Sectorial Mertens and Mirsky formulae for imaginary quadratic number fields

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Abstract

We extend formulae of Mertens and Mirsky on the asymptotic behaviour of the standard Euler function to the Euler functions of principal rings of integers of imaginary quadratic number fields, giving versions in angular sectors and with congruences. [1]

1 Introduction

Let K be a number field of degree n_K, with ring of integers $\mathcal{O}_K$, number of real places r_1, number of complex conjugated places r_2, regulator R_K, class number h_K, number of units ω_K, discriminant D_K and Dedekind zeta function ζ_K (see for instance [Nar]). Let $\mathscr{I}_K^+$ be the semigroup of nonzero ideals of $\mathcal{O}_K$, let $\varphi_K : \mathscr{I}_K^+ \to \mathbb{N}$ be the Euler function of K, and let $\mathtt{N} : \mathscr{I}_K^+ \to \mathbb{N}$ be the norm, with $\varphi_K(a) = \varphi_K(a\mathcal{O}_K)$ and $\mathtt{N}(a) = \mathtt{N}(a\mathcal{O}_K)$ for every $a \in \mathcal{O}_K - \{0\}$. As usual, $\mathfrak{p}$ below ranges over prime ideals in $\mathscr{I}_K^+$. The functions $\operatorname{O}(\cdot)$ below depend only on K.

Our first result (see Section 2) is a Mertens formula with congruences for number fields. Though probably well-known at least when $\mathfrak{m} = \mathcal{O}_K$, we provide a proof for lack of reference (compare with [Gro, Satz 2], [Cos, §4.3], [PP1, Theo. 3.1]) since arguments of its proof will be useful for our next result. For every $\mathfrak{m} \in \mathscr{I}_K^+$, let

$$c_{\mathfrak{m}} = \mathtt{N}(\mathfrak{m}) \prod_{\mathfrak{p}|\mathfrak{m}} \Big(1 + \frac{1}{\mathtt{N}(\mathfrak{p})}\Big) \;.$$

Theorem 1.1 *For every $\mathfrak{m} \in \mathscr{I}_K^+$, if $n_K \geqslant 2$, then as $x \to +\infty$, we have*

$$\sum_{\mathfrak{a} \in \mathscr{I}_K^+ :\, \mathtt{N}(\mathfrak{a}) \leqslant x,\ \mathfrak{m}|\mathfrak{a}} \varphi_K(\mathfrak{a}) = \frac{2^{r_1+r_2-1}\,\pi^{r_2}\,R_K\,h_K}{\omega_K\,\sqrt{|D_K|}\,\zeta_K(2)\,c_{\mathfrak{m}}}\,x^2 + \operatorname{O}\big(x^{2-\frac{1}{n_K}}\big) \;.$$

Assume in the remaining part of this introduction that K is imaginary quadratic and that $\mathcal{O}_K$ is principal. By Dirichlet's unit theorem, these assumptions are more or less necessary (besides $K = \mathbb{Q}$) for the following sums to be well defined and finite.

[1]**Keywords:** Euler function, imaginary quadratic number field, Mertens formula, Mirsky formula. **AMS codes:** 11R04, 11N37, 11R11.

We give in Section 3 a version in angular sectors of the Mertens formula given by Theorem 1.1, that will be needed in [PP3]. For all $z \in \mathbb{C}^\times$, $\theta \in \,]0, 2\pi]$ and $R \geqslant 0$, we consider the truncated angular sector

$$C(z, \theta, R) = \Big\{\rho\, e^{it} z : t \in \,]-\frac{\theta}{2}, \frac{\theta}{2}],\ 0 < \rho \leqslant \frac{R}{|z|}\Big\}\ . \tag{1}$$

It is important that the function $\operatorname{O}(\cdot)$ in the following result is uniform in $\mathfrak{m}$, z and θ.

Theorem 1.2 *Assume that K is imaginary quadratic with $\mathscr{O}_K$ principal. For all $\mathfrak{m} \in \mathscr{I}_K^+$, $z \in \mathbb{C}^\times$ and $\theta \in \,]0, 2\pi]$, as $x \to +\infty$, we have*

$$\sum_{a \in \mathfrak{m} \cap C(z,\theta,x)} \varphi_K(a) = \frac{\theta}{2\sqrt{|D_K|}\, \zeta_K(2)\, c_{\mathfrak{m}}}\, x^4 + \operatorname{O}(x^3)\ .$$

Lastly, we give a uniform asymptotic formula for the sum in angular sectors in $\mathbb{C}$ of angle θ of the products of two shifted Euler functions with congruences, that will be needed in [PP3]. When $K = \mathbb{Q}$ (the sectorial restriction is then meaningless), this formula is due to Mirsky [Mir, Thm. 9, Eq. (30)] without congruences, and to Fouvry [PP2, Appendix] with congruences. For simplicity, we give a version without congruences and without an error term in this introduction, see Section 4 Theorem 4.1 for the general statement.

Theorem 1.3 *For all $z \in \mathbb{C}^\times$, $\theta \in \,]0, 2\pi]$ and $k \in \mathscr{O}_K$, as $x \to +\infty$, we have*

$$\sum_{a \in \mathscr{O}_K \cap C(z,\theta,x)} \varphi_K(a)\, \varphi_K(a+k) \sim \frac{\theta}{3\sqrt{|D_K|}} \prod_{\mathfrak{p}} \big(1 - \frac{2}{\mathtt{N}(\mathfrak{p})^2}\big) \prod_{\mathfrak{p}\,|\,k\mathscr{O}_K} \big(1 + \frac{1}{\mathtt{N}(\mathfrak{p})(\mathtt{N}(\mathfrak{p})^2 - 2)}\big)\, x^6\ .$$

Theorems 1.2 and 1.3 are used in [PP3] in order to study the correlations of pairs of complex logarithms of $\mathbb{Z}$-lattice points in the complex line at various scalings, when the weights are defined by the Euler function, proving the existence of pair correlation functions. We prove in op. cit. that at the linear scaling, the pair correlations exhibit level repulsion, as it sometimes occurs in statistical physics. A geometric application is given in op. cit. to the pair correlation of the lengths of common perpendicular geodesic arcs from the maximal Margulis cusp neighborhood to itself in the Bianchi manifolds $\operatorname{PSL}_2(\mathscr{O}_K)\backslash\mathbb{H}^3_{\mathbb{R}}$.

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2 A Mertens formula with congruences for number fields

Recall that $\mathscr{I}_K^+$ is the semigroup of nonzero (integral) ideals of the Dedekind ring $\mathscr{O}_K$ (with unit $\mathscr{O}_K$). For all $I, J \in \mathscr{I}_K^+$, we write $J \mid I$ if $I \subset J$, we denote by $(I, J) = I + J$ the greatest common ideal divisor of I and J, by $[I, J] = I \cap J$ the least common ideal multiple of I and J, and by IJ the product ideal of I and J.

We denote by $\mathtt{N}(I) = \operatorname{Card}(\mathscr{O}_K/I)$ the (absolute) *norm* of $I \in \mathscr{I}_K^+$, which is completely multiplicative. The *norm* of $a \in \mathscr{O}_K - \{0\}$ is

$$\mathtt{N}(a) = \mathtt{N}(a\mathscr{O}_K)\ .$$

It coincides with the (relative) norm $N_{K/\mathbb{Q}}(a)$ of a (see for instance [Nar]), and in particular is equal to $|a|^2$ if K is imaginary quadratic.

Recall that the *Dedekind zeta function* $\zeta_K : \{s \in \mathbb{C} : \mathrm{Re}(s) > 1\} \to \mathbb{C}$ of K is defined (see for instance [Nar, §7.1]) equivalently by

$$\zeta_K(s) = \sum_{\mathfrak{a} \in \mathscr{I}_K^+} \frac{1}{\mathtt{N}(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})^s}\big)^{-1} .$$

We denote by $\varphi_K : \mathscr{I}_K^+ \to \mathbb{N}$ the *Euler function* of K, defined (see for instance [Nar, page 13]) equivalently by

$$\forall\ \mathfrak{a} \in \mathscr{I}_K^+, \quad \varphi_K(\mathfrak{a}) = \mathrm{Card}\big((\mathscr{O}_K/\mathfrak{a})^\times\big) = \mathtt{N}(\mathfrak{a}) \prod_{\mathfrak{p} | \mathfrak{a}} \big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})}\big) .$$

For every $a \in \mathscr{O}_K - \{0\}$, we define $\varphi_K(a) = \varphi_K(a\mathscr{O}_K)$. Note that the Euler function φ_K is multiplicative[2] by the Chinese remainder theorem. We have

$$\mathtt{N}(\mathfrak{a}) = \sum_{\mathfrak{b} | \mathfrak{a}} \varphi_K(\mathfrak{b}) , \tag{2}$$

as checked by telescopic sum when $\mathfrak{a}$ is a power of a prime ideal, and by multiplicativity.

