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Unramified cohomology of degree 3 and Noether's problem

Emmanuel Peyre

Institut Fourier, UFR de Mathématiques, UMR 5582, Université de Grenoble I et CNRS, BP 74, 38402 Saint-Martin d'Hères CEDEX, France
(e-mail: `Emmanuel.Peyre@ujf-grenoble.fr`)

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Dedicated to Jean-Louis Colliot-Thélène.

Abstract. Let G be a finite group and W be a faithful representation of G over $\mathbf{C}$. The group G acts on the field of rational functions $\mathbf{C}(W)$. The question whether the field of invariant functions $\mathbf{C}(W)^G$ is purely transcendental over $\mathbf{C}$ goes back to Emmy Noether. Using the unramified cohomology group of degree 2 of this field as an invariant, Saltman gave the first examples for which $\mathbf{C}(W)^G$ is not rational over $\mathbf{C}$. Around 1986, Bogomolov gave a formula which expresses this cohomology group in terms of the cohomology of the group G.

In this paper, we prove a formula for the prime to 2 part of the unramified cohomology group of degree 3 of $\mathbf{C}(W)^G$. Specializing to the case where G is a central extension of an $\mathbf{F}_p$-vector space by another, we get a method to construct nontrivial elements in this unramified cohomology group. In this way we get an example of a group G for which the field $\mathbf{C}(W)^G$ is not rational although its unramified cohomology group of degree 2 is trivial.

Contents

1. Introduction

For any function field K over the field of complex numbers $\mathbf{C}$, the unramified cohomology groups $H^i_{\mathrm{nr}}(K, \mathbf{Q}/\mathbf{Z})$ are subgroups of the Galois cohomology groups $H^i(K, \mathbf{Q}/\mathbf{Z})$ which are trivial if K is purely transcendental over $\mathbf{C}$. If K is the function field of a smooth projective variety X over $\mathbf{C}$, then the group $H^1_{\mathrm{nr}}(K, \mathbf{Q}/\mathbf{Z})$ is isomorphic to $\mathrm{Hom}(\pi_1(X(\mathbf{C})), \mathbf{Q}/\mathbf{Z})$, which is trivial if X is unirational, and the group $H^2_{\mathrm{nr}}(K, \mathbf{Q}/\mathbf{Z})$ is isomorphic to the Grothendieck–Brauer group $\mathrm{Br}(X)$. An avatar of this invariant was used by Artin and Mumford in [ArMu] to construct examples of unirational varieties over $\mathbf{C}$ which are not rational. The higher unramified cohomology groups were first introduced by Colliot-Thélène and Ojanguren in [CTO] to produce new examples of such varieties. Other examples based on these unramified cohomology groups of higher degree were produced by the author in [Pe1].

Let G be a finite group and W be a faithful linear representation of G over a field k. The action of G on W induces an action of G on the function field $k(W)$. A natural question, first raised by Emmy Noether [No, p. 222], is to determine whether the field of invariant functions $k(W)^G$ is purely transcendental over k. By the no-name lemma [BK, Lemma 1.3], if W and W' are two faithful representations of G over k, then $k(W \oplus W')^G$ is rational over both $k(W)^G$ and $k(W')^G$. Thus the stable rationality of $k(W)^G$ over k does not depend on the choice of W. In 1969 and 1972, Swan and Voskresenskiĭ constructed examples for which $\mathbf{Q}(W)^G$ is not rational over $\mathbf{Q}$ (see [Sw] and [Vo]). However their method does not work over an algebraically closed field of characteristic 0. In 1984, Saltman gave the first example of a group G such that $\mathbf{C}(W)^G$ is not stably rational over $\mathbf{C}$ using the unramified cohomology group $H^2_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$ as an obstruction. In a subsequent work [Bo1] Bogomolov made an in-depth study of this cohomology group. More precisely he proved that there is a natural isomorphism

$$\bigcap_{B \in \mathscr{B}_G} \mathrm{Ker}(H^2(G, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^2(B, \mathbf{Q}/\mathbf{Z})) \xrightarrow{\sim} H^2_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$$

where $\mathscr{B}_G$ denotes the set of bicyclic subgroups of G, that is the set of subgroups of G which are isomorphic to a quotient of $\mathbf{Z}^2$. Using this isomorphism, he was able to compute explicitly this cohomology group when G is the central extension of an $\mathbf{F}_p$-vector space by another and thus to produce new examples of finite groups G for which $\mathbf{C}(W)^G$ is not stably rational over $\mathbf{C}$.

The aim of this text is to present similar results for the unramified cohomology groups of degree 3. Theorem 1 gives an isomorphism up to 2-torsion from a quotient of a subgroup of $H^3(G, \mathbf{Q}/\mathbf{Z})$ to the unramified cohomology group $H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$. When G is a central extension of an $\mathbf{F}_p$-vector space by another, we use this formula to get a description of a subgroup of this unramified cohomology group in terms of linear algebra (see Theorem 2). Then we use this construction to produce a group G such that $\mathbf{C}(W)^G$ is not stably rational over $\mathbf{C}$ although the unramified cohomology group $H^2_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$ is trivial.

The possibility of extending the work of Bogomolov to higher degrees was first hinted to the author by J.-L. Colliot-Thélène around 1990. The first steps toward this generalization were made by Saltman in [Sa2] where he proved that the unramified cohomology group in degree three is contained in the image of the inflation map

$$H^3(G, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}).$$

In [Bo2], Bogomolov gave a first description of the inverse image of the group $H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$ in $H^3(G, \mathbf{Q}/\mathbf{Z})$ in geometrical terms. In the present paper, we define a subgroup $H^3_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z})$ of the cohomology group $H^3(G, \mathbf{Q}/\mathbf{Z})$ purely in cohomological terms and prove that it coincides with the inverse image of the unramified cohomology group.

One of the main difficulty which remained was to describe the kernel of the inflation map. In [Pe3], we proved, extending ideas of Saltman [Sa2], that there is a natural exact sequence

$$0 \longrightarrow \mathrm{CH}^2_G(\mathbf{C}) \longrightarrow H^3(G, \mathbf{Q}/\mathbf{Z}(2)) \longrightarrow H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}(2))$$

where $\mathrm{CH}^2_G(\mathbf{C})$ denotes the equivariant Chow group of codimension two of a point. The main step of the proof of Theorem 1 relates the image of $\mathrm{CH}^2_G(\mathbf{C})$ with the permutation negligible classes introduced by Saltman in [Sa2].

In Sect. 2 we introduce the notation used in the rest of this paper, Sect. 3 is devoted to the formula describing $H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$ up to 2-torsion and Sect. 4 contains its proof. In Sect. 5 we consider the case of a central extension of an $\mathbf{F}_p$-vector space by another one. The last section is devoted to the construction of an explicit example.

I am very thankful to the referee who pointed out many weaknesses of a former version of this paper.

2. Definitions

Let us fix some notation for the rest of this text.

Notation 1. Let k be a field of characteristic 0, $\overline{k}$ be an algebraic closure of k. For any positive integer n, we denote by μ_n the n-th roots of unity in $\overline{k}$ and for j in $\mathbf{Z}$ we put

$$\mu_n^{\otimes j} = \begin{cases} \mu_n^{\otimes j-1} \otimes \mu_n & \text{if } j > 1, \\ \mathbf{Z}/n\mathbf{Z} & \text{if } j = 0, \\ \mathrm{Hom}\left(\mu_n^{\otimes -j}, \mathbf{Z}/n\mathbf{Z}\right) & \text{if } j < 0. \end{cases}$$

For $i \geqslant 0$, we consider the Galois cohomology groups

$$H^i\left(k, \mu_n^{\otimes j}\right) = H^i\left(\mathrm{Gal}(\overline{k}/k), \mu_n^{\otimes j}\right)$$

as well as their direct limits

$$H^i(k, \mathbf{Q}/\mathbf{Z}(j)) = \varinjlim_n H^i\big(k, \mu_n^{\otimes j}\big).$$

If V is a variety over k, we also consider the étale sheaves $\mu_n^{\otimes j}$ and $\mathbf{Q}/\mathbf{Z}(j)$ on V.

For any function field over k, that is finitely generated as a field over k, we denote by $\mathscr{P}(K/k)$ the set of discrete valuation rings A of rank one such that $k \subset A \subset K$ and such that the fraction field $\mathrm{Fr}(A)$ of A is K. If A belongs to $\mathscr{P}(K/k)$, then let κ_A be its residue field and, for any strictly positive integer i and any j in $\mathbf{Z}$,

$$\partial_A : H^i\big(K, \mu_n^{\otimes j}\big) \longrightarrow H^{i-1}\big(\kappa_A, \mu_n^{\otimes j-1}\big)$$

be the corresponding residue map (see [CTO, p. 142]). They induce residue maps

$$\partial_A : H^i(K, \mathbf{Q}/\mathbf{Z}(j)) \longrightarrow H^{i-1}(\kappa_A, \mathbf{Q}/\mathbf{Z}(j-1)).$$

We then consider the unramified cohomology groups $H^i_{\mathrm{nr}/k}(K, \mathbf{Q}/\mathbf{Z}(j))$ defined as the intersection

$$\bigcap_{A\in\mathscr{P}(K/k)} \mathrm{Ker}\left(H^i(K, \mathbf{Q}/\mathbf{Z}(j)) \xrightarrow{\partial_A} H^{i-1}(\kappa_A, \mathbf{Q}/\mathbf{Z}(j-1))\right).$$

In particular, the unramified Brauer group may be described as

$$\mathrm{Br}_{\mathrm{nr}/k}(K) = H^2_{\mathrm{nr}/k}(K, \mathbf{Q}/\mathbf{Z}(1)).$$

We shall omit k from the notation when the field k is clear from the context.

Let us also recall that two function fields K and L are said to be stably isomorphic over k if there exist indeterminates $U_1, \dots, U_m, T_1, \dots, T_n$ and an isomorphism from $K(U_1, \dots, U_m)$ to $L(T_1, \dots, T_n)$ over k. By [CTO, Proposition 1.2], if K and L are stably isomorphic over k, then

$$H^i_{\mathrm{nr}}\big(K, \mu_n^{\otimes j}\big) \xrightarrow{\sim} H^i_{\mathrm{nr}}\big(L, \mu_n^{\otimes j}\big).$$

In particular, if k is algebraically closed and K stably rational over k then $H^i_{\mathrm{nr}}(K, \mu_n^{\otimes j})$ is trivial.

We shall also use the equivariant Chow groups as defined by Totaro [To, Definition 1.2], Edidin, and Graham [EG, §2.2].

Definition 2. *Let G be a finite group and W a faithful linear representation of G over k. For any strictly positive n, let U_n be the maximal open set in W^n on which G acts freely. We have that* $\mathrm{codim}_{W^n}(W^n - U_n) \geqslant n$. *If Y is* *a quasi-projective smooth geometrically integral variety equipped with an action of G over k, the equivariant Chow group of Y is defined by*

$$\mathrm{CH}^i_G(Y) = \mathrm{CH}^i((Y \times U_{i+2})/G)$$

where $(Y \times U_{i+2})/G$ *is the geometric quotient of* $Y \times U_{i+2}$ *by* G*. We put* $\mathrm{CH}^i_G(k) = \mathrm{CH}^i_G(\mathrm{Spec}\, k)$*, where the action of G on* Spec k *is trivial, and define* $\mathrm{Pic}_G(Y)$ *as* $\mathrm{CH}^1_G(Y)$*.*

By [Pe3, Definition 3.1.3], *if k is algebraically closed, the étale cycle map induces a natural cycle map*

$$\mathrm{cl}_i : \mathrm{CH}^i_G(k) \longrightarrow H^{2i-1}(G, \mathbf{Q}/\mathbf{Z}(i))$$

such that, by [Pe3, Example 3.1.1],

$$\mathrm{cl}_1 : \mathrm{Pic}_G(k) \xrightarrow{\sim} H^1(G, \mathbf{Q}/\mathbf{Z}(1))$$

is an isomorphism.

As indicated in the introduction, one of the main problem to compute the unramified cohomology is to determine the kernel of the inflation map

$$\mathrm{Ker}(H^3(G, \mathbf{Q}/\mathbf{Z}(2)) \longrightarrow H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}(2))).$$

More generally, let us recall the notion of geometrically negligible classes, due to Saltman, which is a variant of the notion introduced by Serre in his lectures at the Collège de France in 1990–1991 [Se1].

