let $K = k(t)$ be the field of rational fractions in the variable t. The following objects are known to be bijectively associated to one another (Iskovskih [6], [7], Lam [8]):

(**a**) Relatively minimal conic bundles X over $\mathbf{P}^1_k$, up to k-birational fibration-preserving isomorphism (recall that a conic bundle $X/\mathbf{P}^1_k$ is a smooth projective k-surface with a k-morphism to $\mathbf{P}^1_k$, the generic fibre of which is a smooth conic over $k(t)$)

(**b**) Smooth complete K-conics, up to K-isomorphism

(**c**) Quaternion algebras over K, up to K-isomorphism

(**d**) Rank 4 Pfister forms over K, up to K-isomorphism

(**e**) Rank 3 quadratic forms over K, with determinant equal to 1, up to K-isomorphism.

That (**c**), (**d**), (**e**) are equivalent is in [8] (proposition 2.5, p. 57): to the quaternion algebra $A = (\frac{a,b}{K})$ one associates the Pfister form $q = \langle\langle -a, -b\rangle\rangle = \langle 1, -a, -b, ab\rangle$ which is the reduced norm of A, and one defines $q_0 = \langle -a, -b, ab\rangle$ as the restriction of q to the pure quaternions. This form in turn defines the conic C with equation $-bx^2 - ay^2 + abz^2 = 0$, which is K-isomorphic to $z^2 - ax^2 - by^2 = 0$. Given X as in (**a**), one defines C as the generic fibre X_K. That any C comes from an X as in (**a**) can be proved by simple direct computations (cf. [6] §3). The structure of degenerate fibres of a relatively minimal conic bundle is given in [9] (theorem 1.6, where a known lemma ([11] p. 91–95) is used without warning), in [6] (lemma 0.9 or, better, explicit proof of theorem 3.2) and in [7] (lemma 6, corollary). Any such fibre over a closed point y of $\mathbf{P}^1_k$ is a $k(y)$-conic X_y with a single $k(y)$-rational point, i.e., over a certain quadratic extension $k(x)/k(y)$ it is isomorphic to the union of two transversal lines $\mathbf{P}^1_{k(x)}$ which the Galois group of $k(x)/k(y)$ swaps. Given such a relatively minimal conic bundle $X/\mathbf{P}^1_k$, let $\{y_i\}$ be the finite set of closed points of $\mathbf{P}^1_k$ over which the fibre is degenerate. Let d_i and $\tilde{d}_i$ be the irreducible components of the fibre of $\bar{X}$ over one (fixed) geometric point of $\mathbf{P}^1_k$ over y_i. We thus fix imbeddings $k(y_i) \subset k(x_i) \subset \bar{k}$; let $\mathfrak{g}_i$ and $\mathfrak{g}'_i$ be the respective Galois groups of $\bar{k}/k(y_i)$ and $\bar{k}/k(x_i)$. Let f be the divisor on $\bar{X}$ defined by the fibre of X above some rational point of $\mathbf{P}^1_k$. It is an easy matter to check the exactness of the following sequence of $\mathfrak{g}$-modules

$$0 \to \bigoplus_i \mathbf{Z}[\mathfrak{g}/\mathfrak{g}_i] \cdot e_i \to \mathbf{Z}f \oplus \bigoplus_i \mathbf{Z}[\mathfrak{g}/\mathfrak{g}'_i] \cdot d_i \to \operatorname{Pic} \bar{X} \xrightarrow{\lambda} \operatorname{Pic} X_{\bar{K}} \to 0. \quad (2.1)$$

Here $\bar{K} = \bar{k}(t)$, the map λ is induced by the restriction to the generic fibre $X_{\bar{K}}$, and the first two terms are $\mathfrak{g}$-submodules of $\operatorname{Div} \bar{X}$, the first one being generated by $\{e_i = f - d_i - \tilde{d}_i\}$, the second one by $\{f, d_i\}$. Since $\operatorname{Pic} X_{\bar{K}}$ is the trivial $\mathfrak{g}$-module $\mathbf{Z}$, upon tensoring over $\mathbf{Z}$ by $\bar{k}^*$, we transform (2.1) into the exact sequence of k-tori (more precisely into the sequence of their $\bar{k}$-points)

$$1 \to \prod_i R_{k(y_i)/k}\mathbf{G}_m \xrightarrow{\omega} \mathbf{G}_m \times \prod_i R_{k(x_i)/k}\mathbf{G}_m \longrightarrow S \longrightarrow \mathbf{G}_m \to 1 \quad (2.2)$$

where we have used the canonical identification of S with S' (cf. §1, proof of (ii)), and where $\omega((a_i)) = (\prod_i N_{k(y_i)/k}(a_i), \dots, a_i^{-1}, \dots)$. Let N_0 be the kernel of λ, and let S_0 be the k-torus defined by $S_0(\bar{k}) = N_0 \otimes_{\mathbf{Z}} \bar{k}^*$. One then deduces from (2.2) the exact sequence

$$0 \to H^1(k, S_0) \to \bigoplus_i \operatorname{Br} k(y_i) \xrightarrow{\mu} \operatorname{Br} k \oplus \bigoplus_i \operatorname{Br} k(x_i) \quad (2.3)$$

where $\mu((\alpha_i)) = (\sum_i \operatorname{Cor}_{k(y_i)/k}(\alpha_i), \dots, -\operatorname{Res}_{k(y_i)/k(x_i)}(\alpha_i), \dots)$. It can therefore be rewritten as

$$0 \to H^1(k, S_0) \to \prod_i k(y_i)^*/N_i(k(x_i)^*) \xrightarrow{\sum_i \operatorname{Cor}_i} \operatorname{Br} k \quad (2.4)$$

where $N_i = N_{k(x_i)/k(y_i)}$ and $\operatorname{Cor}_i = \operatorname{Cor}_{k(y_i)/k}$. This shows $H^1(k, S_0)$ to be annihilated by 2. So is a fortiori $H^1(k, S)$, in view of the exact sequence

$$S(k) \xrightarrow{\pi} k^* \longrightarrow H^1(k, S_0) \longrightarrow H^1(k, S) \to 0 \quad (2.5)$$

which one also deduces from (2.2). On using the birational invariance of $H^1(k, S)$ (cf. theorem 3 (iii)) to restrict oneself to the case of a relatively minimal conic bundle, one gets:

Proposition 1. *Let X be any conic bundle over $\mathbf{P}^1_k$, and let S be the k-torus dual to* $\operatorname{Pic} \bar{X}$. *The group $H^1(k, S)$ is annihilated by* 2. □

There are various algebraic invariants attached to the data (**a**)–(**e**):

(a_1) the collection of quadratic extensions $k(x_i)/k(y_i)$ defined by the degenerate fibres of X;

(a_2) the class ζ_X of the 2-extension of $\mathfrak{g}$-modules defined by (2.1)

$$0 \to \bigoplus_i \mathbf{Z}[\mathfrak{g}/\mathfrak{g}_i] \to \mathbf{Z} \oplus \bigoplus_i \mathbf{Z}[\mathfrak{g}/\mathfrak{g}'_i] \to \operatorname{Pic} \bar{X} \to \mathbf{Z} \to 0; \tag{2.6}$$

upon using the identifications $\operatorname{Ext}^2_{\mathfrak{g}}(\mathbf{Z}, \mathbf{Z}[\mathfrak{g}/\mathfrak{h}]) = H^2(\mathfrak{g}, \mathbf{Z}[\mathfrak{g}/\mathfrak{h}]) = H^2(\mathfrak{h}, \mathbf{Z}) = H^1(\mathfrak{h}, \mathbf{Q}/\mathbf{Z})$, this 2-extension gives rise to elements $\zeta_i \in H^1(k(x_i)/k(y_i), \mathbf{Z}/2) \hookrightarrow H^1(\mathfrak{g}_i, \mathbf{Z}/2)$, and lemma 1 will show these ζ_i to be non-zero;

(b) the class $\alpha_C \in \operatorname{Br} K$ of the canonical 2-extension of $\mathfrak{g}$-modules (recall $\bar{K} = \bar{k}(t)$)

$$0 \to \bar{K}^* \to \bar{K}(C)^* \to \operatorname{Div} C_{\bar{K}} \to \operatorname{Pic} C_{\bar{K}} \to 0$$

which lies in $H^2(\mathfrak{g}, \bar{K}^*)$—since $\operatorname{Pic} C_{\bar{K}}$ is the trivial $\mathfrak{g}$-module $\mathbf{Z}$;