We denote by $\mu_K : \mathscr{I}_K^+ \to \mathbb{Z}$ the *Möbius function* of K, defined by

$$\forall\ \mathfrak{a} \in \mathscr{I}_K^+, \quad \mu_K(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{a} = \mathscr{O}_K \\ 0 & \text{if } \mathfrak{p}^2 \mid \mathfrak{a} \text{ for some prime ideal } \mathfrak{p} \\ (-1)^m & \text{if } \mathfrak{a} = \mathfrak{p}_1 \dots \mathfrak{p}_m \text{ for pairwise distinct prime ideals} \\ & \mathfrak{p}_1, \dots, \mathfrak{p}_m \text{ and } m \in \mathbb{N} - \{0\} . \end{cases}$$

For every $a \in \mathscr{O}_K - \{0\}$, we define $\mu_K(a) = \mu_K(a\mathscr{O}_K)$. We have (see for instance [Sha]) the *Möbius inversion formula*: for all $f, g : \mathscr{I}_K^+ \to \mathbb{C}$,

$$f(\mathfrak{a}) = \sum_{\mathfrak{b} | \mathfrak{a}} g(\mathfrak{b}) \quad \text{if and only if} \quad g(\mathfrak{a}) = \sum_{\mathfrak{b} | \mathfrak{a}} \mu_K(\mathfrak{b}) f(\mathfrak{a}\mathfrak{b}^{-1}) . \tag{3}$$

In particular, since the norm is completely multiplicative and by Equation (2), we have

$$\forall\ \mathfrak{a} \in \mathscr{I}_K^+, \quad \frac{\varphi_K(\mathfrak{a})}{\mathtt{N}(\mathfrak{a})} = \sum_{\mathfrak{b} | \mathfrak{a}} \frac{\mu_K(\mathfrak{b})}{\mathtt{N}(\mathfrak{b})} . \tag{4}$$

Proof of Theorem 1.1. In this proof, all functions $\mathrm{O}(\cdot)$ depend only on K. Let

$$\rho_K = \frac{2^{r_1}\,(2\pi)^{r_2}\,R_K\,h_K}{\omega_K\,\sqrt{|D_K|}} . \tag{5}$$

Recall (see for instance [MO, Theo. 5]) that, as $x \to +\infty$, we have

$$\mathrm{Card}\{\mathfrak{a} \in \mathscr{I}_K^+ : \mathtt{N}(\mathfrak{a}) \leqslant x\} = \rho_K\, x + \mathrm{O}(x^{1-\frac{1}{n_K}}) . \tag{6}$$

By Abel's summation formula, as $y \to +\infty$, we have

$$\sum_{\mathfrak{a} \in \mathscr{I}_K^+ :\, \mathtt{N}(\mathfrak{a}) \leqslant y} \mathtt{N}(\mathfrak{a}) = \sum_{1 \leqslant n \leqslant y} n\,\mathrm{Card}\{\mathfrak{a} \in \mathscr{I}_K^+ : \mathtt{N}(\mathfrak{a}) = n\} = \frac{\rho_K}{2}\, y^2 + \mathrm{O}(y^{2-\frac{1}{n_K}}) . \tag{7}$$

[2] Recall that a function $f : \mathscr{I}_K^+ \to \mathbb{C}^\times$ is *multiplicative* if $f(\mathscr{O}_K) = 1$ and if for all coprime integral ideals $\mathfrak{a}, \mathfrak{b}$ in $\mathscr{I}_K^+$, we have $f(\mathfrak{a}\mathfrak{b}) = f(\mathfrak{a}) f(\mathfrak{b})$.

Furthermore, we have

$$\mathrm{Card}\{\mathfrak{a} \in \mathscr{I}_K^+ : \mathtt{N}(\mathfrak{a}) = y\} = \mathrm{O}(y^{1-\frac{1}{n_K}}) .$$

This formula implies since $\mathtt{N}((\mathfrak{b}, \mathfrak{m})) \leqslant \mathtt{N}(\mathfrak{m})$ that

$$\Big| \sum_{\mathfrak{b}\in\mathscr{I}_K^+ :\, \mathtt{N}(\mathfrak{b})\geqslant x} \mu_K(\mathfrak{b}) \frac{\mathtt{N}((b,\mathfrak{m}))}{\mathtt{N}(\mathfrak{b})^2} \Big| = \mathrm{O}\Big(\mathtt{N}(\mathfrak{m}) \sum_{n\geqslant x} \frac{n^{1-\frac{1}{n_K}}}{n^2}\Big) = \mathrm{O}(\mathtt{N}(\mathfrak{m})\, x^{-\frac{1}{n_K}}) . \tag{8}$$

Let us denote by $S_\mathfrak{m}(x)$ the sum on the left hand side in the statement of Theorem 1.1. Note that by the Gauss lemma, for all $\mathfrak{m}, \mathfrak{b}, \mathfrak{c} \in \mathscr{I}_K^+$, we have $\mathfrak{m} \mid \mathfrak{b}\mathfrak{c}$ if and only if $\mathfrak{m}(\mathfrak{m}, \mathfrak{b})^{-1} \mid \mathfrak{c}$. Then by Equation (4), by the change of variable $\mathfrak{c} = \mathfrak{m}(\mathfrak{m}, \mathfrak{b})^{-1}\mathfrak{a}$, by the complete multiplicativity of the norm, by Equation (7) with $y = \frac{\mathtt{N}((\mathfrak{b},\mathfrak{m}))x}{\mathtt{N}(\mathfrak{b})\,\mathtt{N}(\mathfrak{m})}$, since $\mathtt{N}((\mathfrak{b}, \mathfrak{m})) \leqslant \mathtt{N}(\mathfrak{m})$, and by Equation (8), we have

$$\begin{aligned}
S_\mathfrak{m}(x) &= \sum_{\mathfrak{a}\in\mathscr{I}_K^+ :\, \mathtt{N}(\mathfrak{a})\leqslant x,\ \mathfrak{m}|\mathfrak{a}} \ \sum_{\mathfrak{b},\mathfrak{c}\in\mathscr{I}_K^+ :\ \mathfrak{b}\mathfrak{c}=\mathfrak{a}} \mu_K(\mathfrak{b})\ \mathtt{N}(\mathfrak{c}) \\
&= \sum_{\mathfrak{b}\in\mathscr{I}_K^+ :\, \mathtt{N}(\mathfrak{b})\leqslant x} \mu_K(\mathfrak{b}) \sum_{\mathfrak{c}\in\mathscr{I}_K^+ :\, \mathtt{N}(\mathfrak{c})\leqslant \frac{x}{\mathtt{N}(\mathfrak{b})},\ \mathfrak{m}|\mathfrak{b}\mathfrak{c}} \mathtt{N}(\mathfrak{c}) \\
&= \sum_{\mathfrak{b}\in\mathscr{I}_K^+ :\, \mathtt{N}(\mathfrak{b})\leqslant x} \mu_K(\mathfrak{b}) \sum_{\mathfrak{a}\in\mathscr{I}_K^+ :\, \mathtt{N}(\mathfrak{a})\leqslant \frac{\mathtt{N}((\mathfrak{b},\mathfrak{m}))x}{\mathtt{N}(\mathfrak{b})\,\mathtt{N}(\mathfrak{m})}} \frac{\mathtt{N}(\mathfrak{m})}{\mathtt{N}((\mathfrak{b},\mathfrak{m}))}\ \mathtt{N}(\mathfrak{a}) \\
&= \Big(\sum_{\mathfrak{b}\in\mathscr{I}_K^+ :\, \mathtt{N}(\mathfrak{b})\leqslant x} \mu_K(\mathfrak{b}) \frac{\mathtt{N}((\mathfrak{b},\mathfrak{m}))}{\mathtt{N}(\mathfrak{b})^2}\Big) \frac{\rho_K}{2\ \mathtt{N}(\mathfrak{m})}\ x^2 + \mathrm{O}(\,x^{2-\frac{1}{n_K}}) \\
&= \Big(\sum_{\mathfrak{b}\in\mathscr{I}_K^+} \mu_K(\mathfrak{b}) \frac{\mathtt{N}((\mathfrak{b},\mathfrak{m}))}{\mathtt{N}(\mathfrak{b})^2}\Big) \frac{\rho_K}{2\ \mathtt{N}(\mathfrak{m})}\ x^2 + \mathrm{O}(\,x^{2-\frac{1}{n_K}}) .
\end{aligned} \tag{9}$$

