Université Paris Sud XI Orsay

THE LOCAL FLATTENING THEOREM

Master 2 Memoire by
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Introduction

The aim of this text is to give a proof of the local flattening theorem of analytic geometry. The essential content of this result is that given a morphism of complex spaces $f : V \to W$ and a compact subset L of a fibre $f^{-1}(y)$ of f, we can pick a finite number of finite compositions of local blowings up $\sigma_\alpha : W_\alpha \to W$ such that the strict transform of f by the σ_α is flat at each point of W_α corresponding to L, and the union of images of the σ_α is a neighborhood of y in W. Thus the theorem provides a means of transforming f into to a finite set of mappings that are flat at the points corresponding to L, and which, up to a restriction of the domain, are the base change of f by a finite composition of local blowings up. This finds application for example in the study of subanalytic sets and subanalytic functions (those functions with subanalytic graphs), where the theorem allows to reduce the problem of rectilinearization of sets to the semianalytic case [1, 2].

The first proof of the local flattening theorem was given in [3] using generalized Newton polygon techniques. Hironaka published separately a different proof the same year [1]. Both proofs use the notion of an *étoile* over W: an étoile collects groups together some (though not all) of the finite compositions of local blowings up with a common point of W in their image. Etoiles provide a means of access to the topological requirements of the local flattening theorem, namely the requirement that the images of the σ_α form a neighborhood of y in W. In order to complete the proof of the theorem, we then select from each étoile corresponding to the point y a well-chosen finite sequence of local blowings up. The key to this last choice is the notion of a *flatificator*. The flatificator for the given f and L of the local flattening theorem is a maximal locally closed subspace P of W containing the point y such that the restriction of f above P is flat. The second proof of Hironaka is the exposition that we follow for this step of the proof. A very brief outline of Hironaka's proof of the existence of the flatificator is as follows: first one reduces to the case when $L = \{z\}$ is a single point of V; the result then rests on a generalization of the Weierstrass division theorem, which when f is flat at z provides a decomposition of $\mathcal{O}_{V,z}$ as a free $\mathcal{O}_{W,y}$ module, and when f is not flat provides a system of generators for the flatificator for f and z. Once we have the existence of the flatificator for a given map, these locally closed subspaces then provide us with the centres of the local blowings up in the local flattening theorem.

We can take the centres of the σ_α asserted by the local flattening theorem to be nowhere dense subspaces. This is important in order that we "capture" all of the information given by the morphism $f : V \to W$. More precisely, a closed subspace of an open set $U \subseteq W$ that is dense in a neighborhood of some point contains an entire irreducible component of U, and when we blow up with centre equal to an entire component this component transforms via the blowing up to the empty set, so we lose information. Another important property is that we can also ensure that the centres of the blowings up are smooth. We do not discuss this somewhat deeper addition to the proof, which depends on desingularization of analytic sets.

The exposition is organized as follows. In chapter 1 we briefly recall without proof some elementary definitions and results regarding coherent sheaves and complex spaces. All of these results can be found in [4, 5]; in general our use of complex spaces is closer to the more algebraic exposition of Grauert and Remmert. Further constructions on complex spaces are stated in later chapters as they are required; the same references apply in these cases. For the remainder of the text we follow the exposition of [1]. In chapter 2 we define the blowing up of a complex space along a closed subspace, establish existence of the blowing up, and look at the strict transform and a few further properties of this construction. In chapter 3 we define an étoile over a complex space, and we make precise the way in which an étoile captures only blowings up about a particular point. We also give a topological structure to the set of étoiles and use this to study the

mapping from an étoile to the common image point of the blowings up it contains. At this point we admit two results of Hironaka regarding étoiles; we hope our exposition remains sufficiently detailed to give a clear idea of how the notion of an étoile applies to the topological requirements of the local flattening theorem. Finally in chapter 4 we achieve the proof of the local flattening theorem following the procedure discussed above.

Chapter 1

Complex Spaces

In this chapter we recall without proof some standard facts about coherent sheaves, local properties of analytic sets and complex spaces.

1.1 Sheaves and coherence

Let X and Y be topological spaces, let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X and Y respectively, and let $f : X \to Y$ be a continuous map. The direct image of $\mathcal{F}$ by f is

$$f_*\mathcal{F} : U \mapsto \mathcal{F}(f^{-1}(U)), \quad U \subseteq Y \text{ open},$$

and the inverse image of $\mathcal{G}$ by f is

$$f^{-1}\mathcal{G} : U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V), \quad U \subseteq X \text{ open}.$$

If $\mathcal{A}$ is a sheaf of rings on X and $\mathcal{F}, \mathcal{F}'$ are sheaves of $\mathcal{A}$-modules (briefly, $\mathcal{A}$-modules) then we define

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{F}') : U \mapsto \mathrm{Hom}_{\mathcal{A}_{|U}}(\mathcal{F}_{|U}, \mathcal{F}'_{|U}).$$

We have the adjunction property:

$$\mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}).$$

From now on, let $\mathcal{A}$ be a sheaf of rings on X and let $\mathcal{F}$ be an $\mathcal{A}$-module. We say that $\mathcal{F}$ is *locally free of rank m* if for all $x \in X$ there exists an open neighborhood U of x such that

$$\mathcal{F}_{|U} \simeq \mathcal{A}^m_{|U}.$$

We say that $\mathcal{F}$ is *locally finitely generated* if for all $x \in X$ there exists an open neighborhood U of x and a finite number of sections $s_1, \ldots, s_q \in \mathcal{F}(U)$ such that for all $y \in U$, the germs $s_{1,y}, \ldots, s_{q,y}$ generate the stalk $\mathcal{F}_y$. If $U \subseteq X$ is open and $s_1, \ldots, s_q \in \mathcal{F}(U)$ are any sections, then we define an $\mathcal{A}(U)$-module

$\mathcal{R}(s_1, \ldots, s_q)(U)$ to be the kernel of the morphism

$$(g_1, \ldots, g_q) \in \mathcal{A}_x^q \to g_1 s_{1,x} + \cdots + g_q s_{q,x} \quad x \in U.$$

In this way we define a *subsheaf of relations* $\mathcal{R}(s_1, \ldots, s_q) \subseteq \mathcal{A}_{|U}^q$. We say that $\mathcal{F}$ is *coherent* if $\mathcal{F}$ is locally finitely generated and for all $U \subseteq X$ and all sections $s_1, \ldots, s_q \in \mathcal{F}(U)$ the sheaf $\mathcal{R}(s_1, \ldots, s_q)$ is locally finitely generated. Equivalently, $\mathcal{F}$ is coherent if for each $x \in X$ there is an open neighborhood U of x and a finite presentation

$$\mathcal{A}_{|U}^q \to \mathcal{A}_{|U}^p \to \mathcal{F}_{|U} \to 0.$$

If we have an exact sequence of sheaves of $\mathcal{A}$-modules

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

and any two of the sheaves $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are coherent then the third is coherent. Moreover if $\mathcal{F}$ and $\mathcal{G}$ are two coherent subsheaves of a coherent sheaf $\mathcal{H}$ then the sheaves

$$\mathcal{F} \cap \mathcal{G} \text{ and } \mathcal{F} + \mathcal{G}$$

are coherent. A sheaf of rings $\mathcal{A}$ is *coherent* if it is coherent as an $\mathcal{A}$-module over itself.

1.2 Local properties of analytic sets

First we introduce some general notation. If M is a complex manifold, we write $\mathcal{O}_M$ for the sheaf of holomorphic functions on M. If $U \subseteq M$ is an open set, we write $\mathcal{O}_M(U)$, or just $\mathcal{O}(U)$ for the $\mathbb{C}$-algebra of holomorphic functions on U. Now let $U \subseteq \mathbb{C}^n$ be an open neighborhood of $0 \in \mathbb{C}^n$. Let $(z_1, \ldots, z_n)$ be coördinates on $\mathbb{C}^n$. We write

$$\mathcal{O}_n = \mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}\{z_1, \ldots, z_n\}$$

for the ring of germs of holomorphic functions at zero, with the latter notation indicating that this is equal to the ring of power series convergent in some neighborhood of $0 \in \mathbb{C}^n$. The fundamental theorem is the following:

Theorem 1.2.1 (Weierstrass preparation theorem)**.** *Let U be an open neighborhood of $0 \in \mathbb{C}^n$, and let $(z_1, \ldots, z_n) = (z', z_n)$ be coördinates on $\mathbb{C}^n$. Let $f \in \mathcal{O}(U)$ be a non-zero holomorphic function. Then after a generic, $\mathbb{C}$-linear change of coördinates there exists an integer $s \geq 0$ such that*

$$\lim_{z_n \to 0} f(0, z_n)/z_n^s \tag{1.2.1}$$

is non-zero and finite. For this choice of coördinates, after possibly shrinking the set U there exist

$$u \in \mathcal{O}(U) \text{ and } a_1, \ldots, a_s \in \mathcal{O}(U \cap \mathbb{C}^{n-1} \times \{0\})$$

with u non-vanishing on U and $a_1(0) = \cdots = a_s(0) = 0$ such that

$$f(z) = u(z)(z_n^s + a_1(z')z_n^{s-1} + \cdots + a_s(z')). \tag{1.2.2}$$

A consequence of this is:

Theorem 1.2.2 (Weierstrass division theorem)**.** *Let $f \in \mathcal{O}(U)$, and choose an open neighborhood U of $0 \in \mathbb{C}^n$ with coördinates $(z_1, \ldots, z_n)$ be coördinates on $\mathbb{C}^n$ such that* (1.2.1) *and* (1.2.2) *hold for some $s \geq 0$. Then for every bounded function $g \in \mathcal{O}(U)$ there exist unique $q, r \in \mathcal{O}(U)$ with r polynomial in z_n of degree at most $s - 1$ such that $f = qg + r$.*

Using these theorems it can be shown that $\mathcal{O}_n$ is a noetherian unique factorization domain. Moreover the embedding $\mathcal{O}_{n-1}[z_n] \hookrightarrow \mathcal{O}_n$ is inert, in the sense that $f \in \mathcal{O}_{n-1}[z_n]$ is irreducible if and only if it is irreducible in $\mathcal{O}_n$. Now let U be any open neighborhood of $0 \in \mathbb{C}^n$. We say that a subset $A \subseteq U$ is *analytic* if for each $x \in A$ there exists an open neighborhood V of x in U and holomorphic functions $f_1, \ldots, f_r \in \mathcal{O}(V)$ such that

$$A \cap V = \{z \in V : f_1(z) = \cdots = f_r(z) = 0\}.$$

We define an equivalence relation on the power set of $\mathbb{C}^n$ as follows: $A \sim B$ if and only if there exists a neighborhood V of 0 in $\mathbb{C}^n$ such that $A \cap V = B \cap V$. We write $(A, 0)$ for the equivalence class of a subset, called the *germ* of the subset at 0. We write $(A, 0) \subseteq (B, 0)$ if for some neighborhood V of 0 we have $A \cap V \subseteq B \cap V$, and we define the union of germs $(A, 0) \cup (B, 0)$ to be the germ of the union $(A \cup B, 0)$. Let $(A, 0)$ be a germ of an analytic subset of an open subset $U \subseteq \mathbb{C}^n$. If $f_0 \in \mathcal{O}_n$ then we say that f_0 vanishes on $(A, 0)$ if there exists an open neighborhood V of 0 in $\mathbb{C}^n$ such that f_0 and $(A, 0)$ have representatives f and A respectively in V and

$$f(z) = 0, \quad \forall z \in A \subseteq V.$$

We then associate to $(A, 0)$ an ideal $\mathcal{I}_{A,0} \subseteq \mathcal{O}_n$, which is the ideal of all the germs in $\mathcal{O}_n$ that vanish on $(A, 0)$. In the other direction, if $\mathcal{I} \subseteq \mathcal{O}_n$ is an ideal, then since $\mathcal{O}_n$ is noetherian we can choose generators $f_{1,0}, \ldots, f_{r,0}$. Then there exists some neighborhood V of 0 in $\mathbb{C}^n$ such that these germs have representatives $f_1, \ldots, f_r$ in V, and we define an analytic subset of V by

$$V(\mathcal{I}) = \{z \in V : f_1(z) = \cdots = f_r(z) = 0\}.$$

Now the germ $(V(\mathcal{I}), 0)$ does not depend on the choice of V or the generators of $\mathcal{I}$. We call $(V(\mathcal{I}), 0)$ the *zero locus* of $\mathcal{I}$. We call a germ of an analytic set $(A, 0)$ *irreducible* if whenever we have an expression

$$(A, 0) = (A_1, 0) \cup (A_2, 0)$$

there exists $i \in \{1, 2\}$ such that $(A, 0) = (A_i, 0)$. Then every germ $(A, 0)$ can be uniquely written as a finite unordered union of irreducible germs

$$(A, 0) = \bigcup_{i=1}^{r} (A_i, 0)$$

provided we demand that $(A_i, 0) \nsubseteq (A_j, 0)$ for $j \neq i$. These are called the *components* of $(A, 0)$, or, if A is a

representative of $(A, 0)$ in some open set, they are the *local components* of A at 0 (to be compared with the global components of a complex space below). A germ of an analytic set $(A, 0)$ is irreducible if and only if the ideal $\mathcal{I}_{A,0}$ is prime. An analytic description of irreducible germs of analytic sets is given by the following:

Theorem 1.2.3 (Local parametrization theorem)**.** *Let A be an irreducible analytic subset of an open neighborhood of 0 in $\mathbb{C}^n$. After a non-singular, $\mathbb{C}$-linear change of coördinates on $\mathbb{C}^n$ there exist $d \leq n$, polydisks $\Delta' \subseteq \mathbb{C}^d \times \{0\}$, $\Delta'' \subseteq \{0\} \times \mathbb{C}^{n-d}$ and a proper analytic subset $S \subseteq \Delta'$ such that*

(i) $A_S := A \cap ((\Delta' \backslash S) \times \Delta'')$ is a smooth, d-dimensional complex manifold, dense in $A \cap (\Delta' \times \Delta'')$;

(ii) The projection $A_S \to \Delta' \backslash S$ is a connected covering with a finite number of sheets.

We also have the nullstellensatz:

Theorem 1.2.4 (Nullstellensatz)**.** *If $\mathcal{I} \subseteq \mathcal{O}_n$ is an ideal and $(A, 0)$ is the zero locus of $\mathcal{I}$ then the ideal of A is given by $\mathcal{I}_{A,0} = \sqrt{\mathcal{I}}$, the radical of $\mathcal{I}$.*

A corollary of local parametrization is that if A is an analytic subset of an open set $U \subseteq \mathbb{C}^n$ then the set of regular points, i.e. the set of points at which A is locally a complex manifold, is a open dense subset of A. In fact, the singular points of A, i.e. all the non-regular points, form an analytic subset of A.

1.3 Complex Spaces

Topologically we define complex spaces by gluing together analytic subsets of open sets in $\mathbb{C}^n$. Our definition is that of [5]; what we call a complex space is the *complex analytic scheme* of [4]. That is, we allow for nilpotent elements in the structure sheaf of complex spaces. We suppose throughout that complex spaces are Hausdorff; this is mainly needed in chapter 3. First we fix our notation for ringed spaces.

Definitions 1.3.1. A *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. In the main text we usually write X in place of $(X, \mathcal{O}_X)$ and write X^{top} for the topological space when we wish to make the distinction. We say that $(X, \mathcal{O}_X)$ is a *locally ringed space* if for all $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring, and a *locally $\mathbb{C}$-ringed space* if in addition the sheaf of rings $\mathcal{O}_X$ is a sheaf of $\mathbb{C}$-algebras and the residue field of $\mathcal{O}_{X,x}$ is $\mathbb{C}$. A *morphism of ringed spaces* is given by

$$(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

where $f : X \to Y$ is a continuous map and $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a morphism of sheaves of rings. If moreover for each $x \in X$, if $\mathfrak{m}_{X,x}$ and $\mathfrak{m}_{Y,f(x)}$ are the maximal ideals of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,f(x)}$ respectively then

$$f_x^{\#}(\mathfrak{m}_{Y,f(x)}) \subseteq \mathfrak{m}_{X,x}$$

where $f_x^{\#}$ denotes the morphism induced on stalks by $f^{\#}$, then $(f, f^{\#})$ is a *morphism of locally ringed spaces*. In general we write $f : X \to Y$ for a morphism of complex spaces, reserving the notation $f^{\#}$ for when it is needed, and if $x \in X$ we write $f(x)$ for image point in Y.

A morphism of ringed spaces between locally $\mathbb{C}$-ringed spaces is also a morphism of locally ringed spaces: the induced $\mathbb{C}$-algebra homomorphisms are always homomorphisms of local rings. The basis of our construction of complex spaces is the following two theorems.

Theorem 1.3.2 (Oka's coherency theorem)**.** *If M is a complex manifold, then the sheaf of rings $\mathcal{O}_M$ is coherent.*

Theorem 1.3.3. *If $U \subseteq \mathbb{C}^n$ is an open set, and $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module, then*

$$\mathrm{supp}(\mathcal{F}) = \{x \in X : \mathcal{F}_x \neq 0\}$$

is an analytic subset of U.

Let $U \subseteq \mathbb{C}^n$ be an open set, and let $\mathcal{I} \subseteq \mathcal{O}_U$ be a coherent sheaf of ideals. Let A be the analytic set

$$A = \mathrm{supp}(\mathcal{O}_U/\mathcal{I}).$$

Then we call the locally $\mathbb{C}$-ringed space

$$(A, (\mathcal{O}_U/\mathcal{I})_{|A})$$

a *local model of complex analytic space.*

Definition 1.3.4. A *complex space* $(X, \mathcal{O}_X)$ is a ringed space over a Hausdorff topological space X such that there exists an open cover (Ω_λ) of X and isomorphisms of $\mathbb{C}$-ringed spaces

$$\Phi_\lambda : (\Omega_\lambda, \mathcal{O}_{X|\Omega_\lambda}) \to (A_\lambda, (\mathcal{O}_{U_\lambda}/\mathcal{I}_\lambda)_{|A_\lambda})$$

where $(A_\lambda, (\mathcal{O}_{U_\lambda}/\mathcal{I}_\lambda)_{|A_\lambda})$ is a local model of complex analytic space.

Note that a complex space is necessarily a locally ringed space. Locally a morphism of complex spaces is determined be the morphism of sheaves in the following sense. If $\phi : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a $\mathbb{C}$-algebra homomorphisms between local rings of complex spaces X and Y, then there exists an open set $U \subseteq X$ containing x and a morphism of complex spaces $h : X_{|U} \to Y$ such that $h_x^\# = \tau$, and if $g : X_{|V} \to Y$ is another morphism with this property then there is a neighborhood $W \subseteq U \cap V$ of x such that $h_{|W} = g_{|W}$. In general the morphism of sheaves is not determined by the map of topological spaces unless the domain is reduced. From the coherency theorem of Oka above we deduce the general result: for any complex space X the structure sheaf $\mathcal{O}_X$ is coherent. Let us state:

Theorem 1.3.5. *Let X be a reduced complex space. The set of regular points X_{reg} of X is a dense open subset of X, which is a disjoint union of connected complex manifolds:*

$$X_{reg} = \bigsqcup_\alpha X'_\alpha$$

If X_α is the closure of X'_α in X then

$$X = \bigcup_\alpha X_\alpha$$

and (X_α) is locally finite. We call the X_α the global irreducible components *of X.*

Further constructions on complex spaces are recalled in later chapters as they are encountered.

Chapter 2

Blowings up

2.1 Blowing up plane curves

The *blowing up* X_0 of $\mathbb{C}^2$ at 0 is defined by

$$X_0 = \{(p, \ell) : p \in \ell\} \subseteq \mathbb{C}^2 \times \mathbb{P}^1.$$

We have a natural map

$$\pi : X_0 \to \mathbb{C}^2$$

given by the restriction of the projection. This is called the *blowing up map*. Let (x, y) be coördinates on $\mathbb{C}^2$, and for $(u, v) \neq (0, 0)$ let us identify the line $vx - yu = 0$ with the point $[u : v] \in \mathbb{P}^1$. Then we see that:

$$X_0 = \{((x, y), [u : v]) \in \mathbb{C}^2 \times \mathbb{P}^1 : xv = yu\}.$$

Now we write $\mathbb{P}^1 = U_1 \cup U_2$ where $U_1 = \{[1 : v] : v \in \mathbb{C}\}$ and $U_2 = \{[u : 1] : u \in \mathbb{C}\}$. Then we have:

$$X^1 := X_0 \cap (\mathbb{C}^2 \times U_1) = \{((x, xv), [1 : v])\}$$

and

$$X^2 := X_0 \cap (\mathbb{C}^2 \times U_2) = \{((yu, y), [u : 1])\}$$

are isomorphic to $\mathbb{C}^2$ with coördinates (x, v) and (y, u) respectively. Moreover we see directly that π is an isomorphism outside $\pi^{-1}(0)$ and $E = \pi^{-1}(0) = \{0\} \times \mathbb{P}^1$; the latter is called the *exceptional divisor*. If $C \subseteq \mathbb{C}^2$ is a curve then the *strict transform* of C by π is

$$C' = \overline{\pi^{-1}(C \backslash \{0\})}.$$

The idea is that the blowing up π pulls apart the directions through the origin, so if we take two lines with gradients a and b passing through 0, their strict transforms will be two disjoint lines meeting E at the points

$((0,0),[1:a])$ and $((0,0),[1:b])$ respectively. To see this, assume for simplicity that $a \neq 0$, and consider

$$\pi^{-1}(\{y - ax = 0 : x \neq 0\}).$$

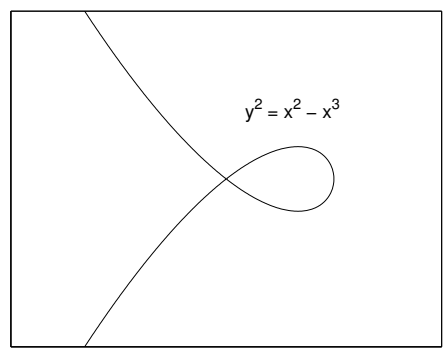
In the chart X^1 the set of points projecting to the line minus zero is:

$$\{((x, ax), [1:a]) : x \in \mathbb{C}\backslash\{0\}\}$$

and in the chart X^2 it is:

$$\{((y/a, y), [1:a]) : y \in \mathbb{C}\backslash\{0\}\}.$$

Therefore the strict transform C' of C is a line intersecting E at the point $[1:a]$. In this way blowing up allows us to desingularize curves. For example, take the curve $C = \{y^2 - x^2 + x^3 = 0\} \subseteq \mathbb{C}^2$, with a double point at $(0,0)$. We factorize C near 0 into $y = \pm x\sqrt{1-x}$, where $\sqrt{1-x}$ is a fixed branch of the square root. We can locally invert each branch to give $x = f_+(y)$ and $x = f_-(y)$ with f_+ and f_- holomorphic functions. Now, considering for example f_+ we have $y = f_+(y)\sqrt{1 - f_+(y)}$, and dividing through by y and letting $y \to 0$ we have that $f_+(y)/y \to 1$ as $y \to 0$, since $f_+(0) = 0$. Now in the chart X^2 we have:



$$\pi^{-1}_{|X^2}(C\backslash\{0\}) = \{((f_+(y), y), [f_+(y)/y, 1]) : y \neq 0\}.$$

Letting $y \to 0$ in this expression we see that the intersection of the strict transform of this branch with the exceptional divisor can be seen in the chart X^1. We see in the same way that this is also true for the branch $x = f_-(y)$. Therefore we regard the chart X^1, where we have:

$$\pi^{-1}_{|X^1}(C\backslash\{0\}) = \{((x, \pm x\sqrt{1-x}), [1, \pm\sqrt{1-x}]) : x \neq 0\}.$$

We see that C' intersects E at two points, $[1:1]$ and $[1:-1]$ (which is what we showed before in the chart X^2); viewing C as a union of two graphs, we have one point of intersection for each branch. With $v = \pm\sqrt{1-x}$, in the (x,v)-plane the curve C is transformed to the curve $C' = \{x = 1 - v^2\}$, with the two points of intersection with E corresponding to $x = 0$, $v = \pm 1$, which are smooth points of C'. In this example, the blowing up provides a mechanism by which we can realize the curve C as the 'shadow' of a smooth curve in X_0, which is itself embedded in the higher dimensional space $\mathbb{C}^2 \times \mathbb{P}^1$.

2.2 Blowing up complex spaces

Let

$$\mathbb{X} = \big\{(z, \ell) \in \mathbb{C}^n \times \mathbb{P}^{n-1} : z \in \ell\big\}.$$

We find that

$$\mathbb{X} = \{((z_1, \dots, z_n), [\ell_1 : \cdots : \ell_n]) : z_i\ell_j - \ell_i z_j = 0,\ 0 \le i < j \le n\}.$$

This is a set of $\binom{n+1}{2}$ equations, so $\mathbb{X}$ is a complex space. In fact $\mathbb{X}$ is a complex manifold of dimension n. Let $U_j = \{\ell_j = 1\}$ be an affine chart of $\mathbb{P}^{n-1}$. Then we have local trivializations

$$\begin{aligned} &\mathbb{X}_{|U_j} \xrightarrow{\sim} U_j \times \mathbb{C} \\ &(z, \ell) \mapsto (\ell, z_j) \end{aligned} \tag{2.2.1}$$

Here we describe a point of $\mathbb{X}$ by the direction of the line ℓ, and the displacement along this line in the z_j direction; by our choice of chart we cannot have z_j constant in the direction ℓ. Now let $\pi : \mathbb{X} \to \mathbb{C}^n$ be the restriction of the projection. Then

$$\pi_{|\mathbb{X}\backslash\pi^{-1}(0)} : \mathbb{X}\backslash\pi^{-1}(0) \to \mathbb{C}^n\backslash\{0\}$$

is an isomorphism, and we have $\pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}$. In the same way as for the two dimensional case, the exceptional divisor parametrizes the directions through the origin in $\mathbb{C}^n$.