(c) to any closed point $y \in \mathbf{P}^1_k$ and any imbedding $k(y) \hookrightarrow \bar{k}$, letting $\mathfrak{g}_y = \operatorname{Gal}(\bar{k}/k(y))$, one can associate the Witt invariant $\chi_y(A) \in H^1(k(y), \mathbf{Q}/\mathbf{Z})$ which is defined by means of the valuation v_y at y:

$$H^2(\mathfrak{g}, \bar{k}(t)^*) \xrightarrow{\text{Res}} H^2(\mathfrak{g}_y, \bar{k}(t)^*) \xrightarrow{v_y} H^2(\mathfrak{g}_y, \mathbf{Z})$$

(cf. [5] proposition 2.1 p. 93 and compléments p. 188);

(d) the collection $(d_\pm(\partial_{2,y}(q)) \in \coprod_y k(y)^*/k(y)^{*2}$, where $\partial_{2,y}$ denotes the second residue homomorphism ([8] chap. 6, cor. 1.6, p. 145) taken with respect to a uniformizing parameter π_y at y, and $d_\pm$ is the signed determinant ([8] p. 38); an easy computation shows that the definition does not depend on the choice of the π_y; moreover, $\partial_{2,y}(q)$ is in the fundamental ideal $I(k(y)) \subset W(k(y))$, and it is non-zero if and only if its signed determinant is not 1;

(e) the same collection, with q_0 in place of q.

LEMMA 1. *The various invariants defined above are related as follows*:

(i) α_C *is the class of* A *in* $\operatorname{Br} k(t)$;

(ii) *the local invariants in* (c), (d), (e) *are all trivial unless* y *is one of the* y_i; *at* y_i, *they are all equal and they coincide with those in* (a_1) *and* (a_2)—*through the usual dictionaries*: $\{$*quadratic and trivial extensions of* $k(y)\} \leftrightarrow H^1(\mathfrak{g}_y, \mathbf{Z}/2) \leftrightarrow k(y)^*/k(y)^{*2}$.

Proof. (i) is [1] lemma 3.14. That the invariants in (a_1) and (c) coincide at all y_i is in [6] (theorem 3.5). Easy residue computations show the invariants in (c), (d), (e) to coincide—and to vanish at $y \neq y_i$. Here is a typical example. Take $a, b \in k[t]$, and assume $\pi = \pi_y$ exactly divides $b = \pi b'$ but does not divide a. The signed determinant of $\partial_{2,\pi}(\langle\langle -a, -b\rangle\rangle) = b'(y)\langle -1, a(y)\rangle$ is $a(y) \in k(y)^*/k(y)^{*2}$; one finds the same result starting from $q_0 = \langle -a, -b, ab\rangle$; finally, proceeding as in [1] (lemma 3.15), one shows the invariant $\chi_y(({}^{a,b}_{K}))$ to be the quadratic extension $k(y)(\sqrt{a(y)})/k(y)$. Now it only remains to show that the ζ_i in (a_2) are non-zero.

Let $U = U_i$ be the complement in X of the union of f (the fixed fibre over a rational point) and all degenerate fibres except X_{y_i}. It is clear that ζ_i can also be defined by the 2-extension

$$0 \to \mathbf{Z}[\mathfrak{g}/\mathfrak{g}_i] \cdot e_i \to \mathbf{Z}[\mathfrak{g}/\mathfrak{g}'_i] \cdot d_i \to \operatorname{Pic} \overline{U} \to \operatorname{Pic} X_{\overline{K}} \to 0 \tag{2.7}$$

defined just as (2.1), with $\operatorname{Pic} X_{\overline{K}} = \mathbf{Z}$ and the first non-trivial arrow sending e_i to $-(d_i + \tilde{d}_i)$. Let M_i be the image of the middle arrow. Now, ζ_i is the image of the extension

$$0 \to M_i \to \operatorname{Pic} \overline{U} \to \mathbf{Z} \to 0 \tag{2.8}$$

by the map $\operatorname{Ext}^1_{\mathfrak{g}}(\mathbf{Z}, M_i) \xrightarrow{\rho} \operatorname{Ext}^2_{\mathfrak{g}}(\mathbf{Z}, \mathbf{Z}[\mathfrak{g}/\mathfrak{g}_i])$ deduced from the exact sequence

$$0 \to \mathbf{Z}[\mathfrak{g}/\mathfrak{g}_i] \to \mathbf{Z}[\mathfrak{g}/\mathfrak{g}'_i] \to M_i \to 0.$$

Since $\operatorname{Ext}^1_{\mathfrak{g}}(\mathbf{Z}, \mathbf{Z}[\mathfrak{g}/\mathfrak{g}'_i]) = H^1(\mathfrak{g}'_i, \mathbf{Z}) = 0$, ρ is an injection, and $\zeta_i = 0$ if and only if (2.8) splits. Let us show it does not. If the restriction $\operatorname{Pic} \overline{U} \to \operatorname{Pic} X_{\overline{K}}$ had a section, there would exist a divisor D on $\overline{X}$ such that $(D.f) = 1$ and such that, for all $\sigma \in \mathfrak{g}$, the divisor ${}^\sigma D - D$ is linearly equivalent to a divisor with support in the complement of $\overline{U}$. Take σ so that ${}^\sigma d_i = \tilde{d}_i$. From

$$(D.\tilde{d}_i) = (D.{}^\sigma d_i) = ({}^{\sigma^{-1}}D.d_i) = (D.d_i)$$

we would get $1 = (D.f) = (D.(d_i + \tilde{d}_i))$ even! □

Remark. From (a_2), we see that the element $(\zeta_i) \in \bigoplus_i H^1(\mathfrak{g}_i, \mathbf{Q}/\mathbf{Z})$ goes to zero under the transfer map to $H^1(\mathfrak{g}, \mathbf{Q}/\mathbf{Z})$: not all local invariants are possible. In the simplest case, where all y_i are k-rational and all extensions $k(x_i)/k$ coincide, this implies that the number of degenerate fibres is even; more generally, if all y_i are rational, the sum (in $H^1(\mathfrak{g}, \mathbf{Z}/2)$) of the quadratic extensions corresponding to the degenerate fibres is zero (Exercise: build a complete model of $y^2 - tz^2 = (t-a)(t-b)(t-c)$ and check the result). This is surely related to Scharlau's reciprocity formula ([8] chap. 9, theorem 4.2, p. 270).

§3. Proof of theorems 1 and 2. The notation will be as in the statement of theorem 2 (cf. introduction) and as in §2. We shall assume $X/\mathbf{P}^1_k$ to be relatively minimal. This is legitimate, since $\Phi : A_0(X) \to H^1(k, S)$ is a k-birational invariant (§1, theorem 3 (iii)), and since condition (i) in theorem 2 is also a k-birational invariant (as shown by Chow's moving lemma, or more simply as in [3] lemme 3.1.1).

Let us quickly recall some facts used by Bloch to establish the exact sequence (*). Let $\mathfrak{Z} = Z^1(\overline{X}, \mathcal{K}_2)$ be the kernel of the map

$$\operatorname{div} : \coprod_{\gamma \in \overline{X}^1} \bar{k}(\gamma)^* \to \coprod_{\overline{X}^2} \mathbf{Z}$$

where $\bar{X}^i$ denotes the set of points on $\bar{X}$ with codimension i. The natural homomorphisms $K_2(\bar{k}) \to H^0(\bar{X}, \mathcal{K}_2)$ and $\operatorname{Pic} \bar{X} \otimes_{\mathbf{Z}} \bar{k}^* \to H^1(\bar{X}, \mathcal{K}_2)$ are shown to be isomorphisms. One therefore has the exact sequence of $\mathfrak{g}$-modules

$$0 \to K_2\bar{F}/K_2\bar{k} \to \mathfrak{Z} \to \operatorname{Pic} \bar{X} \otimes_{\mathbf{Z}} \bar{k}^* \to 0. \tag{3.1}$$

On the other hand, Milnor's exact sequence ([10] theorem 2.3) applied first to the base, then to the generic fibre of $\bar{X}/\mathbf{P}^1_{\bar{k}}$, gives rise to the two exact sequences of $\mathfrak{g}$-modules

$$0 \to \bigoplus_y{}^{0}\, \bar{k}(y)^* \to K_2\bar{F}/K_2\bar{k} \to K_2\bar{F}/K_2\bar{K} \to 0 \tag{3.2}$$

$$0 \to K_2\bar{F}/K_2\bar{K} \to \bigoplus_\gamma \bar{k}(\gamma)^* \xrightarrow{\;N\;} \bar{K}^* \to 0. \tag{3.3}$$

Here y in (3.2) runs through all closed points of $\mathbf{P}^1_k$ (and $\bar{k}(y) = \bar{k} \otimes_k k(y)$), the symbol $\bigoplus^0$ denotes the kernel of the sum of norm maps $\bar{k}(y)^* \to \bar{k}^*$, and γ in (3.3) runs through all closed points of the $\bar{K}$-conic $X_{\bar{K}}$ (they correspond to integral non-vertical divisors on $\bar{X}$).

