Cyclic covers that are not stably rational

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Abstract

Using the methods developed by Kollár, Voisin, ourselves, Totaro, we prove that a cyclic cover of $\mathbb{P}^n_{\mathbb{C}}, n \geq 3$ of prime degree p, ramified along a very general hypersurface of degree mp is not stably rational if $n+1 \leq mp$. In small dimensions, we recover double covers of $\mathbb{P}^3_{\mathbb{C}}$, ramified along a quartic (Voisin), and double covers of $\mathbb{P}^3_{\mathbb{C}}$ ramified along a sextic (Beauville), and we also find double covers of $\mathbb{P}^4_{\mathbb{C}}$ ramified along a sextic. This method also allows one to produce examples over a number field.

Keywords: stable rationality, Chow group of zero-cycles, cyclic covers.

1 Introduction

A projective variety X over a field k is stably rational if for some n the variety $X \times \mathbb{P}^n_k$ is rational. There exist stably rational but not rational varieties [1]. In [10], Claire Voisin introduced a method to show that a variety X is not stably rational. It is based on an integral decomposition of the diagonal in the Chow group $CH^{dim\, X}(X \times X)$ and on a specialization argument. It enabled her to show that a double cover of $\mathbb{P}^3_{\mathbb{C}}$, branched along a very general surface of degree 4 is not stably rational. In [4], we considered the property of CH_0-universal triviality, which is equivalent to the integral decomposition of the diagonal for smooth projective varieties and which made the specialization method more flexible, in particular, allowing specializations over a discrete valuation ring of positive characteristic. We showed that for a very general choice of coefficients, a smooth complex quartic threefold is not stably rational.

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Definition 1.1. *Let $f : X \to Y$ be a projective morphism between varieties over a field k. We say that f is CH_0-universally trivial, if for any field extension L/k the map $f_* : CH_0(X_L) \to CH_0(Y_L)$ is an isomorphism. If $Y = Spec\,k$ and f is the structure morphism, then we say that X is CH_0-universally trivial.*

In particular, a smooth, projective stably rational variety is CH_0-universally trivial.

In [2, 3], A. Beauville considered the case of double covers of $\mathbb{P}^3_{\mathbb{C}}$, branched along a very general surface of degree 6, as well as the case of double covers of $\mathbb{P}^4_{\mathbb{C}}$ and $\mathbb{P}^5_{\mathbb{C}}$, branched along a very general hypersurface of degree 4. In [5], A. Kresch, B. Hassett, and Y. Tschinkel consider the case of certain conic bundles over surfaces.

B. Totaro [9] proved that a very general hypersurface of degree d in $\mathbb{P}^{n+1}_{\mathbb{C}}$ is not stably rational, provided $d \geq 2\lceil(n+2)/3\rceil$ and $n \geq 3$; in the proof he uses Kollár's results [7, 8] on double covers in characteristic 2 and a specialization property of the CH_0-universal triviality [4, Thm. 1.14] over a discrete valuation ring with function field of characteristic zero and residue field of positive characteristic. B. Totaro pointed out in [9] that the methods above also apply to more general covers: in this paper we continue investigations of cyclic covers in positive characteristic and show the following result (see Theorem 4.1):

Theorem 1.2. *Let X be a cyclic, degree p cover of $\mathbb{P}^n_{\mathbb{C}}$, with $n \geq 3$, branched along a very general hypersurface $f(x_0, \ldots, x_n) = 0$ of degree mp. Assume that $m(p-1) < n+1 \leq mp$. Then X is a Fano variety that is not stably rational.*

As in [9], we also obtain examples over number fields.

Note that for $n = 3, m = p = 2$ we get double covers of $\mathbb{P}^3_{\mathbb{C}}$, branched along a quartic (more general results are obtained in the work [10]), for $n = 3, m = 3, p = 2$ we obtain another proof of the results in [3].

For $n + 1 < mp$ Kollár proved that the covers we consider are not ruled [6]. Hovewer, this does not lead to results on stable rationality, since there exist stably rational varieties of dimension 3 that are not rational [1].

