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HILBERT'S THEOREM ON POSITIVE TERNARY QUARTICS: A REFINED ANALYSIS

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Dedicated to Jean-Louis Colliot-Thélène on the occasion of his 60th birthday

Abstract

Let X be an integral plane quartic curve over a field k, let f be an equation for X. We first consider representations $(*)$ $cf = p_1p_2 - p_0^2$ (where $c \in k^*$ and the p_i are quadratic forms), up to a natural notion of equivalence. Using the general theory of determinantal varieties we show that equivalence classes of such representations correspond to nontrivial globally generated torsion-free rank one sheaves on X with a self-duality which are not exceptional, and that the exceptional sheaves are in bijection with the k-rational singular points of X. For $k = \mathbb{C}$, the number of representations $(*)$ (up to equivalence) depends only on the singularities of X, and is determined explicitly in each case. In the second part we focus on the case where $k = \mathbb{R}$ and f is nonnegative. By a famous theorem of Hilbert, such f is a sum of three squares of quadratic forms. We use the Brauer group and Galois cohomology to relate identities $(**)$ $f = p_0^2 + p_1^2 + p_2^2$ to $(*)$, and we determine the number of equivalence classes of representations $(**)$ for each f. Both in the complex and in the real definite case, our results are considerably more precise since they give the number of representations with any prescribed base locus.

Contents

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Introduction

In 1888, David Hilbert published an influential paper [H] which became fundamental for real algebraic geometry, and which remains an inspiring source for research even today (see, for example, [CL], [Re2], and [Pf]). It addresses the problem whether a real polynomial $f(x_1, \dots, x_n)$ which takes nonnegative values on all of $\mathbb{R}^n$ is necessarily a sum of squares of real polynomials. Hilbert proves that the answer is negative in general. As is well known, his results go much beyond this fact and contain a surprising positive aspect as well. Namely, for any pair (n, d) of integers with $n \geq 2$ and even $d \geq 4$, except for $(n, d) = (2, 4)$, he shows that there exists a nonnegative real polynomial of (total) degree d in n variables which is not a sum of squares of polynomials. In the exceptional case, however, he proves that every nonnegative real polynomial of degree four in two (inhomogeneous) variables is a sum of three squares of real polynomials. Modern expositions of Hilbert's arguments are available, e. g. [Re1] and [Re2] for the general negative result and [Ru] and [Sw] for the positive result in the exceptional case. An elementary and constructive approach to the latter result was recently started by Pfister [Pf].

The subject of this paper is a refined analysis of this exceptional case. Let us switch to homogeneous polynomials (as did Hilbert), so we have a real ternary form $f = f(x_0, x_1, x_2)$ of degree four which is positive semidefinite (*psd*), i. e., takes nonnegative values. By Hilbert's theorem, there exists an identity

$$f = p_0^2 + p_1^2 + p_2^2 \tag{1}$$

in which the p_i are quadratic forms with real coefficients. Regarding two such representations as equivalent if one is deduced from the other by an orthogonal transformation in the p_i, we ask for the number of equivalence classes of such representations. This question was first raised in [PR]. A simple dimension count shows that this number is finite for generically chosen f. In [PRSS], to which this paper forms a sequel, it was proved that there exist precisely eight inequivalent representations (1) if the curve $f = 0$ is nonsingular. Here we treat the general singular case: Assuming that f is irreducible over $\mathbb{C}$, we prove that the number of inequivalent representations (1) is finite and depends only on the singularities of the curve $f = 0$. Moreover, we calculate this number in each case. In fact, our analysis results in a refined count which also respects the base loci of representations (1), i. e., the common zeros of p_0, p_1 and p_2.

A key idea for carrying this out is to regard (1) as a twisted form of a symmetric 2×2-determinant, and thus to relate the question to the subject of determinantal representations. This is a very classical and well-studied chapter of algebraic geometry. Thus, in the first part of this paper, we study representations

$$f = \begin{vmatrix} p_1 & p_0 \\ p_0 & p_2 \end{vmatrix} = p_1 p_2 - p_0^2 \tag{2}$$

of irreducible ternary forms f over $\mathbb{C}$ of even degree $2d$, in which the p_i have degree d, again up to a natural notion of equivalence. Under the assumption that the curve $X = \{f = 0\}$ has ADE singularities, we associate with a given identity (2) a line bundle $\mathcal{L}$ on some partial normalization X' of X, such that $\mathcal{L}$ satisfies a certain self-duality, and we show that $\mathcal{L}$ determines (2) up to equivalence (Theorems 1.15 and 2.17). Conversely, the line bundles $\mathcal{L}$ arising from some representation (2) can be characterized by an abstract condition. If $\deg(f) \geq 6$, then in general no representation (2) exists. On the other hand, the case of quartics ($\deg(f) = 4$) is more favorable. First, the ADE hypothesis holds automatically. Second, a very satisfactory characterization of the line bundles $\mathcal{L}$ is possible which arise from an identity (2). It only involves the list of singularities of X (see Corollary 3.9 and 3.11). In this way we achieve a complete count of determinantal representations (2) in the case of quartics (4.4). In fact, for each fixed base locus, we obtain the precise number of corresponding representations.

The idea of associating a line bundle $\mathcal{L}$ to an identity (2) was inspired by Wall [W], and was already used in [PRSS] in the smooth case. The main result of [W] says that for any form f of degree four (over $\mathbb{C}$), possibly reducible, there exists at least one representation (2) (over $\mathbb{C}$), except when f is the product of a cuspidal cubic and its cuspidal tangent. In this case there is no representation (2). Thus, our results give a quantitative refinement of Wall's theorem in the irreducible case. For example, when the curve X is nonsingular, there are precisely 63 inequivalent representations (2) over $\mathbb{C}$. This result was already known to Hesse in 1855 ([He] p. 261, see also [Co] p. 36). For singular X, our results are new.

As an aside, we remark that the negative part of Wall's result corrects a statement erroneously made by Hilbert ([H], middle of second page). Hilbert had claimed that every ternary quartic form over $\mathbb{C}$ is a sum of three squares of quadratic forms.

Our findings show that a key argument in the proof of [W] fails in certain cases. This argument concerns the number of base-point free representations. Our approach covers these cases (even with precise quantitative information), thereby confirming the correctness of the main result of [W]; see 4.5 for more details.

In the second part of this paper, we turn to non-closed base fields k (with $\mathrm{char}(k) = 0$, and with $k = \mathbb{R}$ primarily on our mind). We try to analyze twisted versions of representations (2) over k. That is, given a form $f \in k[x_0, x_1, x_2]$ of degree $2d$, we are looking for identities

$$f = q(p_0, p_1, p_2) \tag{3}$$

in which q is a nondegenerate quadratic form over k in three variables, and the $p_i \in k[x_0, x_1, x_2]$ are forms of degree d. Over the algebraic closure $\bar{k}$, (3) is equivalent to a determinantal representation (2). Therefore, Galois cohomology comes as the main tool to complement the methods of the first part. Assuming that the curve $X = \{f = 0\}$ is geometrically integral and has simple singularities, the data associated over $\bar{k}$ to the representation (3) is Galois-invariant. Thus we have a partial normalization X' of X (defined over k) and a Galois-invariant line bundle $\mathcal{L}$ on $X' \otimes_k \bar{k}$. This bundle $\mathcal{L}$ has a natural invariant $\partial(\mathcal{L})$ in $\mathrm{Br}(k)$, the Brauer group of k, and we show (Proposition 5.16) that this invariant coincides with the so-called Witt invariant of the quadratic form q. Hence X' and $\mathcal{L}$ determine q up to a factor in k^*.

Assume now that $2d = \deg(f) = 4$. When $k = \mathbb{R}$, the field of real numbers, and when f is positive semidefinite, this suffices to express the number of quadratic representations (3) of f (over $\mathbb{R}$) in terms of nothing more than the list of singularities of X (see Scholium 7.2 and 7.3). In fact we get a much more precise statement, since for every fixed base locus we find both the number of definite (sum of squares) and indefinite representations with the given base locus.

In principle, the same can be done when f is indefinite. Of course, there are no definite representations. However, the precise account (of the indefinite representations) depends in this case not only on the singularities of X, but (in general) also on the number of loops, both of X and of its partial normalizations. This makes the analysis considerably more complicated. To keep this paper at a reasonable size, and since our primary focus is on Hilbert's theorem anyway, we did not attempt to cover the indefinite case here in detail. Likewise, we largely ignored the case where f is reducible. In 9.2 we provide elementary proofs for Hilbert's three squares theorem for reducible psd quartics. But in fact a detailed analysis seems possible, similar to the one given here in the irreducible case. We plan to come back to these points in a future publication.

Although the main results of this paper ultimately concern quartic forms over $\mathbb{R}$ or $\mathbb{C}$, there would have been little point in narrowing down so much from the beginning. Thus, a large part of the paper is devoted to a systematic setup for the study of representations (3) over an arbitrary ground field k, where f may be a ternary form of arbitrary even degree. This greater generality does not cause essential additional complications. On the other hand, it has the benefit of making more transparent what is special about the case considered by Hilbert.

Organization of the paper

Here is a brief and more structured overview. The paper is divided into two parts, the first of which deals with (symmetric 2×2) determinantal representations (2). Here we work over arbitrary base fields (sometimes of zero characteristic), and in the first two sections with plane curves X of arbitrary even degree. Section 1 relates identities (2) to certain torsion-free rank one sheaves $\mathcal{F}$ which are not "exceptional". Its content is summarized in Theorem 1.15. The sheaves $\mathcal{F}$ that have come up in Section 1 are related in Section 2 to partial normalizations X' of X, and it is shown that they correspond to certain line bundles on these X' when X has simple singularities (see Theorem 2.17). It is also shown how the base locus of the associated determinantal representations is determined by X' (see Corollary 2.22). In Section 3 we restrict to quartic curves. We prove that the "exceptional" sheaves $\mathcal{F}$ are in one-to-one correspondence with the k-rational singular points of X (see Corollary 3.9), and determine the associated partial normalization in terms of the singularity (see 3.11). Section 4 summarizes the results of Part I for quartics, giving (in principle) the complete account of their determinantal representations (2) (see 4.2). For $k = \bar{k}$ algebraically closed, the results are completely explicit (see 4.4).

The second part studies representations of plane curves by arbitrary (ternary) quadratic forms q over k, that is, by twisted versions (3) of (2). After setting up the proper notion of equivalence, Section 5 relates identities (3) to those determinantal representations (2) over $\bar{k}$ which are Galois-invariant up to equivalence. The latter have a natural Galois cohomological invariant in $\mathrm{Br}(k)$, the Brauer group of the ground field (see 5.14 and 5.15), and we show that this invariant allows to recover the representing quadratic form q (as in (3)) up to a scalar factor (Proposition 5.16). Section 6 provides required information on generalized Jacobians of curves over $\mathbb{R}$. With its help, we combine in Section 7 the Galois cohomological approach from Section 5 with the results of Part I over $\mathbb{C}$. We obtain the full analysis of quadratic representations (3) of psd ternary quadric forms over $\mathbb{R}$ (see Scholium 7.2 and 7.3). Section 8 contains a few selected examples for which the necessary arguments are carried out in detail; some of them illustrate earlier points in the paper. Finally, Section 9 gives an elementary proof of Hilbert's three squares theorem in the reducible case, thus completing the picture.

Notation and preliminaries

0.1. In the entire paper, k is a field of characteristic not two, and $\bar{k}$ denotes an algebraic closure of k. By a k-variety we mean a separated k-scheme of finite type, neither necessarily irreducible nor reduced. If X is a k-variety and E is a k-algebra, then $X(E)$ denotes the set of E-valued points of X.

Given a homogeneous ideal I in $k[x_0,\dots,x_n]$, we write $V_+(I)$ for the closed subscheme of $\mathbb{P}^n_k$ defined by I. The notation $V_+(f_1,\dots,f_r)$ (for homogeneous polynomials f_i) has a similar meaning. For convenience of notation, we use the abbreviation

$$\mathcal{P}_n(k) := H^0(\mathbb{P}^2_k,\, \mathcal{O}(n))$$

throughout this paper for the vector space of ternary forms of degree n over k.

0.2 (Cf. [Be], 1.5)**.** If M is a finitely generated graded module over $S = k[x_0,\dots,x_n]$, then M has a finite graded-free resolution

$$0 \to F_{n+1} \to \cdots \to F_0 \to M \to 0.$$

This resolution is minimal iff $\mathrm{im}(F_{i+1} \to F_i) \subset \mathfrak{m}F_i$ for all i, where $\mathfrak{m} = (x_0,\dots,x_n)$ is the irrelevant ideal. The minimal graded-free resolution for M is unique up to isomorphism. If $\mathcal{F}$ is a coherent $\mathcal{O}_{\mathbb{P}^n}$-module, we can take the sheaf sequence associated to the minimal resolution of the graded S-module $\Gamma_*(\mathcal{F}) = \bigoplus_i H^0(\mathbb{P}^n,\, \mathcal{F}(i))$. This is a twisted-free resolution

$$0 \to \mathcal{L}_{n+1} \to \cdots \to \mathcal{L}_0 \to \mathcal{F} \to 0$$

which will be called the *minimal resolution* of $\mathcal{F}$. Its components $\mathcal{L}_i$ are direct sums of sheaves $\mathcal{O}_{\mathbb{P}^n}(d_{ij})$, with suitable $d_{ij} \in \mathbb{Z}$.

0.3. Given an abelian group A and an integer $n \geq 1$, we write A_n (respectively A/n) for the kernel (respectively cokernel) of multiplication by n on A.

Part I: Determinantal representations

In Part I we are exclusively concerned with representations of plane curves by symmetric 2×2-determinants. In order to simplify language, we will nevertheless simply speak of (symmetric) determinantal representations.

1. Symmetric determinantal representations and self-dual sheaves

The results of this section are essentially a very particular case of the theory of determinantal hypersurfaces, which is a classical topic of algebraic geometry. Its modern formulation is due to Catanese ([Ca1], [Ca2], and [CC]) and Beauville [Be]. We refer to [Be] for an excellent exposition, on which we shall rely heavily in the sequel.

Let k be a field (always with $\mathrm{char}(k) \neq 2$). We consider homogeneous polynomials ("forms") in three variables (x_0, x_1, x_2) over k. Recall (see 0.1) that $\mathcal{P}_n(k)$ denotes the space of such forms of degree n.

1.1. Given a nonzero form $f \in \mathcal{P}_{2d}(k)$ of even degree $2d > 0$, we consider the problem of finding identities

$$cf = \det(P) = p_1 p_2 - p_0^2 \tag{4}$$

(often simply referred to as "determinantal representations" in the sequel), where $c \in k^*$ and

$$P = \begin{pmatrix} p_1 & p_0 \\ p_0 & p_2 \end{pmatrix}$$

is a symmetric 2×2-matrix with p_0, p_1, $p_2 \in \mathcal{P}_d(k)$. With (4) one associates the sheaf $\mathcal{F}_P$ on $\mathbb{P}^2 = \mathbb{P}^2_k$ defined by the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d)^2 \xrightarrow{P} \mathcal{O}_{\mathbb{P}^2}^2 \to \mathcal{F}_P \to 0. \tag{5}$$

Then we have the following proposition.

Proposition 1.2. *Let $d \ge 1$, and assume that $f \in \mathcal{P}_{2d}(k)$ is square-free, i. e., that the curve $X = V_+(f)$ is reduced.*

(a) *Given an identity (4), let $\mathcal{F} := \mathcal{F}_P$ as in (5). Then $\mathcal{F}$ has support X and is a torsion-free coherent $\mathcal{O}_X$-module of rank one. Moreover, $\mathcal{F}$ is generated by its global sections and satisfies the following self-duality:*

$$\mathcal{F} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X(d)). \tag{6}$$

In addition, $\mathcal{F}$ is not isomorphic to $\mathcal{O}_X(\frac{d}{2})$ (in the case where d is even).

(b) *Conversely, let $\mathcal{F}$ be a torsion-free coherent $\mathcal{O}_X$-module of rank one which is generated by its global sections, satisfies (6) and is not isomorphic to $\mathcal{O}_X(\frac{d}{2})$ (in the case where d is even). Then there exists an exact sequence (5) with $\mathcal{F} \cong \mathcal{F}_P$ and $\det(P) = cf$ for some $c \in k^*$.*

1.3. We explain some terminology. Let A be a noetherian ring, and let K be its total ring of fractions. An A-module M is *torsion-free* if $M \to M \otimes_A K$ is injective. We say that M has *rank r* if $M \otimes_A K$ is a free K-module of rank r.

If X is a reduced noetherian scheme, a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is called *torsion-free* if the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ is torsion-free for every $x \in X$. Equivalently, the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is torsion-free for every open affine $U \subset X$. Similarly, $\mathcal{F}$ is said to have *rank r* if the stalk of $\mathcal{F}$ in the generic point of every irreducible component of X has dimension r (over the function field of that component).

A substantial part of Proposition 1.2 follows from previous work by Catanese, Casnati-Catanese and Beauville; see, in particular, [Ca1], Theorem 2.16, [CC], Theorem 0.3 and [Be], Theorem B. We shall give detailed arguments for those parts of the proof that are specific to the present situation, and shall otherwise give references to [Be].

We start with the following lemma (see also [Be], 1.7).

Lemma 1.4. *Let A be an integrally closed noetherian domain. Let M be a free A-module of finite rank $r \ge 1$, and let $p \in \mathrm{End}_A(M)$ such that $\det(p) \ne 0$. Write $f := \det(p)$ and $N := \mathrm{coker}(p)$.*

(a) *N is annihilated by f, and is a torsion-free $A/(f)$-module.*
(b) *If the ring $A/(f)$ is reduced, the $A/(f)$-module N has rank one.*
(c) *Conversely, if N is annihilated by $\sqrt{(f)}$ and is an $A/\sqrt{(f)}$-module of rank one, then $\sqrt{(f)} = (f)$, i. e., the ring $A/(f)$ is reduced.*

Proof. The proof of (a) is a straightforward exercise. For (b) and (c) one uses arguments similar to those in [Be], 1.7. □

1.5. We start the proof of Proposition 1.2 by showing (a). So assume $cf = \det(P)$, and put $\mathcal{F} = \mathcal{F}_P$. It is clear that $\mathcal{F}$ is generated by H^0 and that (5) is the minimal resolution of $\mathcal{F}$ (cf. 0.2). Moreover, $\mathcal{F}$ has support X, and $\mathcal{F}$ is a torsion-free $\mathcal{O}_X$-module of rank one by Lemma 1.4. Self-duality (6) follows from Grothendieck duality (compare [CC] or [Be]).

