BANFF WORKSHOP ON "DIOPHANTINE METHODS, LATTICES, AND ARITHMETIC THEORY OF QUADRATIC FORMS": TITLES AND ABSTRACTS

1. Hour-long talks

Eva Bayer-Fluckiger (École Polytechnique Fédérale de Lausanne)

Title: *Galois algebras, Hasse principle and induction-restriction methods*
Abstract: The aim of this talk is to survey old and new results on self-dual normal bases and more generally invariants of Galois algebras. Particular attention will be given to local-global principles. Some of the results make use of an induction-restriction method for G-quadratic forms that is of independent interest.

J.-L. Colliot-Thélène (CNRS, Université Paris-Sud, France)

Titre de l'exposé: *Sur l'équation* $q(x, y, z) = P(t)$ *en entiers*
Résumé: Soit $q(x, y, z)$ une forme quadratique sur un corps de nombres k, isotrope en une place v, et soit $P(t)$ un polynôme non nul à coefficients dans k. Si $P(t)$ est séparable, on établit l'approximation forte en dehors de la place v pour les solutions de $q(x, y, z) = P(t)$. Pour $P(t)$ quelconque, on montre que sur le lieu lisse de $q(x, y, z) = P(t)$ l'obstruction de Brauer-Manin entière est la seule obstruction à l'approximation forte hors de v. Ceci est un travail en commun avec Fei Xu (Capital Normal University, Beijing, Chine).

Sinnou David (Université Pierre et Marie Curie - Paris 6)

Title: *A journey through heights: the Lehmer problem*
Abstract: After a brief description of the original Lehmer problem and the work that has been done around it till the seventies, we shall concentrate on the last decade. We shall show how the Lehmer problem can now be seen in a general geometric setup, how it interacts with aspects of global analysis, how the variation of the base field can be taken into account. We shall describe a few general conjectures that contain the original Lehmer problem as a special case and present a few recent results.

Roger Heath-Brown (University of Oxford)

Title: *p-adic zeros of systems of quadratic forms*
Abstract: This will be a survey of results on p-adic zeros of systems of quadratic forms, covering different strategies for proving such results, and applications of these results.

Gabriele Nebe (RWTH Aachen University)

Title: *Extremal lattices and codes*
Abstract: One major aim in the theory of lattices is the construction of dense lattice sphere packings. The densest lattices are known up to dimension 8 and in dimension 24. In dimension 8 and 24 these densest lattices are extremal even

unimodular lattices. In the "jump dimensions" $n = 24k$ there are only 5 extremal lattices known, the Leech lattice in dimension 24, 3 extremal lattices in dimension 48 and, since 2010, one extremal lattice Γ in dimension 72. All these lattice realise locally densest lattice sphere packings and provide the densest known packings in their dimension. In the talk I will comment on the construction of Γ as a Hermitian tensor product which makes it possible to compute the minimum with a computer.

I will also comment on extremal self-dual binary doubly even codes. The existence of an extremal code of length 72 is currently one of the major open questions in coding theory. A sequence of papers studies the automorphism group of such a putative code, which is shown to be quite small, of order 5, 10 or a divisor of 24.

Jeffrey Thunder (Northern Illinois University)

Title: *Counting certain points of given height in the function field setting*

Abstract: Fix a dimension n and a degree d. It is a well-known result of Northcott that there are only finitely many points of bounded height in projective n-space that generate a number field of degree d over the rationals (or any other number field, for that matter). The question then becomes to estimate the number of such points. Such estimates essentially all amount to some sort of elaboration on counting lattice points in some region of Euclidean space.

If one replaces the field of rational numbers with a field of rational functions in one variable over a finite field, then the analog of Northcott's result is equally valid, and again one may try to estimate the number of such points. We will explain how one can go about counting points in this situation, how the Riemann-Roch theorem can replace the lattice point estimates above, how that makes some things much simpler than the number field case, and what aspects are perhaps less clear. We will also discuss how the estimates change qualitatively depending on the relative sizes of the dimension and degree.

2. 30-minute talks

Ricardo Baeza (Universidad de Talca)

Title: *Levels and sublevels of Dedekind rings*

Abstract: We prove a representation result for quadratic forms over polynomial rings and we apply it to show, that there exist Dedekind rings A with level $s(A) = s(K) + 1$, where K is the quotient field of A. This is joint work with Jon Arason.

Tim Browning (University of Bristol)

Title: *Square-free hyperplanes on quadrics*

Abstract: The frequency of square-free values of polynomials is investigated as the variables run over integral points on an affine quadric. The proof combines dynamical systems with a refined uniform dimension growth estimate for affine quadrics. This is joint work with A. Gorodnik.

