# Asymptotics of Resonances for Homoclinic Orbits

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# Outline

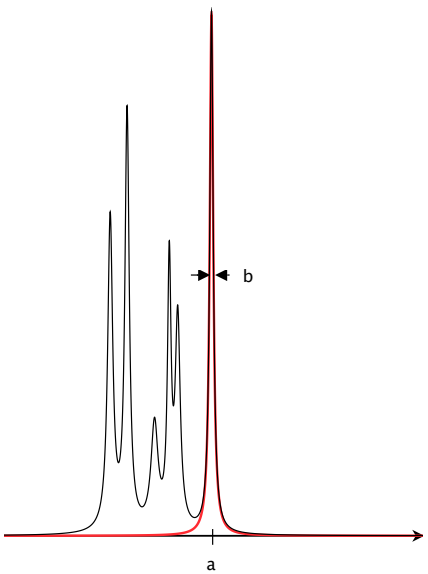
Resonances in the semiclassical regime
- Resonances
- The semiclassical regime
- Some known results

The Homoclinic case
- The geometrical setting
- Main result
- Some comments

Sketch of proof (1 curve)

# Breit-Wigner peaks

- In physics, the introduction of the notion of quantum resonance was motivated by the behavior of various quantities related to scattering experiments, as scattering cross sections, or time delays, ...




- At certain energies, these quantities present peaks, which were modelized by a Lorentzian shaped function

$$w_{a,b} : \lambda \mapsto \frac{1}{\pi} \frac{b/2}{(\lambda - a)^2 + \left(b/2\right)^2}.$$

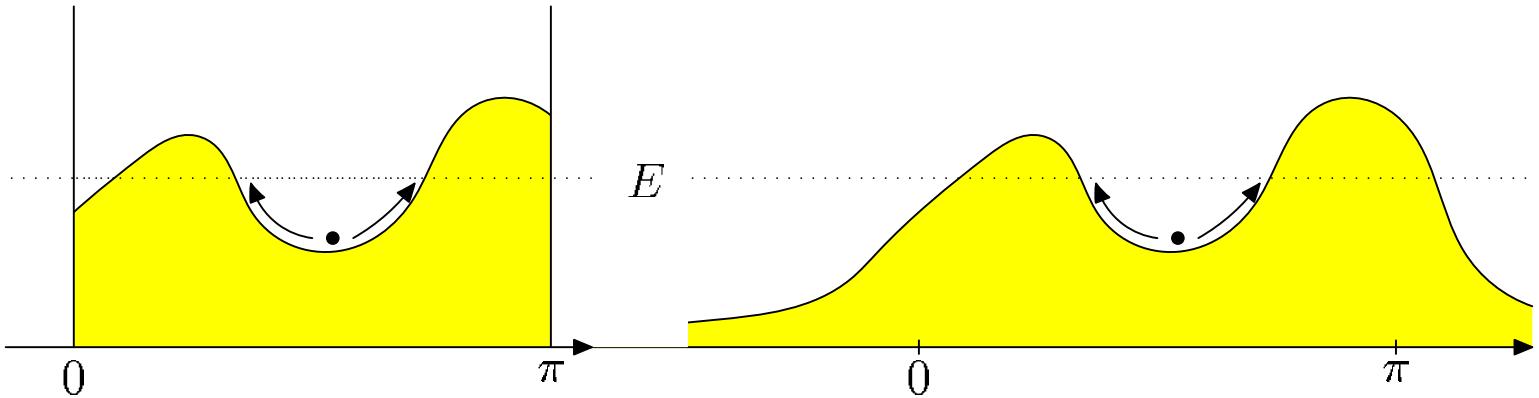
- Note that for $\rho = a - ib/2 \in \mathbb{C}$, one has

$$w_{a,b}(\lambda) = \frac{1}{\pi} \frac{\operatorname{Im} \rho}{|\lambda - \rho|^2},$$

and the complex number $\rho$ was called a resonance.

# From eigenvalues to resonances

- Suppose that a (1d) classical particle with energy $E$ moves in one of the potential below. Its trajectory would be the same in both cases.
- For a (1d) quantum particle, which is associated to an $L^2$ eigenfunction of the Schrödinger operator $-h^2\Delta + V$, the situation is drastically different: there could be no $L^2$ eigenfunction in the case at the right.




- However the existence of a "classical trap" will give rise to the existence of resonant states and corresponding resonant energies, or resonances.

# The Schrödinger operator

- We consider the Schrödinger operator on $\mathbb{R}^n$, $n \geq 1$,

$$P = -h^2\Delta + V(x) = \mathrm{Op}_h^w(p) = \frac{1}{(2\pi h)^n} \iint e^{i(x-y)\cdot\xi/h} p\Big(\frac{x+y}{2}, \xi\Big)\, dy\, d\xi$$

where $p(x,\xi) = \xi^2 + V(x)$, $V(x) \in C^\infty(\mathbb{R}^n; \mathbb{R})$.

- For the talk, we assume that $V \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$.
- Then $P = -h^2\Delta + V$ is self-adjoint on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$.

Its spectrum consists in negative eigenvalues (possibly), and in its essential spectrum $[0, +\infty[$.