(A) *The basic diagram.*

This subsection will be devoted to the construction and study of the following diagram

$$\begin{array}{ccccccc}
 & & S(k) & \xrightarrow{\;\pi\;} & k^* & \xrightarrow{\;\partial\;} & \bigoplus_i \operatorname{Br} k(y_i) \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle j} \\
0 & \longrightarrow & H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{k}) & \xrightarrow{\;\theta\;} & K^*/NA^* & \xrightarrow{\;\psi\;} & \bigoplus_y \operatorname{Br} k(y) \\
 & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\eta} & & \uparrow{\scriptstyle c} \\
0 & \longrightarrow & I^3k & \xrightarrow{\;i\;} & I^3K & \xrightarrow{\;\partial_2\;} & \bigoplus_y I^2k(y)
\end{array} \tag{3.4}$$

where η and κ will only be maps (not necessarily homomorphisms).

The first line is an exact sequence deduced from (2.3) and (2.5) (y_i runs through the closed points of $\mathbf{P}^1_k$ where the fibre of $X/\mathbf{P}^1_k$ is degenerate). The last line is an exact sequence of Milnor's for powers of the fundamental ideals in the Witt ring of $k(t)$ ([10] lemma 5.7, [8] theorem 3.1, p. 265); y runs through all closed points of $\mathbf{P}^1_k$, and ∂_2 is defined as the collection of second residues at all y, with respect to a fixed choice of uniformizing parameters. The maps ι and j are the obvious ones. The homomorphism c is the Clifford invariant, which on I^2 coincides with the Witt invariant of quadratic forms (cf. [8] chap. 5, §3, pp. 116 and 120). The map δ is the one appearing in Bloch's sequence (*); it comes from

(3.1). The *map* η is defined as the one sending $f \in K^*$ to the class of $fq \perp -q$ in $W(K)$, where q is any rank 4 Pfister form $q = \langle\langle -a, -b \rangle\rangle$ corresponding to $X/\mathbf{P}^1_k$ (cf. beginning of §2): one easily checks that this defines a map from K^*/NA^* (where N is the reduced norm Nrd of the quaternion algebra A, actually defined by q) to I^3K, and that this map induces a *homomorphism* $K^*/NA^* \to I^3K/I^4K$. The middle line is an exact sequence obtained by improving on lemma 3.10 of [1]:

LEMMA 2. *Let $N : \coprod_\gamma K(\gamma)^* \to K^*$ be the "sum" of the norm homomorphisms corresponding to all closed points γ of the K-conic $X_K = C$. The image of N is precisely the subgroup $Nrd(A^*)$.*

Proof. Let $\mathfrak{N}$ be this image. That $Nrd(A^*) \subset \mathfrak{N}$ is in [1] (lemma 3.10). Let γ be a closed point of X_K, let $L = K(\gamma)$, and $\alpha = N_{L/K}(\beta)$ with $\beta \in L^*$. Since A_L is split, β, as any other element of L, is represented over L by its reduced norm q_L. Applying either Knebusch's or Scharlau's norm principle (cf. [8] chap. 7, §§4–5), we deduce that α is represented over K by q, i.e., it belongs to $Nrd(A^*)$. □

Remark. One can give a slightly different proof, based purely on properties of central simple algebras. Namely, for any such algebra A with center k, the subgroup $Nrd(A^*) \subset k^*$ coincides with the subgroup generated by all $N_{L/K}(L^*)$, for L running through the finite extensions of k which split A. The proof uses: (*a*) if A and B are two similar simple central algebras over k, an argument involving Dieudonné determinants shows $Nrd(A^*) = Nrd(B^*)$; (*b*) for L/k a finite field extension splitting A, there is an algebra B similar to A in which L is a maximal commutative subring; (*c*) in the situation of (*b*), the reduced norm of B induces on L the usual norm $N_{L/k}$; (*d*) A is similar to a skew field D, and any maximal commutative subfield of D is a splitting field of D. One can in turn deduce from the above identification a "norm principle" for the image of the reduced norm Nrd.

On applying cohomology to (3.3), lemma 2 gives an isomorphism

$$K^*/NA^* \xrightarrow{\sim} H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{K}),$$

which, combined with the cohomology sequence associated to (3.2), yields the horizontal middle exact sequence in (3.4). All maps but κ have now been defined. That η induces (via θ) such a map will be shown at the end of the proof of:

LEMMA 3. *Diagram* (3.4) *is a commutative diagram whose lines are exact sequences. The map η is injective; composing it with reduction modulo I^4K even yields an injective homomorphism $K^*/NA^* \hookrightarrow I^3K/I^4K$ which makes $H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{k})$ into a subgroup of I^3k/I^4k. The map c is injective on the image of $\partial_2 \circ \eta$.*

Proof. We already know that the lines are exact sequences. From $fq \simeq fq'$, with $f, f' \in K^*$ and q a quaternion form, follows $f/f' \in NA^*$: hence η is injective. The composed map $K^*/NA^* \to I^3K/I^4K$ is a homomorphism; let $f \in K^*$ with $q \perp -fq = \langle\langle -f, -a, -b\rangle\rangle \in I^4K$; according to Arason and Pfister ([8] chap. 10, §3, cor. 3.4, p. 290), this 3-fold Pfister form must be hyperbolic, hence $f \in NA^*$. We are left with showing commutativities and the assertion on c.

Restriction to the generic fibre defines a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_2\bar{F}/K_2\bar{k} & \longrightarrow & \mathfrak{X} & \longrightarrow & \operatorname{Pic}\bar{X}\otimes_{\mathbf{Z}}\bar{k}^* & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow \searrow \;\; \bar{k}^* \;\; \swarrow & & \\ 0 & \longrightarrow & K_2\bar{F}/K_2\bar{K} & \longrightarrow & \bigoplus_{\gamma}\bar{K}(\gamma)^* & \longrightarrow & \bar{K}^* & \longrightarrow & 0. \end{array} \tag{3.5}$$

In this diagram, the first line is exact sequence (3.1), the second one exact sequence (3.3); the left vertical arrow is the obvious one, the middle one is obtained by forgetting vertical divisors (recall $\mathfrak{X} \subset \coprod_{\gamma\in\bar{X}^1}\bar{k}(\gamma)^*$); the map $\operatorname{Pic}\bar{X}\otimes_{\mathbf{Z}}\bar{k}^* \to \bar{k}^*$ is induced by $\lambda:\operatorname{Pic}\bar{X}\to\operatorname{Pic}X_{\bar{K}} = \mathbf{Z}$, and the right vertical arrow is obtained by composition with the inclusion $\bar{k}\hookrightarrow\bar{K}$. The left square commutes, because both horizontal arrows are defined by tame symbols: one can therefore restrict oneself to $\operatorname{Div}\bar{X}\otimes_{\mathbf{Z}}\bar{k}^* \subset \mathfrak{X}$ to check the commutativity of the right square, and going both ways sends $D \otimes \alpha$ to $\alpha^{(D\cdot f)}$. On taking cohomology, we get the commutativity of the rectangle in

$$\begin{array}{ccc} S(k) & \xrightarrow{\quad\delta\quad} & H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{k}) \\ \Big\downarrow \iota\circ\pi \quad \searrow^{\pi} & & \swarrow^{\theta} \quad \Big\downarrow \\ \quad k^* \quad \swarrow^{\iota} & & \\ K^*/NA^* & \xrightarrow{\quad\sim\quad} & H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{K}) \end{array}$$

where π and ι have already been defined, and where the very definition of θ shows the lower triangle to commute. The left upper square in (3.4) therefore commutes.