By decomposing a nonzero integral ideal $\mathfrak{b}$ into powers of prime ideals, by the definition of the Möbius function, and by the Euler product formula for the Dedekind zeta function, we have

$$\sum_{\mathfrak{b}\in\mathscr{I}_K^+} \mu_K(\mathfrak{b}) \frac{\mathtt{N}((\mathfrak{b},\mathfrak{m}))}{\mathtt{N}(\mathfrak{b})^2} = \prod_{\mathfrak{p}\nmid\mathfrak{m}} (1 - \frac{1}{\mathtt{N}(\mathfrak{p})^2}) \prod_{\mathfrak{p}\,|\,\mathfrak{m}} (1 - \frac{1}{\mathtt{N}(\mathfrak{p})}) = \frac{1}{\zeta_K(2)} \prod_{\mathfrak{p}\,|\,\mathfrak{m}} \frac{\mathtt{N}(\mathfrak{p})}{1 + \mathtt{N}(\mathfrak{p})} .$$

Equations (9) and (5) hence imply Theorem 1.1. □

3 A sectorial Mertens formula

Assume in the remaining part of this paper that K is imaginary quadratic and that $\mathscr{O}_K$ is principal (or equivalently factorial (UFD)). By Dirichlet's unit theorem, the group of units $\mathscr{O}_K^\times$, whose order we denote by $|\mathscr{O}_K^\times|$, is finite if and only if (r_1, r_2) is equal to $(1, 0)$ or $(0, 1)$. This justifies our restriction, the case $K = \mathbb{Q}$ being well-known. With the notation of the beginning of the introduction, we then have (see for instance [Nar]) $D_K \in \{-4, -8, -3, -7, -11, -19, -43, -67, -163\}$, and

$$r_1 = 0, \quad r_2 = 1, \quad n_K = 2, \quad R_K = 1, \quad \omega_K = |\mathscr{O}_K^\times| \quad \text{and} \quad h_K = 1 . \tag{10}$$

Given a $\mathbb{Z}$-*lattice* $\vec{\Lambda}$ in the Euclidean space $\mathbb{C}$ (that is, a discrete (free abelian) subgroup of $(\mathbb{C}, +)$), we denote by $\mathrm{covol}_{\vec{\Lambda}} = \mathrm{Vol}(\mathbb{C}/\vec{\Lambda})$ the area of a fundamental parallelogram $\mathscr{F}_{\vec{\Lambda}}$ for $\vec{\Lambda}$ and by $\mathrm{diam}_{\vec{\Lambda}}$ the diameter of $\mathscr{F}_{\vec{\Lambda}}$. Note that every element $\mathfrak{m} \in \mathscr{I}_K^+$ is a $\mathbb{Z}$-lattice in $\mathbb{C}$ with

$$\mathrm{covol}_\mathfrak{m} = \mathtt{N}(\mathfrak{m})\, \mathrm{covol}_{\mathscr{O}_K} = \frac{\mathtt{N}(\mathfrak{m})\sqrt{|D_K|}}{2} \quad \text{and} \quad \mathrm{diam}_\mathfrak{m} = \mathrm{O}(\sqrt{|D_K|\ \mathtt{N}(\mathfrak{m})}\,) \tag{11}$$

since $\mathrm{diam}_{\mathscr{O}_K} = |1 + \frac{\sqrt{D_K}}{2}|$ if $D_K \equiv 0 \bmod 4$ and $\mathrm{diam}_{\mathscr{O}_K} = |\frac{3+\sqrt{D_K}}{2}|$ if $D_K \equiv 1 \bmod 4$.

With the notation of Equation (1), note that for every $z' \in \mathbb{C}^\times$, we have

$$z'C(z,\theta,R) = C(zz',\theta,R\,|z'|)\ . \tag{12}$$

Proof of Theorem 1.2. Let $z \in \mathbb{C}^\times$, $\theta \in \,]0,2\pi]$ and $y > 0$. Since $\operatorname{Area}(C(z,\theta,y)) = \frac{\theta}{2}\,y^2$, the standard Gauss counting argument, the finiteness of the number of imaginary quadratic number fields with class number 1, and the equality on the left of Formula (11) give

$$\begin{aligned}\operatorname{Card}\big(\mathscr{O}_K \cap C(z,\theta,y)\big) &= \frac{\operatorname{Area}(C(z,\theta,y))}{\operatorname{covol}_{\mathscr{O}_K}} + \operatorname{O}\big(\frac{\operatorname{diam}_{\mathscr{O}_K}\,y}{\operatorname{covol}_{\mathscr{O}_K}}\big)\\ &= \frac{\theta}{\sqrt{|D_K|}}\,y^2 + \operatorname{O}(y)\ .\end{aligned}$$

Since the map $z' \mapsto |z'|^2 = \mathtt{N}(z')$ takes only integral values on $\mathscr{O}_K$, by Abel's summation formula, as $y \to +\infty$, we have

$$\begin{aligned}\sum_{d\in\mathscr{O}_K\cap C(z,\theta,y)} |d|^2 &= \sum_{1\leqslant n\leqslant y^2} n\operatorname{Card}\{d\in\mathscr{O}_K\cap C(z,\theta,y) : |d|^2 = n\}\\ &= \frac{\theta}{2\sqrt{|D_K|}}\,y^4 + \operatorname{O}(y^3)\ .\end{aligned} \tag{13}$$

For all $x \geqslant 1$ and $\mathfrak{b} \in \mathscr{I}_K^+$, let us fix $b, m, (b,m) \in \mathscr{O}_K - \{0\}$ such that $\mathfrak{b} = b\mathscr{O}_K$, $\mathfrak{m} = m\mathscr{O}_K$ and $(\mathfrak{b},\mathfrak{m}) = (b,m)\mathscr{O}_K$. Since for every $c \in \mathscr{O}_K - \{0\}$ we have $m \mid bc$ if and only if $\frac{m}{(b,m)} \mid c$, by the change of variable $c = \frac{m}{(b,m)}\,d$, by Equation (12) and by Equation (13) applied with $y = \frac{x|(b,m)|}{|m|\,|b|}$, if

$$S_\mathfrak{b} = \sum_{\mathfrak{c}\in\mathscr{I}_K^+,\, a\in\mathfrak{m}\cap C(z,\theta,x)\,:\,\mathfrak{b}\mathfrak{c}=a\mathscr{O}_K} \mathtt{N}(\mathfrak{c})\ ,$$

we have

$$\begin{aligned}S_\mathfrak{b} &= \sum_{c\in\mathscr{O}_K-\{0\},\,a\in\mathfrak{m}\cap C(z,\theta,x)\,:\,bc=a} |c|^2\\ &= \sum_{c\in\mathscr{O}_K-\{0\}\,:\,bc\in C(z,\theta,x),\ m\mid bc} |c|^2 = \sum_{d\in\mathscr{O}_K-\{0\}\,:\,d\in C(\frac{z(b,m)}{m\,b},\,\theta,\,\frac{x|(b,m)|}{|m|\,|b|}),} \frac{\mathtt{N}(\mathfrak{m})}{\mathtt{N}((\mathfrak{b},\mathfrak{m}))}\,|d|^2\\ &= \frac{\theta\ \mathtt{N}((\mathfrak{b},\mathfrak{m}))}{2\sqrt{|D_K|}\ \mathtt{N}(\mathfrak{m})\ \mathtt{N}(\mathfrak{b})^2}\,x^4 + \operatorname{O}\Big(\frac{x^3}{\mathtt{N}(\mathfrak{b})^{3/2}}\Big)\ .\end{aligned}$$