# The resolvent

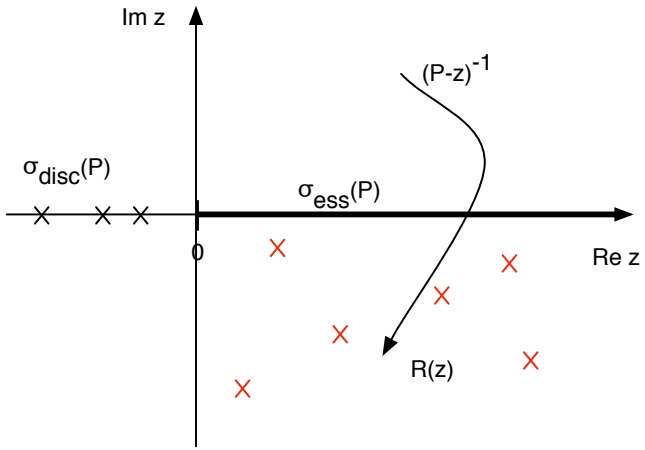
- Since $P$ is self-adjoint, for $\operatorname{Im} z > 0$ in particular, the bounded operator $R(z) := (P - z)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is well defined.

## Theorem and Definition:

The resolvent

$$\{\operatorname{Re} z > 0, \operatorname{Im} z > 0\} \ni z \mapsto R(z) : L^2_{comp}(\mathbb{R}^n) \to L^2_{loc}(\mathbb{R}^n)$$

can be meromorphicaly continued to the lower half-plane $\{\operatorname{Re} z > 0, \operatorname{Im} z < 0\}$. Its poles are called resonances of $P$, and we denote $\operatorname{Res}(P)$ their set.



# Analytic distortion

- Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be such that $F(x) = 0$ for $|x| < C$ and $F(x) = x$ for $|x| \gg 1$.

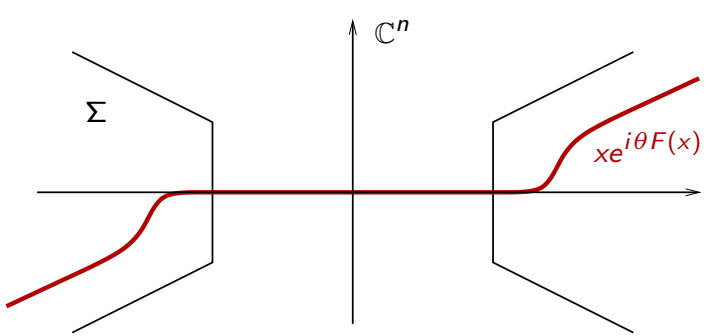
For $\mu \in \mathbb{R}$ small enough, $U_\mu : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the unitary operator

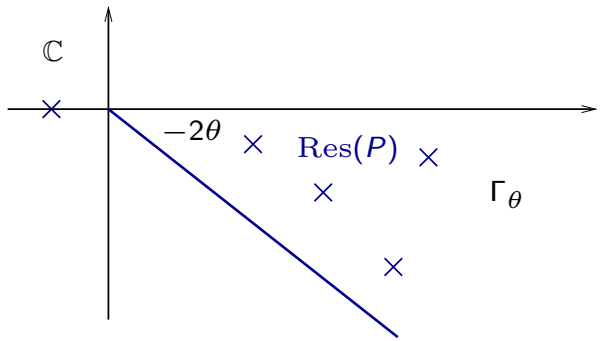
$$U_\mu \varphi(x) = \left| \det \left( d\, x e^{\mu F(x)} \right) \right|^{1/2} \varphi\left( x e^{\mu F(x)} \right)$$

- For $\theta \in \mathbb{R}$ small enough, we denote $P_\theta = U_{i\theta} P U_{i\theta}^{-1}$ the distorted operator.

The set of resonances of $P$ in $\Gamma_\theta = \{-2\theta < \operatorname{Im} E < 0\}$ is

$$\mathrm{Res}_\theta(P) = \sigma(P_\theta) \cap \Gamma_\theta$$



# The truncated resolvent

- For any $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, the truncated resolvent $\chi R(z)\chi$ is well-defined as an operator on $L^2(\mathbb{R}^n)$, and the study of $R(z)$ can be reduced to that of its (suitably) truncated version.
- For $\chi \in C_0^\infty(\mathbb{R}^n)$, the cut-off resolvent

$$\chi(P-z)^{-1}\chi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

has a meromorphic extension to $\Gamma_\theta$, and its poles are the resonances. Moreover,

$$\chi(P-z)^{-1}\chi = \chi(P_\theta - z)^{-1}\chi,$$

when $F = 0$ on $\operatorname{supp}\chi$.

# The trapped set

## Definition:

The trapped set at energy $E > 0$ is

$$K(E_0) = \left\{(x,\xi) \in p^{-1}(E_0);\ t \mapsto \exp(tH_p)(x,\xi) \text{ is bounded}\right\}$$

We recall that $p(x,\xi) = \xi^2 + V(x)$ is the semiclassical symbol of $P = \mathrm{Op}_h^w(p)$, and that $H_p$ is the associated Hamiltonian vector field on $T^*\mathbb{R}^n = \mathbb{R}^{2n}$, given by $H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi = \begin{pmatrix} 2\xi \\ -\nabla V(x) \end{pmatrix}$.

- Examples:
  - An empty trapped set.
  - A hyperbolic trajectory.
  - Le puits dans l'isle (shape resonances).
  - A barrier top.