1.6. For the proof of Proposition 1.2(b), let $\mathcal{F}$ be a torsion-free $\mathcal{O}_X$-module of rank one which satisfies self-duality (6). Proceeding as in [CC], proof of Theorem 0.3, or as in [Be], proof of Theorem B, one sees that the minimal resolution of $\mathcal{F}$ has the shape

$$0 \to \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^2}(-d_i - d) \xrightarrow{P} \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^2}(d_i) \to \mathcal{F} \to 0 \tag{7}$$

with suitable integers d_i, in which the matrix P is symmetric. By Lemma 1.4(c), $\det(P) = 0$ is a reduced equation for X. So $\det(P) = cf$ with some $c \in k^*$. Comparing degrees we see that $2d = \deg(f) = \deg(\det P) = rd + 2\sum_i d_i$, i. e., $2\sum_i d_i = (2-r)d$.

To complete the proof we will use the hypothesis that $\mathcal{F}$ is generated by global sections. It implies $d_i \geq 0$ for $i = 1, \dots, r$; see Lemma 1.7 below. This implies $r \leq 2$. The case $r = 1$ is impossible, since then $d = 2d_1$, and $\mathcal{F}$ would be the cokernel of $\mathcal{O}_{\mathbb{P}^2}(-3d_1) \xrightarrow{f} \mathcal{O}_{\mathbb{P}^2}(d_1)$ which is $\mathcal{O}_X(d_1) = \mathcal{O}_X(\frac{d}{2})$, the sheaf that was excluded. Therefore $r = 2$, and this implies $d_1 = d_2 = 0$. So we have an exact sequence (5). $\square$

To complete the proof of Proposition 1.2, it remains to settle the following lemma:

Lemma 1.7. *Let*

$$0 \to \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^2}(-d_i - d) \xrightarrow{M} \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^2}(d_i) \to \mathcal{F} \to 0$$

be an exact sheaf sequence on $\mathbb{P}^n$, $n \geq 2$, where all non-zero entries of the matrix M have strictly positive degree, and where $\det(M)$ *is a form of degree $2d$. Then $\mathcal{F}$ is generated by global sections if and only if $d_i \geq 0$ for $i = 1, \dots, r$.*

Proof. The "if" being obvious, assume that $\mathcal{F}$ is generated by global sections. The entries of the matrix $M = (m_{ij})_{i,j=1,\dots,r}$ are forms with $\deg(m_{ij}) = d_i + d_j + d$, and $m_{ij} = 0$ whenever $d_i + d_j + d \leq 0$. From $\deg(\det M) = 2d$ we see

$$2\sum_{i=1}^{r} d_i = (2-r)d. \tag{8}$$

We arrange the summands such that

$$d_1 \geq d_2 \geq \cdots \geq d_r. \tag{9}$$

Then for $i = 1, \dots, r$ we have

$$d_i + d_{r+1-i} \geq 1 - d. \tag{10}$$

Indeed, otherwise all entries of M below and to the right of position $(i,\, r+1-i)$ (including this position) would be zero, by assumption (9). The last i columns of M would therefore be linearly dependent (over $k(\mathbf{x})$), contradicting $\det(M) \neq 0$.

We will lead the assumption $d_r < 0$ to a contradiction. Let $1 \leq p < r$ be the largest index with $d_p \geq 0$. By assumption, $\mathcal{F}$ is generated by global sections. This means that the $(r-p) \times p$ matrix,

$$M' := \begin{pmatrix} m_{p+1,1} & \cdots & m_{p+1,p} \\ \vdots & & \vdots \\ m_{r,1} & \cdots & m_{r,p} \end{pmatrix},$$

has rank $r - p$ everywhere on $\mathbb{P}^n$. We conclude $r - p \leq p$, hence $p \geq \frac{r}{2}$.

First assume $p \geq \frac{r}{2} + 1$, and hence $r \geq 3$ since $p < r$. From (10) we get

$$\sum_{i=1}^{r} d_i \;=\; \sum_{i=1}^{r-p} (d_i + d_{r+1-i}) + \sum_{i=r-p+1}^{p} d_i \;\geq\; (r-p)(1-d).$$

The assumption $p \geq \frac{r}{2} + 1$ means $2(r-p) \leq r - 2$. From (8) we therefore conclude $(2-r)d = 2\sum_i d_i \geq (r-2)(1-d)$, which means $r \leq 2$; a contradiction.

The only remaining possibility is $p = \frac{r}{2}$ (for r even), respectively $p = \frac{r+1}{2}$ (for r odd). Here, M' is a $p \times p$ matrix, respectively a $(p-1) \times p$ matrix, whose entries are forms of strictly positive degree and which has maximal rank everywhere on $\mathbb{P}^n$. This is impossible, e.g. by the Eagon-Northcott theorem [EN] in the second case. The lemma is proved. $\square$

In the situation of Proposition 1.2 it remains to clarify to what extent P (respectively, the determinantal representation (4)) is determined by $\mathcal{F}$. To do so we need a few notational preparations.

1.8. On the k-vector space $\mathrm{Sym}_2(k)$ of symmetric 2×2-matrices, the determinant $s \mapsto \det(s)$ is a nondegenerate quadratic form. We shall denote the associated symmetric bilinear form by δ, so

$$\delta(s,t) := \frac{1}{2}\Big(\det(s+t) - \det(s) - \det(t)\Big)$$

for $s,\, t \in \mathrm{Sym}_2(k)$. The orthogonal group of δ will be denoted O_δ, and the action of $\sigma \in \mathrm{O}_\delta(\bar{k})$ on $s \in \mathrm{Sym}_2(\bar{k})$ will be written ${}^{\sigma}s$.

Let the general linear group GL_2 act on Sym_2 by

$$(h, s) \;\mapsto\; \frac{1}{\det(h)} \cdot hsh^t$$

($h \in \mathrm{GL}_2(\bar{k})$, $s \in \mathrm{Sym}_2(\bar{k})$). This action preserves $\det(s)$, hence it defines a homomorphism $\mathrm{GL}_2 \to \mathrm{O}_\delta$ of algebraic groups over k, and in fact an isomorphism from $\mathrm{PGL}_2 = \mathrm{PSL}_2$ to the special orthogonal group SO_δ. Since the natural map $\mathrm{GL}_2(k) \to \mathrm{PGL}_2(k) = \mathrm{PSL}_2(k)$ is surjective, this gives a way of representing the elements of $\mathrm{O}_\delta(k)$: Every $\sigma \in \mathrm{O}_\delta(k)$ acts as

$$^{\sigma}s = \frac{\varepsilon}{\det(h)} \cdot hsh^t \quad (s \in \mathrm{Sym}_2(k))$$

with $h \in \mathrm{GL}_2(k)$ and $\varepsilon = \pm 1$; here $\varepsilon = \det(\sigma)$ is determined by σ, and h is determined up to a factor in k^*. (To avoid possible confusion, note that the map $\mathrm{SL}_2(k) \to \mathrm{PSL}_2(k)$ is not necessarily surjective when $k \neq \bar{k}$.)

1.9. We fix an integer $d \geq 1$ and write $\mathrm{Sym}_2 \mathcal{P}_d(k) := \mathrm{Sym}_2(k) \otimes_k \mathcal{P}_d(k)$ for the vector space of symmetric 2×2-matrices with entries in $\mathcal{P}_d(k)$. The action of O_δ on Sym_2 extends to a determinant-preserving action on $\mathrm{Sym}_2 \mathcal{P}_d$ in the obvious way, written again $(\sigma, P) \mapsto {}^{\sigma}P$.

Given $P \in \mathrm{Sym}_2 \mathcal{P}_d(k)$ with $\det(P) \neq 0$, we have associated with P the sheaf $\mathcal{F}_P$ on $\mathbb{P}^2$ defined by (5). For any $a \in k^*$ and any $\sigma \in \mathrm{O}_\delta(k)$, if we put $P' := a \cdot {}^{\sigma}P$, then it is clear that $\mathcal{F}_{P'} \cong \mathcal{F}_P$. (Indeed, there are $b \in k^*$ and $h \in \mathrm{GL}_2(k)$ with $P' = b \cdot hPh^t$, from which one gets an isomorphism between the defining resolutions of P' and P.) In other words, $P \mapsto \mathcal{F}_P$ is invariant under the action of similitudes of δ on P. Except when f is degenerate, we have the following converse:

Proposition 1.10. *Let $f \in \mathcal{P}_{2d}(k)$ such that f is not a product of two forms of degree d over $\bar{k}$. Let P, $P' \in \mathrm{Sym}_2 \mathcal{P}_d(k)$ and c, $c' \in k^*$ with*

$$\det(P) = cf, \quad \det(P') = c'f,$$

and assume $\mathcal{F}_P \cong \mathcal{F}_{P'}$ (as $\mathcal{O}_{\mathbb{P}^2}$-modules). Then there are $a \in k^*$ and $z \in \mathrm{PSL}_2(k)$ with $c' = ca^2$ and $P' = a \cdot zPz^t$. Moreover, a and z are uniquely determined.

For the proof, see 1.14 below.

Corollary 1.11. *If $\det(P) = \det(P') = f$ in the preceding proposition, then P and P' are conjugate by a unique element of $\mathrm{O}_\delta(k)$.* $\square$

1.12. Let $P = \begin{pmatrix} p_1 & p_0 \\ p_0 & p_2 \end{pmatrix} \in \mathrm{Sym}_2 \mathcal{P}_d(k)$. If the forms p_0, p_1, p_2 are k-linearly dependent, then $\det(P)$ is a product of two forms of degree d over $\bar{k}$. Therefore, if we exclude this case and write $P = \sum_{|\alpha|=d} P_\alpha \, x^\alpha$ with $P_\alpha \in \mathrm{Sym}_2(k)$, the k-linear span of the matrices P_α is $\mathrm{Sym}_2(k)$. Note that the stabilizer of P in $\mathrm{O}_\delta(\bar{k})$ is trivial, i. e., only the identity in $\mathrm{O}_\delta(\bar{k})$ fixes P.

The proof of the following lemma can be checked immediately:

Lemma 1.13. *Let $h \in \mathrm{GL}_2(k)$ such that the matrix sh is symmetric for every $s \in \mathrm{Sym}_2(k)$. Then $h = bI$ with $b \in k^*$.* $\square$

1.14. *Proof of Proposition* 1.10. Since the minimal resolution is unique up to isomorphism, it follows from $\mathcal{F}_P \cong \mathcal{F}_{P'}$ that there are g, $g' \in \mathrm{GL}_2(k)$ with $gP = P'g'$. Transposing this identity and multiplying with g' on the right gives $Pg^tg' = g'^tP'g' = g'^tgP$, whence $Ph = h^tP$ with $h = g^tg'$. In other words, the matrix Ph is symmetric. Writing $P = \sum_{|\alpha|=d} P_\alpha\, x^\alpha$, the matrices P_α span $\mathrm{Sym}_2(k)$ by the assumption on f (see 1.12), and so the matrix sh is symmetric for every $s \in \mathrm{Sym}_2(k)$. By Lemma 1.13, $h = bI$ with $b \in k^*$. Hence also $g'g^t = bI$, and $gPg^t = P'g'g^t = bP'$, which says $P' = b^{-1} \cdot gPg^t$. Comparing determinants we see $c' = a^2c$ with $a = b^{-1}\det(g)$. Hence

$$P' = a \cdot \frac{1}{\det(g)}\, gPg^t = a \cdot zPz^t,$$

where $z \in \mathrm{PSL}_2(k)$ is represented by the matrix $(\det g)^{-1/2}\, g \in \mathrm{SL}_2(\bar{k})$.

This proves the existence of a and z. As for the uniqueness, it was already remarked in 1.12 that $\sigma \in \mathrm{O}_\delta(k)$ with $P' = a \cdot {}^\sigma P$ is determined once a is fixed. Replacing a by $-a$ in Proposition 1.10 would force σ to be replaced by $-\sigma$, which is not in $\mathrm{SO}_\delta(k) = \mathrm{PSL}_2(k)$. □

We summarize the content of Propositions 1.2 and 1.10:

Theorem 1.15. *Let $d \ge 1$, let $f \in \mathcal{P}_{2d}(k)$ be square-free, not a product of two forms of degree d over $\bar{k}$, and let $X = V_+(f)$. Then the construction of* 1.1 *provides a natural bijection between the following objects:*

(1) *Equivalence classes of matrices $P \in \mathrm{Sym}_2\, \mathcal{P}_d(k)$ with $\det(P) = cf$ for some $c \in k^*$;*

(2) *isomorphism classes of torsion-free coherent $\mathcal{O}_X$-modules $\mathcal{F}$ of rank one satisfying $\mathcal{F} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X(d))$, which are generated by their global sections and are not isomorphic to $\mathcal{O}_X\bigl(\frac{d}{2}\bigr)$ (in case d even).*

Here P and P' are called equivalent in (1) *if there are $a \in k^*$ and $g \in \mathrm{GL}_2(k)$ with $P' = a \cdot gPg^t$.* □

2. Self-dual sheaves and partial normalizations

The aim of this section is to make explicit how, under suitable hypotheses on the singularities of the curve X, self-dual sheaves of rank one on X correspond to self-dual line bundles on partial normalizations of X. I profited here from Piontkowski's work [Pi]. He is studying theta characteristics on singular curves (over $\mathbb{C}$), a concept which (by the results of Section 1) is related to, but different from, our topic (see also the remark 2.19 below). At the end of the section we relate the base loci of determinantal representations of X to the associated partial normalizations of X.

2.1. We start with some preparations of general nature. Let X be a locally noetherian scheme and $\mathcal{B}$ a coherent $\mathcal{O}_X$-algebra. Let $\pi\colon X' \to X$ be the associated finite morphism (so $X' = \operatorname{Spec}(\mathcal{B})$ in the notation of [EGA], II.1.3.1). The direct image functor π_* induces an equivalence of categories

$$\pi_*\colon Qcoh(X') \to Qcoh(\mathcal{B}),$$

and we will write π^o for a quasi-inverse functor. (See also [EGA], II.1.4.3, where the quasi-inverse functor is denoted $M \mapsto \widetilde{M}$.)

If $\mathcal{M}$ is a coherent $\mathcal{B}$-module and $\mathcal{N}$ is a coherent $\mathcal{O}_X$-module, then $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N}$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{M})$ are coherent $\mathcal{B}$-modules via the action of $\mathcal{B}$ on $\mathcal{M}$. Following [Ha], ex. III.6.10, we will write

$$\pi^!\mathcal{N} := \pi^o\,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{B},\mathcal{N}).$$

Here are some basic properties:

Lemma 2.2. *Let $\mathcal{M}$ be a coherent $\mathcal{B}$-module, and let $\mathcal{N}$, $\mathcal{N}'$ be coherent $\mathcal{O}_X$-modules:*

(a) $\pi^o\,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) \cong \mathcal{H}om_{\mathcal{O}_{X'}}(\pi^o\mathcal{M},\pi^!\mathcal{N})$.
(b) *If the $\mathcal{O}_X$-module $\mathcal{N}$ is locally free, then* $\pi^!(\mathcal{N}'\otimes_{\mathcal{O}_X}\mathcal{N}) \cong (\pi^!\mathcal{N}')\otimes_{\mathcal{O}_{X'}}(\pi^*\mathcal{N})$. *In particular, then,*

$$\pi^!\mathcal{N} \cong (\pi^!\mathcal{O}_X)\otimes_{\mathcal{O}_{X'}}(\pi^*\mathcal{N}).$$

Proof. To prove (a), apply π^o to the isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) \cong \mathcal{H}om_{\mathcal{B}}(\mathcal{M},\,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{B},\mathcal{N}))$$

of $\mathcal{B}$-modules. For (b) observe that the canonical homomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{B},\mathcal{N}')\otimes_{\mathcal{O}_X}\mathcal{N} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{B},\,\mathcal{N}'\otimes_{\mathcal{O}_X}\mathcal{N})$$

of $\mathcal{O}_X$-modules is $\mathcal{B}$-linear, and is an isomorphism if the $\mathcal{O}_X$-module $\mathcal{N}$ is locally free. Applying π^o to it gives the desired isomorphism. □

2.3. Let k be a field and X an algebraic curve over k. To make the exposition less technical, we will always assume now that X is integral, although everything in this section can be generalized to the case where X is just reduced. By a *partial normalization* of X we mean any integral curve X' over k together with a finite birational morphism $\pi\colon X' \to X$. This implies that $\pi_*\mathcal{O}_{X'}$ is a coherent $\mathcal{O}_X$-subalgebra of $k(X)$, the function field of X. Conversely, for any coherent $\mathcal{O}_X$-subalgebra $\mathcal{B}$ of $k(X)$, the associated finite morphism $\pi\colon X' = \operatorname{Spec}(\mathcal{B}) \to X$ is a partial normalization of X.

If $h\colon \widetilde{X} \to X$ is the normalization of X then, up to isomorphism over X, the partial normalizations of X correspond to the $\mathcal{O}_X$-algebras $\mathcal{B}$ satisfying $\mathcal{O}_X \subset$

$\mathcal{B} \subset h_* \mathcal{O}_{\widetilde{X}}$, or equivalently, to the finite families $(B_x)_{x \in X_{\mathrm{sing}}}$ of intermediate rings

$$\mathcal{O}_{X,x} \subset B_x \subset \widetilde{\mathcal{O}}_{X,x},$$

where $\widetilde{\mathcal{O}}_{X,x}$ is the integral closure of $\mathcal{O}_{X,x}$.

2.4. Let X be an integral curve over k and $\mathcal{F}$ a torsion-free coherent $\mathcal{O}_X$-module of rank one. Then $\mathcal{B} := \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F})$ is a coherent $\mathcal{O}_X$-subalgebra of $k(X)$. Hence $\mathcal{B}$ gives rise to a partial normalization $\pi\colon X' = \mathrm{Spec}(\mathcal{B}) \to X$ of X with $\pi_* \mathcal{O}_{X'} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F})$, which we call the *partial normalization defined by* $\mathcal{F}$. We will use the notation from 2.1. In particular, we get the coherent $\mathcal{O}_{X'}$-module

$$\mathcal{F}' := \pi^o \mathcal{F}$$

associated to $\mathcal{F}$. These assumptions and notations will be fixed for the following.

2.5. If $\mathcal{F}$ is generated by global sections (on X), then also $\mathcal{F}'$ is generated by global sections (on X'). But the converse is not true in general. (This remark is not used in the sequel.)