Let us denote by $S_{\mathfrak{m},z,\theta}(x)$ the sum on the left hand side in the statement of Theorem 1.2. Then by Equation (4), we have

$$\begin{aligned}S_{\mathfrak{m},z,\theta}(x) &= \sum_{a\in\mathfrak{m}\cap C(z,\theta,x)} \varphi_K(a\mathscr{O}_K) = \sum_{a\in\mathfrak{m}\cap C(z,\theta,x)}\ \sum_{\mathfrak{b},\mathfrak{c}\in\mathscr{I}_K^+:\ \mathfrak{b}\mathfrak{c}=a\mathscr{O}_K} \mu_K(\mathfrak{b})\,\mathtt{N}(\mathfrak{c})\\ &= \sum_{\mathfrak{b}\in\mathscr{I}_K^+:\ \mathtt{N}(\mathfrak{b})\leqslant x^2} \mu_K(\mathfrak{b})\ S_\mathfrak{b}\\ &= \Big(\sum_{\mathfrak{b}\in\mathscr{I}_K^+:\ \mathtt{N}(\mathfrak{b})\leqslant x^2} \mu_K(\mathfrak{b})\ \frac{\mathtt{N}((\mathfrak{b},\mathfrak{m}))}{\mathtt{N}(\mathfrak{b})^2}\Big)\frac{\theta}{2\sqrt{|D_K|}\ \mathtt{N}(\mathfrak{m})}\,x^4 + \operatorname{O}(x^3)\ .\end{aligned}$$

The proof then proceeds exactly as in the proof of Theorem 1.1. □

4 A sectorial Mirsky formula

We now give a uniform asymptotic formula for the sum in angular sectors of the products of shifted Euler functions with congruences. For all $z \in \mathbb{C}^\times$, $\theta \in \,]0, 2\pi]$, $k \in \mathscr{O}_K$, $\mathfrak{m} \in \mathscr{I}_K^+$ and $x \geqslant 1$, let

$$S_{z,\theta,k,\mathfrak{m}}(x) = \sum_{a \in \mathfrak{m} \cap C(z,\theta,x)} \varphi_K(a)\, \varphi_K(a+k)\ . \tag{14}$$

Theorem 4.1 *Assume that K is imaginary quadratic with $\mathscr{O}_K$ principal. There exists a universal constant $C > 0$ such that for all $k \in \mathscr{O}_K$ and $\mathfrak{m} \in \mathscr{I}_K^+$, there exists $c_{\mathfrak{m},k} \in \,]0, 1]$ such that for all $z \in \mathbb{C}^\times$, $\theta \in \,]0, 2\pi]$ and $x \geqslant 1$, we have*

$$\Big|\, S_{z,\theta,k,\mathfrak{m}}(x) - \frac{\theta\, c_{\mathfrak{m},k}}{3\sqrt{|D_K|}}\, x^6 \,\Big| \leqslant C\big((1 + \sqrt{\mathtt{N}(k)}\,)\, x^5 + \mathtt{N}(k)\, x^4\big)\ .$$

We will prove Theorem 4.1 at the end of this Section after giving a number of Lemmas required for the proof. We fix $k \in \mathscr{O}_K$ and $\mathfrak{m} = m\mathscr{O}_K \in \mathscr{I}_K^+$, and we define $\mathfrak{h} = k\mathscr{O}_K$, which is a possibly zero integral ideal. We start by giving the first definition and a simpler formula for the constant $c_{\mathfrak{m},k}$ that appears in the statement of Theorem 4.1. We define

$$c_{\mathfrak{m},k} = \sum_{\substack{\mathfrak{b},\mathfrak{c} \in \mathscr{I}_K^+ \\ (\mathfrak{b},\mathfrak{c}) \,|\, \mathfrak{h},\ (\mathfrak{c}(\mathfrak{b},\mathfrak{m}),\mathfrak{m}(\mathfrak{b},\mathfrak{c})) \,|\, \mathfrak{h}\mathfrak{b}}} \mu_K(\mathfrak{b})\, \mu_K(\mathfrak{c})\, \frac{\mathtt{N}\big((\mathfrak{c}(\mathfrak{b},\mathfrak{m}),\ \mathfrak{m}(\mathfrak{b},\mathfrak{c}))\big)}{\mathtt{N}(\mathfrak{b})^2\ \mathtt{N}(\mathfrak{c})^2\ \mathtt{N}(\mathfrak{m})}\ , \tag{15}$$

and

$$c'_{\mathfrak{m}} = \inf_{k \in \mathscr{O}_K} c_{\mathfrak{m},k}\ .$$

Lemma 4.2 *The series in Equation* (15) *defining $c_{\mathfrak{m},k}$ converges absolutely. We have $c_{\mathfrak{m},k} \leqslant 1$ and $c'_{\mathfrak{m}} > 0$. Furthermore, we have*

$$c_{\mathfrak{m},k} = \frac{1}{\mathtt{N}(\mathfrak{m})} \prod_{\substack{\mathfrak{p} \\ (\mathfrak{p},\mathfrak{m}) \,|\, \mathfrak{h}}} \big(1 - \frac{\mathtt{N}((\mathfrak{p},\mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2}\big) \prod_{\mathfrak{p}} \big(1 - \frac{\kappa_{\mathfrak{m},\mathfrak{h}}(\mathfrak{p})\ \kappa'_{\mathfrak{h}}(\mathfrak{p})\ \mathtt{N}((\mathfrak{p},\mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2}\big)\ , \tag{16}$$

where

$$\kappa_{\mathfrak{m},\mathfrak{h}}(\mathfrak{p}) = \begin{cases} (1 - \frac{\mathtt{N}((\mathfrak{p},\mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2})^{-1} & \text{if} \quad (\mathfrak{p},\mathfrak{m}) \,|\, \mathfrak{h} \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad \kappa'_{\mathfrak{h}}(\mathfrak{p}) = \begin{cases} 1 - \frac{1}{\mathtt{N}(\mathfrak{p})} & \text{if} \quad \mathfrak{p} \,|\, \mathfrak{h} \\ 1 & \text{otherwise.} \end{cases} \tag{17}$$

In the special case $\mathfrak{m} = \mathscr{O}_K$, Equation (16) becomes

$$\begin{aligned} c_{\mathscr{O}_K,k} &= \prod_{\mathfrak{p}} \big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})^2}\big) \prod_{\mathfrak{p} \,|\, \mathfrak{h}} \Big(1 - \frac{(1 - \frac{1}{\mathtt{N}(\mathfrak{p})^2})^{-1}(1 - \frac{1}{\mathtt{N}(\mathfrak{p})})}{\mathtt{N}(\mathfrak{p})^2}\Big) \prod_{\mathfrak{p} \,\nmid\, \mathfrak{h}} \Big(1 - \frac{(1 - \frac{1}{\mathtt{N}(\mathfrak{p})^2})^{-1}}{\mathtt{N}(\mathfrak{p})^2}\Big) \\ &= \prod_{\mathfrak{p}} \big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})^2}\big) \prod_{\mathfrak{p} \,|\, \mathfrak{h}} \Big(1 - \frac{\mathtt{N}(\mathfrak{p}) - 1}{\mathtt{N}(\mathfrak{p})(\mathtt{N}(\mathfrak{p})^2 - 1)}\Big) \prod_{\mathfrak{p}} \big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})^2 - 1}\big) \prod_{\mathfrak{p} \,|\, \mathfrak{h}} \Big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})^2 - 1}\Big)^{-1} \\ &= \prod_{\mathfrak{p}} \big(1 - \frac{2}{\mathtt{N}(\mathfrak{p})^2}\big) \prod_{\mathfrak{p} \,|\, \mathfrak{h}} \big(1 + \frac{1}{\mathtt{N}(\mathfrak{p})(\mathtt{N}(\mathfrak{p})^2 - 2)}\big)\ . \end{aligned} \tag{18}$$

Theorem 1.3 in the introduction follows from Theorem 4.1 and the above computation.