2 CH_0-universal triviality of singular varieties

Lemma 2.1. *Let k be an algebraically closed field and X an integral projective variety over k. Let $U \subset X$ be a nonempty Zariski open set. Then for* *any point $z \in X(k)$ there exists a cycle $\xi \in Z_0(U)$ rationally equivalent to z in $CH_0(X)$.*

Proof. If $X = C$ is an integral curve with normalization D, then the statement follows from the triviality of the Picard group of the semilocal rings of D. In the general case, it suffices to observe that there exists an integral curve C, such that $z \in C$ and $C \cap U \neq \emptyset$. $\square$

Lemma 2.2. *Let k be an algebraically closed field and X an integral projective k-rational variety. If X is smooth on the complement of a finite number of closed points, then X is universally CH_0-trivial.*

Proof. Let $\emptyset \neq U \subset X$ be an open subset isomorphic to an open subvariety of $\mathbb{P}^n_k$. Let F/k be some field extension. Any smooth point $z \in X_F(F)$ is rationally equivalent on X_F to a zero-cycle in $Z_0(U_F)$. Using the lemma above, the same holds for any k-point of X. As in [4, Lemma 1.5], we find that every cycle in $Z_0(X_F)$ is rationally equivalent to a cycle Nx, for some N and a (fixed) point $x \in U(k) \subset U(F) \subset X(F)$.

$\square$

Lemma 2.3. *Let k be an algebraically closed field and X a connected projective variety over k. If each reduced component of X is a k-rational variety with isolated singular points, then X is universally CH_0-trivial.*

Proof. It suffices to invoke the previous lemma and [4, Example 1.3.] $\square$

In the next section we apply Lemma 2.3 to exceptional divisors of resolutions of singularities. We also give a more general statement for the union of CH_0-universally trivial varieties. In this article, we will only need Lemma 2.3.

Lemma 2.4. *Let X be a projective reduced geometrically connected variety over a field k and $X = \bigcup_{i=1}^N X_i$ its decomposition into irreducible components. Assume that*

(i) each X_i is geometrically irreducible and CH_0-universally trivial;

(ii) each intersection $X_i \cap X_j$ is either empty or contains a 0-cycle z_{ij} of degree 1.

Then X is CH_0-universally trivial.

Proof. Let L/k be a field extension and $z \in CH_0(X_L)$ a cycle of degree 0. Since X is geometrically connected, the dual graph of geometric irreducible components contains a path through all its vertices: there exists a sequence of indices $i_1, \dots i_m$, $1 \leq i_j \leq N$ (where m could be greater than N), such that $\{i_1, \dots, i_m\} = \{1, \dots, N\}$ and $X_{i_j,L} \cap X_{i_{j+1},L}$ is not empty for any $1 \leq j \leq m$.

There exists a decomposition $z = \sum z_{i_j}$, where $z_{i_j} \in CH_0(X_{i_jL})$ is of degree d_j, $\sum d_j = 0$ (with an arbitrary choice of z_{i_j} on the intersections, some z_{i_j} could be trivial). Then $z_{i_1} = d_1 z_{i_1 i_2 L}$ in $CH_0(X_{i_1 L})$, so that $z_{i_1} + z_{i_2} = (d_1 + d_2) z_{i_2 i_3 L}$ in $CH_0(X_{i_1 L} \cup X_{i_2 L})$. Proceeding in this way, we obtain $z = \sum z_{i_j} = (\sum d_i) z_{i_{m-1}, i_m L} = 0$ in $CH_0(X_L)$. □

Remark. Condition (i) holds if there exists a resolution of singularities $\pi_i : \tilde{X}_i \to X_i$, such that $\tilde{X}_i$ is CH_0-universally trivial and all (scheme) fibres of π_i are CH_0-universally trivial (see. [4], Prop. 1.8.)

3 Cyclic covers and singularities

We first recall some properties of cyclic covers [7, Section V], [8].
Let p be a prime and $f(x_0, \dots, x_n)$ a homogeneous polynomial of degree mp with coefficients in a field k. *A cyclic cover of* $\mathbb{P}^n_k$, branched along $f(x_0, \dots, x_n) = 0$, is a subvariety of $\mathbb{P}(m, 1, 1, \dots, 1)$ given by

$$y^p - f(x_0, \dots, x_n) = 0.$$

If $char\, k = p$, such a cyclic cover is almost never smooth, its singularities correspond to the critical points of f.

Definition 3.1. *A critical point of a polynomial $g(x_1, \dots, x_n)$ over a field k is a point P, such that $\partial g/\partial x_i(P) = 0$, for all i. A critical point P of a polynomial g is nondegenerate if the determinant $|\frac{\partial^2 g}{\partial x_i \partial x_j}(P)|$ is nonzero. A critical point of a homogeneous polynomial $f(x_0, \dots, x_n)$ is a critical point of one of the polynomials $f(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$.*

If $char\, k = 2$ and n is odd, then all critical points of a polynomial $g \in k[x_1, \dots, x_n]$ are degenerate.