# The non-trapping case

## Theorem (Martinez '02)

*Assume that $K(E_0) = \emptyset$. Then, $P$ has no resonance in*

$$\mathrm{Box}(\varepsilon, Ch \ln h) = [E_0 - \varepsilon, E_0 + \varepsilon] + i[-Ch|\ln h|, 0]$$

*for some $\varepsilon > 0$ and all $C > 0$. Moreover, for all $\chi \in C_0^\infty(\mathbb{R}^n)$, there exists $N = N(C)$ such that*

$$\|\chi(P - z)^{-1}\chi\| \lesssim h^{-N}$$

*in this set.*

Moreover, if $V$ is analytic in a whole neighborhood of $\mathbb{R}^n$ of the form

$$\Sigma = \{x \in \mathbb{C}^n;\ |\operatorname{Im} x| < \delta\langle \operatorname{Re} x\rangle\}$$

then Briet–Combes–Duclos '87 and Helffer–Sjöstrand '86 (implicitly) have proved that $P$ has no resonance in a fixed neighborhood of $E_0$.

# A hyperbolic closed trajectory

## Theorem (C.Gérard, Sjöstrand '87)

If $V$ is analytic in $\Sigma$, and $K(E_0)$ consists of a simple closed trajectory whose Poincaré map is hyperbolic, then, in $\mathrm{Box}\,(\varepsilon, Ch)$, there is a bijection $b_h$ between $\mathrm{Res}\,(P)$ (counted with their multiplicity) and the set of roots of the equation

$$A(E) = 2k\pi h + ih\log\rho(E) - ih\sum_{j=1}^{n-1}\alpha_j \log\theta_j(E), \quad k\in\mathbb{Z},\ \ \alpha_j\in\mathbb{N}$$

such that $b_h(E) = E + o(h)$. Here $A(E)$ is the action, $\theta_1,\dots,\theta_{n-1}$ are the eigenvalues of the Poincaré map with modulus $> 1$ and $\rho(E)$ is some analytic function of $E$ satisfying

$$|\rho(E)| = |\theta_1(E)\cdots\theta_{n-1}(E)|^{-\frac{1}{2}}.$$

# A non-degenerate critical point

## Theorem (Sjöstrand '87)

Suppose $V$ is analytic in $\Sigma$, and $K(E_0)$ consists of a non-degenerate critical point $(0,0)$. Denote

$$V(x) = E_0 - \sum_{j=1}^{d} \frac{\lambda_j^2}{4} x_j^2 + \sum_{j=d+1}^{n} \frac{\mu_j^2}{4} x_j^2 + \mathcal{O}(|x|^3) \quad \text{as } x \to 0$$

with $E_0 > 0$, $\lambda_j, \mu_j > 0$, and

$$E_{\alpha,\beta} = E_0 + h \sum_{j=d+1}^{n} \mu_j(\beta_j + \frac{1}{2}) - ih \sum_{j=1}^{d} \lambda_j(\alpha_j + \frac{1}{2})$$

for each $(\alpha, \beta) = (\alpha_1, \ldots, \alpha_d, \beta_{d+1}, \ldots, \beta_n) \in \mathbb{N}^n$. Then for any $C > 0$, there exists a bijection $b_h$ in $\mathrm{Box}\,(Ch, Ch)$ between $\mathrm{Res}\,(P)$ and the set of pseudo-resonances such that $b_h(E) = E + \mathcal{O}(h^{3/2})$.

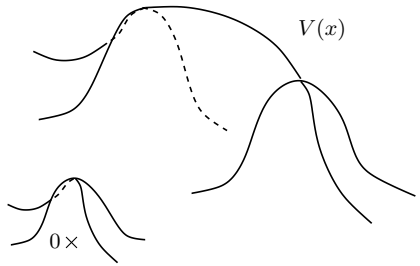
- The case $d = n$ (a barrier top) was also studied by Briet, Combes and Duclos under a virial condition. We denote $\Gamma(h)$ the set of the $E_{\alpha,\beta}$ in that case.
- In the case $d = 0$ (a well in an island), much more precise results have been proved. The imaginary part of resonances is exponentially small and the decay rate is given by the Agmon distance from the well to the sea.

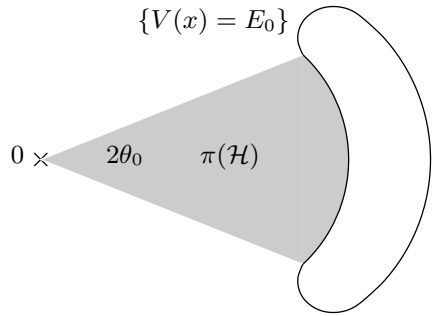
# The potential

- We assume that $V$ has a local non-degenerate maximum at 0

$$V(x) = E_0 - \sum_{j=1}^{n} \frac{\lambda_j^2}{4} x_j^2 + \mathcal{O}(x^3)$$

with $0 < \lambda_1 \leq \cdots \leq \lambda_n$.




- We set

$$F_p = d_{(0,0)} H_p = \begin{pmatrix} 0 & 2Id \\ \frac{1}{2}\,\mathrm{diag}(\lambda_1^2, \cdots, \lambda_n^2) & 0 \end{pmatrix}$$

Notice that $\sigma(F_p) = \{-\lambda_n, \cdots, -\lambda_1, \lambda_1, \cdots, \lambda_n\}$.