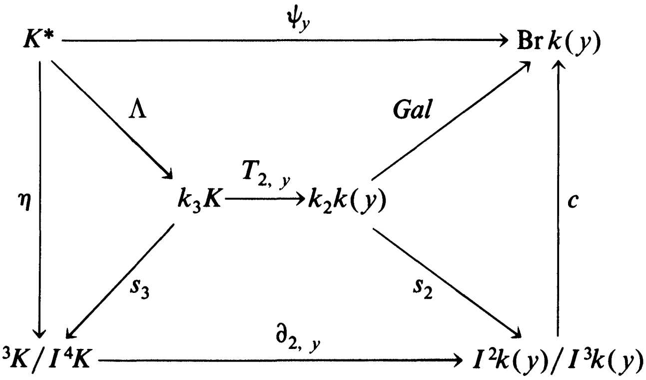
The map $\partial: k^* \to \bigoplus_i \operatorname{Br} k(y_i)$ is the coboundary (∂_i) coming from the 2-extension (2.2), itself obtained from (2.1) by tensoring over $\mathbf{Z}$ with $\bar{k}^*$. Letting $\zeta_i \in H^1(\mathfrak{g}_i, \mathbf{Q}/\mathbf{Z})$ be the quadratic character corresponding to the extension $k(x_i)/k(y_i)$, and letting $\delta\zeta_i \in H^2(\mathfrak{g}_i, \mathbf{Z})$ be its coboundary, we then see from lemma 1 that for $f \in k^*$ the cup-product $\delta\zeta_i \cup f$ coincides with $\partial_i(f)$. Now, for $f \in k^*$, the computations in lemma 3.15 of [1] show $\psi_y(f) = 0$ if the closed point

y is not among the y_i, whereas

$$\psi_{y_i}(f) = T_1\{\pi_{y_i} \cup \delta\zeta_i, f\} = \delta\zeta_i \cup f.$$

This deals with the commutativity of the right upper square in (3.4).

Let y be a closed point of $\mathbf{P}^1_k$. The commutativity of the right lower square in (3.4) follows from that of the big rectangle in the diagram



(3.6)

where the maps with the same label as in (3.4) are induced by the corresponding maps in (3.4), $T_{2,\,y}$ is the tame symbol ([10] 2.1), $\partial_{2,\,y}$ is the second residue at y ([10] 5.1), s_2 and s_3 are as in [10] (4.1), *Gal* is the Galois symbol ([10] 6.1) and $\Lambda(f) = \ell(f).\ell(a).\ell(b)$ for $q = \langle\langle -a, -b\rangle\rangle$ or $A = (\frac{a,\,b}{K})$. The commutativity of the two triangles and of the lower trapezium follows from the definitions of the maps. That of the upper trapezium is dealt with in [1] (lemma 3.15). The big rectangle therefore commutes.

Finally we show that the map c is injective on the image of $\partial_2 \circ \eta$. Indeed, a computation shows that, for any y, the class of $\partial_{2,\,y}(fq \perp -q)$, where $q = \langle\langle -a, -b\rangle\rangle$, is the class of a rank 4 quadratic form with determinant 1. But the formula $c(\langle u, v, w, uvw\rangle) = (\frac{-uv,\,-uw}{k(y)})$ (see [8] chap. 5, lemma 3.2, p. 116) ensures injectivity of c on such forms. Hence $\eta \circ \theta$ factorizes through a map κ into the kernel I^3k of the homomorphism $\partial_2 = \bigoplus_y \partial_{2,y}$. □

(B) *Exploiting the basic diagram.*

Given the $k(t)$-quadratic form $q = \langle\langle -a, -b\rangle\rangle$ the class of which lies in $I^2K \subset W(K)$, let us denote

$$K_q^* = \{f \in K^* \mid fq \perp -q \in i(W(k))\}$$

$$k_q^* = \{\alpha \in k^* \mid \alpha q \perp -q \in i(W(k))\}$$

where $i: W(k) \to W(K)$ is the natural injection. Clearly, $k_q^* = K_q^* \cap k^*$, and $K_q^* \supset NA^*$. Moreover, k_q^* is a subgroup of k^*, and the next proposition shows K_q^* to be a subgroup of K^*.

Proposition 2. *The following equalities hold, where notations are as in the whole section:*

$$k_q^* = \operatorname{im}\left(S(k) \xrightarrow{\pi} k^*\right) = \ker\left(k^* \to H^1(k, S_0)\right)$$
$$= \ker\left(k^* \to \prod_i k(y_i)^*/N_i(k(x_i)^*)\right) \qquad (3.7)$$
$$H^1\left(\mathfrak{g}, K_2\bar{F}/K_2\bar{k}\right) = K_q^*/NA^*, \qquad (3.8)$$

and there is an exact sequence

$$0 \to K_q^*/k_q^*NA^* \to A_0(X) \xrightarrow{\Phi} H^1(k, S). \qquad (3.9)$$

Proof. According to lemma 3, the map c in (3.4) is injective on the image of $\partial_2 \circ \eta$. Hence the two conditions $\partial_2 \circ \eta(f) = 0$ and $\psi(f) = 0$ are equivalent for $f \in K^*$. But the first condition precisely describes K_q^* (exactness of the lower line in (3.4)). Whence (3.8), upon using the middle exact sequence in (3.4). The group k_q^* now appears as the kernel of $\psi \circ \iota$, hence of ∂, whence (3.7) upon using (2.3), (2.5), (2.4). We finally deduce (3.9) from the commutative diagram

$$\begin{array}{ccccccc}
S(k) & \xrightarrow{\delta} & H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{k}) & \longrightarrow & A_0(X) & \xrightarrow{\Phi} & H^1(k, S) \\
\Big\downarrow \pi & & \wr\Big\downarrow \theta & & & & \\
k_q^* & \longrightarrow & K_q^*/NA^* & & & &
\end{array}$$

the first line of which is Bloch's exact sequence (*), deduced from (3.1) by taking cohomology and identifying $H^1(\mathfrak{g}, \mathfrak{X})$ with $A_0(X)$. □

Remark. The proof of this proposition, the full strength of which will only be used in the proof of theorem 5, uses all the painfully established commutativities of lemma 3.

An open question: Is K_q^* always equal to $k_q^*.NA^*$? In other words, given the quadratic form $q = \langle\langle -a, -b\rangle\rangle$ over $k(t)$ and any $f \in k(t)^*$ such that the class of $fq \perp -q$ in $W(k(t))$ comes from $W(k)$, does there exist $\alpha \in k^*$ such that f/α is represented by q over $k(t)$?

After proposition 2, an affirmative answer to this question is equivalent to Φ being injective. Theorem 4 will show the answer to be "yes" in case (i) of theorem 2, and theorem 5 will show it to be "yes" in cases (ii) and (iii). In all these cases, proposition 2 will even yield the value of $H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{k})$, thus answering a question of Bloch ([1] 3.4).

LEMMA 4. *Let y be a closed point of $\mathbf{P}^1_k$, let π be a uniformizing parameter at y, let a and b be in $k(t)$ with $v_\pi(a) = 0$ and $v_\pi(b) = 1$, and let $q = \langle\langle -a, -b\rangle\rangle$. There exists $u \in k(y)^*$ such that for any $g \in k(t)^*$, the following relation holds in $W(k(y))$:*

$$\partial_{1,\pi}(gq) = u.\partial_{2,\pi}(gq). \tag{3.10}$$

Proof. Recall $\partial_{1,\pi}$ denotes the first residue ([8] chap. 6, §1, pp. 145–146). The proof is by two computations: $v_\pi(g)$ odd or even. Letting $b = \pi b'$, one finds in both cases that $u = -b'(y)$ will do. □

THEOREM 4. *Let k be a perfect field of characteristic $\neq 2$ and let X be any conic bundle over $\mathbf{P}^1_k$. If X has a point in an odd degree extension of k, then $H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{k}) = 0$ and the map $\Phi: A_0(X) \to H^1(k, S)$ is injective.*

Proof. Since there obviously exist closed points of degree at most 2 on X, the assumption is tantamount to the existence of a degree one 0-cycle on X. As mentioned at the beginning of §3, we may assume $X/\mathbf{P}^1_k$ to be relatively minimal. Let x be a closed point of odd degree on X, and let y be its projection in $\mathbf{P}^1_k$. Clearly, $k(x)/k(y)$ and $k(y)/k$ are odd degree extensions. Denote by R the local ring of $\mathbf{P}^1_k$ at y, by π a uniformizing parameter of R. Changing variables if necessary, we may assume that the generic fibre is given in $\mathbf{P}^2_K$ by an equation

$$X^2 - aY^2 - bZ^2 = 0 \tag{3.11}$$

where either (*i*) a and b are units at π or (*ii*) a is a unit and $v_\pi(b) = 1$. We can take $q = \langle\langle -a, -b\rangle\rangle$, with the same a and b.