Definition 3. *If G is a finite group, M a G-module and k a field, then a class* λ *in* $H^i(G, M)$ *is said to be totally k-negligible if and only if for any extension K of k and any morphism*

$$\rho : \mathrm{Gal}(K^s/K) \longrightarrow G$$

where K^s *is a separable closure of K, the image of* λ *by* ρ^* *is trivial in* $H^i(K, M)$*. The class* λ *is said to be geometrically negligible if* $k = \mathbf{C}$*.*

Remark 1. As was proven by Serre (see also [Sa2, Proposition 4.5]), the group of geometrically negligible classes in $H^i(G, M)$ coincides with the kernel of the map

$$H^i(G, M) \longrightarrow H^i(\mathbf{C}(W)^G, M).$$

In what follows, we shall be interested by the case where $i = 3$ and $M = \mathbf{Q}/\mathbf{Z}(2)$. We shall also assume that $k = \mathbf{C}$. The map $e : \mathbf{Q} \to \mathbf{C}$ defined by $e(x) = \exp(2i\pi x)$ induces an isomorphism from $\mathbf{Q}/\mathbf{Z}$ to $\mathbf{Q}/\mathbf{Z}(1)$. In the rest of this paper, we shall write explicitly the twist for the cohomology of the fields but use $\mathbf{Q}/\mathbf{Z}$ as coefficients for the cohomology of finite groups, since the definition of a G-module $\mathbf{Q}/\mathbf{Z}(i)$ depends on the choice of a morphism from $\mathrm{Gal}(K^s/K)$ to G for some field K.

Definition 4. *We define the group $H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$ of permutation negligible classes as the group*

$$\sum_{H \subset G} \mathrm{Cores}_H^G(\mathrm{Im}(H^1(H, \mathbf{Q}/\mathbf{Z})^{\otimes 2} \xrightarrow{\cup} H^3(H, \mathbf{Q}/\mathbf{Z}))). \tag{1}$$

where the cup-product on the right is given by the commutative diagram

$$\begin{tikzcd} H^i(G, \mathbf{Q}/\mathbf{Z}) \times H^j(G, \mathbf{Q}/\mathbf{Z}) \arrow[r, "\sim"] \arrow[d, "\cup"] & H^{i+1}(G, \mathbf{Z}) \times H^{j+1}(G, \mathbf{Z}) \arrow[d, "\cup"] \\ H^{i+j+1}(G, \mathbf{Q}/\mathbf{Z}) \arrow[r, "\sim"] & H^{i+j+2}(G, \mathbf{Z}) \end{tikzcd} \tag{2}$$

for any $i \geqslant 1$ and $j \geqslant 1$.

Remark 2. This group was first introduced by Saltman who defined it using permutation modules [Sa2, p. 190] and proved that it may be described as

$$H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z}) = \mathrm{Ker}(H^3(G, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^3(G, \mathbf{C}(W)^*)),$$

where the map on the right-hand side is induced by the natural injection $\mathbf{Q}/\mathbf{Z}(1) \to \mathbf{C}(W)^*$ [Sa2, Proposition 4.7]. In [Pe3, pp. 196–197], we prove that this group coincides with the one defined by the formula (1).

Finally we shall also need to pull back the residue maps to the cohomology of G.

Definition 5. *For any subgroup H of G and any element g of the centralizer $Z_G(H)$ of H in G, we define a map*

$$\partial_{H,g} : H^3(G, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^2(H, \mathbf{Q}/\mathbf{Z})$$

as follows: let I be the subgroup generated by g. The natural morphism $m : (h, i) \mapsto hi$ from $H \times I$ to G induces a map

$$m^* : H^3(G, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^3(H \times I, \mathbf{Q}/\mathbf{Z}).$$

But the projection $\mathrm{pr}_2 : H \times I \to I$ *induces a section of the restriction map*

$$H^3(H \times I, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^3(I, \mathbf{Q}/\mathbf{Z})$$

defined by the map $i_2 : I \to H \times I$ *mapping* i *on* (e, i). *This yields a morphism*

$$H^3(H \times I, \mathbf{Q}/\mathbf{Z}) \xrightarrow{s_{H,I}} \mathrm{Ker}(H^3(H \times I, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^3(I, \mathbf{Q}/\mathbf{Z}))$$

given by $s_{H,I}(\xi) = \xi - \mathrm{pr}_2^* \circ i_2^*(\xi)$. *Using Hochschild–Serre's spectral sequence* [HS]

$$E_2^{p,q} = H^p(H, H^q(I, \mathbf{Q}/\mathbf{Z})) \Rightarrow H^{p+q}(H \times I, \mathbf{Q}/\mathbf{Z})$$

and the fact that $H^2(I, \mathbf{Q}/\mathbf{Z}) = 0$ *we get a map*

$$H^3(H \times I, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^2(H, H^1(I, \mathbf{Q}/\mathbf{Z})). \tag{3}$$

But evaluation at g *defines an injection*

$$H^1(I, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} \mathrm{Hom}(I, \mathbf{Q}/\mathbf{Z}) \hookrightarrow \mathbf{Q}/\mathbf{Z}$$

which yields

$$\partial : H^3(H \times I, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^2(H, \mathbf{Q}/\mathbf{Z}).$$

The map $\partial_{H,g}$ *is then defined as the composite* $\partial \circ m^*$. *We define*

$$H^3_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z}) = \bigcap_{\substack{H \subset G \\ g \in Z_G(H)}} \mathrm{Ker}(\partial_{H,g}).$$

Remark 3. In [Bo2, p. 10, Definition], Bogomolov defines the notion of unramified elements relatively to an element g of G. Our definition of the residue map may be seen as an algebraic version of his construction.

Remark 4. With similar arguments, one can easily define for any subgroup H of G and any g in $Z_G(H)$ a morphism

$$\partial_{H,g} : H^2(G, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^1(H, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} \mathrm{Hom}(H, \mathbf{Q}/\mathbf{Z})$$

and

$$H^2_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z}) = \bigcap_{\substack{H \subset G \\ g \in Z_G(H)}} \mathrm{Ker}(\partial_{H,g}).$$

Let us show that

$$H^2_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z}) = \bigcap_{B \in \mathscr{B}_G} \mathrm{Ker}(H^2(G, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^2(B, \mathbf{Q}/\mathbf{Z}))$$

where $\mathscr{B}_G$ denotes the set of bicyclic subgroups of G.

If γ belongs to the right hand side, let H be a subgroup of G, let g belong to $Z_G(H)$, and let $x \in H$; the group $B = \langle g, x\rangle$ is a bicyclic group of G and the functoriality of the Hochschild–Serre spectral sequence provides a commutative diagram

$$\begin{tikzcd}
H^2(G, \mathbf{Q}/\mathbf{Z}) \arrow[r, "\partial_{H,g}"] \arrow[d, "\mathrm{Res}^G_B"] & H^1(H, \mathbf{Q}/\mathbf{Z}) \arrow[d, "\mathrm{Res}^H_{\langle x\rangle}"] \\
H^2(B, \mathbf{Q}/\mathbf{Z}) \arrow[r, "\partial_{\langle x\rangle,g}"] & H^1(\langle x\rangle, \mathbf{Q}/\mathbf{Z}).
\end{tikzcd}$$

Since $\mathrm{Res}^G_B(\gamma) = 0$, for any x in H we have $\mathrm{Res}^H_{\langle x\rangle}(\partial_{H,g}(\gamma)) = 0$. But $H^1(H, \mathbf{Q}/\mathbf{Z})$ is canonically isomorphic to $\mathrm{Hom}(H, \mathbf{Q}/\mathbf{Z})$ and we get that $\partial_{H,g}(\gamma) = 0$.

Conversely, if γ belongs to $H^2_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z})$ and B is a bicyclic subgroup of G, then $\mathrm{Res}^G_B(\gamma)$ belongs to $H^2_{\mathrm{nr}}(B, \mathbf{Q}/\mathbf{Z})$. But

$$H^2(B, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} \mathrm{Hom}(\Lambda^2 B, \mathbf{Q}/\mathbf{Z})$$

(see [Bro, p. 127]). The group $\Lambda^2 B$ is either trivial or cyclic generated by an element of the form $u \wedge v$. In the latter case, one has that $\partial_{\langle u\rangle,v}$ is injective and $\mathrm{Res}^G_B(\gamma) = 0$.

3. Description of the unramified cohomology group

The first aim of this paper is to prove the following theorem:

Theorem 1. *If G is a finite group and if W is a faithful representation of G over $\mathbf{C}$ then the inflation map induces a surjective map*

$$H^3_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z})/H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z}) \longrightarrow\!\!\!\!\!\rightarrow H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}),$$

the kernel of which is killed by a power of 2.

Remarks 5. (i) If G is of odd order, the above map is an isomorphism. However, in [Sa2, Theorem 4.14], Saltman gave an example of a 2-group for which the kernel of this map is not trivial.

(ii) Using Remark 4, Bogomolov's theorem for the unramified Brauer group [Bo1, Theorem 3.1] could be stated as

$$H^2_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} H^2_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}).$$

4. Proof of the main theorem

We shall first recall the result relating the geometrically negligible classes to the equivariant Chow group of codimension 2.

Notation 6. If V is a variety over a field k of characteristic 0, $V^{(p)}$ denotes the set of points of codimension p in V. For any x in $V^{(p)}$, let $\kappa(x)$ be its residue field. We also denote by $\mathscr{H}^i_{\text{ét}}(\mu_n^{\otimes j})$ the Zariski sheaf corresponding to the presheaf mapping U to $H^i_{\text{ét}}(U, \mu_n^{\otimes j})$. We define similarly the sheaf $\mathscr{H}^i_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(j))$ and $\mathscr{K}_j$ the Zariski sheaf corresponding to the presheaf mapping U to $K_i(U)$, the i-th group of Quillen K-theory.

We denote by $|X|$ the cardinal of a set X.

The following proposition follows from [Pe3, Theorem 2.3.1], but we shall now give a direct proof of it which is due to Colliot-Thélène.

Proposition 1. *Let G be a finite group, W be a faithful representation of G over* $\mathbf{C}$*, and U be an open subset in W on which G acts freely. We assume that* $\operatorname{codim}_W(W-U)$ *is bigger than* 4. *Then there is a canonical exact sequence*

$$0 \longrightarrow \mathrm{CH}^2_G(\mathbf{C}) \longrightarrow H^3(G, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^0_{\mathrm{Zar}}\big(U/G, \mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))\big) \longrightarrow 0.$$

Proof. Let $X = U/G$. By [CT, (3.10)], the Bloch–Ogus spectral sequence [BO] yields an exact sequence

$$\begin{aligned} 0 \longrightarrow H^1_{\mathrm{Zar}}\big(X, \mathscr{H}^2_{\text{ét}}\big(\mu_n^{\otimes 2}\big)\big) &\longrightarrow H^3_{\text{ét}}\big(X, \mu_n^{\otimes 2}\big) \\ \longrightarrow H^0_{\mathrm{Zar}}\big(X, \mathscr{H}^3_{\text{ét}}\big(\mu_n^{\otimes 2}\big)\big) &\longrightarrow \mathrm{CH}^2(X)/n \longrightarrow H^4_{\text{ét}}\big(X, \mu_n^{\otimes 2}\big). \end{aligned} \tag{4}$$

By [CT, (3.2)], there also is an exact sequence

$$0 \longrightarrow H^1(X, \mathscr{K}_2)/n \longrightarrow H^1\big(X, \mathscr{H}^2_{\text{ét}}\big(\mu_n^{\otimes 2}\big)\big) \longrightarrow \mathrm{CH}^2(X)_n \longrightarrow 0 \tag{5}$$

which follows from an argument of Bloch and Ogus and from the Merkur′ev–Suslin theorem. Since we assumed that $\operatorname{codim}_W(W-U) \geqslant 4$, the definition of the Chow groups gives the equality

$$\mathrm{CH}^2(U) = \mathrm{CH}^2(W) = \{0\},$$

the Brown–Gersten–Quillen spectral sequence [Q, §7.5] yields that

$$H^1(U, \mathscr{K}_2) = H^1(W, \mathscr{K}_2) = \{0\},$$

and the Bloch–Ogus spectral sequence implies that

$$H^0_{\mathrm{Zar}}\big(U, \mathscr{H}^3_{\text{ét}}(\mu_n^{\otimes 2})\big) = H^0_{\mathrm{Zar}}\big(W, \mathscr{H}^3_{\text{ét}}(\mu_n^{\otimes 2})\big) = 0.$$

But using a restriction-corestriction argument (see e.g. [Ro, p. 330 and the proof of Theorem 7.1]) for the map $\pi : U \to U/G$, we get that the corresponding groups for X are killed by $|G|$. In particular, this implies that

$$\varinjlim_n \mathrm{CH}^2(X)/n = 0 \quad \text{and} \quad \varinjlim_n H^1(X, \mathscr{K}_2)/n = 0.$$

Taking inductive limits, the exact sequence (4) provides an exact sequence

$$\begin{aligned} 0 \longrightarrow H^1_{\mathrm{Zar}}\big(X, \mathscr{H}^2_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))\big) &\longrightarrow H^3_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(2)) \\ \longrightarrow H^0_{\mathrm{Zar}}\big(X, \mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))\big) &\longrightarrow 0 \end{aligned}$$

and the exact sequence (5) yields an isomorphism

$$H^1_{\mathrm{Zar}}\big(X, \mathscr{H}^2_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))\big) \xrightarrow{\sim} \mathrm{CH}^2(X).$$

By definition of the equivariant Chow groups we have that $\mathrm{CH}^2(X)$ is $\mathrm{CH}^2_G(\mathbf{C})$. We get an exact sequence

$$0 \longrightarrow \mathrm{CH}^2_G(\mathbf{C}) \longrightarrow H^3_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(2)) \longrightarrow H^0_{\mathrm{Zar}}\big(X, \mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))\big) \longrightarrow 0.$$

By [Pe3, Lemma 3.1.1], the Hochschild–Serre spectral sequence yields an isomorphism

$$H^3_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(2)) \xrightarrow{\sim} H^3(G, \mathbf{Q}/\mathbf{Z}(2)).$$

□

To get the group of geometrically negligible classes in $H^3(G, \mathbf{Q}/\mathbf{Z})$, it remains to compute the image of $\mathrm{CH}^2_G(\mathbf{C})$ in that group.

