Fixed points of actions of building groups on $CAT(0)$ spaces

P. Pansu

October 23, 2006

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If $\mathcal{Y}$ is a class of metric spaces, $F\mathcal{Y}$ is the class of finitely generated groups Γ such that every isometric action of Γ on a space $Y \in \mathcal{Y}$ fixes a point.

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Examples

- $\mathcal{A}$ ={simplicial trees}. Then $F\mathcal{A}$ is the class of groups which do not split as amalgamated sums (Bass-Serre).
- $\mathcal{H}$ ={Hilbert spaces}. Then $F\mathcal{H} \Leftrightarrow (T)$ (Delorme-Guichardet).
- $\mathcal{L}^p = \{L^p(X, \mu)\}$. Then $F\mathcal{L}^p$ is useful in connection with local rigidity of smooth actions on compact manifolds (Fisher-Margulis).
- $\mathcal{SI}$ ={symmetric spaces and Euclidean buildings of type $\tilde{A}_n$, all n}. Then $F\mathcal{SI} \Leftrightarrow$ finite representation type (Bass ?).

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This talk : (yet) unsuccessful attempt to prove this.

Combinatorial harmonic maps

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For a map $g : C \to Y$ sending vertices of C to a metric space Y, define the energy

$$E(g) = \sum_{edges\ e} m(e)d(g(ori(e)), g(end(e)))^2 = \frac{1}{2}\sum_{c}\sum_{c'\sim c} m(c,c')d(g(c), g(c'))^2.$$

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Let X be a simplicial 2-complex with a cocompact action of a group Γ.
If $f : X \to Y$ is equivariant, define

$$E(g) = \sum_{e\ edge\ of\ \Gamma\backslash X} m(e)d(f(ori(\tilde{e})), f(end(\tilde{e})))^2,$$

where $\tilde{e}$ denotes a lift of e to X. Say f is harmonic *if it minimizes energy among equivariant maps.*

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Proposition

Let Y be CAT(0). Then an equivariant map $f : X \to Y$ is harmonic if and only if for each vertex x of X, $f(x)$ coincides with the barycenter *of $f_{|link(x)}$, i.e. the unique point of Y which minimizes the weighted sum of squares of distances to the images of the neighbours of X.*

Theorem

(Stated by Gromov, 2002). Let $\mathcal{Y} \subset CAT(0)$ be closed under taking asymptotic cones. Then a finitely generated group Γ has $F\mathcal{Y}$ if and only if every Γ-equivariant harmonic map $X \to Y$ has to be constant.

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Proof. *(Following U. Mayer, H. Izeki/T. Kondo/S. Nayatani).*

1. *(Jost, Mayer).* Gradient ∇E makes sense. Heat flow f_t exists for all $t \geq 0$. It satisfies $\frac{\partial E(f_t)}{\partial t} = -|\nabla E|^2(f_t)$.
2. If there exists a constant C such that for all $t > 0$, $E(f_t) \leq C\,|\nabla E|^2(f_t)$, then f_t converges to a constant map.
3. Otherwise, along a subsequence, $E(f_t)^{-1/2}|\nabla E|(f_t) \to 0$. Pick nonprincipal ultrafilter ω. Rescale (Y, d) to $Y_t = (Y, E(f_t)^{-1/2} d)$. Then $\lim_\omega f_t = f_\omega : X \to Y_\omega$ is non constant, harmonic and equivariant for a limiting action.

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(M.T. Wang, 1998). *Let C be a finite weighted graph, $g : C \to Y$ a nonconstant map. Let*

$$d(g, bar(g))^2 = \inf_{y \in Y} \sum_{c \in C} m(c) d(g(c), y)^2 .$$

(If Y is $CAT(0)$, it is the L^2 distance of map g to its barycenter).

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Example

When $Y = \mathbb{R}$, the bottom of spectrum equals the smallest positive eigenvalue of the combinatorial Laplacian $\Delta g(c) = \sum_{neighbours\ c'\ of\ c} m(c, c')(g(c) - g(c'))$.