Case (i). Since $\partial_{2,\pi}(q) = 0$, the fibre of $X/\mathbf{P}^1_k$ is a smooth conic (see §2, lemma 1). Let $X_R = X \times_{\mathbf{P}^1_k} \operatorname{Spec} R$ be the restriction of $X/\mathbf{P}^1_k$ over $\operatorname{Spec} R$. Equation (3.11) defines a smooth R-curve in $\mathbf{P}^2_R$, call it F_R, and the generic fibres of these two R-curves are K-isomorphic: $X_K \simeq F_K$. Such a K-birational isomorphism induces an R-rational map $X_R \overset{\theta}{\dashrightarrow} F_R$, which on the special fibres induces a $k(y)$-rational map $X_y \dashrightarrow F_y$ (since X_R is regular of dimension 2, and F_R is proper over R, θ is defined outside finitely many closed points). Since X_y is a smooth $k(y)$-curve, and F_y is a proper $k(y)$-curve, this induced $k(y)$-rational map extends to a $k(y)$-morphism (which can be the contraction to one point!): the $k(x)$-valued point of X_y now defines a $k(x)$-valued point of the conic defined by $X^2 - a(y)Y^2 - b(y)Z^2 = 0$ in $\mathbf{P}^2_{k(y)}$. Since $k(x)/k(y)$ is an odd degree extension, this conic also has a $k(y)$-point (special easy case of Springer's theorem), and $\partial_{1,\pi}(q) = \langle 1, -a(y), -b(y), a(y)b(y)\rangle$ is zero in $W(k(y))$. A fortiori $\partial_{1,\pi}(fq) = 0$ for any $f \in k(t)^*$, hence $\partial_{1,\pi}(fq \perp -q) = 0$.

Case (ii). Take $f \in K_q^*$, i.e., $fq \perp -q \in i(W(k))$. From (3.10) in lemma 4, we find

$$\partial_{1,\pi}(fq \perp -q) = u.\partial_{2,\pi}(fq \perp -q) = 0 \in W(k(y))$$

(second residues of constant forms are trivial).

End of proof. Take $f \in K_q^*$. We know that in all cases, $\partial_{1,\pi}(fq\perp -q)$ is zero in $W(k(y))$. By definition, $fq\perp -q$ belongs to $i(W(k))$, and $\partial_{1,\pi} \circ i = i_y$ is the natural map $W(k) \to W(k(y))$, which is an injection by Springer's theorem ([8] chap. 7, cor. 2.2, p. 198). Hence $fq \simeq q$ over $k(t)$, and $f \in NA^*$. We thus have $K_q^* = NA^*$, and the proof is completed on using (3.8) and (3.9) in proposition 2. □

Remark. Under the assumption of the theorem, Chow's moving lemma guarantees the existence of a 0-cycle of degree one, hence of a closed point of odd degree, in any non-empty open set of X: we could do without the study of the degenerate fibres. Note also that the proof of this theorem uses only the lower half of diagram (3.4) and the fact that $\partial_{2,y}(q) = 0$ implies that the fibre of the relatively minimal model $X/\mathbf{P}_k^1$ at y is a smooth conic, which is only a small part of §2.

Proof of theorem 1 (cf. introduction). The case in which k is a finite type extension of $\mathbf{Q}$ immediately follows from the conjunction of theorem 4 and theorem 3 (iv). If k is of cohomological dimension 1, *any* fibre of a conic bundle $X/\mathbf{P}_k^1$ over a k-point of $\mathbf{P}_k^1$ has a k-point! To see this, one may assume $X/\mathbf{P}_k^1$ relatively minimal (since blowing-up and blowing-down preserve existence of k-points); if such a fibre is smooth, it is a conic, hence has a k-point; if it is degenerate, its (unique, cf. beginning of §2) singular point is rational. The assumption of theorem 4 is certainly met, and $H^1(k, S) = 0$, since k is of cohomological dimension 1. □

THEOREM 5. *Let k be a field of characteristic* 0, *and let X be any conic bundle over* $\mathbf{P}_k^1$. *When k is a local or global field, the map* $\Phi : A_0(X) \to H^1(k, S)$ *is injective. Moreover*:

(i) *if k is a* $\mathfrak{p}$*-adic field, then* $H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{k}) = 0$;

(ii) *if* $k = \mathbf{R}$, *then* $H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{k}) = 0$ *if* $X(\mathbf{R}) \neq \emptyset$ *and* $H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{k}) = \mathbf{Z}/2$ *if* $X(\mathbf{R}) = \emptyset$;

(iii) *if k is a number field,* $H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{k}) = (\mathbf{Z}/2)^s$, *where s is the number of real places v of k for which* $X(k_v) = \emptyset$.

Proof. (i) This case is due to Bloch [1]. Apply lemma 3 and note that $I^3k = 0$ for such a field.

(ii) The case $X(\mathbf{R}) \neq \emptyset$ follows from theorem 4. We may assume $X/\mathbf{P}_\mathbf{R}^1$ relatively minimal to deal with the case $X(\mathbf{R}) = \emptyset$ (cf. theorem 3 (iii)). Since $\mathbf{C}$ is the only proper finite extension of $\mathbf{R}$, there cannot exist any degenerate fibre (after §2, such a fibre would lie over a real point, and its singular point would be $\mathbf{R}$-rational). Lemma 1 then shows that $\partial_{2,y}(q) = 0$ for any closed point y of $\mathbf{P}_\mathbf{R}^1$, which implies that $q \in i(W(\mathbf{R})) \subset W(\mathbf{R}(t))$. Whence $q = i(q_0)$ with $q_0 = \langle\langle 1, 1\rangle\rangle$, and X is $\mathbf{R}(t)$-birationally equivalent to $C \times_\mathbf{R} \mathbf{P}_\mathbf{R}^1$, where C is the real conic without real point. Since q is a scalar form, $\mathbf{R}_q^* = \mathbf{R}^*$. Let f be in $\mathbf{R}(t)_q^*$, and let $y_0 \in \mathbf{R}$ be neither a zero nor a pole of f. From

$$fq\perp -q = i \circ \partial_{1,y_0}(fq\perp -q) = f(y_0)q\perp -q \in W(\mathbf{R}(t))$$

there follows $f/f(y_0) \in NA^*$ and $\mathsf{R}(t)_q^* = \mathsf{R}^*.NA^* = \mathsf{R}_q^*.NA^*$. According to (3.9) in proposition 2, the map Φ is therefore injective. Since $\mathsf{R}_+^* \subset NA^*$ and (specialization argument) $-1 \notin NA^*$, we find $\mathsf{R}(t)_q^*/NA^* = \mathsf{Z}/2$, which, together with (3.8), gives the last result.

The argument is general: if $q = i(q_0)$, i.e., if X is $k(t)$-birationally equivalent to the product of a k-conic C with P_k^1, then $H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{k}) = k^*/D_k(q_0)$, where $D_k(q_0)$ is the subgroup of k^* consisting of the elements represented over k by q_0. Since $H^1(k, S) = 0$, one can thus check $A_0(X) = 0$—a fact which is easy to establish directly!