Proof. Let us prove that uniformly in $x \geqslant 1$, we have

$$\sum_{\substack{\mathfrak{b},\mathfrak{c} \in \mathscr{I}_K^+ \,:\, \mathtt{N}(\mathfrak{b}) \geqslant x, \\ (\mathfrak{b},\mathfrak{c}) \,|\, \mathfrak{h},\ (\mathfrak{c}(\mathfrak{b},\mathfrak{m}),\mathfrak{m}(\mathfrak{b},\mathfrak{c})) \,|\, \mathfrak{h}\mathfrak{b}}} \frac{\mathtt{N}\big((\mathfrak{c}(\mathfrak{b},\mathfrak{m}),\ \mathfrak{m}(\mathfrak{b},\mathfrak{c}))\big)}{\mathtt{N}(\mathfrak{b})^2\ \mathtt{N}(\mathfrak{c})^2\ \mathtt{N}(\mathfrak{m})} = \mathrm{O}\big(\frac{1}{\sqrt{x}}\big)\ . \tag{19}$$

This implies, by taking $x = 1$, that the first claim of Lemma 4.2 is satisfied, since the Möbius function has values in $\{0, \pm 1\}$. Let us denote by $Z_{\mathfrak{m},\mathfrak{h}}(x)$ the above sum. Since $\mathtt{N}\big((\mathfrak{c}(\mathfrak{b},\mathfrak{m}),\ \mathfrak{m}(\mathfrak{b},\mathfrak{c}))\big) \leqslant \mathtt{N}(\mathfrak{m}(\mathfrak{b},\mathfrak{c}))$, we have

$$Z_{\mathfrak{m},\mathfrak{h}}(x) \leqslant \sum_{\mathfrak{b},\mathfrak{c}\in\mathscr{I}_K^+\,:\,\mathtt{N}(\mathfrak{b})\geqslant x} \frac{\mathtt{N}((\mathfrak{b},\mathfrak{c}))}{\mathtt{N}(\mathfrak{b})^2\ \mathtt{N}(\mathfrak{c})^2} \leqslant \sum_{\substack{\mathfrak{a},\mathfrak{b}',\mathfrak{c}'\in\mathscr{I}_K^+\\ \mathtt{N}(\mathfrak{b}')\geqslant x/\mathtt{N}(\mathfrak{a})}} \frac{\mathtt{N}(\mathfrak{a})}{\mathtt{N}(\mathfrak{a}\mathfrak{b}')^2\ \mathtt{N}(\mathfrak{a}\mathfrak{c}')^2}$$

$$= \sum_{\mathfrak{c}'\in\mathscr{I}_K^+} \frac{1}{\mathtt{N}(\mathfrak{c}')^2} \sum_{\mathfrak{a}\in\mathscr{I}_K^+} \frac{1}{\mathtt{N}(\mathfrak{a})^{5/2}} \sum_{\substack{\mathfrak{b}'\in\mathscr{I}_K^+\\ \mathtt{N}(\mathfrak{a})\,\mathtt{N}(\mathfrak{b}')\geqslant x}} \frac{1}{\mathtt{N}(\mathfrak{b}')^{3/2}\ (\mathtt{N}(\mathfrak{a})\,\mathtt{N}(\mathfrak{b}'))^{1/2}} \leqslant \zeta_K(2)\,\zeta_K(\frac{5}{2})\,\zeta_K(\frac{3}{2})\,\frac{1}{\sqrt{x}}\ .$$

Equation (19) follows, since there are only finitely many fields K satisfying the assumptions of Theorem 4.1.

The proof of Equation (16) that we now give is similar to Fouvry's proof of Equation (21) in [PP2, Appendix].

For every $\mathfrak{b} \in \mathscr{I}_K^+$, let $\chi_\mathfrak{b} : \mathscr{I}_K^+ \to \{0, 1\}$ be the characteristic function of the set of elements $\mathfrak{c} \in \mathscr{I}_K^+$ such that $(\mathfrak{c}, \mathfrak{b}) \mid \mathfrak{h}$. Let us define a map $\psi_\mathfrak{b} : \mathscr{I}_K^+ \to \mathscr{I}_K^+$ by

$$\psi_\mathfrak{b} : \mathfrak{c} \mapsto \big(\mathfrak{c},\, \frac{\mathfrak{m}}{(\mathfrak{b},\mathfrak{m})}\,(\mathfrak{b},\mathfrak{c})\big)\ . \tag{20}$$

Note that the assertion $(\mathfrak{c}(\mathfrak{b},\mathfrak{m}), \mathfrak{m}(\mathfrak{b},\mathfrak{c})) \mid \mathfrak{b}\,\mathfrak{h}$ is equivalent to the assertion

$$\psi_\mathfrak{b}(\mathfrak{c})\ \Big|\ \frac{\mathfrak{b}}{(\mathfrak{b},\mathfrak{m})}\,\mathfrak{h}\ .$$

For every $\mathfrak{b} \in \mathscr{I}_K^+$, let $\chi^*_\mathfrak{b} : \mathscr{I}_K^+ \to \{0, 1\}$ be the characteristic function of the set of elements $\mathfrak{c} \in \mathscr{I}_K^+$ such that the above divisibility assertion is satisfied. Let us finally define a map $C^* : \mathscr{I}_K^+ \to \mathbb{R}$ (which depends on $\mathfrak{m}$ and $\mathfrak{h}$) by

$$C^* : \mathfrak{b} \mapsto \sum_{\mathfrak{c}\in\mathscr{I}_K^+} \frac{\mu_K(\mathfrak{c})}{\mathtt{N}(\mathfrak{c})^2}\ \chi_\mathfrak{b}(\mathfrak{c})\ \chi^*_\mathfrak{b}(\mathfrak{c})\ \mathtt{N}(\psi_\mathfrak{b}(\mathfrak{c}))\ . \tag{21}$$

By the absolute convergence property, Equation (15) then becomes

$$c_{\mathfrak{m},k} = \frac{1}{\mathtt{N}(\mathfrak{m})} \sum_{\mathfrak{b}\in\mathscr{I}_K^+} \frac{\mu_K(\mathfrak{b})}{\mathtt{N}(\mathfrak{b})^2}\ \mathtt{N}((\mathfrak{b},\mathfrak{m}))\ C^*(\mathfrak{b})\ . \tag{22}$$

In order to transform the series $C^*(\mathfrak{b})$ defined by Formula (21) into an Eulerian product and in order to analyse it, we will use the following two lemmas.

Lemma 4.3 *For every $\mathfrak{b} \in \mathscr{I}_K^+$, the maps $\chi_\mathfrak{b}$, $\chi^*_\mathfrak{b}$ and $\psi_\mathfrak{b}$ on $\mathscr{I}_K^+$ are multiplicative.*

Proof. We have $\psi_\mathfrak{b}(\mathscr{O}_K) = \mathscr{O}_K$ and $\chi_\mathfrak{b}(\mathscr{O}_K) = \chi^*_\mathfrak{b}(\mathscr{O}_K) = 1$. Let $I, J \in \mathscr{I}_K^+$ be coprime.

The equality $(IJ, \mathfrak{b}) = (I, \mathfrak{b})(J, \mathfrak{b})$ and the fact that $(I, \mathfrak{b})$ and $(J, \mathfrak{b})$ are coprime imply that $\chi_\mathfrak{b}(IJ) = \chi_\mathfrak{b}(I)\chi_\mathfrak{b}(J)$.

In order to prove the multiplicativity of the map $\psi_\mathfrak{b}$, we write

$$\psi_\mathfrak{b}(IJ) = \big(IJ,\, \frac{\mathfrak{m}}{(\mathfrak{b},\mathfrak{m})}\,(\mathfrak{b}, IJ)\big) = \big(I,\, \frac{\mathfrak{m}}{(\mathfrak{b},\mathfrak{m})}\,(I,\mathfrak{b})(J,\mathfrak{b})\big)\big(J,\, \frac{\mathfrak{m}}{(\mathfrak{b},\mathfrak{m})}\,(I,\mathfrak{b})(J,\mathfrak{b})\big)\ .$$

Since I is coprime to $(J, \mathfrak{b})$ and since J is coprime to $(I, \mathfrak{b})$, we obtain as wanted the equality $\psi_\mathfrak{b}(IJ) = \psi_\mathfrak{b}(I)\,\psi_\mathfrak{b}(J)$.