Definition 2.2.1. Let X be a complex space, and let Y be a closed subspace with ideal sheaf $\mathcal{I}$. Then $\pi : X' \to X$ is the blowing up of X with centre Y if

(i) $\mathcal{I}\mathcal{O}_{X'}$ is an invertible $\mathcal{O}_{X'}$-module;

(ii) For any morphism of complex spaces $f : T \to X$, such that $\mathcal{I}\mathcal{O}_T$ is an invertible $\mathcal{O}_T$-module, there exists a unique $h : T \to X'$ such that $\pi \circ h = f$

$$\begin{tikzcd} T \arrow[d, "\exists! h"'] \arrow[dr, "f"] & \\ X' \arrow[r, "\pi"'] & X \end{tikzcd}$$

Remark 2.2.2. Here $\pi^{-1}\mathcal{I}$ is a $\pi^{-1}\mathcal{O}_X$-module as a sheaf on X'. Moreover there is a natural morphism $\pi^{-1}\mathcal{O}_X \to \mathcal{O}_{X'}$, and we let $\mathcal{I}\mathcal{O}_{X'}$ be the ideal generated by the image of $\pi^{-1}\mathcal{I}$. Recall that $\pi^*\mathcal{I}$ is defined to be $\pi^{-1}\mathcal{I} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_{X'}$. Now we have an inclusion $\mathcal{I} \subseteq \mathcal{O}_X$ which gives a morphism $\pi^*\mathcal{I} \to \pi^*\mathcal{O}_X = \mathcal{O}_{X'}$ such that $\mathcal{I}\mathcal{O}_{X'}$ is the image of $\pi^*\mathcal{I}$ under this morphism. Now the map $\pi^*\mathcal{I} \to \mathcal{I}\mathcal{O}_{X'}$ is surjective by definition so if $\sigma : X'' \to X'$ is another morphism of complex spaces,

$$\sigma^{-1}\pi^*\mathcal{I} \to \sigma^{-1}\mathcal{I}\mathcal{O}_{X'}$$

is surjective, since σ^{-1} is an exact functor. Tensoring over $\sigma^{-1}\mathcal{O}_{X'}$ we have a surjective map

$$\sigma^*\pi^*\mathcal{I} \to \sigma^*(\mathcal{I}\mathcal{O}_{X'}).$$

Now we find:

$$(\mathcal{I}\mathcal{O}_{X'})\mathcal{O}_{X''} = \mathrm{im}(\sigma^*(\mathcal{I}\mathcal{O}_{X'}) \to \mathcal{O}_{X''})$$

$$= \mathrm{im}(\sigma^*\pi^*\mathcal{I} \to \mathcal{O}_{X''})$$
$$= \mathrm{im}((\pi \circ \sigma)^*\mathcal{I} \to \mathcal{O}_{X''})$$
$$= \mathcal{I}\mathcal{O}_{X''}$$

where in the last line $\mathcal{I}\mathcal{O}_{X''}$ is defined via the map $\pi \circ \sigma$. Thus this construction commutes with composition of mappings. Moreover we see easily that if $\mathcal{I}_1, \mathcal{I}_2$ are coherent sheaves of ideals of $\mathcal{O}_X$ then:

$$(\mathcal{I}_1\mathcal{I}_2)\mathcal{O}_{X'} = (\mathcal{I}_1\mathcal{O}_{X'})(\mathcal{I}_2\mathcal{O}_{X'}).$$

Indeed, both are locally generated as ideals of $\mathcal{O}_{X'}$ by products of the form $\pi^*w_1\pi^*w_2$.

If follows from the definition that if the blowing up exists, then it is unique up to unique isomorphism. We call $\pi^{-1}(Y)$ the *exceptional divisor*. From the definition of the blowing up $\pi : X' \to X$ with centre Y, it follows that if $U \subseteq X$ is an open subset, then $\pi_{|\pi^{-1}U} : \pi^{-1}(U) \to U$ is the blowing up with centre $Y \cap U$.

Proposition 2.2.3. *The morphism $\pi : \mathbb{X} \to \mathbb{C}^n$ defined above is the blowing up with centre*

$$0 = \mathrm{supp}(\mathcal{O}_{\mathbb{C}^n}/(z_1, \ldots, z_n)).$$

Proof. (i) The ideal of 0 in $\mathbb{C}^n$ is $\mathcal{I} = (z_1, \ldots, z_n)$, and we want to see that $(z_1, \ldots, z_n) \cdot \mathcal{O}_{\mathbb{X}}$ is invertible (here π is just a projection, so we write z_i also for the image of $z_i \in \mathcal{O}_{\mathbb{C}^n}$ in $\mathcal{O}_{\mathbb{X}}$). Since $\mathbb{X}$ is defined by the equations $z_i\ell_j - z_j\ell_i$, in the chart $\mathbb{X}_k = \{\ell_k = 1\}$, we have for each $j \neq k$ that $z_j = \ell_j z_k$, and therefore $\mathcal{I}$ is generated by z_k in $\mathbb{X}_k$. The isomorphism (2.2.1) shows that z_k is a nonzerodivisor in $\mathcal{O}_{\mathbb{X}_k}$.

(ii) Let $f : T \to \mathbb{C}^n$ any map such that $\mathcal{I}\mathcal{O}_T$ is an invertible $\mathcal{O}_T$-module. Let $f^\# : \mathcal{O}_{\mathbb{C}^n} \to f_*\mathcal{O}_T$ be the comorphism. At a given point $w \in T$, the ideal $(\mathcal{I}\mathcal{O}_T)_w$ in the local ring $\mathcal{O}_{T,w}$ is generated by $f^\#(z_1), \ldots, f^\#(z_n)$. In a local ring, if a finite number of elements generate a principal ideal, then some element among this set of generators is a generator of the ideal. Therefore there exists i such that $f^\#(z_i)$ generates $(\mathcal{I}\mathcal{O}_T)_w$. By coherence, it follows that $f^\#(z_i)$ generates $\mathcal{I}\mathcal{O}_T$ in a neighborhood of w. Thus we have an open cover $U_1, \ldots, U_n$ of T such that $f^\#(z_i)$ generates $\mathcal{I}\mathcal{O}_{U_i}$. By uniqueness of the blowing up and the statement preceding this proposition, it suffices to establish existence of the required factorization on the sets U_i and the intersections $U_i \cap U_j$. In particular we have reduced to the case when there exists i such that $f^\#(z_i)$ generates $\mathcal{I}\mathcal{O}_T$. Now by hypothesis $\mathcal{I}\mathcal{O}_T$ is invertible, so there exist sections ξ_{ij} of $\mathcal{O}_T$ for each i, j with $j \neq i$ such that $f^\#(z_j) = f^\#(z_i)\xi_{ij}$. We define $\xi_{ii} = f^\#(z_i)$ and a map $h : T \to \mathbb{X}_i = \{\ell_i = 1\}$ by:

$$h^\#(z_j) = f^\#(z_j), \qquad h^\#(\ell_j) = \xi_{ij}.$$

First we see that $h^\#$ does indeed define a morphism of $\mathbb{C}$-algebras: in $\mathbb{X}_i$ we have the equations $z_j = \ell_j z_i$, and h satisfies:

$$h^\#(z_j) = f^\#(z_j) = f^\#(z_i)\xi_{ij} = h^\#(\ell_j)h^\#(z_i).$$

We deduce that:

$$h^\#(\ell_k)h^\#(z_j) = h^\#(\ell_j)h^\#(\ell_k)h^\#(z_i) = h^\#(\ell_j)h^\#(z_k)$$

in $\mathbb{X}_i$ for each j and k. By construction, we have the identity:

$$h^\# \circ \pi^\# = f^\#$$

since $\pi^\#$ maps z_j to the class of z_j in $\mathcal{O}_\mathbb{X}$ for each j. This establishes existence of h. Now let $g : T \to \mathbb{X}$ satisfying the same property. We want to see that $g = h$. First we claim that the image of g is contained in $\mathbb{X}_i$. If not, then there is $t \in T$ such that $g^\#(\ell_i)(t) = 0$ and for some j, $g^\#(\ell_j)(t) \neq 0$, where we regard sections as functions via the morphism $\mathcal{O}_W \to \mathcal{C}_W$ from the sheaf of rings to the sheaf of continuous functions defined for any complex space W. That is, $g^\#(\ell_j) \notin \mathfrak{m} = \mathfrak{m}(\mathcal{O}_{T,t})$ and $g^\#(\ell_i) \in \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{T,t}$. We have the relation

$$g^\#(z_i)g^\#(\ell_j) = g^\#(z_j)g^\#(\ell_i) \tag{2.2.2}$$

which, since $g^\#(z_j) = f^\#(z_j)$ for all j, shows that $f^\#(z_i) \in f^\#(z_j)\mathfrak{m}$. Now $f^\#(z_i)$ generates $(f^\#(z_1), \ldots, f^\#(z_n))\mathcal{O}_{T,t}$, so this implies that $f^\#(z_j) \in f^\#(z_j)\mathfrak{m}$, i.e. $f^\#(z_j) = 0$ by the Krull intersection theorem. Now by (2.2.2) this means that $f^\#(z_i)g^\#(\ell_j) = 0$ in a neighborhood of t, and since $g^\#(\ell_j) \notin \mathfrak{m}$ this means that also $f^\#(z_i) = 0$ in a neighborhood of t, which is impossible. Thus g maps into $\mathbb{X}_i$. Now the relations $z_j = z_i\ell_j$ in $\mathbb{X}_i$ mean that $g^\#(\ell_j) = \tilde{\xi}_{ij}$ with $f^\#(z_j) = f^\#(z_i)\tilde{\xi}_{ij}$ so we have:

$$f^\#(z_i)(\xi_{ij} - \tilde{\xi}_{ij}) = 0.$$

Now by assumption $f^\#(z_i)$ generates the invertible ideal $\mathcal{I}\mathcal{O}_T$, so this implies that $\xi_{ij} = \tilde{\xi}_{ij}$, and therefore $g^\# = h^\#$. □

Remark 2.2.4. It follows from this proposition that the blowing up $\mathbb{X} \to \mathbb{C}^n$ constructed above is not dependent on the choice of coördinates on the linear space $\mathbb{C}^n$.

Let us recall now the universal property for the fibre product of two complex spaces via morphisms $\phi_1 : X \to S$, $\phi_2 : Y \to S$. This is the complex space $X \times_S Y$ with canonical projections $\mathrm{pr}_1 : X \times_S Y \to X$, $\mathrm{pr}_2 : X \times_S Y \to Y$ such that if Z is any complex space, and $f : Z \to X$, $g : Z \to Y$ are two mappings such that $\phi_1 \circ f = \phi_2 \circ g$, then there exists a unique map $h : Z \to X \times_S Y$ such that $f = \mathrm{pr}_1 \circ h$ and $g = \mathrm{pr}_2 \circ h$.

$$\begin{tikzcd}
Z \arrow[dr, "h"] \arrow[drr, "g"] \arrow[ddr, "f"'] & & \\
& X \times_S Y \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"] & Y \arrow[d, "\phi_2"] \\
& X \arrow[r, "\phi_1"'] & S
\end{tikzcd}$$

We recall the following particular cases of fibre products: if S is a single point then the fibre product is the direct product. If $Y = S'$ is an open (resp. closed) subset of S and ϕ_2 is the inclusion then $X \times_S S'$ is the inverse image $\phi_1^{-1}(S')$ of S' in X. For any X, Y, S, the fibre product $X \times_S Y$ is a closed subspace of the direct product $X \times Y$.

Proposition 2.2.5. *If $\pi : X' \to X$ is the blowing up with centre Y, and V is any complex space, then $\pi \times \mathrm{id}_V : X' \times V \to X \times V$ is the blowing up with centre $Y \times V$.*

Proof. Denote by pr_1 and pr_2 the canonical projections for $X \times V$, and $\mathrm{pr}_1', \mathrm{pr}_2'$ those for $X' \times V$. Let $\mathcal{I}$ be the ideal sheaf of Y in $\mathcal{O}_X$. The ideal $\mathcal{I}\mathcal{O}_{X\times V}$ is the ideal of $Y \times V$ in the product space, so to verify the first part of the definition of the blowing up we want to see that $(\mathcal{I}\mathcal{O}_{X\times V})\mathcal{O}_{X'\times V}$ is invertible. We have $\mathrm{pr}_1 \circ (\pi \times \mathrm{id}_V) = \pi \circ \mathrm{pr}_1'$, so:

$$(\mathcal{I}\mathcal{O}_{X\times V})\mathcal{O}_{X'\times V} = (\mathcal{I}\mathcal{O}_{X'})\mathcal{O}_{X'\times V}. \tag{2.2.3}$$

By hypothesis $\mathcal{I}\mathcal{O}_{X'}$ is invertible, and $X' \times V \to X'$ is simply a projection, so the right hand side of (2.2.3) is invertible. To see this, we identify V locally with an analytic subset of a neighborhood of $0 \in \mathbb{C}^r$. Then we can write the local ring of $X' \times V$ at the point $x' \times 0$ as a quotient of $\mathcal{O}_{X',x}\{u_1, \ldots, u_r\}$ such that $\mathcal{O}_{X',x}$ injects into this quotient. Then the comorphism of the projection is the embedding

$$\mathcal{O}_{X',x} \hookrightarrow \mathcal{O}_{X',x}\{u_1, \ldots, u_r\}$$

followed by the canonical projection to the quotient defining $X' \times V$ at $x \times 0$. It is immediate that a nonzerodivisor maps to a nonzerodivisor via this mapping. Now let $f : T \to X \times V$ be any morphism such that $(\mathcal{I}\mathcal{O}_{X\times V})\mathcal{O}_T = \mathcal{I}\mathcal{O}_T$ is invertible (here $\mathcal{I}\mathcal{O}_T$ is generated with respect to the morphism $\mathrm{pr}_1 \circ f$). Then by the universal property for the blowing up $\pi : X' \to X$ applied to $\mathrm{pr}_1 \circ f$, there exists a unique morphism $g : T \to X'$ such that $\mathrm{pr}_1 \circ f = \pi \circ g$. By the universal property for the product $X' \times V$, there is a unique $h : T \to X' \times V$ such that $\mathrm{pr}_1' \circ h = g$ and $\mathrm{pr}_2' \circ h = \mathrm{pr}_2 \circ f$. That is, we have a diagram:

$$\begin{tikzcd}
T \arrow[dr, "h"] \arrow[drr, "g"] \arrow[ddr, "\mathrm{pr}_2 \circ f"'] & & \\
& X' \times V \arrow[r, "\mathrm{pr}_1'"'] \arrow[d, "\mathrm{pr}_2'"] & X' \\
& V &
\end{tikzcd}$$

Now in the following diagram, noting that $\mathrm{pr}_1 \circ f = \pi \circ g$, we see that placing both f and $(\pi \times \mathrm{id}) \circ h$ along the arrow $T \to X \times V$ in the following diagram make it commute:

$$\begin{tikzcd}
T \arrow[dr, "h"] \arrow[drr, "g"] \arrow[ddr, "\mathrm{pr}_2 \circ f"'] & & & \\
& X' \times V \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, "\pi \times \mathrm{id}"] & X' \arrow[dr, "\pi"] & \\
& V \arrow[dr, "\mathrm{id}"'] & X \times V \arrow[r] \arrow[d] & X \\
& & V &
\end{tikzcd}$$

Therefore by the uniqueness statement of the universal property of $X \times V$ we have $(\pi \times \mathrm{id}) \circ h = f$. $\square$

2.3 The strict transform

Let us recall the definition of the inverse image of a complex space. If $f : X' \to X$ is a morphism of complex spaces, and $Z \subseteq U \subseteq X$ is a locally closed subspace of X, then $\pi^{-1}(Z)$ is the closed subspace of $\pi^{-1}(U)$ defined by the coherent sheaf of ideals $(\mathcal{I}\mathcal{O}_{X'})_{|\pi^{-1}(U)} = \mathcal{I}\mathcal{O}_{\pi^{-1}(U)}$. This is compatible with the point-set theoretic inverse image under a mapping, and we have an induced morphism of complex spaces $f : \pi^{-1}(Z) \to Z$.

Proposition 2.3.1. *Let $\pi : X' \to X$ be the blowing up with centre Y. Let Z be any locally closed complex subspace of X. Then there exists a unique smallest closed subspace $Z' \subseteq \pi^{-1}(Z)$ such that*

$$Z' \backslash \pi^{-1}(Y) = \pi^{-1}(Z) \backslash \pi^{-1}(Y).$$

Moreover, π induces the blowing up $\pi : Z' \to Z$ with center $Z \cap Y$.

Definition 2.3.2. The space Z' in the proposition is called the *strict transform* of Z by the blowing up π.

We will prove this proposition after a series of lemmas. The first two are coherency theorems for analytic sheaves. The third is a simple consequence of the nullstellensatz, a proof is indicated in [1].

Lemma 2.3.3. *Let X be a complex space, let $\mathcal{O} = \mathcal{O}_X$, and let $\mathcal{A}$ be a coherent $\mathcal{O}$-module. Then there is a canonical homomorphism*

$$\begin{aligned} \alpha : \mathcal{O} &\to \operatorname{Hom}_{\mathcal{O}}(\mathcal{A}, \mathcal{A}) \\ : f_x &\mapsto [s_x \mapsto f_x s_x] \end{aligned}$$

In particular we have:

$$\ker \alpha = \operatorname{Ann}(\mathcal{A}) := \bigcup_x \operatorname{Ann}(\mathcal{A}_x),$$

and $\operatorname{Ann}(\mathcal{A})$ *is coherent.*

Proof. Let $x \in X$. An element $s_x \in \operatorname{Hom}_{\mathcal{O}}(\mathcal{A}, \mathcal{A})_x$ is the germ of a section $s \in \operatorname{Hom}_{\mathcal{O}}(\mathcal{A}, \mathcal{A})(U)$ for some neighborhood U of x. Since two such sections induce the same morphism $\mathcal{A}_x \to \mathcal{A}_x$ we obtain a morphism $\tau : \operatorname{Hom}_{\mathcal{O}}(\mathcal{A}, \mathcal{A})_x \to \operatorname{Hom}_{\mathcal{O}_x}(\mathcal{A}_x, \mathcal{A}_x)$. We show that τ is an isomorphism. Let $\phi : \mathcal{A}_{|U} \to \mathcal{A}_{|U}$ such that the germ ϕ_x at x is zero. Since $\mathcal{A}$ is coherent, we have that ϕ is zero in a neighborhood of x, so τ is injective. Now let $\sigma : \mathcal{A}_x \to \mathcal{A}_x$ be a homomorphism. Take a neighborhood U of x and sections $s_1, \ldots, s_p$ that generate $\mathcal{A}_{|U}$. Possibly shrinking U, let $t_1, \ldots, t_p$ sections such that we have $\sigma(s_j) = t_j$ for $j = 1, \ldots, p$. Now let $\mathcal{R}(s_1, \ldots, s_p)$ be the sheaf of relations between the s_j. By coherency, this is generated by relations:

$$(f_{ij})_{i=1,\ldots,p} \in \mathcal{O}(U)^p, \qquad j = 1, \ldots, q$$

where again we possibly shrink U. Now we claim that these generators belong to $\mathcal{R}(t_1, \ldots, t_p)(U)$ for some possibly smaller U. Indeed, at the point x, since σ is an $\mathcal{O}_x$-homomorphism $\mathcal{A}_x \to \mathcal{A}_x$

$$\left(\sum_{i=1}^{p} f_{ij} t_i \right)_x = \sigma \left(\sum_{i=1}^{p} f_{ij} s_i \right)_x = \sigma(0) = 0$$

Thus we have that the homomorphism $\phi_U : \mathcal{A}_{|U} \to \mathcal{A}_{|U}$ defined by

$$\phi_y \left(\sum_{i=1}^{p} a_i s_i \right)_y = \left(\sum_{i=1}^{p} a_i t_i \right)_y$$

This gives the required map, i.e. we have $\phi_x = \sigma$. This shows that τ is an isomorphism. Now we can show that $\mathrm{Hom}(\mathcal{A}, \mathcal{A})$ is coherent. In a neighborhood of each point we have a resolution

$$\mathcal{O}_U^p \to \mathcal{O}_U^q \to \mathcal{A}_{|U} \to 0.$$

Then we have:

$$0 \to \mathrm{Hom}(\mathcal{A}_{|U}, \mathcal{A}_{|U}) \to \mathrm{Hom}(\mathcal{O}_U^q, \mathcal{A}_{|U}) \to \mathrm{Hom}(\mathcal{O}_U^p, \mathcal{A}_{|U}).$$

Taking stalks we have that this is exact, since $\mathrm{Hom}_{\mathcal{O}}(\mathcal{A}, \mathcal{A})_x = \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{A}_x, \mathcal{A}_x)$, so we may apply the result of commutative algebra to this effect. This sequence is:

$$0 \to \mathrm{Hom}(\mathcal{A}_{|U}, \mathcal{A}_{|U}) \to \mathcal{A}_{|U}^q \to \mathcal{A}_{|U}^p.$$

Therefore $\mathrm{Hom}(\mathcal{A}, \mathcal{A})$ is coherent. It now follows that $\mathrm{Ann}\,(\mathcal{A})$ is coherent, as it is the kernel of a morphism of coherent sheaves. □

Lemma 2.3.4. *Let X be a complex space and let $\mathcal{A}$ be a coherent $\mathcal{O}_X$-module. Let $(\mathcal{A}_i)_{i \in I}$ be a directed family of coherent $\mathcal{O}_X$-submodules of $\mathcal{A}$, that is, such that for all $i, j \in I$, there exists $k \in I$ such that*

$$\mathcal{A}_i + \mathcal{A}_j \subseteq \mathcal{A}_k.$$

Then

$$\mathcal{B} = \bigcup_{i \in I} \mathcal{A}_i$$

is a coherent sheaf, locally isomorphic to some $\mathcal{A}_i$ in a neighborhood of each point.

This is a direct result of the strong noetherian property of $\mathcal{A}$: every increasing sequence of coherent subsheaves of $\mathcal{A}$ is eventually stationary on every compact subset of X [4].

Lemma 2.3.5. *Let S be a closed subspace of a complex space X, with ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$. Suppose further that $\mathrm{Ann}_{\mathcal{O}_X} \mathcal{I} = \{0\}$. Then the following property is satisfied:*

If X' is any closed subspace of X such that $X' \backslash S = X \backslash S$, then $X' = X$.

Proof of proposition 2.3.1. Let $\mathcal{I}$ be the ideal sheaf of Y in X. Let $U \subseteq X$ be open such that Z is closed in U, and let $\mathcal{J} \subseteq \mathcal{O}_U$ be the ideal sheaf of Z. Let $T = \pi^{-1}(Z)$, and:

$$\mathcal{B} = \bigcup_{m \geq 1} \mathrm{Ann}_{\mathcal{O}_T} \left(\mathcal{I}^m \mathcal{O}_T \right).$$

By the first two lemmas, this is a coherent sheaf. Let $Z' \subseteq T$ be the closed subspace defined by $\mathcal{B}$. First we show that $T \backslash \pi^{-1}(Y) = Z' \backslash \pi^{-1}(Y)$. Indeed, let $y \in Z$; if $y \notin Y$ then we must have $\mathcal{I} = \mathcal{O}_X$ in a

neighborhood of y. We have by definition that $\mathcal{J}\mathcal{O}_{\pi^{-1}(U)}$ is the ideal sheaf of T, and by the above we have that $\mathcal{I}\mathcal{O}_T$ is the ideal generated by $\mathcal{O}_X$ in $\mathcal{O}_T$ in a neighborhood of each point of $T\backslash\pi^{-1}(Y)$. In particular it is the unit ideal (1). Thus by the definition of $\mathcal{B}$, the stalk of $\mathcal{B}$ is zero at each $y' \in T\backslash\pi^{-1}(Y)$ so we have

$$T\backslash\pi^{-1}(Y) = Z'\backslash\pi^{-1}(Y).$$

Next we show that $\mathrm{Ann}_{\mathcal{O}_{Z'}}(\mathcal{I}\mathcal{O}_{Z'})$ is zero. Let $j : Z' \to T$ be the closed immersion of Z' in T. The ideal $\mathcal{I}\mathcal{O}_{Z'}$ is by definition the ideal generated by the image of $j^{-1}(\mathcal{I}\mathcal{O}_T) = (\mathcal{I}\mathcal{O}_T)_{|Z'}$ along the mapping:

$$j^{-1}\mathcal{O}_T = (\mathcal{O}_T)_{|Z'} \to \mathcal{O}_{Z'} = (\mathcal{O}_T/\mathcal{B})_{|Z'}$$

which on stalks is given by the canonical projection. Therefore the preimage of $\mathrm{Ann}_{\mathcal{O}_{Z'}}(\mathcal{I}\mathcal{O}_{Z'})$ in $\mathcal{O}_T$ is contained in the union defining $\mathcal{B}$, so $\mathrm{Ann}_{\mathcal{O}_{Z'}}(\mathcal{I}\mathcal{O}_{Z'}) = (0)$. Now we apply the third lemma above (taking $S = \pi^{-1}(Y) \cap Z'$) to see that Z' is the smallest subspace equal to T outside of $\pi^{-1}(Y)$. Now let us show that $\pi : Z' \to Z$ is the blowing up with centre $Z \cap Y$. By hypothesis, $\mathcal{I}\mathcal{O}_{X'}$ is locally principal, so mapping this to $\mathcal{I}\mathcal{O}_{Z'}$ via

$$Z' \hookrightarrow T \hookrightarrow \pi^{-1}(U) \hookrightarrow X'$$

we have that $\mathcal{I}\mathcal{O}_{Z'}$ is locally principal. We have just shown that $\mathrm{Ann}_{\mathcal{O}_{Z'}}(\mathcal{I}\mathcal{O}_{Z'}) = 0$, so $\mathcal{I}\mathcal{O}_{Z'}$ is generated by a non-zerodivisor, i.e. it is invertible. Now let $f : F \to Z$ be any morphism such that $\mathcal{I}\mathcal{O}_F$ is invertible. Composing f with $Z \hookrightarrow U \hookrightarrow X$ we obtain $g : F \to X'$ such that $\pi \circ g = f$. To conclude we show that g in fact maps into Z'. Since f is a mapping to Z, we must have that $g(F) \subseteq \pi^{-1}(Z) =: T$. We want to factor g through the inclusion $j : Z' \to T$, that is, we seek h making the following commute:

$$\begin{tikzcd}
Z' \arrow[r, "j"] & T \arrow[r, "\pi"] & Z \\
 & F \arrow[ul, "h"] \arrow[u, "g"] \arrow[ur, "f"'] &
\end{tikzcd}$$

The ideal of the inverse image of Z' is $\mathcal{B}\mathcal{O}_F$ so we want to see that $\mathcal{B}\mathcal{O}_F = 0$ (for example the fibre product diagram of $g : F \to T$ and $Z' \hookrightarrow T$ then gives the required factorization since we then have $g^{-1}(Z') = F$). But by hypothesis $\mathcal{I}\mathcal{O}_F$ is invertible, so $\mathrm{Ann}_{\mathcal{O}_F}(\mathcal{I}^m\mathcal{O}_F) = 0$ for each $m \in \mathbb{N}$, and we have

$$\bigcup_m \mathrm{Ann}_{\mathcal{O}_F}(\mathcal{I}^m\mathcal{O}_F) \supseteq \bigcup_m \mathrm{Ann}_{\mathcal{O}_T}(\mathcal{I}^m\mathcal{O}_T)\mathcal{O}_F \qquad \square$$

2.4 Existence of blowings up

First let $n = d + m > d$ be an integer, and let

$$\mathbb{C}^d = \mathbb{C}^d \times \{0\} \subseteq \mathbb{C}^d \times \mathbb{C}^m = \mathbb{C}^n$$

be the inclusion. Let $\pi : U \to \mathbb{C}^m$ be the blowing up of $\mathbb{C}^m$ at zero constructed above. By proposition 2.2.5, we have that

$$\Pi = \mathrm{id} \times \pi : \mathbb{C}^d \times U \to \mathbb{C}^n \tag{2.4.1}$$

is the blowing up of $\mathbb{C}^n$ at $\mathbb{C}^d = \mathbb{C}^d \times \{0\}$. Moreover as observed in remark 2.2.4, the blowing up is independent of the choice of coördinates, as is the product mapping id $\times\pi$, so this construction depends only on the inclusion $\mathbb{C}^d \hookrightarrow \mathbb{C}^n$ as complex spaces.