Garland's formula

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Theorem

(H. Garland, 1972, M.T. Wang, 1998). *Let X be a simplicial complex, Γ a uniform lattice in X, acting isometricly on a metric space Y. Let $f : X \to Y$ be an equivariant map. For $x \in X$, denote by*

$$ED(f,x) = \frac{1}{2} d(f_{|link(x)}, f(x))^2,$$

(where links inherit weights from X). Then

$$E(f) = \sum_{x \in \Gamma \backslash X} ED(f, \tilde{x}).$$

If furthermore f is harmonic, then

$$E(f) = 2 \sum_{x \in \Gamma \backslash X} RQ(f_{|link(x)}) ED(f,x).$$

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In particular, if, for all $x \in X$, $\lambda(link(x), Y) > \frac{1}{2}$, every equivariant harmonic map $X \to Y$ is constant.

Proof of Garland's formula

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If f is harmonic, for each $x \in X$,

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Definition

Let Y be geodesic and CAT(0). If s, s' are geodesics emanating from $y \in Y$, let

$$d(s, s') = \lim_{t \to 0} \frac{d(s(t), s(t'))}{t},$$

(nondecreasing limit). Identify s and s' if $d(s, s') = 0$. This gives a metric space, denoted by $T_y Y$, with a distance nonincreasing map $\pi_y : Y \to T_y Y$.

Comparison to tangent cones

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Corollary

Let Γ act cocompactly on X. Let $\mathcal{Y}_\epsilon$ denote the family of $CAT(0)$ spaces Y such that for all $x \in X$ and $y \in Y$,

$$\lambda(link(x), T_y Y) \geq \frac{1}{2} + \epsilon.$$

Then $\Gamma \in F\mathcal{Y}_\epsilon$.

Examples of bottoms of spectra

Theorem

(H. Izeki/S. Nayatani, 2004).

1. *If Y is a tree, then for every finite weighted graph C, $\lambda(C, Y) = \lambda(C, \mathbb{R})$.*
2. *If Y is the building associated with $Sl(3, \mathbb{Q}_2)$, then for every finite weighted graph C, $\lambda(C, Y) \geq 0.5878\, \lambda(C, \mathbb{R})$.*

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Corollary

If $\lambda(link(x), \mathbb{R}) > \frac{1}{2}$ for all vertices $x \in X$, $\Gamma \in F\mathcal{M}$ where $\mathcal{M}$ is the class obtained by taking products of trees, Hilbert spaces and simply connected nonpositively curved manifolds.
To add the building associated with $Sl(3, \mathbb{Q}_2)$, one needs require $\lambda(link(x), \mathbb{R}) > 0.8507$.

Random groups

Observe that every group presentation can be modified, by adding generators, so that all relators have length 3.

Consider presentations on m fixed generators and $(2m-1)^{3d}$ relators chosen independently at random among the $(2m-1)^3$ possibilities. We are interested in properties which are satisfied with overwhelming probability as m tends to infinity. Such a property is said to be *satisfied by a random group in density d*.

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Proof. (very rough idea)
The Cayley complex has links which look like random graphs. Such graphs (M. Broder and E. Shamir, 1987) have bottom of spectra which tend to 1 as m tends to infinity.

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Theorem

(M. Gromov, 2001). *Let C_k denote the k-cycle. Then, for every $CAT(0)$ space Y,*

$$\lambda(C_k, Y) = \lambda(C_k, \mathbb{R}) = \frac{1}{2}|1 - e^{2i\pi/k}|^2.$$

In particular, $\lambda(C_6, Y) = \frac{1}{2}$.

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Proof. Introduce

$$F(g) = \frac{1}{2\sum m(c)} \sum_{c,\, c' \in C} m(c)m(c')d(g(c), g(c'))^2.$$

Then $d(g, bar(g))^2 \leq F(g)$ with equality when Y is a Hilbert space.

Given $g : C_k \to Y$, extend g to a geodesic polygon, then to a ruled disk $f : D \to Y$. Since D has nonpositive curvature, there exists an embedding $g' : D \to \mathbb{R}^2$ which is isometric on the boundary and does not decrease other distances (Yu. Reshetnyak, 1968). Thus $E(g') = E(g)$ and

$$d(g', bar(g'))^2 = F(g') \geq F(g) \geq d(g, bar(g))^2,$$

thus $RQ(g') \leq RQ(g)$.

