Higher reciprocity laws and rational points

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Conference on Algebraic Geometry

Amsterdam, 8th-12th July, 2013

The classical set-up

K number field, Ω the set of place of K, K_v completion of K at the place v

$\mathrm{Br}\ K_v \hookrightarrow \mathbb{Q}/\mathbb{Z}$, isomorphism if v finite place

The reciprocity law in class field theory
There is a *complex*

$$0 \to \mathrm{Br}\ K \to \oplus_{v \in \Omega} \mathrm{Br}\ K_v \to \mathbb{Q}/\mathbb{Z} \to 0$$

which as a matter of fact is an exact sequence.

G a connected linear algebraic group over K

$$Ш^1(K,G) := \mathrm{Ker}[H^1(K,G) \to \prod_{v \in \Omega} H^1(K_v,G)]$$

This is the set of isomorphism classess of principal homogeneous spaces (torsors) E/K under G with $E(K_v) \neq$ for all $v \in \Omega$

Theorem (Kneser, Harder, Chernousov)
(i) If G is semisimple and simply connected, then $Ш^1(K,G) = 0$: the Hasse principle holds for torsors under G.
(ii) If Z/K is a projective variety which is a homogeneous space of a connected linear algebraic group, the Hasse principle holds for rational points on Z.

Brauer-Manin pairing

X/K smooth, projective, geometrically connected. Let $X(\mathbb{A}_K) = \prod_v X(K_v)$. Let $X(\mathbb{A}_K)^{\mathrm{Br}\ X}$ be the left kernel of the pairing

$$X(\mathbb{A}_K) \times (\mathrm{Br}\ X / \mathrm{Br}\ K) \to \mathbb{Q}/\mathbb{Z}$$

$$(\{P_v\}, A) \mapsto \sum_{v \in \Omega} A(P_v) \in \mathbb{Q}/\mathbb{Z}$$

Reciprocity obstruction to the local-global principle (Manin 1970) :
The reciprocity law implies

$$X(K) \subset X(\mathbb{A}_K)^{\mathrm{Br}\ X} \subset X(\mathbb{A}_K)$$

Theorem (Sansuc, 1981)
E/K torsor under G/K connected linear, $E \subset X$ a smooth compactification. Then $X(K)$ is dense in $X(\mathbb{A}_K)^{\mathrm{Br}\ X}$.

Corollary (Sansuc) *In each of the following cases :*
(i) G is an adjoint group
(ii) G is absolutely almost simple
(iii) The underlying variety of G is K-rational
we have $Ш^1(K, G) = 0$ and weak approximation holds for G.

Indeed, under these assumtions, $\mathrm{Br}\ X = \mathrm{Br}\ K$.

Tchebotarev's theorem yields $\text{Ш}^1(K, \mathbb{Z}/n) = 0$. However :

There exists $G = T$ a K-torus with $\text{Ш}^1(K, T) \neq 0$ (Hasse)
There exists μ a finite Galois module with $\text{Ш}^1(K, \mu) \neq 0$
There exists μ a finite Galois module with $\text{Ш}^2(K, \mu) \neq 0$
There exists a semisimple K-group G with $\text{Ш}^1(K, G) \neq 0$ (Serre)

Sansuc 1981 : The examples with T et G may be interpreted in terms of the Brauer-Manin obstruction.

Set-up for this talk

$\mathcal{X}$ regular connected scheme of dimension 2
K field of rational functions on $\mathcal{X}$
R local *henselian* integral domain, k its residue field
$p : \mathcal{X} \to \mathrm{Spec}\, R$ projective, surjective morphism

local case $\mathrm{dim} R = 2$, $p : \mathcal{X} \to \mathrm{Spec}\, R$ birational. $0 \in \mathrm{Spec}\, R$ closed point, $\mathcal{X}_0/k$ special fibre.
Example : $R = k[[x, y]]$, $\mathcal{X}$ blow-up of $\mathrm{Spec}\, R$ at 0.

semi-global case R discrete valuation ring, F field of fractions of R, generic fibre $\mathcal{X}_\eta/F$ smooth, projective, geometrically connected curve
Example : $K = k((t))(x)$, $\mathcal{X} = \mathbb{P}^1_{k[[t]]}$ or blow-up at points of the special fibre.

