Quantum resonances in the semiclassical regime: a brief survey

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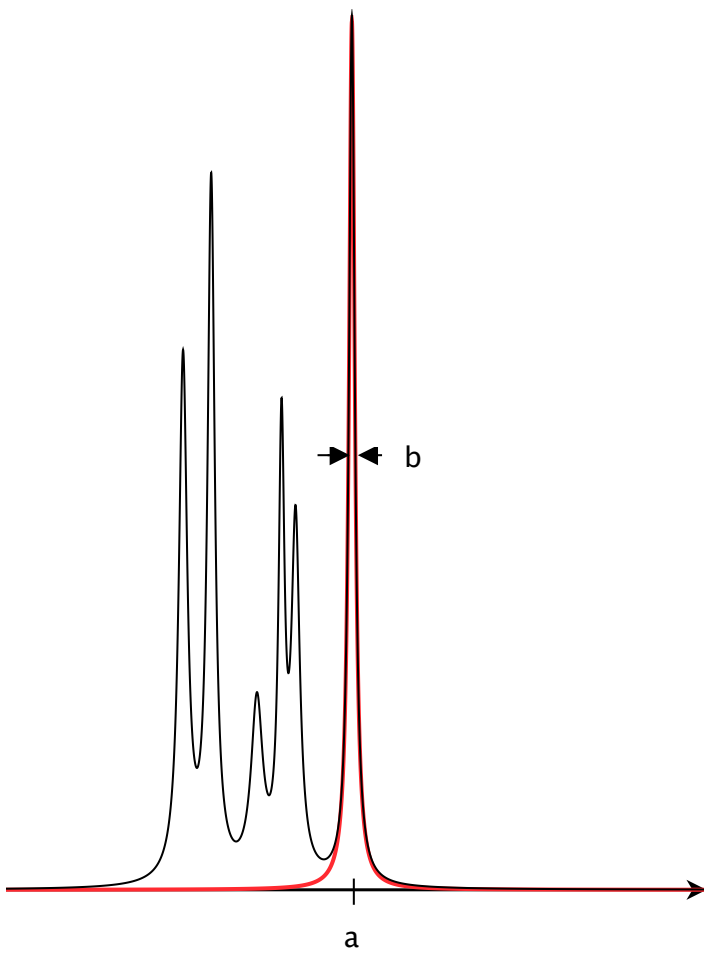
Plan of the Lecture

Introduction

- In physics, the introduction of the notion of quantum resonance was motivated by the behavior of various quantities related to scattering experiments, as scattering cross sections, or time delays, ...



- At certain energies, these quantities present peaks, which were modelized by a Lorentzian shaped function

$$w_{a,b} : \lambda \mapsto \frac{1}{\pi} \frac{b/2}{(\lambda - a)^2 + \left(b/2\right)^2}.$$

- Note that for $\rho = a - ib/2 \in \mathbb{C}$, one has

$$w_{a,b}(\lambda) = \frac{1}{\pi} \frac{\operatorname{Im} \rho}{|\lambda - \rho|^2},$$

and the complex number ρ was called a resonance.

- Such complex values for energies had also appeared for example in the work by Gamow, to explain α-radioactivity, and the notion of resonance was therefore also associated to some energy decay.

The Schrödinger operator

- A quantum particle of mass m, moving in a potential $V \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, is described by a solution of the time-dependent Schrödinger equation

$$\begin{cases} i\hbar\partial_t\phi(t,x) = P(x,\hbar D_x)\phi(t,x), \\ P(x,\hbar D_x) = -\frac{\hbar^2}{2m}\Delta + V(x). \end{cases}$$

- When $E \in \sigma_{disc}(P)$, and $P\phi_0 = E\phi_0$, $\|\phi_0\|_{L^2} = 1$, then

$$\phi(t,x) = e^{-iEt/\hbar}\phi_0(x,\hbar)$$

is a solution, which satisfies the normalization condition

$$\|\phi(t,.)\|_{L^2} = \Big(\int |\phi(t,x)|^2 dx\Big)^{1/2} = 1.$$

$\phi(t,x)$ is called the wave function of the particle, with energy E, and $|\phi(t,x)|^2$ is its density of probability of presence at time t.

- Now suppose ψ_0 is a resonant state (not in L^2) corresponding to the resonance $\rho = a - ib/2$. Its time evolution should be written

$$\psi(t) = e^{-ita-tb/2}\psi_0,$$

so that its density of probability of presence at time t is

$$\frac{|\psi(t,x)|^2}{|\psi_0(x)|^2} = e^{-bt},$$

and b is the decay rate of that probability.

- In other words : a resonance $\rho \in \mathbb{C}$ corresponds to a quasi-particle with lifetime $1/\operatorname{Im}\rho$. The more a resonance is close to the real axis, the more its presence leads to physically observable effects (can also be seen on Breit-Wigner peaks).

Poles of the resolvent

- We suppose that $V \in \mathcal{C}^\infty(\mathbb{R}^n)$ is real valued, and that

$$\partial^\alpha V(x) = \mathcal{O}(\langle x \rangle^{-\sigma-|\alpha|}), \text{ for some } \sigma > 0.$$

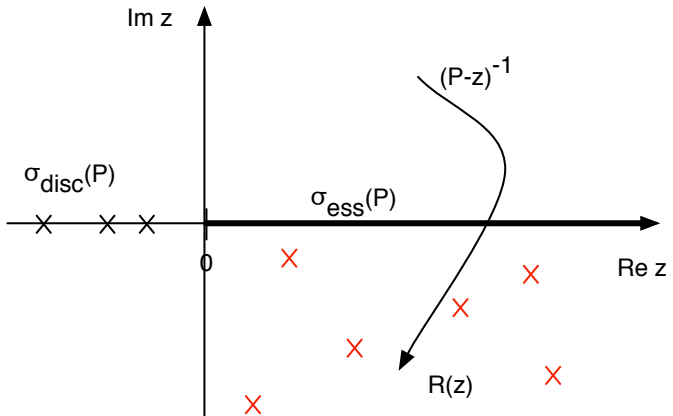
- Then $P = -h^2\Delta + V$ is self-adjoint with domain $H^2(\mathbb{R}^n)$. In particular for $\operatorname{Im} z > 0$, $R(z) = (P-z)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is well defined.