Definition

Let C be a weighted graph, Y a metric space. Define

$$\lambda^{Gro}(C,Y) = \inf_{g:C\to Y} RQ^{Gro}(g) \quad \mathit{where} \quad RQ^{Gro}(g) = \frac{E(g)}{F(g)}.$$

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Theorem

Let C be the incidence graph of a finite projective plane. Let Y be an arbitrary geodesic CAT(0) space. Then

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where $\iota : C\to I$ is the embedding of C in the cone over C, for instance, as the link of a vertex in a Euclidean building of type $\tilde{A}_2$.

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Proof. In the incidence graph of a finite projective plane, the number of 6-cycles containing two given vertices depends only on their distance. Sum up Gromov's estimate on F for all 6-cycles. q.e.d.

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Unfortunately, $RQ^{Gro}(\iota)<\frac{1}{2}$. Note that $RQ(\iota)=\frac{1}{2}$.

Proposition

Let Y be complete geodesic $CAT(0)$. Let C be a finite graph. Let $\mathcal{L}$ be a family of loops of length k (i.e. maps of the k-cycle to C). Denote by

- *p the smallest number of loops in $\mathcal{L}$ which contain a given pair of vertices of C.*
- *Q the largest number of loops in $\mathcal{L}$ which contain a given edge.*
- *A the number of edges of C.*

Then

$$\frac{Q}{p} < \frac{A}{k}\frac{1}{2}|1 - e^{2i\pi/k}|^2 \quad \Longrightarrow \quad \lambda(C, Y) \geq \lambda^{Gro}(C, Y) > \frac{1}{2}.$$

Conjecture

A random graph with n vertices and degree $\sim n^{1/3}$ admits a family of loops of length 6 that satisfies the above assumptions.

With some work, this might imply Gromov's conjecture for the density model of random groups in density $> 1/3$.

Proof of coarse integralgeometric estimate

$$\begin{aligned}\sum_{\ell\in\mathcal{L}} F(g\circ\ell) &= \frac{1}{4k}\sum_{\ell}\sum_{z,\, z'\in C_k} d(g\circ\ell(z), g\circ\ell(z'))^2 \\ &\geq \frac{p}{4k}\sum_{c,\, c'\in C} d(g(c), g(c'))^2 = \frac{pA}{k}F(g).\end{aligned}$$

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$$\begin{aligned}\lambda^{Gro}(C_k,Y)\sum_{\ell\in\mathcal{L}} F(g\circ\ell) &\leq \sum_{\ell\in\mathcal{L}} E(g\circ\ell)\\ &= \sum_{\ell\in\mathcal{L}}\sum_{e \text{ edge of } C_k} d(g\circ\ell(ori(e)),g\circ\ell(end(e)))^2\\ &\leq Q\sum_{e' \text{ edge of } C} d(g(ori(e')),g(end(e'))^2\\ &= Q\,E(g).\end{aligned}$$

Proof of coarse integralgeometric estimate

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$$\begin{aligned}\lambda^{Gro}(C_k,Y)\sum_{\ell\in\mathcal{L}} F(g\circ\ell) &\leq \sum_{\ell\in\mathcal{L}} E(g\circ\ell)\\ &= \sum_{\ell\in\mathcal{L}}\sum_{e \text{ edge of } C_k} d(g\circ\ell(ori(e)),g\circ\ell(end(e)))^2\\ &\leq Q\sum_{e' \text{ edge of } C} d(g(ori(e')),g(end(e'))^2\\ &= Q\,E(g).\end{aligned}$$

$$RQ(g)\geq RQ^{Gro}(g)\geq \frac{pA}{Qk}\lambda^{Gro}(C_k,Y)=\frac{pA}{Qk}\frac{1}{2}|1-e^{2i\pi/k}|^2.$$

Definition

Given $g : C \to Y$, *let*

$$E_j(g) = \sum_{\{c,\, c' \in C,\, d(c,c')=j\}} d(g(c), g(c'))^2.$$

Wirtinger inequalities

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Theorem

(Gromov's generalization of Wirtinger inequality, 2001). Let Y *be* $CAT(0)$*. Consider maps* $g : C_k \to Y$*. For every* j*, the ratio* $\frac{E(g)}{E_j(g)}$ *is minimum when* $Y = \mathbb{R}^2$ *and* g *maps onto a regular* k*-gon.*

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(Gromov's generalization of Wirtinger inequality, 2001). Let Y be $CAT(0)$. Consider maps $g : C_k \to Y$. For every j, the ratio $\frac{E(g)}{E_j(g)}$ is minimum when $Y = \mathbb{R}^2$ and g maps onto a regular k-gon.