Ω set of discrete, rank one, valuations on K, T_v henselization of T at v, K_v field of fractions of T_v. The valuations are centered on $\mathcal{X}$, for $v \in \Omega$ we have $R \subset T_v$.

Theorem (Grothendieck, Artin 1968; ...) *Both in the local and in the semi-global case,*

$$\mathrm{Br}\ K \hookrightarrow \prod_{v \in \Omega} \mathrm{Br}\ K_v.$$

There is no such theorem in a global situation. Let Y be a smooth projective surface over the complex field and $K = \mathbb{C}(Y)$ be its function field. Then

$$Ker[\mathrm{Br}\ K \hookrightarrow \prod_{v \in \Omega} \mathrm{Br}\ K_v] = \mathrm{Br}\ (Y),$$

and it is easy to produce examples where $\mathrm{Br}\ (Y)$ is infinite.

Let us go back once and for all to the local or semi-global situation.

G a linear algebraic group over K.

$$Ш^1(K,G) := \mathrm{Ker}[H^1(K,G) \to \prod_{v\in\Omega} H^1(K_v,G)]$$

Question. Let G/K be a connected linear algebraic group. Do we have $Ш^1(K,G) = 0$?

Question. Let μ/K be a finite Galois module. For $i = 1, 2...$, do we have $Ш^i(K,\mu) = 0$?

The local case, $k = \overline{k}$

• G/K connected linear. If G is simply connecred, or adjoint, or *K-rational*, then $Ш^1(K, G) = 0$ (CT, Gille, Parimala 2004 for G semisimple; Borovoi, Kunyavskiĭ 2004)

• $Ш^1(K, \mathbb{Z}/2) \neq 0$ possible (Jaworski 2001)

The question $Ш^2(K, \mu) = 0$? was already mentioned in CTGiPa 2004.

The semi-global case

The work of Harbater, Hartmann and Krashen (2009-present), based on a new theory of field patching (Harbater, Hartmann 2007)

$\mathcal{X}$ regular connected scheme of dimension 2, K its field of functions
R a complete DVR, t a uniformizing parameter, residue field k nearly arbitrary
$p : \mathcal{X} \to \mathrm{Spec}\, R$ a projective flat morphism.
$\mathcal{X}_0/k$ the special fibre.
A finite set T of points $P \in \mathcal{X}_0$, including all singular points of the reduced special fibre.
$\mathcal{X}_0 \setminus T = \cup_{i \in I} U_i$ with $U_i \subset \mathcal{X}_0$ Zariski open
Given an open $U \subset \mathcal{X}_0$, one defines R_U to be the completion along t of the ring of rational functions on $\mathcal{X}$ which are defined at each point of U. This is an integral domain, one lets K_U be its fraction field.
Given a point $P \in T$, one lets K_P denote the field of fractions of the completed local ring of P on $\mathcal{X}$.

Theorem (Harbater, Hartmann, Krashen 2009)
Let notation be as above.
Let G/K be a connected linear algebraic group. Let E be a homogeneous space of G such that for any field L containing K, the group $G(L)$ acts transitively on $E(L)$.
If G is K-rational, i.e. if its function field is purely transcendental over K, then the following local-global principle holds :
If each $E(K_U)$ and each $E(K_P)$ is not empty, then $E(K)$ is not empty.

The transitivity hypothesis is satisfied in the following two cases :
(i) E is a principal homogeneous space (torsor) of G
(ii) E/K is a projective variety.

In a number of cases, one may pass from the local-global theorems with respect to the K_U's and K_P's to local-global theorems with respect to the completions K_v with respect to the discrete valuations of rank one on K.

• Local-global principle for isotropy of quadratic forms of rank at least 3 (CT-Parimala-Suresh 2009)

• Theorem (Harbater, Hartmann, Krashen 2012)
Let notation be as above. Assume R is equicharacteristic. Let $m > 0$ be an integer invertible in R.
Then for any positive integer $n > 1$, the natural map

$$H^n(K, \mu_m^{\otimes n-1}) \to \prod_{v \in \Omega} H^n(K_v, \mu_m^{\otimes n-1})$$

is injective. (For $n > 3$, the proof uses the Bloch-Kato conjecture, now a theorem of Rost and Voevodsky.)

• G/K connected, linear, *K-rational*, R complete DVR, $k = \overline{k}$, then $Ш^1(K, G) = 0$ (Harbater, Hartmann, Krashen 2012, via CT-Gille-Parimala 2004)

However

• $\text{Ш}^1(K, \mathbb{Z}/2) \neq 0$ possible in the semi-global case.