Proposition 2. *If G is a finite group, then the prime to 2 part of the group of geometrically negligible classes in $H^3(G, \mathbf{Q}/\mathbf{Z})$ is contained in the group $H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$ of permutation negligible classes.*

Remark 6. The fact that the group $H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$ is contained in the group of geometrically negligible classes was first noted by Saltman [Sa2, Proposition 4.7(b)]. We give a new proof of this fact for self-completeness.

Let H be a subgroup of G and W be a faithful linear representation of G. There is a commutative diagram

$$\begin{tikzcd}
H^1(H,\mathbf{Q}/\mathbf{Z})\otimes H^2(H,\mathbf{Z}) \arrow[r, "\cup"] \arrow[dd] & H^3(H,\mathbf{Q}/\mathbf{Z}) \arrow[d] \\
 & H^3(\mathbf{C}(W)^H,\mathbf{Q}/\mathbf{Z}(1)) \arrow[d] \\
H^1(H,\mathbf{C}(W)^*)\otimes H^2(H,\mathbf{Z}) \arrow[r, "\cup"] & H^3\big(\mathbf{C}(W)^H,\mathbf{G}_m\big).
\end{tikzcd}$$

Let μ_∞ be the group of the roots of unity in $\mathbf{C}$. Since the group $\overline{\mathbf{C}(W)}^*/\mu_\infty$ is a $\mathbf{Q}$-vector space, the vertical map on the bottom right of the previous diagram is an isomorphism. By Hilbert's theorem 90 the group $H^1(H,\mathbf{C}(W)^*)$ is trivial. Thus the image of the cup-product in $H^3(H,\mathbf{Q}/\mathbf{Z})$ is contained in the subgroup of geometrically negligible classes. Using the commutativity of

$$\begin{tikzcd}
H^3(H,\mathbf{Q}/\mathbf{Z}) \arrow[r] \arrow[d, "\mathrm{Cores}_H^G"] & H^3(\mathbf{C}(W)^H,\mathbf{Q}/\mathbf{Z}) \arrow[d, "\mathrm{Cores}"] \\
H^3(G,\mathbf{Q}/\mathbf{Z}) \arrow[r] & H^3(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z})
\end{tikzcd}$$

we get the inclusion

$$H^3_{\mathrm{p}}(G,\mathbf{Q}/\mathbf{Z}) \subset \mathrm{Ker}\big(H^3(G,\mathbf{Q}/\mathbf{Z}) \longrightarrow H^3(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z})\big).$$

Proof of Proposition 2. Let p be a prime factor of $|G|$ and G_p be a p-Sylow subgroup of G. By the definition of permutation negligible classes (1), we have that

$$\mathrm{Cores}_{G_p}^G\left(H^3_{\mathrm{p}}(G_p,\mathbf{Q}/\mathbf{Z})\right) \subset H^3_{\mathrm{p}}(G,\mathbf{Q}/\mathbf{Z}).$$

We also have commutative diagrams

$$\begin{tikzcd}
H^3(G,\mathbf{Q}/\mathbf{Z}) \arrow[r] \arrow[d, "\mathrm{Res}_{G_p}^G"] & H^3(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z}) \arrow[d, "\mathrm{Res}"] \\
H^3(G_p,\mathbf{Q}/\mathbf{Z}) \arrow[r] & H^3(\mathbf{C}(W)^{G_p},\mathbf{Q}/\mathbf{Z})
\end{tikzcd}$$

and

$$\begin{array}{ccccc} H^3_{\mathrm{p}}(G_p, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^3(G_p, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^3(\mathbf{C}(W)^{G_p}, \mathbf{Q}/\mathbf{Z}) \\ \Big\downarrow{\scriptstyle \mathrm{Cores}^G_{G_p}} & & \Big\downarrow{\scriptstyle \mathrm{Cores}^G_{G_p}} & & \Big\downarrow{\scriptstyle \mathrm{Cores}} \\ H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^3(G, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}). \end{array}$$

If the top sequence is exact, then the fact that $\mathrm{Cores}^G_{G_p} \circ \mathrm{Res}^G_{G_p}$ is the multiplication by the integer $[G : G_p]$ and a simple chase in the diagrams gives that the p-primary part of the bottom sequence is exact. Hence we are reduced to the case where G is a p-group for p an odd prime.

It is well known that the group $\mathrm{CH}^2_G(\mathbf{C})$ is generated by Chern classes of representations of G (see [EKLV, Appendix C.3], [To, p. 257], and [Pe3, Corollary 3.1.9]).

Till the end of the proof, if $i \geqslant 1$, we identify the group $H^i(G, \mathbf{Q}/\mathbf{Z})$ with the group $H^{i+1}(G, \mathbf{Z})$ via the coboundary map. For any x, y in the group $H^1(G, \mathbf{Q}/\mathbf{Z})$ we denote by xy the cup-product $x \cup y$ defined by the commutative diagram (2).

By [CTSS, Corollaire 1, p. 772], the map $\mathrm{CH}^2_G(\mathbf{C}) \to H^3(G, \mathbf{Q}/\mathbf{Z})$ which appears in the exact sequence of Proposition 1 coincides with the cycle map cl_2. Since $\mathrm{cl}_2 \circ c_2 = c_2$ (see e.g. [Fu, Proposition 19.1.2]), we get that the group $H^3_{\mathrm{n}}(G, \mathbf{Q}/\mathbf{Z})$ of geometrically negligible classes is generated by Chern classes of representations of G. By the Whitney formula (see e.g. [Fu, Theorem 3.2(e)]), if x and y belong to $\mathscr{R}(G)$, one has

$$c_2(x + y) = c_2(x) + c_1(x)c_1(y) + c_2(y).$$

By Definition (1), we have that $c_1(x)c_1(y) \in H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$. Thus the induced map

$$\mathscr{R}(G) \xrightarrow{\bar{c}_2} H^3(G, \mathbf{Q}/\mathbf{Z})/H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$$

is a morphism of groups. We want to show that this morphism is trivial.

By Brauer's theorem (see [Se3, §10.5, Theorem 20]), $\mathscr{R}(G)$ is generated as a group by the representations induced from characters of subgroups. It remains to show that for any subgroup H of G and any character χ of H, one has

$$c_2\big(\mathrm{Ind}^G_H \chi\big) \in H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z}).$$

Let $\mathrm{Cores}^{(2)} : H^1(G, \mathbf{Q}/\mathbf{Z}) \to H^3(G, \mathbf{Q}/\mathbf{Z})$ be the map induced by the intermediate transfer map

$$\mathscr{N}_2 : H^2(G, \mathbf{Z}) \longrightarrow H^4(G, \mathbf{Z})$$

defined by Evens in [Ev1, Theorem 1, p. 63]. More generally, in [FMP, p. 2] Fulton and MacPherson defined transfer maps $f_*^{(n)}$ for finite étale coverings which they used to give an expression without denominators for Chern classes of direct images. Using one of their result [FMP, Corollary 5.3] we get the formula

$$c_2\big(\operatorname{Ind}_H^G \chi\big) = \operatorname{Cores}(c_2(\chi)) + \operatorname{Cores}^{(2)}(c_1(\chi)) \\ + c_1\big(\operatorname{Ind}_H^G 1\big).\operatorname{Cores}(c_1(\chi)) + c_2\big(\operatorname{Ind}_H^G 1\big).$$

Since χ is a representation of dimension 1, $c_2(\chi) = 0$. By [FMP, p. 4], for any z in $H^1(H, \mathbf{Q}/\mathbf{Z})$, one has

$$\operatorname{Cores}(z^2) - \operatorname{Cores}(z)^2 + 2\operatorname{Cores}^{(2)}(z) = 0.$$

Since $p \neq 2$, we get the relation

$$\operatorname{Cores}^{(2)}(z) = \frac{1}{2}(\operatorname{Cores}(z)^2 - \operatorname{Cores}(z^2))$$

and therefore the relation

$$c_2\big(\operatorname{Ind}_H^G \chi\big) = \frac{1}{2}\big(\operatorname{Cores}_H^G(c_1(\chi))^2 - \operatorname{Cores}_H^G(c_1(\chi)^2)\big) \\ + c_1\big(\operatorname{Ind}_H^G 1\big).\operatorname{Cores}_H^G(c_1(\chi)) + c_2\big(\operatorname{Ind}_H^G 1\big).$$

It therefore remains to show that for any subgroup H of G, one has

$$c_2\big(\operatorname{Ind}_H^G 1\big) \in H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z}).$$

We shall proceed by induction on $[G : H]$. If $[G : H] = 1$, then $c_2(1) = 0$ and the result is proven. Let us assume the result for subgroups of index strictly smaller than p^m for $m \geqslant 1$. Let H be a subgroup of G with $[G : H] = p^m$. There exists a subgroup H_1 of G such that H is a normal subgroup of H_1 of index p [Su, Theorem 1.6, p. 88]. We have

$$c_2\big(\operatorname{Ind}_H^G 1\big) = c_2\big(\operatorname{Ind}_{H_1}^G \big(\operatorname{Ind}_H^{H_1} 1\big)\big).$$

We may choose $\chi \in \operatorname{Hom}(H_1, \mathbf{C}^*)$ such that $H = \operatorname{Ker}\chi$. Then the induced representation $\operatorname{Ind}_H^{H_1} 1$ coincides with $\mathbf{C}[H_1/H]$ and its class in $\mathscr{R}(H_1)$ is given by

$$\operatorname{Ind}_H^{H_1} 1 = 1 + \chi + \cdots + \chi^{p-1}.$$

Using the Whitney formula, we get

$$c_2\big(\operatorname{Ind}_H^G 1\big) = c_2\big(\operatorname{Ind}_{H_1}^G(1) + \cdots + \operatorname{Ind}_{H_1}^G(\chi^{p-1})\big) \\ \equiv c_2\big(\operatorname{Ind}_{H_1}^G(1)\big) + \cdots + c_2\big(\operatorname{Ind}_{H_1}^G(\chi^{p-1})\big) \bmod H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z}).$$

By induction, we obtain that $c_2(\operatorname{Ind}_H^G 1)$ belongs to $H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$. $\square$

Let us now describe the inverse image in $H^3(G, \mathbf{Q}/\mathbf{Z})$ of the unramified cohomology group of $\mathbf{C}(W)^G$.

Proposition 3. *The group* $H^3_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z})$ *is the inverse image in the group* $H^3(G, \mathbf{Q}/\mathbf{Z})$ *of the group* $H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$.