Proposition

Let Y be complete geodesic $CAT(0)$. Let C be a finite graph. Let $\mathcal{L}$ be a family of isometricly embedded k-cycles in C. Assume that for each pair c, $c' \in C$, the number N_j of cycles from $\mathcal{L}$ passing through c and c' depends only on $j = d(c, c')$. Assume that C admits an isometric embedding $\iota : C \to I$ to a metric space I, such that each cycle of $\mathcal{L}$ is mapped into a Euclidean plane isometricly embedded in I. Then

$$\lambda^{Gro}(C, Y) = RQ^{Gro}(\iota).$$

Proof of sharp integralgeometric estimate

For each j,

$$\begin{aligned}\sum_{\ell\in\mathcal{L}} E_j(g\circ\ell) &= \frac{1}{4k}\sum_{\ell}\sum_{z,\,z'\in C_k,\,d(z,z')=j} d(g\circ\ell(z),g\circ\ell(z'))^2\\ &= \frac{N_j}{4k}\sum_{c,\,c'\in C} d(g(c),g(c'))^2 = \frac{N_j A}{k}E_j(g).\end{aligned}$$

Proof of sharp integralgeometric estimate

For each j,

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\sum_{\ell\in\mathcal{L}} E_j(g\circ\ell) &= \frac{1}{4k}\sum_{\ell}\sum_{z,\,z'\in C_k,\,d(z,z')=j} d(g\circ\ell(z),g\circ\ell(z'))^2 \\
&= \frac{N_j}{4k}\sum_{c,\,c'\in C} d(g(c),g(c'))^2 = \frac{N_j A}{k}E_j(g).
\end{aligned}$$

$$\begin{aligned}
\frac{E(C_k,Y)}{E_j(C_k,Y)}\sum_{\ell\in\mathcal{L}} E_j(g\circ\ell) &\leq \sum_{\ell\in\mathcal{L}} E(g\circ\ell) \\
&= \sum_{\ell\in\mathcal{L}}\sum_{e \text{ edge of } C_k} d(g\circ\ell(ori(e)),g\circ\ell(end(e)))^2 \\
&= N_1 \sum_{e' \text{ edge of } C} d(g(ori(e')),g(end(e'))^2 \\
&= N_1\, E(g).
\end{aligned}$$

Proof of sharp integralgeometric estimate

For each j,

$$\begin{aligned}\sum_{\ell\in\mathcal{L}} E_j(g\circ\ell) &= \frac{1}{4k}\sum_{\ell}\sum_{z,\,z'\in C_k,\,d(z,z')=j} d(g\circ\ell(z),g\circ\ell(z'))^2\\ &= \frac{N_j}{4k}\sum_{c,\,c'\in C} d(g(c),g(c'))^2 = \frac{N_j A}{k}E_j(g).\end{aligned}$$

$$\begin{aligned}\frac{E(C_k,Y)}{E_j(C_k,Y)}\sum_{\ell\in\mathcal{L}} E_j(g\circ\ell) &\leq \sum_{\ell\in\mathcal{L}} E(g\circ\ell)\\ &= \sum_{\ell\in\mathcal{L}}\sum_{e\text{ edge of }C_k} d(g\circ\ell(ori(e)),g\circ\ell(end(e)))^2\\ &= N_1\sum_{e'\text{ edge of }C} d(g(ori(e')),g(end(e'))^2\\ &= N_1\,E(g).\end{aligned}$$

$$RQ^{Gro}(g)\geq\sum_j\frac{AN_j}{kN_1}\frac{E(C_k,Y)}{E_j(C_k,Y)} = RQ^{Gro}(\iota,Y_0).$$

Definition

(H. Izeki and S. Nayatani, 2004). *Let Y be a geodesic CAT(0) space. Given a finite weighted subset $Z \in Y$ (sum of weights $= 1$), let $\phi : Z \to \mathcal{H}$ be a 1-Lipschitz map to Hilbert space such that for all $z \in Z$, $|\phi(z)| = d(z, bar(Z))$. Define*

$$\delta(Z) = \inf_{\phi} \frac{|bar(\phi)|^2}{\| \phi \|^2}.$$

The IN invariant of Y is $\delta(Y) = \sup_{Z \subset Y} \delta(Z) \in [0,1]$.

