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Mappings Between Degenerate Real Analytic Hypersurfaces in $\mathbf{C}^n$

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Dedicated to Professor L. Ehrenpreis

§. 1 Introduction

Let M be a smooth real hypersurface in $\mathbf{C}^n$. Its holomorphic tangent bundle $\mathrm{T}^{(1,0)}M$ is defined to be $C\mathrm{T}M \cap i(C\mathrm{T}M)$, where $C\mathrm{T}M$ is the complexified tangent bundle of M. A (complex) vector field L along M is called a CR vector field if $\overline{L}$ is a cross section of $\mathrm{T}_M^{(1,0)}$. A function f over M is said to be a CR function if it is annihilated by all CR vector fields along M in the sense of distribution. An interesting question in Complex Analysis is the regularity problem of CR functions (or mappings). In particular, one would like to know when a CR mapping between two real analytic hypersurfaces is also real analytic. By now, there have appeared many studies when the hypersurfaces are assumed to be sufficiently non-degenerate or when the maps are assumed to be sufficiently smooth (see [BJT], [BR1], [BR3], [DF], [Me], [BHR], [Hu1] and [Hu2], as well as the survey paper [Fr] and their references). In this paper, we carry out a study for the analyticity problem of C^1-regular CR mappings between certain degenerate real analytic hypersurfaces in $\mathbf{C}^n$, called weakly essential finite hypersurfaces. Our result provides a regularity theorem for CR

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mappings between quite degenerate hypersurfaces with a little initial regularity (C^1-smooth) to start with.

We first introduce some notation and definitions to state precisely our main theorem.

Let M be a non Levi-flat real analytic hypersurface at p in $\mathbf{C}^n$. After a holomorphic change of coordinates [Me], we may assume that $p = 0$ and there exists a sufficiently small open neighborhood Ω of 0 in $\mathbf{C}^n$ such that M is given by an equation of the following form:

$$(1.1) \qquad Im\ w = (\mathrm{Re} w)^m \phi(z, \bar{z}, \mathrm{Re} w), \quad (z, w) \in \mathbf{C}^{n-1} \times \mathbf{C},$$

where ϕ is a real valued convergent power series in $z, \overline{z}, \mathrm{Re}(w)$ such that

$$(1.2) \quad \phi(z, 0, \mathrm{Re}(w)) \equiv \phi(0, \overline{z}, \mathrm{Re}(w)) \equiv 0, \ \phi(z, \bar{z}, 0) \not\equiv 0 \ \text{ and } m \in \mathbf{N}.$$

Such a choice of coordinates is called normal coordinates. It is shown in [Me] that the integer m is an invariant for normal coordinates. Note that $m = 0$ if and only if M is of finite type at 0 in the sense of Bloom-Graham [BG]. Write

$$\phi(z, \zeta, 0) = \sum_{\alpha} a_\alpha(z) \zeta^\alpha.$$

Definition: M is called *m-essential finite at* 0 if the ideal $(a_\alpha(z))$ in the ring of formal power series $\mathbb{C}[[z]]$ generated by all of the $a_\alpha(z)$ is of finite codimension, i.e.

$$(1.3) \qquad \dim_{\mathbf{C}} \mathbf{C}[[z]]/(a_\alpha(z)) < \infty.$$

Note that M is 0-essential at 0 if and only if M is essentially finite in the sense of [BJT].

It is shown in [Me] that the above definition is independent of the choice of normal coordinates. In what follows, for brevity, by saying *infinite type*, we always mean *infinite type in the sense of Bloom-Graham* [BG]. When M is m-essential at $p \in M$ with $m > 0$, we will also call M is weakly-essential at p.

Main Theorem *Let M and M' be real analytic hypersurfaces of infinite type near* 0 *in* $\mathbf{C}^n$. *Suppose that M and M' are weakly essential at* 0. *Let D be a certain side of M. Let $f : M \to M'$ be a C^1 mapping which is the restriction of a certain continuous mapping over $\overline{D}$, holomorphic in D and with $f(0) = 0$. Either suppose that the Jacobian of f is not identically zero over M and $f^{-1}(0) \cap M$ does not contain any non-trivial holomorphic curves; or suppose that f, as a map from M into M', is finite to one. Then f extends holomorphically to a neighborhood of* 0.