Proof. We denote by

$$\rho : \mathrm{Gal}(\overline{\mathbf{C}(W)}/\mathbf{C}(W)^G) \longrightarrow G$$

the natural surjection. Let γ in $H^3_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z})$. We want to prove that its image $\rho^*(\gamma)$ in $H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$ is unramified. Let A be a ring of the set $\mathscr{P}(\mathbf{C}(W)^G/\mathbf{C})$, as defined in Sect. 2. Let B be an element of the set $\mathscr{P}(\mathbf{C}(W)/\mathbf{C})$ above A. We denote by K the field $\mathbf{C}(W)^G$, by L the field $\mathbf{C}(W)$, by $\widehat{L}_B$ the completion of L at B, by $\widehat{K}_A$ the completion of K in $\widehat{L}_B$, by $\overline{L}_B$ an algebraic closure of $\widehat{L}_B$, and by $\widehat{K}_A^{\mathrm{nr}}$ (resp. $\widehat{L}_B^{\mathrm{nr}}$) the maximal unramified extension of K_A (resp. L_B) in $\overline{L}_B$. Let D be the decomposition group of B in G and I be the inertia group. We also put $\mathscr{G}_A = \mathrm{Gal}(\overline{L}_B/\widehat{K}_A)$, $\mathscr{G}_B = \mathrm{Gal}(\overline{L}_B/\widehat{L}_B)$, $I_A = \mathrm{Gal}(\overline{L}_B/\widehat{K}_A^{\mathrm{nr}})$, and $I_B = \mathrm{Gal}(\overline{L}_B/\widehat{L}_B^{\mathrm{nr}})$. We have the following diagram of fields

$$\begin{tikzcd}[column sep=small]
 & & \overline{L}_B \arrow[dl, equal] \arrow[dr, dash, "I_B"] & & & \\
 & \overline{K}_A \arrow[dr, dash, "I_A"'] & & \widehat{L}_B^{\mathrm{nr}} \arrow[dl, dash, "I"'] \arrow[dr, dash, "\mathscr{G}_B/I_B"] & & \\
 & & \widehat{K}_A^{\mathrm{nr}} \arrow[dr, dash, "\mathscr{G}_A/I_A"'] & & \widehat{L}_B \arrow[dl, dash, "D"'] \arrow[dr, dash] & \\
 & & & \widehat{K}_A \arrow[dr, dash] & & L \arrow[dl, dash, "G"'] \\
 & & & & K &
\end{tikzcd}$$

which yields a commutative diagram of groups

$$\begin{tikzcd}
0 \arrow[r] & I_A \arrow[r, "j_A"] \arrow[d, "f_I"] & \mathscr{G}_A \arrow[r, "\pi_A"] \arrow[d, "f_{\mathscr{G}}"] & \mathscr{G}_A/I_A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \arrow[r, "j"] & D \arrow[r, "\pi"] & D/I \arrow[r] & 0.
\end{tikzcd} \tag{6}$$

On the other hand the residue map

$$H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}(2)) \xrightarrow{\partial_A} H^2(\kappa_A, \mathbf{Q}/\mathbf{Z}(1))$$

is defined as the composite of the maps

$$\begin{aligned} H^3(K, \mathbf{Q}/\mathbf{Z}(2)) &\longrightarrow H^3(\widehat{K}_A, \mathbf{Q}/\mathbf{Z}(2)) \\ &\longrightarrow H^2\big(\mathscr{G}_A/I_A, H^1(I_A, \mathbf{Q}/\mathbf{Z}(2))\big) \xrightarrow{\sim} H^2(\kappa_A, \mathbf{Q}/\mathbf{Z}(1)) \end{aligned} \tag{7}$$

where the second map is induced be the Hochschild–Serre spectral sequence

$$H^p\big(\mathscr{G}_A/I_A, H^q(I_A, \mathbf{Q}/\mathbf{Z}(2))\big) \Rightarrow H^{p+q}(\mathscr{G}_A, \mathbf{Q}/\mathbf{Z}(2))$$

(see [CTO, p. 142]). Indeed I_A, which is isomorphic to $\widehat{\mathbf{Z}}(1)$ is of cohomological dimension 1, and the group $H^1(I_A, \mathbf{Q}/\mathbf{Z}(n))$ is canonically isomorphic to $\mathbf{Q}/\mathbf{Z}(n-1)$. The latter fact gives the last morphism in (7).

We are going to relate the Hochschild–Serre spectral sequence for the exact sequence

$$0 \longrightarrow I_A \xrightarrow{j_A} \mathscr{G}_A \xrightarrow{\pi_A} \mathscr{G}_A/I_A \longrightarrow 0 \tag{8}$$

to the one for the sequence

$$0 \longrightarrow I \xrightarrow{i_2} D \times I \xrightarrow{\mathrm{pr}_1} D \longrightarrow 0,$$

where $i_2(a) = (e, a)$ for any a in I. The extension given by (8) is central since the roots of unity are in $\mathbf{C}$. By [Se2, II, §4, Théorème 2], the field $\widehat{K}_A$ is isomorphic to the field of formal series $\kappa_A((T))$, the extension $\widehat{K}_A^{\mathrm{nr}}/\widehat{K}_A$ is isomorphic to the extension $\overline{\kappa}_A((T))/\kappa_A((T))$, and by [Se2, IV, §2, Proposition 8] the field $\overline{K}_A$ is isomorphic to the direct limit of the fields $\varinjlim \overline{\kappa}_A((T^{1/n}))$. Such isomorphisms induce a splitting of the central extension described in (8).

Using (6), we get that I is central in D and the morphism $f_{\mathscr{G}}$ factorises through a morphism $\psi : \mathscr{G}_A \to D \times I$: let r be a retraction of the map j_A, then the following diagram commutes

$$\begin{tikzcd}
\mathscr{G}_A \arrow[r, "\phi"] \arrow[d, "f_{\mathscr{G}}"] & \mathscr{G}_A \times I_A \arrow[d, "f_{\mathscr{G}} \times f_I"] \\
D & D \times I \arrow[l, "m"]
\end{tikzcd}$$

where we denote by ϕ the morphism sending g to $(gr(g)^{-1}, r(g))$ and by $m : D \times I \to D$ the morphism sending (d, i) to di. Thus we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I_A \arrow[r, "j_A"] \arrow[d, "f_I"] & \mathscr{G}_A \arrow[r, "\pi_A"] \arrow[d, "\psi"] & \mathscr{G}_A/I_A \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & I \arrow[r, "i_2"] \arrow[d, equal] & D \times I \arrow[r, "\mathrm{pr}_1"] \arrow[d, "m"] & D \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & I \arrow[r, "j"] & D \arrow[r, "\pi"] & D/I \arrow[r] & 0
\end{tikzcd} \tag{9}$$

which has exact lines and where τ is the only map making the diagram commutative. For the cohomology groups we have the diagram

$$\begin{tikzcd}
H^3(G,\mathbf{Q}/\mathbf{Z}) \arrow[r, "\mathrm{Res}"] \arrow[d] & H^3(D,\mathbf{Q}/\mathbf{Z}) \arrow[r, "m^*"] \arrow[d, "f_{\mathscr{G}}^*"] & H^3(D\times I,\mathbf{Q}/\mathbf{Z}) \arrow[d, "\psi^*"] \\
H^3(K,\mathbf{Q}/\mathbf{Z}(2)) \arrow[r] & H^3(\widehat{K}_A,\mathbf{Q}/\mathbf{Z}(2)) \arrow[r, equal] & H^3(\mathscr{G}_A,\mathbf{Q}/\mathbf{Z}(2))
\end{tikzcd} \tag{10}$$

which commutes by the definition of ψ and the diagram

$$\begin{tikzcd}
H^3(I,\mathbf{Q}/\mathbf{Z}) \arrow[r, "\mathrm{pr}_2^*"] \arrow[d, "f_I^*"] & H^3(D\times I,\mathbf{Q}/\mathbf{Z}) \arrow[d, "\psi^*"] \\
0 = H^3(I_A,\mathbf{Q}/\mathbf{Z}) \arrow[r, "r^*"] & H^3(\mathscr{G}_A,\mathbf{Q}/\mathbf{Z})
\end{tikzcd}$$

which commutes since $\mathrm{pr}_2\circ\psi = f_I\circ r$. Thus we get a commutative diagram

$$\begin{tikzcd}
H^3(D\times I,\mathbf{Q}/\mathbf{Z}) \arrow[r, "\psi^*"] \arrow[d, "s_{D,I}"] & H^3(\mathscr{G}_A,\mathbf{Q}/\mathbf{Z}(2)) \arrow[d, equal] \\
\mathrm{Ker}(H^3(D\times I,\mathbf{Q}/\mathbf{Z}) \longrightarrow H^3(I,\mathbf{Q}/\mathbf{Z})) \arrow[r, "\psi^*"] \arrow[d] & H^3(\mathscr{G}_A,\mathbf{Q}/\mathbf{Z}(2)) \arrow[d] \\
H^2(D, H^1(I,\mathbf{Q}/\mathbf{Z})) \arrow[r] & H^2\big(\mathscr{G}_A/I_A, H^1(I_A,\mathbf{Q}/\mathbf{Z}(2))\big)
\end{tikzcd} \tag{11}$$

where the map $s_{D,I}$ has been introduced in Definition 5 and the vertical maps in the bottom square come from the Hochschild–Serre spectral sequence. Using Diagrams (10) and (11) we may choose a generator g of I so that the diagram

$$\begin{tikzcd}
H^3(G,\mathbf{Q}/\mathbf{Z}) \arrow[r, "\partial_{D,g}"] \arrow[d] & H^2(D,\mathbf{Q}/\mathbf{Z}) \arrow[d] \\
H^3(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z}(2)) \arrow[r, "\partial_A"] & H^2(\kappa_A,\mathbf{Q}/\mathbf{Z}(1))
\end{tikzcd} \tag{12}$$

commutes. Therefore $\partial_A(\gamma) = 0$ whenever γ belongs to $H^3_{\mathrm{nr}}(G,\mathbf{Q}/\mathbf{Z})$ and

$$H^3_{\mathrm{nr}}(G,\mathbf{Q}/\mathbf{Z}) \subset \rho^{*-1}\big(H^3_{\mathrm{nr}}(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z}(2))\big).$$

We now want to prove the reverse inclusion. For any positive integer i, let $H^i_{\mathrm{gnr}}(G, \mathbf{Q}/\mathbf{Z})$ be the inverse image in $H^i(G, \mathbf{Q}/\mathbf{Z})$ of the group $H^i_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}(2))$. For any morphism of groups $\pi : H \to G$, we have

$$\pi^*\big(H^i_{\mathrm{gnr}}(G, \mathbf{Q}/\mathbf{Z})\big) \subset H^i_{\mathrm{gnr}}(H, \mathbf{Q}/\mathbf{Z}).$$

Indeed let W be a faithful representation of G and V be a faithful representation of H. Then W is a representation of H via π and $V \oplus W$ a faithful representation of H. But we have the following field inclusions

$$\mathbf{C}(W)^G \subset \mathbf{C}(W)^H \subset \mathbf{C}(V \oplus W)^H.$$

Therefore, we get a commutative diagram

$$\begin{array}{ccc} H^3(G, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\quad \pi^* \quad} & H^3(H, \mathbf{Q}/\mathbf{Z}) \\ \downarrow & & \downarrow \\ H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}(2)) & \xrightarrow{\ i\ } & H^3(\mathbf{C}(V \oplus W)^H, \mathbf{Q}/\mathbf{Z}(2)) \end{array}$$

and by [CTO, p. 143] the image by i of $H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}(2))$ is contained in the unramified cohomology group $H^3_{\mathrm{nr}}(\mathbf{C}(V \oplus W)^H, \mathbf{Q}/\mathbf{Z}(2))$. This implies the claim.

We have to show that for any γ in $H^3_{\mathrm{gnr}}(G, \mathbf{Q}/\mathbf{Z})$, for any subgroup H of G, and for any g in $Z_G(H)$ generating a subgroup I of G, we have $\partial_{H,g}(\gamma) = 0$. By the last claim and the definition of $\partial_{H,g}$, we can restrict ourselves to the case where $G = H \times I$. In that particular case, let W be a faithful representation of H and χ be the representation of dimension 1 of I defined by the injection $I \hookrightarrow \mathbf{C}^*$ sending g to $\boldsymbol{e}(1/|I|)$. Then $W \oplus \chi$ is a faithful representation of G. We may consider $\mathbf{C}(W \oplus \chi)$ as $\mathbf{C}(W)(T)$ where T is an indeterminate and define $B \in \mathscr{P}(\mathbf{C}(W \oplus \chi)/\mathbf{C})$ as the valuation defined by the divisor $T = 0$. Let A be the intersection of B with

$$\mathbf{C}(W \oplus \chi)^G = \mathbf{C}(W)^H(T^{|I|}).$$

We now are precisely in the situation described in the first part of the proof and the commutative diagram from (12) may be written as

$$\begin{array}{ccc} H^3(G, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\quad \partial_{H,g} \quad} & H^2(H, \mathbf{Q}/\mathbf{Z}) \\ \downarrow & & \downarrow \\ H^3(\mathbf{C}(W \oplus \chi)^G, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\ \partial_A\ } & H^2(\mathbf{C}(W)^H, \mathbf{Q}/\mathbf{Z}). \end{array} \tag{13}$$

But it is well known that the group of geometrically negligible classes in $H^2(G, \mathbf{Q}/\mathbf{Z})$ is trivial. Indeed, let μ_∞ be the group of roots of unity in $\mathbf{C}$, the group $\mathbf{C}^*/\mu_\infty$ being a $\mathbf{Q}$-vector space, $H^2(G, \mathbf{Q}/\mathbf{Z})$ is isomorphic to $H^2(G, \mathbf{C}^*)$ and there is an exact sequence

$$0 \longrightarrow \mathbf{C}^* \longrightarrow \mathbf{C}(W)^* \longrightarrow \mathrm{Div}(W) \longrightarrow 0$$

where $\mathrm{Div}(W)$ is the free abelian group of the divisors of codimension 1 on W. Its natural basis is globally invariant under the action of G. Thus, using Shapiro's lemma, we have that $H^1(G, \mathrm{Div}(W)) = 0$ and the morphism

$$H^2(G, \mathbf{C}^*) \longrightarrow H^2(G, \mathbf{C}(W)^*)$$

is injective. The claim then follows from Hilbert's theorem 90 (see [Se2, X, §4, Proposition 6]). Therefore the vertical map on the right of Diagram (13) is injective and, if γ belongs to $H^3_{\mathrm{gnr}}(G, \mathbf{Q}/\mathbf{Z})$, then $\partial_{H,g}(\gamma) = 0$. □

5. Central extensions of vector spaces

5.1. The setting

By a result of Fischer [Fi, p. 78], if G is abelian and W a faithful linear representation of G over $\mathbf{C}$, then $\mathbf{C}(W)^G$ is rational over $\mathbf{C}$. Therefore the first interesting groups are central extensions of an $\mathbf{F}_p$-vector space by another one. The example considered by Saltman in [Sa1] is of this type. The unramified Brauer group of $\mathbf{C}(W)^G$ was computed for such groups G by Bogomolov in [Bo1, Lemma 5.1]. A few preliminary results in degree 3 were given in [Pe2].