Definition 3.2. *Let char $k = 2$ and n odd. A critical point P of a polynomial $g(x_1, \dots, x_n)$ is almost nondegenerate, if*

$$\operatorname{length} \mathcal{O}_{\mathbb{A}^n,P}/(\partial g/\partial x_1(P), \dots, \partial g/\partial x_n(P)) = 2.$$

In order to investigate stable rationality we need some results on resolutions of singularities of cyclic covers (see also [6]).

Lemma 3.3. *Let k be an algebraically closed field of characteristic 2 and let*

$$X : \; y^2 = f(x_1, \dots, x_n)$$

be an affine cyclic cover singular at $P = (y, x_1, \dots, x_n) = (0, 0, \dots, 0)$, where $n \geq 2$ is even and $(0, \dots, 0)$ is a nondegenerate critical point of f. Let $\tilde{X} \to X$ be the blow-up of P. Then:

(i) Locally around P the cover X is given by the equation

$$y^2 = x_1x_2 + x_3x_4 + \ldots + x_{n-1}x_n + g(x_1, \dots, x_n),$$

where each monomial of $g(x_1, \dots, x_n)$ is of degree at least three.

(ii) E is universally CH_0-trivial and $\tilde{X}$ is smooth in a neighbourhood of E.

Proof. See Exercise V.5.6.6 in [7] for property (i) (and the proof of Theorem 3.7 below).

Let us prove (ii). It suffices to consider the following charts of the blow-up:

1. $x_i = yz_i$, $1 \leq i \leq n$, and $\tilde{X}$ is given by the equation

$$1 = z_1z_2 + z_3z_4 + \ldots + z_{n-1}z_n + \frac{1}{y^2} g(yz_1, \dots, yz_n)$$

   in affine coordinates $y, z_1, \dots z_n$.

   Note that the polynomial $\frac{1}{y^2} g(yz_1, \dots, yz_n)$ is divisible by y. The exceptional divisor E of the blow-up $\tilde{X} \to X$ is defined by the condition $y = 0$. We obtain the equation of E in this chart:

$$z_1z_2 + z_3z_4 + \ldots + z_{n-1}z_n = 1,$$

   which gives a smooth quadric. The variety $\tilde{X}$ is smooth at every point of E (it follows from the equation of $\tilde{X}$).

2. $y = wx_1$, $x_i = x_1 z_i$, $i \neq 1$. The exceptional divisor E is defined by the condition $x_1 = 0$. We obtain the following equations for $\tilde{X}$ and E in this chart:

$$w^2 = z_2 + z_3 z_4 + \ldots + z_{n-1} z_n + \frac{1}{x_1^2} g(x_1, x_1 z_2, \ldots, x_1 z_n)$$

and

$$z_2 = -(z_3 z_4 + \ldots + z_{n-1} z_n) + w^2,$$

respectively. Hence E is smooth and rational (since E is isomorphic to an affine space with coordinates $z_3, \ldots z_n, w$). The variety $\tilde{X}$ is smooth at every point of E.

We obtain that the exceptional divisor E is a smooth rational variety, hence, universally CH_0-trivial (see [4, Lemma 1.5]).

□

Lemma 3.4. *Let k be an algebraically closed field of characteristic 2 and let*

$$X:\ y^2 = f(x_1, \ldots, x_n)$$

be an affine cyclic cover, singular at $P = (y, x_1, \ldots, x_n) = (0, 0, \ldots, 0)$, where $n \geq 3$ is odd and $(0, \ldots, 0)$ is an almost nondegenerate critical point of f. Let $\tilde{X} \to X$ be the blow-up of P. Then:

(i) Locally around P the cover X is given by the equation

$$y^2 = a x_1^2 + x_2 x_3 + x_4 x_5 + \ldots + x_{n-1} x_n + g(x_1, \ldots, x_n),$$

where each monomial of $g(x_1, \ldots, x_n)$ is of degree at least three, and the coefficient b of the polynomial g at x_1^3 is nonzero.

(ii) E is universally CH_0-trivial and $\tilde{X}$ is smooth in a neighbourhood of E.

Proof. See Exercise V.5.7 in [7] for property (i) (and the proof of Theorem 3.7 below).

Let us prove (ii). It suffices to consider the following charts of the blow-up:

1. $x_i = y z_i$, $1 \leq i \leq n$, and $\tilde{X}$ is given by the equation

$$1 = a z_1^2 + z_2 z_3 + z_4 z_5 + \ldots + z_{n-1} z_n + \frac{1}{y^2} g(y z_1, \ldots, y z_n)$$

in affine coordinates $y, z_1, \ldots z_n$.