# Stable/unstable manifolds

- We denote $\Lambda_-$ and $\Lambda_+$ the stable incoming and outgoing manifold at $(0,0)$ respectively:

$$\Lambda_\pm = \left\{(x,\xi);\ \exp(tH_p)(x,\xi) \to 0 \text{ as } t \to \mp\infty\right\}$$

- $\Lambda_-$ and $\Lambda_+$ are smooth, Lagrangian manifolds, stable under the $H_p$ flow, given near $(0,0)$ by

$$\Lambda_\pm = \left\{(x,\xi);\ \xi = \nabla\varphi_\pm(x)\right\} \quad \text{with} \quad \varphi_\pm(x) = \pm\sum_{j=1}^{n}\frac{\lambda_j}{4}x_j^2 + \mathcal{O}(x^3)$$

- For $\rho_\pm \in \Lambda_\pm$, one can see that

$$\Pi_x \exp(tH_p)(\rho_\pm) = g_\pm(\rho_\pm)e^{\pm\lambda_1 t} + \mathcal{O}\left(e^{\pm(\lambda_1+\varepsilon)t}\right) \quad \text{as } t \to \mp\infty.$$

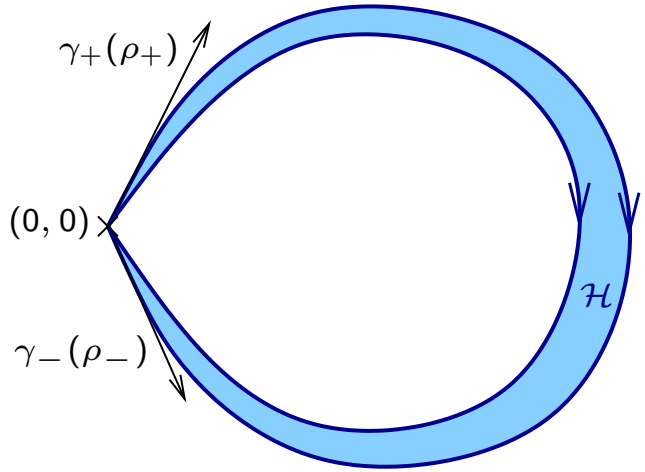
# Assumptions

**(A1)** The trapped set at energy $E_0$ is

$$K(E_0) = \{(0,0)\} \cup \mathcal{H}, \text{ with } \mathcal{H} = \Lambda_- \cap \Lambda_+ \setminus \{(0,0)\} \neq \emptyset.$$

$\mathcal{H}$ is the set of the homoclinic curves.

**(A2)** $\forall \rho_-, \rho_+ \in \mathcal{H},\ g_-(\rho_-) \cdot g_+(\rho_+) \neq 0.$



# Assumptions and notations

- We focus here on the particular case where

**(A3)** $\mathcal{H} = \bigcup_{j=1}^{J} \gamma_j$, where $\gamma_j$ are homoclinic trajectories

**(A4)** The intersection $\Lambda_+ \cap \Lambda_-$ is transverse along each of the $\gamma_j$'s.

- For $j \in \{1, \dots, J\}$, we denote:
  - $g_j^{\pm}$ the asymptotic directions of the trajectory $\gamma_j$,
  - $A_j = \int_{\gamma_j} \xi \cdot dx$ the classical action along $\gamma_j$,
  - $\nu_j$ the Maslov index along $\gamma_j$.
- Morever, we set

$$E - E_0 = hz, \quad L := \frac{1}{2} \sum_{j=2}^{n} \lambda_j \quad \text{and } \zeta = \zeta(z) = \frac{1}{\lambda_1}(L - iz).$$

# The main result

- Consider the $J \times J$ matrix $\mathcal{M} = \mathcal{M}(\zeta, h) = (m_{jk})_{j,k}$ where

$$m_{jk}(\zeta) = K_{jk} e^{iA_j/h} \frac{\Gamma(\zeta + \frac{1}{2})}{(i\lambda_1 g_+^j \cdot g_-^k)^{\zeta+\frac{1}{2}}},$$

$$K_{jk} = \sqrt{\frac{2\pi}{\lambda_1}} e^{-\frac{\pi}{2}(\nu_j+\frac{1}{2})i} |g_-^k| \lim_{t\to-\infty} \frac{\sqrt{D_j(t)}}{e^{(\frac{\lambda_1}{2}+L)t}} \lim_{t\to+\infty} \frac{e^{(\frac{\lambda_1}{2}-L)t}}{\sqrt{D_k(t)}}.$$

## Theorem (Bony, Fujiie, R., Zerzeri)

Let $\mu_1, \ldots, \mu_J$ be the eigenvalues of $\mathcal{M}$, and

$$\Psi\mathrm{Res}\,(P) = \{E \in \mathbb{C}; \exists j \in \{1, \ldots, J\} \text{ s.t. } h^\zeta \mu_j(\zeta, h) = 1\}.$$

For any small $\epsilon$, we have, in $B_1(\epsilon) = \mathrm{Box}\,(Ch, (\Lambda + \lambda_1 - \epsilon)h) \backslash \{\Gamma(h) + D(\epsilon h)\}$,