Finally, the multiplicativity property $\chi^*_\mathfrak{b}(IJ) = \chi^*_\mathfrak{b}(I)\chi^*_\mathfrak{b}(J)$ of the function $\chi^*_\mathfrak{b}$ is a consequence of the multiplicativity of the map $\psi_\mathfrak{b}$ and of the fact that $\psi_\mathfrak{b}(I)$ and $\psi_\mathfrak{b}(J)$ are coprime. □

Lemma 4.4 *For every prime ideal* $\mathfrak{p}$ *and every* $\mathfrak{b} \in \mathscr{I}_K^+$*, we have*

$$\psi_{\mathfrak{b}}(\mathfrak{p}) = \begin{cases} \mathfrak{p} & \text{if} \quad \mathfrak{p} \mid \mathfrak{b}, \\ (\mathfrak{p}, \mathfrak{m}) & \text{otherwise}, \end{cases}$$

and

$$\chi_{\mathfrak{b}}(\mathfrak{p})\ \chi_{\mathfrak{b}}^*(\mathfrak{p}) = 1 \Leftrightarrow \begin{cases} \mathfrak{p} \mid (\mathfrak{b}, \mathfrak{h}) \\ \text{or} \\ \mathfrak{p} \nmid \mathfrak{b} \quad \text{and} \quad (\mathfrak{p}, \mathfrak{m}) \mid \mathfrak{h}\ . \end{cases}$$

Proof. The first formula follows from the definition of $\psi_{\mathfrak{b}}(\mathfrak{p})$ (see Formula (20)) by considering the three cases

- $\mathfrak{p} \mid \mathfrak{b}$,
- $\mathfrak{p} \nmid \mathfrak{b}$ and $\mathfrak{p} \mid \mathfrak{m}$, and
- $\mathfrak{p} \nmid \mathfrak{b}$ and $\mathfrak{p} \nmid \mathfrak{m}$.

The second formula follows from the first one, from the definitions of $\chi_{\mathfrak{b}}(\mathfrak{p})$ and $\chi_{\mathfrak{b}}^*(\mathfrak{p})$, and from the fact that $\chi_{\mathfrak{b}}(\mathfrak{p})\ \chi_{\mathfrak{b}}^*(\mathfrak{p}) = 1$ if and only if $\chi_{\mathfrak{b}}(\mathfrak{p}) = \chi_{\mathfrak{b}}^*(\mathfrak{p}) = 1$, by considering the two cases

- $\mathfrak{p} \mid \mathfrak{b}$ and
- $\mathfrak{p} \nmid \mathfrak{b}$. □

The arithmetic function $\mathfrak{c} \mapsto \mu_K(\mathfrak{c})\,\chi_{\mathfrak{b}}(\mathfrak{c})\ \chi_{\mathfrak{b}}^*(\mathfrak{c})\ \mathtt{N}(\psi_{\mathfrak{b}}(\mathfrak{c}))$ being multiplicative by Lemma 4.3 and the complete multiplicativity of the norm, and vanishing on the nontrivial powers of primes, the series defining $C^*(\mathfrak{b})$ in Formula (21) may be written as an Eulerian product

$$C^*(\mathfrak{b}) = \prod_{\mathfrak{p}} \big(1 - \frac{\chi_{\mathfrak{b}}(\mathfrak{p})\ \chi_{\mathfrak{b}}^*(\mathfrak{p})\ \mathtt{N}(\psi_{\mathfrak{b}}(\mathfrak{p}))}{\mathtt{N}(\mathfrak{p})^2}\big) = \prod_{\substack{\mathfrak{p} \\ \chi_{\mathfrak{b}}(\mathfrak{p})\ \chi_{\mathfrak{b}}^*(\mathfrak{p})=1}} \big(1 - \frac{\mathtt{N}(\psi_{\mathfrak{b}}(\mathfrak{p}))}{\mathtt{N}(\mathfrak{p})^2}\big)\ . \tag{23}$$

By Equations (22) and (23), and by Lemma 4.4, we have

$$c_{\mathfrak{m},k} = \frac{1}{\mathtt{N}(\mathfrak{m})} \sum_{\mathfrak{b} \in \mathscr{I}_K^+} \frac{\mu_K(\mathfrak{b})}{\mathtt{N}(\mathfrak{b})^2}\ \mathtt{N}((\mathfrak{b}, \mathfrak{m})) \prod_{\mathfrak{p} \nmid \mathfrak{b},\ (\mathfrak{p},\mathfrak{m}) \mid \mathfrak{h}} \big(1 - \frac{\mathtt{N}((\mathfrak{p}, \mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2}\big) \prod_{\mathfrak{p} \mid (\mathfrak{b},\mathfrak{h})} \big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})}\big)\ .$$

Let us define $\Gamma_{\mathfrak{m},\mathfrak{h}} = \prod_{\substack{\mathfrak{p} \\ (\mathfrak{p},\mathfrak{m}) \mid \mathfrak{h}}} \big(1 - \frac{\mathtt{N}((\mathfrak{p},\mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2}\big)$, so that

$$c_{\mathfrak{m},k} = \frac{\Gamma_{\mathfrak{m},\mathfrak{h}}}{\mathtt{N}(\mathfrak{m})} \sum_{\mathfrak{b} \in \mathscr{I}_K^+} \frac{\mu_K(\mathfrak{b})}{\mathtt{N}(\mathfrak{b})^2}\ \mathtt{N}((\mathfrak{b}, \mathfrak{m})) \prod_{\substack{\mathfrak{p} \\ \mathfrak{p} \mid \mathfrak{b},\ (\mathfrak{p},\mathfrak{m}) \mid \mathfrak{h}}} \big(1 - \frac{\mathtt{N}((\mathfrak{p}, \mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2}\big)^{-1} \prod_{\substack{\mathfrak{p} \\ \mathfrak{p} \mid (\mathfrak{b},\mathfrak{h})}} \big(1 - \frac{1}{\mathtt{N}(\mathfrak{p})}\big)\ .$$

This equation writes $c_{\mathfrak{m},k}$ as a series $\frac{\Gamma_{\mathfrak{m},\mathfrak{h}}}{\mathtt{N}(\mathfrak{m})} \sum_{\mathfrak{b} \in \mathscr{I}_K^+} \frac{f(\mathfrak{b})}{N(\mathfrak{b})^2}$ where $f : \mathscr{I}_K^+ \to \mathbb{R}$ is a multiplicative function, which vanishes on the nontrivial powers of prime ideals. By Eulerian product, we have therefore proved Equation (16).

Let us now prove that $0 \leqslant c_{\mathfrak{m},k} \leqslant 1$. Note that for every prime ideal $\mathfrak{p}$, we have

$$1 \leqslant \kappa_{\mathfrak{m},\mathfrak{h}}(\mathfrak{p}) \leqslant 2 \quad \text{and} \quad \frac{1}{2} \leqslant \kappa'_{\mathfrak{h}}(\mathfrak{p}) \leqslant 1\ . \tag{24}$$

In particular all the factors of the two products over $\mathfrak{p}$ in Equation (16) belong to $[0, 1]$, hence $0 \leqslant c_{\mathfrak{m},k} \leqslant \frac{1}{\mathtt{N}(\mathfrak{m})} \leqslant 1$.

Let us finally prove that $c'_{\mathfrak{m}} > 0$. For every prime ideal $\mathfrak{p}$, let $w_{\mathfrak{p}} = \frac{\kappa_{\mathfrak{m},\mathfrak{h}}(\mathfrak{p})\ \kappa'_{\mathfrak{h}}(\mathfrak{p})\ \mathtt{N}((\mathfrak{p},\mathfrak{m}))}{\mathtt{N}(\mathfrak{p})^2}$. By Formula (17), if $\mathtt{N}(\mathfrak{p}) = 2$, we have

$$w_{\mathfrak{p}} = \begin{cases} 1/2 & \text{if} \quad \mathfrak{p} \mid \mathfrak{h} \quad \text{and} \quad \mathfrak{p} \mid \mathfrak{m} \\ 1/6 & \text{if} \quad \mathfrak{p} \mid \mathfrak{h} \quad \text{and} \quad \mathfrak{p} \nmid \mathfrak{m} \\ 1/2 & \text{if} \quad \mathfrak{p} \nmid \mathfrak{h} \quad \text{and} \quad \mathfrak{p} \mid \mathfrak{m} \\ 1/3 & \text{if} \quad \mathfrak{p} \nmid \mathfrak{h} \quad \text{and} \quad \mathfrak{p} \nmid \mathfrak{m} \end{cases}$$