Note that the polynomial $\frac{1}{y^2}g(yz_1, \dots, yz_n)$ is divisible by y. The exceptional divisor E of the blow-up $\tilde{X} \to X$ is given by the condition $y = 0$. We obtain the equation of E in this chart:

$$az_1^2 + z_2z_3 + z_4z_5 + \dots + z_{n-1}z_n = 1.$$

For $a = 0$ we obtain the product of $\mathbb{A}^1$ and a smooth quadric. For $a \neq 0$ we obtain an irreducible quadric with one singularity at the point $z_i = 0, i > 1$, and $az_1^2 = 1$.

Then $\tilde{X}$ is smooth at every point of E: a singular point of $\tilde{X}$ should satisfy the conditions: $z_2 = \dots = z_n = 0$, $y = 0$, $bz_1^3 = 0$ and $az_1^2 = 1$, which is impossible.

2. $y = wx_2$, $x_i = x_2z_i$, $i \neq 2$. The exceptional divisor E is given by the condition $x_2 = 0$. In this chart, we obtain the following equations for $\tilde{X}$ and E:

$$w^2 = az_1^2 + z_3 + z_4z_5 + \dots + z_{n-1}z_n + \frac{1}{x_2^2}g(x_2z_1, x_2, x_2z_3 \dots, x_2z_n)$$

and

$$z_3 = -(az_1^2 + z_4z_5 + \dots + z_{n-1}z_n) + w^2$$

respectively. As above, the polynomial $\frac{1}{x_2^2}g(x_2z_1, x_2, x_2z_3 \dots, x_2z_n)$ is divisible by x_2. Hence E is smooth and rational (it is isomorphic to an affine space) and $\tilde{X}$ is smooth at every point of E.

3. $y = wx_1$, $x_i = x_1z_i$, $i \neq 1$. The exceptional divisor E is given by the condition $x_1 = 0$. In this chart, we obtain the following equations for $\tilde{X}$ and E:

$$w^2 = a + z_2z_3 + z_4z_5 + \dots + z_{n-1}z_n + \frac{1}{x_1^2}g(x_1, x_1z_2, \dots, x_1z_n)$$

and

$$a + z_2z_3 + z_4z_5 + \dots + z_{n-1}z_n - w^2 = 0$$

respectively. The variety E is an irreducible quadric with one singularity at the point $z_i = 0$ for all i and $a - w^2 = 0$. Since the coefficient of g at x_1^3 is nonzero, $\tilde{X}$ is smooth at every point of E (similarly as in 1).

We obtain that E is irreducible and has an isolated singular point $(c : 1 : 0 : \dots 0)$, with $c^2 = a$, and an open subvariety of E is smooth and rational. By Lemma 2.3, E is universally CH_0-trivial.

□

Lemma 3.5. *Let k be an algebraically closed field of characteristic $p > 2$ and*

$$X:\ y^p = f(x_1, \dots, x_n)$$

an affine cyclic cover singular at $P = (y, x_1, \dots, x_n) = (0, 0, \dots, 0)$, where $(0, \dots, 0)$ is a nondegenerate critical point of f. Assume that n is even. Then:

(i) Locally around P the cover X is given by the condition $y^p = x_1x_2 + x_3x_4 + \ldots + x_{n-1}x_n + g(x_1, \dots, x_n)$, where each monomial of $g(x_1, \dots, x_n)$ is of degree at least three.

(ii) After blowing up P and a finite number of isolated singular points above P one obtains a variety $\tilde{X}$ which is smooth in a neighbourhood of $\tilde{X}_P$ and such that the fibre $\tilde{X}_P$ is universally CH_0-trivial (but $\tilde{X}_P$ is not irreducible in general).

Proof. See Exercise V.5.6.6 in [7] for property (i) (and the proof of Theorem 3.7 below).

Let us prove (ii). Let $X' \to X$ be the blow-up of X at P. It suffices to consider the following charts:

1. $y = wx_1$, $x_i = x_1z_i$, $i \neq 1$. The exceptional divisor E is given by the condition $x_1 = 0$. In this chart, we obtain the following equations for X' and E:

$$x_1^{p-2}w^p = z_2 + z_3z_4 + \ldots + z_{n-1}z_n + \frac{1}{x_1^2}g(x_1, x_1z_2, \dots, x_1z_n)$$

(where the polynomial $\frac{1}{x_1^2}g(x_1, x_1z_2, \dots, x_1z_n)$ is divisible by x_1) and

$$z_2 = -(z_3z_4 + \ldots + z_{n-1}z_n).$$

Hence, E is smooth and rational and X' is smooth at every point of E.