Now let us pass to the general case. We take a complex space X and a closed subspace Y. Since existence is a local question, we may assume that there is an open set $\Omega \subseteq \mathbb{C}^d$ and $f_1, \ldots, f_m \in \mathcal{O}(\Omega)$ such that

$$X = \{z \in \Omega : f_1(z) = \cdots = f_m(z) = 0\}$$

and the ideal sheaf $\mathcal{I}_Y = (f_1, \ldots, f_m)$. Now we define an embedding

$$X \hookrightarrow \mathbb{C}^n = \mathbb{C}^d \times \mathbb{C}^m$$

as follows. Write

$$z = (z_1, \ldots, z_n) = ((z_1, \ldots, z_d), (z_{d+1}, \ldots, z_{d+m})) = (z', z'').$$

Let

$$A = \left\{(z', z'') \in \mathbb{C}^d \times \mathbb{C}^m : z' \in X, z_{d+j} = f_j(z'),\ j = 1, \ldots, m\right\}$$

and define the embedding of X as the composition of the inclusions $X \subseteq A$ and $A \subseteq \mathbb{C}^d \times \mathbb{C}^m$. In this way we have $Y = X \cap \mathbb{C}^d$, where we view all spaces as their embeddings in $\mathbb{C}^n$. Letting Π be the mapping (2.4.1), and X' the strict transform of X by the blowing up Π. By proposition 2.3.1, we obtain that $\pi' = \Pi_{|X'} : X' \to X$ is the blowing up with center Y. We have nearly proved

Theorem 2.4.1. *For every (Hausdorff) complex space X and every closed subspace Y of X, there exists a mapping $\pi : X' \to X$ with the following properties:*

(i) π is the blowing up of X with centre Y;

(ii) π is an isomorphism $X' \backslash \pi^{-1}(Y) \to X \backslash Y$;

(iii) π is proper;

(iv) X' is Hausdorff.

Proof. It remains to prove (ii)-(iv).

For (ii), we recall that the existence of the blowing up is local on X, so we have that $\pi_{|X' \backslash \pi^{-1}Y}$ is the blowing up of $X \backslash S$ with centre $\emptyset$. Moreover the identity mapping also satisfies the universal property for the blowing up of $X \backslash S$ with centre $\emptyset$, so we have that π is an isomorphism outside $\pi^{-1}Y$.

To prove (iii) and (iv), we note that in the local construction preceding the theorem, the blowing up is a proper mapping from a Hausdorff space (since the blowing up of $\mathbb{C}^n$ at a point is proper, so the product map (2.4.1) is proper, and therefore so is the restriction to a closed subspace). Since X' is Hausdorff if and only if X' is locally Hausdorff, (iv) follows. It remains therefore to show that π is proper, assuming the result in a neighborhood of each point. Let $K \subseteq X$ be compact. Choose finite open covers (U_j) and (V_j), $j = 1, \ldots, \ell$ of K such that $U_j \subseteq V_j$ for all j, the U_j are relatively compact in the V_j and $\pi_{|V_j}$ is proper for each j. Then we can write

$$\pi^{-1}(K) = \bigcup_{j=1}^{\ell} \pi^{-1}(K \cap \overline{U}_j)$$

which is compact. □

Remark 2.4.2. Note that by our construction, if the centre Y of the blowing up $\pi : X' \to X$ is the whole space X, then X' is empty, since the smallest closed subspace Z' of X' such that

$$Z' \backslash \pi^{-1}(X) = \pi^{-1}(X) \backslash \pi^{-1}(X) = \emptyset$$

is of course the empty set.

2.5 Further properties of blowings up

Lemma 2.5.1. *Let A be a local ring, and let I_1, I_2 be ideals in A. If $I_1 I_2$ is principal, generated by a non-zero divisor in A, then the same is true for each of I_1, I_2.*

Proof. Pick generators g_{ij}, of I_i, $i = 1, 2$. Then the $g_{1j} g_{2k}$ generate $I_1 I_2$, and since A is local, we can choose $g_1 g_2$ among these elements that generates $I_1 I_2$. By our hypotheses g_1, g_2 are non-zerodivisors. We have

$$I_1 I_2 = g_1 g_2 A \subseteq g_2 I_1 \subseteq I_1 I_2$$

so $g_1 g_2 A = g_2 I_1$, and therefore $g_1 A = I_1$, and similarly $g_2 A = I_2$. □

Corollary 2.5.2. *Let Y_1, Y_2 be closed subspaces of a complex space X, with ideal sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ respectively. Then*

(i) For each morphism $f : T \to X$, the ideal sheaf $\mathcal{I}_1 \mathcal{I}_2 \mathcal{O}_T$ is invertible as an $\mathcal{O}_T$-module if and only if $\mathcal{I}_1 \mathcal{O}_T$ and $\mathcal{I}_2 \mathcal{O}_T$ are invertible as $\mathcal{O}_T$-modules;

(ii) If Y_3 is the closed subspace defined by $\mathcal{I}_1 \mathcal{I}_2$ and $\pi_\alpha : X_\alpha \to X$ is the blowing up with centre Y_α for $\alpha \in \{1, 2, 3\}$ then we have a diagram:

$$\begin{tikzcd}
 & X_2 \arrow[dr, "\pi_2"] & \\
X_3 \arrow[ur, "q_2"] \arrow[rr, "\pi_3"] \arrow[dr, "q_1"'] & & X \\
 & X_1 \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

Proof. (i) follows immediately from the lemma.

(i) $\implies$ (ii): By the definition of the blowing up, $\mathcal{I}_1 \mathcal{I}_2$ is invertible as an $\mathcal{O}_{X_3}$-module, so by the lemma $\mathcal{I}_1 \mathcal{O}_{X_3}$ and $\mathcal{I}_2 \mathcal{O}_{X_3}$ are invertible $\mathcal{O}_{X_3}$-modules. Now by the universal properties for the blowings up π_1 and π_2 we have mappings q_1 and q_2 making the diagram commute. □

Let us admit the following property relating the blowing up along a closed subspace to the blowing up along the inverse image of this subspace by a morphism, with the resulting complex space sitting inside the fibre product. A proof, in the algebraic case, is given in [6].

Proposition 2.5.3. *Let $\pi : X' \to X$ be the blowing up of a complex space X along a closed subspace Y. Let $f : V \to X$ be a morphism of complex spaces. Let*

$$pr_1 : V \times_X X' \to V$$
$$pr_2 : V \times_X X' \to X'$$

be the projections. Let $Z = f^{-1}(Y)$, and let V' be a minimal complex subspace of $V \times_X X'$ containing $pr_1^{-1}(V \backslash Z)$. Then $\tau : V' \to V$ obtained by restricting pr_1 is the blowing up of V along Z.

$$\begin{tikzcd}
\tau^{-1}(Z) = F \arrow[r, hook] \arrow[dr] & V' \arrow[r] \arrow[dr, "\tau"] & V \times_X X' \arrow[r, "pr_2"] \arrow[d, "pr_1"] & X' \arrow[d, "\pi"] \\
& Z \arrow[r, hook] & V \arrow[r, "f"] & X
\end{tikzcd}$$

We will see below that the minimal complex subspace V' in this proposition is unique.

2.6 Local blowings up

Let $U \subseteq X$ be an open subset of a complex space X. Let $E \subseteq U$ be a closed subset of U. Let $\pi : U' \to U$ be the blowing up with centre E. Then the composition $\sigma : U' \to U \hookrightarrow X$ is called a *local blowing up*, and is denoted by the triple $\sigma = (U, E, \pi)$. By a finite sequence of local blowings up we mean a composition:

$$W_m \xrightarrow{\sigma_{m-1}} W_{m-1} \longrightarrow \cdots \longrightarrow W_1 \xrightarrow{\sigma_0} W_0$$

where each σ_i is a local blowing up. Now let $f : V \to W$ be any morphism of complex spaces and let $\sigma : W' \to W$ be a local blowing up. We have a local blowing up $\sigma' : V' \to V$ defined by $(f^{-1}(U), f^{-1}(E), \pi')$. We claim that there exists a unique $f' : V' \to W'$ making the following diagram commute:

$$\begin{tikzcd}
V' \arrow[r, "\sigma'"] \arrow[d, "f'"'] & V \arrow[d, "f"'] \\
W' \arrow[r, "\sigma"] & W
\end{tikzcd}$$

To see this, let $\mathcal{I}$ be the ideal sheaf of E in U. Then $\mathcal{I}\mathcal{O}_{f^{-1}(U)}$ is the ideal sheaf of $f^{-1}(E)$. Since σ' is the blowing up with centre $f^{-1}(E)$, the ideal sheaf $\mathcal{I}\mathcal{O}_{V'}$ is an invertible $\mathcal{O}_{V'}$-module. Therefore by the universal property for σ there exists a unique $f' : V' \to W'$ such that the diagram above commutes. We call f' the *strict transform* of f by σ. If $\sigma = \sigma_0 \circ \cdots \circ \sigma_{m-1}$ is a finite sequence of local blowings-up, $\sigma_i : W_{i+1} \to W_i$ then the strict transform of f by σ is given by the sequence of strict transforms:

$$\begin{tikzcd}
V_m \arrow[r, "\sigma'_{m-1}"] \arrow[d, "f_m"] & \cdots \arrow[r] & V_1 \arrow[r, "\sigma'_0"] \arrow[d, "f_1"] & V = V_0 \arrow[d, "f = f_0"] \\
W_m \arrow[r, "\sigma_{m-1}"] & \cdots \arrow[r] & W_1 \arrow[r, "\sigma_0"] & W = W_0
\end{tikzcd}$$

where at each step we take the strict transform of f_i by σ_i to obtain f_{i+1}, and if $\sigma_i = (U_i, E_i, \pi_i)$ then $\sigma'_i = (f_i^{-1}(U_i), f_i^{-1}(E_i), \pi'_i)$. Now by the universal property for the fibre product, we have a mapping $\kappa_m : V_m \to V \times_W W_m$ such that:

$$\begin{tikzcd}
V_m \arrow[dr, "\kappa_m"] \arrow[drr, "f_m"] \arrow[ddr, "\sigma'_0\circ\cdots\circ\sigma'_{m-1}"'] & & \\
& V \times_W W_m \arrow[r] \arrow[d] & W_m \arrow[d, "\sigma"] \\
& V \arrow[r, "f"'] & W
\end{tikzcd}$$

We claim that κ_m is a closed immersion onto the smallest analytic subspace of $V \times_W W_m$ containing $V \times_W (W \backslash E)$, where E is the union of the exceptional divisors in W_m. To see this, first suppose that $\sigma = \sigma_0 = (U_0, E_0, \pi_0)$ is a single local blowing up. Since $\mathrm{im}(\sigma) \subseteq U_0$ we have $V \times_W W_1 = f^{-1}(U) \times_U W_1$, so to prove the result we may as well assume that $\sigma_0 = \pi_0$ is a global blowing up with centre E_0. Now by proposition 2.5.3, up to isomorphism $V_m = V_1$ is any minimal complex subspace of $V \times_W W_1$ containing $\mathrm{pr}_1^{-1}(V \backslash f^{-1}(E_0))$ and σ'_1, f_1 are the restrictions of the projections. Moreover if $V \times_W W_1$ contains several isomorphic minimal closed subspaces containing $\mathrm{pr}_1^{-1}(V \backslash f^{-1}(E_0))$ then the morphism $\kappa_m = \kappa_1$ is not unique, so we may identify V_1 with the unique smallest closed subspace of the fibre product containing $\mathrm{pr}_1^{-1}(V \backslash f^{-1}(E_0))$. Now we may take as κ_1 the inclusion $V_1 \subseteq V \times_W W_1$, so the result follows. The general case $\sigma = \sigma_0 \circ \cdots \circ \sigma_{m-1}$ now follows by induction on $m \geq 1$.

Chapter 3

La voûte étoilée

3.1 The category $\mathcal{C}(W)$

First of all we define the notion of a strict morphism. This is a generalization of the notion of a finite sequence of local blowings-up, which has the advantage of a concrete definition that proves easier to work with.

Definition 3.1.1. A morphism $f : V \to W$ of complex spaces if *strict* if there exists a closed complex subspace $F \subseteq V$ such that

(i) f is an isomorphism outside the set F;

(ii) If $V' \subseteq V$ is a closed subspace such that $V' \backslash F = V \backslash F$, then $V' = V$.

The following property is central to our use of strict morphisms:

Lemma 3.1.2. *Let $Y \subseteq X$ be a closed subspace of a complex space X. Let $\mathcal{I}$ be the ideal sheaf of Y in $\mathcal{O}_X$. Let $\pi : X' \to X$ be the blowing up with centre Y. If $f : V \to X$ is any strict morphism of complex spaces then*

(i) there exists a most one morphism $f' : V \to X'$ such that $\pi \circ f' = f$

$$\begin{tikzcd} X' \arrow[r, "\pi"] & X \\ V \arrow[u, "f'"] \arrow[ur, "f"'] & \end{tikzcd}$$

(ii) if such an f' exists, then $\mathcal{I}\mathcal{O}_V$ is an invertible $\mathcal{O}_V$-module and f' is also strict

Proof. First we note that by the universal property of the blowing up π, in order to show uniqueness of f', it suffices to show that $\mathcal{I}\mathcal{O}_V$ is an invertible $\mathcal{O}_V$-module. First let us apply the definition of a strict morphism to f, denoting by F the closed subspace of V thus obtained. Let $v \in V \backslash F$ be any point outside F. By hypothesis f is an isomorphism in a neighborhood of v, so we can choose an open subset $v \in U \subseteq V \backslash F$ such that $f_{|U} : U \to f(U)$ is an isomorphism onto an open subset $f(U)$ of X. We claim that $f'_{|U}$ is a locally

closed embedding. Since $f(U)$ is open in X, $M := \pi^{-1}(f(U))$ is open in X'. Moreover $f'_{|U}$ is proper: if $K \subseteq X'$ is compact then $\pi(K)$ is compact (a continuous image of a compact set is compact) and since $f_{|U}$ is an isomorphism we have that $f^{-1}\pi(K)$ is compact in U. Now

$$(f')^{-1}(K) \subseteq (f')^{-1}\pi^{-1}\pi(K) = f^{-1}\pi(K) \subseteq U$$

is a closed subset of a compact subset of U, hence compact. Therefore $f'_{|U}$ is proper. Now by Remmert's proper mapping theorem [5] we have that $f'(U)$ is an analytic subset of $M = \pi^{-1}(f(U))$; in particular it is closed. To show that $f'_{|U}$ is a locally closed embedding we want to show that f' is a bijection onto its image. But we have that

$$f^{-1}(z) = (f')^{-1}\pi^{-1}(z), \qquad z \in f(U)$$

is a single point; so f' is a bijection onto its image.

Next let us show that $f'(U) = M$. We have $f'(U)\backslash\pi^{-1}(Y) = M\backslash\pi^{-1}(Y)$ since f is an isomorphism on U and π is an isomorphism outside the preimage of Y. Moreover the ideal sheaf $\mathcal{I}\mathcal{O}_{X'}$ of $\pi^{-1}(Y)$ is locally generated by a non-zerodivisor since π is the blowing up with centre Y, so its annihilator is (0). Thus by lemma 2.3.5, we have $f'(U) = M$. Therefore f' is an isomorphism outside F, and F satisfies the minimality condition (ii) by assumption, so we have that f' is strict.

It remains to show that $\mathcal{I}\mathcal{O}_V$ is invertible. We know that $\mathcal{I}\mathcal{O}_V$ is locally principal, since this is true of $\mathcal{I}\mathcal{O}_{X'}$. Moreover $\mathcal{I}\mathcal{O}_V$ is invertible outside of $V\backslash F$, since f' is an isomorphism in this set. Therefore the annihilator sheaf $\mathrm{Ann}_{\mathcal{O}_V}(\mathcal{I}\mathcal{O}_V)$ defines a closed subspace $V' \subseteq V$ such that $V'\backslash F = V\backslash F$, since $\mathrm{Ann}_{\mathcal{O}_V}(\mathcal{I}\mathcal{O}_V)$ vanishes outside F. Now by the minimality assumption on F we have that $V' = V$ i.e. $\mathrm{Ann}_{\mathcal{O}_V}(\mathcal{I}\mathcal{O}_V) = (0)$. □

Remark 3.1.3. We saw in the proof of lemma 3.1.2 that if f' exists, then outside F, both f and f' are isomorphisms. Therefore if π is not an isomorphism at any point of $\pi^{-1}(Y)$ then we have $f^{-1}(Y) \subseteq F$, i.e. $Y \subseteq f(F)$. In particular if f is the blowing up with centre Y' then $Y \subseteq Y'$.

Corollary 3.1.4. *Let $f : V \to X$ be a strict morphism of complex spaces. Let $\mathcal{I}$ be a coherent sheaf of ideals in $\mathcal{O}_X$, which is invertible as an $\mathcal{O}_X$-module. Then $\mathcal{I}\mathcal{O}_V$ is an invertible $\mathcal{O}_V$-module.*

Proof. Let $Y \subseteq X$ be the complex subspace defined by $\mathcal{I}$. Since $\mathcal{I}$ is already invertible, the blowing up with centre Y is just $\mathrm{id} : X \to X$. Therefore there exists a diagram as in the lemma with $f' = f$:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] & X \\ V \arrow[u, "f"] \arrow[ur, "f"'] & \end{tikzcd}$$

Now by the statement (ii) of the lemma we have that $\mathcal{I}\mathcal{O}_V$ is an invertible $\mathcal{O}_V$-module. □

Proposition 3.1.5. *Let $\sigma = \sigma_0 \circ \cdots \circ \sigma_{m-1} : W_m \to W_0$ be a finite sequence of local blowings-up with $\sigma_i : W_{i+1} \to W_i$. Then:*

(i) σ is strict;

(ii) if $f : V \to W_0$ is any strict morphism, then there exists at most one $f' : V \to W_m$ such that $\sigma \circ f' = f$;

(iii) if f' exists, then f' is strict.

Proof. (i) Recall that if $\sigma : X' \to U \hookrightarrow X$ is a local blowing up then we write $\sigma = (U, E, \pi)$, where $\pi : X' \to U$ is the blowing up with centre E. In the situation of the proposition, let $\sigma_i = (U_i \subseteq W_i, E_i \subseteq U_i, \pi_i : W_{i+1} \to U_i)$. Let $\mathcal{I}_i$ be the ideal sheaf of E_i in U_i, and let $\mathcal{J} = \prod_i (\mathcal{I}_i \mathcal{O}_{W_m})$. Let F be the closed subspace of W_m defined by $\mathcal{J}$. Then clearly σ is isomorphic outside F since each σ_i is an isomorphism outside E_i, and F is the union of the inverse images of the E_i. Moreover, $\mathcal{I}_i \mathcal{O}_{W_{i+1}}$ is invertible as an $\mathcal{O}_{W_{i+1}}$-module, and the σ_i are each strict morphisms (the blowing up is strict, and composing with the inclusion does not change this) so by the corollary $\mathcal{I}_i \mathcal{O}_{W_m}$ are invertible $\mathcal{O}_{W_m}$-modules, and therefore so is $\mathcal{J}\mathcal{O}_{W_m}$. This means that the annihilator of $\mathcal{J}\mathcal{O}_{W_m}$ is zero, so by lemma 2.3.5 σ is strict.

(ii) and (iii). Suppose that $f : V \to W_0$ is strict, and that there exists $f' : V \to W_m$ such that $\sigma \circ f' = f$. We have a diagram:

$$\begin{tikzcd} W_m \arrow[r, "\sigma_{m-1}"] & W_{m-1} \arrow[r, "\sigma_{m-2}"] & \cdots \arrow[r, "\sigma_1"] & W_1 \arrow[r, "\sigma_0"] & W_0 \\ & V \arrow[ul, "f'"] \arrow[urrrr, "f"] & & & \end{tikzcd}$$

Let $f_1' : V \to W_1$ be the morphism obtained as $f_1' = \sigma_1 \circ \cdots \circ \sigma_{m-1} \circ f'$. Then we have a commutative triangle

$$\begin{tikzcd} W_1 \arrow[r, "\sigma_0"] & W_0 \\ V \arrow[u, "f_1'"] \arrow[ur, "f"'] & \end{tikzcd}$$

By commutativity of this diagram we have that $f(V) \subseteq U_0$ so we may apply lemma 3.1.2 to the blowing up $\pi_0 : W_1 \to U_1 \subseteq W_0$ to see that f_1' is strict and f_1' is unique assuming that f' exists. Thus we have reduced the length of the diagram we must consider by 1:

$$\begin{tikzcd} W_m \arrow[r, "\sigma_{m-1}"] & W_{m-1} \arrow[r, "\sigma_{m-2}"] & \cdots \arrow[r, "\sigma_1"] & W_1 \\ & V \arrow[ul, "f'"] \arrow[urrr, "f_1'"] & & \end{tikzcd}$$

Therefore we may repeat the process inductively with the morphisms $f_i' = \sigma_i \circ \cdots \circ \sigma_{m-1} \circ f'$, $i = 1, \ldots, m-1$, and at the last step we see that f' is unique assuming f' exists, and f' is a strict morphism. $\square$

An important corollary of this proposition is that if $\sigma_1 : W_1 \to W$ and $\sigma_2 : W_2 \to W$ are two finite sequences of local blowings-up, then there is at most one morphism $q : W_2 \to W_1$ such that $\sigma_1 \circ q = \sigma_2$.

Definition 3.1.6. Let W be a complex space. We define a category $\mathcal{C}(W)$ as follows. An object of $\mathcal{C}(W)$ is a finite sequence of local blowings-up $\sigma : W' \to W$, and a morphism $q \in \mathrm{Hom}(\sigma_1, \sigma_2)$ where $\sigma_1 : W_1 \to W$ and $\sigma_2 : W_2 \to W$ is a morphism $q : W_1 \to W_2$ of complex spaces such that $\sigma_1 = \sigma_2 \circ q$.

We therefore have that for any $\sigma_1, \sigma_2 \in \mathcal{C}(W)$ that $\#\,\mathrm{Hom}(\sigma_1, \sigma_2) \leq 1$. Our aim now is to show that products exist in the category $\mathcal{C}(W)$. That is, given $\sigma_1, \sigma_2 \in \mathcal{C}(W)$ we want to show that there exists $\sigma_3 \in \mathcal{C}(W)$ together with $q_i \in \mathrm{Hom}(\sigma_3, \sigma_i)$, $i = 1, 2$ such that if $\tau \in \mathcal{C}(W)$ and $h_i \in \mathrm{Hom}(\tau, \sigma_i)$, $i = 1, 2$ then there exists a unique $h \in \mathrm{Hom}(\tau, \sigma_3)$ such that:

$$\begin{tikzcd}
\sigma_1 & \sigma_3 \arrow[l, "q_1"'] \arrow[r, "q_2"] & \sigma_2 \\
& \tau \arrow[ul, "h_1"] \arrow[u, "h"] \arrow[ur, "h_2"'] &
\end{tikzcd}$$

is a commutative diagram. The picture becomes somewhat harder to read if we do not use the simplified notation of the category $\mathcal{C}(W)$. Keeping the same notation suppose that $\sigma_i : W_i \to W$, $i = 1, 2, 3$, $\tau : V \to W$, so the diagram is:

$$\begin{tikzcd}
& & W_1 \arrow[dr, "\sigma_1"] & \\
V \arrow[urr, "h_1"] \arrow[r, "h"] \arrow[drr, "h_2"'] & W_3 \arrow[ur, "q_1"] \arrow[rr, "\sigma_3"] \arrow[dr, "q_2"] & & W \\
& & W_2 \arrow[ur, "\sigma_2"'] &
\end{tikzcd}$$

where the morphism τ is not shown. The idea of the proof is quite simple: if σ_1 and σ_2 are each a single local blowing up, then we have already seen a construction of this kind in lemma 2.5.2. In the general case we take the square formed by W_3, W_1, W_2 and W in the picture above for each blowing up in the composition, glue them together, and from this define a commutative lattice of such squares. That allows us to apply the case of a single local blowing up inductively to establish the result.