In particular $1-w_{\mathfrak p}\neq 0$ if $\mathtt{N}(\mathfrak p)=2$. From the inequalities (24) and by Equation (16), we have

$$c_{\mathfrak m,k}\geqslant \frac{1}{\mathtt{N}(\mathfrak m)}\prod_{\substack{\mathfrak p\\ (\mathfrak p,\mathfrak m)\,|\,\mathfrak h}}\big(1-\frac{\mathtt{N}((\mathfrak p,\mathfrak m))}{\mathtt{N}(\mathfrak p)^2}\big)\prod_{\mathfrak p\,:\,\mathtt{N}(\mathfrak p)\geqslant 3}\big(1-\frac{2\;\mathtt{N}((\mathfrak p,\mathfrak m))}{\mathtt{N}(\mathfrak p)^2}\big)\prod_{\mathfrak p\,:\,\mathtt{N}(\mathfrak p)=2}(1-w_{\mathfrak p})\;.$$

Since there are only finitely many primes ideals $\mathfrak p$ dividing $\mathfrak m$, the term on the right hand side is bounded from below by a positive constant $c'_{\mathfrak m}=\min_{k\in\mathscr O_K}c_{\mathfrak m,k}>0$. This concludes the proof of Lemma 4.2. □

Now that we understand the constant $c_{\mathfrak m,k}$, we continue towards the proof of Theorem 4.1 by giving an asymptotic formula for the sum

$$\widetilde S(x)=\sum_{a\in\mathfrak m\cap C(z,\theta,x)}\frac{\varphi_K(a)}{\mathtt{N}(a)}\;\frac{\varphi_K(a+k)}{\mathtt{N}(a+k)}\;. \tag{25}$$

Lemma 4.5 *Uniformly in $\mathfrak m\in\mathscr I_K^+$, $k\in\mathscr O_K$, $z\in\mathbb C^\times$, $\theta\in\,]0,2\pi]$ and $x\geqslant 1$, we have*

$$\widetilde S(x)=\frac{\theta\;c_{\mathfrak m,k}}{\sqrt{|D_K}}\;x^2+\mathrm O(x)\;. \tag{26}$$

Proof. For all nonzero elements a and b in the factorial ring $\mathscr O_K$, we denote by (a,b) any fixed choice of gcd of a and b, and by $[a,b]$ any fixed choice of lcm of a and b.

By Equation (4), for every $a\in\mathscr O_K-\{0\}$, we have

$$\frac{\varphi_K(a)}{\mathtt{N}(a)}=\frac{1}{|\mathscr O_K^\times|}\sum_{b\in\mathscr O_K-\{0\}\,:\,b\,|\,a}\frac{\mu_K(b)}{\mathtt{N}(b)}\;.$$

Let $x\geqslant 1$. Applying twice this equality, since $\mathtt{N}(b)\leqslant\mathtt{N}(a)$ when $b\mid a$, we have by Fubini's theorem

$$\begin{aligned}\widetilde S(x)&=\frac{1}{|\mathscr O_K^\times|^2}\sum_{a\in\mathfrak m\cap C(z,\theta,x)}\;\sum_{b\in\mathscr O_K-\{0\}\,:\,b\,|\,a}\frac{\mu_K(b)}{\mathtt{N}(b)}\sum_{c\in\mathscr O_K-\{0\}\,:\,c\,|\,a+k}\frac{\mu_K(c)}{\mathtt{N}(c)}\\&=\frac{1}{|\mathscr O_K^\times|^2}\sum_{b\in\mathscr O_K-\{0\}\,:\,|b|\leqslant x}\frac{\mu_K(b)}{\mathtt{N}(b)}\sum_{c\in\mathscr O_K-\{0\}}\frac{\mu_K(c)}{\mathtt{N}(c)}\sum_{\substack{a\in\mathfrak m\cap C(z,\theta,x)\\ b\,|\,a,\;c\,|\,a+k}}1\;.\end{aligned} \tag{27}$$

Let $b,c\in\mathscr O_K-\{0\}$. The system of three congruences $\left\{\begin{array}{l}a\equiv 0 \bmod m\\ a\equiv 0\bmod b\\ a\equiv -k\bmod c\end{array}\right.$ has a solution $a\in\mathscr O_K-\{0\}$ such that $|a|\leqslant x$ if and only if there exists an element $n\in\mathscr O_K-\{0\}$ such that $a=bn$, $|n|\leqslant\frac{x}{|b|}$ and

$$\left\{\begin{array}{l}bn\equiv 0\;\bmod m\\ bn\equiv -k\;\bmod c\;.\end{array}\right. \tag{28}$$

When $(b,c)\nmid k$, no solution exists.

Assume that $(b,c)\mid k$. Since $\frac{b}{(b,c)}$ is invertible modulo $\frac{c}{(b,c)}$, we denote by $\overline{\frac{b}{(b,c)}}$ a multiplicative inverse of $\frac{b}{(b,c)}$ modulo $\frac{c}{(b,c)}$. Then the system of congruences (28) is equivalent to

$$\left\{\begin{array}{l}\frac{b}{(b,m)}n\equiv 0\;\bmod\frac{m}{(b,m)}\\ \frac{b}{(b,c)}n\equiv-\frac{k}{(b,c)}\;\bmod\frac{c}{(b,c)}\end{array}\right.\Leftrightarrow\left\{\begin{array}{l}n\equiv 0\;\bmod\frac{m}{(b,m)}\\ n\equiv-\frac{k}{(b,c)}\,\overline{\frac{b}{(b,c)}}\;\bmod\frac{c}{(b,c)}\;.\end{array}\right. \tag{29}$$

Recall that a system of two congruences $\left\{\begin{array}{l}n\equiv\alpha_0\bmod\alpha\\ n\equiv\beta_0\bmod\beta\end{array}\right.$ with unknown $n\in\mathscr O_K$, where $\alpha,\beta,\alpha_0,\beta_0\in\mathscr O_K$ and $\alpha,\beta\neq 0$, has a solution if and only if $\alpha_0-\beta_0\equiv 0\bmod(\alpha,\beta)$. Furthermore, if this congruence condition is satisfied, that is, if there exists $n_0, m_0 \in \mathscr{O}_K$ such that $\alpha_0 - \beta_0 = \beta m_0 - \alpha n_0$, then n is a solution if and only if

$$n - \alpha_0 - \alpha n_0 \in \alpha\mathscr{O}_K \cap \beta\mathscr{O}_K = [\alpha, \beta]\mathscr{O}_K \ .$$

This is equivalent to asking n to belong to the translate $\Lambda_{\alpha,\beta,\alpha_0,\beta_0} = \alpha_0 + \alpha n_0 + \vec{\Lambda}_{\alpha,\beta}$ of the $\mathbb{Z}$-lattice $\vec{\Lambda}_{\alpha,\beta} = [\alpha, \beta]\mathscr{O}_K$.

Applying this with $\alpha = \frac{m}{(b,m)}$, $\beta = \frac{c}{(b,c)}$, $\alpha_0 = 0$ and $\beta_0 = -\frac{k}{(b,c)} \overline{\frac{b}{(b,c)}}$, since the elements $\frac{b}{(b,c)}$ and $\frac{b}{(b,m)}$ are both coprime with $\big(\frac{m}{(b,m)}, \frac{c}{(b,c)}\big)$, the system (29) has a solution if and only if the following divisibility condition holds

$$\Big(\frac{m}{(b,m)}, \frac{c}{(b,c)}\Big) \mid \frac{k}{(b,c)} \overline{\frac{b}{(b,c)}} \Leftrightarrow \Big(\frac{m}{(b,m)}, \frac{c}{(b,c)}\Big) \mid \frac{k}{(b,c)}$$
$$\Leftrightarrow \Big(\frac{m}{(b,m)}, \frac{c}{(b,c)}\Big) \mid \frac{k}{(b,c)} \frac{b}{(b,m)} \Leftrightarrow \big(m(b,c), c(b,m)\big) \mid k\, b \ .$$