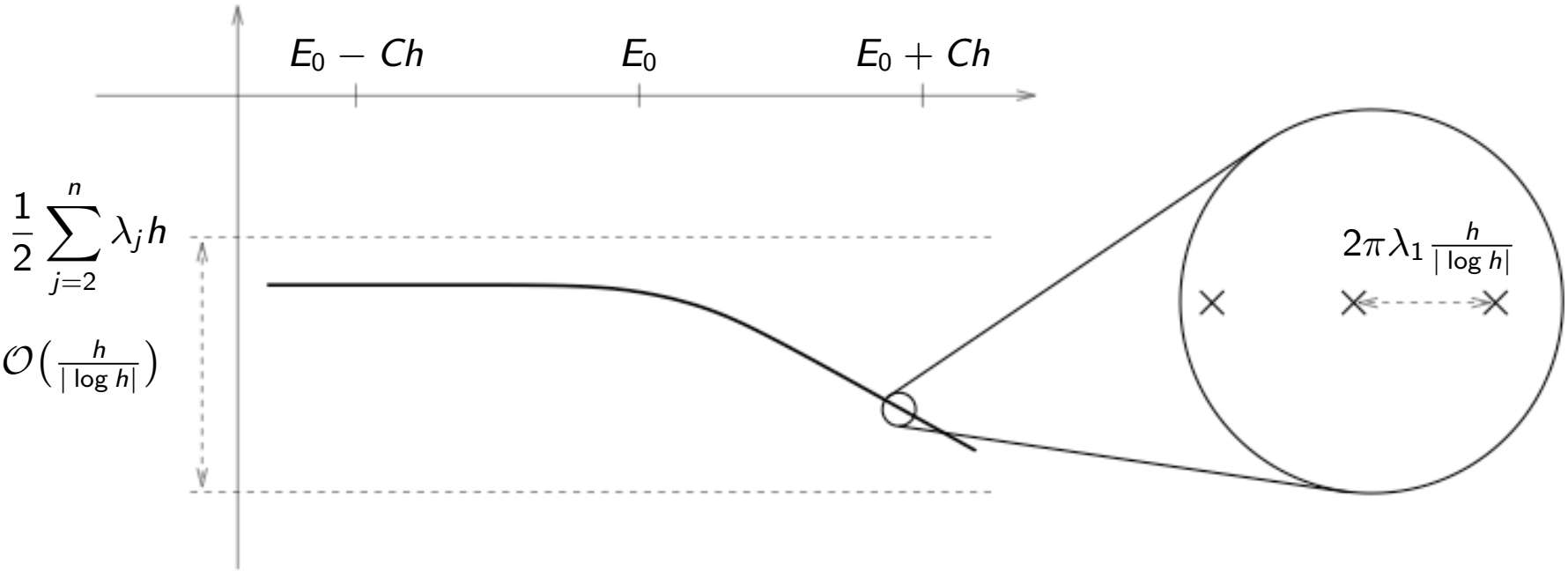
$$\mathrm{dist}(\mathrm{Res}(P), \Psi\mathrm{Res}(P)) = o\left(\frac{h}{|\log h|}\right).$$

# The case of 1 homoclinic cure

- Let $\sigma \in \mathbb{R}$. The pseudo-resonances $E$, such that $\operatorname{Re} E = E_0 + \sigma h + o(h)$, are given by

$$E = E_0 + \lambda_1\Big(2k\pi - \frac{A}{h}\Big)\frac{h}{|\log h|} - i\bigg(Lh - \lambda_1 \log m\Big(-i\frac{\sigma}{\lambda_1}\Big)\frac{h}{|\log h|}\bigg) + o\Big(\frac{h}{|\log h|}\Big),$$

for some $k \in \mathbb{Z}$.



# Behavior of the resonances at the boundary of the box

- The behaviour of the "resonance curve" when $(\operatorname{Re} E - E_0)/h$ goes to $\pm\infty$ can easily be seen. On that curve one has

$$\operatorname{Im} E = -Lh + \frac{h}{|\log h|} \begin{cases} K - \pi \dfrac{\operatorname{Re} E - E_0}{h} + o(1) & \text{as } \dfrac{\operatorname{Re} E - E_0}{h} \to +\infty, \\ K + o(1) & \text{as } \dfrac{\operatorname{Re} E - E_0}{h} \to -\infty, \end{cases}$$

for some explicit constant $K$. Thus we see the transition from the geometric situation for energies $E$ below $E_0$ (which is generally trapping) to the geometric situation above $E_0$ (which is generally non-trapping).

# Two homoclinic curves

- Suppose $g_{\pm}^1 \cdot g_{\pm}^2 < 0$ (i.e. the homoclinic trajectories are on the opposite side).

– When $\sigma << -1$, $\mathcal{M}$ is almost diagonal, hence $|\mu_1| \sim |m_{11}|$, $|\mu_2| \sim |m_{22}|$, and the "resonance curve" consists of two lines on each of which the distance between two neighboring resonances is $2\pi h$.

– When $\sigma >> 1$, $\mathcal{M}$ is almost anti-diagonal and $|\mu_1| \sim |\mu_2| \sim \sqrt{|m_{12} m_{21}|}$, and the "resonance curve" consists of a unique line in which the distance between two neighboring resonances is $\pi h$.

# The general case

## Proposition

Let $\sigma \in \mathbb{R}$. The pseudo-resonances $E$, such that $\operatorname{Re} E = E_0 + \sigma h + o(h)$, are given by

$$E = E_0 + 2k\pi\lambda_1 \frac{h}{|\log h|} - iLh + i\lambda_1 \log \mu_j \frac{h}{|\log h|} + o\Big(\frac{h}{|\log h|}\Big),$$

for some $k \in \mathbb{Z}$.