Corollary: *Let M and M' be real analytic hypersurfaces of infinite type near* 0 *in* $\mathbb{C}^n$. *Suppose that M and M' are weakly essential at* 0. *Let D be a certain side of M. Let $f : M \to M'$ be a C^1 mapping which is the restriction of a certain continuous mapping over $\overline{D}$, holomorphic in D with $f(0) = 0$. Suppose that $f = (f_1, \cdots, f_{n-1}, g) : M \to M'$ is a finite to one map. Then f extends as a local proper holomorphic map from a neighborhood of* 0. *Moreover there is an integer $k > 0$ such that $g(z, w) = w^k g^*(z, w)$ with $g^*(0) \neq 0$, in case M and M' are defined by equations of the forms in (1.1) and (1.2).*

It should be mentioned that when the map f is apriori assumed to be infinitely smooth, then the hypothesis in the theorem can be replaced by the analytic hypothesis that "the Jacobian of f does not vanish to infinite order at any boundary point", as proved in the deep work of Baouendi-Rothschild [BR3] in the case of complex dimension two and the later generalization by the second author [Me]. However, in view of the following example, this is no longer the case for only finitely many times differentiable CR mappings.

Example : Let M and M' be given in $\mathbb{C}^2$ by

$$M = \{(z, w) \in \mathbb{C}^2 | Re\ w = |z|^{2k} |w|^{2k^2+k} (Im\ w)^2\},$$

$$M' = \{(z', w') \in \mathbb{C}^2 | Re\ w' = |z'|^{2k} (Imw')^2\},$$

$$f(z, w) = (zw^{k+\frac{1}{2}}, w),$$

where k is even. One can check that h is of class $C^{k,\frac{1}{2}}$, h extends holomorphically to one side of M, $h(M) \subset M'$, but h is not of class C^{k+1}. Notice that the Jacobian of h has a finite order of vanishing at 0 and $h^{-1}(0) \cap M$ is the $w = 0$.

We should also mention that when the hypersurfaces are essentially finite in the Corollary, a result of Baouendi and Rothschild [BR2] indicates that the integer k has to be 1, from which it follows that f preserves the side. However, this is no longer the case in our situation, as the following example demonstrates (see also [BR3] for similar examples):

Example: Let $M_1 = \{(z, w = s + it) : t = |z|^2 s\}$, $M_2 = \{(z, w = s + it) : t = 2s\frac{|z|^2}{1-|z|^4}\}$ and $f(z, w) = (z, w^2)$. Then $f(M_1) \subset M_2$ and $k = 2$. Notice that f does not preserve sides.

We now say a few words about the proof of the main theorem. The first step is to show that the map, though only assumed to be C^1, behaves like a smooth function in the sense that it has an integer vanishing order. (This can be regarded as a much stronger version of the unique continuation phenomenon for CR mappings). By applying the famous Hanges-Treves [HT] propagation theorem, we then can work near a point where the map is a kind of C^1-diffeomorphism. Next, we show that the Baouendi-Jacobowitz-Treves reflection principle [BJT] is usable there. Lastly, we prove that the map extends along an open dense subset of the complex hypersurface contained in M, and thus it extends everywhere too. Some of the approaches used in this paper (especially, the idea of proving the extension at some thin set and then applying the Hanges-Treves theorem to get the extension everywhere) were also later used by the second author in [Me] for some other related purposes.

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§2. More Notation and Preliminary Facts

In what follows, we always assume that M and M' are real analytic hypersurfaces near the origin defined by equations of the following form, respectively:

$$Im\ w = (Re\ w)^m \phi(z, \bar{z}, Re\ w), \quad (z, w) \in \mathbf{C}^{n-1} \times \mathbf{C},$$

$$(2.1) \qquad Im\ w' = (Re\ w')^m \psi(z', \bar{z}', Re\ w'), \quad (z', w') \in \mathbf{C}^{n-1} \times \mathbf{C},$$

Here, ψ satisfies the normalization condition as imposed in (1.2).

Let $f = (f^*, g) = (f_1, \cdots, f_{n-1}, g)$ be a continuous CR mapping from M_1 into M_2, which extends continuously to a certain side D of M_1. g is then called the normal component of f. Write E for the $w = 0$ in $\mathbf{C}^n$. We also recall that a real analytic hypersurface in $\mathbf{C}^n$ is of finite type at p in the sense of Bloom-Graham if and only if it does not contain a complex analytic variety of pure codimension one passing through p.

The following lemma seems standard and its proof can be easily achieved by studying the uniqueness of the Bishop equation describing the attached analytic disks. (Hence we omit its proof).

Lemma 2.1: Let M be a real analytic hypersurface with $0 \in M$ a point of infinite type. Assume that E is the complex hypersurface passing through 0 and is contained in M. Let $\xi(\cdot)$ be a holomorphic disk attached to M and $\xi(\cdot)$ is continuous up to the boundary. When $\xi(\cdot)$ is small enough and $\xi(1) = 0$, then $\xi(\Delta) \subset E$, where Δ is the unit disk in the complex plane.

Lemma 2.2: Suppose that f is a continuous CR mapping from M into M' with $f(0) = 0$. If M is of infinite type at 0 and if $f^{-1}(0) \cap M$ does not contain any non-trivial holomorphic curves, then M' must be of infinite type at 0. Conversely, if M' is of infinite type at 0 and M is of finite type, then the normal component g of f has to be identically 0.