(iii) The case where k is totally imaginary is due to Bloch [1]. Let us show that for k a number field, and $f \in K_q^*$, there exists $\alpha \in k^*$ such that $fq = \alpha q$ in $W(k(t))$. Assume X/P_k^1 relatively minimal (use theorem 3 (iii)). Since $I^3 k_v = 0$ at the non-real places v, we deduce from the last line in (3.4) that for such v's, $fq = q$ in $W(k_v(t))$, and moreover $fq = \alpha q$ in $W(k_v(t))$ for any $\alpha \in k_{v,q}^*$. We have just seen in (ii) that, for any real place v, there exists $\alpha_v \in k_{v,q}^*$ such that $fq = \alpha_v q$ in $W(k_v(t))$. Let ∞ denote the set of *real* places of k. The k-torus S is certainly split over a cyclic extension at the places of ∞: hence the diagonal image of $S(k)$ in the product $\prod_{v \in \infty} S(k_v)$ is dense for the real topology. Note that for any extension k' of k, the group $k_q'^*$ is still the image of the homomorphism $S(k') \xrightarrow{\pi} k'^*$ (this is not entirely obvious, since X/P_k^1 might cease being relatively minimal after going over to k'; however the new S differs from the old one by factors which are sent by π to $1 \in k'^*$: the image of π does not depend on which model has been taken). One can therefore approximate $(\alpha_v)_{v \in \infty}$ in the product $\prod_{v \in \infty} k_{v,q}^*$ by $\alpha \in k_q^* = \pi(S(k))$ in such a way that $fq = \alpha q$ in each $W(k_v(t))$ for $v \in \infty$. We know that $fq = \alpha q$ holds in every $W(k_v(t))$ with v non-real, hence at all v. This is known to imply $fq \simeq \alpha q$ over $k(t)$ (see [3] corollaire 1.1.1, p. 154). We have thus shown $K_q^* = k_q^*.NA^*$, which implies that Φ is injective (proposition 2, (3.9)). From (3.8), we then get:

$$H^1(\mathfrak{g}, K_2\bar{F}/K_2\bar{k}) = K_q^*/NA^* = k_q^*/k^* \cap NA^*.$$

We shall show that the diagonal map $k_q^*/k^* \cap NA^* \to \prod_{v \in \infty} k_{v,q}^*/k_v^* \cap NA_v^*$ is an isomorphism: combined with part (ii), this gives the last statement. From the above approximation argument, we already know the map to be surjective; if $\alpha \in k_q^*$ satisfies $\alpha q \simeq q$ in $k_v(t)$ for $v \in \infty$, this is true at all places v of k (since $I^3 k_v = 0$ at the non-real places v), hence ([3] loc. cit.) this is true over $k(t)$, i.e., $f \in NA^*$: the map is injective. □

This theorem complements [1] (corollary 3.3).

Remark. In the case where $X(k_v) \neq \emptyset$ for all $v \in \infty$, one need not use the approximation property for tori: using theorem 4, we find that $fq \simeq q$ over $k_v(t)$, for any $f \in K_q^*$ and any place $v \in \infty$. As this holds also at the other v's, one uses [3] (loc. cit.) to show $K_q^* = NA^*$.

§4. Remarks and conjectures. In this section, we indicate the connection between the results of this paper and the problems listed in [4] (§V, pp. 232–233).

(a) *Geometrical questions.*

Let X/k be a conic bundle over $\mathbf{P}^1_k$, char.$k=0$ and $X(k)\neq\emptyset$. If $\operatorname{Pic}\overline{X}$ is a direct summand of a permutation $\mathfrak{g}$-module, it is an open question whether there exists a k-variety Y such that $X\times_k Y$ is k-rational (cf. [4] loc. cit., questions **(a)**, **(b)**, **(c)**). Now, under this assumption, theorem 2 implies $A_0(X_L)$ is zero for any extension L/k. Taking $L=k(X)$, and fixing $P\in X(k)$, we deduce that the generic point of X is rationally equivalent to P over $k(X)$. Can any information on the geometry of X be deduced from this fact? More generally, under the single assumption $X(k)\neq\emptyset$, can any geometric information be deduced from the injection $A_0(X_L)\hookrightarrow H^1(L,S)$?

In a different direction, theorem 4 raises the question: is $H^1(\mathfrak{g},K_2\overline{F}/K_2\bar{k})=0$ for any rational surface X with $X(k)\neq\emptyset$?

(b) *An explicit upper bound for $A_0(X)$.*

In the remainder of this section, k will denote a *number field*. Given a k-torus S, with character group $\hat{S}$, and a finite set Σ of places of k, let

$$Ш^1_\Sigma(k,S)=\ker\Bigl(H^1(k,S)\to\prod_{v\notin\Sigma}H^1(k_v,S)\Bigr)$$

and

$$Ш^i(k,U)=\ker\Bigl(H^i(k,U)\to\prod_{\text{all }v}H^i(k_v,U)\Bigr)$$

for $U=S$ or $\hat{S}$, and $i=1,2$. Note that $Ш^1(k,S)=Ш^1_\emptyset(k,S)$. All these (finite abelian) groups can also be computed at the finite level K/k, for K/k any finite Galois extension which splits S (use Hilbert's theorem 90, the Brauer-Hasse-Noether theorem and Čebotarev's theorem). Moreover, class field theory (Tate [12]) identifies $Ш^1(k,S)$ with the dual of $Ш^2(k,\hat{S})$, and $Ш^1_\Sigma(k,S)/Ш^1(k,S)$ with the dual of the cokernel of the restriction map

$$H^1(k,\hat{S})\to\prod_{v\in\Sigma}H^1(k_v,\hat{S}).$$

This permits an effective calculation of the group $Ш^1_\Sigma(k,S)$.

Let X be a *conic bundle* over $\mathbf{P}^1_k$, let $\hat{S}=\operatorname{Pic}\overline{X}$, and let Σ be the union of the finite places v of k at which X does not have good reduction and of the real places v of k such that $X(k_v)$ has at least two (real) components.

PROPOSITION 3. (i) *$A_0(X)$ is a subgroup of the finite group $Ш^1_\Sigma(k,S)$.*

(ii) *If X is split by a cyclic extension of k, the diagonal map*

$$A_0(X)\xrightarrow{\ \delta\ }\prod_{\text{all }v}A_0(X_{k_v})$$

is an injection.

Proof. (i) Bloch ([1] theorem 0.4) has shown that $A_0(X_{k_v}) = 0$ if X has good reduction at the finite place v. When $k_v = \mathbf{R}$, and $X(k_v) = \emptyset$, one easily shows $A_0(X_{k_v}) = 0$, using $A_0(X_{k_v, \mathbf{C}}) = 0$. When $X(k_v) \neq \emptyset$ admits s real components for the real topology, it is an easy consequence of recent results of Ischebeck that $A_0(X_{k_v}) = (\mathbf{Z}/2)^{s-1}$. The first assertion now follows from comparing global and local maps $A_0(X) \xrightarrow{\Phi} H^1(k, S)$, and using the injectivity of the global one (theorem 2 (iii)).

(ii) Recall that X is said to be split by the Galois extension K/k if X_K can be obtained from $\mathbf{P}^2_K$ by successive blowing up and blowing down at K-points. For such an extension K/k, the group $\mathrm{Gal}(\bar{k}/K)$ acts trivially on $\mathrm{Pic}\,\bar{X}$: the extension K/k splits the k-torus S. Let $G = \mathrm{Gal}(K/k)$. After theorem 2 (iii), the kernel of δ is a subgroup of $Ш^1(k, S)$, and one knows that this group is dual to the kernel $Ш^2(K/k, \hat{S})$ of the restriction map $H^2(G, \hat{S}) \to \prod_{\text{all } v} H^2(G_v, \hat{S})$. Since G is cyclic, there exists a v such that $G_v = G$ (Čebotarev's theorem), and this restriction map is an injection. □

Example. An instance of surfaces as in (ii) is provided by smooth proper models of the equation $y^2 - dz^2 = \prod_{i=1}^{N} (x - e_i)$ as given in [4] (§IV p. 230).

(c) *Conjectures on the exact value of* $A_0(X)$.

First recall that class field theory (Tate [12]) gives a long exact sequence

$$0 \to Ш^1(k, S) \to H^1(k, S) \to \coprod_{\text{all } v} H^1(k_v, S) \xrightarrow{\sum i_v} H^1(k, \hat{S})^{\sim} \to Ш^1(k, \hat{S})^{\sim} \to 0 \tag{4.1}$$

and an isomorphism

$$Ш^1(k, S) \simeq Ш^2(k, \hat{S})^{\sim}. \tag{4.2}$$

Here $\tilde{M}$ denotes the dual $\mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$ of the finite abelian group M. The maps $i_v : H^1(k_v, S) \to H^1(k, \hat{S})^{\sim}$ are obtained by composition of the local isomorphisms

$$H^1(k_v, S) \xrightarrow{\sim} H^1(k_v, \hat{S})^{\sim} \tag{4.3}$$

and of the duals

$$H^1(k_v, \hat{S})^{\sim} \longrightarrow H^1(k, \hat{S})^{\sim} \tag{4.4}$$

of the restriction maps. Although one does not yet know whether $A_0(X)$ is finite for all rational surfaces over a number field k, there is no harm in stating the following conjectures for all rational surfaces. Let X be a (smooth, projective) rational surface over the number field k. Let $j_v : A_0(X_{k_v}) \to H^1(k, \hat{S})^{\sim}$ be the composition of the local map $\Phi : A_0(X_{k_v}) \to H^1(k_v, S)$, with (4.3) and (4.4).