# Strategy

- Our results rely heavily on the study of what we call a microlocal Cauchy problem, of the form

$$\begin{cases} (P-E)u = v & \text{microlocally in } \Omega, \\ u = u_- & \text{microlocally near } S_-, \end{cases}$$

where $\Omega$ is some neighborhood of the trapped set, and $S_-$ is a suitable hypersurface in the incoming region of $\Omega$.

  - Our main resolvent estimate (and therefore the existence of a resonance free region) in some region of the complex plane of energies follows from a uniqueness result.
  - The existence, and precise estimates on resonances follows from the existence of solution to the inhomogeneous microlocal Cauchy problem.

- We obtain that way a Bohr-Sommerfeld like quantization condition for the resonances, which implies in particular that they accumulate on precisely defined curves as $h \to 0$.

# A resolvent estimate

- We start by proving a polynomial estimate for the resolvent, away from the pseudo-resonances.

## Proposition

$\forall C > 0$, $\forall \epsilon > 0$, $\exists M > 0$ such that if $E \in B_1(\epsilon) \setminus \left( \Psi\mathrm{Res}(P) + D(\frac{\epsilon h}{|\log h|}) \right)$, then, for $\theta = h \log(1/h)$, one has

$$(P_\theta - E)^{-1} = \mathcal{O}(h^{-M}).$$

- We prove this theorem by contradiction. We assume that there exist $\epsilon > 0$, $C > 0$, $E \in B_1(\epsilon) \setminus \{\Psi\mathrm{Res}(P) + D(\frac{\epsilon h}{|\log h|})\}$, $u \in L^2$ with $||u|| = 1$ satisfying

$$(P_\theta - E)u = \mathcal{O}(h^\infty), \tag{1}$$

and deduce $\|u\|| = \mathcal{O}(h^\infty)$, which is a contradiction.

# Step 0

It is enough to show that $u = 0$ microlocally on the trapped set.

## Definition

*Let $\rho \in \mathbb{R}^{2n}$. We say that $u = 0$ microlocally near $\rho$ iff*

$$\| \mathrm{Op}(\psi) u \| = \mathcal{O}(h^\infty)$$

*for some $\psi \in C_0^\infty(\mathbb{R}^{2n})$ with $\psi = 1$ near $\rho$.*

- Localization on the energy level $E_0$

Since $P_\theta - z \approx P - z$ is elliptic outside of the energy level $E_0$, we have

$$u = \varphi(P) u + \mathcal{O}_{H^2}(h^\infty)$$

for all $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi = 1$ near $E_0$.

- Spatial localization

Let $(x, \xi)$ close to the energy level $E_0$.

$$-\operatorname{Im} p_\theta(x,\xi) \approx \begin{cases} \sin(2\theta)\xi^2 \approx 2\theta E_0 \gtrsim \theta & \text{for } |x| > R \\ \mathcal{O}(\theta) \end{cases}$$

for $R \gg 1$. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi = 1$ for $|x| < R$. This implies

$$\mathcal{O}(h^\infty) = -\operatorname{Im}\langle (P_\theta - z)u, u\rangle \gtrsim \theta \|(1-\chi)u\|^2 - \mathcal{O}(\theta)\|\chi u\|^2$$

Then

$$\|(1-\chi)u\| \lesssim \|\chi u\| + \mathcal{O}(h^\infty)$$

- Therefore, it is enough to prove that

$$\|\varphi(P)\chi u\| = \mathcal{O}(h^\infty)$$

By a compactness argument, it is enough to prove that

$$u = 0 \text{ microlocally near } \rho$$

for all $\rho \in \operatorname{supp}(\varphi(p(x,\xi))\chi(x))$.

# Step 1

- Suppose $\rho \in \Lambda_- \setminus \Lambda_+$.

**Lemma**

$$\exp(tH_p)(\rho) \to \infty \text{ as } t \to -\infty \iff \rho \notin \Lambda_+$$

- Therefore, there exists $T > 0$ such that

$$\exp(-TH_p)(\rho) \in \Gamma^- = \left\{(x,\xi) \in p^{-1}(E_0);\ |x| \gg 1, \cos(x,\xi) = \frac{x \cdot \xi}{|x||\xi|} < -1/2\right\}$$

**Theorem (Bony–Michel '04)**

$$u = 0 \text{ microlocally near each } \rho \in \Gamma^-$$

- Standard propagation of singularities

If for some $T >> 1$,

$$\begin{cases} (P_\theta - z)u = \mathcal{O}(h^\infty) \\ u = 0 \text{ microlocally near } \exp(-TH_p)(\rho) \end{cases}$$

Then, by standard propagation of singularities, $u = 0$ microlocally near $\rho$. □

# Propagation through the critical point

## Theorem (BFRZ'07)

*For $\varepsilon \ll 1$, let $S_\varepsilon = \{(x,\xi) \in \Lambda_-;\ |x| = \varepsilon\}$ and $\rho_- \in S_\varepsilon$ be such that $g_-(\rho_-) \neq 0$. Let $u_-$, with $\|u_-\| \leq 1$, be such that*

$$\begin{cases} (P-z)u_- = 0 & \text{microlocally near } S_\varepsilon \\ u_- = 0 & \text{microlocally near each point of } S_\varepsilon \setminus \{\rho_-\} \end{cases}$$

*Then, the problem*

$$\begin{cases} (P-z)u = 0 & \text{microlocally near } (0,0) \\ u = u_- & \text{microlocally near } S_\varepsilon \end{cases}$$