Proof of Lemma 2.2: Since M contains the complex hypersurface $E = \{w = 0\}$, and since the CR mapping f, when restricted to E, is a holomorphic map, the hypothesis of the first part of the lemma indicates that f has to be proper from E near 0 to its image $E' = f(E)$. Hence, M' contains a complex analytic variety of complex codimension one at the origin. Therefore, M' has to be of infinite Bloom-Graham type at 0.

Conversely, suppose that M' is of infinite type at 0. If M does not contain a complex analytic hypersurface passing through 0, then, it is known (see, f.g, [T] [Tu]) that there exists a family of small complex analytic disks ξ_t attached to M such that their images fill an a uniqueness set for holomorphic functions in a certain side D of M near 0 and $\xi_t(1) = 0$. Also, by the approximation theorem of Baouendi-Treves, we can assume that f extends holomorphically to each small disk attached to M. Now, notice that for each t, $f \circ \xi_t$ is also a complex analytic disk attached to M' through 0. Therefore, by Lemma 2.1, $f \circ \xi_t$ has image inside $E' = \{w' = 0\}$ in M. Therefore, we see that f maps $M \cup D$ into E'. This is equivalent to saying that the normal component g of f has to be identically 0. ∎

Remark 2.3: Notice that even if f is smooth, $f^{-1}(0) \cap M = \{0\}$ and M' is of infinite type, M does not have to be of infinite type at 0. To see this, we can simply take $M = \{(z, u+iv) \in \mathbf{C}^2 : \ v = |z|^2\}$, $M' = \{(z, u+iv) : \ v = u|z|^2\}$ and $f = (e^{-\frac{1}{w^{1/3}}}, 0)$. Of course, in all such examples, $g \equiv 0$ by the above lemma.

Applying the implicit function theorem to (2.1), we can assume that M is also defined by an equation of the following form:

$$\rho^*((z,w), \overline{(z,w)}) = -w + \overline{w} + \sum_{j \geq m} \phi_j^*(z, \overline{z})\overline{w}^j, \tag{2.2}$$

where ϕ_j^* does not contain any harmonic terms and the idea generated by $\{a_\alpha(z)\}$ has finite codimension in $\mathbf{C}[z]$. Here $\phi_m^*(z,\zeta) = \sum a_\alpha(\zeta) z^\alpha$. We recall that for each point $(a,b) \in \mathbf{C}^{n-1} \times \mathbf{C}$ close to the origin, the Segre variety $Q_{(a,b)}$ of M corresponding to (a,b) is defined by $Q_{(a,b)} = \{(z,w) \approx 0 : \rho^*((z,w), \overline{(a,b)}) = 0\}$. As usual, we write $A_{(a,b)} = \{(z,w) \approx 0 : \ Q_{(z,w)} = Q_{(a,b)}\}$. (See [We], for instance). It is clear from the definition concerning the essential finite type that $(a,b) \in M$ is a point of essential finite type if and only if $A_{(a,b)}$ is discrete near (a,b).

Next, we give a different invariant characterization of the m - essential property.

Lemma 2.4: Assume that M is of infinite type at 0. If M is weakly essential at $0 \in M$, then $A_{(z,w)}$ is discrete for $(z,w) \notin E$. Moreover, when M is weakly essential at 0, then M is weakly essential at any other point $p(\approx 0) \in E$ and M is essentially finite at $M \setminus E$.

Proof of Lemma 2.4: Write $\phi_j^* = \sum_\alpha c_{j,\alpha}(\overline{z}) z^\alpha$. For any $(a,b) \in M \setminus E$. Notice that $b \neq 0$, and

$$A_{(a,b)} = \{(z,w) : w = b, \ \sum_{j \geq m} c_{j,\alpha}(\overline{z})\overline{w}^j = \sum_j c_{j,\alpha}(\overline{a})\overline{b}^j\}$$

$$= \{(z,w) : \ w = b, \ \sum_{j \geq m} c_{j,\alpha}(\overline{z})\overline{b}^{j-m} = \sum_{j \geq m} c_{j,\alpha}(\overline{a})\overline{b}^{j-m}\}.$$

Therefore

$$A_{(a,b)} = \{(z,w) : w = b,\ c_{m,\alpha}(\overline{z}) = c_{m,\alpha}(\overline{a}) + O(b|z|),\ \text{for all } \alpha\}.$$

By the weak essential property, we see that $A_{(a,b)}$ is discrete near 0 if M is weakly essential at 0, where $b \neq 0$ is sufficiently close to 0. Since $A_{(a,0)} = E$, we easily conclude the proof of Lemma 2.4. ∎

Finally, we prove the following lemma which will be used later.