Notation 7. Let U and V be two $\mathbf{F}_p$-vector spaces for p an odd prime number and let

$$0 \longrightarrow V \xrightarrow{\iota} G \xrightarrow{\pi} U \longrightarrow 0$$

be a central extension of U by V such that $\exp(G) = p$. Without loss of generality, we may assume that $V = [G, G]$ or in other words, that the map $\gamma : \Lambda^2 U \to V$ defined by

$$[g_1, g_2] = \iota \circ \gamma(\pi(g_1) \wedge \pi(g_2))$$

for any g_1, g_2 in G is surjective. By [Bro, §IV.3, Exercise 8], this map γ determines this extension up to isomorphism. More precisely, we may choose a set-theoretic section $s : U \to G$ of π such that

$$\forall u_1, u_2 \in U, \quad s(u_2)s(u_1u_2)^{-1}s(u_1) = \iota\left(\frac{1}{2}\gamma(u_1 \wedge u_2)\right). \qquad (14)$$

Remark 7. If $Z(G) \neq [G, G]$ then G is isomorphic to a product $E \times H$ where E is the $\mathbf{F}_p$-vector space $Z(G)/[G, G]$. Let W (resp. W') be a faithful linear representation of H (resp. E) over $\mathbf{C}$. Then $W \oplus W'$ is a faithful representation of G and the field $\mathbf{C}(W \oplus W')^G$ is the compositum of the fields $\mathbf{C}(W)^H$ and $\mathbf{C}(W')^E$ over $\mathbf{C}$. By Fisher's result, it is rational over $\mathbf{C}(W)^H$. Thus we may assume that $Z(G) = [G, G]$.

Notation 8. For any $\mathbf{F}_p$-vector space E we denote by $E^\vee$ its dual. For any positive integer i there is a natural isomorphism (see [Bki1, A III, p. 154, Proposition 7])

$$\begin{aligned} \Lambda^i(E^\vee) &\longrightarrow (\Lambda^i E)^\vee \\ f_1 \wedge \cdots \wedge f_i &\longmapsto \Big(v_1 \wedge \cdots \wedge v_i \longmapsto \sum_{\sigma \in \mathfrak{S}_i} \epsilon(\sigma) f_1(v_{\sigma(1)}) \ldots f_i(v_{\sigma(i)})\Big). \end{aligned}$$

From now on, we identify $\Lambda^i(E^\vee)$ with $(\Lambda^i E)^\vee$ and denote it by $\Lambda^i E^\vee$. For any subset F of $\Lambda^i E$ (resp. $\Lambda^i E^\vee$) we denote by $F^\perp$ its orthogonal in $\Lambda^i E^\vee$ (resp. $\Lambda^i E$).

The surjective linear map γ induces an injection

$$\gamma^\vee : V^\vee \longrightarrow \Lambda^2 U^\vee.$$

We also put

$$K^2 = \gamma^\vee(V^\vee) \subset \Lambda^2 U^\vee \quad \text{and} \quad K^3 = \gamma^\vee(V^\vee) \wedge U^\vee \subset \Lambda^3 U^\vee.$$

We put $S^i = (K^i)^\perp \subset \Lambda^i U$ if $i = 2$ or 3. Let S^3_{dec} (resp. S^2_{dec}) be the subgroup of S^3 (resp. S^2) generated by the elements of the form $u \wedge v$ for $u \in \Lambda^2 U$ (resp. U) and $v \in U$. We define $K^i_{\max} \supset K^i$ as the orthogonal in $\Lambda^i U^\vee$ of S^i_{dec} for $i = 2$ or 3.

We consider the map

$$\eta_G : \Lambda^i U^\vee \longrightarrow H^i(G, \mathbf{Q}/\mathbf{Z})$$

defined as the composite map

$$\begin{aligned} \Lambda^i U^\vee \xrightarrow{\sim} \Lambda^i H^1(U, \mathbf{F}_p) &\xrightarrow{\cup} H^i(U, \mathbf{F}_p) \\ &\longrightarrow H^i(U, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\pi^*} H^i(G, \mathbf{Q}/\mathbf{Z}) \end{aligned} \tag{15}$$

where $\cup$ is the cup-product. It induces a map

$$\Lambda^i U^\vee \longrightarrow H^i(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}(i-1)) \tag{16}$$

obtained by composing the maps

$$\Lambda^i U^\vee \xrightarrow{\eta_G} H^i(G, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^i(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}(i-1)).$$

5.2. The result

Let us recall the result of Bogomolov: by [Bo1, Lemma 5.1], the map defined by (16) induces an isomorphism

$$K^2_{\mathrm{max}}/K^2 \xrightarrow{\sim} \mathrm{Br}_{\mathrm{nr}}(\mathbf{C}(W)^G) = H^2_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}(1)).$$

Our aim in this paragraph is to prove the following result:

Theorem 2. *With notation as above, the map defined in* (16) *induces an injection*

$$K^3_{\mathrm{max}}/K^3 \hookrightarrow H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}).$$

Remark 8. In [Pe2, §9.3], we construct an example of a 2-group for which

$$\mathrm{Ker}(\Lambda^3 U^\vee \longrightarrow H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})) \not\subset K^3.$$

This shows that the condition $p \neq 2$ is necessary.

5.3. The cohomology of an $\mathbf{F}_p$-vector space

Let us first prove a few basic facts about the cohomology groups of an $\mathbf{F}_p$-vector space.

Lemma 1. *If p is a prime number and E an $\mathbf{F}_p$-vector space, then for any strictly positive integer i, one has*

$$pH^i(E, \mathbf{Q}/\mathbf{Z}) = \{0\}.$$

Proof. We prove the lemma by induction on the dimension n of E. The result is true if $n = 0$. If $n \geqslant 1$, let E' be a subgroup of index p in E. We may identify E with $E' \oplus \mathbf{F}_p$. Let m_p (resp. m'_p) be the multiplication by p in $H^i(E, \mathbf{Q}/\mathbf{Z})$ (resp. $H^i(E', \mathbf{Q}/\mathbf{Z})$). Then $m_p = \mathrm{Cores}^E_{E'} \circ \mathrm{Res}^E_{E'}$. But we have

$$\mathrm{Id}_{H^i(E', \mathbf{Q}/\mathbf{Z})} = \mathrm{Res}^E_{E'} \circ \mathrm{pr}_1^*$$

where pr_1 is the projection from E to E'. We get

$$\begin{aligned} m_p &= \mathrm{Cores}^E_{E'} \circ \mathrm{Res}^E_{E'} \circ \mathrm{pr}_1^* \circ \mathrm{Res}^E_{E'} \\ &= m_p \circ \mathrm{pr}_1^* \circ \mathrm{Res}^E_{E'} \\ &= \mathrm{pr}_1^* \circ m'_p \circ \mathrm{Res}^E_{E'}\,. \end{aligned}$$

By induction, $m'_p = 0$ and we get that m_p is trivial. □

Notation 9. Let p be a prime number. For any group G and any integer $i \geqslant 0$, let

$$\delta_i : H^i(G, \mathbf{Z}/p\mathbf{Z}) \longrightarrow H^{i+1}(G, \mathbf{Z}/p\mathbf{Z})$$

be the Bockstein operator defined as the boundary map associated to the short exact sequence

$$0 \longrightarrow \mathbf{Z}/p\mathbf{Z} \longrightarrow \mathbf{Z}/p^2\mathbf{Z} \longrightarrow \mathbf{Z}/p\mathbf{Z} \longrightarrow 0.$$

We denote by $\delta : H^*(G, \mathbf{Z}/p\mathbf{Z}) \to H^*(G, \mathbf{Z}/p\mathbf{Z})$ the map defined by the maps δ_i. Note that δ is a derivation of the ring $H^*(G, \mathbf{F}_p)$ and that $\delta^2 = 0$ (see [Ev2, p. 28]).

Let p be an odd prime and E be an $\mathbf{F}_p$-vector space. We denote by $S^*(E^\vee)$ the symmetric algebra on the $\mathbf{F}_p$-vector $E^\vee$. There is a unique homomorphism of algebras

$$\rho : \Lambda^*(E^\vee) \otimes S^*(E^\vee) \longrightarrow H^*(E, \mathbf{F}_p)$$

mapping $S^i(E^\vee) \otimes \Lambda^j(E^\vee)$ in $H^{2i+j}(E, \mathbf{F}_p)$, so that the map

$$\Lambda^1(E^\vee) \otimes \mathbf{F}_p \longrightarrow H^1(E, \mathbf{F}_p)$$

is induced by the natural isomorphism $\tau : E^\vee \xrightarrow{\sim} H^1(E, \mathbf{F}_p)$ and the map

$$\mathbf{F}_p \otimes S^1(E^\vee) \longrightarrow H^2(E, \mathbf{F}_p)$$

is induced by the composite map $\delta \circ \tau$. By [AM, Corollary II.4.3], the map ρ is an isomorphism of algebras. We also denote by δ the derivation $\rho^{-1} \circ \delta \circ \rho$ of the algebra $\Lambda^*(E^\vee) \otimes S^*(E^\vee)$. It follows from the constructions that for any x in $E^\vee$,

$$\delta(x \otimes 1) = 1 \otimes x \quad \text{and} \quad \delta(1 \otimes x) = 0. \tag{17}$$

The projection $r_p : \mathbf{Z} \to \mathbf{F}_p$ induces a ring homomorphism

$$r_{p*} : H^*(E, \mathbf{Z}) \longrightarrow H^*(E, \mathbf{F}_p).$$

We also denote by $j_p : \mathbf{Z}/p\mathbf{Z} \to \mathbf{Q}/\mathbf{Z}$ the natural injection.

Proposition 4. *With the previous notation, if $i > 0$, then the map*

$$r_{p*} : H^i(E, \mathbf{Z}) \longrightarrow H^i(E, \mathbf{F}_p)$$

is injective and the groups $\mathrm{Im}(r_{p*})$, $\mathrm{Ker}(\delta_i)$ *and* $\mathrm{Im}(\delta_{i-1})$ *coincide in the group* $H^i(E, \mathbf{F}_p)$.

Proof. Since the multiplication by p is trivial in $H^i(E, \mathbf{Q}/\mathbf{Z})$ for $i > 0$, it is trivial in $H^i(E, \mathbf{Z})$ as well. The short exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\times p} \mathbf{Z} \xrightarrow{r_p} \mathbf{F}_p \longrightarrow 0$$

induces for $i > 0$ short exact sequences

$$0 \longrightarrow H^i(E, \mathbf{Z}) \xrightarrow{r_{p*}} H^i(E, \mathbf{F}_p) \xrightarrow{\beta} H^{i+1}(E, \mathbf{Z}) \longrightarrow 0;$$

therefore r_{p_*} is injective.

We have a natural commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z} \arrow[r, "\times p"] \arrow[d, "r_p"] & \mathbf{Z} \arrow[r, "r_p"] \arrow[d, "r_{p^2}"] & \mathbf{Z}/p\mathbf{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{Z}/p\mathbf{Z} \arrow[r] & \mathbf{Z}/p^2\mathbf{Z} \arrow[r, "r_p^{p^2}"] & \mathbf{Z}/p\mathbf{Z} \arrow[r] & 0
\end{tikzcd}$$

with exact lines which induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^i(E, \mathbf{Z}) \arrow[r, "r_{p*}"] \arrow[d, "r_{p^2_*}"] & H^i(E, \mathbf{F}_p) \arrow[r, "\beta"] \arrow[d, equal] & H^{i+1}(E, \mathbf{Z}) \arrow[r] \arrow[d, hook, "r_{p*}"] & 0 \\
 & H^i(E, \mathbf{Z}/p^2\mathbf{Z}) \arrow[r, "r_{p*}^{p^2}"] & H^i(E, \mathbf{F}_p) \arrow[r, "\delta_i"] & H^{i+1}(E, \mathbf{F}_p) &
\end{tikzcd} \tag{18}$$

with exact lines. The commutativity of the left hand square yields the inclusion of $\mathrm{Im}(r_{p_*})$ in $\mathrm{Im}(r_{p_*}^{p^2})$, the exactness of the lines implies that

$$\mathrm{Ker}(\beta) = \mathrm{Im}(r_{p_*}) \subset \mathrm{Im}\left(r_{p_*}^{p^2}\right) = \mathrm{Ker}(\delta_i).$$

But the injectivity of r_{p_*} and the commutativity of the right hand square imply that $\mathrm{Ker}(\delta_i)$ is contained in $\mathrm{Ker}(\beta)$. Thus $\mathrm{Im}(r_{p_*}) = \mathrm{Ker}(\delta_i)$. Using the corresponding diagram for $i - 1$ and the surjectivity of β we get that $\mathrm{Im}(r_{p_*}) = \mathrm{Im}(\delta_{i-1})$. □

Corollary 1. *If* $i \geqslant 1$, *the morphism*

$$j_{p_*} : H^i(E, \mathbf{F}_p) \longrightarrow H^i(E, \mathbf{Q}/\mathbf{Z})$$

is surjective and induces an isomorphism from $\mathrm{Coker}(\delta_{i-1})$ *to the group* $H^i(E, \mathbf{Q}/\mathbf{Z})$.