Theorem 3.1.7. *Let $\sigma_i : W_i \to W \in \mathcal{C}(W)$, $i = 1, 2$. Then there exists $\sigma_3 : W_3 \to W$ such that:*

(i) there exists $q_i \in \mathrm{Hom}(\sigma_3, \sigma_i)$, $i = 1, 2$;

(ii) if $f : V \to W$ is strict, and if $h_i : V \to W_i$ such that $f = \sigma_i \circ h_i$, $i = 1, 2$ then there exists a unique $h_3 : V \to W_3$ with $q_i \circ h_3 = h_i$, $i = 1, 2$.

Moreover $q_i \in \mathcal{C}(W_i)$, $i = 1, 2$.

Let us consider the case where σ_1, σ_2 each consist of a single local blowing up, say $\sigma_1 = (U_1, E_1, \pi_1 : W_1 \to W)$ and $\sigma_2 = (U_2, E_2, \pi_2 : W_2 \to W)$. Let $\mathcal{I}_i$ be the ideal sheaf of E_i in $\mathcal{O}_{U_i}$, $i = 1, 2$.

Lemma 3.1.8. *Let $U_3 = U_1 \cap U_2$ and let $E_3 \subseteq U_3$ be the closed subspace defined by $\mathcal{I}_1\mathcal{I}_2$. Let $\sigma_3 = (U_3, E_3, \pi_3 : W_3 \to W)$ be the local blowing up with centre E_3. Then σ_3 has the properties (i) and (ii) of the theorem.*

Proof. We apply lemma 2.5.2 to the restrictions of π_1, π_2 to U_3 to obtain the following diagram:

$$\begin{tikzcd}
& \pi_1^{-1}(U_2) = \pi_1^{-1}(U_3) \arrow[dr, "\pi_1"] & \\
W_3 \arrow[ur, "\tilde{q}_1"] \arrow[rr, "\pi_3"] \arrow[dr, "\tilde{q}_2"'] & & U_3 \\
& \pi_2^{-1}(U_1) = \pi_2^{-1}(U_3) \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

Now composing $\tilde{q}_i$ with the inclusion $\pi_i^{-1}(U_3) \hookrightarrow W_i$, and denoting the resulting mapping by q_i, we obtain q_1, q_2 with the property (i) of the theorem. Now take any strict morphism $f : V \to W$ and $h_i : V \to W_i$ such that $f = \sigma_1 \circ h_i$, $i = 1, 2$. We have

$$\begin{aligned} f(V) &= \sigma_i \circ h_i(V) \\ &\subseteq \sigma_i(W_i) = U_i \end{aligned}$$

for $i = 1, 2$, so $f(V) \subseteq U_1 \cap U_2 = U_3$. Therefore we have two commutative diagrams:

$$\begin{tikzcd} W_i \arrow[r, "\pi_i"] & U_i \\ V \arrow[u, "h_i"] \arrow[ur, "f"'] & \end{tikzcd}$$

Since f is strict and π_i is a blowing up, by lemma 3.1.2 the ideal $\mathcal{I}_i\mathcal{O}_V$ is an invertible $\mathcal{O}_V$-module, $i = 1, 2$. Therefore $\mathcal{I}_1\mathcal{I}_2\mathcal{O}_V$ is an invertible $\mathcal{O}_V$-module. By the universal property for the blowing up $\pi_3 : W_3 \to U_3$, there exists a unique $h : V \to W_3$ such that $f = \pi_3 \circ h_3$. Now we have:

$$\pi_i \circ h_i = f = \pi_3 \circ h_3 = \pi_i \circ q_i \circ h_3.$$

By lemma 3.1.2 the functions $h_i : V \to W_i$ such that $\pi_i \circ h_i = f$ are unique, so we have $h_i = q_i \circ h_3$. Thus the property (ii) of the theorem is satisfied. □

In the next lemma we verify the last remaining statement of the theorem in the case of single local blowings-up. That is, we show that $q_i \in \mathcal{C}(W)$, $i = 1, 2$. In the statement we retain the same notation: $\sigma_i = (U_i, E_i, \pi_i : W_i \to U_i)$, $i = 1, 2, 3$ with $U_3 = U_1 \cap U_2$ and E_3 defined by the product of the ideal sheaves of E_1 and E_2, $q_i : W_3 \to \pi_i^{-1}(U_3) \hookrightarrow W_i$, $i = 1, 2$. This allows us to proceed by induction in the general case.

Lemma 3.1.9. *We have a local blowing up $q_1 = (\pi_1^{-1}(U_2), \pi_1^{-1}(E_2), \tilde{q}_1 : W_3 \to \pi_1^{-1}(U_2))$ where $\tilde{q}_i, q_i$ are defined as in the proof of 3.1.8. In particular, $q_1 \in \mathcal{C}(W_1)$.*

Since the problem is symmetrical, it follows that we also have a local blowing up $q_2 = (\pi_2^{-1}(U_1), \pi_2^{-1}(E_1), \tilde{q}_2 : W_3 \to \pi_2^{-1}(U_1))$.

Proof. Let $W' = \pi_1^{-1}(U_2)$. We have that $\mathcal{I}' := \mathcal{I}_2\mathcal{O}_{W'}$ is the ideal sheaf of $\pi_1^{-1}(E_2)$ in $\mathcal{O}_{W'}$. Now

$$\mathcal{I}'\mathcal{O}_{W_3} = (\mathcal{I}_2\mathcal{O}_{W'})\mathcal{O}_{W_3} = \mathcal{I}_2\mathcal{O}_{W_3}$$

so this is an invertible $\mathcal{O}_{W_3}$-module. Let $q' : W'' \to W'$ be the blowing up with centre $\pi_1^{-1}(E_2)$, so by the universal property for the blowing up there exists a unique $g : W_3 \to W''$ such that $q' \circ g = q_1$. Now as a blowing up, q' is strict, and $\mathcal{I}_1\mathcal{O}_{W_1}$ is an invertible $\mathcal{O}_{W_1}$-module, we have that $\mathcal{I}_1\mathcal{O}_{W''}$ is an invertible $\mathcal{O}_{W''}$-module. Moreover $\mathcal{I}_2\mathcal{O}_{W''} = \mathcal{I}'\mathcal{O}_{W''}$ is an invertible $\mathcal{O}_{W''}$-module, so by the universal property for π_3 there exists a unique $g' : W'' \to W_3$ such that $\pi_3 \circ g' = \pi_1 \circ q'$:

$$\begin{tikzcd}
W_1 \arrow[r, "\pi_2"] & W \\
W_3 \arrow[u, "q_2"] \arrow[ur, "\pi_3"] \arrow[r, "q_1"] \arrow[d, shift left, "g"] & W_1 \arrow[u, "\pi_1"'] \\
W'' \arrow[u, shift left, "g'"] \arrow[r, "q'"'] & W' = \pi_1^{-1}(U_2) \arrow[u, "\subseteq"']
\end{tikzcd}$$

If we show that $g' = g^{-1}$ then we have that W_3 is isomorphic to the local blowing up $W'' \to \pi_1^{-1}(U_2) \hookrightarrow W_1$, and we are done. We have

$$\begin{aligned}
\pi_3 \circ (g' \circ g) &= \pi_1 \circ (q' \circ g) \\
&= \pi_1 \circ q_1 = \pi_3.
\end{aligned}$$

Since π_3 is strict, there exists at most one map α such that $\pi_3 \circ \alpha = \pi_3$, and the identity has this property, so $g' \circ g = \mathrm{id}$. On the other hand,

$$\begin{aligned}
(\pi_1 \circ q') \circ (g \circ g') &= \pi_1 \circ q_1 \circ g' && [q' \circ g = q_1] \\
&= \pi_1 \circ q'.
\end{aligned}$$

Since $\pi_1 \circ q'$ is strict, this shows that $g \circ g' = \mathrm{id}$. $\square$

Now we proceed with the proof of the theorem in the general case. Namely we suppose that σ_1, σ_2 are given as finite compositions of local blowings-up:

$$\begin{aligned}
&\{(U_{i0}, E_{i0}, \pi_{i0} : W_{i+1,0} \to U_{i0}) : i = 0, \dots, m-1\} && [\text{for } \sigma_1] \\
&\{(U'_{0j}, E'_{0j}, \pi'_{0j} : W_{0,j+1} \to U'_{0j}) : j = 0, \dots, n-1\} && [\text{for } \sigma_2]
\end{aligned}$$

Here $W_1 = W_{m0}$, $W_2 = W_{0n}$ and $W_{00} = W$. We write σ_{i0} for the composition of the blowing up π_{i0} with the inclusion $U_{i0} \hookrightarrow W_{i0}$, and similarly for σ'_{0j}, so:

$$W_1 = W_{m0} \xrightarrow{\sigma_{m-1,0}} W_{m-1,0} \to \cdots \to W_{10} \xrightarrow{\sigma_{00}} W_{00} = W$$

and

$$W_2 = W_{0n} \xrightarrow{\sigma'_{0,n-1}} W_{0,n-1} \to \cdots \to W_{01} \xrightarrow{\sigma'_{00}} W_{00} = W$$

and

$$\sigma_1 = \sigma_{00} \circ \cdots \circ \sigma_{m-1,0}, \quad \sigma_2 = \sigma'_{00} \circ \cdots \circ \sigma'_{0,n-1}.$$

Now we construct a lattice of local blowings up:

$$\{(U_{ij}, E_{ij}, \pi_{ij} : W_{i+1,j} \to U_{ij}) : i = 0, \dots, m,\ j = 0, \dots, n\}$$
$$\{(U'_{ij}, E'_{ij}, \pi'_{ij} : W_{i,j+1} \to U'_{ij}) : j = 0, \dots, n,\ i = 0, \dots, m\}$$

Suppose that $(U_{ij}, E_{ij}, \pi_{ij} : W_{i+1,j} \to U_{ij})$ and $(U'_{ij}, E'_{ij}, \pi'_{ij} : W_{i,j+1} \to U'_{ij})$ have been constructed for some $0 \le i < m$, $0 \le j < n$. Let $\tilde{U}_{ij} = U_{ij} \cap U'_{ij}$, let $\mathcal{I}_{ij}$ and $\mathcal{I}'_{ij}$ be the ideal sheaves of E_{ij} and E'_{ij} in U_{ij} and U'_{ij} respectively. Let $\tilde{E}_{ij}$ be the closed subspace of $\tilde{U}_{ij}$ defined by $\mathcal{I}_{ij}\mathcal{I}'_{ij}$. Let $\tilde{\sigma}_{ij} : W_{i+1,j+1} \to W_{ij}$ be the local blowing up defined by

$$W_{i+1,j+1} \xrightarrow{\tilde{\pi}_{ij}} \tilde{U}_{ij} \hookrightarrow W_{ij}$$

where $\tilde{\pi}_{ij}$ is the blowing up with centre $\tilde{E}_{ij}$. By the first lemma above, there exist morphisms as depicted:

$$\begin{tikzcd}
 & W_{i+1,j} \arrow[dr, "\sigma_{ij}"] & \\
W_{i+1,j+1} \arrow[ur, "\sigma'_{i+1,j}"] \arrow[rr, "\tilde{\sigma}_{ij}"] \arrow[dr, "\sigma_{i,j+1}"'] & & W_{ij} \\
 & W_{i,j+1} \arrow[ur, "\sigma'_{ij}"'] &
\end{tikzcd}$$

By the second lemma above, the morphisms $\sigma'_{i+1,j}$ and $\sigma_{i,j+1}$ each consist of a single local blowing up. Thus we obtain a commutative lattice of local blowings up (see page 26).

We define $W_3 := W_{mn}$, and let $\sigma_3 : W_3 \to W$ be the composition of the maps along the commutative lattice. Let

$$q_1 = \sigma'_{m0} \circ \cdots \circ \sigma'_{m,n-1}, \quad q_2 = \sigma_{0n} \circ \cdots \circ \sigma_{m-1,n}.$$

With these definitions the property (i) of the theorem is satisfied, and $q_i \in \mathcal{C}(W_i)$ since each of the σ_{ij}, σ'_{ij} are local blowings-up. Thus it remains only to prove the property (ii) of the theorem. Let $f : V \to W$ be strict, and let $h_i : V \to W_i$ such that $f = \sigma_i \circ h_i$, $i = 1, 2$. Now we apply the first case inductively to each of the squares just obtained to obtain a system of morphisms $h_{ij} : V \to W_{ij}$, $i = 0, \dots, m$, $j = 0, \dots, n$. By the strictness assertion of proposition 3.1.5 we conclude that we may indeed define the h_{ij} inductively. By the uniqueness assertion of 3.1.5, and since the lattice of σ_{ij}, σ'_{ij} commutes, we conclude that the definition of h_{ij} does not depend on the path taken through the lattice. Moreover the strictness assertion of 3.1.5 we conclude that h_1 and h_2 are strict, then by uniqueness we have that $h_1 = h_{m0}$ and $h_2 = h_{0n}$. Now following the compositions down the lattice we have that $q_i \circ h_3 = h_1$, $i = 1, 2$ as required. This completes the proof of the theorem.

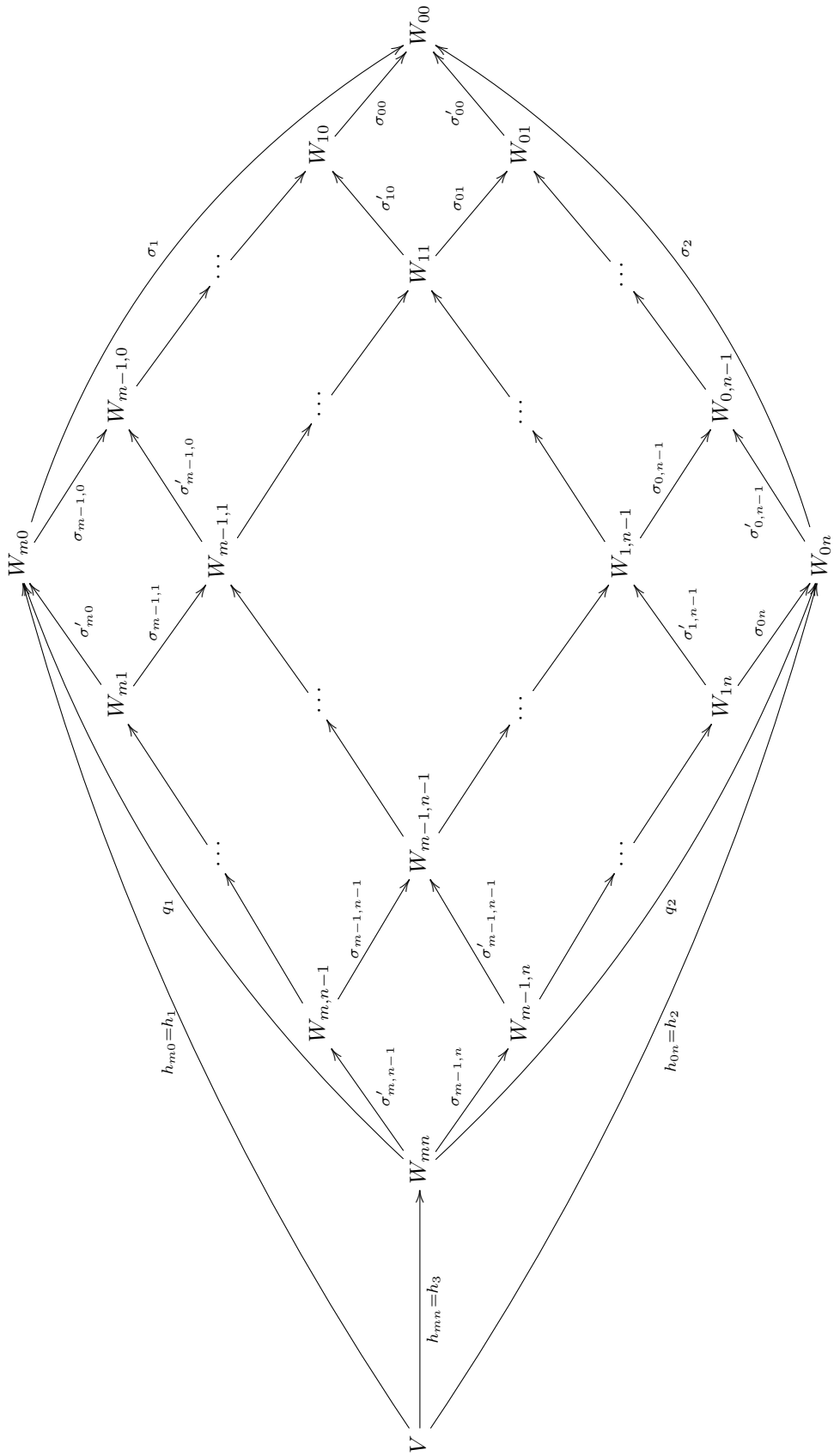
Remarks 3.1.10.

(i) We write $\sigma_3 = \sigma_1 \wedge \sigma_2$ for the product in $\mathcal{C}(W)$;

(ii) The product $\wedge$ is associative and commutative;

(iii) If $\mathrm{Hom}(\sigma_1, \sigma_2) \neq \emptyset$ then $\sigma_1 \wedge \sigma_2 = \sigma_1$.

Figure 3.1. Construction of the product in $\mathcal{C}(W)$



Remark 3.1.11. Let $\sigma_1 \wedge \sigma_2 : W_3 \to W$ be the product of $\sigma_i : W_i \to W \in \mathcal{C}(W)$, $i = 1, 2$, with morphisms $q_i : W_3 \to W_i$. Then taking q_1 and q_2 in the universal property of the fibre product $W_1 \times_W W_2$ we obtain a mapping $W_3 \to W_1 \times_W W_2$:

$$\begin{tikzcd}
W_3 \arrow[dr, "\exists!"] \arrow[drr, "q_1"] \arrow[ddr, "q_2"'] & & \\
& W_1 \times_W W_2 \arrow[r] \arrow[d] & W_1 \arrow[d, "\sigma_1"] \\
& W_2 \arrow[r, "\sigma_2"'] & W
\end{tikzcd}$$

It can be shown that this mapping is in fact a closed immersion [1].

3.2 Étoiles

We recall that all complex spaces are assumed Hausdorff, so by theorem 2.4.1 all the complex spaces that appear in the category $\mathcal{C}(W)$ are Hausdorff.

Definition 3.2.1. Let $\{\mathcal{C}_i(W) : i \in I\}$ be the set of all subcategories of $\mathcal{C}(W)$ satisfying:

(E) For every pair of $\sigma_\alpha : W_\alpha \to W$, $\alpha = 1, 2$ there exists $\sigma_3 : W_3 \to W$ belonging to $\mathfrak{e}$ such that:

(i) there exists $q_\alpha \in \mathrm{Hom}\,(\sigma_3, \sigma_\alpha)$, $\alpha = 1, 2$ so that we have a commutative diagram:

$$\begin{tikzcd}
& W_1 \arrow[dr, "\sigma_1"] & \\
W_3 \arrow[ur, "q_1"] \arrow[rr, "\sigma_3"] \arrow[dr, "q_2"'] & & W \\
& W_2 \arrow[ur, "\sigma_2"'] &
\end{tikzcd}$$

(ii) $W_3 \neq \emptyset$ and $q_\alpha(W_3)$ is relatively compact in W_α, $\alpha = 1, 2$.

Then $\{\mathcal{C}_i(W) : i \in I\}$ is ordered by inclusion. An *étoile* over W is a maximal element of this set with respect to inclusion.

By Zorn's lemma, given any subcategory $\mathfrak{e}_0$ of $\mathcal{C}(W)$ satisfying (E) there exists at least one étoile over W containing $\mathfrak{e}_0$ (this incurs a minor set-theoretic issue, since in a given complex space W' we may replace any given point x of the topological space with any given set T, and define a complex space in which T plays the role of x. To apply Zorn's lemma we can locally embed any complex space in $\mathbb{C}^n$, and construct blowings up by gluing patches of the local construction. From here we can extract a set of representatives of the isomorphism classes of local blowings up over W without applying the axiom of choice to proper classes). Moreover if W is any complex space the set of étoiles over W is non-empty: fix $t \in W$ and let $\mathcal{A}$ be the set of inclusions $U \hookrightarrow W$, where U is a neighborhood of t in W. Then we see easily that $\mathcal{A}$ satisfies (E). Roughly speaking, an étoile is a collection of finite sequences of local blowings up with a unique common image point. This is stated in proposition 3.2.7. By remark 3.1.3 we know that if $\pi_1 : X_1 \to X$

and $\pi_2 : X_2 \to X$ are blowings up with centres Y_1, Y_2 such that there exists $f : X_1 \to X_2$ with $\pi_2 \circ f = \pi_1$, and π_2 is not an isomorphism at any point of $\pi_2^{-1}(Y_2)$ then we must have $Y_2 \subseteq Y_1$ (the condition that π_2 is not an isomorphism on its exceptional divisor accounts for centres that give trivial blowings up, e.g. smooth centres with codimension one). Thus we may think of an étoile as defining a tree of blowings up, with σ_3 in the definition of an étoile corresponding roughly to "localizing on W to an open subset and taking the union of centres in this set".

Proposition 3.2.2. *Let $\mathfrak{e}$ be an étoile over W.*

(i) If $\sigma' \in \mathfrak{e}$, $\sigma \in \mathcal{C}(W)$ and $\mathrm{Hom}(\sigma', \sigma) \neq \emptyset$ then $\sigma \in \mathfrak{e}$;

(ii) For any $\sigma_1, \sigma_2 \in \mathcal{C}(W)$ we have $\sigma_1 \wedge \sigma_2 \in \mathfrak{e}$ if and only if $\sigma_1, \sigma_2 \in \mathfrak{e}$.

Proof.

(i) Let $\mathfrak{e}'$ be the subcategory of $\mathcal{C}(W)$ obtained from $\mathfrak{e}$ by adding all those $\sigma \in \mathcal{C}(W)$ with $\mathrm{Hom}\,(\sigma', \sigma) \neq \emptyset$. We show that $\mathfrak{e}'$ satisfies (E). By maximality of $\mathfrak{e}$ this shows that $\mathfrak{e}' = \mathfrak{e}$. Let $\sigma_1, \sigma_2 \in \mathfrak{e}'$. If $\sigma_i \notin \mathfrak{e}$, then choose $\sigma_i' \in \mathfrak{e}$ such that $\mathrm{Hom}\,(\sigma_i', \sigma_i) \neq \emptyset$. If $\sigma_i \in \mathfrak{e}$, then let $\sigma_i' = \sigma_i$. Then we have a commutative diagram:

$$\begin{tikzcd}
 & \tilde{W}_1 \arrow[r, "\tilde{\sigma}_1"] \arrow[dr, "\sigma_1'"] & W_1 \arrow[dr, "\sigma_1"] & \\
W_3 \arrow[ur, "q_1"] \arrow[rr, "\sigma_3"] \arrow[dr, "q_2"'] & & & W \\
 & \tilde{W}_2 \arrow[r, "\tilde{\sigma}_2"'] \arrow[urr, "\sigma_2'"] & W_2 \arrow[ur, "\sigma_2"'] &
\end{tikzcd}$$

where $W_3 \neq \emptyset$ and the image of q_i in $\tilde{W}_i$ is relatively compact. Since σ_i' is continuous, we have that the image of $\sigma_i' \circ q_i$ is relatively compact in W_i. Therefore the space W_3 with the morphisms $\tilde{\sigma}_i \circ q_i$ show the property (E) for σ_1, σ_2.

(ii) It follows directly from (i) that if $\sigma_1 \wedge \sigma_2 \in \mathfrak{e}$, then $\sigma_1, \sigma_2 \in \mathfrak{e}$, since $\mathrm{Hom}\,(\sigma_1 \wedge \sigma_2, \sigma_i) \neq \emptyset$, $i = 1, 2$. Now suppose that $\sigma_1, \sigma_2 \in \mathfrak{e}$, and choose σ_3, q_1 and q_2 as in the property (E) for σ_1, σ_2. Then since $\sigma_1 \wedge \sigma_2$ is the product in $\mathcal{C}(W)$, we have a unique $s \in \mathrm{Hom}\,(\sigma_3, \sigma_1 \wedge \sigma_2)$ such that if $\mathrm{pr}_i : \sigma_1 \wedge \sigma_2 \to \sigma_i$ are the projections we have $\mathrm{pr}_1 \circ s = q_1$ and $\mathrm{pr}_2 \circ s = q_2$; in particular, $\mathrm{Hom}\,(\sigma_3, \sigma_1 \wedge \sigma_2) \neq \emptyset$, so by (i), $\sigma_1 \wedge \sigma_2 \in \mathfrak{e}$. □

It follows from this proposition that if $\mathfrak{e}$ is an étoile over W then $\mathrm{id}_W \in \mathfrak{e}$.

Remark 3.2.3. Let $S \subseteq \mathcal{C}(W)$ such that if $\sigma_1, \sigma_2 \in S$ then $\sigma_1 \wedge \sigma_2 \in S$. Let $\mathfrak{e}$ be an étoile over W. Then $S \subseteq \mathfrak{e}$ if and only if the following condition is satisfied:

(E') For every pair $(\sigma_\alpha, \sigma_i) \in \mathfrak{e} \times S$, there exists $(\sigma_\alpha', \sigma_i') \in \mathfrak{e} \times S$ such that:

$$\mathrm{Hom}\,(\sigma_\alpha', \sigma_\alpha) \neq \emptyset \neq \mathrm{Hom}\,(\sigma_i', \sigma_i)$$

and if q is the element of $\mathrm{Hom}\,(\sigma_\alpha' \wedge \sigma_i', \sigma_\alpha \wedge \sigma_i)$ then $\mathrm{im}\, q$ is relatively compact and non-empty.