Lemma 2.5: Let $\{a_k(z)\}_{k\geq 1}$ be a family of holomorphic functions defined near $0 \in \mathbf{C}^n$ and vanishing at 0. Suppose the idea generated by these functions has finite codimension in the ring of power series $C[z]$. Then there is a complex subvariety V_0 of positive codimension in $\mathbb{C}^n$ such that for each $z \notin V_0$ close to 0, there are $k_1, \cdots, k_n$ with

$$\det(\frac{\partial a_{k_j}}{\partial z_l})_{1\leq l,j\leq n}|_z \neq 0.$$

Proof of Lemma 2.5: First, by the Hilbert Zero theorem, there is an integer N such that the common zero of $a_k(z)$ is 0 where $k \leq N$. Define $\pi(z) = (a_k(z))_{k=1}^N$ to be the map from $\mathbf{C}^n$ into $\mathbf{C}^N$ near the origin. Then $\pi(z)$ is locally finite to one and thus is locally proper near 0. Hence, the image V of a neighborhood of 0 in $\mathbf{C}^n$ under π is a complex analytic subvariety of dimension n in $\mathbb{C}^n$ near 0. Therefore, there is a V_0 of positive codimension such that for each $z(\approx 0) \notin V_0$, π is locally biholomorphic from a neighborhood of z to V near $\pi(z)$. The proof of the lemma nows follows easily from the rank theorem. ∎

§3. An Integer Vanishing Order of the Normal Component

One of the key steps for the proof of our Main Theorem is to show that the normal component g of f behaves like a smooth function along a certain wedge, although it is only assumed to be C^1 or even Hölder continuous. For this purpose, we first prove the following integer-vanishing order property for g along a wedge, which plays a crucial role in the whole discussion. We would like to mention that results of this type are also related to the more general framework on the boundary unique continuation problems of holomorphic functions in [ABR] [BL] [HK] [BR3] [E], etc, where the non-infinite vanishing order problem of the normal component is investigated. In the following, we use $O(a)$ to denote a small neighborhood of a.

Lemma 3.1: Let $\Delta^+ = \{z \in \mathbf{C} : \ |z| < 1, \mathrm{Im}(z) > 0\}$. Let $f(z,t) = u(z,t) + iv(z,t)$ be a holomorphic function over Δ^+ in z, which is continuous over $\overline{\Delta^+} \times O(t_0)$, where $O(t_0)$ is a small neighborhood of t_0 in the parameter t-space. If for $x \in (-1,1)$, $v(x,t) = u(x,t)P(x,t)$ with $P(x,t) \in C^\alpha((-1,1) \times O(t))$, then there exists a non-negative integer k_t, depending upper semi-continuously on t, such that $f(z,t) = z^{k_t} g(z,t)$, where $g(0,t) \neq 0$ and $g(z,t)$ depends Hölder continuously on z up to $(-1,1)$. Moreover when k_t is constant for $t \approx t_0$, then $g(z,t)$ depends C^α-continuously on $(z,t) \in (\Delta^+ \cup (-1,1)) \times O(t_0)$. Here α is a certain positive non-integer number.

Proof of Lemma 3.1: Write $f^+(z,t) = f(z,t)$ and $f^-(z,t) = \overline{f(\overline{z},t)}$ for $z \in \overline{\Delta^-} = \{z \in \mathbf{C} : \ |z| \leq 1, \ \mathrm{Im}(z) < 0\}$. Then for $x \in (-1,1)$, one has $v(x,t) = \frac{1}{2i}(f^+(x,t) - f^-(x,t))$ and $u(x,t) = \frac{1}{2}(f^+(x,t) + f^-(x,t))$. Substituting these into the given equation $v = uP(x,t)$, it follows that $\frac{1+iP(x,t)}{1-iP(x,t)} f^-(x,t) = f^+(x,t)$. Let

$$\Gamma(z,t) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\log \frac{1+iP(\xi,t)}{1-iP(\xi,t)}}{\xi - z} d\xi.$$

Then by the well-known Plemej formula, one has $e^{\Gamma^+(x,t) - \Gamma^-(x,t)} = \frac{1+iP(x,t)}{1-iP(x,t)}$. Here for $z \notin (-1,1)$, $\Gamma^\pm(z,t)$ is defined to be the value of $\Gamma(z,t)$ for $z \in \Delta^\pm$, respectively; and the values on $(-1,1)$ are understood as the boundary values of $\Gamma^{\pm}(z)$ from inside, respectively. By the Privalow theorem, they are C^{α} on t and on z up to the real axis $(-1,1)$. Therefore, one obtains $\frac{f^+(x,t)}{e^{\Gamma^+(x,t)}} = \frac{f^-(x,t)}{e^{\Gamma^-(x,t)}}$. Hence, $\frac{f^+(x,t)}{e^{\Gamma^+(x,t)}}$ extends holomorphically in z to the unit disk Δ. Denote its extension by $g(z,t)$ and write $g(z,t) = z^{k_t} g^*(z,t)$ with $g(0,t) \neq 0$. Then, again by the Privalow theorem, it follows that $e^{\Gamma^+(z,t)} g(z,t)$ depends Hölder continuously on t and $z \in \Delta^+ \cup (-1,1)$. Applying also the Hurwitz theorem, we conclude that k_t is upper semi-continuous. Moreover, in case k_t is constant for $t \approx t_0$, $g^*(z,t)$ depends C^{α} on (z,t) over $(\Delta^+ \cup (-1,1)) \times O(t_0)$, once one makes $O(t_0)$ sufficiently close to t_0. This completes the proof of the lemma. ■