Proof. Since multiplication by p is trivial in $H^i(E, \mathbf{Q}/\mathbf{Z})$, the exact sequence

$$0 \longrightarrow \mathbf{F}_p \xrightarrow{j_p} \mathbf{Q}/\mathbf{Z} \xrightarrow{\times p} \mathbf{Q}/\mathbf{Z} \longrightarrow 0$$

gives an exact sequence

$$H^{i-1}(E, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\delta'_{i-1}} H^i(E, \mathbf{F}_p) \xrightarrow{j_{p*}} H^i(E, \mathbf{Q}/\mathbf{Z}) \longrightarrow 0$$

which proves that the map j_{p_*} is surjective and that the cokernel of δ'_{i-1} is isomorphic to $H^i(E, \mathbf{Q}/\mathbf{Z})$. On the other hand, let $\gamma : \mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$ be the map sending x on the class of x/p; we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z} \arrow[r] \arrow[d, "r_p"] & \mathbf{Q} \arrow[r] \arrow[d, "\gamma"] & \mathbf{Q}/\mathbf{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{F}_p \arrow[r, "j_p"] & \mathbf{Q}/\mathbf{Z} \arrow[r, "\times p"] & \mathbf{Q}/\mathbf{Z} \arrow[r] & 0
\end{tikzcd}$$

with exact lines which yields a commutative diagram

$$\begin{tikzcd}
& H^i(E, \mathbf{Z}) \arrow[d, "r_{p*}"] \\
H^{i-1}(E, \mathbf{Q}/\mathbf{Z}) \arrow[ur] \arrow[r, "\delta'_{i-1}"'] & H^i(E, \mathbf{F}_p).
\end{tikzcd}$$

For any $i > 0$ the map $H^{i-1}(E, \mathbf{Q}/\mathbf{Z}) \to H^i(E, \mathbf{Z})$ is surjective. Thus $\mathrm{Im}(\delta_{i-1}) = \mathrm{Im}(r_{p_*})$ coincides with $\mathrm{Im}(\delta'_{i-1})$. □

Notation 10. We denote by ϕ_i the natural map $\Lambda^i E^\vee \hookrightarrow H^i(E, \mathbf{Q}/\mathbf{Z})$ defined as the composite map

$$\Lambda^i E^\vee \xrightarrow{\rho} H^i(E, \mathbf{F}_p) \longrightarrow H^i(E, \mathbf{Q}/\mathbf{Z})$$

and by ψ_i the map from $S^i(E^\vee)$ to $H^{2i-1}(E, \mathbf{Q}/\mathbf{Z})$ given as the composite map

$$S^i(E^\vee) \xrightarrow{\sim} S^i H^2(E, \mathbf{Z}) \xrightarrow{\cup} H^{2i}(E, \mathbf{Z}) \xrightarrow{\sim} H^{2i-1}(E, \mathbf{Q}/\mathbf{Z})$$

(this corresponds to the cup-product defined by (2)).

Corollary 2. *The following maps are isomorphisms*

$$E^\vee \xrightarrow{\phi_1} H^1(E, \mathbf{Q}/\mathbf{Z}),$$
$$\Lambda^2 E^\vee \xrightarrow{\phi_2} H^2(E, \mathbf{Q}/\mathbf{Z}),$$

and

$$\Lambda^3 E^\vee \oplus S^2(E^\vee) \xrightarrow{\phi_3+\psi_2} H^3(E, \mathbf{Q}/\mathbf{Z}).$$

Proof. The map ϕ_1 is the composite of the isomorphisms

$$E^\vee \xrightarrow{\sim} \mathrm{Hom}(E, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} H^1(E, \mathbf{Q}/\mathbf{Z}).$$

By Corollary 1, the map ρ induces an isomorphism

$$\sigma_2 : \mathrm{Coker}(E^\vee \xrightarrow{\delta} \Lambda^2 E^\vee \oplus E^\vee) \longrightarrow H^2(E, \mathbf{Q}/\mathbf{Z})$$

where, by (17), one has $\delta(x) = (0, x)$ and $\sigma_2(x, 0) = \phi_2(x)$.

Similarly, we have an isomorphism σ_3 from the group

$$\mathrm{Coker}(\Lambda^2 E^\vee \oplus S^1(E^\vee) \longrightarrow \Lambda^3 E^\vee \oplus \Lambda^1 E^\vee \otimes S^1(E^\vee))$$

to the group $H^3(E, \mathbf{Q}/\mathbf{Z})$, where, since δ is a derivation with $\delta^2 = 0$, we get from (17) that

$$\delta((u \wedge v, t)) = (0, u \otimes v - v \otimes u)$$

for any $u, v \in E^\vee$ and any t in $S^1(E^\vee)$. Since $p \neq 2$, the quotient $E^\vee \otimes E^\vee / \Lambda^2 E^\vee$ is canonically isomorphic to $S^2(E^\vee)$ and we get that σ_3 induces an isomorphism

$$\overline{\sigma}_3 : \Lambda^3 E^\vee \oplus S^2(E^\vee) \longrightarrow H^3(E, \mathbf{Q}/\mathbf{Z}).$$

We have that the restriction of $\overline{\sigma}_3$ to $\Lambda^3 E^\vee$ coincides with ϕ_3. Using the diagram (18), for any $u, v \in E^\vee$, we have

$$\begin{aligned}\overline{\sigma}_3(uv) &= j_{p_*}(\rho(u \otimes v)) = j_{p_*}(\tau(u) \cup \delta(\tau(v))) \\ &= j_{p_*}(\tau(u) \cup \beta(\tau(v))) = \psi_1(u) \cup \beta(\tau(v)) = \psi_2(uv)\end{aligned}$$

where the three cup-products correspond respectively to the natural product maps

$$\mathbf{F}_p \otimes \mathbf{F}_p \longrightarrow \mathbf{F}_p, \quad \mathbf{F}_p \otimes \mathbf{Z} \longrightarrow \mathbf{F}_p \quad \text{and} \quad \mathbf{Q}/\mathbf{Z} \otimes \mathbf{Z} \longrightarrow \mathbf{Q}/\mathbf{Z}.$$ □

5.4. The inverse image of the unramified cohomology group

We now turn back to the proof of Theorem 2. To begin with, we shall prove the following proposition:

Proposition 5. *The inverse image in $\Lambda^3 U^\vee$ of the unramified cohomology group $H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}(2))$ coincides with $K^3_{\max}$.*

Remark 9. When K^2 is generated by elements of the form $u \wedge v$ with u in $U^\vee$ and v in $U^\vee$, this proposition may be deduced from [Pe1, Theorem 2] and [Pe2, Proposition 9.4 and Lemma 9.3]. For the general case, we shall give a direct proof based upon Proposition 3.

Notation 11. Let E be a vector space of finite dimension over $\mathbf{F}_p$ and let u be any non zero element of E. The complex

$$0 \longrightarrow \mathbf{F}_p \xrightarrow{\times u} E \xrightarrow{u\wedge\cdot} \Lambda^2 E \xrightarrow{u\wedge\cdot} \cdots \xrightarrow{u\wedge\cdot} \Lambda^{\dim(E)} E \longrightarrow 0$$

is exact and induces by duality an exact sequence

$$0 \longrightarrow \Lambda^{\dim(E)} E^\vee \xrightarrow{\mathrm{d}_u} \Lambda^{\dim(E)-1} E^\vee \xrightarrow{\mathrm{d}_u} \cdots \xrightarrow{\mathrm{d}_u} E^\vee \xrightarrow{\mathrm{d}_u} \mathbf{F}_p \longrightarrow 0.$$

If $u = 0$, we put $\mathrm{d}_u = 0$.

Remarks 10. The morphism d_u is characterised by the fact that

$$d_u(u^\vee \wedge v) = v \quad \text{and} \quad d_u(w) = 0$$

for any $v, w \in \Lambda^*(u^\perp)$ and any $u^\vee \in E^\vee$ such that $u^\vee(u) = 1$.

The complex $(\Lambda^* U^\vee, \mathrm{d}_u)$ is the Koszul complex for u considered as an element of $U^{\vee\vee}$ (see [Bki2, §9]).

Notation 12. For any subgroup H of G, let V_H be $V \cap H$ and U_H be $\pi(H) \subset U$. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & V_H \arrow[r] \arrow[d, "\kappa_H"] & H \arrow[r] \arrow[d] & U_H \arrow[r] \arrow[d, "\lambda_H"] & 0 \\
0 \arrow[r] & V \arrow[r, "\iota"] & G \arrow[r, "\pi"] & U \arrow[r] & 0.
\end{tikzcd}$$

For any $i \geqslant 0$, we denote by $\lambda_H^* : \Lambda^i U^\vee \to \Lambda^i U_H^\vee$ the map induced by the injection λ_H.

Lemma 2. *Let H be a subgroup of G and g be an element of $Z_G(H)$. Let $u = \pi(g)$. The following diagram*

$$\begin{tikzcd}
\Lambda^3 U^\vee \arrow[r, "\mathrm{d}_u"] \arrow[d] & \Lambda^2 U^\vee \arrow[r, "\lambda_H^*"] & \Lambda^2 U_H^\vee \arrow[d] \\
H^3(G, \mathbf{Q}/\mathbf{Z}) \arrow[rr, "\partial_{H,g}"] & & H^2(H, \mathbf{Q}/\mathbf{Z})
\end{tikzcd}$$

is commutative.

Proof. If g is trivial, then $\partial_{H,g}$ and d_u are trivial. Otherwise, let I be the subgroup of G generated by g. The group I is an $\mathbf{F}_p$-vector space of dimension 1 and there is a canonical isomorphism

$$\begin{aligned} \Lambda^3 U_H^\vee \oplus \Lambda^2 U_H^\vee \otimes I^\vee &\longrightarrow \Lambda^3\big(U_H^\vee \oplus I^\vee\big) \\ x + y \otimes i &\longmapsto x + y \wedge i. \end{aligned} \tag{19}$$

We denote by $\mathrm{pr}_2 : \Lambda^3 U_H^\vee \to \Lambda^2 U_H^\vee \otimes I^\vee$ the projection on the second factor and by $g^\vee$ the unique element of $I^\vee$ such that $g^\vee(g) = 1$. Let

$$\overline{m} : U_H \times I \to U$$

be the morphism sending (u, i) on $\lambda_H(u) + \pi(i)$ and let

$$\overline{m}^* : \Lambda^3 U^\vee \to \Lambda^3\big(U_H^\vee \oplus I^\vee\big)$$

be the induced map. Then we have commutative diagrams

$$\begin{tikzcd} H \times I \arrow[r] \arrow[d] & G \arrow[d, "\pi"] \\ U_H \times I \arrow[r, "\overline{m}"] & U \end{tikzcd}$$

and

$$\begin{tikzcd} \Lambda^3 U^\vee \arrow[r, "\mathrm{d}_u"] \arrow[d, "\overline{m}^*"] & \Lambda^2 U^\vee \arrow[d] \\ \Lambda^3(U_H^\vee \oplus I^\vee) \arrow[r, "\mathrm{pr}_2"] & \Lambda^2 U_H^\vee \otimes I^\vee \end{tikzcd}$$

where the vertical map on the right maps x onto $\lambda_H^*(x) \otimes g^\vee$. Therefore, by definition of $\partial_{H,g}$, it remains to prove the commutativity of the following diagram

$$\begin{tikzcd} \Lambda^3(U_H^\vee \oplus I^\vee) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\eta_{H\times I}"] & \Lambda^2 U_H^\vee \otimes I^\vee \arrow[d] \\ H^3(H \times I, \mathbf{Q}/\mathbf{Z}) \arrow[r] & H^2(H, H^1(I, \mathbf{Q}/\mathbf{Z})) \end{tikzcd}$$

where the map at the bottom was defined in (3) using the Hochschild–Serre spectral sequence. But this commutativity follows from the fact that for any element of $\Lambda^3(U_H^\vee \oplus I^\vee)$ written as $x + y \wedge g^\vee$ with $x \in \Lambda^3 U_H^\vee$ and $y \in \Lambda^2 U_H^\vee$, one has

$$s_{H,I}\big(\eta_{H\times I}(x + y \wedge g^\vee)\big) = \eta_{H\times I}(x + y \wedge g^\vee)$$

and the compatibility of the Hochschild–Serre spectral sequence with the cup-product. $\square$