Renaud Coulangeon (University of Bordeaux)

Title: *The unreasonable effectiveness of tensor product*

Abstract: An extremal even unimodular lattice in dimension 72 was recently constructed by Gabriele Nebe, elaborating on ideas of Robert Griess. This lattice appears rather naturally as a tensor product over an imaginary quadratic ring of two lattices of lower dimension. Similar tensor constructions of extremal modular lattices in dimension 40 and 80 had already been proposed by Christine Bachoc and Gabriele Nebe in 1998. In this talk, I will first explain why one should not expect, in general, to produce dense lattices using tensor product and try then to analyse the very special features of the above mentionned constructions which explain their "unreasonnable effectiveness". In a different direction, I will also report on an intriguing conjecture by Jean-Benoît Bost which predicts a very rigid behavior of so-called semi-stable lattices with respect to tensor product.

Rainer Dietmann (Royal Holloway, University of London)

Title: *Weyl's inequality and systems of forms*
Abstract: Whereas for one single quadratic form or a pair of quadratic forms a variety of approaches from areas such as modular forms or arithmetic geometry have been successfully applied, for systems of more than two quadratic forms Hardy-Littlewood's circle method still seems to be the most powerful tool. In this talk we want to explain how one can give the basic form of Weyl's inequality in Birch's seminal work on forms in many variables a more efficient interpretation for systems of forms. This way we can improve a result of W.M. Schmidt to the effect that the expected Hardy-Littlewood asymptotic formula for integer zeros of a system of r integral quadratic forms holds true, provided that each form in the rational pencil has rank exceeding $2r^2 + 2\mathrm{r}$, and we obtain a similar improvement for systems of cubic forms. Finally, we also briefly want to discuss work in progress about representations of quadratic forms by quadratic forms, which is closely related to this topic.

Jonathan Hanke (University of Georgia)

Title: *TBA*
Abstract: TBA.

Ben Kane (University of Cologne)

Title: *Representations by triangular, square, and pentagonal sums*
Abstract: Fermat claimed that all positive integers are represented by 3 triangular numbers, 4 squares, 5 pentagonal, . . . , and m m-gonal numbers. Its determination in the cases $m = 4$ (resp. $m = 3$) was celebrated work of Lagrange (resp. Gauss) and the full conjecture was finally resolved by Cauchy in 1813. In this talk, we will discuss the related question of which "weighted sums" represent all but finitely many positive integers, with a focus on complications which first arise in the $m = 5$ case. This is based on ongoing joint work with W.K. Chan and A. Haensch.

Abhinav Kumar (Massachusetts Institute of Technology)

Title: *Energy minimization for lattices and periodic configurations, and formal duality*
Abstract: Closely related to the sphere packing problem is the energy minimization problem for discrete sets of points in Euclidean space, which can be reformulated as an optimization problem for the (average) theta function. I will describe some numerical experiments (joint work with Henry Cohn and Achill Schuermann) which gives evidence for the energy minimizing periodic configurations in low dimensions, for Gaussian potential energy. Surprisingly, the putative optimizers are not necessarily lattices, but appear in families exhibiting formal duality.

Byeong-Kweon Oh (Seoul National University)

Title: *Class numbers of ternary quadratic forms*

Abstract: In this talk, we investigate the behavior of class numbers of ternary quadratic forms under the Watson's transformation. We compute the class numbers of arbitrary ternary quadratic forms by using some data on maximal ternary quadratic forms.

Bruce Reznick (University of Illinois at Urbana Champaign)

Title: *Linear dependence among powers of quadratic forms*

Abstract: Let $F(k)$ be the smallest integer r so that there exist r distinct binary quadratic forms $\{q_i\}_{i=1}^r$ over $\mathbb{C}$ with $\{q_i^k\}_{i=1}^r$ linearly dependent. We compute $F(k)$ for $k \leq 7$ and give all minimal sets $\{q_i\}_{i=1}^r$ for $k \leq 5$. For example, $F(3) = 3$ and up to changes of variable and scaling, the unique minimal set is the familiar Pythagorean parameterization: $\{x^2 + y^2, x^2 - y^2, 2xy\}$. As might be expected, minimal sets usually have other nice properties as well.