2.6. In view of Theorem 1.15, we now translate a self-duality condition $\mathcal{F} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$ for $\mathcal{F}$, in which $\mathcal{L}$ is an invertible sheaf on X, into a similar condition for $\mathcal{F}' = \pi^o \mathcal{F}$ on X'. Thus let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, and assume we have an isomorphism

$$\phi\colon \mathcal{F} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L}) \tag{11}$$

of $\mathcal{O}_X$-modules. Both are in fact $\mathcal{B}$-modules, and ϕ is automatically $\mathcal{B}$-linear, as one sees by looking at ϕ in the generic point. Applying π^o to (11) and using Lemma 2.2(a), we get an isomorphism

$$\phi'\colon \mathcal{F}' \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{F}', \pi^! \mathcal{L}) \tag{12}$$

of $\mathcal{O}_{X'}$-modules. Obviously, the existence of isomorphisms (11) and (12) is equivalent. Note that, according to Lemma 2.2(b),

$$\pi^! \mathcal{L} \cong (\pi^! \mathcal{O}_X) \otimes_{\mathcal{O}_{X'}} (\pi^* \mathcal{L}). \tag{13}$$

2.7. The sheaf $\pi^! \mathcal{O}_X$ is the *conductor sheaf* of X' over X, that is, the largest ideal sheaf $\mathcal{I}'$ of $\mathcal{O}_{X'}$ for which $\pi_* \mathcal{I}'$ is contained in $\mathcal{O}_X$. If X is projective (which is our case of interest), $\pi^! \mathcal{O}_X$ is related to the dualizing sheaves ω (respectively ω') of X (respectively X') as follows. One has $\pi^! \omega = \omega'$ ([Ha], ex. III.7.2). If X is a Gorenstein curve, then ω is an invertible sheaf, and (13), for $\mathcal{L} := \omega$, gives

$$\pi^! \mathcal{O}_X \cong \omega' \otimes_{\mathcal{O}_{X'}} (\pi^* \omega)^{-1}. \tag{14}$$

2.8. Assuming $\mathcal{F} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$ as in 2.6, the $\mathcal{O}_{X'}$-module $\mathcal{F}' = \pi^o\mathcal{F}$ is torsion-free of rank one, self-dual in the sense of (12), and satisfies $\mathcal{E}nd_{\mathcal{O}_{X'}}(\mathcal{F}') = \mathcal{O}_{X'}$. Without additional hypotheses, $\mathcal{F}'$ need not be an invertible sheaf on X', and neither need $\pi^!\mathcal{L}$ be invertible. The situation gets more accessible, though, if we assume that X has simple singularities, as we are now going to explain. For the rest of this section we assume $\mathrm{char}(k) = 0$.

2.9. Let A be a complete noetherian local ring of dimension one, with algebraically closed residue field $k = \bar{k}$ of characteristic zero, and assume that A is a *simple* (or *ADE*) *curve singularity*. Thus A is isomorphic to one of the singularities A_n $(n \geq 0)$, D_n $(n \geq 4)$ or E_n $(n = 6, 7, 8)$, where A_0 denotes the case where $A = k[[t]]$ is regular. Let $\widetilde{A}$ denote the normalization of A, i. e., the integral closure of A in its total ring of fractions. For later reference we recall some well-known facts:

(1) A is a reduced Gorenstein ring.
(2) There are only finitely many self-dual torsion-free A-modules M of rank one, up to isomorphism.
(3) Let M be as in (2), and put $B := \mathrm{End}_A(M)$. Then M is free (of rank one) as a B-module.
(4) For M and B as in (3), the semilocal ring B is a finite direct product of simple curve singularities. In particular, B is a Gorenstein ring.

In particular, the self-dual torsion-free A-modules of rank one are, up to A-module isomorphism, precisely the intermediate rings $A \subset B \subset \widetilde{A}$ with $B \cong \mathrm{Hom}_A(B, A)$ (as A-modules). There are only finitely many such B, and for any given A, their list is precisely known. For these facts we refer to [GK], [Y] (ch. 9) and [Pi] (Theorem 3.1). For the singularities occuring on plane quartics, we'll include the explicit list in 3.10 below.

Definition 2.10. Let k be a field, $\mathrm{char}(k) = 0$, and let X be an integral curve over k. We say that X has *simple singularities* if the following is true: For any closed point y of the curve $\bar{X} := X \otimes_k \bar{k}$, the completion $\widehat{\mathcal{O}_{\bar{X},y}}$ is a simple singularity (as in 2.9).

2.11. Let x be a closed point of X, let $y_1, \dots, y_r$ be the points of $\bar{X}$ lying over x. Then $\mathcal{O}_{X,x} \otimes_k \bar{k}$ is the semilocal ring of $\bar{X}$ in $y_1, \dots, y_r$, and consequently its completion satisfies

$$\big(\mathcal{O}_{X,x} \otimes_k \bar{k}\big)^\frown \cong \widehat{\mathcal{O}_{\bar{X},y_1}} \times \cdots \times \widehat{\mathcal{O}_{\bar{X},y_r}}.$$

Thus, X has simple singularities in the sense of Definition 2.10 if and only if, for every closed point x of X, the completion of the semilocal ring $\mathcal{O}_{X,x} \otimes_k \bar{k}$ is a (finite) direct product of complete local rings of *ADE* type (with residue field $\bar{k}$).

We keep assuming $\mathrm{char}(k) = 0$. Our goal is to prove Proposition 2.15 below.

Lemma 2.12. *Let X be an integral curve over k with simple singularities. Let x be a closed point of X and $A = \mathcal{O}_{X,x}$, let M be a torsion-free A-module of rank one with $M \cong \mathrm{Hom}_A(M, A)$, and let $B = \mathrm{End}_A(M)$. Then*

(a) *B is a Gorenstein ring.*
(b) *M is free (of rank one) as a B-module.*

In particular, $B \cong \mathrm{Hom}_A(B, A)$ as A-modules.

Proof. Let $A_1 := (A \otimes_k \bar{k})\hat{}$. Then A_1 is a direct product of finitely many complete local rings of ADE type, and $A \to A_1$ is faithfully flat. Let $M_1 = M \otimes_A A_1$, a torsion-free A_1-module of rank one (cf. 1.3). Since $A \to A_1$ is flat, one has $\mathrm{Hom}_A(N, N') \otimes_A A_1 \cong \mathrm{Hom}_{A_1}(N \otimes_A A_1, N' \otimes_A A_1)$ for any two finitely generated A-modules N, N'. It follows that $\mathrm{End}_{A_1}(M_1) = B \otimes_A A_1 =: B_1$, and that M_1 is self-dual as an A_1-module. By the facts recalled in 2.9(3) and (4), M_1 is free of rank one as a B_1-module, and B_1 is a Gorenstein ring. Since $B \to B_1$ is faithfully flat, it follows that B is Gorenstein as well (e. g., by [Bo], ch. X, §3, no. 8, Corollary 1), and that M is locally free as a B-module ([EGA] IV.2.5.2), hence free since B is semilocal. □

Lemma 2.13. *Let A be a local Gorenstein ring of dimension one, and let $A \subset B$ be a finite ring extension such that B is reduced. Then the following are equivalent:*

(i) *The B-module $\mathrm{Hom}_A(B, A)$ is free (of rank one);*
(ii) *B is a Gorenstein ring.*

Proof. Since A is Gorenstein, $\mathrm{Hom}_A(B, A)$ is a canonical module for B, e. g. by [Bo], §9, no. 3, Proposition 6. (Note that the A-module B is Cohen-Macaulay since B is reduced.) Hence B is Gorenstein iff the B-module $\mathrm{Hom}_A(B, A)$ is locally free, iff it is free (since B is semilocal). □

Corollary 2.14. *Assume that the integral curve X over k has simple singularities. Let $\pi\colon X' \to X$ be a partial normalization (cf.* 2.3*). The following conditions are equivalent:*

(i) *X' is a Gorenstein curve;*
(ii) *$\pi^!\mathcal{O}_X$ is an invertible sheaf on X';*
(iii) *the $\mathcal{O}_X$-modules $\mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_{X'}, \mathcal{O}_X)$ and $\pi_*\mathcal{O}_{X'}$ are locally isomorphic.*

When these conditions hold we shall simply say that $X' \to X$ is a *Gorenstein partial normalization* of X.

Proof. (i) $\Leftrightarrow$ (ii) follows from Lemma 2.13, (ii) $\Rightarrow$ (iii) is trivial since $\mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_{X'}, \mathcal{O}_X) = \pi_*\pi^!\mathcal{O}_X$, and (iii) $\Rightarrow$ (i) follows from Lemma 2.12. □

Proposition 2.15. (char$(k) = 0$) *Assume that the integral curve X over k has simple singularities. Let $\mathcal{F}$ be a torsion-free $\mathcal{O}_X$-module of rank one with $\mathcal{F} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$ for some invertible sheaf $\mathcal{L}$ on X, and let $\pi\colon X' \to X$ be the partial normalization of X defined by $\mathcal{F}$ (see 2.4). Then*

(a) *X' is a Gorenstein curve (with simple singularites);*
(b) *$\pi^!\mathcal{O}_X$, $\pi^!\mathcal{L}$ and $\mathcal{F}' = \pi^o\mathcal{F}$ are invertible $\mathcal{O}_{X'}$-modules.*

Proof. Lemma 2.12 implies (a), and also that $\mathcal{F}'$ is an invertible $\mathcal{O}_{X'}$-module. Since X' is Gorenstein, $\pi^!\mathcal{O}_X$ is invertible (see Corollary 2.14). By (13), $\pi^!\mathcal{L} = (\pi^!\mathcal{O}_X) \otimes (\pi^*\mathcal{L})$ is invertible as well. □

2.16. Using 2.6, it follows from Proposition 2.15 that $\mathcal{F}'$ satisfies $\mathcal{F}' \otimes_{\mathcal{O}_{X'}} \mathcal{F}' \cong \pi^!\mathcal{L}$. Provided that $\pi^!\mathcal{L}$ is a double in $\mathrm{Pic}(X')$, the number of non-isomorphic torsion-free rank one $\mathcal{O}_X$-modules $\mathcal{F}$ satisfying $\mathcal{F} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$ and $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}) = \pi_*\mathcal{O}_{X'}$ is therefore equal to the cardinality of the 2-torsion subgroup $\mathrm{Pic}_2(X')$ of $\mathrm{Pic}(X')$.

Assume now $k = \bar{k}$, and that X is projective. If $\deg(\mathcal{L})$ is even, then $\pi^!\mathcal{L}$ is a double in $\mathrm{Pic}(X')$. Indeed, by (13), and since the generalized Jacobian of X' is a divisible group, this amounts to saying that the invertible sheaf $\pi^!\mathcal{O}_X$ on X' has even degree. This is indeed true since, according to (14), we have

$$\deg(\pi^!\mathcal{O}_X) = 2\,p_a(X') - 2\,p_a(X). \tag{15}$$

The following corollary summarizes the discussion so far:

Theorem 2.17. *Let $\mathrm{char}(k) = 0$, and let X be a plane projective integral curve of degree $2d > 0$ over k, with simple singularities. Then there is a natural bijection between the following objects (each up to isomorphism):*

(1) *Torsion-free coherent $\mathcal{O}_X$-modules $\mathcal{F}$ of rank one satisfying $\mathcal{F} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X(d))$;*
(2) *pairs $(X', \mathcal{F}')$, where $\pi\colon X' \to X$ is a Gorenstein partial normalization of X and $\mathcal{F}' \in \mathrm{Pic}(X')$ satisfies $\mathcal{F}' \otimes \mathcal{F}' \cong \pi^!\mathcal{O}_X(d) \cong \omega_{X'} \otimes \pi^*\mathcal{O}_X(3-d)$.*

If $k = \bar{k}$ is algebraically closed, the total number of these objects is $\sum_{X'} |\,\mathrm{Pic}_2(X')|$ (sum over all X' as in (2)).

Proof. This is clear from the previous results. The isomorphism $\pi^!\mathcal{O}_X(d) \cong \omega_{X'} \otimes \pi^*\mathcal{O}_X(3-d)$ in (2) follows from $\omega_X = \mathcal{O}_X(2d-3)$ and (13) and (14). Since $\deg \mathcal{O}_X(d) = 2d^2$ is even, $\pi^!\mathcal{O}_X(d)$ is a double in $\mathrm{Pic}(X'_{\bar{k}})$ by 2.16, which gives the additional statement for $k = \bar{k}$. □

2.18. Given X as in Theorem 2.17, the Gorenstein partial normalizations X' of X correspond to families $\mathcal{O}_{X,x} \subset B_x \subset \widetilde{\mathcal{O}}_{X,x}$ $(x \in X)$ of intermediate rings (see 2.3), for which B_x is self-dual as an $\mathcal{O}_{X,x}$-module (see Corollary 2.14). Knowing the singularities $\mathcal{O}_{X,x}$, these B_x can be listed explicitly, by descending the known list (cf. 2.9 and 3.10) from $(\mathcal{O}_{X,x} \otimes_k \bar{k})\widehat{\ }$ to $\mathcal{O}_{X,x}$.

Namely, let X be an integral curve over k (where $\mathrm{char}(k)=0$), and let x be a closed point of X. Let $A=\mathcal{O}_{X,x}$, write $A_0=A\otimes_k\bar{k}$ and $A_1=\widehat{A_0}$. Let $\widetilde{A}$ be the normalization of A. Then $\widetilde{A}_1:=A_1\otimes_A\widetilde{A}$ is the normalization of A_1 (cf. [EGA], IV.7.8.3). The Galois group $G_k=\mathrm{Gal}(\bar{k}/k)$ acts on A_1 over A, and hence also on $\widetilde{A}_1$ over $\widetilde{A}$. Let $\mathcal{R}$ be the set of intermediate rings of $A\subset\widetilde{A}$ and $\mathcal{R}_1$ the set of intermediate rings of $A_1\subset\widetilde{A}_1$. Then $B\mapsto B\otimes_A A_1$ is a bijection from $\mathcal{R}$ onto $\mathcal{R}_1^{G_k}$ (the set of G_k-invariant elements of $\mathcal{R}_1$), with inverse $B_1\mapsto\widetilde{A}\cap B_1$. This bijection is compatible with the property of being self-dual, in the following sense: $B\in\mathcal{R}$ satisfies $B\cong\mathrm{Hom}_A(B,A)$ (as A-modules) if and only if $B_1:=B\otimes_A A_1$ satisfies $B_1\cong\mathrm{Hom}_{A_1}(B_1,A_1)$ (as A_1-modules).

This can be seen as follows. As before, we continuously use that $A\to A_1$ is faithfully flat. Let $\mathfrak{f}=\{a\in A\colon a\widetilde{A}\subset A\}$, the conductor of $\widetilde{A}$ over A. Then $\mathfrak{f}_1:=\mathfrak{f}\otimes_A A_1$ is the conductor of $\widetilde{A}_1$ over A_1. The elements of $\mathcal{R}$ correspond to the intermediate rings of $A/\mathfrak{f}\subset\widetilde{A}/\mathfrak{f}$, and similarly for $\mathcal{R}_1$. Now $\widetilde{A}/\mathfrak{f}$ is a finite k-algebra, and $\widetilde{A}_1/\mathfrak{f}_1=(\widetilde{A}/\mathfrak{f})\otimes_k\bar{k}$. From this the first assertion is clear. Let $B\in\mathcal{R}$, put $M=\mathrm{Hom}_A(B,A)$, $B_1=B\otimes_A A_1$ and $M_1=M\otimes_A A_1=\mathrm{Hom}_{A_1}(B_1,A_1)$. If $B_1\cong M_1$ as A_1-modules, then $B\cong M$ as A-modules; see [EGA], IV.2.5.8. The converse is clear anyway.

2.19. This is a remark on the relation to theta characteristics. Let X be an integral plane curve of degree $2d$, as before. We are studying torsion-free $\mathcal{O}_X$-modules $\mathcal{F}$ of rank one satisfying $\mathcal{F}\cong\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{O}_X(d))$ (see Proposition 1.2 and Theorem 2.17). A theta characteristic on X is a torsion-free rank one $\mathcal{O}_X$-module $\mathcal{G}$ with $\mathcal{G}\cong\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G},\omega_X)$ ([Hr], p. 618, [Pi]). Assuming that X has simple singularities, theta characteristics on X correspond to locally free theta characteristics on Gorenstein partial normalizations of X. For $k=\bar{k}$, theta characteristics on X are in bijection with our sheaves $\mathcal{F}$, but there exists no natural bijection in general. Over a non-closed field, there need not even be a bijection. Since, moreover, we need to identify the condition that $\mathcal{F}$ is generated by H^0, and this condition is not preserved by a bijection with theta characteristics, it appears that the relation with theta characteristics is too loose to be useful. An exception would be the case of sextics, where the sheaves $\mathcal{F}$ are precisely the theta characteristics on X.

Definition 2.20. Let $cf=p_1p_2-p_0^2=\det(P)$ be a determinantal representation of $f\in\mathcal{P}_{2d}(k)$ as in 1.1, let $X=V_+(f)$. The closed subscheme $V_+(p_0,p_1,p_2)$ of X is called the *base locus* of the representation (or of P). The representation is *base-point free* if the base locus is empty.

Clearly, the base locus is contained in the singular locus of X. It may be non-reduced. We show now that the ideal sheaf of the base locus coincides with the conductor sheaf of the partial normalization defined by $\mathcal{F}=\mathcal{F}_P$.

Lemma 2.21. *Let $(R, \mathfrak{m})$ be a regular local ring, $\dim(R) = 2$, let $0 \neq f \in \mathfrak{m}$ such that $A = R/fR$ is reduced, and let M be a finitely generated torsion-free A-module of rank one. Then $M \cong \mathrm{Fitt}_1(M)$, and $M = \mathrm{Fitt}_1(M)$ if M is an ideal of A.*

Here $\mathrm{Fitt}_1(M)$ is the first Fitting ideal of M, i. e., the ideal generated by the $(n-1)$-minors of φ in a free presentation $A^m \xrightarrow{\varphi} A^n \to M \to 0$.