2. $x_i = yz_i$, $1 \leq i \leq n$ and X' is given by the equation

$$y^{p-2} = z_1z_2 + z_3z_4 + \ldots + z_{n-1}z_n + \frac{1}{y^2}g(yz_1, \dots, yz_n).$$

The exceptional divisor E of the blow-up $X' \to X$ is given by the condition $y = 0$. In this chart, we obtain the equation of E:

$$z_1z_2 + z_3z_4 + \ldots + z_{n-1}z_n = 0.$$

This defines a quadric which is singular at $(z_1, \ldots, z_n) = (0, \ldots, 0)$.

Note that X' is singular at one point $P' = (y, z_1, \ldots, z_n) = (0, \ldots 0)$ if $p > 3$ and smooth if $p = 3$. If $p > 3$, let $X'' \to X'$ be the blow-up of X' at P'. Similarly, consider the following charts:

(a) $y = z_1 w$, $z_i = t_i z_1$, $i \neq 1$. In this chart, the exceptional divisor E' is given by the condition $z_1 = 0$. In this chart, we obtain the following equations for X'' and E':

$$w^{p-2} z_1^{p-4} = t_2 + t_3 t_4 + \ldots + t_{n-1} t_n + \frac{1}{z_1^2} h(z_1 w, z_1, z_1 t_2, \ldots, z_1 t_n),$$

$$t_2 + t_3 t_4 + \ldots + t_{n-1} t_n = 0,$$

respectively, where we denote $\frac{1}{y^2} g(yz_1, \ldots, yz_n) = h(y, z_1, \ldots, z_n)$. Note that the polynomial $h(z_1 w, z_1, z_1 t_2, \ldots, z_1 t_n)$ is divisible by z_1^3. We obtain that E' is smooth and rational: it is the product of $\mathbb{A}^1$ (corresponding to the coordinate w), and the variety fiven by

$$t_2 = -(t_3 t_4 + \ldots + t_{n-1} t_n).$$

Also X'' is smooth at every point of E' in this chart.

(b) $z_i = y t_i$, $1 \leq i \leq n$. In this chart, the exceptional divisor E' is given by the condition $y = 0$ and X'' is given by the condition

$$y^{p-4} = t_1 t_2 + t_3 t_4 + \ldots + t_{n-1} t_n + \frac{1}{y^4} g(y^2 t_1, \ldots, y^2 t_n).$$

The polynomial $\frac{1}{y^4} g(y^2 t_1, \ldots, y^2 t_n)$ is divisible by y and the exceptional divisor E' is a quadric given by the equation

$$t_1 t_2 + t_3 t_4 + \ldots + t_{n-1} t_n = 0.$$

Similarly, X'' is singular at one point $(y, t_1, \ldots, t_n) = (0, \ldots 0)$ if $p > 5$ and smooth if $p = 5$. If X'' is singular, we repeat the previous construction. After a finite number of such operations we obtain $\tilde{X} \to X$ with $\tilde{X}$ smooth at every point above P and such that all the exceptional divisors are rational varieties which are either smooth or singular at one isolated point, as described above.

From the description of the exceptional divisors and Lemma 2.3 we obtain that the fibre $\tilde{X}_P$ is a (connected) CH_0-universally trivial variety.

□

Lemma 3.6. *Let k be an algebraically closed field of characteristic $p > 2$ and let*

$$X : \ y^p = f(x_1, \ldots, x_n)$$

be an affine cyclic cover singular at $P = (y, x_1, \ldots, x_n) = (0, 0, \ldots, 0)$, where $(0, \ldots, 0)$ is a nondegenerate critical point of f. Assume that n is odd. Then:

(i) locally around P the cover X is given by the equation $y^p = x_1^2 + x_2x_3 + x_4x_5 + \ldots + x_{n-1}x_n + g(x_1, \ldots, x_n)$, where each monomial of $g(x_1, \ldots, x_n)$ has degree at least three.

(ii) after blowing up P and a finite number of isolated singular points above P one obtains a variety $\tilde{X}$ that is smooth in a neighbourhood of $\tilde{X}_P$ and such that the fibre $\tilde{X}_P$ is universally CH_0-trivial (but $\tilde{X}_P$ is not irreducible in general).

Proof. See Exercise V.5.6.6 in [7] for property (i) (and the proof of Theorem 3.7 below).