Proof. First we remark that $\mathrm{Hom}\,(\sigma'_\alpha \wedge \sigma'_i, \sigma_\alpha \wedge \sigma_i) \neq \emptyset$ by the universal property for $\sigma_\alpha \wedge \sigma_i$:

$$\begin{tikzcd}
\sigma_\alpha & \sigma_\alpha \wedge \sigma_i \arrow[l] \arrow[r] & \sigma_i \\
\sigma'_\alpha \arrow[u] & \sigma'_\alpha \wedge \sigma'_i \arrow[l] \arrow[r] \arrow[u, "\exists!"] \arrow[ul] \arrow[ur] & \sigma'_i \arrow[u]
\end{tikzcd}$$

where the diagonal morphisms are obtained by composition. Suppose that the condition (E') holds. Let $\mathfrak{e}'$ be the subcategory of $\mathcal{C}(W)$ obtained by adding to $\mathfrak{e}$ all the $\sigma_\alpha \wedge \sigma_i$ with $\sigma_\alpha \in \mathfrak{e}$, $\sigma_i \in S$. As remarked above we have $\mathrm{id}_W \in \mathfrak{e}$, so $\mathfrak{e}' \supseteq S$, and it suffices to prove that $\mathfrak{e}' \subseteq \mathfrak{e}$. Suppose we show that $\mathfrak{e}'$ satisfies the property (E). Then there exists an étoile $\mathfrak{e}_0$ containing $\mathfrak{e}'$, and therefore containing $\mathfrak{e}$ (since $\sigma_\alpha \wedge \sigma_i \in \mathfrak{e}_0$ implies $\sigma_\alpha \in \mathfrak{e}_0$). By maximality of $\mathfrak{e}$ this shows that $\mathfrak{e}' \subseteq \mathfrak{e}_0 = \mathfrak{e}$.

Let us show that $\mathfrak{e}'$ satisfies (E). Let $(\sigma_\alpha, \sigma_i), (\sigma_\beta, \sigma_j) \in \mathfrak{e} \times S$. By (E') there exists $(\sigma_\gamma, \sigma_k) \in \mathfrak{e} \times S$ such that:

$$\mathrm{Hom}\,(\sigma_\gamma, \sigma_\alpha \wedge \sigma_\beta) \neq \emptyset \neq \mathrm{Hom}\,(\sigma_k, \sigma_i \wedge \sigma_j)$$

and

$$q \in \mathrm{Hom}\,(\sigma_\gamma \wedge \sigma_k, (\sigma_\alpha \wedge \sigma_\beta) \wedge (\sigma_i \wedge \sigma_j))$$

has a non-empty, relatively compact image. Let $\sigma_3 = \sigma_\gamma \wedge \sigma_k$. Using the associativity and commutativity of $\wedge$, we have projections

$$(\sigma_\alpha \wedge \sigma_\beta) \wedge (\sigma_i \wedge \sigma_j) \to \sigma_\alpha \wedge \sigma_i, \quad (\sigma_\alpha \wedge \sigma_\beta) \wedge (\sigma_i \wedge \sigma_j) \to \sigma_\beta \wedge \sigma_j.$$

We compose q with each of these to obtain

$$q_1 \in \mathrm{Hom}\,(\sigma_3, \sigma_\alpha \wedge \sigma_i), \quad q_2 \in \mathrm{Hom}\,(\sigma_3, \sigma_\beta \wedge \sigma_j)$$

with non-empty, relatively compact images. This proves the condition (E) for $\mathfrak{e}'$.

Conversely, suppose that $S \subseteq \mathfrak{e}$. Suppose that $(\sigma_\alpha, \sigma_i) \in \mathfrak{e} \times S \subseteq \mathfrak{e} \times \mathfrak{e}$, and choose $\sigma_3 \in \mathfrak{e}$, $q_1 \in \mathrm{Hom}(\sigma_3, \sigma_\alpha)$, $q_2 \in \mathrm{Hom}(\sigma_3, \sigma_i)$ as in the property (E). We take $\sigma'_\alpha = \sigma'_i = \sigma_3$, and let

$$q \in \mathrm{Hom}(\sigma_3 \wedge \sigma_3, \sigma_\alpha \wedge \sigma_i) = \mathrm{Hom}(\sigma_3, \sigma_\alpha \wedge \sigma_i).$$

We know that $\mathrm{im}\, q_1$ and $\mathrm{im}\, q_2$ are non-empty and relatively compact; we want to see the same for q. By the universal property for $\wedge$ we have a diagram:

$$\begin{tikzcd}
\sigma_\alpha & \sigma_\alpha \wedge \sigma_i \arrow[l] \arrow[r] & \sigma_i \\
& \sigma_3 \arrow[ul, "q_1"] \arrow[u, "q"] \arrow[ur, "q_2"'] &
\end{tikzcd}$$

By remark 3.1.11 there is a closed embedding from the domain of $\sigma_\alpha \wedge \sigma_i$ into the fibre product of the domains of σ_α and σ_i over W, so the fact that q_1 and q_2 have relatively compact image implies the same for q. $\square$

Definition 3.2.4. For a complex space W, we denote by $\mathcal{E}_W$ the set of all étoiles over W. We introduce a topology in $\mathcal{E}_W$ by taking as a basis of open sets:

$$\mathcal{E}_\sigma = \{\mathfrak{e} \in \mathcal{E}_W : \sigma \in \mathfrak{e}\}, \qquad \sigma \in \mathcal{C}(W).$$

Note that for each $\mathfrak{e} \in \mathcal{E}_W$ there must be some σ such that $\mathfrak{e} \in \mathcal{E}_\sigma$ since an étoile is maximal, hence non-empty. If $\sigma, \sigma' \in \mathcal{C}(W)$ and $\mathfrak{e} \in \mathcal{E}_\sigma \cap \mathcal{E}_{\sigma'}$ then by proposition 3.2.2 we have:

$$\mathfrak{e} \in \mathcal{E}_{\sigma\wedge\sigma'} = \mathcal{E}_\sigma \cap \mathcal{E}_{\sigma'}.$$

Therefore these sets do indeed form a basis for a topology on $\mathcal{E}_W$.

Proposition 3.2.5. *Let $\sigma : W' \to W$ be an element of $\mathcal{C}(W)$. Then $\mathcal{E}_W$ and $\mathcal{E}_{W'}$ are related as follows:*

(i) Given $\mathfrak{e}' \in \mathcal{E}_{W'}$, there exists one and only one $\alpha(\mathfrak{e}') = \mathfrak{e} \in \mathcal{E}_W$ such that:

$$\mathfrak{e} \supseteq \{\sigma \circ \sigma' : \sigma' \in \mathfrak{e}'\};$$

(ii) Given $\mathfrak{e} \in \mathcal{E}_W$, let $\mathfrak{e}''$ be the full subcategory of $\mathcal{C}(W')$ with objects:

$$\mathfrak{e}'' = \{q \in \mathrm{Hom}\,(\sigma \wedge \sigma_\alpha, \sigma) : \sigma_\alpha \in \mathfrak{e}\};$$

Then $\mathfrak{e}''$ has the property (E) if and only if $\mathfrak{e} \in \mathcal{E}_\sigma$

(iii) Given $\mathfrak{e} \in \mathcal{E}_\sigma$, the $\mathfrak{e}''$ of (ii) is an étoile over W', and consists of:

$$\mathfrak{e}'' = \{q \in \mathcal{C}(W') : \sigma \circ q \in \mathfrak{e}\}.$$

Moreover $\mathfrak{e}' \mapsto \alpha(\mathfrak{e}')$ and $\mathfrak{e} \mapsto \mathfrak{e}''$ are mutually inverse bijections $\mathcal{E}_\sigma \to \mathcal{E}_{W'}$.

Proof. First we prove (ii). Suppose that $\mathfrak{e}''$ has the property (E). Take $S = \{\sigma\}$. Then the property (E') for $\mathfrak{e}$ reads:

> For all $\sigma_\alpha \in \mathfrak{e}$ there exists $\sigma'_\alpha \in \mathfrak{e}$ such that $\mathrm{Hom}\,(\sigma'_\alpha, \sigma_\alpha) \neq \emptyset$ and the $q \in \mathrm{Hom}\,(\sigma'_\alpha \wedge \sigma, \sigma_\alpha \wedge \sigma)$ has relatively compact, non-empty image.

If we show that this holds then $\{\sigma\} \subseteq \mathfrak{e}$ so $\mathfrak{e} \in \mathcal{E}_\sigma$. Let $\sigma_\alpha \in \mathfrak{e}$ and apply the property (E), replacing both σ_1 and σ_2 with the unique $q \in \mathrm{Hom}\,(\sigma \wedge \sigma_\alpha, \sigma)$. Then we obtain $q' \in \mathrm{Hom}\,(\sigma \wedge \sigma'_\alpha, \sigma \wedge \sigma_\alpha)$ for some $\sigma \wedge \sigma'_\alpha \in \mathfrak{e}''$ with non-empty, relatively compact image. That is, the property (E') for $\mathfrak{e}$ and S is satisfied, and by remark 3.2.3 we have $S \subseteq \mathfrak{e}$, that is, $\mathfrak{e} \in \mathcal{E}_\sigma$. Conversely suppose that $\mathfrak{e} \in \mathcal{E}_\sigma$, that is, $\sigma \in \mathfrak{e}$. Then for $\sigma_\alpha \in \mathfrak{e}$, we have $\sigma \wedge \sigma_\alpha \in \mathfrak{e}$, so the property (E) for $\mathfrak{e}''$ follows immediately from this property for $\mathfrak{e}$.

Now we prove (i) and (iii) together. First we define $\alpha(\mathfrak{e}')$ non-uniquely: take $\mathfrak{e}' \in \mathcal{E}_{W'}$. Let $\mathfrak{e}_0$ be the subcategory of $\mathcal{C}(W)$ consisting of those $\sigma \circ \sigma'$ with $\sigma' \in \mathfrak{e}'$. Then $\mathfrak{e}_0$ satisfies (E) since $\mathfrak{e}'$ does, and there exists an étoile $\alpha(\mathfrak{e}')$ containing $\mathfrak{e}_0$.

Since $\mathrm{id}_{W'} \in \mathfrak{e}'$, we have $\sigma \in \alpha(\mathfrak{e}')$, i.e. $\alpha(\mathfrak{e}') \in \mathcal{E}_\sigma$. Therefore $\alpha(\mathfrak{e}')''$ defined by (ii) satisfies (E). We claim that $\mathfrak{e}' \subseteq \alpha(\mathfrak{e}')''$. Let $\sigma' \in \mathfrak{e}'$ so $\sigma \circ \sigma' \in \mathfrak{e}_0 \subseteq \mathfrak{e}$. Since $(\sigma \circ \sigma') \wedge \sigma = \sigma \circ \sigma'$ we recover $\sigma' \in \mathfrak{e}'$ as

$$\sigma' \in \mathrm{Hom}(\sigma \wedge (\sigma \circ \sigma'), \sigma) \in \alpha(\mathfrak{e}')''.$$

Now since $\mathfrak{e}'$ is an étoile, it follows that $\alpha(\mathfrak{e}')'' = \mathfrak{e}'$. That is, the mapping

$$\mathfrak{e}' \in \mathcal{E}_{W'} \overset{\alpha}{\longmapsto} \alpha(\mathfrak{e}') \in \mathcal{E}_\sigma \subseteq \mathcal{E}_W \mapsto \alpha(\mathfrak{e}')'' \in \mathcal{E}_{W'} \tag{3.2.1}$$

is the identity.

Now pick any $\mathfrak{e} \in \mathcal{E}_\sigma \subseteq \mathcal{E}_W$ and define $\mathfrak{e}''$ as in (ii). Then $\mathfrak{e}''$ satisfies (E), and we can choose an étoile $\mathfrak{e}' \in \mathcal{E}_{W'}$ containing $\mathfrak{e}''$. Then we take any image point $\alpha(\mathfrak{e}')$ as defined before. Now we claim that $\mathfrak{e} \subseteq \alpha(\mathfrak{e}')$. If $\sigma_\alpha \in \mathfrak{e}$, then we have that $q \in \mathrm{Hom}(\sigma \wedge \sigma_\alpha, \sigma)$ belongs to $\mathfrak{e}'' \subseteq \mathfrak{e}'$. Moreover $\alpha(\mathfrak{e}')$ is an étoile containing $\sigma \circ q = \sigma \wedge \sigma_\alpha$ so $\sigma_\alpha \in \alpha(\mathfrak{e}')$. By maximality of $\mathfrak{e}$ we conclude that $\mathfrak{e} = \alpha(\mathfrak{e}')$. Thus the mapping

$$\mathfrak{e} \in \mathcal{E}_\sigma \mapsto \mathfrak{e}' \in \mathcal{E}_{W'} \overset{\alpha}{\longmapsto} \alpha(\mathfrak{e}') \in \mathcal{E}_W \tag{3.2.2}$$

is the identity, and we have $\mathfrak{e}' \supseteq \mathfrak{e}''$. By (3.2.2) we have $\alpha(\mathfrak{e}') = \mathfrak{e}$ and by (3.2.1) we have $\alpha(\mathfrak{e}')'' = \mathfrak{e}'$, so $\mathfrak{e}' = \mathfrak{e}''$, and (3.2.2) reads:

$$\mathfrak{e} \in \mathcal{E}_\sigma \mapsto \mathfrak{e}'' \in \mathcal{E}_{W'} \overset{\alpha}{\longmapsto} \alpha(\mathfrak{e}') \in \mathcal{E}_W. \tag{3.2.3}$$

Moreover (3.2.1) shows that $\mathfrak{e} \mapsto \mathfrak{e}''$ is surjective; it then follows from (3.2.3) that α is well defined, i.e. has a unique image point. Now (3.2.1) and (3.2.3) together show that $\mathfrak{e}' \mapsto \alpha(\mathfrak{e}')$ and $\mathfrak{e} \mapsto \mathfrak{e}''$ are inverse to each other, so they are mutually inverse bijections $\mathcal{E}_\sigma \leftrightarrow \mathcal{E}_{W'}$. □

Proposition 3.2.6. *Given $\sigma : W' \to W \in \mathcal{C}(W)$, there exists a unique $j_\sigma : \mathcal{E}_{W'} \to \mathcal{E}_W$ such that*

$$j_\sigma(\mathfrak{e}') \supseteq \{\sigma \circ \sigma' : \sigma' \in \mathfrak{e}'\}.$$

Moreover j_σ induces an isomorphism of topological spaces $\mathcal{E}_{W'} \to \mathcal{E}_\sigma$.

Proof. By (i) of the proposition and the final assertion, $j_\sigma : \mathcal{E}_{W'} \to \mathcal{E}_\sigma$ exists and is bijective. It suffices to check bicontinuity of j_σ on a basis. Moreover by the proposition, for any $\mathcal{E}_{\sigma'} \subseteq \mathcal{E}_{W'}$ we have $j_\sigma(\mathcal{E}_{\sigma'}) = \mathcal{E}_{\sigma\circ\sigma'}$, so j_σ^{-1} is continuous. If $\tilde{\sigma} \in \mathcal{C}(W)$ then $\mathcal{E}_{\tilde{\sigma}} \cap \mathcal{E}_\sigma = \mathcal{E}_{\tilde{\sigma}\wedge\sigma}$, and if $q \in \mathrm{Hom}(\tilde{\sigma} \wedge \sigma, \sigma)$ then we have

$$j_\sigma(\mathcal{E}_q) = \mathcal{E}_{\sigma\circ q} = \mathcal{E}_{\tilde{\sigma}\wedge\sigma}$$

Since j_σ is a bijection this shows that $j_\sigma^{-1}(\mathcal{E}_{\tilde{\sigma}\wedge\sigma})$ is open, so j_σ is continuous. □

Proposition 3.2.7. *Take any $\sigma : W' \to W \in \mathfrak{e} \in \mathcal{E}_W$. Then*

$$\bigcap_{\substack{\sigma' \in \mathcal{C}(W') \\ \sigma\circ\sigma' \in \mathfrak{e}}} \mathrm{im}\, \sigma'$$

is a single point of W'. Moreover for every open neighborhood U' of this point in W' we have $\sigma_{|U'} \in \mathfrak{e}$.

Proof. Suppose that the result holds for all choices of W and $\sigma = \mathrm{id}_W$. By (iii) of proposition 3.2.5, writing $\mathfrak{e}'' = j_\sigma^{-1}(\mathfrak{e})$ we have that:

$$\bigcap_{\substack{\sigma' \in \mathcal{C}(W') \\ \sigma\circ\sigma' \in \mathfrak{e}}} \mathrm{im}\, \sigma' = \bigcap_{\sigma' \in \mathfrak{e}''} \mathrm{im}\, \sigma'$$

$$= \bigcap_{\substack{\sigma' \in \mathcal{C}(W') \\ \mathrm{id}_{W'} \circ \sigma' \in \mathfrak{e}''}} \mathrm{im}\, \sigma'.$$

Now by our assumption applied to W' and $\mathrm{id}_{W'}$, this last intersection is equal to a single point of W', and $\mathrm{id}_{|U'} \in \mathfrak{e}''$ for all neighborhoods of this point. Thus the first intersection is a single point and by (iii) of proposition 3.2.5, $\sigma_{|U'} \in \mathfrak{e}$ for all neighborhoods U' of this point. Thus we may as well assume that $\sigma = \mathrm{id}_W$. Let $S = \left\{\mathrm{id}_{W|U}\right\}$ where U runs through the set of all open neighborhoods of some fixed point

$$y \in \bigcap_{\sigma_\alpha \in \mathfrak{e}} \overline{\mathrm{im}\, \sigma_\alpha}$$

provided this intersection is non-empty. Then S verifies the property (E'): given $(\sigma_\alpha, \sigma_i) \in \mathfrak{e} \times S$, we can find $(\sigma'_\alpha, \sigma'_i) \in \mathfrak{e} \times S$ such that the images of σ'_α and σ'_i in σ_α and σ_i are relatively compact; then using the closed embedding of $\sigma_\alpha \wedge \sigma_i$ into the fibre product we have the property (E'). Thus $S \subseteq \mathfrak{e}$. Since W is Hausdorff, it follows that:

$$\bigcap_{\sigma_\alpha \in \mathfrak{e}} \overline{\mathrm{im}\, \sigma_\alpha} = \{y\}.$$

But for each $\sigma_\alpha \in \mathfrak{e}$ there exists $\sigma_\beta \in \mathfrak{e}$ and $q \in \mathrm{Hom}\,(\sigma_\beta, \sigma_\alpha)$ with $\mathrm{im}\, q$ relatively compact in $\mathrm{dom}\, \sigma_\alpha$, so $\overline{\mathrm{im}\, \sigma_\beta} \subseteq \mathrm{im}\, \sigma_\alpha$, and we obtain

$$\bigcap_{\sigma_\alpha \in \mathfrak{e}} \overline{\mathrm{im}\, \sigma_\alpha} \supseteq \bigcap_{\sigma_\alpha \in \mathfrak{e}} \mathrm{im}\, \sigma_\alpha \supseteq \bigcap_{\sigma_\alpha \in \mathfrak{e}} \overline{\mathrm{im}\, \sigma_\alpha}.$$

Thus it suffices to show that $\bigcap_{\sigma_\alpha \in \mathfrak{e}} \overline{\mathrm{im}\, \sigma_\alpha} \neq \emptyset$. It follows from the property (E) that for some $\sigma_\alpha \in \mathfrak{e}$ we have $\overline{\mathrm{im}\, \sigma_\alpha}$ is compact. Moreover after an inductive application of (E) we have that every finite set $A = \{\alpha_i\}$ we have $\bigcap_{\alpha_i \in A} \overline{\mathrm{im}\, \sigma_{\alpha_i}} \neq \emptyset$. It follows that the whole intersection is non-empty. □

Definition 3.2.8. We write $p_W : \mathcal{E}_W \to W$ for the map defined by:

$$\{p_W(\mathfrak{e})\} = \bigcap_{\sigma \in \mathfrak{e}} \mathrm{im}\, \sigma.$$

Remark 3.2.9. In the proof of the last proposition we saw that for an open set $U \subseteq W$, if $\iota_U : U \hookrightarrow W$ is the inclusion then $\iota_U \in \mathfrak{e}$ if and only if $p_W(\mathfrak{e}) \in U$.

Proposition 3.2.10. *The map p_W is continuous and surjective. For $\sigma : W' \to W$ we have $p_W \circ j_\sigma = \sigma \circ p_{W'}$.*

Proof. By remark 3.2.9, if $U \subseteq W$ is open we have $p_W^{-1}(U) = \mathcal{E}_{\iota_U}$, so p_W is continuous. Now let $y \in W$ and take $\mathfrak{e}_0 = \{\iota_U\}$ where U runs through the open neighborhoods of y in W. It is clear that $\mathfrak{e}_0$ satisfies (E), so we can choose an étoile $\mathfrak{e}$ containing $\mathfrak{e}_0$. Then we have $p_W(\mathfrak{e}) = y$ so p_W is surjective. For the last part, if $\mathfrak{e}' \in \mathcal{E}_{W'}$,

$$\begin{aligned} p_W(j_\sigma(\mathfrak{e}')) &= \bigcap_{\sigma \in j_\sigma(\mathfrak{e}')} \mathrm{im}\, \sigma \\ &\subseteq \bigcap_{\sigma' \in \mathfrak{e}'} \mathrm{im}(\sigma \circ \sigma') \end{aligned}$$

$$= \sigma\left(\bigcap_{\sigma' \in \mathfrak{e}'} \operatorname{im} \sigma'\right)$$
$$= \sigma(p_{W'}(\mathfrak{e}')). \qquad \square$$

We denote $p_\sigma : \mathcal{E}_W \to W'$ the map defined by $p_\sigma \circ j_\sigma = p_{W'}$, where $\sigma : W' \to W$ is any object in $\mathcal{C}(W)$.

3.3 Further properties of étoiles

Lemma 3.3.1. *Let $\mathfrak{e} \in \mathcal{E}_W$ be an étoile, let $\pi : W' \to W$ be a blowing up with nowhere dense centre E. Then $\pi \in \mathfrak{e}$.*

Proof. Let $S = \{\pi\}$; we will show that $(S, \mathfrak{e})$ satisfy the condition (E'). That is, for all $\sigma_\alpha \in \mathfrak{e}$ there exists $\sigma_\beta \in \mathfrak{e}$ such that $\operatorname{Hom}(\sigma_\beta, \sigma_\alpha) \neq \emptyset$ and

$$q \in \operatorname{Hom}(\sigma_\beta \wedge \pi, \sigma_\alpha \wedge \pi)$$

has non-empty, relatively compact image. Let our notation be as follows:

$$\begin{aligned} \sigma_\alpha &: W_\alpha \to W \in \mathfrak{e} \\ \sigma_\alpha \wedge \pi &: W'_\alpha \to W \\ \tau &: W'_\alpha \to W_\alpha \in \operatorname{Hom}(\sigma_\alpha \wedge \pi, \sigma_\alpha) \end{aligned}$$

Then we have the following diagram:

$$\begin{tikzcd} W'_\alpha \arrow[r] \arrow[d, "\tau"'] \arrow[dr, "\sigma_\alpha \wedge \pi"] & W' \arrow[d, "\pi"] \\ W_\alpha \arrow[r, "\sigma_\alpha"] & W \end{tikzcd}$$

Since σ_α is an isomorphism in an open, dense subset, $\sigma_\alpha^{-1}(E)$ is nowhere dense in W_α. In the case where σ_α is a single local blowing up, lemma 3.1.9 shows that τ is the blowing up with centre $\sigma_\alpha^{-1}(E)$, and when σ_α is composed of several local blowings up, we apply lemma 3.1.9 inductively to show the same result. Since $\sigma_\alpha^{-1}(E)$ is nowhere dense, by remark 2.4.2 we must have that W'_α is non-empty. Now using the condition (E), choose $\sigma_\beta \in \mathfrak{e}$ such that $\operatorname{Hom}(\sigma_\beta, \sigma_\alpha) \neq \emptyset$ and $q \in \operatorname{Hom}(\sigma_\beta, \sigma_\alpha)$ has non-empty, relatively compact image. Let us choose notation as follows:

$$\begin{tikzcd} W'_\beta \arrow[r, "q'"] \arrow[d, "\sigma'_\beta"'] \arrow[dr, "\tau \wedge q"] & W'_\alpha \arrow[r] \arrow[d, "\tau"'] \arrow[dr, "\sigma_\alpha \wedge \pi"] & W' \arrow[d, "\pi"] \\ W_\beta \arrow[r, "q"'] & W_\alpha \arrow[r, "\sigma_\alpha"'] & W \end{tikzcd}$$

Claim: $\sigma_\alpha \circ (\tau \wedge q) = (\sigma_\alpha \wedge \pi) \wedge \sigma_\beta = \sigma_\beta \wedge \pi$

Proof: The second equality is immediate, since $\operatorname{Hom}(\sigma_\beta, \sigma_\alpha) \neq \emptyset$, and $\wedge$ is commutative. For the first, we observe that:

$$\sigma_\alpha \circ \tau = \sigma_\alpha \wedge \pi, \text{ and } \sigma_\alpha \circ q = \sigma_\beta$$

thus

$$(\sigma_\alpha \wedge \pi) \wedge \sigma_\beta = (\sigma_\alpha \circ \tau) \wedge (\sigma_\alpha \circ q).$$

So the first equality follows from the fact that $\circ$ distributes over $\wedge$ in the sense that:

$$(\sigma_\alpha \circ \tau) \wedge (\sigma_\alpha \circ q) = \sigma_\alpha \circ (\tau \wedge q).$$

To see this, we just add the arrow $\sigma_\alpha : W_\alpha \to W$ to the diagram for the universal property of the product $\wedge$ in $\mathcal{C}(W_\alpha)$.