Proof. Since M is isomorphic to an ideal of A, it suffices to prove $I = \mathrm{Fitt}_1(I)$ for every ideal I of A which does not entirely consist of zero-divisors of A. Let $J \subset R$ be the ideal with $I = J/fR$. By the Auslander-Buchsbaum formula, $\mathrm{projdim}_R(R/J) = 2$, and hence J is perfect of grade two. So $J = \mathrm{Fitt}_1^R(J)$ by the Hilbert-Burch Theorem, which proves the lemma. □

Corollary 2.22. *Let $0 \neq f \in \mathcal{P}_{2d}(k)$ such that $X = V_+(f)$ is integral. Let $P \in \mathrm{Sym}_2\mathcal{P}_d(k)$ with $\det(P) = cf$, and let $\mathcal{I}$ be the ideal sheaf of the base locus of P (see Definition 2.20). Let $\mathcal{F} = \mathcal{F}_P$ be the $\mathcal{O}_X$-module associated to P, and let $\pi\colon X' \to X$ be the partial normalization defined by $\mathcal{F}$ (see 2.4). Assume that X has simple singularities. Then $\mathcal{I}$ is equal to the conductor sheaf (see 2.7) of X' over X, i. e., $\mathcal{I} = \pi_*\pi^!\mathcal{O}_X$.*

Proof. Let x be a closed point of X, let $A = \mathcal{O}_{X,x}$, $M = \mathcal{F}_x$ and $I = \mathcal{I}_x$. Then $I = \mathrm{Fitt}_1(M)$ by the definition of I. On the other hand, let $B = \mathrm{End}_A(M)$, and let $(A : B)$ be the conductor of B over A. Then $M \cong (A : B)$ as A-modules by Lemma 2.12, and therefore $I = \mathrm{Fitt}_1(M) = \mathrm{Fitt}_1(A : B)$. By Lemma 2.21, this implies $I = (A : B)$. □

In particular, we see:

Corollary 2.23. *In the situation of Corollary 2.22, the representation P is base-point free if and only if the $\mathcal{O}_X$-module $\mathcal{F}$ is locally free.* □

3. Exceptional sheaves for curves of degree four

3.1. Let $f \in k[x_0, x_1, x_2]$ be square-free, homogeneous of degree $2d > 0$, and let $X = V_+(f)$. By Theorem 1.15, the equivalence classes of representations

$$cf = p_1p_2 - p_0^2 \tag{16}$$

(with $p_i \in \mathcal{P}_d(k)$) are in bijective correspondence with the isomorphism classes of coherent $\mathcal{O}_X$-modules $\mathcal{F}$ satisfying the following four properties:

(a) $\mathcal{F} \ncong \mathcal{O}_X(\frac{d}{2})$ (if d is even);
(b) $\mathcal{F}$ is torsion-free of rank one;
(c) $\mathcal{F} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X(d))$;
(d) $\mathcal{F}$ is generated by its global sections.

Assuming that X is irreducible and has ADE singularities, the classification of these $\mathcal{F}$ was almost reduced in Theorem 2.17 to the classification of line bundles $\mathcal{F}'$ on partial normalizations X' of X which satisfy a self-duality. Only "almost" since condition (d) was ignored in Theorem 2.17.

If $2d \ge 6$, it is not possible to make general statements about the $\mathcal{F}$ which not only satisfy condition (1) of Theorem 2.17, but are also globally generated. Indeed, for generic f of degree $2d \ge 6$ there does not exist any identity (16), as a simple dimension count shows. If $2d = 4$, on the other hand, it is indeed possible to control the global generation condition. This is the subject of this section. Briefly, the $\mathcal{F}$ which satisfy (a)–(c) but violate (d) are in canonical bijection with the k-rational singular points of X. As a consequence we will obtain, at least for $k = \mathbb{C}$, the complete classification of representations (16) in terms of nothing more than the list of singularities of X (see Section 4).

3.2. In the sequel, k can be any field of characteristic $\ne 2$. Fix a nonzero form $f \in k[x_0, x_1, x_2]$ of degree four, let $X = V_+(f)$, and assume that f has no factor of degree two in $k[x_0, x_1, x_2]$ (so, in particular, X is reduced). Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module which satisfies (a)–(c) of 3.1 (with $d = 2$). By the proof of Proposition 1.2 (see (7)), $\mathcal{F}$ has a minimal resolution

$$0 \to \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^2}(-d_i - 2) \xrightarrow{M} \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^2}(d_i) \to \mathcal{F} \to 0 \tag{17}$$

with suitable $d_i \in \mathbb{Z}$, in which $M = (m_{ij})$ is a symmetric $r \times r$-matrix whose entries are homogeneous of degrees $\deg(m_{ij}) = d_i + d_j + 2$, and $\det(M) = cf$ with $c \in k^*$. Rearranging the summands we assume

$$d_1 \ge d_2 \ge \cdots \ge d_r. \tag{18}$$

Proposition 3.3. *Keep the assumptions of* 3.2 *(in particular, f has no quadratic factor over k). Then exactly one of the following two possibilities happens:*

(1) *$r = 2$ and $d_1 = d_2 = 0$, hence $M = \begin{pmatrix} p_1 & p_0 \\ p_0 & p_2 \end{pmatrix}$ with quadratic forms p_i; so $cf = p_1 p_2 - p_0^2$, and* (17) *is*

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^2 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^2 \to \mathcal{F} \to 0.$$

(2) *$r = 3$ and $d_1 = d_2 = 0$, $d_3 = -1$, hence $M = \begin{pmatrix} p_1 & p_0 & l_2 \\ p_0 & p_2 & l_1 \\ l_2 & l_1 & 0 \end{pmatrix}$ with quadratic forms p_i and linear forms l_j; so $cf = 2l_1 l_2 p_0 - l_1^2 p_1 - l_2^2 p_2$, and* (17) *is*

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^2 \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^2 \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{F} \to 0.$$

$\mathcal{F}$ is generated by its global sections in case (1)*, but not in case* (2)*. In case* (2) *we say that $\mathcal{F}$ is an* exceptional $\mathcal{O}_X$-module.

Conversely, any sheaf $\mathcal{F}$ with a resolution as in (1) *or* (2) *satisfies* (a)–(c) *of* 3.1.

Proof. The last statement is clear, see the proof of Proposition 1.2(a) in 1.5. Therefore let $\mathcal{F}$ be as in 3.2, with minimal resolution (17). Comparing degrees in $\det(M) = cf$ gives $2r + 2\sum_i d_i = 4$, i. e. $\sum_i d_i = 2 - r$. Minimality of (17) means $m_{ij} = 0$ whenever $d_i + d_j \le -2$ (see 0.2 or [Be], 1.5). As in the proof of Lemma 1.7, we have $d_i + d_{r+1-i} \ge -1$ for all i, and we conclude $2\sum_i d_i \ge -r$. Since $\sum_i d_i = 2 - r$, this already gives $r \le 4$. In the case $r = 4$ we would necessarily have $d_1 + d_4 = d_2 + d_3 = -1$, and hence $d_3 \le -1$. Then M would have the shape

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}.$$

Such a symmetric determinant is a square, contradicting the assumption. Hence $r \le 3$.

If $r = 3$ we have $d_1 + d_3 \ge -1$, $d_2 \ge 0$ and $d_1 + d_2 + d_3 = -1$, which imply $(d_1, d_2, d_3) = (e,\, 0,\, -1-e)$ with some $e \ge 0$. If $e \ge 1$, then M would have the shape,

$$\begin{pmatrix} * & * & * \\ * & * & 0 \\ * & 0 & 0 \end{pmatrix},$$

which would imply that f factors as a product of a quadratic form and two linear forms; a contradiction. Therefore, $r = 3$ implies $(d_1, d_2, d_3) = (0, 0, -1)$, and this gives case (2) of the proposition. By Lemma 1.7, $\mathcal{F}$ is not generated by H^0.

If $r = 2$, then $(d_1, d_2) = (e, -e)$ with $e \ge 0$. Assuming $e \ge 1$ gives for M the shape,

$$\begin{pmatrix} * & * \\ * & 0 \end{pmatrix},$$

which would imply that f is a product of two quadratic factors, again contradicting our assumption. Therefore, $r = 2$ implies $(d_1, d_2) = (0, 0)$, which is case (1). Finally, the case $r = 1$ is excluded by the hypothesis $\mathcal{F} \ne \mathcal{O}_X(1)$. $\square$

Lemma 3.4. *Let p_i be quadratic forms and l_j linear forms in $k[x_0, x_1, x_2]$, let*

$$M = \begin{pmatrix} p_1 & p_0 & l_2 \\ p_0 & p_2 & l_1 \\ l_2 & l_1 & 0 \end{pmatrix}$$

and

$$f = \det(M) = 2p_0 l_1 l_2 - p_1 l_1^2 - p_2 l_2^2.$$

Further let $\mathcal{F}$ *be the cokernel of*

$$M\colon \mathcal{O}_{\mathbb{P}^2}(-2)^2 \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2}^2 \oplus \mathcal{O}_{\mathbb{P}^2}(-1).$$

Assume that f *is square-free, let* X *be the quartic curve* $f = 0$*. Then the lines* $l_j = 0$ *(*$j = 1, 2$*) are distinct, and their intersection* z *is a singular (*k*-rational) point of* X*. The global sections* $H^0(X, \mathcal{F})$ *generate* $\mathcal{F}$ *in every point* $x \neq z$*, but do not generate* $\mathcal{F}$ *in* z*. Away from* z*, the* $\mathcal{O}_X$*-module* $\mathcal{F}$ *is locally free.*

Proof. It is clear that the lines l_1 and l_2 intersect in a singular point z of X. Let $x \in X$, and let e_ν ($\nu = 1, 2, 3$) denote the images of the three canonical local generators of $\mathcal{O}^2 \oplus \mathcal{O}(-1)$ in x (each e_ν being determined up to a unit of $\mathcal{O}_{X,x}$). The submodule generated by $H^0(X, \mathcal{F})$ in $\mathcal{F}_x$ is the submodule generated by e_1 and e_2. If $x \neq z$, then $l_j(x) \neq 0$ for at least one $j \in \{1, 2\}$, and from the shape of M one concludes that $\mathcal{F}_x = \mathcal{O}_{X,x}\, e_j$. On the other hand, $x = z$ implies $e_3 \notin \mathcal{O}_{X,x}\, e_1 + \mathcal{O}_{X,x}\, e_2$. □

Definition 3.5. Assume that the quartic form f has no factor of degree two, and let $X = V_+(f)$. Given an exceptional $\mathcal{O}_X$-module $\mathcal{F}$ (see Proposition 3.3) there is, according to Lemma 3.4, a unique singular k-rational point z on X such that $\mathcal{F}$ is generated by $H^0(X, \mathcal{F})$ away from z. We say that z is the singular point *associated to* $\mathcal{F}$.

We shall now show that, conversely, the associated singular point z determines $\mathcal{F}$ up to isomorphism. As before, let f be a form of degree four and $X = V_+(f)$.

Proposition 3.6. *Assume that* f *has no factor of degree two. Let* z *be a* k*-rational singular point on* X*. Then there exists an exceptional* $\mathcal{O}_X$*-module* $\mathcal{F}$ *with associated point* z *(see Definition* 3.5*). Moreover, any two such* $\mathcal{F}$ *are isomorphic.*

Proposition 3.6 justifies the following definition:

Definition 3.7. Given f and z as in Proposition 3.6, we call $\mathcal{F}$ the *exceptional sheaf associated with* z. The partial normalization $\pi\colon X' \to X$ defined by $\mathcal{F}$ (with $\pi_*\mathcal{O}_{X'} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F})$, see 2.4) will be called the *exceptional partial normalization associated with* z.

Proof of Proposition 3.6. The existence of such $\mathcal{F}$ is immediate: Choosing affine coordinates (u, v) centered at z, f has the affine form

$$f(u, v) = f_4(u, v) + f_3(u, v) + f_2(u, v)$$

with f_ν homogeneous of degree ν; it is obvious that such f can be written in the form

$$f(u, v) = 2\, uv\, q_0 - u^2 q_1 - v^2 q_2$$

with polynomials $q_i(u,v)$ of degree ≤ 2. Homogenizing, this gives $\mathcal{F}$ as desired (Lemma 3.4).

It remains to prove uniqueness of $\mathcal{F}$. If M_1, M_2 are two injective homomorphisms $\mathcal{O}_{\mathbb{P}^2}(-2)^2 \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2}^2 \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$, and if $\mathcal{F}_1$, $\mathcal{F}_2$ are their cokernels, then $\mathcal{F}_1 \cong \mathcal{F}_2$ provided there are invertible 3×3-matrices A, B of the shape,

$$\begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ m_1 & m_2 & c_{33} \end{pmatrix},$$

with $M_2B = A^tM_1$, for which $c_{ij} \in k$ and m_1, m_2 are linear forms. Let us write $M_1 \sim M_2$ in this case. Let $\mathcal{F}$ be an exceptional $\mathcal{O}_X$-module with associated singular point z, and let

$$M = \begin{pmatrix} p_1 & p_0 & l_2 \\ p_0 & p_2 & l_1 \\ l_2 & l_1 & 0 \end{pmatrix}$$

be a matrix with $\mathcal{F} = \operatorname{coker}(M)$. Then the lines $l_1 = 0$, $l_2 = 0$ intersect in z. One immediately sees that

$$M \sim M' := \begin{pmatrix} ap_1 & ap_0 & l_2 \\ ap_0 & ap_2 & l_1 \\ l_2 & l_1 & 0 \end{pmatrix}$$

for any $a \in k^*$, and $\det(M') = a\,\det(M)$. Thus we can assume $\det(M) = f$. Given any other pair of distinct lines l_1', l_2' which intersect in z, we can write $l_j' = a_{1j}l_1 + a_{2j}l_2$ $(j = 1, 2)$ with $\det(a_{ij}) \neq 0$. Using the transformation

$$S := \begin{pmatrix} a_{22} & a_{12} & 0 \\ a_{21} & a_{11} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

SMS^t becomes a matrix of the same shape as M in which l_j is replaced by l_j' for $j = 1, 2$ (and the p_i by other quadratics p_i'). Therefore, given a second exceptional $\mathcal{O}_X$-module $\mathcal{F}'$ with associated singular point z, we can assume $\mathcal{F}' = \operatorname{coker}(M')$, where

$$M' = \begin{pmatrix} p_1' & p_0' & l_2 \\ p_0' & p_2' & l_1 \\ l_2 & l_1 & 0 \end{pmatrix}$$

has the same linear forms l_1, l_2 as M, and for which $\det(M') = f$ as well. Writing $q_i := p_i' - p_i$ $(i = 0, 1, 2)$, we have

$$2l_1l_2q_0 - l_1^2q_1 - l_2^2q_2 = 0$$

since $\det(M') = \det(M)$. So there exist linear forms m_1, m_2 with $q_1 = 2m_1l_2$, $q_2 = 2m_2l_1$ and $q_0 = m_1l_1 + m_2l_2$. The matrix,

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m_1 & m_2 & 1 \end{pmatrix},$$

satisfies $S^tMS = M'$, which shows $M' \sim M$, hence $\mathcal{F}' \cong \mathcal{F}$. This proves the proposition. □

For later use we record the following variant, which is shown in a completely similar way:

Lemma 3.8. *Let $(A, \mathfrak{m})$ be a regular local ring of dimension two containing $\frac{1}{2}$. Given $f \in \mathfrak{m}^2$, there is a symmetric matrix,*

$$M = \begin{pmatrix} p_1 & p_0 & x_2 \\ p_0 & p_2 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix},$$

over A with $\det(M) = f$, where x_1, x_2 generate $\mathfrak{m}$. The $A/(f)$-module $N = \operatorname{coker}(M)$ depends, up to isomorphism, only on the ideal (f), but not on the choice of M. □

We summarize:

Corollary 3.9. *Let f be a form of degree four without a factor of degree two, and let $X = V_+(f)$. Then the isomorphism classes of $\mathcal{O}_X$-modules $\mathcal{F}$ which satisfy* (a)–(c)*, but not* (d)*, of* 3.1*, are in natural bijection with the set of k-rational singular points of X. The bijection is given by $\mathcal{F} \mapsto z$, where z is the unique point in which $H^0(X, \mathcal{F})$ fails to generate $\mathcal{F}$.* □

3.10. Let R be a simple curve singularity as in 2.9, i. e., a complete noetherian one-dimensional local ring of ADE type with residue field $k = \bar{k}$ of characteristic zero. We need to refer to specific self-dual R-modules of rank one, and so we need notational conventions to talk about them. Fortunately, the list of all such modules is well documented; see for example [GK] and [Y], ch. 9 or [Pi], Theorem 3.1. To save space, we refer to one of these sources for the complete list, and content ourselves here with listing the necessary information for those singularities that can occur on a plane irreducible quartic curve. These are A_n $(n \le 6)$, D_4, D_5 and E_6.

In each case, a polynomial $f \in k[x, y]$ with $R \cong k[[x, y]]/(f)$ is listed. We give R as a subring of its normalization $\widetilde{R}$ and write $\delta = \dim_k(\widetilde{R}/R)$. We list those intermediate rings $R \subset B \subset \widetilde{R}$ which are self-dual as R-modules (briefly called "self-dual partial normalizations" below); they also represent the isomorphism classes of all self-dual torsion-free R-modules of rank one (cf. 2.9). Since B is again an ADE singularity (possibly reducible), we also mention its type. We write A_0 for the case where R is normal, and A_{-1} for the direct product of two rings of type A_0.

A_n, n even: $f = y^2 - x^{n+1}$, $\widetilde{R} = k[[t]]$, $R = k[[t^2, t^{n+1}]]$, $\delta = \frac{n}{2}$. The self-dual partial normalizations are the $B_i = R[t^{n+1-2i}]$, of type A_{n-2i} $(i = 0, \dots, \delta)$.

A_n, n odd: $f = y^2 - x^{n+1}$, $\widetilde{R} = k[[t]] \times k[[t]]$, $R = k[[(t,t),\ (t^\delta, 0)]]$ where $\delta = \frac{n+1}{2}$. The self-dual partial normalizations are the $B_i = R[(t^{\delta-i}, 0)]$, of type A_{n-2i} $(i = 0, \dots, \delta)$.

D_4: $f = x^2y - y^3$, $\widetilde{R} = k[[t]] \times k[[t]] \times k[[t]]$, $R = k[[(t,t,0),\ (t,0,t)]]$, $\delta = 3$. The self-dual partial normalizations are $B_0 = R$, $B_2 = \widetilde{R}$ (of type $A_0 \times A_0 \times A_0$) and

$$B_1 = R[(1,0,0)],\ B_1' = R[(0,1,0)],\ B_1'' = R[(0,0,1)],$$

each of type $A_0 \times A_1$.

D_5: $f = x^2y - y^4$, $\widetilde{R} = k[[t]] \times k[[t]]$, $R = k[[(t^3,t),\ (t^2,0)]]$, $\delta = 3$. The self-dual partial normalizations are $B_0 = R$, $B_1 = R[(1,0)]$ (of type $A_2 \times A_0$), $B_1' = R[(t,0)]$ (of type A_1) and $B_2 = \widetilde{R}$ (of type $A_0 \times A_0$).

E_6: $f = y^3 - x^4$, $\widetilde{R} = k[[t]]$, $R = k[[t^3,\ t^4]]$, $\delta = 3$. The self-dual partial normalizations are $B_0 = R$, $B_1 = R[t^2]$ (of type A_2) and $B_2 = \widetilde{R}$ (of type A_0).