Let us prove (ii). Let $X' \to X$ be the blow-up of X at P. It suffices to consider the following charts:

1. $y = wx_1$, $x_i = x_1 z_i$, $i \neq 1$. In this chart, the exceptional divisor E is given by the condition $x_1 = 0$. We obtain the following equations for X' and E:

$$x_1^{p-2} w^p = 1 + z_2z_3 + z_4z_5 + \ldots + z_{n-1}z_n + \frac{1}{x_1^2} g(x_1, x_1z_2, \ldots, x_1z_n)$$

(where the polynomial $\frac{1}{x_1^2} g(x_1, x_1z_2, \ldots, x_1z_n)$ is divisible by x_1) and

$$1 + z_2z_3 + z_4z_5 + \ldots + z_{n-1}z_n = 0$$

respectively. Hence E is smooth and rational: it is a product of $\mathbb{A}^1$ (corresponding to the coordinate w) and a smooth quadric given by the equation $1 + z_2z_3 + z_4z_5 + \ldots + z_{n-1}z_n = 0$. Also X' is smooth at every point of E in this chart.

2. $y = wx_2$, $x_i = x_2 z_i$, $i \neq 2$. In this chart, the exceptional divisor E is given by the condition $x_2 = 0$. We obtain the following equations for X' and E:

$$x_2^{p-2} w^p = z_1^2 + z_3 + z_4z_5 + \ldots + z_{n-1}z_n + \frac{1}{x_2^2} g(z_1x_2, x_2, x_2z_3, \ldots, x_2z_n)$$

(where the polynomial $\frac{1}{x_2^2}g(z_1x_2, x_2, x_2z_3, \ldots, x_2z_n)$ is divisible by x_2) and

$$z_3 = -(z_1^2 + z_4z_5 + \ldots + z_{n-1}z_n),$$

respectively. Hence the variety E is smooth and rational (it is isomorphic to an affine space). Also X' is smooth at every point of E in this chart.

3. $x_i = yz_i$, $1 \leq i \leq n$ and X' is given by the equation

$$y^{p-2} = z_1^2 + z_2z_3 + z_4z_5 + \ldots + z_{n-1}z_n + \frac{1}{y^2}g(yz_1, \ldots, yz_n).$$

Note that the polynomial $\frac{1}{y^2}g(yz_1, \ldots, yz_n)$ is divisible by y.

The exceptional divisor E of the blow-up $X' \to X$ is given by the condition $y = 0$. In this chart we obtain the following equation of E:

$$z_1^2 + z_2z_3 + z_4z_5 + \ldots + z_{n-1}z_n = 0.$$

This defines a quadric with one singularity at the point $(z_1, \ldots, z_n) = (0, \ldots, 0)$.

Note that X' also has one singularity at the point $P' = (y, z_1, \ldots, z_n) = (0, \ldots, 0)$ if $p > 3$ and it is smooth in a neighbourhood of the exceptional divisor if $p = 3$. If $p > 3$, let $X'' \to X'$ be a blow-up of X' at the point P'. As in the prevdo lemma, we consider the following charts:

   (a) $y = z_1w$, $z_i = t_iz_1$, $i \neq 1$. The exceptional divisor E' is given by the condition $z_1 = 0$. We obtain the following equations of X'' and E':

$$w^{p-2}z_1^{p-4} = 1 + t_2t_3 + t_4t_5 + \ldots + t_{n-1}t_n + \frac{1}{z_1^2}h(z_1w, z_1, z_1t_2, \ldots, z_1t_n),$$

$$1 + t_2t_3 + t_4t_5 + \ldots + t_{n-1}t_n = 0,$$

   where the polynomial $\frac{1}{y^2}g(yz_1, \ldots, yz_n) = h(y, z_1, \ldots, z_n)$ is divisible by z_1^3. We obtain that E' is smooth and rational: it is a product of $\mathbb{A}^1$ (corresponding to the coordinate w) and a variety given by the equation $1 + t_2t_3 + t_4t_5 + \ldots + t_{n-1}t_n = 0$. Also X' is smooth at every point of E' in this chart.

   (b) $y = z_2w$, $z_i = t_iz_2$, $i \neq 2$. The exceptional divisor E' is given by the condition $z_2 = 0$. We obtain the following equations for X'' and E' respectively:

$$w^{p-2}z_2^{p-4} = t_1^2 + t_3 + t_4t_5 + \ldots + t_{n-1}t_n + \frac{1}{z_2^2}h(z_2w, z_2t_1, z_2, z_2t_3, \ldots, z_2t_n),$$

$$t_1^2 + t_3 + t_4t_5 + \ldots + t_{n-1}t_n = 0,$$

where we denote $\frac{1}{y^2}g(yz_1, \ldots, yz_n) = h(y, z_1, \ldots, z_n)$. We obtain that E' is smooth and rational, and that X'' is smooth at every point of E' in this chart.