Therefore in our diagram,

$$q' \in \mathrm{Hom}(\sigma_\beta \wedge \pi, \sigma_\alpha \wedge \pi).$$

Moreover, we know that $\mathrm{im}(q)$ is relatively compact and non-empty, and τ is a blowing up, so by proposition 2.4.1 τ is proper. Therefore $\mathrm{im}(q')$ is also relatively compact (and non-empty). This shows that (E') is satisfied for $S = \{\pi\}$, and we are done. □

Remark 3.3.2. Before we proceed we note that "belonging to an étoile" is a local property, in the sense that if $\mathfrak{e}$ is an étoile over a complex space W, and U is any neighborhood of $p_W(\mathfrak{e})$ in W, then for any $\sigma \in \mathcal{C}(W)$ we have the equivalence:

$$\sigma \in \mathfrak{e} \iff \sigma_{|\sigma^{-1}(U)} \in \mathfrak{e}.$$

To see this, let $\iota : U \to W$ be the inclusion, so by remark 3.2.9, our hypotheses imply that $\iota \in \mathfrak{e}$, i.e. $\mathfrak{e} \in \mathcal{E}_\iota$. Therefore

$$\begin{aligned}
\sigma \in \mathfrak{e} &\Leftrightarrow \mathfrak{e} \in \mathcal{E}_\sigma \\
&\Leftrightarrow \mathfrak{e} \in \mathcal{E}_\sigma \cap \mathcal{E}_\iota = \mathcal{E}_{\sigma\wedge\iota} \\
&\Leftrightarrow \sigma \wedge \iota \in \mathfrak{e} \\
&\Leftrightarrow \sigma_{|\sigma^{-1}(U)} \in \mathfrak{e},
\end{aligned}$$

where the last equivalence follows from the fact that ι is just the inclusion, so $\sigma \wedge \iota = \sigma_{|\sigma^{-1}(U)}$.

Corollary 3.3.3. *Let $\sigma : W' \to W \in \mathcal{C}(W)$ and let $\mathfrak{e}' \in \mathcal{E}_W$. If (U', E', σ') is a local blowing up over W' such that $y' = p_{W'}(\mathfrak{e}') \in U'$ and E' is nowhere dense in some neighborhood of y' in U', then $\sigma \circ \sigma' \in j_\sigma(\mathfrak{e}')$.*

Proof. By (iii) of proposition 3.2.5 the assertion is equivalent to the statement $\sigma' \in \mathfrak{e}'$. Thus the statement does not depend on σ, so we may as well assume that $W' = W$, $\sigma = \mathrm{id}_W$. Thus it suffices to prove:

Let W be a complex space, let $\mathfrak{e} \in \mathcal{E}_W$. If $\sigma = (U, E, \pi)$ is a local blowing up over W such that $y = p_W(\mathfrak{e}) \in U$, and E is nowhere dense in a neighborhood of y in U, then $\sigma \in \mathfrak{e}$ (i.e. $\mathfrak{e} \in \mathcal{E}_\sigma$).

Now by remark 3.3.2 we have $\sigma \in \mathfrak{e}$ if and only if $\sigma_{|\sigma^{-1}(U')} \in \mathfrak{e}$ for any neighborhood U' of y in U. Therefore we may assume that E is nowhere dense and σ is a (global) blowing up. In particular, the result follows from lemma 3.3.1. □

Lemma 3.3.4. *Let $\sigma : W' \to W \in \mathfrak{e} \in \mathcal{E}_W$, and let $\sigma' = (U', E', \pi')$ be a local blowing up over W such that $\sigma \circ \sigma' \in \mathfrak{e}$. Let $y' = p_\sigma(\mathfrak{e}) \in U'$ and let U'' be any neighborhood of y' in U'. Let E'' be any closed complex subspace of $E' \cap U''$. Let $\sigma'' = (U'', E'', \pi'')$ be the local blowing up defined by these data. Then $\sigma \circ \sigma'' \in \mathfrak{e}$.*

Proof. We have a homeomorphism $j_\sigma : \mathcal{E}_{W'} \to \mathcal{E}_\sigma$, and there exists $\mathfrak{e}' \in \mathcal{E}_{W'}$ such that $j_\sigma(\mathfrak{e}') = \mathfrak{e}$. Moreover j_σ induces a homeomorphism

$$j_\sigma : \mathcal{E}_{\sigma'} \to \mathcal{E}_{\sigma\circ\sigma'}$$

Thus the hypothesis $\sigma \circ \sigma' \in \mathfrak{e}$ is equivalent to $j_\sigma^{-1}(\mathfrak{e}) = \mathfrak{e}' \in \mathcal{E}_{\sigma'}$, and we want to show that $\sigma'' \in \mathfrak{e}'$. So the lemma does not depend on σ. We choose notation as in the following diagram:

$$\begin{tikzcd}
 & W_2' \arrow[dr, "\sigma''"] & \\
W_3' \arrow[ur, "q_2"] \arrow[rr, "\sigma'\wedge\sigma''"] \arrow[dr, "q_1"'] & & W' \\
 & W_1' \arrow[ur, "\sigma'"'] &
\end{tikzcd}$$

Let

$$y_1' = p_{\sigma'}(\mathfrak{e}') = p_{W_1'} \circ j_{\sigma'}^{-1}(\mathfrak{e}') \in W_1'$$

so

$$\begin{aligned}
\sigma'(y_1') &= \sigma' \circ p_{W_1'} \circ j_{\sigma'}^{-1}(\mathfrak{e}') \\
&= p_{W'} \circ j_{\sigma'} \circ j_{\sigma'}^{-1}(\mathfrak{e}') && \text{[proposition 3.2.10]} \\
&= p_{W'}(\mathfrak{e}') \\
&= p_{W'} \circ j_\sigma^{-1}(\mathfrak{e}) = p_\sigma(\mathfrak{e}) = y'.
\end{aligned}$$

Let $V' = (\sigma')^{-1}(U')$ and $V'' = (\sigma')^{-1}(U'')$. We suppose that $y' \in U''$, so the computation above shows that $y_1' \in V''$. Now by lemma 3.1.9 we know that

$$q_1 = (V'', (\sigma')^{-1}(E''), q_{1|q_1^{-1}(V'')})$$

is a local blowing up over W_1'. Moreover $(\sigma')^{-1}(E')$ has an invertible ideal sheaf in $\mathcal{O}_{V'}$ by the definition of σ', so $(\sigma')^{-1}(E')$ is nowhere dense in V'. Therefore the subset $(\sigma')^{-1}(E'') \subseteq (\sigma')^{-1}(E')$ is nowhere dense in V''. Therefore by corollary 3.3.3, we have $\sigma' \circ q_1 \in j_{\sigma'}(\mathfrak{e}_0)$ for all étoiles $\mathfrak{e}_0$ with $\mathfrak{e}_0 \in \mathcal{E}_{W_1'}$ such that $p_{W_1'}(\mathfrak{e}_0) \in V''$. We have seen that:

$$p_{W_1'} \circ j_{\sigma'}^{-1}(\mathfrak{e}') = p_{\sigma'}(\mathfrak{e}') = y_1' \in V''$$

so we have $\sigma' \circ q_1 \in j_{\sigma'} \circ j_{\sigma'}^{-1}(\mathfrak{e}') = \mathfrak{e}'$. Finally we have $\sigma' \circ q_1 = \sigma' \wedge \sigma''$, so this shows that $\sigma'' \in \mathfrak{e}'$ as required. □

Before we continue, we recall some facts about meromorphic functions and fractional ideals on complex

spaces. Let $(X, \mathcal{O}_X)$ be a complex space. We have a subsheaf $\mathcal{S}_X \subseteq \mathcal{O}_X$ of nonzerodivisors

$$U \mapsto \mathcal{S}_X(U) = \{\text{nonzerodivisors of } \mathcal{O}_X(U)\}.$$

We define the sheaf $\mathcal{M}_X$ of meromorphic functions on X to be the sheaf associated to the presheaf

$$U \mapsto \mathcal{O}_X(U)[\mathcal{S}_X(U)^{-1}]$$

where $\mathcal{O}_X(U)[\mathcal{S}_X(U)^{-1}]$ is the total ring of fractions of $\mathcal{O}_X(U)$, i.e. the localization of $\mathcal{O}_X(U)$ at the set of all nonzerodivisors. We also have the sheaf $\mathcal{M}_X^*$ of groups of units in $\mathcal{M}_X$, which by construction is the sheaf of nonzerodivisors in $\mathcal{M}_X$. We call a sub-$\mathcal{O}_X$-module $\mathcal{J} \subseteq \mathcal{M}_X$ a *fractional ideal*. Now a fractional ideal $\mathcal{J} \subseteq \mathcal{M}_X$ is invertible if and only if for each $x \in X$ there exists an open neighborhood U of x such that $\mathcal{J}_{|U} = \mathcal{O}_U \cdot f$ for some $f \in \mathcal{M}_X^*(X)$. We see easily that f is unique up to multiplication by elements of $\mathcal{O}_X^*(U)$. From the above we have

Proposition 3.3.5. *Let $\mathcal{J}$ be an invertible fractional ideal.*

(i) If $\mathcal{J}_{|U} = \mathcal{O}_U \cdot f$, and we define $\mathcal{J}'_{|U} = \mathcal{O}_U \cdot f^{-1}$ then $\mathcal{J}' \simeq \mathcal{J}^{-1}$;

(ii) If $\mathcal{J}_1, \mathcal{J}_2$ are invertible fractional ideals, then $\mathcal{J}_1 \otimes \mathcal{J}_2 \simeq \mathcal{J}_1\mathcal{J}_2$.

Since the set of (isomorphism classes of) invertible sheaves form a group under $\otimes$, the proposition shows that the set of invertible fractional ideals form a group under multiplication. We use this in the following:

Lemma 3.3.6. *Let $\sigma_i = (U_i, E_i, \pi_i)$ be a local blowing up over a complex space W, $i = 1, 2$. Let*

$$E = E_1 \cap E_2,\ U = U_1 \cap U_2, \quad \sigma = (U, E, \pi)$$

where $\pi : W' \to U$ is the blowing up over U with centre E. Let $q_i \in \mathrm{Hom}(\sigma \wedge \sigma_i, \sigma)$. Then there exist closed complex subspaces E'_1, E'_2 of W' such that $E'_1 \cap E'_2 = \emptyset$ and q_i is the blowing up with centre E'_i, $i = 1, 2$.

Proof. We choose notation as follows:

$$\begin{tikzcd}
W'_1 \arrow[dd] \arrow[dr, "q_1"] & & W'_2 \arrow[dl, "q_2"'] \arrow[dd] \\
& W' \arrow[d, "\sigma"] & \\
W_1 \arrow[dr, "\sigma_1"'] & & W_2 \arrow[dl, "\sigma_2"] \\
& W &
\end{tikzcd}$$

Let $\mathcal{I}_i$ be the ideal sheaf of E_i in $\mathcal{O}_{U_i}$, so $\mathcal{I}_1 + \mathcal{I}_2$ is the ideal sheaf of E in $\mathcal{O}_U$, and $(\mathcal{I}_1 + \mathcal{I}_2)\mathcal{O}_{W'}$ is an invertible $\mathcal{O}_{W'}$-module. Let $\mathcal{Q}'$ be an invertible fractional ideal of $\mathcal{O}_{W'}$ such that

$$\mathcal{Q}'(\mathcal{I}_1 + \mathcal{I}_2)\mathcal{O}_{W'} = \mathcal{O}_{W'}.$$

Then we have

$$\mathcal{I}_i\mathcal{Q}'(\mathcal{I}_1 + \mathcal{I}_2)\mathcal{O}_{W'} = \mathcal{I}_i\mathcal{O}_{W'}$$

and

$$\mathcal{I}_i\mathcal{Q}' \subseteq \mathcal{Q}'(\mathcal{I}_1+\mathcal{I}_2)\mathcal{O}_{W'} = \mathcal{O}_{W'}$$

so $\mathcal{Q}_i := \mathcal{I}_i\mathcal{Q}'$ is a (coherent) sheaf of ideals of $\mathcal{O}_{W'}$ such that $\mathcal{Q}_i(\mathcal{I}_1+\mathcal{I}_2)\mathcal{O}_{W'} = \mathcal{I}_i\mathcal{O}_{W'}$. Now

$$\begin{aligned}(\mathcal{I}_1+\mathcal{I}_2)\mathcal{O}_{W'} &= \mathcal{I}_1\mathcal{O}_{W'} + \mathcal{I}_2\mathcal{O}_{W'}\\ &= \mathcal{Q}_1(\mathcal{I}_1+\mathcal{I}_2)\mathcal{O}_{W'} + \mathcal{Q}_2(\mathcal{I}_1+\mathcal{I}_2)\mathcal{O}_{W'}\\ &= (\mathcal{Q}_1+\mathcal{Q}_2)(\mathcal{I}_1+\mathcal{I}_2)\mathcal{O}_{W'}\end{aligned}$$

so we have $Q_1 + Q_2 = \mathcal{O}_{W'}$. Let E_i' be the closed complex subspace of W' defined by Q_i. Then we have $E_1' \cap E_2' = \emptyset$. By lemma 3.1.9 we have that q_i is the blowing up with centre $\sigma^{-1}(E_i)$. Moreover if $\mathcal{I}_{\sigma^{-1}(E_i)}$ and $\mathcal{I}_{E_i'}$ are the ideal sheaves of $\sigma^{-1}(E_i)$ and E_i' respectively then

$$\begin{aligned}\mathcal{I}_{\sigma^{-1}(E_i)} = \mathcal{I}_i\mathcal{O}_{W'} &= \mathcal{Q}_1(\mathcal{I}_1+\mathcal{I}_2)\mathcal{O}_{W'}\\ &= \mathcal{I}_{E_i'}(\mathcal{I}_1+\mathcal{I}_2)\mathcal{O}_{W'}.\end{aligned}$$

Since $(\mathcal{I}_1+\mathcal{I}_2)\mathcal{O}_{W'}$ is already invertible, it follows that q_i is also the blowing up with centre E_i'. This last statement is true because by lemma 2.5.2, a product of ideals is invertible if and only if each of the ideals is separately invertible, and by lemma 3.1.2 the inverse image of the invertible sheaf of ideals $(\mathcal{I}_1+\mathcal{I}_2)\mathcal{O}_{W'}$ by the strict morphism given by the blowing up of E_i' is again invertible. □

Now we show that $\mathcal{E}_W$ is Hausdorff.

Proposition 3.3.7. *Let W be a complex space, let $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathcal{E}_W$, with $\mathfrak{e}_1 \neq \mathfrak{e}_2$. Then there exists $\sigma \in \mathfrak{e}_1 \cap \mathfrak{e}_2$ such that $p_\sigma(\mathfrak{e}_1) \neq p_\sigma(\mathfrak{e}_2)$.*

The fact that $\mathcal{E}_W$ is Hausdorff follows from this: we have $\sigma \in \mathfrak{e}_1 \cap \mathfrak{e}_2$ if and only if $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathcal{E}_\sigma$. Write $\sigma : W' \to W$. Now W' is Hausdorff by proposition 2.4.1, so there exist open sets $U_1, U_2 \subseteq W'$ such that $U_1 \cap U_2 = \emptyset$ and $p_\sigma(\mathfrak{e}_i) \in U_i$. Let $\sigma_i = \sigma_{|U_i}$, $i = 1, 2$. Then we have

$$p_\sigma^{-1}(U_i) = j_\sigma \circ p_{W'}^{-1}(U_i) = j_\sigma(\mathcal{E}_{\mathrm{id}\,|U_i}) = \mathcal{E}_{\sigma\circ\mathrm{id}\,|U_i} = \mathcal{E}_{\sigma_i}$$

so we must have $\mathcal{E}_{\sigma_1} \cap \mathcal{E}_{\sigma_2} = \emptyset$, and by proposition 3.2.7 applied to $j_\sigma^{-1}(\mathfrak{e}_i)$, we have $\mathfrak{e}_i \in \mathcal{E}_{\sigma_i}$, $i = 1, 2$. Before proving proposition 3.3.7 we need a lemma. The lemma establishes the case when there exist single local blowings up $\sigma_i \in \mathfrak{e}_i$ with $\sigma_1 \notin \mathfrak{e}_2$ and $\sigma_2 \notin \mathfrak{e}_1$. We will then deduce the general case using the diagram on page 26.

Lemma 3.3.8. *Let W be a complex space, let $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathcal{E}_W$. Suppose that there exists $\sigma_i = (U_i, E_i, \pi_i) \in \mathfrak{e}_i$, $i = 1, 2$ such that $\sigma_1 \notin \mathfrak{e}_2$ and $\sigma_2 \notin \mathfrak{e}_1$. Then there exists $\sigma \in \mathfrak{e}_1 \cap \mathfrak{e}_2$ such that $p_\sigma(\mathfrak{e}_1) \neq p_\sigma(\mathfrak{e}_2)$.*

Proof of lemma 3.3.8. If $p_W(\mathfrak{e}_1) \neq p_W(\mathfrak{e}_2)$ then we may take $\sigma = \mathrm{id}$. Therefore suppose that $p_W(\mathfrak{e}_1) = p_W(\mathfrak{e}_2)$, and let

$$U = U_1 \cap U_2,\ E = E_1 \cap E_2, \quad \sigma = (U, E, \pi : W' \to W).$$

Then $p_W(\mathfrak{e}_1) \in U$, so by lemma 3.3.4 (with $\sigma = \mathrm{id}$, $\mathfrak{e} = \mathfrak{e}_1$, $U'' = U$, $E'' = E$ and $y' = p_W(\mathfrak{e}_1)$) we have that

$\sigma \in \mathfrak{e}_1$, and in the same way $\sigma \in \mathfrak{e}_2$. Thus $\sigma \in \mathfrak{e}_1 \cap \mathfrak{e}_2$. Now if

$$q_i \in \mathrm{Hom}(\sigma_i \wedge \sigma, \sigma)$$

by lemma 3.3.6 there exist closed subspaces $E_i' \subseteq W'$ such that q_i is the blowing up with centre E_i', $i = 1, 2$, and $E_1' \cap E_2' = \emptyset$. If $y_i = p_\sigma(\mathfrak{e}_i)$, it then suffices to show that $y_i \in E_j'$, $j \neq i$. Indeed suppose that $y_1 \in E_2'$, since E_2' is the (closed) centre of q_2 there exists a neighborhood V of y_1 in W' such that $q_{2|q_2^{-1}(V)}$ is an isomorphism. Moreover we know from remark 3.3.2 that $\sigma \circ q_2 \in \mathfrak{e}_1$ if and only if some restriction of $\sigma \circ q_2$ to the preimage of a neighborhood of y_1 belongs to $\mathfrak{e}_1$, which is the case for $\sigma \circ q_{2|q_2^{-1}(V)}$. Moreover by the definition of q_2 we have $\sigma \circ q_2 = \sigma_2 \wedge \sigma$, from which we deduce that $\sigma_2 \in \mathfrak{e}_1$, which contradicts our hypothesis on σ_2. Therefore $y_1 \in E_2'$. In the same way we show that $y_2 \in E_1'$, so we are done. □

Proof of proposition 3.3.7. Let $\mathfrak{e}_1 \neq \mathfrak{e}_2$, and choose $\sigma_i \in \mathfrak{e}_i$ with $\sigma_1 \notin \mathfrak{e}_2$ and $\sigma_2 \notin \mathfrak{e}_1$. We recall here the notation of the proof of theorem 3.1.7, in particular that of the diagram on page 26. We write σ_1 and σ_2 as a finite sequence of local blowings up:

$$\sigma_1 = \sigma_{00} \circ \cdots \circ \sigma_{m-1,0},$$

$$\sigma_2 = \sigma'_{00} \circ \cdots \circ \sigma'_{n-1,0}$$

and we define in the same way a commutative lattice of local blowings up

$$\sigma_{ij} : W_{i+1,j} \to W_{ij},\ \sigma'_{ij} : W_{i,j+1} \to W_{ij}$$

for $i = 0, \ldots, m$, $j = 0, \ldots, n$. Let $\sigma^{ij} : W_{ij} \to W$ be the composition of these morphisms along the lattice. Now there must exist $i < m$ and $j < n$ such that

$$\sigma^{ij} \in \mathfrak{e}_1 \cap \mathfrak{e}_2,\ \sigma^{i+1,j} \notin \mathfrak{e}_2 \text{ and } \sigma^{i,j+1} \notin \mathfrak{e}_1. \tag{3.3.1}$$

If this were not the case, say $\sigma^{mj} \in \mathfrak{e}_2$ then we would have $\mathrm{Hom}(\sigma^{mj}, \sigma_1) \neq \emptyset$, since we have the morphism obtained along the lattice. This implies $\sigma_1 \in \mathfrak{e}_2$, a contradiction. Now let

$$\mathfrak{e}_1' = j_{\sigma^{ij}}^{-1}(\mathfrak{e}_1),\ \mathfrak{e}_2' = j_{\sigma^{ij}}^{-1}(\mathfrak{e}_2).$$

Then by (3.3.1) we have $\sigma_{ij} \notin \mathfrak{e}_2'$ and $\sigma'_{ij} \notin \mathfrak{e}_1'$. Since σ_{ij} and σ'_{ij} are each a single local blowing up, we may apply the lemma to obtain $\sigma \in \mathfrak{e}_1' \cap \mathfrak{e}_2'$ such that $p_\sigma(\mathfrak{e}_1') \neq p_\sigma(\mathfrak{e}_2')$. Now we have

$$\begin{aligned} p_\sigma(\mathfrak{e}_1') &= p_{W'} \circ j_\sigma^{-1} \circ j_{\sigma^{ij}}^{-1}(\mathfrak{e}_1) \\ &= p_{W'}(j_{\sigma^{ij}} \circ j_\sigma)^{-1}(\mathfrak{e}_1) \\ &= p_{W'} \circ j_{\sigma^{ij} \circ \sigma}^{-1}(\mathfrak{e}_1) \\ &= p_{\sigma^{ij} \circ \sigma}(\mathfrak{e}_1) \end{aligned}$$

and similarly $p_\sigma(\mathfrak{e}_2') = p_{\sigma^{ij} \circ \sigma}(\mathfrak{e}_2)$, so we have

$$p_{\sigma^{ij} \circ \sigma}(\mathfrak{e}_1) \neq p_{\sigma^{ij} \circ \sigma}(\mathfrak{e}_2)$$ □

We omit the proof of the following two propositions of [1].

Proposition 3.3.9. *Let W be a complex space. The canonical map $p_W : \mathcal{E}_W \to W$ is proper.*

Proposition 3.3.10. *Let W be a complex space, and let $y \in W$. Let $\sigma_\alpha : W_\alpha \to W \in \mathcal{C}(W)$, $\alpha \in \Delta$ be a family of finite sequences of local blowings up. The following are equivalent:*

(i) $p_W^{-1}(y) \subseteq \bigcup_{\alpha \in \Delta} \mathcal{E}_{\sigma_\alpha}$,

(ii) There exists a finite subset $\Delta' \subseteq \Delta$ and for each $\alpha \in \Delta'$ a relatively compact open subset $V_\alpha \subseteq W_\alpha$ such that

$$\bigcup_{\alpha \in \Delta'} \sigma_\alpha(V_\alpha)$$

is a neighborhood of y in W.

Chapter 4

The local flattening theorem

In this chapter we prove the local flattening theorem of Hironaka [1]. First we discuss some elementary properties of flat modules.

4.1 Flatness

Let A be a commutative ring with unity, let M be a unitary A-module. Then M is *A-flat*, if for all exact sequences of A-modules

$$N \to L \to P$$

the sequence

$$N \otimes_A M \to L \otimes_A M \to P \otimes_A M$$

is exact. If N is an A-module and

$$\cdots \to F_2 \xrightarrow{\alpha_1} F_1 \xrightarrow{\alpha_0} F_0 \to N$$

is a free resolution of N, then we have a sequence obtained by tensoring:

$$\cdots \to F_2 \otimes M \xrightarrow{\alpha_1 \otimes \mathrm{id}} F_1 \otimes M \xrightarrow{\alpha_0 \otimes \mathrm{id}} F_0 \otimes M \to N \otimes M$$

and the left derived functor of $- \otimes M$ is

$$\mathrm{Tor}_i^A(N, M) = \frac{\ker(\alpha_i \otimes \mathrm{id})}{\mathrm{im}(\alpha_{i+1} \otimes \mathrm{id})}.$$

Then if:

$$0 \to N \to L \to P \to 0$$

is a short exact sequence, we have a long exact sequence:

$$\cdots \to \mathrm{Tor}_1^A(N, M) \to \mathrm{Tor}_1^A(L, M) \to \mathrm{Tor}_1^A(P, M) \to N \to L \to P \to 0.$$

Proposition 4.1.1. *Let A be a commutative ring and let M be an A-module. Then the following are equivalent:*

(i) M is A-flat;

(ii) If $N \to L$ is an injective morphism of A-modules, then so is $N \otimes M \to L \otimes M$;

(iii) For every A-module N, $\mathrm{Tor}_i(M, N) = 0$ for $i \geq 1$;

(iv) For every A-module N, $\mathrm{Tor}_1(M, N) = 0$.

We do not prove this proposition in detail. Essentially we have (i) $\Leftrightarrow$ (ii) since $- \otimes M$ is always right exact, (i) $\Leftrightarrow$ (iv) due to the long exact sequence obtained from the derived functors and (iii) $\Leftrightarrow$ (iv) since a free module is flat (see below) and Tor_i^A can be computed by a free resolution. Let A be a commutative ring. We have the following elementary examples:

(i) The ring A is a flat A-module, since $M \otimes A \simeq M$ for every A-module M;

(ii) If $M = \bigoplus_{i \in I} M_i$ is a direct sum of A-modules then M if flat if and only if each M_i is flat. This is true since the tensor product distributes over the direct sum;

(iii) A corollary of the above is that for any commutative ring A, a free A-module is A-flat.

In fact it is true that an A-module M is flat if and only if for every ideal I of A, $I \otimes M \simeq IM$.

Definition 4.1.2. Let $f : X \to Y$ be a morphism of complex spaces. Then f is *flat* at $\xi \in X$ if $\mathcal{O}_{X,\xi}$ is a flat $\mathcal{O}_{Y,f(\xi)}$-module via

$$\phi_\xi^\# : \mathcal{O}_{Y,f(\xi)} \to \mathcal{O}_{X,\xi}.$$

Flat morphisms have the following properties:

(i) The composite of two flat morphisms are flat;

(ii) Flatness is an open property, i.e. the set of points at which a morphism is flat is open;

(iii) Flatness is preserved by base change.