Next, let M be of infinite type at 0. Let $0 \in M$ and $\mathcal{S} \subset M$ be a totally real submanifold of real dimension n passing through p such that $\mathcal{S} \cap E$ is totally real of dimension $n-1$ near 0. In what follows, we will assume, without loss of generality, that E is still the $w = 0$ but $\mathcal{S} = \mathbf{R}^n \cap O(0)$. Also, Im$w$ is the outward normal direction of M along $\mathcal{S}$ near 0. Write

$$\mathcal{W}_{\delta}^{\pm} = \{(z,w) \mid \pm \operatorname{Im}(w) > \delta(|Imz_1| + \cdots + |Im\ z_{n-1}|)\}.$$

Then $\mathcal{W}_{\delta}^{+} \subset O(0) \setminus D \cup M$ and $(\overline{z}, \overline{w}) \in D$ for $(z,w) \in \mathcal{W}_{\delta}^{+}$ near the origin. Let

$$\beta = (\beta_1 \ldots \beta_n) \in \mathbb{R}^n, \beta' = (\beta_1', \ldots, \beta_{n-1}') \in \mathbb{R}^{n-1},$$

$$\Delta_{\epsilon} = \{\zeta \in \mathbb{C} |\ \ |\zeta| < \epsilon\}, {\Delta_{\epsilon}}^{+} = \{\zeta \in \Delta_{\epsilon} |\ \ Im\ \zeta > 0\}$$

For $\beta_n > \delta(|\beta_1| + \cdots + |\beta_{n-1})|$ and for ϵ sufficiently small, we define $\psi_{\beta,\beta'}$ by

$$\zeta \longrightarrow (\beta_1\zeta + \beta_1', \ldots, \beta_{n-1}\zeta + \beta_{n-1}', \beta_n\zeta).$$

Notice that $\psi_{\beta,\beta'}(\Delta_{\epsilon} \setminus (\Delta_{\epsilon}^{+} \cup (-\epsilon,\epsilon)) \subset D$.

Let M' be a real analytic hypersurface at 0 given in the normal coordinate by $\operatorname{Im}w' = (\operatorname{Re}w')^{m'} \chi(z', \bar{z}', \operatorname{Re}w')$ where $\chi(z', \bar{z}', 0) \not\equiv 0$,,

$\chi(z', 0, w') \equiv 0$. Let $f = (f^*, g) : M \longrightarrow M'$ be a Hölder continuous CR mapping, which is the restriction of a certain continuous mapping from $\overline{D}$ holomorphic in D, with $f(0) = 0$.

Proposition 3.2: Assume the above notation. Suppose that g is not identically zero along the normal direction in the sense that for each fixed β', $g(\beta', t) \not\equiv 0$ for $t < 0$. Then if δ is sufficiently small, there is a positive integer k, and a point $z_0 = (\beta'_0, 0)$ such that for $(z, w) \in \mathcal{W}^+_\delta \cap O(z_0)$, $g(\overline{z}, \overline{w}) = \overline{w^k} g^*(z, w)$ with $g^*(\beta'_0, 0) \neq 0$ and g^* Hölder continuously in $\overline{\mathcal{W}^+_\delta} \cap O(z_0)$. Here β'_0 is a certain fixed real $(n-1)$-tuple sufficiently close to 0.