Proof of Proposition 5. In this proof, we denote by $\Lambda^3 U^\vee_{\mathrm{nr}}$ the inverse image of the group $H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}(2))$ in $\Lambda^3 U^\vee$. Let H be a subgroup of G. The Hochschild–Serre spectral sequence for the extension

$$0 \longrightarrow V_H \longrightarrow H \longrightarrow U_H \longrightarrow 0$$

gives an exact sequence

$$H^1(V_H, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^2(U_H, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^2(H, \mathbf{Q}/\mathbf{Z})$$

that is

$$V_H^\vee \longrightarrow \Lambda^2 U_H^\vee \longrightarrow H^2(H, \mathbf{Q}/\mathbf{Z}).$$

By [Pe2, p. 135], the map $V_H^\vee \to \Lambda^2 U_H^\vee$ is given by $-\gamma^t_{|\Lambda^2 U_H}$. Its image coincides with $\lambda_H^*(K^2)$. Since

$$\operatorname{Ker}\bigl(\lambda_H^* : \Lambda^2 U^\vee \longrightarrow \Lambda^2 U_H^\vee\bigr) = \bigl(\Lambda^2 U_H\bigr)^\perp,$$

for any element y of $\Lambda^2 U^\vee$, one has that $\eta_H(\lambda_H^*(y))$ is trivial in the group $H^2(H, \mathbf{Q}/\mathbf{Z})$ if and only if y belongs to $K^2 + (\Lambda^2 U_H)^\perp$. Let us also remark that the condition $g \in Z_G(H)$ is equivalent to $\gamma(u \wedge U_H) = \{0\}$. Using Lemma 2, Proposition 3 and the definition of $H^3_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z})$, we see that an element y of $\Lambda^3 U^\vee$ belongs to $\Lambda^3 U^\vee_{\mathrm{nr}}$ if and only if for any u in U and any subgroup U' of U such that $\gamma(U' \wedge u) = \{0\}$, one has that

$$\mathrm{d}_u(x) \in K^2 + (\Lambda^2 U')^\perp. \tag{20}$$

For a given u in U, it is enough to check this condition for a maximal U', that is for

$$U' = \{v \in U \mid v \wedge u \in S^2\} = \mathrm{d}_u(K^2)^\perp.$$

But for any subgroup F of U one has $(\Lambda^2 F)^\perp = F^\perp \wedge U^\vee$. Therefore

$$\bigl(\Lambda^2\bigl(\mathrm{d}_u(K^2)^\perp\bigr)\bigr)^\perp = \mathrm{d}_u(K^2) \wedge U^\vee.$$

We have proven that the group $\Lambda^3 U^\vee_{\mathrm{nr}}$ may be described as

$$\bigcap_{u \in U} \mathrm{d}_u^{-1}\bigl(K^2 + \mathrm{d}_u(K^2) \wedge U^\vee\bigr).$$

Let $u^\vee$ be an element of $U^\vee$ such that $u^\vee(u) = 1$. One has

$$\begin{aligned}\mathrm{d}_u^{-1}\bigl(K^2 + \mathrm{d}_u(K^2) \wedge U^\vee\bigr) &= \bigl(K^2 + \mathrm{d}_u(K^2) \wedge U^\vee\bigr) \wedge u^\vee + \Lambda^3(u^\perp)\\ &= K^2 \wedge u^\vee + \mathrm{d}_u(K^2) \wedge u^\vee \wedge u^\perp + \Lambda^3(u^\perp).\end{aligned}$$

But $\mathrm{d}_u(K^2) \wedge u^\vee + \Lambda^2(u^\perp) = K^2 + \Lambda^2(u^\perp)$. Therefore

$$\mathrm{d}_u^{-1}\bigl(K^2 + \mathrm{d}_u(K^2) \wedge U^\vee\bigr) = K^2 \wedge U^\vee + \Lambda^3(u^\perp) = K^3 + \Lambda^3(u^\perp)$$

and

$$\Lambda^3 U_{\mathrm{nr}}^\vee = \bigcap_{u \in U} (K^3 + \Lambda^3(u^\perp)).$$

By definition,

$$\begin{aligned} K^3_{\max} &= (\{u \wedge v \in S^3 \mid u \in U, v \in \Lambda^2 U\})^\perp \\ &= \bigcap_{u \in U} \left(\mathrm{d}_u(K^3)^\perp \wedge u\right)^\perp = \bigcap_{u \in U} \mathrm{d}_u^{-1}\left(\mathrm{d}_u(K^3)\right) \\ &= \bigcap_{u \in U} (K^3 + \Lambda^3(u^\perp)) \end{aligned}$$

which concludes the proof of the proposition. □

5.5. Weights on the cohomology

To prove Theorem 2 it remains to prove that

$$K^3 = \mathrm{Ker}(\Lambda^3 U^\vee \longrightarrow H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}))$$

or, using Theorem 1, that K^3 is the inverse image of $H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$ in $\Lambda^3 U^\vee$. As a first step we introduce a notion of weights on the cohomology of subgroups of G.

Proposition 6. *There exists an action* $(\lambda, g) \mapsto \lambda.g$ *of* $\mathbf{F}_p^*$ *on the group* G *which verifies*

$$\forall g \in G, \forall \lambda \in \mathbf{F}_p^*, \quad \pi(\lambda.g) = \lambda\pi(g). \tag{21}$$

Moreover, for any such action, one has

$$\forall v \in V, \forall \lambda \in \mathbf{F}_p^*, \quad \lambda.\iota(g) = \iota(\lambda^2 g).$$

Remark 11. This action is not unique. But since $\iota(V) = [G, G]$, any automorphism of G induces an automorphism of U and we get a morphism $\mathrm{Aut}(G) \to \mathrm{Aut}(U)$. Any element in its kernel is an automorphism of the form $g \mapsto \iota(f(\pi(g)))g$ for an element f of $\mathrm{Hom}(U, V)$. In other words we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}(U, V) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(U).$$

The group $\mathbf{F}_p^*$ acts on $\mathrm{Hom}(U, V)$ by multiplication and the actions on G which satisfy the condition of the proposition form an affine space under the vector space of cocycles $Z^1(\mathbf{F}_p^*, \mathrm{Hom}(U, V))$. But these cocycles are of the form $\lambda \mapsto (\lambda - 1)f$ for f in $\mathrm{Hom}(U, V)$.

Proof of Proposition 6. To prove the proposition we use the fact that, by (14), G is isomorphic to $V \times U$ equipped with the group law given by

$$(v, u)(v', u') = \left(v + v' + \frac{1}{2}\gamma(u \wedge u'), u + u'\right)$$

for any u, u' in U and any v, v' in V. We define an action of $\mathbf{F}_p^*$ on $V \times U$ by

$$\forall \lambda \in \mathbf{F}_p^*, \forall v \in V, \forall u \in U, \quad \lambda.(v, u) = (\lambda^2 v, \lambda u).$$

This action is compatible with the group law above. Indeed

$$\begin{aligned}\lambda.((v, u)(v', u')) &= \lambda.\left(v + v' + \frac{1}{2}\gamma(u \wedge u'), u + u'\right)\\ &= \left(\lambda^2 v + \lambda^2 v' + \frac{\lambda^2}{2}\gamma(u \wedge u'), \lambda u + \lambda u'\right)\end{aligned}$$

and

$$\begin{aligned}\lambda.(v, u)\lambda.(v', u') &= (\lambda^2 v, \lambda u)(\lambda^2 v', \lambda u')\\ &= \left(\lambda^2 v + \lambda^2 v' + \frac{1}{2}\gamma(\lambda u \wedge \lambda u'), \lambda u + \lambda u'\right)\end{aligned}$$

for any λ in $\mathbf{F}_p^*$, any v, v' in V, and any u, u' in U.

The last assertion of the proposition follows from the fact that $\iota(V)$ is $[G, G]$ and the relation $[g, g'] = \iota \circ \gamma(\pi(g) \wedge \pi(g'))$ for any g, g' in G. $\square$

Definition 13. *We now fix an action as in Proposition 6. If H is a subgroup of G, such that $\iota(V) \subset H$, then H is the kernel of the natural surjection $G \mapsto U/U_H$ and, thanks to* (21)*, is invariant under the action of $\mathbf{F}_p^*$. This action induces an action of $\mathbf{F}_p^*$ on the cohomology groups $H^i(H, \mathbf{Q}/\mathbf{Z})$.*

There is an isomorphism of rings

$$\chi : \mathbf{Z}/(p-1)\mathbf{Z} \longrightarrow \mathrm{End}_{\mathbf{Z}}(\mathbf{F}_p^*)$$

which sends the class of k to the endomorphism $\lambda \mapsto \lambda^k$.

Let $i \geqslant 0$ be an integer. Let $H^i(H, \mathbf{Q}/\mathbf{Z})[p]$ be the p-torsion part of the cohomology group $H^i(H, \mathbf{Q}/\mathbf{Z})$. For any k in $\mathbf{Z}/(p-1)\mathbf{Z}$ we define

$$H^i(H, \mathbf{Q}/\mathbf{Z})[p]_{(k)} = \left\{x \in H^i(H, \mathbf{Q}/\mathbf{Z})[p] \middle| \forall \lambda \in \mathbf{F}_p^*, \lambda.x = \lambda^k x\right\}.$$

Lemma 3. *The sum $\sum_{k \in \mathbf{Z}/(p-1)\mathbf{Z}} H^i(H, \mathbf{Q}/\mathbf{Z})[p]_{(k)}$ is a direct sum.*

Proof. If ξ is a generator for $\mathbf{F}_p^*$ then $H^i(H, \mathbf{Q}/\mathbf{Z})[p]_{(k)}$ is the eigenspace for the eigenvalue ξ^k with respect to the operator defined by ξ. □

Most of the rest of this section is devoted to the proof of the fact that $H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$ is contained in $H^3(G, \mathbf{Q}/\mathbf{Z})[p]_{(-2)} + H^3(G, \mathbf{Q}/\mathbf{Z})[p]_{(-4)}$ and therefore does not meet the image of $\Lambda^3 U^\vee$, contained in the group $H^3(G, \mathbf{Q}/\mathbf{Z})[p]_{(-3)}$.

5.6. Triviality of the corestriction

By definition, the group $H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$ is generated by elements of the form $\mathrm{Cores}_H^G(u \cup v)$ for H a subgroup of G and u, v in $H^1(H, \mathbf{Q}/\mathbf{Z})$. We first want to prove that we only have to consider subgroups H which contain $\iota(V)$.

Lemma 4. *With notation as above, if H is a subgroup of G such that the center of G is not contained in H and if u, v belong to $H^1(H, \mathbf{Q}/\mathbf{Z})$, then*

$$\mathrm{Cores}_H^G(u \cup v) = 0.$$

Proof. Let H' be the subgroup of G generated by H and $Z(G)$. Then

$$\mathrm{Cores}_H^G(u \cup v) = \mathrm{Cores}_{H'}^G \circ \mathrm{Cores}_H^{H'}(u \cup v).$$

Let us choose a decomposition

$$Z(G) = (Z(G) \cap H) \oplus E$$

we get an isomorphism $H \times E \xrightarrow{\sim} H'$. Then

$$\mathrm{Cores}_H^{H'} = \mathrm{Cores}_H^{H'} \circ \mathrm{Res}_H^{H'} \circ \mathrm{pr}_1^* = |E| \, \mathrm{pr}_1^* .$$

But p divides $|E|$ and $pu \cup v = 0$. Therefore $\mathrm{Cores}_H^G(u \cup v) = 0$. □

When $\iota(V) \subset H$, then the cohomology groups of H and G are equipped with an action of $\mathbf{F}_p^*$ as described in Definition 13. The corestriction is compatible with these actions, since the action on H is the restriction of the action on G.