In other words, an element in K may be a square in each completion K_v without being a square in K.

This is a reinterpretation (CT, Parimala, Suresh 2009) of a computation by Shuji Saito 1985.

Main theorem of the talk (CT, Parimala, Suresh, jan. 2013)

Theorem

Let $k = \mathbb{C}$. *Over* $K = \mathbb{C}((x))(t)$, *and over* $K = \mathbb{C}((x, y))$,

(a) there exists a connected, linear algebraic K*-group* G *with* $Ш^1(K, G) \neq 0$;

(b) there exists a finite Galois module μ/K *with* $Ш^2(K, \mu) \neq 0$.

For (a), we have examples with G a K-torus and with G a semi-simple K-group.

(Known) reduction steps

By Weil restriction of scalars, it is enough to prove $Ш^1(K, G) \neq 0$ and $Ш^2(K, \mu) \neq 0$ for K the field of functions of a suitable curve over $\mathbb{C}((t))$ and for a suitable finite extension of $\mathbb{C}((x, y))$.

It is enough to produce an example with T a K-torus, indeed an example on one of the following lines generates an example on the following line (over a number field, the analogue occurs in Serre's book Cohomologie galoisienne)

• An example of a K-torus T with $Ш^1(K, T) \neq 0$
• An example of a finite Galois module μ with $Ш^2(K, \mu) \neq 0$
• An example of a connected semisimple group G/K with $Ш^1(K, G) \neq 0$.

Which obstruction to the local-global principle ?

Local or semi-global situation, $\mathcal{X}$ a regular surface, $n \in O_{\mathcal{X}}^{\times}$, we assume $\mathbb{Z}/n \simeq \mu_n$.
Reciprocity law : Bloch-Ogus *complex*

$$0 \to H^2(K, \mathbb{Z}/n) \xrightarrow{\{\partial_\gamma\}} \oplus_{\gamma \in \mathcal{X}^{(1)}} H^1(\kappa(\gamma), \mathbb{Z}/n) \xrightarrow{\{\partial_{\gamma,x}\}} \oplus_{x \in \mathcal{X}^{(2)}} \mathbb{Z}/n \to 0$$

The homology of this complex (under the Gersten conjecture) :
• degree 0 : $\mathrm{Br}\ \mathcal{X}[n] \simeq \mathrm{Br}\ \mathcal{X}_0[n]$, zero if $k = \overline{k}$,
• degree 1 : subgroup of $H^3(\mathcal{X}, \mathbb{Z}/n) \simeq H^3(\mathcal{X}_0, \mathbb{Z}/n)$, zero if $k = \overline{k}$,
• degree 2 : $CH_0(\mathcal{X})/n$ zero, indeed $CH_0(\mathcal{X}) = 0$

"Analogue" of the class field theory exact sequence

Reciprocity obstruction

Z/K smooth, projective, geometrically connected
$\alpha \in \mathrm{Br}\ Z[n]$, $\gamma \in \mathcal{X}^{(1)}$
The composite map

$$\sigma_\alpha : Z(K_\gamma) \xrightarrow{\alpha} \mathrm{Br}\ K_\gamma[n] \xrightarrow{\partial_\gamma} H^1(\kappa(\gamma), \mathbb{Z}/n)$$

vanishes for almost all $\gamma \in \mathcal{X}^{(1)}$.
The composite map

$$\rho_\alpha : \prod_{\gamma \in \mathcal{X}^{(1)}} Z(K_\gamma) \xrightarrow{\sigma_\alpha} \oplus_\gamma H^1(\kappa(\gamma), \mathbb{Z}/n) \xrightarrow{\{\partial_{\gamma,x}\}} \oplus_{x \in \mathcal{X}^{(2)}} \mathbb{Z}/n$$

vanishes on the diagonal image of $Z(K)$ in $\prod_{\gamma \in \mathcal{X}^{(1)}} Z(K_\gamma)$.

Let

$$[\prod_{\gamma} Z(K_{\gamma})]^{\mathrm{Br}\ Z} = \bigcap_{\alpha \in \mathrm{Br}\ Z} \mathrm{Ker} \rho_{\alpha}.$$

Reciprocity obstruction

$$Z(K) \subset [\prod_{\gamma} Z(K_{\gamma})]^{\mathrm{Br}\ Z} \subset \prod_{\gamma} Z(K_{\gamma}).$$

This is an analogue of the Brauer-Manin obstruction over number fields.