We recall here that the fibre product of complex spaces does not locally correspond to tensor products of commutative rings as in the algebraic case. In the analytic case, since we have convergence problems to deal with, the fibred product of complex spaces corresponds to an 'analytic tensor product' of analytic algebras (which enjoys many of the properties of the algebraic tensor product). In general the analytic tensor product and usual tensor product of modules over analytic algebras are different in the non-finitely generated case, and therefore give rise to different left derived functors. Details on the analytic tensor product and fibre products can be found in [7]. Another consequence of flatness is equidimensionality: if $f : X \to Y$ is a morphism of complex spaces and f is flat at $x \in X$, then:

$$\dim(X, x) = \dim(f^{-1} f(x), x) + \dim(Y, f(x)).$$

4.2 The local flattening theorem

Our aim is the following result.

Theorem 4.2.1 (Local flattening theorem)**.** *Let $f : V \to W$ be a morphism of complex spaces. Assume that W is reduced. Let $y \in W$ and $L \subset\subset f^{-1}(y)$ be compact. Then there exists a finite number of finite sequences of local blowings up $\sigma_\alpha : W_\alpha \to W$ such that:*

(i) For each α, the centres of the blowings up in σ_α are nowhere dense in their respective spaces;

(ii) For each α, there exists $K_\alpha \subset\subset W_\alpha$ compact such that

$$\bigcup_\alpha \sigma_\alpha(K_\alpha)$$

is a neighborhood of y in W;

(iii) For each α, the strict transform $f_\alpha : V_\alpha \to W_\alpha$ of f by σ_α is flat at each point of V_α corresponding to a point of L, i.e. at each point of $(\sigma'_\alpha)^{-1}(L)$ where $f \circ \sigma'_\alpha = \sigma_\alpha \circ f_\alpha$:

$$\begin{tikzcd} V_\alpha \arrow[r, "\sigma'_\alpha"] \arrow[d, "f_\alpha"'] & V \arrow[d, "f"'] \\ W_\alpha \arrow[r, "\sigma_\alpha"] & W \end{tikzcd}$$

To this end, we make the following definition:

Definition 4.2.2. Let $f : V \to W$ be a morphism of complex spaces. Let $y \in W$ and $L \subset\subset f^{-1}(y)$ compact. A germ of a complex space $(P, y) \subseteq (W, y)$ is the *flatificator* for (f, L) if:

(F) For each morphism of complex spaces $h : T \to W$ and each $t \in T$ such that $h(t) = y$, the projection $\mathrm{pr}_1 : T \times_W V \to T$ is flat at each point of $\{t\} \times_W L$ if and only if h induces a morphism $(T, t) \to (P, y)$.

Taking the inclusion $P \hookrightarrow W$ in the universal property property, where P is a representative of (P, y) we have that $f^{-1}(P) \to P$ is flat at each point of $L \cap f^{-1}(P)$. Moreover if Q is any locally closed subspace of W such that $f^{-1}(Q) \to Q$ is flat at each point of $L \cap f^{-1}(Q)$, then we must have $(Q, y) \subseteq (P, y)$. Thus the flatificator the maximal germ with respect to this property. Therefore the flatificator is unique. First let us show a useful property of the flatificator.

Lemma 4.2.3. *With notation as in definition 4.2.2, let (P, y) be the flatificator for (f, L). If $\sigma : W' \to W$ is a morphism of complex spaces and $y \in W'$ with $\sigma(y') = y$, then $(\sigma^{-1}(P), y')$ is the flatificator for (pr_1, L'') where $pr_1 : W' \times_W V \to W'$ is the projection of the fibre product and $L'' = \{y'\} \times_W L$. In particular if $W' \subseteq W$ is a closed subspace containing y and $P' = W' \cap P$ then (P', y) is the flatificator for $(f^{-1}(W') \to W', f^{-1}(W') \cap L)$.*

Proof. Let $h : T \to W'$ be a morphism and let $t \in T$ such that $h(t) = y'$. We have a diagram

$$\begin{tikzcd} T \times_W V \arrow[r] \arrow[d, "\mathrm{pr}'_1"] & W' \times_W V \arrow[r] \arrow[d, "\mathrm{pr}_1"] & V \arrow[d, "f"] \\ T \arrow[r, "h"] & W' \arrow[r, "\sigma'"] & W \end{tikzcd}$$

Then we have $\sigma \circ h(t) = y$, so by associativity of the fibre product if pr_1' is flat at each point of $\{t\} \times_W L$, then by the definition of (P, y) we have $\sigma \circ h(T, t) \subseteq (P, y)$. Therefore $h(T, t) \subseteq (\sigma^{-1}(P), y')$. Conversely if $h(T, t) \subseteq (\sigma^{-1}(P), y')$ then we have $\sigma \circ h(T, t) \subseteq (P, y)$ so again by the property of (P, y) it follows that pr_1' is flat at each point of $\{t\} \times L$. The statement for a closed subspace $W' \subseteq W$ containing y follows if we take σ to be the inclusion $W' \hookrightarrow W$, so $W' \times_W V = f^{-1}(W')$ and $\{y\} \times_W L = f^{-1}(W') \cap L$. □

Now we prove:

Proposition 4.2.4. *Let $f : V \to W$ be a morphism of complex spaces, let $y \in W$ and let $L \subset\subset f^{-1}(y)$ be compact. Then there exists a flatificator (P, y) for (f, L).*

First we need a:

Lemma 4.2.5. *Let $A \subseteq V \subseteq \mathbb{C}^m$ and $B \subseteq W \subseteq \mathbb{C}^n$ be two analytic subsets of open sets V, W. Let $\mathcal{I}_A$ and $\mathcal{I}_B$ denote their ideal sheaves. Let $\phi : A \to B$ be a morphism, and let Γ_ϕ be its graph, defined by the ideal*

$$\mathcal{J} = \mathcal{I}_A + (z_1 - \phi_1(x), \ldots, z_n - \phi_n(x))$$

where $\phi_1, \ldots, \phi_n$ are the coördinate functions of ϕ, and $(z_1, \ldots, z_n)$ are coördinates on $\mathbb{C}^n$. Then there exists a natural biholomorphism $\gamma : A \to \Gamma_\phi$ such that $\phi = \pi \circ \gamma$, where $\pi : V \times W \to W$ is the projection.

Proof. The map between topological spaces is given by

$$\begin{aligned} \gamma^{\mathrm{top}} &: A^{\mathrm{top}} \to \Gamma_\phi^{\mathrm{top}} \\ &: x \mapsto (x, \phi_1(x), \ldots, \phi_n(x)) \end{aligned}$$

and the map of sheaves of rings by

$$\begin{aligned} \gamma^{\#} &: \mathcal{O}_{\Gamma_\phi} \to \gamma_*^{\mathrm{top}} \mathcal{O}_A \\ &: f(x, z) \mapsto f(x, \phi(x)) \end{aligned}$$

which of course has a two-sided inverse, so γ thus defined is an isomorphism. □

Thus if $f : V \to W$ is a morphism of complex spaces, since flatness is a local property we may as well assume that f factors as a locally closed embedding followed by a projection:

$$\begin{tikzcd} z = y \times 0 \in V \arrow[r, hook] \arrow[dr, "f"'] & W \times \mathbb{C}^n \arrow[d] \\ & W \end{tikzcd}$$

where here and in what follows we fix $z \in V$ such that $f(z) = y$ and identify $z = y \times 0$ with its image in $W \times \mathbb{C}^n$. Let $A = \mathcal{O}_{W,y}$ so that

$$\mathcal{O}_{W \times \mathbb{C}^n, y \times 0} = A\{t_1, \ldots, t_n\} = A\{\mathbf{t}\}$$

where $\mathbf{t} = (t_1, \ldots, t_n)$ are coördinates of $\mathbb{C}^n$. We write $\mathbf{t}_i = (t_1, \ldots, t_i)$, with the convention that $\mathbf{t}_0 = ()$ and

$A\{\mathbf{t}_0\} = A$. Suppose that

$$B = \bigoplus_{i=0}^{n} A\{\mathbf{t}_i\}^{\nu_i}, \qquad \text{for some } \nu_i \geq 0.$$

Let $\Phi : B \to A\{\mathbf{t}\}^m$ be an A-homomorphism. Since finite direct sums and products agree in the category of modules, we have $\nu_0 + \cdots + \nu_n$ canonical injections and m canonical projections:

$$A\{\mathbf{t}_i\} \to B, \qquad A\{\mathbf{t}\}^m \to A\{\mathbf{t}\}.$$

We say that Φ is *natural* if the morphisms $A\{\mathbf{t}_i\} \to A\{\mathbf{t}\}$ obtained by composing the $A\{\mathbf{t}_i\} \to B$, Φ and the projections $A\{\mathbf{t}\}^m \to A\{\mathbf{t}\}$, are all $A\{\mathbf{t}_i\}$-module homomorphisms. We need the following lemma:

Lemma 4.2.6. *Let $\mathbf{u} = (u_1, \ldots, u_r)$, let $A = \mathbb{C}\{u_1, \ldots, u_r\}/I$ be an analytic algebra. Let $S = A\{\mathbf{t}\}^m$ with $\mathbf{t} = (t_1, \ldots, t_n)$ and $m \geq 0$ any integer, let J be a proper $A\{\mathbf{t}\}$-submodule of S. Then, after a non-singular $\mathbb{C}$-linear transformation of the t_i we can find $\nu_0, \ldots, \nu_n \geq 0$ and a natural A-homomorphism*

$$\Phi : B = \bigoplus_{i=0}^{n} A\{\mathbf{t}_i\}^{\nu_i} \to S$$

such that

(i) Φ induces a surjective A-homomorphism $\phi : B \to S/J$;

(ii) $\ker \phi \subseteq \mathfrak{m}B$ where $\mathfrak{m} = \mathfrak{m}(A)$ is the maximal ideal of A.

This lemma can be thought of as a version of the Hironaka division theorem. The Weierstrass division theorem corresponds to the case $A = \mathbb{C}$, $m = 1$ and J a principal ideal. In this instance we have $B = A\{\mathbf{t}_{n-1}\}^{\nu}$ for some ν and $\ker \phi = \{0\}$. To see this let $g \in A\{\mathbf{t}\}$ be a non-unit and let $J = (g)$. After a non-singular, $\mathbb{C}$-linear transformation of the t_i, the Weierstrass division theorem states that for any $f \in A\{\mathbf{t}\}$ we can uniquely write $f = qg + r$ with r polynomial in t_n of degree at most ν for some integer ν independent of f. Therefore we have an isomorphism

$$A\{\mathbf{t}_{n-1}\}^{\nu} \xrightarrow{\sim} \frac{A\{\mathbf{t}\}}{J}$$

where multiplication in $A\{\mathbf{t}_{n-1}\}^{\nu}$ is defined using the Weierstrass division theorem. Returning to the general case, geometrically the homomorphism ϕ corresponds to a locally closed embedding of the zero locus of J into the space with local ring B, i.e. into

$$\bigsqcup_{i=0}^{n} \bigsqcup_{j=1}^{\nu_i} W \times \mathbb{C}^i \tag{4.2.1}$$

where $A = \mathcal{O}_{W,y}$. The condition $\ker \phi \subseteq \mathfrak{m}B$ ensures that the image contains

$$\bigsqcup_{i=0}^{n} \bigsqcup_{j=1}^{\nu_i} \{y\} \times \mathbb{C}^i$$

Proof. If $J \subseteq \mathfrak{m}S$ then we can take $\nu_0 = \cdots = \nu_{n-1} = 0$, $\nu_n = m$ and $\Phi = \mathrm{id}$. Therefore suppose that $J \nsubseteq \mathfrak{m}S$ and let $(e_1, \ldots, e_m)$ be an $A\{\mathbf{t}\}$-basis of S. We may reorder the e_i such that there exists $d \in \{1, \ldots, m\}$ and:

a. For $i = 1, \dots, d$ there exists

$$h_i = h_{ii}e_i + \sum_{j=i+1}^{m} h_{ij}e_j \in J$$

with $h_{ii} \notin \mathfrak{m}A\{\mathbf{t}\}$;

b. We have

$$J \cap \left(\sum_{j=d+1}^{m} A\{\mathbf{t}\}e_j \right) \subseteq \mathfrak{m} \sum_{j=d+1}^{m} A\{\mathbf{t}\}e_j.$$

This is tautological: since $J \nsubseteq \mathfrak{m}S$ there exists an element $\sum_{j=1}^{m} h'_j e_j$ with $h'_j \notin \mathfrak{m}A\{\mathbf{t}\}$ for some j. We then exchange $e_j \leftrightarrow e_1$. Then either we have

$$J \cap \left(\sum_{j=2}^{m} A\{\mathbf{t}\}e_j \right) \subseteq \mathfrak{m} \sum_{j=2}^{m} A\{\mathbf{t}\}e_j$$

or we may repeat the process for indices $2, \dots, d$, to obtain a. and b.. Now we have

$$\frac{A\{\mathbf{t}\}}{\mathfrak{m}A\{\mathbf{t}\}} = \frac{A}{\mathfrak{m}A}\{\mathbf{t}\} \simeq \mathbb{C}\{\mathbf{t}\}.$$

Let $\overline{h}_{ii}$ be the class of h_{ii} in $\mathbb{C}\{\mathbf{t}\}$. There is an open, dense set of vectors $v \in \mathbb{C}^n$ such that $\mathbb{C} \ni \xi \mapsto h_{ii}(\xi v)$ is not identically zero, so after transforming the t_i by an element of $\mathrm{GL}_n(\mathbb{C})$ we may assume that $(0, \dots, 0, t_n)$ is some such vector. Then we have $\overline{h}_{ii} \notin (t_1, \dots, t_{n-1})\mathbb{C}\{\mathbf{t}\}$, so

$$h_{ii} \notin (t_1, \dots, t_{n-1})A\{\mathbf{t}\} + \mathfrak{m}A\{\mathbf{t}\}$$

Let $\tilde{h}_{ii}$ be any representative of h_{ii} in $\mathbb{C}\{\mathbf{u}, \mathbf{t}\}$. Now by the Weierstrass preparation theorem there exist $r_i \geq 0$, $\tilde{g}_{i1}, \dots, \tilde{g}_{ir_i} \in \mathbb{C}\{\mathbf{u}, \mathbf{t}_{n-1}\}$ with $g_{ik}(0) = 0$, $k = 1, \dots, r_i$ and $\tilde{u}_i$ units in $\mathbb{C}\{\mathbf{u}, \mathbf{t}\}$ such that

$$\tilde{h}_{ii} = \tilde{u}_i(t_n^{r_i} + \tilde{g}_{i1}t_n^{r_1-1} + \cdots + \tilde{g}_{ir_i}).$$

Taking the quotient by I and letting u_i, g_{ik} be the classes of $\tilde{u}_i$ and $\tilde{g}_{ik}$ respectively, we have

$$h_{ii} = u_i(t_n^{r_i} + g_{i1}t_n^{r_1-1} + \cdots + g_{ir_i}) \tag{4.2.2}$$

with $u_i \in A\{\mathbf{t}\}$ units and

$$g_{ik} \in (\mathbf{u}, t_1, \dots, t_{n-1})A\{\mathbf{t}_{n-1}\} = (t_1, \dots, t_{n-1})A\{\mathbf{t}_{n-1}\} + \mathfrak{m}A\{\mathbf{t}\}.$$

Next, let $f \in A\{\mathbf{t}\}$ and let $\tilde{f}$ be a representative of f in $\mathbb{C}\{\mathbf{u}, \mathbf{t}\}$. Then by the Weierstrass division theorem there exist unique $\tilde{q}, \tilde{R} \in \mathbb{C}\{\mathbf{u}, \mathbf{t}\}$ such that

$$\tilde{f} = \tilde{h}_{ii}\tilde{q} + \tilde{R} \tag{4.2.3}$$

where $\tilde{R}$ is polynomial in t_n of degree at most $r_i - 1$. Therefore, taking the quotient first by I and then by $h_{ii}A\{\mathbf{t}\}$ we have

$$\frac{A\{\mathbf{t}\}}{h_{ii}A\{\mathbf{t}\}} \simeq \frac{A\{\mathbf{t}_{n-1}\}}{z_n^{r_i}A\{\mathbf{t}\}} = A\{\mathbf{t}_{n-1}\} \oplus \cdots \oplus t_n^{r_i-1}A\{\mathbf{t}_{n-1}\}$$

where we understand that the product in this last direct sum is computed by means of the Weierstrass division theorem as in (4.2.3). Therefore we have an isomorphism of $A\{\mathbf{t}_{n-1}\}$-modules

$$\begin{aligned}\psi_i : A\{\mathbf{t}_{n-1}\}^{r_i} &\to \frac{A\{\mathbf{t}\}}{h_{ii}A\{\mathbf{t}\}} \\ : (0,\ldots,0,1,0,\ldots,0) &\mapsto t_n^{r_i-j}\end{aligned} \tag{4.2.4}$$

where the 1 is in the jth position. Let $m' = r_1 + \cdots + r_d$, let $S_1 = \sum_{j=d+1}^{m} A\{\mathbf{t}\}e_j$, and let $S' = A\{\mathbf{t}_{n-1}\}^{m'}$. We define a map $\Psi : S' \oplus S_1 \to S$ as follows. Since the direct sum comes equipped with canonical injections, it suffices to define Ψ on each factor. On S_1 we let Ψ be the identity. On each factor $A\{\mathbf{t}_{n-1}\}^{r_i}$ we define Ψ by

$$\Psi_{|A\{\mathbf{t}_{n-1}\}^{r_i}} : (0,\ldots,1,\ldots,0) \mapsto t_n^{r_i-j}$$

where the 1 is in the jth position. Therefore by (4.2.4), Ψ induces an isomorphism of $A\{\mathbf{t}_{n-1}\}$-modules

$$S' \oplus S_1 \xrightarrow{\sim} \frac{A\{\mathbf{t}\}}{h_{11}A\{\mathbf{t}\}} \oplus \cdots \oplus \frac{A\{\mathbf{t}\}}{h_{dd}A\{\mathbf{t}\}} \oplus A\{\mathbf{t}\} \oplus \cdots \oplus A\{\mathbf{t}\}$$

and therefore an isomorphism

$$S' \oplus S_1 \xrightarrow{\sim} \frac{S}{\sum_{i=1}^{d} h_{ii}A\{\mathbf{t}\}e_i}.$$

Moreover we have a natural isomorphism between the submodule generated by $h_{ii}e_i$ and the submodule generated by h_i (since $h_{ii} \notin \mathfrak{m}A\{\mathbf{t}\}$ means h_{ii} is a nonzerodivisor), so Ψ induces an isomorphism:

$$S' \oplus S_1 \xrightarrow{\sim} \frac{S}{\sum_{i=1}^{d} h_i A\{\mathbf{t}\}}. \tag{4.2.5}$$

and therefore a surjection $S' \oplus S_1 \to S/J$. Since Ψ is the identity on S_1 and we suppose that

$$J \cap S_1 \subseteq \mathfrak{m}S_1$$

we have that $\Psi^{-1}(J) \cap S_1 \subseteq \mathfrak{m}S_1$. We now prove the result by induction on $n \geq 0$. First suppose that $n = 0$. By our choice of the h_{ii}, we have $h_{ii} \notin \mathfrak{m}A = \mathfrak{m}$ for $i = 1,\ldots,d$, so $h_{11},\ldots,h_{dd}$ are all units of A. Then the integers r_i are all zero since in (4.2.2) we can take $u_i = h_{ii}$. Therefore $S' = \{0\}$ and in this case

$$\Psi : S_1 = \sum_{j=d+1}^{m} Ae_j \to \sum_{j=1}^{m} Ae_j = S$$

is just the inclusion. Moreover Ψ induces and isomorphism as in (4.2.5), and we have $h_i \in J$ for $i = 1,\ldots,d$

so $\sum_{i=1}^{d} h_i A\{\mathbf{t}\} \subseteq J$, i.e. Ψ induces a surjection $\psi : S_1 \to S/J$. Using the facts that $\Psi^{-1}(J) \cap S_1 \subseteq \mathfrak{m}S_1$ and $S' = \{0\}$ we have

$$\ker \psi = \Psi^{-1}(J) = \Psi^{-1}(J) \cap S_1 \subseteq \mathfrak{m}S_1.$$

Therefore if we take $\nu_0 = m - d$ and $\Phi = \Psi$, the lemma holds for $n = 0$. Now suppose that the lemma holds for $n - 1$. Let J' be the image of $\Psi^{-1}(J)$ under the projection

$$S' \oplus S_1 \to \frac{S' \oplus S_1}{S_1} = S'.$$

Since $S' = A\{\mathbf{t}_{n-1}\}^{m'}$, by the induction hypothesis there exist $B' = \bigoplus_{i=0}^{n-1} A\{\mathbf{t}_i\}^{\nu_i}$ and a natural A-homomorphism $\Phi' : B' \to S'$ such that Φ' induces a surjective homomorphism $\phi' : B' \to S'/J'$ and $\ker \phi' = (\Phi')^{-1}(J') \subseteq \mathfrak{m}B'$. Let $\nu_n = m - d$ and let

$$\Phi = \Psi \circ (\Phi' \oplus \mathrm{id}_{S_1}) : B = \bigoplus_{i=0}^{n} A\{\mathbf{t}_i\}^{\nu_i} \to S$$

as in the following diagram:

$$\begin{tikzcd}
B' \arrow[r, hook] \arrow[d, "\Phi'"'] & B' \oplus S_1 \arrow[d, "\Phi' \oplus \mathrm{id}_{S_1}"'] & S_1 \arrow[l, hook'] \arrow[d, "\mathrm{id}_{S_1}"'] \\
S' \arrow[r, hook] \arrow[d] & S' \oplus S_1 \arrow[d, "\Psi"'] & S_1 \arrow[l, hook'] \arrow[d] \\
\frac{S'}{\Psi^{-1}(J) \cap S'} & S & \frac{S_1}{\Psi^{-1}(J) \cap S_1}
\end{tikzcd}$$

We have three things to verify:

1. Φ *is natural:* we have that Φ', Ψ and id_{S_1} are all natural, so it follows that Φ has the same property.

2. Φ *induces a surjective homomorphism* $\phi : B \to S/J$*:* we have that Φ', id_{S_1} and Ψ induce surjective maps $\phi' : B' \to S'/(S' \cap \Psi^{-1}(J))$, $\chi : S_1 \to S_1/(S_1 \cap \Psi^{-1}(J))$ and $\psi : S' \oplus S_1 \to S/J$ respectively. Since ψ is determined by its values on S' and S_1, and $\ker \psi = \Psi^{-1}(J)$, we have that $\phi = \psi \circ (\phi' \oplus \chi)$ is a well defined A-homomorphism, and since ψ, ϕ' and χ are all surjective, so is ϕ.

3. $\ker \phi \subseteq \mathfrak{m}B$*:* Let $f \in \Phi^{-1}(J)$. We have

$$\begin{aligned}
\Phi^{-1}(J) &= (\Phi' \oplus \mathrm{id}_{S_1})^{-1}\Psi^{-1}(J) \\
&= [(\Phi')^{-1}(\Psi^{-1}(J) \cap S')] \oplus [\mathrm{id}_{S_1}^{-1}(\Psi^{-1}(J) \cap S_1)].
\end{aligned}$$

Since $J' = \Psi^{-1}(J) \cap S'$, if f' is the image of f in B', then $f' \in (\Phi')^{-1}(J') \subseteq \mathfrak{m}B'$. Let

$$S_2 = \sum_{i=1}^{d} A\{\mathbf{t}\}e_i$$

so $\Phi(f) \in J \cap (\mathfrak{m}S_2 + S_1)$. We can write $\Phi(f) = \sum_{j=1}^{m} f_j e_j$ where $f_j \in \mathfrak{m}A\{\mathbf{t}\}$ for $j = 1, \ldots, d$. We find

that:

$$\left(\prod_{i=1}^{d} h_{ii}\right)\Phi(f) - \sum_{j=1}^{d} f_j \left(\prod_{i\neq j} h_{ii}\right) h_j = \sum_{j=1}^{m}\left(\prod_{i=1}^{d} h_{ii}\right) f_j e_j - \sum_{j=1}^{d} f_j \left(\prod_{i\neq j} h_{ii}\right) h_j$$
$$= \sum_{j=1}^{d}\sum_{k=j+1}^{m} f_j \left(\prod_{i\neq j} h_{ii}\right) h_{jk} e_k.$$

Since the f_j belong to $\mathfrak{m}A\{\mathbf{t}\}$, $j = 1, \ldots, d$, comparing the coefficients of each basis element e_j we have that $f_j \prod_{i=1}^{d} h_{ii} \in \mathfrak{m}A\{\mathbf{t}\}$, $j = 1, \ldots, m$. Since $h_{ii} \notin \mathfrak{m}A\{\mathbf{t}\}$, $i = 1, \ldots, d$ and $\mathfrak{m}A\{\mathbf{t}\}$ is prime, this shows that $f_j \in \mathfrak{m}A\{\mathbf{t}\}$ for $j = d+1, \ldots, m$. Since Φ is the identity on S_1, together with the fact that $f' \in \mathfrak{m}B'$ this shows that $f \in \mathfrak{m}B$. Therefore we have $\Phi^{-1}(J) \subseteq \mathfrak{m}B$. □

We retain the notation introduced in the paragraph preceding this lemma, and taking $m = 1$, $J = \mathcal{I}_V$ the ideal of the image of V in $W \times \mathbb{C}^n$ in the statement, we have coördinates $(t_1, \ldots, t_n)$ on $\mathbb{C}^n$ such that there exists Φ satisfying (i) and (ii) of this result. We have $\Phi^{-1}(\mathcal{I}_V) = \ker\phi$. If $g \in B$ then we have a unique expression:

$$g = \sum_{i=0}^{n}\sum_{j=1}^{\nu_i}\sum_{\alpha\in\mathbb{N}_0^i} g_{ij\alpha} e_{ij} \mathbf{t}_i^{\alpha}$$

where $\alpha = (\alpha_1, \ldots, \alpha_i)$, $\mathbf{t}_i^{\alpha} = t_1^{\alpha_1}\cdots t_i^{\alpha_i}$, $(e_{i1}, \ldots, e_{i\nu_i})$ is an $A\{\mathbf{t}_i\}$-basis for $A\{\mathbf{t}_i\}^{\nu_i}$ and $g_{ij\alpha} \in A = \mathcal{O}_{W,y}$. We define an ideal $\mathcal{I}(f,z)$ of A to be that generated by all the $g_{ij\alpha}$, $i = 0, \ldots, n$, $j = 1, \ldots, \nu_i$, $\alpha \in \mathbb{N}_0^i$ for all $g \in \Phi^{-1}(\mathcal{I}_V) = \ker\phi$. Then by property (ii) we have $\mathcal{I}(f,z) \subseteq \mathfrak{m}(A) \subsetneqq A$, so the locally closed subspace of W defined by $\mathcal{I}(f,z)$ contains $y \in W$.