CONJECTURE A. *For any rational surface X over a number field k, there is an exact sequence of finite abelian groups*

$$0\to Ш^1(k,S)\to A_0(X)\xrightarrow{\delta}\coprod_{\text{all } v}A_0(X_{k_v})\xrightarrow{\sum j_v}H^1(k,\hat{S})^{\sim} \tag{4.5}$$

where δ is the diagonal map.

Comments. (1) Underlying this conjecture is the fact that $A_0(X_{k_v})=0$ for almost all v. This we know for conic bundles $X/\mathbf{P}^1_k$ ([1] theorem 0.4). That the same holds true for all rational surfaces has been shown to us by Bloch (letter). On the other hand, we do not yet know whether $A_0(X_{k_v})$ is finite for all rational surfaces and all v-except for v archimedean (the above result of Ischebeck holds in fact for any smooth projective rational variety).

(2) Let K/k be a finite Galois extension which splits S, and let $G=\operatorname{Gal}(K/k)$. Conjecture A would give bounds of an algebraic nature for the arithmetic quantity $\tau=(\prod_v \# A_0(X_{k_v}))/\# A_0(X)$:

$$\frac{1}{\# H^2(G,\hat{S})}\leqslant\tau\leqslant\# H^1(G,\hat{S}). \tag{4.6}$$

When K/k is cyclic, τ should be an integer and the more precise bounds

$$\frac{1}{\# H^1(G,\hat{S})}\prod_v\# A_0(X_{k_v})\leqslant\# A_0(X)\leqslant\prod_v\# A_0(X_{k_v}) \tag{4.7}$$

would be available.

(3) There is a K-theoretical version of the above conjecture. The notation is as in §3. Let $\mathfrak{N}(X)$ be the subgroup of $H^2(\mathfrak{g},K_2\bar{F}/K_2\bar{k})$ consisting of the elements which go to 0 under *all* maps $H^2(\mathfrak{g},K_2\bar{F}/K_2\bar{k})\to H^2(\mathfrak{g},\bar{k}(C)^*)$ deduced from the tame symbols attached to the integral curves $C\subset X$, *and* which vanish in all $H^2(\mathfrak{g}_v,K_2\bar{F}_v/K_2\bar{k}_v)$, v running through the places of k. It is a consequence of [1] and of the vanishing of almost all $A_0(X_{k_v})$ that this group is *finite* for any rational surface X.

CONJECTURE B. *The group $\mathfrak{N}(X)$ is zero for any rational surface X.*

Granted that $\Phi: A_0(X)\to H^1(k,S)$ is an injection at the local and global levels, it can easily be shown that this conjecture implies conjecture A. They are even equivalent in the case of conic bundles over $\mathbf{P}^1_k$ which have a 0-cycle of degree one. In the case of arbitrary conic bundles over $\mathbf{P}^1_k$, conjecture B also implies:

CONJECTURE C. *Let k be a number field, and let X/k be a rational surface with $H^1(k,\operatorname{Pic}\bar{X})=0$. If there is a 0-cycle of degree one on each X_{k_v} (v running through all places of k), then there is such a 0-cycle on X.*

This applies in particular to smooth proper models $X_{d,P}$ of $k(X)=k(x,y,z)$

where $y^2 - dz^2 = P(x)$ with $d \in k$, $d \notin k^2$, and with P an irreducible polynomial of even degree. In the case when this degree is 4, this would imply the usual Hasse principle for such equations, in view of Theorem C of [2].

(4) Let us consider $X_{d,P}$ as above, with P irreducible of the fourth degree. *Assume* now $X(k) \neq \emptyset$. In this case, $Ш^1(k,S) = H^1(k,\hat{S}) = 0$, and conjecture A boils down to: $A_0(X) \tilde{\rightarrow} \prod_v A_0(X_{k_v})$. In view of theorem C of [2], this would indeed hold if we knew that $X_{d,P}$ satisfies weak approximation, which is an open question. As a matter of fact, when $k = \mathbf{Q}$, Schinzel's hypothesis (H) (cf. [4] p. 236, and the references therein) does imply weak approximation for $X_{d,P}$.

(5) The last map $\prod_{\text{all } v} A_0(X_{k_v}) \xrightarrow{\sum j_v} H^1(k,\hat{S})^{\sim}$ in (4.5) need not be surjective onto the kernel of the natural map $H^1(k,\hat{S})^{\sim} \rightarrow Ш^1(k,\hat{S})^{\sim}$, as exemplified by the first two numerical examples in the Table below.

(d) *On the kernel of the diagonal map* $A_0(X) \xrightarrow{\delta} \prod_{\text{all } v} A_0(X_{k_v})$.

Here X is any rational surface over k. The map $\Phi : A_0(X) \rightarrow H^1(k,S)$ clearly sends $\ker\delta$ to $Ш^1(k,S)$. It certainly induces an *injection* $\ker\delta \hookrightarrow Ш^1(k,S)$ when X is a conic bundle over $\mathbf{P}^1_k$ (theorem 2 (iii)). *Assume* $X(k) \neq \emptyset$.

ASSERTION. *If the universal torseurs* ([4] *p.* 226) *over X satisfy the Hasse principle, the map* $\Phi : \ker\delta \rightarrow Ш^1(k,S)$ *is surjective.*

Recall it is an open question whether such torseurs satisfy the Hasse principle (see [4] §§II et IV). Let us establish the above assertion. Take $P \in X(k)$ and let $\mathcal{T} = \mathcal{T}^P$ be the universal torseur with trivial fibre at P. Let α be in $Ш^1(k,S)$, and let $\mathcal{T}^\alpha$ be the universal torseur over X obtained by twisting $\mathcal{T}$ by $-\alpha$. Since by definition the fibre of $\mathcal{T}^\alpha$ at P is $-\alpha \in H^1(k,S)$, hence $0 \in H^1(k_v,S)$ for any v, the universal torseur $\mathcal{T}^\alpha$ has points in all completions of k. If such torseurs satisfy the Hasse principle, there exists a $P_\alpha \in X(k)$ such that $\mathcal{T}^\alpha(P_\alpha) = 0 \in H^1(k,S)$ (take for P_α the projection of a k-point of $\mathcal{T}^\alpha$). Hence $\mathcal{T}(P_\alpha) = \alpha \in H^1(k,S)$, and α is the image under Φ of the 0-cycle $P_\alpha - P$ (use theorem 3 (i)), which proves the assertion. □

(e) *On the cokernel of* $\delta : A_0(X) \rightarrow \prod_{\text{all } v} A_0(X_{k_v})$ *in the case of Châtelet surfaces.*

The Châtelet surfaces are conic bundles over $\mathbf{P}^1_k$ given in affine coordinates by an equation

$$y^2 - dz^2 = (x - e_1)(x - e_2)(x - e_3) \tag{4.8}$$

with $d \in k$, $d \notin k^2$ and $e_i \neq e_j$ for $i \neq j$. The projection onto $\mathbf{P}^1_k$ is given by the x-coordinate. One can give a simple smooth projective model X, which has 4 degenerate fibres ([4] §IV). The universal torseurs on X can be described by the affine equations

$$0 \neq x - e_i = c_i(u_i^2 - dv_i^2) \qquad (i = 1,2,3; c_1c_2c_3 = 1) \tag{4.9}$$

in the variables (x, u_i, v_i). The computations of [4] (§IV p. 231) imply $Ш^1(k,S) = 0$ and $Ш^1(k,\hat{S}) = 0$—indeed S is split by the cyclic extension

$k(\sqrt{d})/k$. Moreover, $H^1(k,\hat{S}) = (\mathbf{Z}/2)^2$, and $H^1(k_v,\hat{S}) = 0$ or $(\mathbf{Z}/2)^2$, according as d is or is not a square in k_v. The following proposition is proved in [2] (6.7 (iv)):

PROPOSITION 4. *Let O be the rational point of X/k which lies at infinity on the model X mentioned above. For any extension L/k, any 0-cycle of degree nought on X_L is rationally equivalent to some $P - O$, over X_L, with $P \in X(L)$.* □

This proposition and the computations of [4] (§IV) enable us to identify the image of the map $A_0(X) \xrightarrow{\Phi} H^1(k,S) = (k^*/NK^*)^2$ (here $K = k(\sqrt{d})$ and $N = N_{K/k}$) with the image of the map $X(k) \to (k^*/NK^*)^2$ which sends any P with $y^2 - dz^2 \neq 0$ to $(cl(x-e_1), cl(x-e_2))$. The same holds with k_v in place of k. In this case, $\ker\delta$ is obviously 0. Some support in favour of Conjecture A is provided by:

PROPOSITION 5. *Let X be a Châtelet surface over the number field k. The sequence*

$$0 \to A_0(X) \xrightarrow{\delta} \coprod_{\text{all } v} A_0(X_{k_v}) \xrightarrow{\sum j_v} H^1(k,\hat{S})^{\sim} \tag{4.10}$$

is exact if and only if the universal torseurs over X satisfy the Hasse principle.