(c) $z_i = yt_i$, $1 \leq i \leq n$, E' is given by the condition $y = 0$ and X'' is given by the equation

$$y^{p-4} = t_1^2 + t_2t_3 + t_4t_5 + \ldots + t_{n-1}t_n + \frac{1}{y^4}g(y^2t_1, \ldots, y^2t_n).$$

Note that the polynomial $\frac{1}{y^4}g(y^2t_1, \ldots, y^2t_n)$ is divisible by y. The exceptional divisor E' is the quadric

$$t_1^2 + t_2t_3 + t_4t_5 + \ldots + t_{n-1}t_n = 0.$$

Similarly, in this chart X'' is singular in a neighbourhood of E at a single point $(y, t_1, \ldots, t_n) = (0, \ldots 0)$ if $p > 5$ and smooth if $p = 5$. If X'' is singular, we repeat the previous construction. After a finite number of such operations we obtain a morphism $\tilde{X} \to X$ with $\tilde{X}$ smooth at every point above P and such that all the exceptional divisors are rational varieties which are either smooth or singular at one isolated point, as described above.

From the description of the exceptional divisors and Lemma 2.3 we obtain that the fibre $\tilde{X}_P$ is a (connected) CH_0-universally trivial variety.

□

The following statement provides the key nontrivial invariants of cyclic covers.

Recall that the coefficients of polynomials $f \in k[x_0, \ldots, x_n]$ of a given degree are parametrized by points of some projective space. A general choice of coefficients of f means that we consider coefficients in some nonempty open subset (in Zariski topology) of this projective space.

Theorem 3.7. *Let k be an algebraically closed field of characteristic p and $f(x_0, \ldots, x_n)$ a homogeneous polynomial of degree $mp \geq n+1$, with $n \geq 3$. For a general choice of coefficients of f one has the following properties:*

(i) All critical points of f are nondegenerate if $p > 2$ or $p = 2$ and n is even.

(ii) *All critical points of f are almost nondegenerate it $p = 2$ and n is odd.*

(iii) *If $\tilde{X} \to X$ is a resolution of singularities of X, obtained by successive blow-ups of singular points, then the morphism $\tilde{X} \to X$ is CH_0-universally trivial, $H^0(\tilde{X}, \Lambda^{n-1}\Omega_{\tilde{X}}) \neq 0$, and $\tilde{X}$ is not CH_0-universally trivial.*

Proof. Properties (i) and (ii) follow from [7, Section V, Exercises 2.7 and 5.11]. Assume that $P = (b, a_1, \ldots, a_n)$ is a critical point of f. After a linear change of variables $y - c, x_i - a_i$, where $c^p = f(P)$ (k is algebraically closed) we may assume that $P = (0, \ldots, 0)$. Then we can decompose f as a sum of linear terms, quadratic terms, and terms of higher degrees:

$$f = f_1(x_1, \ldots x_n) + f_2(x_1, \ldots x_n) + f_3(x_1, \ldots x_n).$$

Since P is a critical point, we have $f_1 = 0$. Since k is algebraically closed, any quadratic form over k could be written in a diagonal form as a sum of squares (if $char(k) \neq 2$), or as a sum of $\sum x_i y_i$ (regular part) and a sum of squares. Hence one easily verifies that the condition that P is a nondegenerate (resp. almost nondegenerate) point corresponds to the decomposition in Lemmas 3.3, 3.5, 3.6 (resp. 3.4), which holds for a general choice of coefficients of f (see [7, Section V, Exercice 5.6.6.3]).

In order to prove (*iii*), as in the arguments of B. Totaro [9], we use [7, Thm. V.5.11] for $\mathbb{P}^n_k$, with $n \geq 3$, and $L^p = \mathcal{O}_{\mathbb{P}^n}(mp)$. We obtain:

1. $K_{\mathbb{P}^n} \otimes L^p = \mathcal{O}_{\mathbb{P}^n}(mp - n - 1)$,
2. if $mp \geq 4$, the map $H^0(\mathbb{P}^n, L^p) \to \mathcal{O}_{\mathbb{P}^n}/m_x^4 \otimes L^p$ is surjective for every closed point $x \in X$.