3.11. ($\mathrm{char}(k) = 0$) Let now X be a plane irreducible curve of degree 4 over k, and let z be a k-rational singular point on X. Let $\mathcal{F}$ be the exceptional sheaf and $\pi\colon X' \to X$ the partial normalization associated to z (see Definition 3.7). We want to determine π in terms of the singularity z. Since $\mathcal{F}$ is locally free on $X - z$ (see Lemma 3.4), π is an isomorphism over $X - z$. So we only need to determine the isomorphism type of the $\mathcal{O}_{X,z}$-module $\mathcal{F}_z$. By 2.18, it suffices to do so after passing to the completion of $\mathcal{O}_{X,z} \otimes_k \bar{k}$. In other words, we have to determine, for each ADE singularity $R = k[[x,y]]/(f)$ as in 3.10, the R-module $N = \mathrm{coker}(M)$ where M is a matrix as in Lemma 3.8 over $k[[x,y]]$ with $\det(M) = f$. According to Lemma 2.21, N is isomorphic to the ideal $I_2(M)$ generated by the 2×2-minors of the matrix M. This ideal can be directly read off, and the result is as follows. (See 3.10 for notation; in the table we write $B := \mathrm{End}_R(N)$, so $N \cong B$ as an R-module, even as a B-module.) From Table 1 we see that $X' = X$, i.e. $\mathcal{F}$ is a line bundle on X, if and only if z is an A_1-singularity.

sing. z	B	type(B)
A_1	$B_0 = R$	A_1
A_2	$B_1 = \widetilde{R}$	A_0
A_n $(n \geq 3)$	B_2	A_{n-4}
D_4	$B_2 = \widetilde{R}$	$A_0 \times A_0 \times A_0$
D_5	$B_2 = \widetilde{R}$	$A_0 \times A_0$
E_6	$B_2 = \widetilde{R}$	A_0

Table 1

4. Summary of determinantal representations

4.1. Let k be a field, $\operatorname{char}(k) = 0$. Let $f \in k[x_0, x_1, x_2]$ be a form of degree four, irreducible over $\bar{k}$, and let $X = V_+(f)$. From the results of the previous sections it follows that there is a natural bijective correspondence between the following objects:

(1) Determinantal representations

$$cf = p_1 p_2 - p_0^2 \tag{19}$$

(with $p_i \in \mathcal{P}_2(k)$, $c \in k^*$), up to equivalence (see Theorem 1.15);

(2) pairs $(X', \mathcal{F}')$, where $\pi\colon X' \to X$ is a Gorenstein partial normalization of X and $\mathcal{F}' \in \operatorname{Pic}(X')$ satisfies $\mathcal{F}' \otimes \mathcal{F}' \cong \pi^! \mathcal{O}_X(2)$. Moreover, it is required here that the $\mathcal{O}_X$-module $\mathcal{F} = \pi_* \mathcal{F}'$ is not exceptional (see Proposition 3.3), and that $\mathcal{F} \neq \mathcal{O}_X(1)$ if $X' = X$.

The total number of pairs $(X', \mathcal{F}')$ for which $\mathcal{F} = \pi_* \mathcal{F}'$ is exceptional and which otherwise satisfy condition (2) is equal to the number of k-rational singular points of X. In each case, the conductor sheaf of X' over X is equal to the ideal sheaf of the base locus of the corresponding representation (19).

4.2. At least in principle, a complete account can therefore be made as follows. First, list all singular points of X. With the help of the lists in 3.10 (and using 2.18), obtain the list of all Gorenstein partial normalizations $\pi\colon X' \to X$ of X. For each of them, decide whether $\pi^! \mathcal{O}_X$ is a double in $\operatorname{Pic}(X')$. If this is the case, there are $|\operatorname{Pic}_2(X')|$ many $\mathcal{F}$ corresponding to X'. From this list remove the following items:

1. $\mathcal{F} = \mathcal{O}_X(1)$ (corresponding to $X' = X$);
2. for every k-rational singular point x of X, the associated exceptional sheaf $\mathcal{F}$ (see Proposition 3.6); the partial normalization X' corresponding to $\mathcal{F}$ is listed in Table 1 of 3.11.

The remaining $\mathcal{O}_X$-modules $\mathcal{F}$ correspond bijectively to the equivalence classes of representations (19).

Speaking in general, there are two steps which tend to be difficult in this program: Deciding whether $\pi^!\mathcal{O}_X \in 2\,\mathrm{Pic}(X')$, and finding the number of 2-torsion classes in $\mathrm{Pic}(X')$.

4.3. In any case, we see that the number of base-point free (see Definition 2.20) determinantal representations (19) is equal to

$$|\,\mathrm{Pic}_2(X)| - (1+n),$$

where n is the number of k-rational nodes of X.

4.4. When k is algebraically closed, the difficulties disappear and we obtain a complete analysis. Let $X = V_+(f) \subset \mathbb{P}^2$ be an integral quartic curve over $k = \bar{k}$. For any possible configuration of singularities of X, the following table lists the total number of representations

$$f = p_1 p_2 - p_0^2 \tag{20}$$

(up to equivalence), as well as the number of base-point free such representations.

More generally, the recipe given in 4.2 allows us to give, for each such curve and each fixed base locus, the precise number of representations of f corresponding to this base locus. Obvious space limitations prohibit to include full details here, but the complete information is available from the author's web page. See the Examples section for a sample detailed discussion of a few selected cases.

Sing.	total	bp-free
smooth	63	63
A_1	46	30
A_2	30	15
A_3	18	7
A_4	10	3
A_5	5	1
A_6	2	–
D_4	9	3
D_5	4	1
E_6	1	–

Sing.	total	bp-free
$2A_1$	33	13
$A_1 + A_2$	21	6
$A_1 + A_3$	12	2
$A_1 + A_4$	6	–
$2A_2$	13	3
$A_2 + A_3$	7	1
$A_2 + A_4$	3	–
$3A_1$	23	4
$2A_1 + A_2$	14	1
$A_1 + 2A_2$	8	–
$3A_2$	4	–

TABLE 2

4.5. ($k = \bar{k}$) As remarked in 4.3, the number in the last column of Table 2 is equal to $|\operatorname{Pic}_2(X)| - (n+1)$, where n is the number of nodes of X. This shows that a key argument used in the proof of the main theorem of [W] contains a gap, and is actually incorrect in some cases. Namely, it was asserted in *loc. cit.* (p. 420, bottom) that any nonzero 2-torsion class in $\operatorname{Pic}(X)$ gives rise to a determinantal representation (20) of f, which we now see is wrong if X has a node. The argument was used in [W] to reduce the proof of the main theorem (existence of a representation (20)) to the case where X is unibranch at every singular point. From our analysis we see that there would have been two more cases to be considered, namely $A_1 + 2A_2$ and $A_1 + A_4$. Indeed, no class in $\operatorname{Pic}_2(X)$ gives rise to a representation (20) in these cases. Table 2 shows that, nevertheless, there do exist representations in both cases (with base points), showing that the correctness of the main theorem of [W] is not affected. For details in the case of $A_1 + A_4$, see 8.1 in the Examples section; see also 5.12 for how the error was initially discovered.

Part II: Twisted determinantal representations

5. Quadratic representations and Gram tensors

Let k be a field of characteristic not two, and let $\bar{k}$ be an algebraic closure of k. Fix $d \geq 1$, and recall that $\mathcal{P}_m(k)$ denotes the space of homogeneous polynomials of degree m in $k[x_0, x_1, x_2]$.

5.1. Let $f \in \mathcal{P}_{2d}(k)$. By a *quadratic representation* of f (over k) we mean a pair $(S, \boldsymbol{p})$ of a symmetric invertible 3×3-matrix $S = (s_{ij})_{0 \leq i,j \leq 2}$ over k and a triple $\boldsymbol{p} = (p_0, p_1, p_2)^t \in \mathcal{P}_d(k)^3$ (regarded as a column) such that

$$f = \boldsymbol{p}^t S \boldsymbol{p} = \sum_{i,j=0}^{2} s_{ij}\, p_i p_j.$$

Let $\operatorname{span}(\boldsymbol{p})$ be the linear span of p_0, p_1, p_2 in $\mathcal{P}_d(k)$, and call $\operatorname{rk}(\boldsymbol{p}) := \dim_k \operatorname{span}(\boldsymbol{p})$ the *rank* of $\boldsymbol{p}$. Two quadratic representations $(S, \boldsymbol{p})$ and $(S', \boldsymbol{p}')$ of f will be called *equivalent*, denoted $(S, \boldsymbol{p}) \sim (S', \boldsymbol{p}')$, if there exists $T \in \mathrm{GL}_3(k)$ with

$$S' = T^t S T \quad \text{and} \quad T\boldsymbol{p}' = \boldsymbol{p}.$$

Note that this implies $\operatorname{span}(\boldsymbol{p}) = \operatorname{span}(\boldsymbol{p}')$.

Alternatively, we could consider $(S, \boldsymbol{p})$ and $(S', \boldsymbol{p}')$ as equivalent if $\operatorname{span}(\boldsymbol{p}) = \operatorname{span}(\boldsymbol{p}')$. While in general this is a strictly coarser relation, it coincides with the above notion of equivalence whenever f is geometrically irreducible, as we show after the next lemma.

Lemma 5.2. *Let $(S, \boldsymbol{p})$ be a quadratic representation of f. If $\mathrm{rk}(\boldsymbol{p}) \le 2$, then f is a product of two forms of degree d over $\bar{k}$.*

Proof. Diagonalizing S we can assume $f = a_1 p_1^2 + a_2 p_2^2$ with $a_i \in k$ and $p_i \in \mathcal{P}_d(k)$, from which the lemma is clear. □

Proposition 5.3. *Let $(S, \boldsymbol{p})$, $(S', \boldsymbol{p}')$ be quadratic representations of f, and assume that f is not divisible by a form of degree d over $\bar{k}$. Then $(S, \boldsymbol{p}) \sim (S', \boldsymbol{p}')$ if and only if $\mathrm{span}(\boldsymbol{p}) = \mathrm{span}(\boldsymbol{p}')$.*

Proof. "Only if" has already been remarked, so assume $\mathrm{span}(\boldsymbol{p}) = \mathrm{span}(\boldsymbol{p}')$. Since $\mathrm{rk}(\boldsymbol{p}) = 3$ by Lemma 5.2, there is a unique matrix $T \in \mathrm{GL}_3(k)$ with $T\boldsymbol{p} = \boldsymbol{p}'$. Hence $\boldsymbol{p}^t(S - T^t S' T)\boldsymbol{p} = 0$. We show that the six forms $p_i p_j$ $(0 \le i \le j \le 2)$ are linearly independent, which implies $S = T^t S' T$. For this we can assume $k = \bar{k}$. Assume that the forms are linearly dependent. Then there is a nonzero quadratic form g in 3 variables with $g(p_0, p_1, p_2) = 0$, and g is irreducible since $\mathrm{rk}(\boldsymbol{p}) = 3$. After a linear change in the p_i we can assume $p_0^2 = p_1 p_2$. Hence we can write $p_0 = hq_1q_2$ and $p_i = hq_i^2$ $(i = 1, 2)$ with suitable forms h, q_1, q_2. It follows that $f = h^2 \phi(q_1, q_2)$ where $\phi(y_1, y_2)$ is a binary form of degree 4. This implies that f is a product of two forms of degree d; a contradiction. □

5.4. With a quadratic representation $(S, \boldsymbol{p})$ of f as above we associate its *Gram tensor*, defined by

$$\gamma(S, \boldsymbol{p}) \;:=\; \sum_{i,j=0}^{2} s_{ij} \cdot p_i \otimes p_j.$$

This is a symmetric tensor in $\mathcal{P}_d(k) \otimes \mathcal{P}_d(k)$. Note that the multiplication map

$$\mu\colon \mathcal{P}_d(k) \otimes \mathcal{P}_d(k) \to \mathcal{P}_{2d}(k) \tag{21}$$

maps $\gamma(S, \boldsymbol{p})$ to f. Equivalent quadratic representations have the same Gram tensor.

Remarks 5.5. 1. Every quadratic representation of f is equivalent to a diagonalized representation $f = \sum_{i=0}^{2} a_i p_i^2$ (with $a_i \in k^*$, $p_i \in \mathcal{P}_d(k)$).

2. Viewing S as a (nondegenerate) symmetric bilinear form on k^3 and $\boldsymbol{p}$ as a linear map $\mathcal{P}_d(k)^\vee \to k^3$, the Gram tensor $\gamma(S, \boldsymbol{p})$ is the pull-back $\boldsymbol{p}^* S$, a symmetric bilinear form on $\mathcal{P}_d(k)^\vee$.

3. Upon choosing a basis of the vector space $\mathrm{Sym}_2(k)$, every determinantal representation $f = \det(P)$ with $P \in \mathrm{Sym}_2 \mathcal{P}_d(k)$ (as considered in Section 1) becomes a quadratic representation of f. Up to equivalence, this quadratic representation is independent of the choice of the basis. So, with a slight abuse, we consider determinantal representations as particular quadratic representations. We simply write $\gamma(P)$ for the associated Gram tensor. Explicitly,

if $P = \begin{pmatrix} p_1 & p_0 \\ p_0 & p_2 \end{pmatrix}$, then

$$\gamma(P) = \frac{1}{2}\Big(p_1 \otimes p_2 + p_2 \otimes p_1\Big) - p_0 \otimes p_0.$$

4. Two determinantal representations $f = \det(P) = \det(P')$ of f are equivalent in the sense of 5.1 if and only P and P' are conjugate under $\mathrm{O}_\delta(k)$ (as in 1.9).

5. Over $\bar{k}$, every quadratic representation is equivalent to a determinantal representation, since any two nondegenerate quadratic forms of the same rank are isometric over $\bar{k}$. Therefore, we can view quadratic representations over k as twisted determinantal representations.

5.6. The *rank* $\mathrm{rk}(t)$ of a symmetric tensor $t \in \mathcal{P}_d(k) \otimes \mathcal{P}_d(k)$ is defined as the rank of the symmetric bilinear form t on $\mathcal{P}_d(k)^\vee$. Note that $\mathrm{rk}\,\gamma(S, \boldsymbol{p}) \le \mathrm{rk}(\boldsymbol{p})$ for every quadratic representation $(S, \boldsymbol{p})$, with equality if $\mathrm{rk}(\boldsymbol{p}) = 3$.

Lemma 5.7. *Let $t \in \mathcal{P}_d(k) \otimes \mathcal{P}_d(k)$ be a symmetric tensor of rank ≤ 3.*

(a) *There exists a quadratic representation $(S, \boldsymbol{p})$ of $f := \mu(t)$ with $t = \gamma(S, \boldsymbol{p})$.*

(b) *If $\mathrm{rk}(t) = 3$, then $(S, \boldsymbol{p})$ is uniquely determined up to equivalence.*

Proof. (a) is clear since t can be diagonalized. (b) Assume $t = \gamma(S, \boldsymbol{p}) = \gamma(S', \boldsymbol{p}')$, where $\mathrm{rk}(t) = 3$. Then $\boldsymbol{p}$ and $\boldsymbol{p}'$ have rank 3 (see 5.6), so the linear maps $\boldsymbol{p}, \boldsymbol{p}' \colon \mathcal{P}_d(k)^\vee \to k^3$ are surjective. The assertion follows from the following elementary lemma:

Let V, W be finite-dimensional k-vector spaces. For $i = 1, 2$, let b_i be a nondegenerate symmetric bilinear form on W, and let $\phi_i \colon V \to W$ be a surjective linear map. Assume $\phi_1^* b_1 = \phi_2^* b_2$. Then there exists a unique $\sigma \in \mathrm{GL}(W)$ with $\sigma \circ \phi_1 = \phi_2$ and $\sigma^* b_2 = b_1$. □

Summarizing Proposition 5.3 and 5.6 we get:

Corollary 5.8. *Let $(S, \boldsymbol{p})$, $(S', \boldsymbol{p}')$ be quadratic representations of f, and assume $\mathrm{rk}(\boldsymbol{p}) = 3$. Then*

$$(S, \boldsymbol{p}) \sim (S', \boldsymbol{p}') \quad \Leftrightarrow \quad \mathrm{span}(\boldsymbol{p}) = \mathrm{span}(\boldsymbol{p}') \quad \Leftrightarrow \quad \gamma(S, \boldsymbol{p}) = \gamma(S', \boldsymbol{p}').$$

In this case, moreover, the matrix $T \in \mathrm{GL}_3(k)$ with $\boldsymbol{p} = T\boldsymbol{p}'$ and $T^t S T = S'$ is unique. □

For the case of determinantal representations, the uniqueness part is already contained in Corollary 1.11. Note that the condition $\mathrm{rk}(\boldsymbol{p}) = 3$ is automatically fulfilled if f is irreducible over $\bar{k}$ (see Lemma 5.2). Also note that the equivalent conditions in Corollary 5.8 are independent of the ground field.

5.9. We now give a more conceptual interpretation. As before, $d \ge 1$ is fixed. Consider the vector spaces $\mathrm{Sym}_2 \mathcal{P}_d$ (of symmetric 2×2-matrices over $\mathcal{P}_d$), $\mathcal{P}_d \otimes \mathcal{P}_d$ and $\mathcal{P}_{2d}$ as affine algebraic k-varieties (namely affine spaces). Let Y be the closed reduced subscheme of $\mathcal{P}_d \otimes \mathcal{P}_d$ whose geometric points are the symmetric tensors of rank ≤ 3. For brevity, denote the variety $\mathrm{Sym}_2 \mathcal{P}_d$ by V. All varieties are considered as k-schemes, and morphisms of varieties as k-morphisms, unless otherwise specified.

The orthogonal group O_δ (see 1.8) acts linearly on $V = \mathrm{Sym}_2 \mathcal{P}_d$ in the natural way (this action was considered in 1.9). The Gram tensor construction 5.4 defines a morphism $\gamma\colon V \to Y$ which is nothing but the quotient morphism in the sense of invariant theory. In other words, $k[Y] = k[V]^{\mathrm{O}_\delta}$, the ring of invariants, and Y could be denoted $V/\!/\mathrm{O}_\delta$. The multiplication map μ (see (21)) is a morphism $\mu\colon Y \to \mathcal{P}_{2d}$. Let $Y_0 \subset Y$ be the open subset consisting of the tensors of rank equal to 3.

5.10. Given $f \in \mathcal{P}_{2d}(k)$, let Y_f be the fibre of f under μ. By Lemma 5.7, the map

$$\big\{\text{quadratic representations of } f \text{ mod } \sim\big\} \to Y_f(k), \quad (S, \boldsymbol{p}) \mapsto \gamma(S, \boldsymbol{p})$$

is surjective. This map induces a bijection between the equivalence classes $(S, \boldsymbol{p})$ with $\mathrm{rk}(\boldsymbol{p}) = 3$ and the set $Y_{0,f}(k) := Y_0(k) \cap Y_f(k)$ (see Corollary 5.8). Now assume that f is not a product of forms of degree d over $\bar{k}$. Then $Y_f \subset Y_0$ (see Lemma 5.2), and the map

$$\gamma\colon \big\{\text{quadratic representations of } f \text{ mod } \sim\big\} \xrightarrow{\sim} Y_f(k) \tag{22}$$

is bijective. Observe that $Y_f(k)$ is identified with the set of all $\mathrm{Gal}(k_s/k)$-invariant equivalence classes of determinantal representations of f over the separable closure k_s.

