Appendix A. The translation length of product of hyperbolic isometries of $\mathbb{R}$-trees

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As noticed by the first author of this appendix in the first version of this paper, Assertion (*ii*) of Proposition 1.6 (2) in [16] is incorrect. Explicit counter-examples are given after the proof of Proposition 3.5. This appendix serves as an erratum of the paper [16] where Proposition 1.6 (2)(*ii*) therein should be replaced by Assertion (2)(*ii*) of the following Proposition A.1. Except this replacement, the remainder of the paper [16] is unchanged.

The second author of this appendix is extremely grateful to the first one for finding the mistake and for fixing it.

We keep the notation of [16] in this appendix, in order to facilitate the checking process. In particular, if γ is an hyperbolic isometry of T, then $l(\gamma)$ is its translation length and A_γ is its translation axis. Most of the statements in the following result also follow from [1, Propositions 8.1, 8.3].

Proposition A.1. *Let γ, δ be two hyperbolic isometries of an $\mathbb{R}$-tree T.*

(1) *Assume that $A_\gamma \cap A_\delta = \emptyset$. Let D be the length of the connecting arc S between A_γ and A_δ. Then S is contained in the translation axis of $\gamma\delta$, and the isometry $\gamma\delta$ translates $S \cap A_\delta$ towards $S \cap A_\gamma$. We have*

$$l(\gamma\delta) = l(\gamma) + l(\delta) + 2D .$$

(2) *Assume that $A_\gamma \cap A_\delta \neq \emptyset$. Let $D \in [0, +\infty]$ be the length of the intersection $A_\gamma \cap A_\delta$, with $D = 0$ if this intersection is reduced to a point, and $D = \infty$ if this intersection is noncompact.*

- (*i*) *Either if $D > 0$ and the translation directions of γ and δ on $A_\gamma \cap A_\delta$ coincide, or if $D = 0$, then*

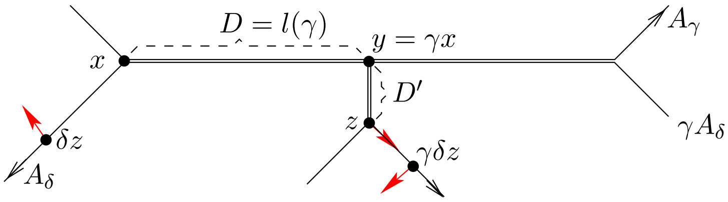
$$l(\gamma\delta) = l(\gamma) + l(\delta) .$$

- (*ii*) *Assume that $D > 0$ and that the translation directions of γ and δ are opposite on $A_\gamma \cap A_\delta$. Let $D' \in [0, +\infty]$ be the length of the (possibly empty or infinite) segment $A_\delta \cap \gamma A_\delta$ (resp. $A_\gamma \cap \delta A_\gamma$) if $l(\delta) > l(\gamma)$ (resp. $l(\delta) < l(\gamma)$), then*
  - $l(\gamma\delta) = l(\gamma) + l(\delta) - 2D$ *if* $\min\{l(\gamma), l(\delta)\} > D$,
  - $l(\gamma\delta) = |l(\gamma) - l(\delta)|$ *if* $\min\{l(\gamma), l(\delta)\} < D < \max\{l(\gamma), l(\delta)\}$ *or* $\max\{l(\gamma), l(\delta)\} \leq D$,
  - $l(\gamma\delta) = 0$ *if* $\min\{l(\gamma), l(\delta)\} = D < \max\{l(\gamma), l(\delta)\} \leq D + 2D'$,
  - $l(\gamma\delta) = \max\{l(\gamma), l(\delta)\} - D - 2D'$ *if* $\min\{l(\gamma), l(\delta)\} = D$ *and* $\max\{l(\gamma), l(\delta)\} > D + 2D'$.

  In all four cases, we have $l(\gamma\delta) < l(\gamma) + l(\delta)$.

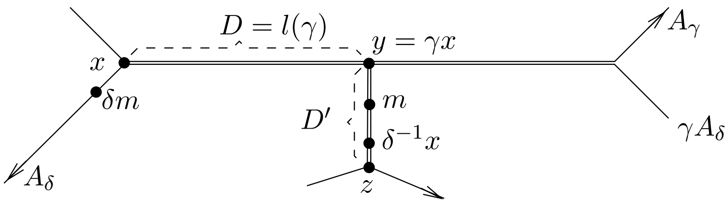
Proof. We may assume that $l(\gamma) \leq l(\delta)$. The proofs of Assertions (1) and (2)(*i*), as well as the first two cases of Assertion (2)(*ii*), are the same ones as in [16], see also [1, Propositions 8.1, 8.3].