Theorem and Definition 1 :

The resolvent

$$\{\operatorname{Re} z > 0, \operatorname{Im} z > 0\} \ni z \mapsto R(z) : L^2_{comp}(\mathbb{R}^n) \to L^2_{loc}(\mathbb{R}^n)$$

can be meromorphically continued to the lower half-plane $\{\operatorname{Re} z > 0, \operatorname{Im} z < 0\}$. Its poles are called resonances of P.



Classical scattering

- A classical particle with mass 1, when placed in the conservative force field $F(x) = -\nabla V(x)$, follows the trajectory $(x(t, x_0, \xi_0), \xi(t, x_0, \xi_0))$ given by

$$\begin{cases} \dot{x}(t) = \xi(t), \\ \dot{\xi}(t) = -\nabla V(x(t)). \end{cases}$$

- This is Newton's Equation written in Hamiltonian form : In the phase space $T^*\mathbb{R}^d$, the trajectory is an integral curve of the vector field

$$H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi$$

associated to the total energy of the particle $p(x, \xi) = \frac{1}{2}\xi^2 + V(x)$

- Note that :

$$Pu(x) = \mathrm{Op}_h^w(p(x,\xi))u(x) = \int e^{i(x-y)\cdot\xi/h} p(\frac{x+y}{2}, \xi) u(y) dy \, \frac{d\xi}{(2\pi h)^n}$$

- Scattering : $V \to 0$ at ∞, and we consider trajectories going to infinity. The aim could be to recover V knowing what goes out of the interaction region when one throw particles towards that region.

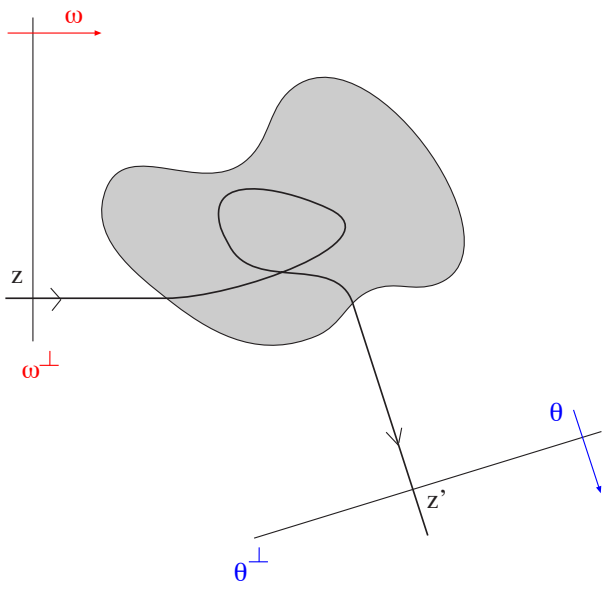
- Let $E > 0$ be fixed. For any $\alpha \in S^{n-1}$, $z \in \alpha^\perp \sim \mathbb{R}^{n-1}$, there is one and only one integral curve for H_p

$$\gamma_\pm(t, z, \alpha, E) = (x_\pm(t, z, \alpha, E), \xi_\pm(t, z, \alpha, E)) \in p^{-1}(E)$$

such that

$$\lim_{t\to\pm\infty} |x_\pm(t, z, \alpha) - \sqrt{2E}\alpha t - z| = 0,$$

$$\lim_{t\to\pm\infty} |\xi_\pm(t, z, \alpha) - \sqrt{2E}\alpha| = 0.$$



- The classical scattering map at energy E is the map $(z, \omega) \mapsto (z', \theta)$.

The scattering amplitude

- We suppose for simplicity that V is a very short range potential :

$$\partial^\alpha V(x) = \mathcal{O}(\langle x\rangle^{-\sigma-|\alpha|}), \text{ for some } \sigma > (n+1)/2.$$

- Let $E > 0$ be fixed. For any $\omega, \theta \neq \omega \in S^{n-1}$, there exists one and only one $u \in L^2_{loc}(\mathbb{R}^n)$ such that

$$\begin{cases} Pu = Eu, \\ u(x,h) = e^{i\sqrt{E}x\cdot\omega/h} + \mathcal{A}(\omega,\theta,E,h)\frac{e^{i\sqrt{E}x\cdot\theta/h}}{|x|^{(n-1)/2}} + o(|x|^{(1-n)/2}), \end{cases}$$

as $|x| \to +\infty$ with $\frac{x}{|x|} = \theta$.

- The scattering amplitude at energy E is the operator $\mathcal{A}(E) : L^2(\mathcal{S}^{n-1}) \to L^2(\mathcal{S}^{n-1})$ with kernel $\mathcal{A}(\omega,\theta,E,h)$. The scattering matrix at energy E is

$$\mathcal{S}(E) = \mathcal{I} - 2i\pi c(E,h)\mathcal{A}(E),$$

for some normalization constant $c(E,h)$.