Lemma

Let Y be a geodesic CAT(0) space, let C be a finite weighted graph. Then

$$\lambda(C, Y) \geq (1 - \delta(Y))\lambda(C, \mathbb{R}).$$

Proof. Given $g : C \to Y$, let $Z = g(C)$. Choose optimal ϕ for Z. Pythagore gives $d(\phi, bar(\phi))^2 = \| \phi \|^2 - |bar(\phi)|^2 = (1 - \delta(Z))\| \phi \|^2 = (1 - \delta(Z)) d(g, bar(g))^2$.

$$\lambda(C, \mathbb{R}) \leq RQ(\phi \circ g) = \frac{E(\phi \circ g)}{d(\phi \circ g, bar(\phi \circ g))^2} \leq \frac{E(g)}{d(\phi, bar(\phi))^2} = \frac{1}{1 - \delta(Z)} RQ(g).$$

Examples of values of IN invariant

Examples

1. *Hilbert spaces have* $\delta = 0$*, by definition.*
2. *For all* Y, $\delta(Y) = \inf_{y \in Y} \delta(T_y Y)$*. Therefore nonpositively curved manifolds have* $\delta(Y) = 0$*.*
3. *Trees have* $\delta = 0$*.*
4. δ *is continuous under ultralimits. Therefore (non proper) Euclidean buildings which are asymptotic cones of symmetric spaces have* $\delta(Y) = 0$*.*
5. *For all* Y *and probability measure spaces* Ω, $\delta(L^2(\Omega, Y)) \leq \delta(Y)$*.*
6. $\delta(Y_1 \times Y_2) \leq \max\{\delta(Y_1), \delta(Y_2)\}$*. Therefore, products of the above have* $\delta = 0$*.*
7. *The Euclidean building of* $Sl(3, \mathbb{Q}_p)$ *has* $\delta \geq \frac{(\sqrt{p}-1)^2}{2(p-\sqrt{p}+1)}$ *(equality conjectured).*
8. *The Euclidean building of* $Sl(3, \mathbb{Q}_2)$ *has* $\delta < \frac{1}{2}$*.*

The fixed point property $F\mathcal{Y}_{\leq\delta_0}$

Definition

Fix $\delta_0 \in [0,1]$. Say a group Γ has property $F\mathcal{Y}_{\leq\delta_0}$ if every isometric action of Γ on a geodesic CAT(0) space Y with $\delta(Y) \leq \delta_0$ has a fixed point.

Proposition

Let $\delta_0 < \frac{1}{2}$. Let X be a simplicial complex, Γ a uniform lattice in X. Assume that for all $x \in X$, $\lambda(link(x), \mathbb{R}) > \frac{1}{2(1-\delta_0)}$. Then Γ has property $F\mathcal{Y}_{\leq\delta_0}$.

Theorem

(A. Zuk, 2003, H. Izeki, T. Kondo and S. Nayatani, 2006). *If $\delta_0 < \frac{1}{2}$, random groups in density $> \frac{1}{3}$ have asymptoticly property $F\mathcal{Y}_{\leq\delta_0}$.*

Theorem

(T. Kondo, 2006). *In the space of marked groups, $F\mathcal{Y}_{\leq\delta_0}$ is an open condition. Furthermore, $F\mathcal{Y}_{<1/2}$ is dense.*

Definition

(H. Bass, 1980). *Say a group* Γ *has* finite representation type *if for all* n*, every homomorphism* $\Gamma \to Gl(n, \mathbb{C})$ *factors through a finite group.*

Theorem

(T. Kondo, 2006). *In the space of marked groups, there is a dense* G_δ *of groups which have property* $F\mathcal{Y}_{<1/2}$ *and finite representation type.*

Proposition

If a group has a fixed point in all its isometric actions on symmetric spaces and classical Euclidean buildings of type $\tilde{A}_n$ *(call this FIS), then it has finite representation type.*

Remark

$F\mathcal{Y}_{\leq\delta_0}$ *does not imply FIS. Indeed,* δ *tends to* $1/2$ *for classical buildings of type* $\tilde{A}_2$*, as* p *tends to infinity.*