It follows from [7, Exercise V.5.7] (see also [7, Thm. V.5.11], [8, Thm. 4.4]), that for a general choice of $f \in H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n}(mp))$ (in particular, f satisfies (*i*) and (*ii*)), if $q : X \to \mathbb{P}^n_k$ is a cyclic, degree p cover of $\mathbb{P}^n_k$, branched along a hypersurface $f = 0$, and $\pi : \tilde{X} \to X$ is a resolution of singularities of X obtained by successive blow-ups of singular points, then $\pi^* q^* \mathcal{O}_{\mathbb{P}^n}(mp - n - 1)$ is a subsheaf of $\Lambda^{n-1}\Omega_{\tilde{X}}$. In particular, if $mp - n - 1 \geq 0$, then $H^0(\tilde{X}, \Lambda^{n-1}\Omega_{\tilde{X}}) \neq 0$.

B. Totaro proved [9, Lemma 2.2] that if $\tilde{X}$ is CH_0-universally trivial, then $H^0(\tilde{X}, \Lambda^{n-1}\Omega_{\tilde{X}}) = 0$. Lemmas 3.3, 3.4, 3.5, 3.6 imply that all the fibres of the map $\tilde{X} \to X$ are CH_0-universally trivial, hence the map $\tilde{X} \to X$ is CH_0-universally trivial as well (see [4, Prop. 1.8]). $\square$

Remark. If $n+1 > mp - m$ and X is normal (in particular, this holds if f has only isolated critical points), then X is a Fano variety: the line bundle $-K_X$ is ample (see [8, Prop. 4.14]).

4 Cyclic covers that are not stably rational

Theorem 4.1. *Let X be a cyclic, degree p cover of $\mathbb{P}^n_{\mathbb{C}}$, with $n \geq 3$, branched along a very general hypersurface $f(x_0, \dots, x_n) = 0$ of degree mp. Assume that $m(p-1) < n+1 \leq mp$. Then X is a Fano variety that is not stably rational. There exists a cyclic, degree p cover, branched along a hypersurface of degree mp, that is not stably rational and is defined over a number field.*

Proof. Let $Y : y^p = f(x_0, \dots, x_n)$ be a cover satisfying the conditions of Theorem 3.7. One may choose Y so that the coefficients of f are defined over some finite field $\mathbb{F}_q$. Since in Theorem 3.7, condition $f = 0$ defines a very general hypersurface, we may assume that the hypersurface $f = 0$ is smooth over $\mathbb{F}_q$. Hence there exists a polynomial H of degree mp with coefficients in some number field, such that f is the reduction of H modulo p, and the cover $X : y^p = H(x_0, \dots, x_n)$ is smooth. Since X degenerates to Y and the resolution $Y' \to Y$, constructed in Lemmas 3.3, 3.4, 3.5, 3.6 and Theorem 3.7, is CH_0-universally trivial, we obtain that $X_{\mathbb{C}}$ is not a CH_0-universally trivial variety by Theorem 3.7 and [4, Theorem 1.14(iii)]. Hence X is not stably rational. Moreover, from the construction we obtain an example over a number field. Since the coefficients of the polynomials of degree mp are parametrized by an irreducible variety (in fact, a projective space), by [4, Thm. 2.3], for a very general choice of such polynomials, the corresponding degree p covers of $\mathbb{P}^n_{\mathbb{C}}$ are not CH_0-universally trivial. □

Remark. We obtain that a cyclic, degree p cover X of the projective space $\mathbb{P}^n_{\mathbb{C}}$, branched along a very general hypersurface $f(x_0, \dots, x_n) = 0$ of degree mp with $n+1 \leq mp$ is not CH_0-universally trivial. As in [4], this implies that X is not retract rational. Recall that a stably rational variety over $\mathbb{C}$ is retract rational, but it is still unknown if these notions are different.

Examples.

1. For $p = 2, n = 3$, and $mp = 6$, we obtain another proof of the results of A. Beauville [3].

2. For $n = 3, m = p = 2$, we get double covers of $\mathbb{P}^3_{\mathbb{C}}$ branched along a quartic (more general results were obtained by Claire Voisin [10]).

3. For $p = 2, n = 4$, and $mp = 6, 8$, we obtain that a double cover of $\mathbb{P}^4_{\mathbb{C}}$ branched along a very general hypersurface of degree 6 or 8 is not stably rational.

4. For $p = 2, n = 5$, we obtain examples for $2m = 8, 10$.

5. For $p = 3, n = 4$, and $mp = 6$, we obtain an example of a Fano variety of dimension 4, that is not stably rational: a degree 3 cover, branched along a very general hypersurface of degree 6.

6. The case of double covers of $\mathbb{P}^n_{\mathbb{C}}$, $n = 4, 5$, branched along a quartic (A. Beauville [2]), is not contained in Theorem 4.1.

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