More general potentials

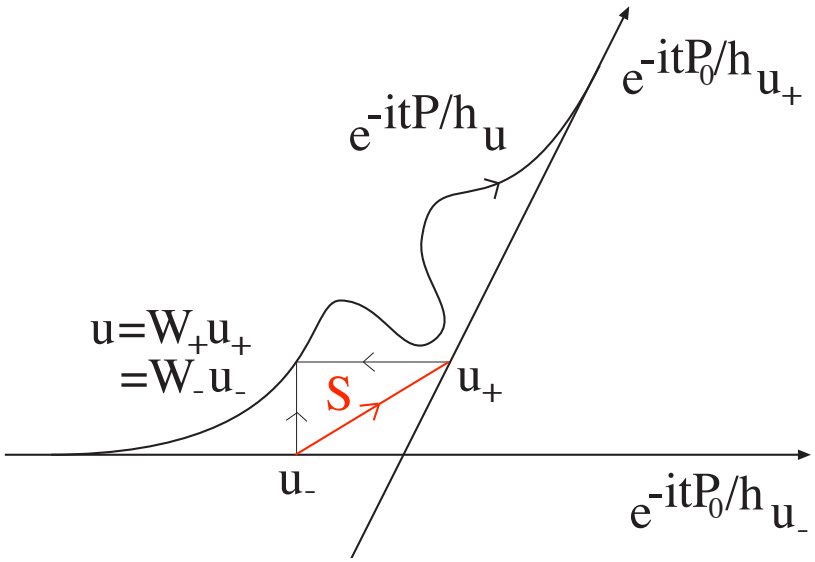
- We suppose now that V is a short range potential :

$$\partial^\alpha V(x) = \mathcal{O}(\langle x\rangle^{-\sigma-|\alpha|}), \text{ for some } \sigma > 1,$$

and we set $P = P_0 + V$, $P_0 = -h^2\Delta$.

- For any given u_-, one can find u, and then u_+, such that

$$\|e^{-itP_0/h}u_- - e^{-itP/h}u\| \to 0 \text{ as } t \to -\infty,$$
$$\|e^{-itP_0/h}u_+ - e^{-itP/h}u\| \to 0 \text{ as } t \to +\infty.$$



- The Wave operators $W_\pm$are defined by $u = W_- u_- = W_+ u_+$.
- Notice the intertwining property : $PW_\pm = W_\pm P_0$.

The scattering amplitude

- The scattering operator for (P, P_0) is defined by

$$\mathcal{S} = (W_+)^{-1} W_- : \ u_- \mapsto u_+.$$

- Since $P_0 \mathcal{S} - \mathcal{S} P_0 = 0$, one has

$$\mathcal{S} = \int_0^{+\infty} \mathcal{S}(E) d\mu_P(E), \ \mathcal{S}(E) : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1}),$$

The operator $\mathcal{S}(E)$ is the scattering matrix at energy E.

- $\mathcal{T}(E) = \frac{1}{2i\pi}(\mathcal{I}d - \mathcal{S}(E))$ is an integral operator, with a smooth kernel out of $\theta = \omega$. Up to a multiplicative constant, $\mathcal{T}(E)$ is the scattering amplitude at energy E.

Meromorphic extension of the scattering amplitude

Theorem and Definition 2 :

Suppose V is a long range potential. The scattering amplitude $\mathbb{R}_+^* \ni E \mapsto \mathcal{A}(E)$ can be meromorphically continued to the lower half-plane $\{\mathrm{Re}\, z > 0, \mathrm{Im}\, z < 0\}$. Its poles are called resonances of P.

- When is V compactly supported, one can prove that for $\chi_1 \prec \chi_2 \in \mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$\mathcal{A}(\omega, \theta, E, h) = i\pi E^{\frac{n-2}{2}} \langle (P - (E + i0))^{-1} [h^2\Delta, \chi_2] e^{i\sqrt{E} x\cdot\omega/h}, [h^2\Delta, \chi_1] e^{i\sqrt{E} x\cdot\theta/h} \rangle$$

Then one can extend $\mathcal{A}(\omega, \theta, E, h)$ to complex values of E by

$$\mathcal{A}(\omega, \theta, z, h) = i\pi z^{\frac{n-2}{2}} \langle R(z)[h^2\Delta, \chi_2] e^{i\sqrt{z} x\cdot\omega/h}, [h^2\Delta, \chi_1] e^{i\sqrt{z} x\cdot\theta/h} \rangle$$

Analytic dilation

- For $\mu \in \mathbb{R}$, we set

$$U_\mu : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), U_\mu \phi(x) = e^{d\mu/2}\phi(xe^\mu), \text{ for } \phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$$

and we see

$$U_\mu P U_\mu^* = e^{-2\mu}(-h^2\Delta) + V(xe^\mu).$$

Definition

The funtion $V : \mathbb{R}^d \to \mathbb{R}$ is dilation-analytic when $\mu \mapsto V(xe^\mu)(-h^2\Delta + 1)^{-1}$ extends as an analytic (w.r.t. μ) family of compact operators on $L^2(\mathbb{R}^d)$.

- For example, if $V \in \mathcal{C}^\infty(\mathbb{R}^d)$ extends as holomorphic function in

$$\Omega_{\theta_0} = \{x \in \mathbb{C}^d, |\operatorname{Im} x| \leq \tan\theta_0 \langle \operatorname{Re} x \rangle\},$$

where it satisfies

$$V(x) \to 0 \text{ as } |x| \to +\infty,$$

then V is dilation-analytic.

Resonances

- For such a V,

$$U_\mu P U_\mu^* = e^{-2\mu}(-h^2\Delta) + [V(xe^\mu)(-h^2\Delta+1)^{-1}](-h^2\Delta+1),$$

so that one can define $P_\theta = U_{i\theta} P U_{i\theta}^*$ for $\theta \in \mathbb{R}^+$ small enough.

- By Weyl's Theorem

$$\sigma_{ess}(P_\theta) = e^{-2i\theta}\sigma_{ess}(-h^2\Delta) = e^{-2i\theta}\mathbb{R}^+.$$

Definition 3

Let V be analytic-dilation in $|\mu| < \delta$. The set $\mathrm{Res}(P)$ of resonances of P is

$$\mathrm{Res}(P,h) = \bigcup_{0<\theta<\delta} \sigma_{disc}(P_\theta) \cap (0, \mathbb{R}^+, e^{-2i\theta}\mathbb{R}^+).$$

Complex distortion

- Now we assume only that $V(x)$ has an analytic extension to

$$\Sigma = \{x \in \mathbb{C}^n;\ |\operatorname{Im} x| < \delta |\operatorname{Re} x| \text{ and } |\operatorname{Re} x| > C\}$$

for some $\delta, C > 0$, and that $V(x) \to 0$ as $|x| \to \infty$, $x \in \Sigma$.

- Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be s. t. $F(x) = 0$ for $|x| < C$ and $F(x) = 1$ for $|x| \gg 1$. For $\mu \in \mathbb{R}$ small enough, we define $U_\mu : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ as the unitary operator

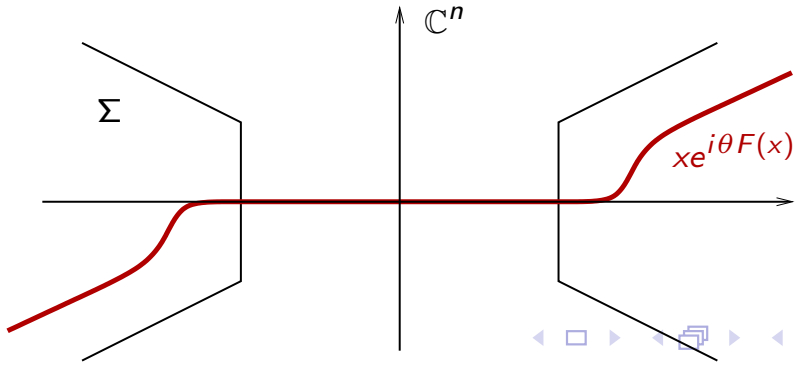
$$U_\mu \varphi(x) = \left|\det\left(d\, x e^{\mu F(x)}\right)\right|^{1/2} \varphi\left(x e^{\mu F(x)}\right)$$

For $\theta \in \mathbb{R}$ small enough, we denote

$$P_\theta = U_{i\theta} P U_{i\theta}^{-1}$$

the distorted operator. As before, the set of resonances of P is

$$\operatorname{Res}(P, h) = \bigcup_{0<\theta<\delta} \sigma_{disc}(P_\theta) \cap (0, \mathbb{R}^+, e^{-2i\theta}\mathbb{R}^+).$$



All these definitons coincide !

Theorem : (Helffer-Martinez)

When their domain of validity overlap, these different definitions of resonance (as well as more sophisticated ones) coïncide.

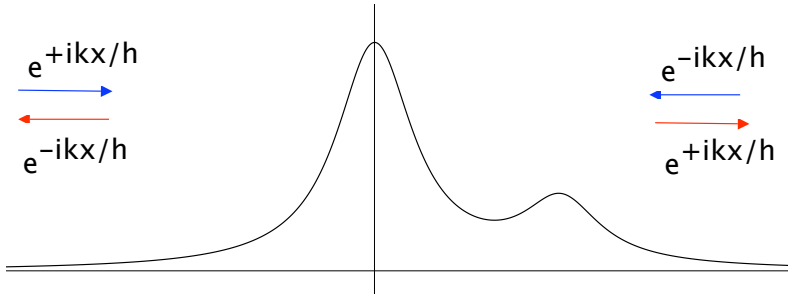
Scattering in 1d

- The (stationary) Schrödinger equation in 1d is

$$-h^2 u'' + Vu = Eu.$$

- The scattering matrix $\mathcal{S}(E,h)$ gives outgoing data from incoming data : for $V \in \mathcal{C}_0^\infty(\mathbb{R},\mathbb{R})$, and with $k = \sqrt{E}$, for any solution u (not in L^2 !) , one has

$$u\mathbf{1}_{x<-R} = p_- e^{ikx/h} + q_- e^{-ikx/h}, \quad u\mathbf{1}_{x>R} = p_+ e^{-ikx/h} + q_+ e^{ikx/h},$$



Then $\mathcal{S}(E)$ is defined by $\begin{pmatrix} q_+ \\ q_- \end{pmatrix} = \mathcal{S}(E,h) \begin{pmatrix} p_+ \\ p_- \end{pmatrix}$.

- On can show that $\mathcal{S}(E,h) = \dfrac{1}{a^*(E,h)} \begin{pmatrix} 1 & -b^*(E,h) \\ b(E,h) & 1 \end{pmatrix}$ where $a(E)$ and $b(E)$ are analytic. The resonances are the zeros of $a^*(E)$.

- Now we suppose $P = h^2 D^2 + V(x)$ with V holomorphic in $\{x \in \mathbb{C}, |\operatorname{Im} x| < \tan\theta_0 |\operatorname{Re} x|\}$, and that V is short range in that set. We have

$$P_\theta u(x) = U_\theta P U_{-\theta} u(x) = (-h^2 u''(x) + V(xe^{i\theta})u(x).$$

- We denote by $J_\pm^{d,g}$ the solutions of $(P - E)u = 0$ characterized by

$$\begin{cases} J_\pm^r(x, E, h) \sim e^{\pm i\sqrt{E}x/h} \text{ as } \operatorname{Re} x \to +\infty, \\ J_\pm^l(x, E, h) \sim e^{\pm i\sqrt{E}x/h} \text{ as } \operatorname{Re} x \to -\infty. \end{cases}$$

Then, for real E's, $J_{\pm,\theta}^{r,l} : x \mapsto J_\pm^{r,l}(xe^{i\theta})$ form a basis of solutions to $P_\theta u = Eu$ in $L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R}^-)$ respectively .

- For $E \in \mathbb{C}$ with $-2\theta_0 < \arg E < 0$, $\theta < \theta_0$ the only solution in $L^2(\mathbb{R}^+)$ is $J_{+,\theta}^r(x, E, h)$, and $J_{-,\theta}^l(x, E, h)$ is the only one in $L^2(\mathbb{R}^-)$.