Proof of Proposition 3.2: Write $g_{\beta,\beta'} = \overline{g \circ \psi_{\beta,\beta'}(\overline{\zeta})}$. Then it depends Hölder continuously on $\beta, \beta', \zeta \in (-\epsilon, \epsilon)$ and extends holomorphically to Δ^+_ϵ in ζ. Notice that for $\zeta \in (-\epsilon, \epsilon)$, $\mathrm{Im}(g_{\beta,\beta'}(\zeta)) = -(\mathrm{Re}(g_{\beta,\beta'}(\zeta)))^m \chi(f \circ \psi_{\beta,\beta'}, \overline{f \circ \psi_{\beta,\beta'}}, \mathrm{Re}(g_{\beta,\beta'}(\zeta))$. Applying Lemma 3.1, we can write $g_{\beta,\beta'}(\zeta) = \zeta^{k_t} g^*_{\beta,\beta'}(\zeta)$ with $g^*_{\beta,\beta'}(0) \neq 0$ and k_t depending upper semi-continuously on $t = (\beta, \beta')$. Since k_t takes only on integer values, we see the existence of a certain parameter $t_0 = (\beta_0, \beta'_0)$ such that $k_t = k$ is constant for t close to t_0. Then by Lemma 3.1, we conclude that $g_{\beta,\beta'}(\zeta) = \zeta^k g^*_{\beta,\beta'}(\zeta)$ with $g^*_{\beta,\beta'}(0) \neq 0$ and Hölder continuously on $\beta(\approx \beta_0), \beta'(\approx \beta'_0)$ and $\zeta \in \overline{\Delta^+_\epsilon}$. Denote by z_0 the point $(\beta'_0, 0)$. Since the image of all such $\psi_{\beta,\beta'}$ fill in $\mathcal{W}^+_{\delta'} \cap O(z_0)$ with $0 < \delta' << \delta$ and since $w^k \circ \psi_{\beta,\beta'}(\zeta) = (\beta_n)^k \zeta^k$, we conclude easily that $h(z) = \overline{g(\overline{z})}/w^k = \frac{1}{\beta_n^k} g^*_{\beta,\beta'}(\zeta)$ is holomorphic and uniformly bounded over $\overline{\mathcal{W}^+_{\delta'}} \cap O(z_0)$. Write $\beta_0 = (\beta^*_0, 0) = (\beta_{0,1}, \cdots, \beta_{0,n})$ and write $\mathrm{Re}(z, w) = (x', x_n) = (x_1, \cdots, x_n)$. We easily see that for $x(\approx (z_0)) \in \mathcal{S}$, when $z \to x$ from $\mathcal{W}^+_{\delta'}$ along the normal direction, then $h(z)$ takes the boundary value

$$\frac{1}{(\beta_{0,n})^k} g^*_{\beta_0,(x' - x_n\beta^*_0/\beta_{0,n})}(x_n/\beta_{0,n}),$$

which, as a function in x, is Hölder continuous over a neighborhood of $z_0 = (\beta'_0, 0)$ in $\mathcal{S}$ by Lemma 3.1. Therefore, we conclude that $h(z)$

extends Hölder continuously up to $\mathcal{W}_{\delta'}^{+} \cup \mathcal{S}$. The proof of Proposition 3.2 is complete.

Remark 3.3: From the proof, it is clear that Proposition 3.2 holds for all functions g with the following property: (a): g does not vanish identically along the normal direction; (b): $\mathrm{Im}(g) = \mathrm{Re}(g)P(z)$ over $\mathcal{S}$ and g is Hölder continuous.

§4. Proof of the Main Theorem

We now present the proof of our Main Theorem. As already showed up in the proof of Proposition 3.2, an important fact in our discussion is that if a holomorphic function defined over a certain wedge blows up polynomially when approaching to the edge and if its distribution limit is continuous, then the function extends continuously up to the edge from the wedge. Another useful fact for our argument is the following propagation theorem of Hanges-Treves [HT]:

Theorem 4.1 (Hanges-Treves): Let M be as in our Main Theorem. Let h be a CR function over M. If for a certain $p \in E$, h extends holomorphically to a neighborhood of p, then h extends holomorphically to a neighborhood of E.

Now, we let M, M' and f be as in the Main Theorem. Since f must be locally proper from E to E', it is locally biholomorphic from E into E', away from a thin set. By Theorem 4.1 and Lemma 2.2, we can assume without loss of generality that f^* is locally biholomorphic from E into E' near the origin and we can assume that Proposition 3.2 holds near the origin $z_0 = 0$. Also, we use the same notation there. And making a change of coordinates, we can let δ in Proposition 3.2 be 1.

Write $\{L_j\}_{j=1}^{n-1}$ for a real analytic basis of the space of all cross sections of $\mathrm{T}^{(1,0)}M$. (Namely, we assume that each L_j has real analytic coefficients). Hence, $\{L_j|_0\}$ forms a basis of $\mathbf{C}^{n-1}$; for $T_0^{(1,0)}M = \mathbf{C}^{n-1}$. Write $\Lambda(z,\overline{z},\overline{Df}) = \det(\overline{L_j f_k})_{1\le j,l\le n-1}$. It thus follows easily that $\Lambda(z,\overline{z},\overline{Df})|_0 \neq 0$. As before, we still use $z=(z_1,\cdots,z_{n-1},w)$ for the coordinates in $\mathbf{C}^n$.