In the local and in the semi-global case, we shall produce $\mathcal{X}/R$, a K-torus T, a torsor E of T, a smooth k-compactification Z of E with $\prod_{\gamma} Z(K_{\gamma}) \neq \emptyset$ and $[\prod_{\gamma} Z(K_{\gamma})]^{\mathrm{Br}\ Z} = \emptyset$, hence $Z(K) = \emptyset$.

Let $a, b, c \in K^{\times}$.
Let T be the K-torus T with equation

$$(x_1^2 - ay_1^2)(x_2^2 - by_2^2)(x_3^2 - aby_3^2) = 1.$$

Let E/k be the torsor under T defined by

$$(x_1^2 - ay_1^2)(x_2^2 - by_2^2)(x_3^2 - aby_3^2) = c.$$

Let Z be a smooth K-compactification of E. Then $\mathrm{Br}\ Z/\mathrm{Br}\ K \subset \mathbb{Z}/2$, a generator being given by the class of the quaternion algebra $\alpha = (x_1^2 - ay_1^2, b)$. As $E(K_\gamma)$ is dense in $Z(K_\gamma)$, it is enough to evaluation α on $E(K_\gamma)$.

For $\{P_\gamma\} \in \prod_\gamma E(K_\gamma)$, we must evaluate

$$\sum_{x \in \gamma} \partial_{\gamma,x} \partial_\gamma(\alpha(P_\gamma)) \in \oplus_{x \in \mathcal{X}^{(2)}} \mathbb{Z}/2.$$

We now assume $k = \overline{k}$. The residue fields $\kappa(\gamma)$ then have cohomological dimension 1, the fields K_γ are similar to "local fields".
Hensel's lemma gives a criterion for $E(K_\gamma) \neq \emptyset$. For each $\gamma \in \mathcal{X}^{(1)}$, the image of the composite map (evaluation of α, then residue)

$$E(K_\gamma) \to \mathrm{Br}\ K_\gamma \to \kappa(\gamma)^\times / \kappa(\gamma)^{\times 2}$$

is an explicit set consisting of at most 2 elements.

Proposition. *Let R be a regular semilocal ring with 3 maximal ideals m_j, $j = 1, 2, 3$, with $m_1 = (\pi_2, \pi_3)$ etc. The elements π_i vanish on the sides of a triangle the vertices of which are the m_j's. Set $a = \pi_2\pi_3$, $b = \pi_3\pi_1$, $c = \pi_1\pi_2\pi_3$. Let E be defined by*

$$(x_1^2 - ay_1^2)(x_2^2 - by_2^2)(x_3^2 - aby_3^2) = c.$$

Then $E(K) = \emptyset$.

Proof. Let R_i be the henselisation of R at π_i, let K_i be its fraction field and κ_i its residue field.

One computes the composite map

$$\prod_i E(K_i) \xrightarrow{\alpha} \oplus_i \mathrm{Br}\ K_i[2] \xrightarrow{\{\partial_i\}} \oplus_i \kappa_i^\times/\kappa_i^{\times 2} \xrightarrow{\{\partial_{i,j}\}} \oplus_{j=1}^3 \mathbb{Z}/2.$$

The image of $E(K_1)$ consists of $(0,0,1)$ and $(0,1,0)$
The image of $E(K_2)$ consists of $(0,0,0)$ and $(1,0,1)$
The image of $E(K_3)$ consists of $(0,0,0)$ and $(1,1,0)$
None of the vertical sums of triplets equals $(0,0,0)$.
For the other points $\gamma \in \mathcal{X}^{(1)}$, the image of $E(K_\gamma) \to \kappa_\gamma^\times/\kappa_\gamma^{\times 2}$ is equal to 1, hence does not contribute to the sums

$$\sum_{m_j \in \gamma} \partial_{\gamma,m_j}\partial_\gamma(\alpha(P_\gamma)) \in \oplus_j \mathbb{Z}/2.$$

Thus $(0,0,0)$ does not lie in the image of the composite map

$$\prod_i E(K_i) \xrightarrow{\alpha} \oplus_i \mathrm{Br}\ K_i[2] \xrightarrow{\{\partial_i\}} \oplus_i\ \kappa_i^\times/\kappa_i^{\times 2} \xrightarrow{\{\partial_{i,j}\}} \oplus_j\ \mathbb{Z}/2$$

Reciprocity on $\mathcal{X} = \operatorname{Spec} R$ then implies $E(K) = \emptyset$.