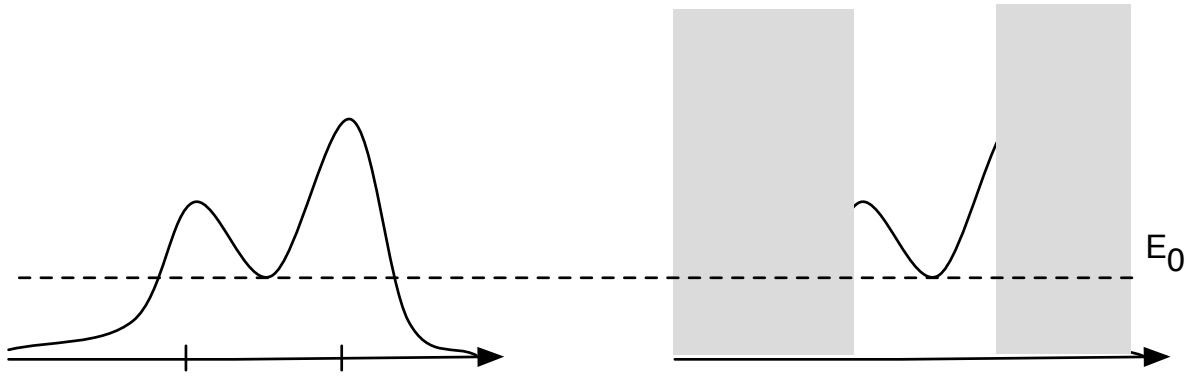
- $E \in \operatorname{Res}(P)$ iff there is an outgoing solution, i.e. $a^*(E, h) = 0$.

Le puits dans l'isle

- We suppose now that $V \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ has an analytic extension to

$$\Sigma = \{x \in \mathbb{C}^n;\ |\operatorname{Im} x| < \delta\langle \operatorname{Re} x\rangle\}$$

for some $\delta > 0$, and that $V(x) = \mathcal{O}(\langle x\rangle^{-\sigma})$, $\sigma > 0$, as $|x| \to \infty$, $x \in \Sigma$. We also assume that V has a non-degenerate local minimum at x_0, and "looks like"



- Let K be a compact set containing a vicinity of x_0. We denote P^D the Dirichlet realization of P in K. The spectrum of P^D is discrete and we denote its eigenvalues by $E_1(h), E_2(h), \ldots$.

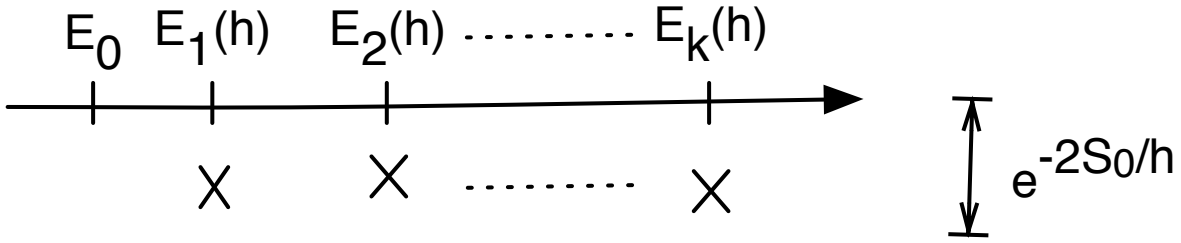
Shape resonances

Theorem (Helffer-Sjöstrand)

There exists a bijection $b(h)$ between the spectrum of P^D in $D(E_0, Ch)$ and the set of resonances of P in $D(E_0, Ch)$. Moreover

$$|b(z) - z| = O(e^{-(2S_0/h)}),$$

where S_0 is the Agmon distance between x_0 and the sea.



Barrier-top resonances

- We suppose that V has an analytic extension to

$$\Sigma = \{x \in \mathbb{C}^n;\ |\operatorname{Im} x| < \delta\langle \operatorname{Re} x\rangle\}$$

for some $\delta > 0$, and that $V(x) \longrightarrow 0$ as $|x| \to \infty$, $x \in \Sigma$. Moreover, we assume

$$V(x) = E_0 - \sum_{j=1}^{n} \lambda_j^2 x_j^2 + O(|x|^3).$$

and that the trapped set at energy E_0 is $\{0\}$.

Theorem (Sjöstrand, Briet-Combes-Duclos)

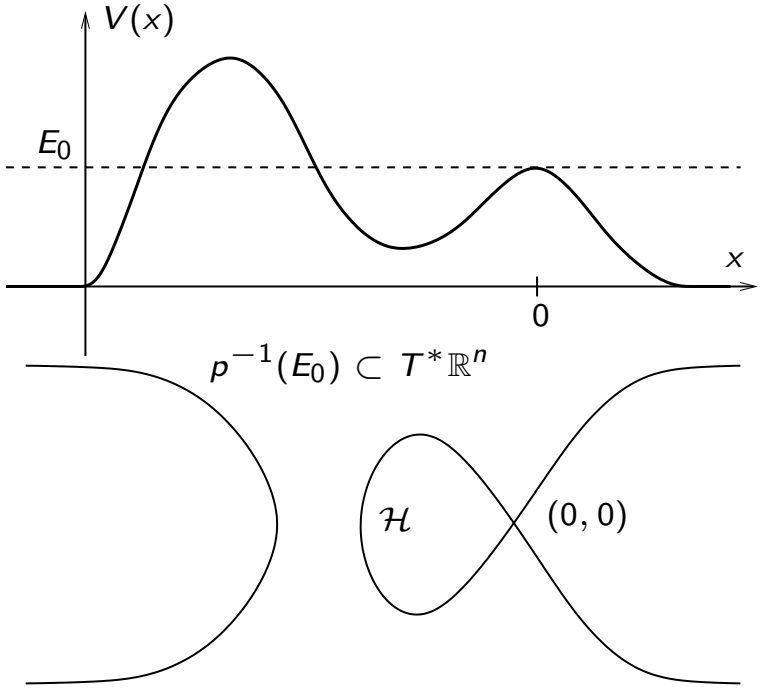
Let $C > 0$ be different from $\sum_{j=1}^{n}(\alpha_j + \frac{1}{2})\lambda_j$ for all $\alpha \in \mathbb{N}^n$. Then, for $h > 0$ small enough, there exists a bijection b_h between the sets $\operatorname{Res}_0(P) \cap D(E_0, Ch)$ and $\operatorname{Res}(P) \cap D(E_0, Ch)$ counted with their multiplicity, where

$$\operatorname{Res}_0(P) = \Big\{ z_\alpha^0 = E_0 - ih \sum_{j=1}^{n} (\alpha_j + \frac{1}{2})\lambda_j;\ \alpha \in \mathbb{N}^n \Big\},$$

such that $b_h(z) - z = o(h)$.