Next, using Lemma 2.2, we see that for $p(\approx 0)\in M_1\backslash E$, $q=f(p)\in M'\backslash E'$ and $df|_p$ gives a linear isomorphism from $T_p^{(1,0)}M$ to $T_q^{(1,0)}M'$. We now need a result proved in [BJT]. (See pp395 , Remark 6.1 of [BJT]: Notice in [BJT], the result is stated for C^1 diffeomorphism, however it works the same way for maps with the property described below. Also notice that it follows from the work in [Hu1] in case $n=2$):

Theorem 4.2 (Baouendi-Jacobowitz-Treves): Let M and M' be two essentially finite hypersurfaces near 0. Let f be a C^1-CR map from M into M'. Assume that $df|_0$ is an isomorphism from $T_0^{(1,0)}M$ to $T_0^{(1,0)}M'$. Then f admits a multiple-valued holomorphic extension near 0.

With Proposition 3.2 and Theorem 4.2 at our disposal, we can proceed the proof of our Theorem as follows. Certain arguments here are motivated by the work done in [BJT] [BR1] [Hu1].

Since $f(M)\subset M'$, we get that

$$\mathrm{Im}(g(z)) = (\mathrm{Re}(g(z)))^m \psi(f(z),\overline{f(z)},\mathrm{Re}(g(z)))$$

for $z\in M$. Applying the implicit function theorem, we can obtain the following functional equation:

$$\overline{g(z)} = g(z) + (g(z))^m H(f^*(z),\overline{f^*(z)},g(z)), \tag{4.1}$$

for $z\in M$, where $S(a,\overline{a},0)$ does not have any harmonic terms and the idea generated by $(h_\alpha(z'))$ is of finite codimension in $C[z']$, where $H(a,b,0)=\sum_\alpha h_\alpha(a)b^\alpha$.

Since f^* is proper from a neighborhood of $0'$ in E to a certain small neighborhood of $0'$ in E', $f^*(\mathbb{R}^{n-1} \cap O(0'))$ fills in a real semi-analytic set B_0, which also serves as a uniqueness set for holomorphic functions over E'. By Lemma 2.5, we may assume that for a certain $a_0(\approx 0) \in B_0$ and $(n-1)$ n-indices $\{\alpha_1, \cdots, \alpha_{n-1}\}$, it holds that

$$\det \frac{\partial(h_{\alpha_1}, \cdots, h_{\alpha_{n-1}})}{\partial(z_1, \cdots, z_{n-1})}|_{a_0} \neq 0. \tag{4.1$'$}$$

Write $z_0 = (x_0', 0) \in \mathcal{S}$ with $f(z_0) = (a_0, 0)$. Notice by Theorem 4.2, f is real analytic over $\mathcal{S}$ at most away from a thin set. Also, from the hypothesis of the main theorem, we can assume that $g \not\equiv 0$ along the normal directions near $(x_0', 0)$. In what follows, we assume that $\mathcal{W}^+$ and $\mathcal{S}$ are sufficiently close to z_0.

Now, applying $\overline{L_j}$ to (4.1) for each $j \in \{1, \cdots, n-1\}$, we obtain

$$\overline{L_j g} = (g(z))^m \sum_{l=1}^{n-1} D_{b_l}^{e_j} H(f^*(z), \overline{f^*(z)}, g(z)) \overline{L_j f_l}, \tag{4.2}$$

where $e_j = (0, \cdots, 0, 1^{jth}, 0, \cdots, 0)$. Write for $z \in \mathcal{W}^+$,

$$\lambda(z) = \Lambda(z, z, \overline{Df(\overline{z})}).$$

By theorem 4.2, we know that $\lambda(z)$ is almost everywhere C^0 up to $\mathcal{S}$. On the other hand, $\lambda(z)$ blows up polynomially when $z \to \mathcal{S}$, by the Cauchy estimates, and $\lambda(z)$ has a continuous boundary limit. Hence, $\lambda(z) \in C^0(\mathcal{W}^+ \cup \mathcal{S})$ with $\lambda(0) \neq 0$.

Regard now (4.2) as a system of linear equations in

$$g^m D_b^{e_j} H(f^*, \overline{f^*}, g).$$

Using the fact that $\lambda|_0 \neq 0$ and the Cramer rule, we see that for each j and $z \in \mathcal{S}$,

$$(g^*(x))^m D_b^{e_j} H(f^*, \overline{f^*}, g) = \chi_{e_j}(z) = \frac{\lambda_{e_j}(z, \overline{z}, \overline{Df(z)})}{w^{km}\lambda(z)},$$

where λ_{e_j} is holomorphic in its variable. Hence, a similar argument as above shows that $\chi_{e_j}(z)$ extends holomorphically to $\mathcal{W}^+$ and continuously over $\overline{\mathcal{W}}^+$.