5.11. Let $t \in Y_0(k)$, and let V_t denote the fibre of t under γ. The group $\mathrm{O}_\delta(\bar{k})$ acts freely and transitively on the nonempty set $V_t(\bar{k})$ (see Lemma 5.7 and Corollary 5.8). Thus, V_t is an O_δ-torsor (over k). The isomorphism classes of O_δ-torsors form the cohomology set $H^1(k, \mathrm{O}_\delta)$. So we have the natural map

$$\eta_0\colon Y_0(k) \to H^1(k, \mathrm{O}_\delta), \quad t \mapsto V_t. \tag{23}$$

The set $H^1(k, \mathrm{O}_\delta)$ classifies the nondegenerate quadratic forms of rank 3 over k up to isometry. The interpretation of η_0 is obvious: If $\eta_0(t)$ is the class of the quadratic form $q = \langle a_0, a_1, a_2 \rangle = a_0 y_0^2 + a_1 y_1^2 + a_2 y^2$ (with $a_i \in k^*$), there exists a quadratic representation $f = q(p_0, p_1, p_2) = a_0 p_0^2 + a_1 p_1^2 + a_2 p_2^2$ which corresponds to t under (22); and every other quadratic representation corresponding to t under (22) is equivalent to this one.

It is clear how to calculate $\eta_0(t)$ from the tensor t: Viewing t as a symmetric bilinear form on $\mathcal{P}_d(k)^\vee$, $\eta_0(t)$ is the class of this form modulo its null space (cf. Remark 5.5.2).

5.12. The approach to quadratic representations via Gram tensors, as in 5.10, is particularly well suited for computations. To fix ideas, let $k \subset \mathbb{C}$. After fixing an enumeration $m_1, \dots, m_N$ of the monomials of degree d in (x_0, x_1, x_2), the symmetric tensors in $\mathcal{P}_d(\mathbb{C}) \otimes \mathcal{P}_d(\mathbb{C})$ correspond to $\mathrm{Sym}_N(\mathbb{C})$, the space of symmetric matrices of size N. The Gram tensors for f form an affine-linear subspace of $\mathrm{Sym}_N(\mathbb{C})$, easily described in terms of the coefficients of f. Its subset $Y_f(\mathbb{C})$ is described by the vanishing of all 4×4-minors. At least for generic f, the set $Y_f(\mathbb{C})$ is finite, and can therefore be determined with the help of suitable computer algebra systems, at least when the data is not too complex.

This approach was carried out successfully during an early stage of this work and of its precursor [PRSS]; see also [PR]. Powers, Reznick, Sottile and the author collected a supply of empirical data by computing the quadratic representations of selected quartic forms. It was actually in this way that the error mentioned in 4.5 was first discovered, before its theoretical explanation was found. The computations were mostly carried out with the help of the system SINGULAR, which proved to be very efficient.

5.13. Recall that a quaternion algebra over k is a central simple k-algebra A of degree two (that is, with $[A : k] = 4$). For a, $b \in k^*$, the quaternion algebra (a, b) has k-basis 1, e_1, e_2, e_3 with the relations $e_1^2 = a$, $e_2^2 = b$ and $e_1 e_2 = e_3 = -e_2 e_1$. Isomorphism classes of quaternion algebras over k are in canonical bijection with isometry classes of 3-dimensional quadratic forms of determinant $1 \in k^*/k^{*2}$. Under this bijection, the quaternion algebra (a, b) corresponds to the quadratic form $\langle -a, -b, ab\rangle$. (Recall that $\langle a_1, \dots, a_n\rangle$ denotes the diagonal quadratic form $\sum_{i=1}^n a_i y_i^2$.) Expressed in a more invariant way, one associates to A the restriction of the reduced norm $N \colon A \to k$ to the subspace of pure quaternions of A; see, e. g., [Sch], ch. 2, §11.

In Galois cohomology these facts are reflected as follows. The set $H^1(k, \mathrm{SO}_\delta)$ can be viewed as the pointed set of all 3-dimensional quadratic forms of determinant 1, up to isometry, with $\langle 1, -1, -1\rangle$ as the distinguished point. The extension $1 \to \mu_2 \to \mathrm{SL}_2 \to \mathrm{PSL}_2 \to 1$ of algebraic groups over k induces an injective map $H^1(k, \mathrm{PSL}_2) \hookrightarrow H^2(k, \mu_2) = \mathrm{Br}_2(k)$, where $\mathrm{Br}_2(k)$ denotes the 2-torsion subgroup of the Brauer group $\mathrm{Br}(k)$ of k. This map identifies $H^1(k, \mathrm{PSL}_2)$ with the set of classes of quaternion algebras in $\mathrm{Br}_2(k)$. Composing it with an isomorphism $\mathrm{SO}_\delta \xrightarrow{\sim} \mathrm{PSL}_2$ (cf. 1.8), the resulting map is

$$c_0 \colon H^1(k, \mathrm{SO}_\delta) \hookrightarrow \mathrm{Br}_2(k), \quad \langle -a, -b, ab\rangle \mapsto (a, b),$$

which is the correspondence indicated before. Using $\mathrm{O}_\delta = \mathrm{SO}_\delta \times \mu_2$ we get $H^1(k, \mathrm{O}_\delta) = H^1(k, \mathrm{SO}_\delta) \times (k^*/k^{*2})$, and composing the projection to the first factor with c_0 above, we get the canonical map

$$c\colon H^1(k, \mathrm{O}_\delta) \to \mathrm{Br}_2(k), \quad \langle a, b, c\rangle \mapsto (-ab, -bc) \tag{24}$$

known as the *Witt invariant* [Sch]. We will write $\eta := c \circ \eta_0$ (see (23)), hence η is the map

$$\eta\colon Y_0(k) \to \mathrm{Br}_2(k), \quad t \mapsto c(\eta_0(t)). \tag{25}$$

So η associates with each tensor $t \in Y_0(k)$ a quaternion algebra $\eta(t) = (a, b)$ over k. Up to scaling, this quaternion algebra encodes the quadratic form q which corresponds to t as in 5.11, via $q = \lambda \cdot \langle a, b, -ab\rangle$ for some $\lambda \in k^*$.

5.14. From now on we assume $\mathrm{char}(k) = 0$. We are going to exhibit a second way of calculating the invariant $\eta(t)$ (Proposition 5.16 below), which will link this invariant to the approach pursued in Part I of this paper.

Let $t \in Y(k)$, put $f = \mu(t)$, and assume that f is irreducible over $\bar{k}$. Let $X = V_+(f)$ and write $\bar{X} := X \otimes_k \bar{k}$. Over $\bar{k}$, t comes from a determinantal representation of f, so there exists $P \in \mathrm{Sym}_2\mathcal{P}_d(\bar{k})$ with $\gamma(P) = t$. Associated with P we have the $\mathcal{O}_{\bar{X}}$-module $\mathcal{F} = \mathcal{F}_P$ (see Proposition 1.2), and thus the partial normalization $\pi_1\colon X_1 \to \bar{X}$ of $\bar{X}$ defined by $\mathcal{F}$ (see 2.4) and characterized by $\pi_*\mathcal{O}_{X_1} = \mathcal{E}nd_{\mathcal{O}_{\bar{X}}}(\mathcal{F})$. Since any other $P' \in \mathrm{Sym}_2\mathcal{P}_d(\bar{k})$ with $\gamma(P') = \gamma(P)$ is conjugate to P under $\mathrm{O}_\delta(\bar{k})$ (see 5.8), $\mathcal{F}$ and X_1 depend only on t.

Let $G_k = \mathrm{Gal}(\bar{k}/k)$. For every $\sigma \in G_k$, $\sigma(P)$ and P are conjugate under $\mathrm{O}_\delta(\bar{k})$, and so ${}^\sigma\mathcal{F} = \mathrm{coker}(\sigma P) \cong \mathrm{coker}(P) = \mathcal{F}$. Hence X_1 is invariant under the action of G_k on the set of partial normalizations of $\bar{X}$ (dominated by $\widetilde{X} \otimes_k \bar{k}$). Arguing as in 2.18, one concludes that there exists a (uniquely determined) partial normalization $\pi\colon X' \to X$ of X (over k) with $X_1 \cong X' \otimes_k \bar{k}$ as $\bar{X}$-schemes. We also write $\bar{X}' = X' \otimes_k \bar{k}$ in the pullback diagram

$$\begin{tikzcd} X_1 = \bar{X}' \arrow[r] \arrow[d, "\pi_1"'] & X' \arrow[d, "\pi"] \\ \bar{X} \arrow[r] & X \end{tikzcd}$$

and call $X' \to X$ the *partial normalization associated to* the Gram tensor t. Assume that X has simple singularities. Then $\mathcal{F}_1 := \pi_1^o\mathcal{F}$ is a line bundle on $X_1 = \bar{X}'$ (see Proposition 2.15), and its class in $\mathrm{Pic}(\bar{X}')$ is G_k-invariant; that is, $\mathcal{F}_1 \in \mathrm{Pic}(\bar{X}')^{G_k}$.

If $(S, \boldsymbol{p})$ is a quadratic representation of f with Gram tensor t, note that its base locus (i.e., the common zero scheme of p_0, p_1, p_2) is equal to the closed subscheme defined by the conductor sheaf of X' over X (see Corollary 2.22).

5.15. As for any complete and geometrically integral k-variety, one has a natural map $\partial\colon\ \mathrm{Pic}(\bar{X}')^{G_k} \to \mathrm{Br}(k)$ making the following sequence exact:

$$0 \to \mathrm{Pic}(X') \to \mathrm{Pic}(\bar{X}')^{G_k} \xrightarrow{\partial} \mathrm{Br}(k) \xrightarrow{\mathrm{res}} \mathrm{Br}(X'). \tag{26}$$

Here $\mathrm{Br}(X') = H^2_{\text{ét}}(X, \mathbb{G}_m)$ is the (cohomological) Brauer group of X', and res is the restriction map. Sequence (26) follows from the Hochschild-Serre spectral sequence $H^i(G_k,\, H^j_{\text{ét}}(\bar{X}', \mathbb{G}_m)) \Rightarrow H^{i+j}_{\text{ét}}(X, \mathbb{G}_m)$ in étale cohomology (e. g. [Mi]), using Hilbert's Theorem 90. In a more elementary way, (26) is deduced from the exact sequence

$$1 \to \bar{k}^* \to \bar{k}(\bar{X}')^* \to \mathrm{Div}(\bar{X}') \to \mathrm{Pic}(\bar{X}') \to 0$$

of G_k-modules, where $\bar{k}(\bar{X}')$ is the function field of $\bar{X}'$.

Proposition 5.16. (char$(k) = 0$) *Assume that $f \in \mathcal{P}_{2d}(k)$ is irreducible over $\bar{k}$, and that $X = V_+(f)$ has simple singularities. Let $t \in Y(k)$ with $\mu(t) = f$, and construct X' and $\mathcal{F}_1 \in \mathrm{Pic}(\bar{X}')^{G_k}$ from t as in* 5.14. *Then*

$$\eta(t) = \partial(\mathcal{F}_1)$$

in $\mathrm{Br}_2(k) \subset \mathrm{Br}(k)$.

Here η and ∂ are as defined in (25) and (26), respectively.

Proof. Let $P \in \mathrm{Sym}_2\mathcal{P}_d(\bar{k})$ with $\gamma(P) = t$. For every $\sigma \in G_k$ there are $\varepsilon_\sigma \in \{\pm 1\}$, $g_\sigma \in \mathrm{SL}_2(\bar{k})$ with $\sigma(P) = \varepsilon_\sigma \cdot g_\sigma^t P g_\sigma$ (see Corollary 5.8), and $(\varepsilon_\sigma,\, \bar{g}_\sigma) \in \{\pm 1\} \times \mathrm{PSL}_2(\bar{k}) = \mathrm{O}_\delta(\bar{k})$ is uniquely determined. The cocycle $(\varepsilon_\sigma,\, \bar{g}_\sigma)_\sigma \in Z^1(G_k, \mu_2 \times \mathrm{PSL}_2)$ represents $\eta_0(t)$. Therefore $\eta(t) \in H^2(G_k, \mu_2)$ is represented by the 2-cocycle $(h_{\sigma,\tau})$, where

$$h_{\sigma,\tau} = g_\sigma \cdot \sigma(g_\tau) \cdot g_{\sigma\tau}^{-1} \ \in \{\pm 1\}.$$

On the other hand, $\partial(\mathcal{F}_1)$ can be calculated as follows. For every $\sigma \in G_k$ there exists an isomorphism $\varphi_\sigma\colon \mathcal{F}_1 \xrightarrow{\sim} \sigma_*\mathcal{F}_1$ of invertible sheaves on $\bar{X}'$. Given σ, $\tau \in G_k$, the composition $c_{\sigma,\tau} := \varphi_{\sigma\tau}^{-1} \circ \sigma_*(\varphi_\tau) \circ \varphi_\sigma$ is an automorphism of $\mathcal{F}_1$, hence an element of $\mathcal{O}_{\bar{X}'}(\bar{X}')^* = \bar{k}^*$. The family $(c_{\sigma,\tau})$ is a cocycle whose class in $H^2(G_k, \bar{k}^*) = \mathrm{Br}(k)$ is $\partial(\mathcal{F}_1)$.

Concretely, we get a system of isomorphisms φ_σ as above from the commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2_{\bar{k}}}(-d)^2 \arrow[r, "P"] & \mathcal{O}^2_{\mathbb{P}^2_{\bar{k}}} \arrow[r] \arrow[d, "g_\sigma^t"', "\sim"] & \mathcal{F} \arrow[r] \arrow[d, "\psi_\sigma"', "\sim"] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2_{\bar{k}}}(-d)^2 \arrow[r, "\sigma(P)"] \arrow[u, "\varepsilon_\sigma g_\sigma", "\sim"'] & \mathcal{O}^2_{\mathbb{P}^2_{\bar{k}}} \arrow[r] & \sigma_*\mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

(for $\sigma \in G_k$), by $\varphi_\sigma := \pi_1^o(\psi_\sigma)$. Therefore, $c_{\sigma,\tau}$ is induced by the automorphism $g_{\sigma\tau}^{-t} \cdot \sigma(g_\tau)^t \cdot g_\sigma^t \in \mathrm{GL}_2(\bar{k})$ of $\mathcal{O}_{\mathbb{P}^2_{\bar{k}}}$. Thus $c_{\sigma,\tau} = h_{\sigma,\tau} \in \{\pm 1\}$, and the proposition is proved. □

6. Real curves and their generalized Jacobians

6.1. Let X always be a projective, geometrically integral curve over $\mathbb{R}$, and let J be the generalized Jacobian of X. In this section we collect some general information, in particular pertaining to the (2-) torsion and cotorsion of $J(\mathbb{R})$. Much of it is well known, but the difficulty of finding suitable direct references (or the lack thereof) makes it preferable to collect these facts here. Our results are complete in the case where $X(\mathbb{R})$ is finite, which is equivalent to the function field $\mathbb{R}(X)$ being nonreal. Note that, in this case, $X(\mathbb{R})$ consists of singular points of X. When $|X(\mathbb{R})| = \infty$, some matters get more complicated, and we plan to deal with these elsewhere.

Always write $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Given an $\mathbb{R}$-scheme Y, let $Y_{\mathbb{C}} := Y \otimes_{\mathbb{R}} \mathbb{C}$. As usual, $\mathrm{Pic}(Y) = H^1(Y, \mathcal{O}_Y^*)$ denotes the Picard group of Y. We will constantly employ the exact sequence (cf. 5.15),

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathbb{C}})^G \xrightarrow{\partial} \mathrm{Br}(\mathbb{R}) \to \mathrm{Br}(X), \tag{27}$$

where $\mathrm{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)$.

For $d \in \mathbb{Z}$ let $\mathrm{Pic}^d(X)$ denote the set of classes of degree d in $\mathrm{Pic}(X)$. Thus $\mathrm{Pic}^0(X_{\mathbb{C}}) = J(\mathbb{C})$, and $\mathrm{Pic}^0(X)$ is a subgroup of $J(\mathbb{R})$. By $J(\mathbb{R})^0$ we denote the identity component of the real Lie group $J(\mathbb{R})$.

Everything in this section (as well as in the next ones) works as well when $\mathbb{R}$ gets replaced by an arbitrary real closed field, upon substituting connectedness with semi-algebraic connectedness where appropriate.

6.2. As is well known, the Picard group of X can be realized via Weil divisors on the regular locus X_{reg}. Let K be the semilocal ring of X in the singular points of X, i. e., the ring of all rational functions f on X which are regular on an open set containing X_{sing}. Then the natural sequence

$$1 \to \mathbb{R}^* \to K^* \xrightarrow{\div} \mathrm{Div}(X_{\mathrm{reg}}) \to \mathrm{Pic}(X) \to 0 \tag{28}$$

is exact, where the map $\div$ sends $f \in K^*$ to the (Weil) divisor of f restricted to X_{reg}. We denote the map $\mathrm{Div}(X_{\mathrm{reg}}) \to \mathrm{Pic}(X)$ by $D \mapsto [D]$, if necessary.

Lemma 6.3. *The group $J(\mathbb{R})^0$ is divisible, the group $J(\mathbb{R})/J(\mathbb{R})^0$ is finite of exponent ≤ 2, and the extension*

$$0 \to J(\mathbb{R})^0 \to J(\mathbb{R}) \to J(\mathbb{R})/J(\mathbb{R})^0 \to 0$$

splits.