A homoclinic orbit in 1d

- We suppose here that $d = 1$ and that



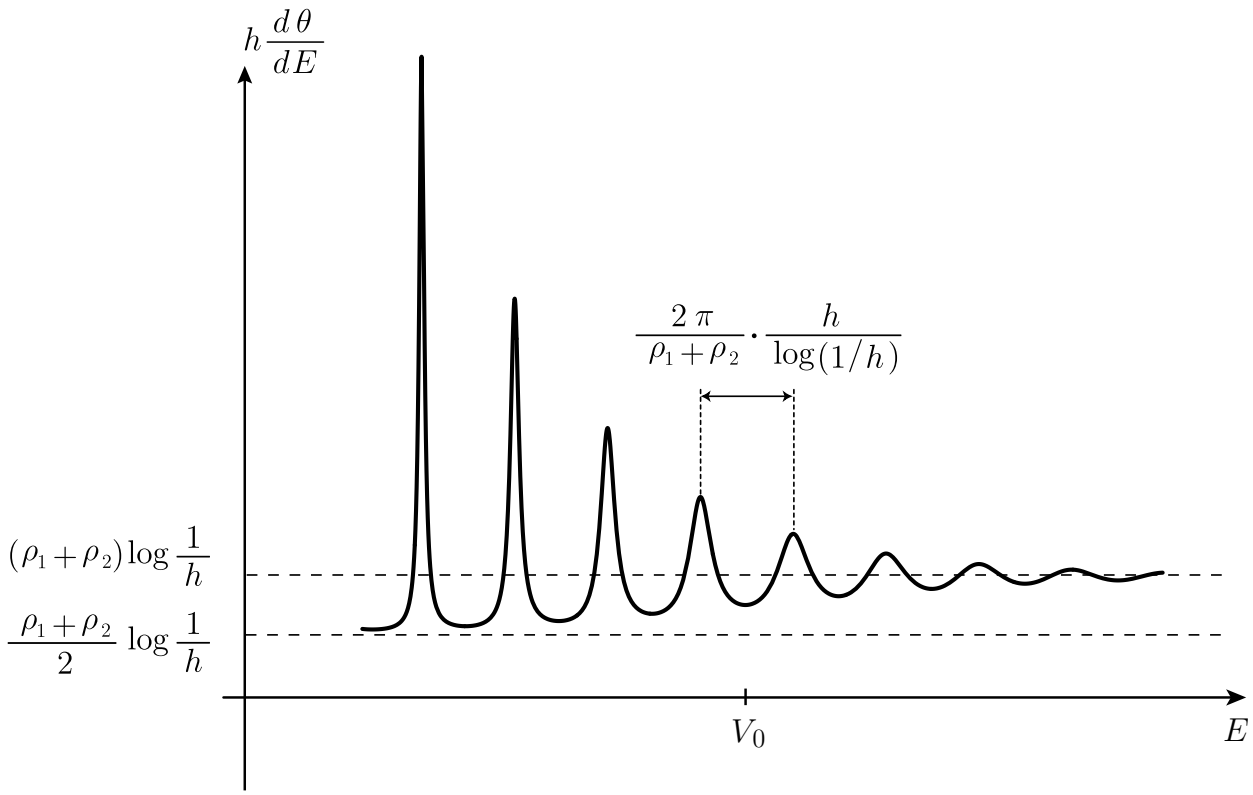
Theorem (Fujiié–R.)

The resonances $(z_k)_{k\in\mathbb{N}}$ of P in $D(E_0, Ch)$ satisfies

$$z_k = E_0 + \lambda \frac{S_0 - (2k+1)\pi h}{2|\ln h|} - i \frac{h}{|\ln h|} \frac{\lambda \ln 2}{2} + \mathcal{O}\big(h/(\ln h)^2\big),$$

where $V(x) = E_0 - \frac{\lambda^2}{2}x^2 + O(x^3)$.

- **The Time delay** :



- For $E \in]0,+\infty[$, $\mathcal{S}(E,h)$ is unitary, and the scattering phase is defined as $\theta(E,h) = \frac{1}{2i}\ln\det\mathcal{S}(E,h)$.

Settings

- We consider the semiclassical Schrödinger operator on $\mathbb{R}^n$, $n \geq 1$,

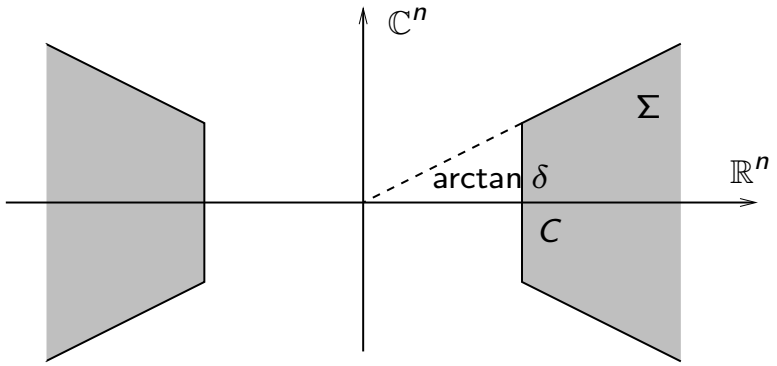
$$P = -h^2\Delta + V(x)$$

where $V(x) \in C^\infty(\mathbb{R}^n; \mathbb{R})$.

- We define the resonances by analytic distortion, assuming that $V(x)$ has an analytic extension to

$$\Sigma = \{x \in \mathbb{C}^n;\ |\operatorname{Im} x| < \delta |\operatorname{Re} x| \text{ and } |\operatorname{Re} x| > C\}$$

for some $\delta, C > 0$, and $V(x) \longrightarrow 0$ as $|x| \to \infty$, $x \in \Sigma$.