*has a unique solution $u$ with $\|u\| \leq h^{-C}$.*

*Moreover, if $\rho_+ \in \Lambda_+$ satisfies $g_-(\rho_-) \cdot g_+(\rho_+) \neq 0$, then*

$$u = h^{\sum \frac{\lambda_j - \lambda_1}{2\lambda_1} - i\frac{z}{h\lambda_1}} \int e^{i(\varphi_+(x) - \varphi_-(y))/h} d(x,y;h) u_-(y)\, dy$$

*microlocally near $\rho_+$.*

# Step 2:

- On $\Lambda_-$, $u$ is supported only on $\gamma \cup (0,0)$, and hence, by (A4), we can apply the previous Theorem for $(x,\xi)$ near each $\rho_+ = (x_+,\xi_+) \in \Lambda_+ \cup \gamma$. In particular, $u$ is of the form

$$u(x,h) = a_+(x,h)e^{i\phi_+(x)/h} \quad \text{microlocally near } \rho_+,$$

with $a_+(x,h) \in S(h^{-c})$.

- The standard Maslov theory tells us that, after continuation along $\gamma$, $u$ is also of the form

$$u(x,h) = a_-(x,h)e^{i\tilde{\phi}_+(x)/h} \quad \text{microlocally near } \rho_- \tag{2}$$

with $a_-(x,h) \in S(h^{-c})$ for some $c \in \mathbb{R}$. Here $\tilde{\phi}_+$ is a generating function of the evolution of $\Lambda_+$ near $\rho_-$.

# Step 3

- Starting from (2), we continue $u$ one more tour along $(0,0) \cup \gamma$ using propagation through the critical point and the Maslov theory. Then we obtain another expression of $u$ microlocally near $(x_-, \xi_-)$:

$$u(x,h) = \tilde{a}_-(x,h) e^{i\tilde{\phi}_+(x)/h} \quad \text{microlocally near } \rho_-. \tag{3}$$

## Lemma

*At the principal level, the value $\tilde{a}_-(x,h)$ is determined by the value $a(x_-,h)$*

$$\tilde{a}_-(x,h) = \mathcal{M}(x,x_-;\zeta,h) a_-(x_-,h)(1+\mathcal{O}(h)),$$

*where $\mathcal{M}(x,x_-;\zeta,h) \in S(h^{\operatorname{Re}\zeta})$ is a symbol satisfying*

$$\mathcal{M}(x_-,x_-;\zeta,h) = h^{\zeta} m(\zeta) e^{iA/h}.$$

- We have, for each point $\rho' = (x', \xi') \in \Lambda_+$ with $g_+(x') \cdot g_-(x_-) \neq 0$,

$$u(x', h) = a_+(x', h) e^{i\phi_+(x)/h}, \quad a_+(x', h) = J(x', x_-; \zeta, h) a_-(x_-, h). \quad (4)$$

- On the other hand, by Maslov theory, $\tilde{a}(x, h)$ can be expressed by the value $a_+(x, h)$ when $\rho = (x, \xi) \in \widetilde{\Lambda}_+$ is connected by a classical trajectory with $\rho' = (x', \xi')$:

$$\tilde{a}_-(x, h) = e^{iA(\rho', \rho)/h} e^{-\frac{\pi}{2}\nu i} I_{\rho' \to \rho} a_+(x', h). \quad (5)$$

Here $I_{\rho' \to \rho}$ is a multiplication operator given by

$$I_{\rho' \to \rho} = \sqrt{\frac{D(t_{\rho'}, y')}{D(t_\rho, y')}}|_{y'=y_0'},$$

where $\rho = (x(t_\rho, y_0'), \xi(t_\rho, y_0'))$ and $\rho' = (x(t_{\rho'}, y_0'), \xi(t_{\rho'}, y_0'))$.
Combining (4) and (5), we obtain the lemma. Note that $\mathcal{M}(x_-, x_-)$ is independent of the choice of $x_-$.

# Step 4

We have obtained that, microlocally near $\rho_-$,

$$a_-(x,h) = \mathcal{M}(x,x_-;\zeta,h)a_-(x_-,h) + S(h^{\operatorname{Re}\zeta - c+1}) \tag{6}$$

At $x = x_-$, one has

$$a_-(x_-,h) = \frac{\mathcal{O}(h^{\operatorname{Re}\zeta - c+1})}{1-\mathcal{M}(x_-,x_-)} = \frac{\mathcal{O}(h^{-c+1})}{h^{-\zeta} - me^{iA/h}}$$

## Lemma

*$E \notin \Psi\operatorname{Res}(P) + D(\epsilon \frac{h}{|\log h|})$ implies $|h^{-\zeta} - me^{iA/h}| \geq \nu$ for an $h$-independent positive constant $\nu$.*

Hence

$$a_-(x_-,h) = \mathcal{O}(h^{-c+1}).$$

Substituting this into (6), one gets

$$a_-(x,h) \in S(h^{\operatorname{Re}\zeta - c+1}).$$

- Eventually, $\operatorname{Im} E > (-L - \lambda_1 + \epsilon)h$ implies $\operatorname{Re} \zeta > -1 + \frac{\epsilon}{\lambda_1}$. Hence

$$a_-(x,h) \in S(h^{-c+\frac{\epsilon}{\lambda_1}}).$$

Thus we obtain $a_- \in S(h^{-c+\frac{\epsilon}{\lambda_1}})$ from the assumption $a_- \in S(h^{-c})$. We conclude by bootstrap that

$$a_-(x,h) \in S(h^{\infty}),$$

which is the contradiction we were looking for.