Proof. $J(\mathbb{R})^0$ is the image of the norm map $N\colon J(\mathbb{C}) \to J(\mathbb{R})$, and therefore is connected and divisible. The quotient $J(\mathbb{R})/J(\mathbb{R})^0$ is the Tate cohomology group $\widehat{H}^0(G, J(\mathbb{C}))$, and hence is finite of exponent ≤ 2. Divisibility of $J(\mathbb{R})^0$ implies that the above extension splits. $\square$

Lemma 6.4. *Assume* $|X(\mathbb{R})| < \infty$. *Then* $\operatorname{Pic}^0(X) = J(\mathbb{R})^0$ *and* $\deg \operatorname{Pic}(X) = 2\mathbb{Z}$. *In particular,* $\operatorname{Pic}^0(X)$ *is a divisible group. Moreover,*

$$2\operatorname{Pic}(X) = \{\alpha \in \operatorname{Pic}(X)\colon\ \deg(\alpha) \equiv 0\ (4)\}.$$

Proof. The norm map $N\colon \operatorname{Pic}(X_{\mathbb{C}}) \to \operatorname{Pic}(X)$ is surjective. Indeed, over any closed point $P \in X_{\mathrm{reg}}$ there lie two points $Q \ne \bar{Q}$ of $X_{\mathbb{C}}$, and so $[P] \in \operatorname{Pic}(X)$ is the norm of $[Q] \in \operatorname{Pic}(X_{\mathbb{C}})$ (cf. 6.2). This implies $\deg \operatorname{Pic}(X) = 2\mathbb{Z}$ and $\operatorname{Pic}^0(X) = N(J(\mathbb{C})) = J(\mathbb{R})^0$. The characterization of $2\operatorname{Pic}(X)$ follows from the divisibility of $\operatorname{Pic}^0(X)$. $\square$

The following result is due to Weichold, who was a student of Felix Klein; see [Ge] for a modern exposition. Recall the notation $M_2 = \ker(M \xrightarrow{2} M)$ for an abelian group M (see 0.3).

Proposition 6.5. *Assume that X is nonsingular of genus g, and let s be the number of connected components of $X(\mathbb{R})$. Then $J(\mathbb{R})^0 \cong (\mathbb{R}/\mathbb{Z})^g$ and $\big[J(\mathbb{R}) : J(\mathbb{R})^0\big] = 2^a$, where*

$$a = \begin{cases} s-1 & X(\mathbb{R}) \ne \varnothing, \\ \ 1 & X(\mathbb{R}) = \varnothing \text{ and } g \text{ is odd}, \\ \ 0 & X(\mathbb{R}) = \varnothing \text{ and } g \text{ is even}. \end{cases}$$

In particular, $|J(\mathbb{R})_2| = 2^{g+a}$. Moreover,

$$\deg\big(\operatorname{Pic}(X_{\mathbb{C}})^G\big) = \begin{cases} \mathbb{Z} & g \text{ is even}, \\ 2\mathbb{Z} & g \text{ is odd}. \end{cases} \qquad \square$$

6.6. In the sequel, let $\pi\colon \widetilde{X} \to X$ be the normalization of the (possibly singular) curve X, and let $\widetilde{J}$ be the Jacobian of $\widetilde{X}$. Then J is an extension of algebraic groups over $\mathbb{R}$

$$0 \to L \to J \xrightarrow{\pi^*} \widetilde{J} \to 0 \qquad (29)$$

of the abelian variety $\widetilde{J}$ by a connected linear algebraic group L. The group L is determined by the singularities of X in the following well-known way. Given a singular point x of X, let $R_x = \mathcal{O}_{X,x} \otimes_{\mathbb{R}} \mathbb{C}$, and let $\widetilde{R}_x$ be the normalization of R_x. Then

$$L = \prod_{x \in X_{\mathrm{sing}}} L_x$$

where L_x is the linear algebraic group over $\mathbb{R}$ satisfying

$$L_x(\mathbb{C}) = \widetilde{R}_x^* / R_x^*$$

as G-modules.

6.7. Assume that X has simple singularities. We follow Gudkov's convention [Gu] for the notation of real ADE singularities: If there is no superscript, the point is real and all branches are real. An asterisk indicates that the point is real and two branches are complex conjugate. A superscript i indicates a pair of (different) complex conjugate singularities. Thus, for example, $2A_1^i$ is a pair $P \neq \bar{P}$ of complex conjugate nodes.

An irreducible plane quartic curve over $\mathbb{C}$ can only have singular points of type A_n $(n \leq 6)$, D_4, D_5 or E_6. It follows that an irreducible plane quartic curve over $\mathbb{R}$ can only have the following singularities:

$$A_n\ (n \leq 6),\quad A_n^*\ (n = 1, 3, 5),\quad 2A_n^i\ (n = 1, 2),\quad D_4,\quad D_4^*,\quad D_5,\quad E_6. \tag{30}$$

Let x be a singular point of X. Depending on the type of the ADE singularity x, the linear algebraic group L_x over $\mathbb{R}$ is as follows. (For brevity we only list the singularities occuring in (30).)

x	L_x
A_{2n+1}	$\mathbb{G}_m \times \mathbb{G}_a^n$
A_{2n+1}^*	$R^1\mathbb{G}_m \times \mathbb{G}_a^n$
$2A_{2n+1}^i$	$R\mathbb{G}_m \times G_a^{2n}$
A_{2n}	$\mathbb{G}_a^n$
$2A_{2n}^i$	$\mathbb{G}_a^{2n}$
D_4	$\mathbb{G}_m^2 \times \mathbb{G}_a$
D_4^*	$R\mathbb{G}_m \times \mathbb{G}_a$
D_5	$\mathbb{G}_m \times \mathbb{G}_a^2$
E_6	$\mathbb{G}_a^3$

TABLE 3

Here R denotes Weil restriction of $\mathbb{C}/\mathbb{R}$; thus $R^1\mathbb{G}_m$ is the anisotropic $\mathbb{R}$-torus of rank one, and $R\mathbb{G}_m$ is the quasisplit, nonsplit $\mathbb{R}$-torus of rank two.

In the following we study the map ∂ from (27). As usual, let $p_g(X)$ (respectively, $p_a(X)$) denote the geometric (respectively, arithmetic) genus of X.

Lemma 6.8. *The map*

$$\partial\colon\ \mathrm{Pic}(X_\mathbb{C})^G \to \mathrm{Br}(\mathbb{R})$$

is zero if and only if $X(\mathbb{R}) \neq \varnothing$. Its restriction

$$\partial_0\colon J(\mathbb{R}) \to \mathrm{Br}(\mathbb{R})$$

to $J(\mathbb{R})$ is zero if and only if $X(\mathbb{R}) \neq \varnothing$ or $p_g(X)$ is even.

Proof. Any point $P \in X(\mathbb{R})$ defines a section of $\mathrm{Br}(\mathbb{R}) \to \mathrm{Br}(X)$ in (27). Therefore, $X(\mathbb{R}) \neq \varnothing$ implies $\partial = 0$. Now assume $X(\mathbb{R}) = \varnothing$. Let us first consider the case where X is nonsingular of genus g. By a theorem of Witt [Wi], every element of $\mathbb{R}(X)$ is a sum of two squares. In particular, $(-1,-1) = 0$ in $\mathrm{Br}\,\mathbb{R}(X)$, which implies $(-1,-1) = 0$ in $\mathrm{Br}(X)$. Hence the map ∂ is surjective. From this, from Lemma 6.4 and from Proposition 6.5 it follows that ∂_0 is surjective if and only if g is odd.

To relate the general case to the nonsingular case, let $\pi\colon \widetilde{X} \to X$ be the normalization of X, and let $\widetilde{\partial}\colon\ \mathrm{Pic}(\widetilde{X}_\mathbb{C})^G \to \mathrm{Br}(\mathbb{R})$ be the boundary map for $\widetilde{X}$. Using the notation of 6.6, we get the exact sequences of G-modules

$$0 \to L(\mathbb{C}) \to \mathrm{Pic}(X_\mathbb{C}) \to \mathrm{Pic}(\widetilde{X}_\mathbb{C}) \to 0$$

and

$$0 \to L(\mathbb{C}) \to J(\mathbb{C}) \to \widetilde{J}(\mathbb{C}) \to 0.$$

Since $X(\mathbb{R}) = \varnothing$, $L(\mathbb{C})$ is an induced G-module, and so $H^1(G, L(\mathbb{C})) = 0$. Thus we get a commutative diagram

$$\begin{array}{ccccc} J(\mathbb{R}) & \subset & \mathrm{Pic}(X_\mathbb{C})^G & \xrightarrow{\partial} & \\ \downarrow & & \downarrow & & \mathrm{Br}(\mathbb{R}) \\ \widetilde{J}(\mathbb{R}) & \subset & \mathrm{Pic}(\widetilde{X}_\mathbb{C})^G & \xrightarrow{\widetilde{\partial}} & \end{array}$$

with surjective vertical arrows. By the nonsingular case, $\widetilde{\partial} \neq 0$, and so $\partial \neq 0$. Similarly, $\partial|_{J(\mathbb{R})} = 0 \Leftrightarrow \widetilde{\partial}|_{\widetilde{J}(\mathbb{R})} = 0$, and so we can again apply the nonsingular case to deduce the statement for ∂_0. □

For later use we record the following by-product of the last proof:

Corollary 6.9. *If $X(\mathbb{R}) = \varnothing$, then $\mathrm{Pic}(X_\mathbb{C})^G \to \mathrm{Pic}(\widetilde{X}_\mathbb{C})^G$ and $J(\mathbb{R}) \to \widetilde{J}(\mathbb{R})$ are surjective.* □

Corollary 6.10. *The restriction of $\partial_0\colon J(\mathbb{R}) \to \mathrm{Br}(\mathbb{R})$ to the 2-torsion subgroup $J(\mathbb{R})_2$ is zero if and only if $X(\mathbb{R}) \neq \varnothing$ or $p_g(X)$ is even.*

Proof. If $X(\mathbb{R}) \neq \varnothing$ or $p_g(X)$ is even, then even $\partial_0 = 0$ by Lemma 6.8. Assume that $X(\mathbb{R}) = \varnothing$ and $p_g(X)$ is odd. Then $\partial_0 \neq 0$ by Lemma 6.8. Since $J(\mathbb{R})^0 \subset \ker(\partial_0)$ (see Lemma 6.4) and every connected component of $J(\mathbb{R})$ contains an involution (see Lemma 6.3), there exists an involution α in $J(\mathbb{R})$ with $\partial_0(\alpha) \neq 0$. □

Corollary 6.11. *Assume that $X(\mathbb{R})$ is finite.*

(a) *If $X(\mathbb{R}) \neq \varnothing$ or $p_g(X)$ is even, then $J(\mathbb{R}) = J(\mathbb{R})^0 = \mathrm{Pic}^0(X)$.*
(b) *If $X(\mathbb{R}) = \varnothing$ and $p_g(X)$ is even, then $\mathrm{Pic}^1(X_{\mathbb{C}})^G \neq \varnothing$.*
(c) *If $X(\mathbb{R}) = \varnothing$ and $p_g(X)$ is odd, then $[J(\mathbb{R}) : J(\mathbb{R})^0] = 2$ and $\mathrm{Pic}^1(X_{\mathbb{C}})^G$ is empty.*

Proof. The sequence $0 \to J(\mathbb{R})^0 \to J(\mathbb{R}) \xrightarrow{\partial_0} \mathrm{Br}(\mathbb{R})$ is exact by Lemma 6.4. This implies (a) and the first statement in (c), using Lemmas 6.8 and 6.4. (b) follows from Proposition 6.5 and Corollary 6.9, and the second statement in (c) follows from Proposition 6.5, applied to $\widetilde{X}$. □

We next discuss the subgroup $2\,\mathrm{Pic}(X_{\mathbb{C}})^G := \mathrm{im}\big(\mathrm{Pic}(X_{\mathbb{C}})^G \xrightarrow{2} \mathrm{Pic}(X_{\mathbb{C}})^G\big)$ of $\mathrm{Pic}(X)$.

Proposition 6.12. *Assume that $X(\mathbb{R})$ is finite. Then the following hold:*

(a) *If $X(\mathbb{R}) \neq \varnothing$, then $\mathrm{Pic}(X_{\mathbb{C}})^G = \mathrm{Pic}(X)$ and*
$$2\,\mathrm{Pic}(X_{\mathbb{C}})^G = \{\alpha \in \mathrm{Pic}(X)\colon\ \deg(\alpha) \equiv 0 \ (4)\}.$$
(b) *If $X(\mathbb{R}) = \varnothing$ and $p_g(X)$ is even, then $\mathrm{Pic}^0(X_{\mathbb{C}})^G = \mathrm{Pic}^0(X)$ and*
$$2\,\mathrm{Pic}(X_{\mathbb{C}})^G = \mathrm{Pic}(X).$$
(c) *Assume that $X(\mathbb{R}) = \varnothing$ and $p_g(X)$ is odd. For every even $d \in \mathbb{Z}$, the set $\mathrm{Pic}^d(X_{\mathbb{C}})^G$ has two connected components, one of which is $\mathrm{Pic}^d(X)$. Moreover,*
$$2\,\mathrm{Pic}(X_{\mathbb{C}})^G = 2\,\mathrm{Pic}(X) = \{\alpha \in \mathrm{Pic}(X)\colon\ \deg(\alpha) \equiv 0 \ (4)\}.$$

Proof. (a) $\mathrm{Pic}(X_{\mathbb{C}})^G = \mathrm{Pic}(X)$ by Lemma 6.8, so the claim follows from Lemma 6.4.

(b) $\mathrm{Pic}^0(X_{\mathbb{C}})^G = \mathrm{Pic}^0(X)$ by Lemma 6.8, and $\mathrm{Pic}^1(X_{\mathbb{C}})^G \neq \varnothing$ by Corollary 6.11. Therefore $\mathrm{Pic}(X) \subset 2\,\mathrm{Pic}(X_{\mathbb{C}})^G$ follows from the divisibility of $\mathrm{Pic}^0(X_{\mathbb{C}})^G = \mathrm{Pic}^0(X)$.

(c) The first assertion follows from Corollary 6.11, which also gives $\deg \mathrm{Pic}(X_{\mathbb{C}})^G = 2\mathbb{Z}$. Using the divisibility of $\mathrm{Pic}^0(X)$, this implies $2\,\mathrm{Pic}(X_{\mathbb{C}})^G = 2\,\mathrm{Pic}(X)$, and we can again use Lemma 6.4. □

6.13. When $|X(\mathbb{R})| = \infty$, there is an elegant characterization of $2\,\mathrm{Pic}(X)$ due to Pedrini and Weibel. Even though we won't use it, it should be mentioned at this point. Topologically, each connected component of $X(\mathbb{R})$ is a bouquet of circles which are called the *loops* of X. Let K be as in 6.2. For every $f \in K^*$ and every loop L, the sum $\sum_{P\in L} \mathrm{ord}_P(f)$ (sum over the non-singular points in L) is even. So there is a well-defined relative degree mod 2 on L, denoted $\deg_L \colon \mathrm{Pic}(X) \to \mathbb{Z}/2$. If $L_1, \dots, L_t$ are the loops of X, then the map

$$(\deg_{L_1}, \dots, \deg_{L_t})\colon \ \mathrm{Pic}(X)/2 \to (\mathbb{Z}/2)^t$$

is an isomorphism [PW].

We also need to discuss the 2-torsion subgroup of $J(\mathbb{R})$. Since this tends to become more subtle when $|X(\mathbb{R})| = \infty$, we leave this case away and restrict to $|X(\mathbb{R})| < \infty$.

Proposition 6.14. *Assume that $X(\mathbb{R})$ is finite. Then* (29) *induces exact sequences*

$$0 \to L(\mathbb{R}) \to J(\mathbb{R}) \to \widetilde{J}(\mathbb{R}) \to B \to 0 \tag{31}$$

and

$$0 \to L(\mathbb{R})_2 \to J(\mathbb{R})_2 \to \widetilde{J}(\mathbb{R})_2 \to B \to 0, \tag{32}$$

where $B = \mathbb{Z}/2$ if $X(\mathbb{R}) \neq \varnothing$ and $p_g(X)$ is odd, and $B = 0$ otherwise.

Proof. We only have to calculate the cokernels in the two sequences. The group L is a direct product of copies of $\mathbb{G}_a$ and of tori $R^1\mathbb{G}_m$ and $R\mathbb{G}_m$ (see 6.7). Therefore $L(\mathbb{R})$ is connected, hence divisible. From the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J(\mathbb{R})_2 & \longrightarrow & J(\mathbb{R}) & \xrightarrow{2} & J(\mathbb{R})^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{J}(\mathbb{R})_2 & \longrightarrow & \widetilde{J}(\mathbb{R}) & \xrightarrow{2} & \widetilde{J}(\mathbb{R})^0 & \longrightarrow & 0
\end{array}$$

it therefore follows that the first and second vertical arrow have the same cokernel, since the last vertical arrow is surjective. Therefore it suffices to study $B := \mathrm{coker}(J(\mathbb{R}) \to \widetilde{J}(\mathbb{R}))$.

If $p_g(X)$ is even, then $B = 0$ follows from the (obvious) surjectivity of $\mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X})$, together with $\mathrm{Pic}^0(X) = J(\mathbb{R})$ and $\mathrm{Pic}^0(\widetilde{X}) = \widetilde{J}(\mathbb{R})$ (see Proposition 6.12, (a) and (b)). If $X(\mathbb{R}) = \varnothing$, then again $B = 0$ by Corollary 6.9. Assume now that $p_g(X)$ is odd and $X(\mathbb{R}) \neq \varnothing$. Then $J(\mathbb{R})$ is connected while $\widetilde{J}(\mathbb{R})$ has two connected components (see Corollary 6.11), which implies $B = \mathbb{Z}/2$. $\square$

7. Positive semidefinite real quartic forms

We continue to write $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. For the final analysis of quadratic representations of quartic forms, we shall need to know whether or not $\pi^!\mathcal{O}_X(2) \in \mathrm{Pic}(X')$ is a double in $\mathrm{Pic}(X'_\mathbb{C})^G$, when $\pi\colon X' \to X$ is a given partial normalization. Since $\pi^!\mathcal{O}_X(2) \cong \pi^!\mathcal{O}_X \otimes \pi^*\mathcal{O}_X(1)^{\otimes 2}$ (see Lemma 2.2(b)), it is equivalent to ask whether $\pi^!\mathcal{O}_X \in 2\,\mathrm{Pic}(X'_\mathbb{C})^G$.

Proposition 7.1. *Let X be an integral curve over $\mathbb{R}$ with simple singularities, and let $\pi\colon X' \to X$ be a Gorenstein partial normalization of X. Assume that $X(\mathbb{R})$ is finite. The invertible sheaf $\pi^!\mathcal{O}_X$ on X' is a double in $\mathrm{Pic}(X'_\mathbb{C})^G$ if and only if (1) or (2) holds:*

(1) *$p_a(X) - p_a(X')$ is even;*
(2) *$X'(\mathbb{R}) = \varnothing$ and $p_g(X)$ is even.*

$\pi^!\mathcal{O}_X$ is a double in $\mathrm{Pic}(X')$ if and only if (1) holds.