Lemma 5. *One has*

$$H^1(H, \mathbf{Q}/\mathbf{Z}) = H^1(H, \mathbf{Q}/\mathbf{Z})[p]_{(-1)} \oplus H^1(H, \mathbf{Q}/\mathbf{Z})[p]_{(-2)}. \tag{22}$$

Moreover the inflation map $H^1(U_H, \mathbf{Q}/\mathbf{Z}) \to H^1(H, \mathbf{Q}/\mathbf{Z})$ and the restriction map $H^1(H, \mathbf{Q}/\mathbf{Z}) \to H^1(V_H, \mathbf{Q}/\mathbf{Z})$ induce isomorphisms

$$U_H^\vee \xrightarrow{\sim} H^1(H, \mathbf{Q}/\mathbf{Z})[p]_{(-1)}$$

and

$$(V_H/[H,H])^\vee \xrightarrow{\sim} H^1(H,\mathbf{Q}/\mathbf{Z})[p]_{(-2)}.$$

Proof. The group $H^1(H,\mathbf{Q}/\mathbf{Z})$ is isomorphic to

$$\mathrm{Hom}(H,\mathbf{Q}/\mathbf{Z}) = (H/[H,H])^\vee.$$

There is a natural exact sequence

$$0 \longrightarrow V_H/[H,H] \longrightarrow H/[H,H] \longrightarrow U_H \longrightarrow 0$$

which induces the upper line of the following commutative diagram

$$\begin{tikzcd}
U_H^\vee \arrow[r] \arrow[d] & (H/[H,H])^\vee \arrow[r] \arrow[d] & V_H^\vee \arrow[d] \\
H^1(U_H,\mathbf{Q}/\mathbf{Z}) \arrow[r] & H^1(H,\mathbf{Q}/\mathbf{Z}) \arrow[r] & H^1(V_H,\mathbf{Q}/\mathbf{Z})
\end{tikzcd}$$

in which vertical maps are isomorphisms. Let ξ be a generator of $\mathbf{F}_p^*$. By construction (see Proposition 6), ξ acts on $U_H^\vee$ by multiplication by ξ^{-1} and on $V_H^\vee$ by multiplication by ξ^{-2}. Let f_ξ be the operator defined by ξ on $H^1(H,\mathbf{Q}/\mathbf{Z})$. Since $f_\xi^{p-1} = \mathrm{Id}$, the endomorphism f_ξ is semi-simple and it follows from the above diagram that the eigenvalues of its action on $H^1(H,\mathbf{Q}/\mathbf{Z}) \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p$ are ξ^{-1} and ξ^{-2}. We get the decomposition

$$H^1(H,\mathbf{Q}/\mathbf{Z}) = H^1(H,\mathbf{Q}/\mathbf{Z})[p]_{(-1)} \oplus H^1(H,\mathbf{Q}/\mathbf{Z})[p]_{(-2)}$$

and the requested isomorphisms. □

Lemma 6. *With the preceding notation,*

$$\mathrm{Cores}_H^G\left(H^1(H,\mathbf{Q}/\mathbf{Z})[p]_{(-1)} \cup H^1(H,\mathbf{Q}/\mathbf{Z})[p]_{(-2)}\right) = \{0\}.$$

Proof. Let x belong to $H^1(H,\mathbf{Q}/\mathbf{Z})[p]_{(-1)}$ and y to $H^1(H,\mathbf{Q}/\mathbf{Z})[p]_{(-2)}$. By Lemma 5, x comes from an element z of $H^1(G,\mathbf{Q}/\mathbf{Z})$. By the transfer formula [Bro, (3.8), p. 112],

$$\mathrm{Cores}_H^G(x \cup y) = \mathrm{Cores}_H^G\left(\mathrm{Res}_H^G(z) \cup y\right) = z \cup \mathrm{Cores}_H^G(y).$$

But $\mathrm{Cores}_H^G(y)$ is in $H^1(G,\mathbf{Q}/\mathbf{Z}) = H^1(G,\mathbf{Q}/\mathbf{Z})[p]_{(-1)}$ since this group is isomorphic to $U^\vee$ and in the group $H^1(G,\mathbf{Q}/\mathbf{Z})[p]_{(-2)}$. Therefore it is trivial. □

5.7. Proof of Theorem 2

We now complete the proof of Theorem 2. Let H be a subgroup of G such that $\iota(V) \subset H$. First note that if x belongs to $H^1(H, \mathbf{Q}/\mathbf{Z})[p]_{(k)}$ and y to $H^1(H, \mathbf{Q}/\mathbf{Z})[p]_{(l)}$ then

$$\mathrm{Cores}_H^G(x \cup y) \in H^3(G, \mathbf{Q}/\mathbf{Z})[p]_{(k+l)}.$$

Using Lemma 5 and Lemma 6, we get that for any x, y in $H^1(H, \mathbf{Q}/\mathbf{Z})$ we have

$$\mathrm{Cores}_H^G(x \cup y) \in H^3(G, \mathbf{Q}/\mathbf{Z})[p]_{(-2)} + H^3(G, \mathbf{Q}/\mathbf{Z})[p]_{(-4)}.$$

Using Lemma 4, and the definition of the $H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$, we get that

$$H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z}) \subset H^3(G, \mathbf{Q}/\mathbf{Z})[p]_{(-2)} + H^3(G, \mathbf{Q}/\mathbf{Z})[p]_{(-4)}.$$

But the image of $\Lambda^3 U^\vee$ in $H^3(G, \mathbf{Q}/\mathbf{Z})$ is contained in $H^3(G, \mathbf{Q}/\mathbf{Z})[p]_{(-3)}$ and does not meet $H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$. Using Proposition 2 we get that the kernel of the map

$$\Lambda^3 U^\vee \longrightarrow H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$$

coincides with the kernel of the map $\Lambda^3 U^\vee \to H^3(G, \mathbf{Q}/\mathbf{Z})$, which is K^3 by [Pe2, Lemma 9.3]. We then apply Proposition 5 to conclude the proof.

6. A special case

If the dimension of U is less than 5 then any λ in $\Lambda^3 U$ may be written as $\lambda = u \wedge v$ with u in U and v in $\Lambda^2 U$ (see [Re, §1.4]). Therefore $K^3 = K^3_{\max}$ whenever $\dim U \leqslant 5$. Let us give an example with $\dim U = 6$.

Theorem 3. *Let U and V be two $\mathbf{F}_p$-vector spaces of dimension 6 for p an odd prime. We denote by $(u_i)_{1\leqslant i\leqslant 6}$ a basis of U and $(v_i)_{1\leqslant i\leqslant 6}$ a basis of V. We denote by $(u_i^\vee)_{1\leqslant i\leqslant 6}$ the dual basis of $U^\vee$. Let γ be the element of $\Lambda^2 U^\vee \otimes V$ defined by*

$$\begin{aligned}\gamma = v_1 \otimes \left(u_1^\vee \wedge u_2^\vee - u_4^\vee \wedge u_5^\vee\right) + v_2 \otimes \left(u_2^\vee \wedge u_3^\vee - u_5^\vee \wedge u_6^\vee\right)\\ + v_3 \otimes u_1^\vee \wedge u_4^\vee + v_4 \otimes u_2^\vee \wedge u_5^\vee + v_5 \otimes u_3^\vee \wedge u_6^\vee + v_6 \otimes u_4^\vee \wedge u_6^\vee.\end{aligned}$$

This defines a map $\gamma : \Lambda^2 U \to V$. Let

$$0 \longrightarrow V \longrightarrow G \longrightarrow U \longrightarrow 0$$

be the corresponding central extension (see Notation 7), then for any faithful representation W of G one has

$$\mathrm{Br}_{\mathrm{nr}}(\mathbf{C}(W)^G) = \{0\}$$

but

$$H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}) \neq \{0\}.$$

In particular, $\mathbf{C}(W)^G$ *is not a rational extension of* $\mathbf{C}$.

Proof. By [Bo1, Lemma 5.1], one has

$$\mathrm{Br}_{\mathrm{nr}}(\mathbf{C}(W)^G) \xrightarrow{\sim} K^2_{\max}/K^2.$$

But

$$K^2 = \langle u_1^\vee \wedge u_2^\vee - u_4^\vee \wedge u_5^\vee, u_2^\vee \wedge u_3^\vee - u_5^\vee \wedge u_6^\vee,$$
$$u_1^\vee \wedge u_4^\vee, u_2^\vee \wedge u_5^\vee, u_3^\vee \wedge u_6^\vee, u_4^\vee \wedge u_6^\vee \rangle$$

and

$$K^{2\perp} = \langle u_1 \wedge u_2 + u_4 \wedge u_5, u_2 \wedge u_3 + u_5 \wedge u_6,$$
$$u_3 \wedge u_4, u_6 \wedge u_1, u_1 \wedge u_3, u_2 \wedge u_4, u_3 \wedge u_5, u_5 \wedge u_1, u_6 \wedge u_2 \rangle.$$

Since

$$u_1 \wedge u_2 + u_4 \wedge u_5 = (u_1 + u_4) \wedge (u_2 + u_5) + u_2 \wedge u_4 + u_5 \wedge u_1$$

and

$$u_2 \wedge u_3 + u_5 \wedge u_6 = (u_2 + u_5) \wedge (u_3 + u_6) + u_6 \wedge u_2 + u_3 \wedge u_5,$$

we have

$$K^{2\perp}_{\mathrm{dec}} = K^{2\perp} \quad \text{and} \quad K^2 = K^2_{\max}.$$

This proves the first assertion. We now compute K^3 and $K^3_{\max}$

$$\begin{aligned}
K^3 = \langle & u_1^\vee \wedge u_4^\vee \wedge u_5^\vee, u_1^\vee \wedge u_2^\vee \wedge u_3^\vee - u_1^\vee \wedge u_5^\vee \wedge u_6^\vee,\\
& u_1^\vee \wedge u_2^\vee \wedge u_5^\vee, u_1^\vee \wedge u_3^\vee \wedge u_6^\vee, u_1^\vee \wedge u_4^\vee \wedge u_6^\vee,\\
& u_2^\vee \wedge u_4^\vee \wedge u_5^\vee, u_2^\vee \wedge u_5^\vee \wedge u_6^\vee, u_1^\vee \wedge u_2^\vee \wedge u_4^\vee,\\
& u_2^\vee \wedge u_3^\vee \wedge u_6^\vee, u_2^\vee \wedge u_4^\vee \wedge u_6^\vee,\\
& u_1^\vee \wedge u_2^\vee \wedge u_3^\vee - u_3^\vee \wedge u_4^\vee \wedge u_5^\vee,\\
& u_3^\vee \wedge u_5^\vee \wedge u_6^\vee, u_1^\vee \wedge u_3^\vee \wedge u_4^\vee, u_2^\vee \wedge u_3^\vee \wedge u_5^\vee,\\
& u_3^\vee \wedge u_4^\vee \wedge u_6^\vee, u_1^\vee \wedge u_2^\vee \wedge u_4^\vee,\\
& u_2^\vee \wedge u_3^\vee \wedge u_4^\vee - u_4^\vee \wedge u_5^\vee \wedge u_6^\vee,\\
& u_2^\vee \wedge u_4^\vee \wedge u_5^\vee, u_3^\vee \wedge u_4^\vee \wedge u_6^\vee, u_1^\vee \wedge u_2^\vee \wedge u_5^\vee,\\
& u_2^\vee \wedge u_3^\vee \wedge u_5^\vee, u_1^\vee \wedge u_4^\vee \wedge u_5^\vee, u_3^\vee \wedge u_5^\vee \wedge u_6^\vee,\\
& u_4^\vee \wedge u_5^\vee \wedge u_6^\vee, u_1^\vee \wedge u_2^\vee \wedge u_6^\vee - u_4^\vee \wedge u_5^\vee \wedge u_6^\vee,\\
& u_2^\vee \wedge u_3^\vee \wedge u_6^\vee, u_1^\vee \wedge u_4^\vee \wedge u_6^\vee, u_2^\vee \wedge u_5^\vee \wedge u_6^\vee \rangle
\end{aligned}$$

$$= \langle u_1^\vee \wedge u_2^\vee \wedge u_3^\vee - u_1^\vee \wedge u_5^\vee \wedge u_6^\vee, u_1^\vee \wedge u_2^\vee \wedge u_3^\vee - u_3^\vee \wedge u_4^\vee \wedge u_5^\vee,$$
$$u_1^\vee \wedge u_2^\vee \wedge u_4^\vee, u_1^\vee \wedge u_2^\vee \wedge u_5^\vee, u_1^\vee \wedge u_2^\vee \wedge u_6^\vee, u_1^\vee \wedge u_3^\vee \wedge u_4^\vee,$$
$$u_1^\vee \wedge u_3^\vee \wedge u_6^\vee, u_1^\vee \wedge u_4^\vee \wedge u_5^\vee, u_1^\vee \wedge u_4^\vee \wedge u_6^\vee, u_2^\vee \wedge u_3^\vee \wedge u_4^\vee,$$
$$u_2^\vee \wedge u_3^\vee \wedge u_5^\vee, u_2^\vee \wedge u_3^\vee \wedge u_6^\vee, u_2^\vee \wedge u_4^\vee \wedge u_5^\vee, u_2^\vee \wedge u_4^\vee \wedge u_6^\vee,$$
$$u_2^\vee \wedge u_5^\vee \wedge u_6^\vee, u_3^\vee \wedge u_4^\vee \wedge u_6^\vee, u_3^\vee \wedge u_5^\vee \wedge u_6^\vee, u_4^\vee \wedge u_5^\vee \wedge u_6^\vee \rangle.$$

Therefore

$$K^{3\perp} = \langle u_1 \wedge u_2 \wedge u_3 + u_3 \wedge u_4 \wedge u_5 + u_5 \wedge u_6 \wedge u_1, u_1 \wedge u_3 \wedge u_5 \rangle.$$

By [Pe1, p. 264, Example 2],

$$S^3_{\mathrm{dec}} = \langle u_1 \wedge u_3 \wedge u_5 \rangle.$$

Therefore $K^3_{\max}/K^3 \xrightarrow{\sim} \mathbf{F}_p$ and by Theorem 2, we get that

$$H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}) \neq \{0\}.$$

□

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