"Semi-global" example

Let $R = \mathbb{C}[[t]]$. Let $\mathcal{X}/R$ be the regular proper minimal model (Kodaira, Néron) of the elliptic curve with affine equation

$$y^2 = x^3 + x^2 + t^3.$$

Its special fibre $\mathcal{X}_0$ consists of 3 lines L_i building up a triangle. One then chooses elements $\pi_i \in K^\times$ with $\mathrm{div}(\pi_i) = L_i + D_i$ in a reasonable fashion, so as to ensure that none of the D_i's contains a vertex of the triangle and that at any point $x \in \mathcal{X}^{(2)}$ one at least of the π_i's is invertible.
Set $a = \pi_2\pi_3$, $b = \pi_3\pi_1$, $c = \pi_1\pi_2\pi_3$. Let E be given by the equation

$$(x_1^2 - ay_1^2)(x_2^2 - by_2^2)(x_3^2 - aby_3^2) = c.$$

Then $E(K_v) \neq \emptyset$ for each $v \in \Omega$, but $E(K) = \emptyset$.

"Local" example

Let

$$R = \mathbb{C}[[x, y, z]]/(xyz + x^4 + y^4 + z^4)$$

and let $\mathcal{X} \to \operatorname{Spec} R$ be a minimal desingularization.
Then take E/K to be given by the equation

$$(X_1^2 - yzY_1^2)(X_2^2 - xzY_2^2)(X_3^2 - xyZ_3^2) = xyz(x + y + z).$$

With some more effort, one produces a semi-global example
• $R = \mathbb{F}[[t]]$, $\mathbb{F}$ a finite field
or
• R the ring of integers of a p-adic field
and $\mathcal{X}$ a proper regular R-curve, K its function field, and E a torsor of a K-torus of the above type.

Harari and Szamuely have very recently produced a duality theory for tori over such fields K which only involves the discrete valuation rings corresponding to the closed points of the generic fibre of $\mathcal{X} \to \mathrm{Spec}\,(R)$. They use the group $H^3_{nr}(K(\mathcal{X})/K, \mathbb{Q}/\mathbb{Z}(2))$ rather than the Brauer group $H^2_{nr}(K(\mathcal{X})/K, \mathbb{Q}/\mathbb{Z}(1))$.

Both in the local and the semi-global case, the following problems remain open.
In the special case where the residue field k is a finite field, they were proposed as conjectures by CT, Parimala, Suresh 2009.

Problem. *Let G/K be a semisimple connected K-group. If G is simply connected, is $Ш^1(K, G) = 0$?*

When the residue field k is finite, this has been shown for many types of groups (Yong Hu ; R. Preeti). There is some relation with the Rost invariant and a local-global principle of K. Kato.

Problem. *Does the local-global principle hold for projective homogeneous spaces of connected linear algebraic K-groupes ?*

For quadrics, this was proved by CT, Parimala, Suresh 2009, as a consequence of the results of Harbater, Hartmann, Krashen.

In analogy with results by Sansuc and by Borovoi over global fields, one may further ask if the obstruction to the local-global principle used in our examples is the only obstruction to the local-global principle.

Here is one special case where we can prove such a result.

Theorem. *Let us consider either the local or the semi-global set up* $p : \mathcal{X} \to \operatorname{Spec} R$. *Assume* R *is a* k*-algebra,* $\operatorname{char}(k) = 0$, *and* $k = \overline{k}$. *Let* $a, b, c \in K^\times$. *Let* E *be the* K*-variety defined by*

$$(X_1^2 - aY_1^2)(X_2^2 - bY_2^2)(X_3^2 - abZ_3^2) = c$$

and let Z *be a smooth* K*-compactification of* E. *Let* $\alpha = (X_1^2 - aY_1^2, b) \in \mathrm{Br}\, Z$. *Assume that the union of the supports of the divisors of* a, b *and* c *on* $\mathcal{X}$ *is a divisor with normal crossings. If there exists a family* $\{P_\gamma\} \in \prod_\gamma E(K_\gamma)$ *such that the family* $\{\partial_\gamma(\alpha(P_\gamma))\}$ *is in the kernel of*

$$\oplus_{\gamma \in \mathcal{X}^{(1)}} H^1(\kappa(\gamma), \mathbb{Z}/2) \xrightarrow{\{\partial_{\gamma,x}\}} \oplus_{x \in \mathcal{X}^{(2)}} \mathbb{Z}/2,$$

then $E(K) \neq \emptyset$.