After solving (4.2) for $g^m(z)D_b^{e_j}H(f^*(z),\overline{f^*(z)},g(z))$, we can still apply $\overline{L_l}$ and reach, by limit, to the points where f does not extend. More generally, inductively repeating such a process, we get the following:

For each multiple index α, there exists a function λ_α holomorphic in its arguments and polynomial in $D^\alpha f$ with $\|\alpha\| > 1$ such that for each $z \in \mathcal{S}$,

$$g^{*m}(z)D_b^\alpha H(f^*(z),\overline{f^*(z)},g(z)) = \frac{\lambda_\alpha(z,\overline{z},\overline{Df},\cdots,\overline{D^\alpha f})}{w^{km}\lambda(z)}.$$

For $z \in \mathcal{W}^+$, writing $\chi_\alpha(z) = \frac{\lambda_\alpha(z,z,\cdots,\overline{D^\alpha f(\overline{z})})}{w^{km}\lambda(z)}$ and repeating the previous argument, we obtain the following

Claim 4.3: For each multiple index α, there exists a function χ_α holomorphic in $\mathcal{W}^+$ and continuous up to $\mathcal{W}^+ \cup \mathcal{S}$ such that

$$(g^*(x))^m D_b^\alpha H(f^*(x),\overline{f^*(x)},g(x)) = \chi_\alpha(x)$$

for $x \in \mathcal{S}$.

Notice that $\frac{1}{\alpha!}D_b^\alpha H(f^*(x),\overline{f^*(x)},g(x))$ can be expanded as

$$h_{\alpha,0}(f^*,\overline{f^*}) + \sum_{\alpha,j} h_{\alpha,j}(f^*,\overline{f^*})g^j.$$

Using (4.1)′, we can apply the implicit function theorem to see that near $(x_0',0) \in \mathcal{S}$ and for each $j < n$,

$$f_j(x) = \sigma_j(x,\overline{f^*},g^*(x),x_n^k g^*(x),\widetilde{\chi}_{\alpha_1},\cdots,\widetilde{\chi}_{\alpha_{n-1}}) \tag{4.3}$$

where σ_j is holomorphic in its argument. Taking the complex conjugate to (4.1), we note that there is a certain holomorphic function σ_n in its arguments such that for $x \in \mathcal{S}$

$$g^* = \sigma_n(x_n,f^*,\overline{f^*},\overline{g^*}) = \overline{g^*} + o(x_n). \tag{4.3$'$}$$

Substituting $(4.3)'$ to (4.3) and applying again the implicit function theorem, we can solve g^* in terms of $(x, \overline{f^*}, \overline{g}, \widetilde{\chi_{\alpha_j}})$ and get:

Claim 4.4: For each j and $x(\subset \mathcal{S}) \approx (x_0', 0)$, there exists a function $\widetilde{\sigma}_j$ holomorphic in its arguments such that

$$f_j(x) = \widetilde{\sigma}_j(x, \overline{f^*(x)}, \overline{g^*(x)}, \widetilde{\chi}_{\alpha_1}, \cdots, \widetilde{\chi}_{\alpha_{n-1}}),$$

$$g(x) = \widetilde{\sigma}_n(x, \overline{f^*(x)}, \overline{g^*(x)}, \widetilde{\chi}_{\alpha_1}, \cdots, \widetilde{\chi}_{\alpha_{n-1}}).$$

Recall that g^* extends anti-holomorphically to $\mathcal{W}^+$. Hence, $f(x)$ extends holomorphically to $\mathcal{W}^-$ and

$$\widetilde{\sigma}_j(x, \overline{f^*(x)}, \overline{g^*(x)}, \widetilde{\chi}_{\alpha_1}, \cdots, \widetilde{\chi}_{\alpha_{n-1}})$$

extends holomorphically to $\mathcal{W}^+$. Making use of the weak version of the edge of the wedge theorem, we get that $f = (f^*, g)$ extends holomorphically across $(x_0', 0)$. By Theorem 4.1, we conclude the proof of our Main Theorem. ∎

Proof of Corollary: Now that the f in the corollary extends across 0, we can conclude that $g = z_n^k g^*(z)$ near the origin with $g^*(0) \neq 0$. Indeed, suppose that $Z_g = E \cup_j V_j$ be the finite decomposition of the zero of g into its irreducible components near 0. Then for each $a \in V_j \setminus E$, since $f(Q_a) \subset E'$ by the invariant property, it follows that $g(Q_a) \equiv 0$. Hence when $a \approx 0$, Q_a has to be a certain component of Z_g. This is a contradiction, because $Q_a \cap E = \emptyset$. Hence, $Z_g = E$ and $g = w^k g^*$ with $g^*(0) \neq 0$.

Therefore, $f^{-1}(0)$ is locally finite near 0 and f is locally proper near 0. This gives the proof of the corollary. ∎

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