Hence we assume that $l(\gamma) = D < l(\delta)$. In particular D is finite and nonzero, and $A_\gamma \cap A_\delta$ is a compact segment which may be written $[x, y]$ with $y = \gamma x$. We denote by z the point in T such that $[y, z] = \gamma A_\delta \cap A_\delta$, if this segment is compact, or the point at infinity of T such that $[y, z[= \gamma A_\delta \cap A_\delta$ otherwise.

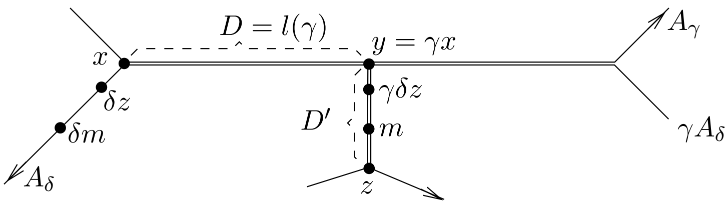


Assume first that $l(\delta) > D + 2D'$, so that in particular D' is finite, $z \in T$ and $D' = d(y, z)$. See the above picture. Since $l(\delta) > D + D'$, the point x belongs to $[z, \delta z]$ and besides $d(x, \delta z) = \ell(\delta) - D - D' > D'$. Therefore $\gamma\delta z$ does not belong to A_δ. The germ at z of the segment from z to $\gamma\delta z$ is hence not sent to the germ at $\gamma\delta z$ of the segment from $\gamma\delta z$ to z. Thus, as wanted,

$$l(\gamma\delta) = d(z, \gamma\delta z) = d(\gamma\delta z, y) - d(y, z) = d(\delta z, x) - d(y, z) = \ell(\delta) - D - 2D'.$$



Assume now that $l(\delta) \leq D + D'$. See the above picture. Note that $\delta^{-1}x$ does not belong to A_γ since $l(\delta) > D$, and that $d(\delta^{-1}x, y) = l(\delta) - D \leq D'$. Let m be the midpoint of the segment $[y, \delta^{-1}x]$, so that $d(\delta m, x) = d(m, \delta^{-1}x) = d(m, y)$. Hence $\gamma\delta m$, which is the point of $[y, z]$ (or $[y, z[$ if $D' = +\infty$) at distance $d(\delta m, x)$ from y, is equal to m and $l(\gamma\delta) = 0$, as wanted.



Assume finally that $D + D' < l(\delta) \leq D + 2D'$. See the above picture. In particular D' is finite, $z \in T$ and $D' = d(y, z)$. Note that δz does not belong to A_γ since $l(\delta) > D + D'$, and that

$$d(\delta z, x) = d(\delta z, z) - d(z, y) - d(y, x) = l(\delta) - D - D' \leq D' \ .$$

Hence $\gamma\delta z \in [y, z]$ and $d(\gamma\delta z, y) = d(\delta z, x) = l(\delta) - D - D'$, so that

$$d(\gamma\delta z, z) = d(z, y) - d(\gamma\delta z, y) = D' - (l(\delta) - D - D') = D + 2D' - l(\delta) \ .$$

Let m be the midpoint of the segment $[\gamma\delta z, z]$, so that $d(m, z) = \frac{1}{2}(D + 2D' - l(\delta))$. Hence

$$d(y, m) = d(y, z) - d(z, m) = \frac{1}{2}(l(\delta) - D) \ .$$

But since m belongs to A_δ and comes after z on A_δ oriented by the translation direction of δ, we have

$$d(\delta m, x) = d(\delta m, \delta z) + d(\delta z, x) = \frac{1}{2}\big(D + 2D' - l(\delta)\big) + \big(l(\delta) - D - D'\big)$$
$$= \frac{1}{2}(l(\delta) - D) = d(y, m) \leq D' \ .$$

Hence $\gamma\delta m$, which is the point of $[y, z]$ at distance $d(\delta m, x)$ from y, is equal to m and $l(\gamma\delta) = 0$, as wanted. □

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