Proof. (a) Take $\{(P_v - O)\} \in \coprod_{\text{all } v} A_0(X_{k_v})$ (proposition 4 allows us to consider only such family of 0-cycles). Assume $\{(P_v - O)\}$ goes to $0 \in H^1(k,\hat{S})^{\sim}$. According to (4.1), there exists $\alpha \in H^1(k,S)$ with $\mathrm{Res}_{k/k_v}(\alpha) = \Phi(P_v - O) = \mathfrak{T}(P_v) \in H^1(k_v,S)$ for each v—here $\mathfrak{T}$ denotes the universal torseur over X with trivial fibre at O (given by $c_i = 1$ in the above explicit model (4.9)). Denote by $\mathfrak{T}^{\alpha}$ the torseur obtained by twisting $\mathfrak{T}$ by $-\alpha$. We find $\mathfrak{T}^{\alpha}(P_v) = 0$, i.e., $\mathfrak{T}^{\alpha}$ has a k_v-point for all v. If the universal torseurs satisfy the Hasse principle, there exists $P \in X(k)$ with $\mathfrak{T}^{\alpha}(P) = 0 \in H^1(k,S)$ (take for P the projection of a k-point of $\mathfrak{T}^{\alpha}$). Hence (theorem (3 (i)) $\alpha = \Phi(P - O)$, and $(P - O) \in A_0(X)$ goes to $(P_v - O) \in A_0(X_{k_v})$ for each v (use theorem 2 (ii)).

(b) Assume now that the sequence (4.10) is exact. Any universal torseur over X is deduced from $\mathfrak{T}$ by twisting by some $-\alpha \in H^1(k,S)$, i.e., is of the form $\mathfrak{T}^{\alpha}$. If such a $\mathfrak{T}^{\alpha}$ has points in all completions of k, there exists for each v some $P_v \in X(k_v)$ with $\mathfrak{T}^{\alpha}(P_v) = 0$ in $H^1(k_v,S)$, hence $\mathrm{Res}_{k/k_v}(\alpha) = \Phi(P_v - O)$. By definition of the j_v, and by the exactness of (4.1), the point $\{(P_v - O)\} \in \coprod_{\text{all } v} A_0(X_{k_v})$ goes to $0 \in H^1(k,\hat{S})^{\sim}$. The vanishing of $Ш^1(k,S)$, the exactness of sequence (4.10) and proposition 4 now imply the existence of $P \in X(k)$ with $\alpha = \Phi(P - O) \in H^1(k,S)$. Therefore $\mathfrak{T}^{\alpha}(P) = 0$, and $\mathfrak{T}^{\alpha}$ has a k-point (above P). □

When $k = \mathbf{Q}$, any counterexample to the equivalent statements of proposition 5 would provide a counterexample to Schinzel's hypothesis (H) (cf. [4] p. 236).

The following Table gives some numerical evidence for the truth of both statements in proposition 5. Here $k = \mathbf{Q}$. We only give those $A_0(X_{k_v})$ which are non-zero. Write $G = H^1(k,\hat{S})^{\sim} = \mathbf{Z}/2 \times \mathbf{Z}/2$, $G_1 = \mathbf{Z}/2 \times 0 \subset G$, $G_2 = 0 \times \mathbf{Z}/2$

Table 1.

Equation	$A_0(X)$	$v \to$	$A_0(X_{k_v})$	$\sum_v j_v(A_0(X_{k_v}))$
$y^2 - 17z^2 = x(x-1)(x-17)$	0	17	G_1	G_1
$y^2 - 17z^2 = x(x-4)(x-17)$	0	17	G_1	G_1
$y^2 - 5z^2 = x(x-1)(x-5)$	0	2	G_2	G
		5	G_1	
$y^2 + z^2 = x(x-1)(x-5)$	$\mathbf{Z}/2$	∞	G_2	G
		2	G	
$y^2 + z^2 = x(x-1)(x-3)$	$(\mathbf{Z}/2)^2$	∞	G_2	G
		2	G	
		3	G_1	
$y^2 + z^2 = x(x-3)(x-7)$	$(\mathbf{Z}/2)^3$	∞	G_2	G
		2	G	
		3	G_3	
		7	G_1	
$y^2 + z^2 = x(x-3)(x-9)$	$(\mathbf{Z}/2)^3$	∞	G_2	G
		2	G	
		3	G	
$y^2 + z^2 = x(x-7)(x-19)$	$(\mathbf{Z}/2)^4$	∞	G_2	G
		2	G	
		3	G_2	
		7	G_3	
		19	G_1	
$y^2 + z^2 = x(x-1)(x-9)$	$\mathbf{Z}/2$	∞	G_2	G
		2	G_2	
		3	G_1	
$y^2 + z^2 = x(x-1)(x-17)$	$\mathbf{Z}/2$	∞	G_2	G_2
		2	G_2	

$\subset G$ and write G_3 for the diagonal $\mathbf{Z}/2$ in G. The equality $A_0(X_{k_v}) = G_i$ means that G_i is the image of the composed injection j_v:

$$A_0(X_{k_v}) \overset{\Phi}{\hookrightarrow} H^1(k_v, S) \xrightarrow{\simeq} H^1(k_v, \hat{S})^{\sim} \hookrightarrow H^1(k, \hat{S})^{\sim} = \mathbf{Z}/2 \times \mathbf{Z}/2 = G,$$

the map

$$A_0(X_{k_v}) \overset{\Phi}{\hookrightarrow} H^1(k_v, S) \simeq (k_v^*/NK_w^*)^2$$

being given by

$$(P - O) \mapsto (cl(x), cl(x-b))$$

for P a rational point on

$$y^2 - dz^2 = x(x-b)(x-c) \neq 0.$$

Let us just mention that the group structure on $A_0(X)$ and the rational points on the degenerate fibres are of great use in drawing up this Table.

(e) *A concluding note on weak approximation.*

The surjections $X(L) \to A_0(X_L)$ $(P \mapsto (P - O))$, which have been used to draw up Table 1, show that *all* surfaces in the Table *fail to satisfy weak approximation*! Indeed, this is so as soon as some $A_0(X_{k_v})$ is non-zero. Fix P_{v_0} in $X(k_{v_0})$ with $(P_{v_0} - O) \neq 0$ in $A_0(X_{k_{v_0}})$, and let $P_v = O$ at all other v's with $A_0(X_{k_v}) \neq 0$. The finite collection $\{P_v\}$ cannot be simultaneously approximated by points in $X(k)$: for a closed enough point $P \in X(k)$, the image of $\Phi(P - O) \in H^1(k, S)$ in $\coprod_{\text{all } v} H^1(k_v, S)$ would have exactly one non-vanishing component (apply theorem 2 (ii) at v_0), but this would contradict the exact sequence (4.1), since for Châtelet surfaces the restriction map $H^1(k, \hat{S}) \to H^1(k_v, \hat{S})$ is the identity as soon as $H^1(k_v, \hat{S}) \neq 0$ (which is certainly the case for $A_0(X_{k_v}) \neq 0$, cf. theorem 2 (ii)). Slightly subtler examples are possible: for the third surface, $X(\mathbf{Q})$ is not dense in $X(\mathbf{Q}_2)$ nor in $X(\mathbf{Q}_5)$.

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