Proof. We have $\deg(\pi^!\mathcal{O}_X) = 2p_a(X') - 2p_a(X)$; see (15). Therefore, $\pi^!\mathcal{O}_X \in 2\,\mathrm{Pic}(X')$ is equivalent to (1) by Lemma 6.4. From Proposition 6.12(b) we see that (2) implies $\pi^!\mathcal{O}_X \in 2\,\mathrm{Pic}(X'_\mathbb{C})^G$. Conversely assume $\pi^!\mathcal{O}_X \in 2\,\mathrm{Pic}(X'_\mathbb{C})^G$, and assume that (1) fails. Then $\deg(\pi^!\mathcal{O}_X) \equiv 2\ (4)$, and Proposition 6.12 shows that (2) holds. □

The question of when $\pi^!\mathcal{O}_X \in 2\,\mathrm{Pic}(X')$ holds has a more complicated answer when $X(\mathbb{R})$ is infinite. We plan to investigate this elsewhere, together with its consequences for quadratic representations of indefinite real quartics.

Scholium 7.2. We now give a summary of how the complete analysis of quadratic representations of a given real quartic form is obtained. For this we assume that the quartic form $f \in \mathbb{R}[x_0, x_1, x_2]$ is psd (i.e., nonnegative) and irreducible over $\mathbb{C}$. Let $X = V_+(f)$. For simplicity, we will refer to general quadratic representations of f over $\mathbb{R}$ (see 5.1) as *signed representations*, since they are equivalent to representations $f = \pm q_0^2 \pm q_1^2 \pm q_2^2$. We will call such a representation *definite* if all signs are $+$, and indefinite otherwise. (Note that at least one sign is $+$ since f is psd.)

Start by making the list of singularities of X. From this obtain the list of all Gorenstein partial normalizations $\pi\colon X' \to X$ of X, using 3.10 and 2.18. Fixing one such X' for the rest of the discussion, we explain how to obtain the number of signed representations of f associated to X' (in the sense of 5.14). Note that these are precisely the signed representations whose base locus is the zero scheme of the conductor sheaf of X' over X.

The sheaf $\pi^!\mathcal{O}_X(2) = \pi^!\mathcal{O}_X \otimes \pi^*\mathcal{O}_X(2)$ lies in $\mathrm{Pic}(X')$. From Proposition 7.1 we can read off whether $\pi^!\mathcal{O}_X$ is a double in $\mathrm{Pic}(X'_\mathbb{C})^G$. If this is not the case, there exist no signed representations associated to X'.

Assume therefore that $\pi^! \mathcal{O}_X \in 2\operatorname{Pic}(X'_{\mathbb{C}})^G$, and let J' be the generalized Jacobian of X'. The number of $\mathcal{F}' \in \operatorname{Pic}(X'_{\mathbb{C}})^G$ for which $\mathcal{F}' \otimes \mathcal{F}'$ is isomorphic to $\pi^! \mathcal{O}_X \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'_{\mathbb{C}}}$ is equal to $|J'(\mathbb{R})_2|$. We have to determine which of these $\mathcal{F}'$ correspond to a determinantal representation of f over $\mathbb{C}$ (see 5.10). By 4.2, this means to discard $\mathcal{O}_{X_{\mathbb{C}}}(1)$ in case $X' = X$, and also those $\mathcal{F}'$ which are exceptional with respect to some singular point of $X_{\mathbb{C}}$. Since $\mathcal{F}'$ is G-invariant, $\mathcal{F}'$ can only be exceptional with respect to an $\mathbb{R}$-rational singular point of X (using Lemma 3.4). Let e be the number of singular points in $X(\mathbb{R})$ whose associated exceptional partial normalization (see Definition 3.7) is X'. This number can be read off from 3.11, Table 1. The number of signed representations associated to X' is then equal to

$$\begin{cases} |J'(\mathbb{R})_2| - (e+1) & \text{if } X' = X, \\ |J'(\mathbb{R})_2| - e & \text{if } X' \neq X. \end{cases}$$

The order of the group $J'(\mathbb{R})_2$ is easily obtained from Proposition 6.5, 6.7 and Proposition 6.14.

It remains to discuss how many of these representations are definite respectively indefinite. Given one such representation, let $\mathcal{F}'$ be its associated line bundle in $\operatorname{Pic}(X'_{\mathbb{C}})^G$. According to Proposition 5.16, the representation is indefinite if and only if $\partial(\mathcal{F}') = 0$, where $\partial\colon \operatorname{Pic}(X'_{\mathbb{C}})^G \to \operatorname{Br}(\mathbb{R})$ is the map explained in 5.15. Let ∂_2 denote the restriction of ∂ to $J'(\mathbb{R})_2$. From Corollary 6.10 we can read off whether or not ∂_2 is the zero map.

First assume that $\partial_2 = 0$. Then the signed representations associated to X' are either all definite or all indefinite. Which one happens can be decided as follows: If $\pi^! \mathcal{O}_X \in 2\operatorname{Pic}(X')$, then all representations are indefinite. Otherwise they are all definite. The information as to whether or not $\pi^! \mathcal{O}_X \in 2\operatorname{Pic}(X')$ is given in Proposition 7.1.

Second, assume $\partial_2 \neq 0$. Then precisely $\frac{1}{2}\,|J'(\mathbb{R})_2|$ many representations are definite, and the remaining ones are indefinite. Indeed, $\partial(\mathcal{F}') = 0$ holds for any exceptional $\mathcal{F}'$, since such $\mathcal{F}'$ lies in $\operatorname{Pic}(X')$; and similarly $\partial(\mathcal{O}_X(1)) = 0$ (in case $X' = X$).

This gives the complete account of all signed representations of f, up to equivalence, sorted by their base locus and by being definite or not.

7.3. Part of the results of this analysis is summarized in the table below. Let f be any geometrically irreducible psd ternary quartic form over $\mathbb{R}$. The first column in Table 4 lists the configuration of singularities of $X = V_+(f)$. The next two columns give the total number of quadratic ("signed") representations of f over $\mathbb{R}$ and the number of base-point free such representations. The last two columns give the total number of definite representations over $\mathbb{R}$ and the number of base-point free definite representations. (As is common, the shortcut *sos* stands for "sum of squares".) Singularities that are not listed in Table 4 cannot occur for f as considered here.

Sing.	total qr	bpf qr	total sos	bpf sos
smooth	15	15	8	8
A_1^*	10	6	4	–
$2A_1^*$	9	5	2	–
$2A_1^i$	11	7	6	4
$3A_1^*$	11	4	1	–
$A_1^* + 2A_1^i$	7	2	3	–
$2A_2^i$	7	3	4	2
$A_1^* + 2A_2^i$	4	–	2	–
A_3^*	6	3	2	–
$A_1^* + A_3^*$	6	2	1	–
A_5^*	3	1	1	–

Table 4

More complete data (containing the explicit numbers of definite and indefinite representations associated to any fixed base locus) can be found on the author's webpage. For a few examples which are more explicit, see the next section.

7.4. Obviously, in any sum of squares representation

$$f = p_0^2 + p_1^2 + p_2^2, \tag{33}$$

the p_i vanish in all real points of X. Table 4 shows that there exists a base-point free representation (33) whenever $X(\mathbb{R}) = \varnothing$. The refined analysis (which is not displayed here) shows actually that there always exists a representation (33) for which all common zeros of the p_i are real. However, it is not always possible to find such a representation for which the base locus is reduced. For example, this is impossible for an A_5^*-singularity (see Example 8.5 below).

8. Examples

We illustrate the general method of determining all quadratic representations of a ternary quartic form by a few selected examples. In Examples 8.1–8.3 we only consider representations over $k = \mathbb{C}$, while in Examples 8.4

and 8.5 we study signed representations over $k = \mathbb{R}$. For the theoretical background, see 4.2 and 4.4 over $\mathbb{C}$ and Scholium 7.2 over $\mathbb{R}$. Always $X = V_+(f)$ where f is a ternary quartic form over k, irreducible over $\mathbb{C}$.

Example 8.1. Let X have an A_1-singularity P and an A_4-singularity Q. We discuss the determinantal representations of f over $\mathbb{C}$. The Gorenstein partial normalizations of X form a diagram

$$\begin{tikzcd} & & X_{12} \arrow[dl] \arrow[dr] & \\ & X_{11} \arrow[dl] \arrow[dr] & & X_{02} \arrow[dl] \\ X_{10} \arrow[dr] & & X_{01} \arrow[dl] & \\ & X_{00} & & \end{tikzcd}$$

where the X_{1j} are nonsingular over P and the X_{i2} are nonsingular over Q. So $X = X_{00}$ and $\widetilde{X} = X_{12}$. Let J_{ij} be the generalized Jacobian of X_{ij}. Then $J_{ij} \cong \mathbb{G}_m^{1-i} \times \mathbb{G}_a^{2-j}$, so $|J_{ij}(\mathbb{C})_2| = 2^{1-i}$. The exceptional partial normalization associated with P is $X = X_{00}$, the one associated with Q is X_{02} (see 3.11, Table 1). We conclude that for each of X_{02} and X_{1j} $(j = 0, 1, 2)$ there is exactly one associated representation, while there are two for X_{01} and none for X_{00}. In particular, there exists no base-point free representation, although X has two branches at P. Compare 4.5.

Example 8.2. $(k = \mathbb{C})$ Let X have a D_5-singularity in Q. The curve X is rational and has two branches at Q, one of which is cuspidal. Besides the normalization $\widetilde{X}$ there are two proper partial normalizations X_j $(j = 1, 2)$ of X. We label them in such a way that X_j has an A_j-singularity $(j = 1, 2)$. Let J respectively J_j be the generalized Jacobian of X respectively X_j $(j = 1, 2)$. We have $J \cong \mathbb{G}_m \times \mathbb{G}_a^2$, $J_1 \cong \mathbb{G}_m$ and $J_2 \cong \mathbb{G}_a$. The exceptional partial normalization is $\widetilde{X}$ (see 3.11, Table 1). It follows that there are altogether four determinantal representations: One of them is base-point free, one has an infinitely near point on the non-cuspidal branch as base locus (it corresponds to X_2), and two have an infinitely near point on the cuspidal branch as base locus (they correspond to X_1).

Example 8.3. $(k = \mathbb{C})$ Let X have an E_6-singularity. There are three Gorenstein partial normalizations $X_3 \to X_2 \to X_0 = X$, where $X_3 = \widetilde{X}$ and X_2 has an A_2-singularity. The generalized Jacobian J_i of X_i is isomorphic to $\mathbb{G}_a^{3-i}$ $(i = 0, 2, 3)$. The exceptional partial normalization is $X_3 = \widetilde{X}$ (see

3.11). Hence there exists only one single quadratic representation of f, and it is associated to X_2.

Example 8.4. ($k = \mathbb{R}$) Let f be psd such that X has two A_1^*-singularities P_1, P_2 and is otherwise smooth. There are four Gorenstein partial normalizations which form a diagram:

$$\begin{tikzcd} & \widetilde{X} \arrow[dl] \arrow[dr] & \\ X_1 \arrow[dr, "\pi_1"'] & & X_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

Here X_i is smooth over P_i ($i = 1, 2$). Let $\widetilde{\pi}$ denote the normalization $\widetilde{X} \to X$ of X.

We have $p_g(X) = 1$ and $p_a(X) = 3$, $p_a(X_i) = 2$ ($i = 1, 2$). We see that $\pi_i^! \mathcal{O}_X(2)$ is not a double in $\mathrm{Pic}(X_{i\mathbb{C}})^G$ for $i = 1, 2$ (see Proposition 7.1). On the other hand, $\widetilde{\pi}^! \mathcal{O}_X(2)$ is a double in $\mathrm{Pic}(\widetilde{X})$, again by Proposition 7.1.

The generalized Jacobian J of X is an extension $1 \to L \to J \to \widetilde{J} \to 1$, where $\widetilde{J}$ (the Jacobian of $\widetilde{X}$) is an elliptic curve and $L = R^1\mathbb{G}_m \times R^1\mathbb{G}_m$. Hence $|\widetilde{J}(\mathbb{R})_2| = 4$ by Proposition 6.5 and $|J(\mathbb{R})_2| = 8$ by Proposition 6.14. We have two exceptional partial normalizations, one corresponding to each P_i, and both are equal to X itself (see 3.11). Hence there are $8 - (1 + 2) = 5$ base-point free signed representations, and 4 signed representations with base locus $\{P_1, P_2\}$.

To determine their signatures we have to see whether ∂ vanishes on the 2-torsion. This is the case for X since $X(\mathbb{R}) \neq \varnothing$, but not for $\widetilde{X}$ (see Corollary 6.10). So 2 of the 4 representations with base locus $\{P_1, P_2\}$ are definite, the other two are indefinite. All 5 base-point free representations are indefinite.

For an explicit example, let $f = x^4 + x^2y^2 + y^4 - 2x^2 + 1$ (in affine coordinates). Here X has A_1^*-singularities in $P_{1,2} = (\pm 1, 0)$. The two definite representations are

$$f = (x^2 - 1)^2 + x^2y^2 + y^4 = y^2 + \frac{3}{4}\,y^4 + \frac{1}{4}\Bigl(2x^2 + y^2 - 2\Bigr)^2.$$

Example 8.5. ($k = \mathbb{R}$) Let f be psd such that X has an A_5^*-singularity. There are four Gorenstein partial normalizations $X_3 \to X_2 \to X_1 \to X_0 = X$ (cf. 3.10), where $X_3 = \widetilde{X}$ is the normalization. Write $\pi_i \colon X_i \to X$.

We have $p_g(X) = 0$ and $p_a(X_i) = 3 - i$. The curve X_i has a singularity of type A_{5-2i}^* ($i = 0, 1, 2$), respectively is smooth ($i = 3$). Since $X_i(\mathbb{R}) \neq \varnothing$ for $i = 0, 1, 2$ (and $X_3(\mathbb{R}) = \varnothing$), we see that $\pi_i^! \mathcal{O}_X(2)$ is a double in $\mathrm{Pic}(X_{i\mathbb{C}})^G$

for $i \neq 1$ (but not for $i = 1$), and is a double in $\mathrm{Pic}(X_i)$ for $i = 0, 2$ (but not for $i = 1, 3$), by Proposition 7.1.

The generalized Jacobian J_i of X_i is a linear group, isomorphic to $R^1\mathbb{G}_m \times \mathbb{G}_a^{2-i}$ (for $i = 0, 1, 2$) (respectively trivial for $i = 3$). Hence $J_i(\mathbb{R})_2$ has order 2 for $i = 0, 1, 2$. The restriction of ∂ to $J_i(\mathbb{R})_2$ is trivial for all i since $p_g = 0$ (see Corollary 6.10). Finally, the exceptional partial normalization (cf. Definition 3.7) is X_2, according to 3.11.

From this we see that for $i = 0, 2, 3$ there is precisely one signed representation associated with X_i, while there is none associated with X_1. For $i = 3$, this representation is definite (since $\pi_3^! \mathcal{O}_X(2)$ is not a double in $\mathrm{Pic}(X_3)$), while for $i = 0, 2$ it is indefinite (since $\pi_i^! \mathcal{O}_X(2)$ is a double in $\mathrm{Pic}(X_i)$); see Proposition 7.1.

For an explicit example, let $f = x^4 + 2x^2y^2 + y^4 + 2xy^2 + x^2$ (with an A_5^*-singularity in the origin). The three signed representations are

$$\begin{aligned} f &= \frac{1}{16}\Big(4x^2 + 5y^2 + 2\Big)^2 - \frac{1}{16}\Big(3y^2 - 2\Big)^2 - \frac{1}{2}\Big(xy - 2y\Big)^2 \\ &= (x^2 + y^2)^2 + (x + y^2)^2 - y^4 \\ &= x^4 + 2x^2y^2 + (x + y^2)^2. \end{aligned}$$

The first is the only base-point free representation, the third is the only definite representation (each up to equivalence).

9. Complements

9.1. Our results (7.3, Table 4) re-prove Hilbert's 1888 theorem, at least in the irreducible case: If $f \in \mathbb{R}[x_0, x_1, x_2]$ is a psd form of degree 4 which is irreducible over $\mathbb{C}$, then f is a sum of three squares of quadratic forms. Of course, this is not the point of this work altogether. It was not our aim to give an alternative proof, but to arrive at a refined understanding and analysis of the existing representations.

9.2. Nevertheless we'd like to indicate here, for the sake of completeness, how the reducible cases of Hilbert's theorem can all be covered by direct arguments. These are elementary, but not all of them are obvious. There are several cases to distinguish. Let $f \in \mathbb{R}[x_0, x_1, x_2]$ be a psd form of degree 4 which is reducible over $\mathbb{C}$.

If f is irreducible over $\mathbb{R}$, then $f = p_1^2 + p_2^2$ is a sum of two squares of quadratic forms. Suppose therefore that f is reducible over $\mathbb{R}$. Then $f = q_1q_2$ where q_1, q_2 are quadratic forms over $\mathbb{R}$. If one of them is indefinite, then $q_2 = cq_1$ with $c > 0$, and so f is a square. Hence we can assume that q_1, q_2

are both psd. If both are sums of two squares, then the same holds for f. If one q_i is a square, we are equally finished.

Assume that q_1 is irreducible over $\mathbb{C}$. Hence q_1 is a sum of three, but not of two squares over $\mathbb{R}$. First let q_2 be a sum of two squares (not a square). By simultaneously diagonalizing q_1 and q_2 we can assume $q_1 = x_0^2 + x_1^2 + x_2^2$ and $q_2 = a_0x_0^2 + a_1x_1^2$ where $0 < a_1 \le a_0$. Then $q_1 = \frac{1}{a_0}q_2 + p$ where $p = \bigl(1 - \frac{a_1}{a_0}\bigr)x_1^2 + x_2^2$ is a sum of two squares, and hence $f = q_1q_2 = \frac{1}{a_0}q_2^2 + pq_2$ is a sum of three squares.

It remains to consider the case where both q_1 and q_2 are irreducible over $\mathbb{C}$, and hence both are sums of three but not of two squares over $\mathbb{R}$. Here one can use the following lemma which is an exercise in linear algebra: *Let b_1, b_2 be two positive definite symmetric bilinear forms on $\mathbb{R}^3$. For $0 \ne x \in \mathbb{R}^3$ put $g(x) := \frac{b_1(x,x)}{b_2(x,x)}$. Then there exist $0 \ne x$, $y \in \mathbb{R}^3$ with $b_i(x,y) = 0$ for $i = 1, 2$ and with $g(x) = g(y)$.*

Using this lemma, one shows that there exist linear forms u_i $(i = 1, \dots, 4)$ and a real number $c > 0$ such that $q_1 = u_1^2 + u_2^2 + u_3^2$ and $q_2 = c(u_1^2 + u_2^2 + u_4^2)$. From the identity

$$(u_1^2 + u_2^2 + u_3^2)(u_1^2 + u_2^2 + u_4^2) \;=\; (u_1^2 + u_2^2 + u_3u_4)^2 + (u_1^2 + u_2^2)(u_3 - u_4)^2$$

one concludes therefore that q_1q_2 is a sum of three squares. (After diagonalizing q_1 and q_2 simultaneously, and after some tedious calculations, one can write down such a representation explicitly.)

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