Damien Roy (University of Ottawa)

Title: *On rational approximation to real points on plane quadratic curves defined over* $\mathbb{Q}$

Abstract: Let $q(x, y, z)$ be a homogeneous polynomial of degree 2 in 3 variables, with rational coefficients. Assume that q admits a non-trivial real zero and that q is irreducible over the field $\mathbb{Q}$ of rational numbers. Denote by U the set of real zeros q having $\mathbb{Q}$-linearly independent coordinates. We show that

a) each point in this set U has an exponent of uniform rational approximation between $1/2$ and $1/g = 0.618...$ where g denotes the golden ratio,

b) the elements of U for which the upper bound is achieved form an infinite countable set.

For $q(x, y, z) = x * z - y^2$, the statement a) is due to Davenport and Schmidt (1967) while b) is due to the author (2003). When q admits a non-trivial rational zero, we are quickly reduced to that case. Otherwise, the proof of a) is simpler, but the existence of "extremal" points in b) requires additional tools.

Rudolf Scharlau (Universität Dortmund)

Title: *Automorphism groups of lattices in large genera*

Abstract: In fixed dimension n, almost all lattices (with primitive integral quadratic forms) of determinant d have trivial automorphism groups when $d \to \infty$. This is a well known, classical consequence of reduction theory. The lattices of any given dimension and determinant split into genera, and in the thesis of J. Biermann (Goettingen, 1981) it had been shown that the result also holds for the lattices of any genus. Since the mass and the class number of genera tend to infinity also with the dimension n, one might expect that the result more sharply holds if $\max(n, d) \to \infty$. That is, only finitely many genera might exceed a specified proportion of lattices with non-trivial group. This is far from being proved. In the talk, we shall be more modest and report on explicit, computational results on the automorphism groups actually occuring for arithmetically interesting genera of dimension up to 20 and small level. Roughly speaking one observes that for these parameters (the level seems to be more appropriate than the absolute size of the determinant), automorphism groups are still a good invariant. On the other hand, when the level 1,2,3,4,5,6,7,11 goes up, the quick increase of the mass is mostly caused by a quick increase of the number of lattices with very small, eventually trivial automorphism group.

Achill Schürmann (Universität Rostock)

Title: *Strictly periodic extreme lattices*

Abstract: A lattice is called periodic extreme if it cannot locally be modified to yield a better periodic sphere packing. It is called strictly periodic extreme if it gives an isolated local optimum among periodic sphere packings. We derive sufficient conditions for periodic extreme and strictly periodic extreme lattices. We hereby in particular show that the root lattice E_8, the Coxeter-Todd lattice K_{12}, the Barnes-Wall lattice BW_{16} and the Leech lattice Λ_{24} are strictly periodic extreme.

Cameron Stewart (University of Waterloo)

Title: *Exceptional units and cyclic resultants*

Abstract: Let a be a nonzero algebraic integer of degree d over the rationals. Put $K = \mathbb{Q}(a)$ and let $\mathcal{O}(K)$ denote the ring of algebraic integers of K. We shall discuss estimates for the number of positive integers n for which $a^n - 1$ is a unit in $\mathcal{O}(K)$ and for the largest positive integer n for which $a^j - 1$ is a unit for j from 1 to n.

Takao Watanabe (Osaka University)

Title: *Polyhedral reduction of Humbert forms over a totally real number field*

Abstract: We construct a polyhedral fundamental domain for the action of some arithmetic subgroup on the space of Humbert forms. To do this, we discuss generalizations of perfect forms, perfect domains and Ryshkov polyhedra.

Mark Watkins (University of Sydney)

Title: *Indefinite LLL and solving quadratic equations*

Abstract: About 5-10 years ago, Denis Simon wrote a couple of papers where he pointed out that LLL can be used for *indefinite* symmetric matrices, via placing absolute values around the comparison condition. One catch is what to do with isotropic vectors (those of norm 0). He then used this algorithm in the solving of quadratic equations, via the use of an idea of Cassels (as enhanced by Bosma and Stevenhagen) We review his work, and then describe related ideas (such as the finding of a maximal totally isotropic subspace) and our implementation in Magma. We also indicate the general practicality of diagonalising a unimodular symmetric matrix of nonconstant signature.

Martin Widmer (Graz University of Technology)

Title: *Integral points of fixed degree and bounded height*

Abstract: Let k be a number field, let n and e be positive rational integers, and let $X > 1$ be real. We consider the algebraic points $(\alpha_1, ..., \alpha_n)$ of affine Weil height at most X such that each coordinate is an algebraic integer, and such that they generate an extension $k(\alpha_1, ..., \alpha_n)$ of k of degree e.

We present a precise asymptotic estimate for their number (as X tends to infinity) involving several main terms of decreasing order. The proof involves a gap principle for the successive minima of a lattice under a certain group of linear transformations. This principle might have applications to other counting problems.