The trapped set

- We have seen three situations where one has been able to show existence and give precise location, in the semiclassical regime, of resonances close to some $E_0 \in \mathbb{R}$. In these three cases, the trapped set at energy E_0 is not empty :

Definition :

The trapped set at energy $E > 0$ is

$$K(E_0) = \left\{(x,\xi) \in p^{-1}(E_0);\ t \mapsto \exp(tH_p)(x,\xi) \text{ is bounded}\right\}$$

We recall that $p(x,\xi) = \xi^2 + V(x)$ is the semiclassical symbol of $P = \mathrm{Op}_h^w(p)$, and that H_p is the vector field on $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ given by $H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi = \begin{pmatrix} 2\xi \\ -\nabla V(x) \end{pmatrix}$.

- Examples :
  - Le puits dans l'isle (shape resonances).
  - A barrier top
  - The homoclinic case

What if the trapped set is empty ?

Theorem (Martinez)

Suppose $K(E_0) = \emptyset$. Then P has no resonance in

$$Z_h(E_0) = [E_0 - \varepsilon, E_0 + \varepsilon] + i[-Ch|\ln h|, 0]$$

for some $\varepsilon > 0$ and all $C > 0$. Moreover, for all $\chi \in C_0^\infty(\mathbb{R}^n)$, there exists $N = N(C)$ such that for $z \in Z_h(E_0)$,

$$\|\chi(P - z)^{-1}\chi\| \lesssim h^{-N}.$$

- If V is analytic in a whole neighborhood of $\mathbb{R}^n$, Briet–Combes–Duclos and Helffer–Sjöstrand have proved that P has no resonance in a fixed neighborhood of E_0.
- Notice that if $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is supported where the distortion opertor does nothing, then $\chi(P - z)^{-1}\chi = \chi(P_\theta - z)^{-1}\chi$, so that the estimate on the resolvent implies that there is no resonance in $Z_h(E_0)$.

Resolvent estimate for real energies

Theorem

For $E_0 > 0$, the following properties are equivalent.

i) $K(E_0) = \emptyset$.

ii) There exists $\varepsilon > 0$ such that, for any $\chi \in C_0^\infty(\mathbb{R}^n)$,

$$\sup_{z \in [E_0 - \varepsilon, E_0 + \varepsilon]} \left\| \chi (P - z)^{-1} \chi \right\| \lesssim h^{-1}.$$

Theorem (Bony-Burq-R)

For $E_0 > 0$, the following properties are equivalent.

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ii) There exists $\chi \in C_0^\infty(\mathbb{R}^n)$ such that, for all $\varepsilon > 0$,

$$\sup_{z \in [E_0 - \varepsilon, E_0 + \varepsilon]} \left\| \chi (P - z)^{-1} \chi \right\| \gtrsim h^{-1} |\ln h|.$$

Moreover, when $K(E_0) \neq \emptyset$, the above estimate is true for any $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi = 1$ near $\pi_x K(E_0)$.

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Moreover, when $K(E_0) \neq \emptyset$, the above estimate is true for any $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi = 1$ near $\pi_x K(E_0)$.

Proof (1)

- (ii)$\Rightarrow$ (i) by the first theorem. We only have to prove that (i)$\Rightarrow$ (ii). We set

$$h^{-1}M(h) = \sup_{z\in[E_0-\varepsilon,E_0+\varepsilon]} \left\|\chi(P-z)^{-1}\chi\right\|.$$

Using Kato smoothness, we have

$$\int_{\mathbb{R}} \left\|\chi \mathbf{1}_{[E_0-\varepsilon,E_0+\varepsilon]}(P)e^{isP}u\right\|_{L^2}^2 ds \lesssim h^{-1}M(h)\|u\|^2.$$

Therefore, if $\varphi \in C_0^\infty([E_0-\varepsilon, E_0+\varepsilon];[0,1])$ with $\varphi(E_0)=1$,

$$\int_{\mathbb{R}} \left\|\chi\varphi(P)e^{itP/h}u\right\|_{L^2}^2 dt \lesssim M(h)\|u\|^2.$$

- Now $\varphi(P)\chi^2\varphi(P) = \mathrm{Op}_h^w(a)$, where

$$a(x,\xi;h) = a_0(x,\xi) + hS_h(1), \quad a_0(x,\xi) = \chi^2(x)\varphi(p(x,\xi))^2.$$

Thus we have

$$\int_{\mathbb{R}} \left(\mathrm{Op}_h^w(a)e^{itP/h}u, e^{itP/h}u\right)dt \lesssim M(h)\|u\|^2.$$

Proof (2)

- Now let $u = u_\rho$ be a normalized gaussian coherent state centered at $\rho \in K(E_0)$ ($u = T_\rho^h(e^{-x^2/2h})$). We have seen in Clotilde's talk that

$$\lim_{h\to 0} \left(\mathrm{Op}_h^w(a) e^{itP/h} u_\rho, e^{itP/h} u_\rho \right) = a_0(\exp(tH_p)(\rho)),$$

for $|t| \leq C|\ln h|$, where $C > 0$.

- Since $\exp(tH_p) \in K(E_0)$ for all t, we have $a_0(\exp(tH_p)(\rho)) = 1$, and

$$\left(\mathrm{Op}_h^w(a) e^{itP/h} u_\rho, e^{itP/h} u_\rho \right) \geq \frac{1}{2},$$

provided h is small enough. We obtain

$$C|\ln h| \leq \int_{|t|\leq C|\ln h|} \left(\mathrm{Op}_h^w(a) e^{itP/h} u, e^{itP/h} u \right) dt \lesssim M(h)\|u\|^2.$$

so that, as stated,

$$|\ln h| \lesssim M(h).$$

A universal upper bound

Theorem (Burq)

For a broad class of operators P, whatever the trapped set is, there is no resonance in

$$[E_0 - \varepsilon, E_0 + \varepsilon] + i\left[e^{-C/h}, 0\right]$$

for some $\varepsilon, C > 0$. Moreover

$$\sup_{z \in [E_0 - \varepsilon, E_0 + \varepsilon]} \left\| \chi (P - z)^{-1} \chi \right\| \lesssim e^{C/h}.$$

- This theorem is optimal, as shown by the "Puits dans l'isle" case.