Thus Equation (27) becomes, using Equation (12),

$$\widetilde{S}(x) = \frac{1}{|\mathscr{O}_K^\times|^2} \sum_{\substack{b,c\in\mathscr{O}_K-\{0\}\,:\,|b|\leqslant x \\ (b,c)\,|\,k,\ (m(b,c),c(b,m))\,|\,k\,b}} \frac{\mu_K(b)\,\mu_K(c)}{\mathtt{N}(b)\ \mathtt{N}(c)} \sum_{n\in\Lambda_{\alpha,\beta,\alpha_0,\beta_0}\cap C(b^{-1}z,\,\theta,\,x/|b|)} 1 \ .$$

Let b, c be as in the index of the first sum above. Using again the standard Gauss counting argument, using Formula (11) for the second equality and the equation $\mathtt{N}([\alpha, \beta]) = \frac{\mathtt{N}(\alpha)\,\mathtt{N}(\beta)}{\mathtt{N}((\alpha,\beta))}$ for the last equality, we have, uniformly in $b, c, m \in \mathscr{O}_K - \{0\}$, $k \in \mathscr{O}_K$, $z \in \mathbb{C}^\times$, $\theta \in \,]0, 2\pi]$ and $y \geqslant 1$,

$$\begin{aligned}
\mathrm{Card}(\Lambda_{\alpha,\beta,\alpha_0,\beta_0} \cap C(b^{-1}z, \theta, y)) &= \frac{\theta}{2\ \mathrm{covol}_{\vec{\Lambda}_{\alpha,\beta}}}\ y^2 + \mathrm{O}\Big(\frac{\mathrm{diam}_{\vec{\Lambda}_{\alpha,\beta}}}{\mathrm{covol}_{\vec{\Lambda}_{\alpha,\beta}}}\ y\Big) \\
&= \frac{\theta}{\sqrt{|D_K|}\,\mathtt{N}([\frac{m}{(b,m)}, \frac{c}{(b,c)}])}\ y^2 + \mathrm{O}\Big(\frac{1}{\sqrt{\mathtt{N}([\frac{m}{(b,m)}, \frac{c}{(b,c)}])}}\ y\Big) \\
&= \frac{\theta\ \mathtt{N}\big((m(b,c), c(b,m))\big)}{\sqrt{|D_K|}\ \mathtt{N}(m)\ \mathtt{N}(c)}\ y^2 + \mathrm{O}\Big(\frac{\mathtt{N}\big((m(b,c), c(b,m))\big)^{1/2}}{\mathtt{N}(m)^{1/2}\ \mathtt{N}(c)^{1/2}}\ y\Big) \ .
\end{aligned}$$

Using this with $y = \frac{x}{|b|}$, which is at least 1 since $|b| \leqslant x$, we have

$$\begin{aligned}
\widetilde{S}(x) = \frac{\theta\, x^2}{\sqrt{|D_K|}} &\sum_{\substack{b,c\in\mathscr{O}_K-\{0\}\,:\,|b|\leqslant x \\ (b,c)\,|\,k,\ (m(b,c),c(b,m))\,|\,k\,b}} \frac{\mu_K(b)\,\mu_K(c)\ \mathtt{N}\big((m(b,c), c(b,m))\big)}{|\mathscr{O}_K^\times|^2\ \mathtt{N}(b)^2\ \mathtt{N}(c)^2\ \mathtt{N}(m)} \\
&+ \mathrm{O}\Big(x \sum_{b,c\in\mathscr{O}_K-\{0\}} \frac{\mathtt{N}\big((m(b,c), c(b,m))\big)^{1/2}}{|\mathscr{O}_K^\times|^2\ \mathtt{N}(b)^{3/2}\ \mathtt{N}(c)^{3/2}\ \mathtt{N}(m)^{1/2}}\Big) \ .
\end{aligned} \tag{30}$$

By Equation (19) (replacing therein x by x^2), completing the first sum of the above equation with the indices $b \in \mathscr{O}_K - \{0\}$ such that $|b| > x$ introduces an error of the form $\mathrm{O}(\frac{1}{x})$ (uniformly in $m \in \mathscr{O}_K - \{0\}$, $k \in \mathscr{O}_K$ and $x \geqslant 1$). A computation similar to the one done for Equation (19) gives that the second sum in Equation (30) is actually bounded by $\frac{1}{|\mathscr{O}_K^\times|^2}\zeta_K(\frac{3}{2})^2\,\zeta_K(2)$, which is uniform since there are only finitely many such fields K.

By the definition of the constant $c_{\mathtt{m},k}$ in Equation (15), this proves Equation (26), hence concludes the proof of Lemma 4.5. □

Proof of Theorem 4.1. For all $a, k \in \mathscr{O}_K$ with $a \neq 0$, we have

$$\mathtt{N}(a + k) = \mathtt{N}(a)\Big|1 + \frac{k}{a}\Big|^2 \leqslant \mathtt{N}(a)\Big(1 + 2\sqrt{\frac{\mathtt{N}(k)}{\mathtt{N}(a)}} + \frac{\mathtt{N}(k)}{\mathtt{N}(a)}\Big) \ ,$$

and similarly $\mathtt{N}(a+k) \geqslant \mathtt{N}(a)\big(1 - 2\sqrt{\frac{\mathtt{N}(k)}{\mathtt{N}(a)}} + \frac{\mathtt{N}(k)}{\mathtt{N}(a)}\big)$. Let us define the maps $f_\pm : [1, +\infty[\; \to \mathbb{R}$ by $t \mapsto t^2 \pm 2\sqrt{\mathtt{N}(k)}\, t^{3/2} + \mathtt{N}(k)\, t$, so that their derivatives are $f_\pm'(t) = 2t \pm 3\sqrt{\mathtt{N}(k)}\, t^{1/2} + \mathtt{N}(k)$ and

$$\frac{f_-(\mathtt{N}(a))}{\mathtt{N}(a)\;\mathtt{N}(a+k)} \leqslant 1 \leqslant \frac{f_+(\mathtt{N}(a))}{\mathtt{N}(a)\;\mathtt{N}(a+k)} \tag{31}$$

For all $z \in \mathbb{C}^\times$, $\theta \in\,]0, 2\pi]$, $x \geqslant 1$ and $n \in \mathbb{N} - \{0\}$, let

$$a_n = \sum_{a \in \mathfrak{m} \cap C(z,\theta,x)\,:\,\mathtt{N}(a)=n} \frac{\varphi_K(a)}{\mathtt{N}(a)}\;\frac{\varphi_K(a+k)}{\mathtt{N}(a+k)}\;,$$

so that by Equation (25), we have $\widetilde{S}(x) = \sum_{1 \leqslant n \leqslant x^2} a_n$.

By the definition (14) of the sum $S_{z,\theta,k,\mathfrak{m}}(x)$ and the inequalities (31), by Abel's summation formula, by applying twice Lemma 4.5, and since $c_{\mathfrak{m},k} \leqslant 1$ by Lemma 4.2, we have

$$\begin{aligned}
&S_{z,\theta,k,\mathfrak{m}}(x) \leqslant \sum_{1 \leqslant n \leqslant x^2} a_n\, f_+(n) = \Big(\sum_{1 \leqslant n \leqslant x^2} a_n\Big)\, f_+(x^2) - \int_1^{x^2} \Big(\sum_{1 \leqslant n \leqslant t} a_n\Big)\, f_+'(t)\; dt \\
&= \Big(\frac{\theta\; c_{\mathfrak{m},k}}{\sqrt{|D_K|}}\; x^2 + \mathrm{O}(x)\Big)\Big(x^4 + 2\sqrt{\mathtt{N}(k)}\, x^3 + \mathtt{N}(k)\, x^2\Big) \\
&\quad - \int_1^{x^2} \Big(\frac{\theta\; c_{\mathfrak{m},k}}{\sqrt{|D_K|}}\; t + \mathrm{O}(t^{1/2})\Big)\Big(2t + 3\sqrt{\mathtt{N}(k)}\, t^{1/2} + \mathtt{N}(k)\Big)\; dt \\
&= \frac{\theta\; c_{\mathfrak{m},k}}{3\sqrt{|D_K|}}\; x^6 + \mathrm{O}\left((1 + \sqrt{\mathtt{N}(k)}\,)\, x^5 + \mathtt{N}(k)\, x^4\right)\,.
\end{aligned}$$

Replacing f_+ by f_- gives the same minoration to $S_{z,\theta,k,\mathfrak{m}}(x)$, hence Theorem 4.1 follows. □

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