# Existence of resonances

- We prove the existence of a resonance close to each pseudo-resonance also by a contradiction argument. We assume that there exist $\varepsilon > 0$ and $\widetilde{E} \in \Psi\mathrm{Res}(P) \cap B_1(C, \varepsilon)$ such that there is no resonance in the interior of the circle $\mathcal{C} := \{z \in \mathbb{C};\ |z - \widetilde{E}| = \frac{\varepsilon h}{|\log h|}\}$.
- For $E \in \mathcal{C}$, we consider the inhomogeneous equation

$$(P_\theta - E)u = v, \tag{7}$$

where $v$ is suitably chosen.

# Construction of $v$

- Let $H_-$ be a hyperplane containing $x = x_-$ and transversal to $\Pi_x\gamma$, and let $\tilde{v}(x) = \tilde{\chi}(x,h)e^{i\tilde{\phi}_+(x)/h}$ be the WKB solution of the microlocal Cauchy problem

$$\begin{cases} (P-E)\tilde{v} = 0, \\ \tilde{v} = \chi(x)e^{i\tilde{\phi}_+(x)/h} \quad \text{on } H_-, \end{cases}$$

where $\chi \in C_0^\infty$ and $\chi = 1$ near $x_-$.

- We set

$$v(x) = \psi[P,\varphi]\tilde{v},$$

where $\psi$ and $\varphi$ are smooth cutoff functions with small compact support such that $\varphi$ is equal to 1 in a neighborhood of $x_-$, and that $\psi$ is equal to 1 on $\operatorname{supp}[P,\varphi]\tilde{v} \cap \gamma_+$, where $\gamma_+ = \bigcup_{t<0} \exp tH_p(\rho_-)$. Obviously, $v$ is holomorphic in $E$ near $E_0$.

- The solution $u = (P_\theta - E)^{-1} v$ of (7) is well-defined and satisfies $\|u\| = \mathcal{O}(h^{-c})$. Moreover, working as in (2), one can prove that $u$ is of the form

$$u = a_-(x,h) e^{i\tilde{\phi}_+(x)/h}, \quad a_- \in S(h^{-c}), \tag{8}$$

microlocally near $x_-$.

- Thanks to (7), $(P - z)u = 0$ microlocally near $(0,0)$. Moreover, on $\Lambda_-$, $u$ is supported microlocally on $\gamma$. Thus $u$ is of the form

$$u_+(x',h) = a_+(x',h) e^{i\phi_+(x)/h},$$

with

$$a_+(x',h) = J(x',x_-;\zeta,h) a_-(x_-,h) + S(h^{\mathrm{Re}\,\zeta - c + 1 - \mu}), \tag{9}$$

microlocally on $\Lambda_+ \cap \gamma$.

- On the other hand, along $\gamma$, $u$ satisfies the following microlocal Cauchy problem:

$$\begin{cases} (P-E)u = v & \text{microlocally along } \gamma, \\ u = u_+ & \text{microlocally near } S^\varepsilon_+ \cap \gamma. \end{cases}$$

The solution of this linear problem can be written as the sum $u_1 + u_2$ of the solutions of the microlocal Cauchy problems

$$\begin{cases} (P-E)u_1 = 0 & \text{along } \gamma, \\ u_1 = u_+ & \text{near } S^\varepsilon_+ \cap \gamma, \end{cases} \quad \text{and} \quad \begin{cases} (P-E)u_2 = v & \text{along } \gamma, \\ u_2 = 0 & \text{near } S^\varepsilon_+ \cap \gamma. \end{cases}$$

As in (5), we have

$$u_1(x,h) = \left( e^{iA/h} e^{-\frac{\pi}{2}\nu i} e^{izT} \mathcal{I}_{\rho' \to \rho}\ a_+(x',h) + S(h^{\operatorname{Re}\zeta - c + 1}) \right) e^{i\tilde{\phi}_+(x)/h}.$$

Moreover, $v$ has been constructed so that $u_2 = \varphi \tilde{v}$ microlocally near $\rho_-$.

- Summing up, we have obtained

$$a_-(x,h) = e^{iA/h} e^{-\frac{\pi}{2}\nu i} e^{izT} \mathcal{I}_{\rho' \to \rho} a_+(x',h) + \tilde{\chi}(x,h) + S(h^{\mathrm{Re}\,\zeta - c + 1 - \mu}), \tag{10}$$

near $x_-$. Combining (9) and (10), we obtain

$$a_-(x,h) = \mathcal{M}(x, x_-; \zeta, h) a_-(x_-,h) + \tilde{\chi}(x,h) + S(h^{\mathrm{Re}\,\zeta - c + 1 - \mu}). \tag{11}$$

and we can show that $a_- \in S(1)$.

- Thus (11) becomes

$$a_-(x) = \frac{h^{-\zeta}}{h^{-\zeta} - m e^{iA/h}} \tilde{\chi}(x,h) + S(h^{1-\mu}).$$

We integrate this identity with respect to $E$ along the contour $\mathcal{C}$, and we obtain

$$\int_{\mathcal{C}} a_-(x) dE = 2\pi\lambda_1 \frac{h}{\log h} \chi + S(h^{2-\mu}).$$

This is a contradiction because the left hand side should vanish if there is no resonance inside the contour $\mathcal{C}$.