The barrier top case

- We suppose that V has an analytic extension to

$$\Sigma = \{x \in \mathbb{C}^n;\ |\operatorname{Im} x| < \delta\langle \operatorname{Re} x\rangle\}$$

for some $\delta > 0$, and that $V(x) \longrightarrow 0$ as $|x| \to \infty$, $x \in \Sigma$. Moreover, we assume

$$V(x) = E_0 - \sum_{j=1}^{n} \lambda_j^2 x_j^2 + O(|x|^3).$$

and that $K(E_0) = \{0\}$.

- We already know that there exists $\delta > 0$ such that P has no resonance in $D(E_0, \delta h)$.

Resolvent estimates in the barrier top case

Theorem (Alexandrova-Bony-R)

There exists $\varepsilon > 0$ such that, for any $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, and for $|z - E_0| \leq \varepsilon$.

$$\|\chi(P-z)^{-1}\chi\| \lesssim h^{-1}|\ln h|,$$

Theorem (Bony-Fujiié-R-Zerzeri)

There exists $\varepsilon > 0$ such that, for all $C > 0$ and h small enough,
$i)$ The operator P has no resonances in

$$[E_0 - \varepsilon, E_0 + \varepsilon] + i[-Ch, 0] \setminus D(E_0, 2Ch).$$

$ii)$ There exists $K > 0$ such that

$$\|\chi(P-z)^{-1}\chi\| \lesssim h^{-K} \prod_{z_\alpha \in \mathrm{Res}(P) \cap D(E_0, 2Ch)} |z - z_\alpha|^{-1},$$

for all $z \in [E_0 - \varepsilon, E_0 + \varepsilon] + i[-Ch, Ch]$.

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The Schrödinger group

- We denote by $e^{-itP/h}u_0$ the solution u to the time-dependent Schrödinger equation

$$\begin{cases} i\hbar\partial_t\phi(t,x) = P(x,\hbar D_x)\phi(t,x), \\ u_{|_{t=0}} = u_0. \end{cases}$$

- If $\lambda_0 \in \mathbb{R}$ is an isolated eigenvalue of P, and $\psi \in \mathcal{C}_0^\infty(]\lambda_0 - \delta, \lambda_0 + \delta[)$ for some $\delta > 0$ small enough, one has

$$e^{-itP/h}\psi(P) = e^{-it\lambda_0/h}\Pi_{\lambda_0}\psi(\lambda_0),$$

where Π_{λ_0} is the spectral (orthogonal) projection onto the eigenspace associated to λ_0.

- Is (how) the behavior of the quantum evolution related to the resonances ?

The spectral projection for a resonance

Definition

Let z be a resonance for the opertor P. The associated (generalized) spectral projection is

$$\Pi_z = -\frac{1}{2i\pi} \oint_{\gamma_z} (P - \zeta)^{-1} d\zeta,$$

as an operator from L^2_{comp} to L^2_{loc}. The multiplicity of a resonance is the rank of Π_z.

- Notice that Π_z is not an orthogonal projection.

The shape resonances case

- When the resonances are close to the real axis, one can expect that they truly influence the quantum evolution. In the case of shape resonances we have the

Theorem : (Nakamura-Stefanov-Zworski)

Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and write $\chi = \chi_1 + \chi_2$ with χ_2 not supported near the well, and χ_1 not supported near the sea. Let $\psi \in C_0^\infty([E_0 - \varepsilon, E_0 + \varepsilon])$ for some $\varepsilon > 0$ small enough. Then for any $\mu > 0$, there exists $T > 0$ such that

$$\chi e^{-itP/h}\chi\psi(P) = \sum_{z_\alpha \in \mathrm{Res}(P)\cap D(E_0,\mu h)} e^{-itz_\alpha/h}\chi_1\Pi_{z_\alpha}\chi_1\psi(P) + \mathcal{O}(h^\infty) + \chi_2\mathcal{O}((\frac{(t-T)_+}{h})^{-\infty})\chi_2,$$

for all $t \geq 0$.

- Notice that this makes sense only for $t > T$.

The barrier top case

- Suppose that we are in the barrier-top situation. The resonances z_α are close to the $z_\alpha^0 = E_0 - ih\sum_{j=1}^n(\alpha_j + \frac{1}{2})\lambda_j$.

Theorem : (Bony-Fujiie-R-Zerzeri)

Let $\mu > 0$ be different from $\sum_{j=1}^n(\alpha_j + \frac{1}{2})\lambda_j$ for all $\alpha \in \mathbb{N}^n$.
Suppose that all the z_α^0 in $D(E_0, \mu h)$ are simple.
Let $\chi \in C_0^\infty(\mathbb{R}^n)$ and $\psi \in C_0^\infty([E_0 - \varepsilon, E_0 + \varepsilon])$ for some $\varepsilon > 0$ small enough.
Then, there exists $K = K(\mu) > 0$ such that

$$\chi e^{-itP/h}\chi\psi(P) = \sum_{z_\alpha \in \mathrm{Res}(P)\cap D(E_0,\mu h)} e^{-itz_\alpha/h}\chi\Pi_{z_\alpha}\chi\psi(P) + \mathcal{O}(h^\infty) + \mathcal{O}(e^{-\mu t}h^{-K}),$$

for all $t \geq 0$.

- This makes sense only for $t > K|\ln h|$. When $t/|\ln h| \to +\infty$ as $h \to 0$, the sum over the resonances is negligible and this result says only $\chi e^{-itP/h}\chi\psi(P) = \mathcal{O}(h^\infty)$.