Proof (sketch)

Since $k = \overline{k}$, the complex

$$0 \to \mathrm{Br}\ K[2] \stackrel{\{\partial_\gamma\}}{\longrightarrow} \oplus_{\gamma \in \mathcal{X}^{(1)}} H^1(\kappa(\gamma), \mathbb{Z}/2) \stackrel{\{\partial_{\gamma,x}\}}{\longrightarrow} \oplus_{x \in \mathcal{X}^{(2)}} \mathbb{Z}/2 \to 0$$

is exact, and $\partial_\gamma : \mathrm{Br}\ K_\gamma[2] \stackrel{\simeq}{\to} H^1(\kappa(\gamma), \mathbb{Z}/2)$ for each γ. There thus exists $\beta \in \mathrm{Br}\ K[2]$ with image $\alpha(P_\gamma) \in \mathrm{Br}\ K_\gamma$ for each $\gamma \in \mathcal{X}^{(1)}$. One shows that β vanishes in $\mathrm{Br}\ K[\sqrt{b}]$. [Idea : this is obvious for $\alpha = (X_1^2 - aY_1^2, b)$, hence for all $\alpha(P_\gamma)$.]
Therefore $\beta = (b, \rho)$, with some $\rho \in K^\times$.

This enables us to perform a *descente* :
The K-variety W with equations

$$X_1^2 - aY_1^2 = \rho.(U^2 - bV^2) \neq 0$$

$$(X_1^2 - aY_1^2)(X_2^2 - bY_2^2) = c.(X_3^2 - abY_3^2) \neq 0,$$

admits a K-morphism $W \to E$, and it has rational points in all K_γ's.
A change of variables $(U + \sqrt{b}V)(X_2 + \sqrt{b}Y_2) = X_4 + \sqrt{b}Y_4)$ transforms this system of equations into the system

$$X_1^2 - aY_1^2 = \rho.(U^2 - bV^2) \neq 0$$

$$\rho.(X_4^2 - bY_4^2) = c.(X_3^2 - abY_3^2) \neq 0,$$

This is the *product* of two K-varieties, each of which a pointed cone over a smooth 3-dimensional quadrics over K, each being given by a diagonal quadratic form the coefficients of which have "normal crossings" on $\mathcal{X}$.
A theorem of CT-Parimala-Suresh 2009 then guarantees that these quadrics satisfy the local-global principle. They thus both have rational K-points, hence also E.

Corollary.

Let us consider either the local or the semi-global set up $p : \mathcal{X} \to \mathrm{Spec}\, R$. *Assume* R *is a* k*-algebra,* $\mathrm{char}(k) = 0$*, and* $k = \overline{k}$. *Let* $a, b, c \in K^{\times}$. *Let* E *be the* K*-variety defined by*

$$(X_1^2 - aY_1^2)(X_2^2 - bY_2^2)(X_3^2 - abZ_3^2) = c$$

and let Z *be a smooth* K*-compactification of* E. *Assume that the union of the supports of the divisors of* a*,* b *and* c *on* $\mathcal{X}$ *is a divisor with normal crossings.*

If the diagram of components of the special fibre is a **tree***, and if* $\prod_{\gamma} E(K_{\gamma}) \neq \emptyset$*, then* $E(K) \neq \emptyset$*.*

This explains why our many earlier attempts at producing semi-global examples with generic fibre a projective line failed, as also failed an attack on the equation

$$(X_1^2 - xY_1^2)(X_2^2 - (x-t)Y_2^2)(X_3^2 - (x-t^2).Y_3^2) = y$$

over the elliptic curve

$$y^2 = x(x-t)(x-t^2).$$

This curve has type I_2^*, the special fibre is a tree with 7 components, a chaine of three lines with multiplicity 2, say E, F, G, two curves of multiplicity 1 meeting E, and two curves of multiplicity 1 meeting G.

Ik heb het einde van mijn lezing bereikt, dank u