Lemma 4.2.7. *Let P be the locally closed subspace of W defined by $\mathcal{I}(f,z)$. Then:*

(1) $\mathcal{O}_{f^{-1}(P),z}$ is a free $\mathcal{O}_{P,y}$-module; in particular $f^{-1}(P) \to P$ is flat at $z = y \times 0 \in f^{-1}(y)$;

(2) If $Q \subseteq W$ is a locally closed subspace containing y such that $f^{-1}(Q) \to Q$ is flat at z, then $(Q,y) \subseteq (P,y)$.

Proof. (1): By the definition of $\mathcal{I}(f,z)$, if $g \in \Phi^{-1}(\mathcal{I}_V) = \ker\phi$ then $g \in \mathcal{I}(f,z)B$, i.e. $\ker\phi \subseteq \mathcal{I}(f,z)B$. Thus Φ induces a surjective map $A\{\mathbf{t}\}/\mathcal{I}_V \to B/\mathcal{I}(f,z)B$ (if $\rho : M \to N$ and $\sigma : M \to L$ are module homomorphisms such that $\ker\rho \subseteq \ker\sigma$ then σ factors uniquely through ρ):

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[dr] & A\{\mathbf{t}\}/\mathcal{I}_V \arrow[d, "\exists!"] \\ & B/\mathcal{I}(f,z)B \end{tikzcd}$$

This map has kernel $\mathcal{I}(f,z)A\{\mathbf{t}\} + \mathcal{I}_V$ (this ideal must be contained in the kernel, and if $g \in A\{\mathbf{t}\}/\mathcal{I}_V$ mapping to $0 \in B/\mathcal{I}(f,z)B$, then there is $g' \in B$ mapping to g, and necessarily $g' \in \mathcal{I}(f,z)B$, so $\Phi(g') \in \mathcal{I}(f,z)A\{\mathbf{t}\}$, whence $g \in \mathcal{I}(f,z)A\{\mathbf{t}\} + \mathcal{I}_V$) so we have an isomorphism

$$\frac{B}{\mathcal{I}(f,z)B} \simeq \frac{A\{\mathbf{t}\}}{\mathcal{I}_V + \mathcal{I}(f,z)A\{\mathbf{t}\}}.$$

Now, $\mathcal{I}_V$ is the ideal of V in $A\{\mathbf{t}\} = \mathcal{O}_{W\times\mathbb{C}^n, y\times 0}$, and $\mathcal{I}(f,z)$ is the ideal of P in A, so

$$\mathcal{I}_V + \mathcal{I}(f,z)A\{\mathbf{t}\} = \mathcal{I}_{f^{-1}(P), y\times 0} = \mathcal{I}_{f^{-1}(P),z}$$

is the ideal of $f^{-1}(P) = (P\times\mathbb{C}^n)\cap V$ in $A\{\mathbf{t}\}$. That is,

$$\frac{A\{\mathbf{t}\}}{\mathcal{I}_V + \mathcal{I}(f,z)A\{\mathbf{t}\}} = \mathcal{O}_{f^{-1}(P),z}.$$

Moreover, since $A/\mathcal{I}(f,z) = \mathcal{O}_{P,y}$ we have an isomorphism

$$\frac{B}{\mathcal{I}(f,z)B} \simeq \bigoplus_{i=0}^{n} \mathcal{O}_{P,y}\{\mathbf{t}_i\}^{\nu_i}.$$

Therefore $\mathcal{O}_{f^{-1}(P),z}$ is a free $\mathcal{O}_{P,y}$-module as required.

(2): Let $Q\subseteq W$ be locally closed containing y such that $f^{-1}(Q)\to Q$ is flat at z. Let $\mathcal{I}_1$ be the ideal of Q in A, so

$$\mathcal{O}_{Q,y} = A/\mathcal{I}_1 =: A_1.$$

Then the mapping

$$B \to \frac{A\{\mathbf{t}\}}{\mathcal{I}_V} \to \frac{A\{\mathbf{t}\}}{\mathcal{I}_V + \mathcal{I}_1 A\{\mathbf{t}\}}$$

induces a surjective map of A_1-modules

$$\psi : B_1 := \frac{B}{\mathcal{I}_1 B} = \bigoplus_{i=0}^{n} A_1\{\mathbf{t}_i\}^{\nu_i} \to \frac{A\{\mathbf{t}\}}{\mathcal{I}_V + \mathcal{I}_1 A\{\mathbf{t}\}}.$$

Suppose we show that $\ker\psi = 0$, then $\ker\phi = \Phi^{-1}(\mathcal{I}_V) \subseteq \mathcal{I}_1 B$, so by the definition of $\mathcal{I}(f,z)$ we have $\mathcal{I}(f,z)\subseteq\mathcal{I}_1$, and therefore $(Q,y)\subseteq(P,y)$. Let us show that $\ker\psi = 0$. We define $\overline{B}_1 = B_1/\ker\psi$. Since $\mathcal{O}_{f^{-1}(Q),z} = A\{\mathbf{t}\}/(\mathcal{I}_V + \mathcal{I}_1 A\{\mathbf{t}\})$ we then have an isomorphism $\overline{B}_1 \simeq \mathcal{O}_{f^{-1}(Q),z}$. By assumption $\mathcal{O}_{f^{-1}(Q),z}$ and therefore $\overline{B}_1$ are $\mathcal{O}_{Q,y}$-flat. Let $\mathfrak{m}_1 = \mathfrak{m}(A_1)$ be the maximal ideal of $A_1 = \mathcal{O}_{Q,y}$. Then by flatness we have

$$\mathfrak{m}_1^k \overline{B}_1 \simeq \mathfrak{m}_1^k \otimes \overline{B}_1$$

(all tensor products are over A_1). Therefore, we have isomorphisms of $A_1/\mathfrak{m}_1$-modules:

$$\begin{aligned}
\frac{\mathfrak{m}_1^k \overline{B}_1}{\mathfrak{m}_1^{k+1}\overline{B}_1} &\simeq \frac{A_1}{\mathfrak{m}_1} \otimes \mathfrak{m}_1^k \overline{B}_1 \\
&\simeq \frac{A_1}{\mathfrak{m}_1} \otimes \mathfrak{m}_1^k \otimes \overline{B}_1 \\
&\simeq \frac{\mathfrak{m}_1^k}{\mathfrak{m}_1^{k+1}} \otimes \overline{B}_1 \\
&\simeq \frac{A_1}{\mathfrak{m}_1} \otimes \frac{\mathfrak{m}_1^k}{\mathfrak{m}_1^{k+1}} \otimes \overline{B}_1
\end{aligned}$$

$$\simeq \frac{\mathfrak{m}_1^k}{\mathfrak{m}_1^{k+1}} \otimes \frac{\overline{B}_1}{\mathfrak{m}_1 \overline{B}_1}. \tag{4.2.6}$$

We have $\ker \psi \subseteq \mathfrak{m}_1 B_1$ (since $\ker \phi \subseteq \mathfrak{m}(A)B$, $\mathcal{I}_V + \mathcal{I}_1 A\{\mathbf{t}\} \subseteq \mathfrak{m}(A)A\{\mathbf{t}\}$, and ψ is obtained by factoring the composite of these mappings through the projection $B \to B_1$) so

$$\frac{\overline{B}_1}{\mathfrak{m}_1 \overline{B}_1} \simeq \frac{B_1}{\mathfrak{m}_1 B_1}$$

and therefore, repeating the series of isomorphisms (4.2.6) for B_1 instead of $\overline{B}_1$ we have:

$$\frac{\mathfrak{m}_1^k \overline{B}_1}{\mathfrak{m}_1^{k+1} \overline{B}_1} \simeq \frac{\mathfrak{m}_1^k}{\mathfrak{m}_1^{k+1}} \otimes \frac{\overline{B}_1}{\mathfrak{m}_1 \overline{B}_1} \simeq \frac{\mathfrak{m}_1^k}{\mathfrak{m}_1^{k+1}} \otimes \frac{B_1}{\mathfrak{m}_1 B_1} \simeq \frac{\mathfrak{m}_1^k B_1}{\mathfrak{m}_1^{k+1} B_1}.$$

Therefore we have $\ker \psi \subseteq \mathfrak{m}_1^k B_1$ for all $k > 0$, so by the Krull intersection theorem, $\ker \psi = 0$ as required. $\square$

Now let us prove the existence of the flatificator. First we consider the case when $L = \{z\}$ is a single point of $f^{-1}(y)$. We claim that (P, y) defined as in the lemma is the flatificator for $(f, \{z\})$. That is, we claim that if $f : T \to W$ and $h(t) = y$ then $\mathrm{pr}_1 : T \times_W V \to T$ is flat at $t \times z$ if and only if $h : (T, t) \to (P, y)$:

$$\begin{tikzcd}
 & z \in V \arrow[rr] \arrow[dr, "f"] & & W \times \mathbb{C}^n \ni y \times 0 \arrow[dl] \\
t \times z \in T \times_W V \arrow[ur] \arrow[dr, "\mathrm{pr}_1"] & & W \ni y & \\
 & t \in T \arrow[ur, "h"] & &
\end{tikzcd}$$

Since the question is local, we may as well assume that h factors as a locally closed embedding followed by a projection, so we have the following picture:

$$\begin{tikzcd}
T \times_W V \arrow[r] \arrow[d] & V \times \mathbb{C}^r \arrow[r] \arrow[d] & V \arrow[r, hook] \arrow[d, "f"] & V \times \mathbb{C}^n \arrow[dl] \\
T \arrow[r, hook] & W \times \mathbb{C}^r \arrow[r] & W &
\end{tikzcd}$$

Suppose we have proven that $(P \times \mathbb{C}^r, y \times 0)$ has the properties *(1)* and *(2)* of lemma 4.2.7 for $(f \times \mathrm{id}_{\mathbb{C}^r}, z \times 0)$, i.e.

- $\star$ $(f \times \mathrm{id}_{\mathbb{C}^r})^{-1}(P \times \mathbb{C}^r) \to P \times \mathbb{C}^r$ is flat at $z \times 0$;
- $\star$ if $W' \subseteq W \times \mathbb{C}^r$ is locally closed, $y \times 0 \in W'$ and $(f \times \mathrm{id}_{\mathbb{C}^r})^{-1}(W') \to W'$ is flat at $z \times 0$ then $(W', y \times 0) \subseteq (P \times \mathbb{C}^r, 0)$.

Then it follows that for W' as in *(2)* we have a morphism

$$(W', y \times 0) \to (P \times \mathbb{C}^r, y \times 0) \to (P, y)$$

obtained by composing with the projection, if and only if $(f \times \mathrm{id}_{\mathbb{C}^r})^{-1}(W') \to W'$ is flat at $z \times 0$. But by the last diagram above this states precisely that (P, y) is the flatificator for $(f, \{z\})$ when we take $W' = T$.

Let us show the properties *(1)* and *(2)*. Applying the lemma to $(f \times \mathrm{id}_{\mathbb{C}^r}, z \times 0)$ we obtain a locally closed subspace $(\tilde{P}, y \times 0)$ of $W \times \mathbb{C}^r$ such that $y \times 0 \in \tilde{P}$ and $\tilde{P}$ has the properties *(1)* and *(2)*. To conclude we show that $(\tilde{P}, y \times 0) = (P \times \mathbb{C}^r, y \times 0)$.

1. $(\tilde{P}, y \times 0) \supseteq (P \times \mathbb{C}^r, y \times 0)$: this is immediate because $f^{-1}(P) \to P$ is flat at z, so $(f \times \mathrm{id}_{\mathbb{C}^r})^{-1}(P \times \mathbb{C}^r) \to P \times \mathbb{C}^r$ is flat at $z \times 0$.

2. $(\tilde{P} \cap (W \times \{0\})), y \times 0) \subseteq (P \times 0, y \times 0)$: to see this we identify $W = W \times \{0\} \subseteq W \times \mathbb{C}^r$, as in:

$$\begin{tikzcd}
V \arrow[r, leftrightarrow] \arrow[d, "f"] & V \times \{0\} \arrow[r, hook] \arrow[d, "f"] & V \times \mathbb{C}^r \arrow[d, "f \times \mathrm{id}_{\mathbb{C}^r}"] \\
W \arrow[r, leftrightarrow] & W \times \{0\} \arrow[r, hook] & W \times \mathbb{C}^r
\end{tikzcd}$$

   Since $(f \times \mathrm{id}_{\mathbb{C}^r})^{-1}(\tilde{P}) \to \tilde{P}$ is flat at $z \times 0$, since the right hand square in the diagram above is the change of base of $f \times \mathrm{id}_{\mathbb{C}^r}$ by $W \times \{0\} \hookrightarrow W \times \mathbb{C}^r$, and since flatness is preserved under base change we have that

$$f^{-1}(\tilde{P} \cap (W \times \{0\})) \to \tilde{P} \cap (W \times \{0\})$$

   is flat at $z \times 0$. Therefore by the properties *(1)* and *(2)* for (P, y) we have the required inclusion.

3. *If τ is a sufficiently small translation of $\mathbb{C}^r$, then $((\mathrm{id} \times \tau)(\tilde{P}), y \times 0) \subseteq (\tilde{P}, y \times 0)$:* Since flatness is an open property, provided τ is a sufficiently small translation, $f \times \mathrm{id}_{\mathbb{C}^r}$ is flat at $y \times \tau(0)$. Thus if P' is the space obtained in the same way as $\tilde{P}$, except replacing the point $y \times 0$ with $y \times \tau(0)$ we have

$$((\mathrm{id} \times \tau)(\tilde{P}), y \times 0) \subseteq (P', y \times \tau(0)).$$

   But by the construction, in a sufficiently small neighborhood the ideal $\mathcal{I}(f \times \mathrm{id}_{\mathbb{C}^r}, z \times 0)$ is represented by an ideal sheaf whose stalk at $y \times \tau(0)$ is $\mathcal{I}(f \times \mathrm{id}_{\mathbb{C}^r}, z \times \tau(0))$. This is true because the A-homomorphism Φ_r of lemma 4.2.6 applied to $f \times \mathrm{id}_{\mathbb{C}^r}$ defines via the induced morphism of local rings a morphism from $V \times \mathbb{C}^r \times \mathbb{C}^n$ to the germ (4.2.1) (with W replaced by $W \times \mathbb{C}^r$) in a neighborhood of $y \times 0$. If this neighborhood is sufficiently small, the same morphism has the properties of lemma 4.2.6. Since the ideal $\mathcal{I}(f \times \mathrm{id}_{\mathbb{C}^r}, z \times \tau(0))$ is defined via the homomorphism of local rings, we have that $P' = \tilde{P}$ in a neighborhood of 0.

Now we prove proposition 4.2.4 in the general case where $L \subset\subset f^{-1}(y)$ is any compact set. We define the ideal

$$\mathcal{I}(f, L) = \sum_{z \in L} \mathcal{I}(f, z).$$

Let $P \subseteq W$ be the locally closed subspace defined by $\mathcal{I}(f, L)$. For each $z \in L$, let (P_z, y) be the flatificator for $(f, \{z\})$ obtained above. Then if $h : T \to W$ is any morphism of complex spaces with $h(t) = y$ then $\mathrm{pr}_1 : T \times_W V \to T$ is flat at each point of $\{t\} \times_W L$ if and only if it is flat at each $t \times z$, $z \in L$ if and only if

$$h(T, t) \subseteq \bigcap_{z \in L} (P_z, y) = (P, y).$$

Our next aim is to prove the so-called fibre cutting lemma. We retain the notation used in the proof of proposition 4.2.4. Namely $f : V \to W$, $y =\in W$, $L \subset\subset f^{-1}(y)$, $\mathcal{I}(f,z)$, B, $A = \mathcal{O}_{W,y}$, $\Phi : B \to A\{\mathbf{t}\}$, $\phi : B \to A\{\mathbf{t}\}/\mathcal{I}_V$ with $\ker\phi = \Phi^{-1}(\mathcal{I}_V) \subseteq \mathfrak{m}(A)B$ are all defined as above.

Proposition 4.2.8. *Let $f : V \to W$ be a morphism of complex spaces. Let $y \in W$ and let $L \subset\subset f^{-1}(y)$ be a compact set. Let (P,y) be the flatificator for (f,L). Choose a representative P of (P,y) in an open neighborhood U of y in W. Let $\sigma = (U, P, \pi : W' \to U)$ be the local blowing up with centre P. Let $f' : V' \to W'$ be the strict transform of f by σ:*

$$\begin{tikzcd} V' \arrow[r, "\sigma'"] \arrow[d, "f'"] & V \arrow[d, "f"] \\ W' \arrow[r, "\sigma"] & W \end{tikzcd}$$

Then for each $y' \in \sigma^{-1}(y)$ there exists at least one $z \in L$ such that if $z' = y' \times z \in V' \hookrightarrow W' \times_W V$ then

$$((f')^{-1}f'(z'), z') \to (f^{-1}f(z), z)$$

induced by σ' is not an isomorphism.

Let $\mathrm{pr}_1 : W' \times_W V \to W'$ be the projection, and let $y = f(z) \in W$, $y' \in \sigma^{-1}(z)$, $y' = f'(z')$ be as above. Since $V' \subseteq W' \times_W V$ we have

$$(f')^{-1}(y') \hookrightarrow \mathrm{pr}_1^{-1}(y') \xrightarrow{\sim} f^{-1}(y) \tag{4.2.7}$$

where the second isomorphism is because the fibre product does not change fibres. With this identification the fibre cutting lemma states that

$$((f')^{-1}f'(z'), z') \subsetneqq (f^{-1}f(z), z).$$

First we prove a

Lemma 4.2.9. *Let $\mathcal{I}(f,z)$ be the ideal defined in the proof of proposition 4.2.4, that is, $\mathcal{I}(f,z)$ is the ideal of (P,y). If $\mathcal{I}(f,z)$ is an invertible $\mathcal{O}_{W,y}$-module then there exists $h \in \mathcal{O}_{V,z}$ such that*

(i) $h \notin \mathfrak{m}(A)\mathcal{O}_{V,z} = \mathcal{I}_{f^{-1}(y)}$ where $\mathfrak{m}(A) = \mathfrak{m}(\mathcal{O}_{W,y})$ is the maximal ideal of $A = \mathcal{O}_{W,y}$ and $\mathcal{I}_{f^{-1}(y)}$ is the ideal of $f^{-1}(y)$;

(ii) $\mathcal{I}(f,z)h = 0$ in $\mathcal{O}_{V,z}$.

Moreover if $\tilde{V}$ is a locally closed subspace of V in a neighborhood of z such that $\mathcal{I}(f,z)\mathcal{O}_{\tilde{V},z}$ is an invertible $\mathcal{O}_{\tilde{V},z}$-module then the inclusion

$$\tilde{V} \cap f^{-1}(y) \subseteq f^{-1}(y)$$

is strict at z.

Proof. By our invertibility assumption on $\mathcal{I}(f,z)$ there exists $g \in \Phi^{-1}(\mathcal{I}_V)$ such that there exists a triple (i_0, j_0, α_0) such that $g_{i_0 j_0 \alpha_0}$ generates $\mathcal{I}(f,z)$ and is a nonzerodivisor in $\mathcal{O}_{W,y}$. Then we may write $g =$

$g' g_{i_0 j_0 \alpha_0}$, where

$$g' = \mathbf{t}_{i_0}^{\alpha_0} e_{i_0 j_0} + \sum_{(i,j,\alpha) \neq (i_0 j_0 \alpha_0)} \frac{g_{ij\alpha} \mathbf{t}_i^{\alpha}}{g_{i_0 j_0 \alpha_0}} e_{ij} \notin \mathfrak{m}(A)B = \ker \phi + \mathfrak{m}(A)B.$$

Thus if h is the class of $\Phi(g')$ mod $\mathcal{I}_V$, then

$$h \in \mathcal{O}_{V,z} \backslash \mathfrak{m}(A)\mathcal{O}_{V,z}.$$

But $\Phi(g) \in \mathcal{I}_V$, so we must have $g_{i_0 j_0 \alpha_0} h = 0 \in \mathcal{O}_{V,z}$. Since $g_{i_0 j_0 \alpha_0}$ generates $\mathcal{I}(f,z)$ we have $\mathcal{I}(f,z)h = 0$. This establishes (i) and (ii). For the last part, we have that $h \notin \mathcal{I}_{f^{-1}(y)}$. If we let (h) denote the ideal generated by h in $\mathcal{O}_{Z,z}$ then by (ii) we have:

$$(0) = (h)\mathcal{O}_{\tilde{V},z} \mathcal{I}(f,z) \mathcal{O}_{\tilde{V},z},$$

but since $\mathcal{I}(f,z)\mathcal{O}_{\tilde{V},z}$ is invertible this means that $h = 0 \in \mathcal{O}_{\tilde{V},z}$ i.e. $h \in \mathcal{I}_{\tilde{V},z}$. Thus

$$h \in \mathcal{I}_{\tilde{V},z} + \mathfrak{m}(\mathcal{O}_{W,y})\mathcal{O}_{V,z}$$

so we have the required inclusion $\tilde{V} \cap f^{-1}(y) \subsetneqq f^{-1}(y)$. □

Now by lemma 4.2.3, if

$$\begin{tikzcd} V' \arrow[r, "\sigma'"] \arrow[d, "f'"] & V \arrow[d, "f"] \\ W' \arrow[r, "\sigma"] & W \end{tikzcd}$$

as above and (P,y) is the flatificator for f, then $(\sigma^{-1}(P), y')$ is the flatificator for (pr_1, L'') where $\mathrm{pr}_1 : W' \times_W V \to W'$ and $L'' = \{y'\} \times_W L$. Now let $\mathcal{I}' = \mathcal{I}_{\sigma^{-1}(P)}$ be the ideal sheaf of $\sigma^{-1}(P)$, an invertible $\mathcal{O}_{W'}$-module by the definition of σ. Since $\mathcal{I}'_{y'}$ is generated by a single nonzerodivisor, and also by the $\mathcal{I}(\mathrm{pr}_1, z')$, $z' \in L''$, we can choose $z' \in L''$ such that

$$\mathcal{I}'_{y'} = \mathcal{I}(\mathrm{pr}_1, z')\mathcal{O}_{W'}, \qquad z' = z \times y'$$

We prove the proposition 4.2.8 for this choice of z'. We have that $\mathcal{I}(\mathrm{pr}_1, z')\mathcal{O}_{V',z'}$ is an invertible $\mathcal{O}_{V',z'}$-module. Indeed if $\mathcal{I}(f,L)$ is the ideal of (P,y) in $\mathcal{O}_{W,y}$ then

$$\mathcal{I}(\mathrm{pr}_1, z')\mathcal{O}_{V',z'} = \mathcal{I}'\mathcal{O}_{V',z'} = \mathcal{I}(f,L)\mathcal{O}_{V',z'}$$

and this last is invertible since σ' is the local blowing up with centre $f^{-1}(P)$, and $\mathcal{I}(f,L)\mathcal{O}_{V',z'}$ is generated via the morphism $\sigma \circ f' = f \circ \sigma'$. Thus we may apply the lemma 4.2.9 above taking in the statement $\mathcal{I}(f,z) = \mathcal{I}(\mathrm{pr}_1, z')$ and $\tilde{V} = V'$, to obtain:

$$V' \cap \mathrm{pr}_1^{-1}(y') \subsetneqq \mathrm{pr}_1^{-1}(y').$$

By (4.2.7) this means precisely that

$$(f')^{-1}(y') \subsetneq f^{-1}(y).$$

This completes the proof of proposition 4.2.8. We now proceed to the proof of the local flattening theorem. The existence of the flatificator and the fibre cutting lemma are the key components of the proof of the local flattening theorem. We will choose an infinite sequence of local blowings up $(\sigma_0, \sigma_1, \ldots)$ in such a way that if the strict transform of f by $\sigma_0 \circ \cdots \circ \sigma_m$ is not flat at some point corresponding to L then we are placed in the situation of proposition 4.2.8. Then, since we cannot have an infinite, strictly decreasing sequence of inclusions of the fibres, it must be that after a certain rung the strict transform of f is flat at each point corresponding to L. The flatificator is key to choosing the blowings up σ_i.

4.3 Choosing the sequence of blowings up

Let $f : V \to W$ be a morphism of complex spaces, with W reduced. Let $y \in W$ and let $L \subset\subset f^{-1}(y)$. Pick any étoile $\mathfrak{e}$ such that $p_W(\mathfrak{e}) = y$ (recall that p_W is the surjective map that takes an étoile $\mathfrak{e}$ to the unique image point common to all the local blowings up in $\mathfrak{e}$). We define an infinite sequence of local blowings up associated to $\mathfrak{e}$:

$$S(\mathfrak{e}) = \{\sigma_m = (U_m, E_m, \pi_m : W_{m+1} \to U_m) : m = 0, 1, \ldots\}.$$

We use the following notation for the strict transform of f by the blowings up in $S(\mathfrak{e})$:

$$\begin{tikzcd} V_{m+1} \arrow[r, "\sigma'_m"] \arrow[d, "f_{m+1}"] & V_m \subseteq V \times_W W_m = V_{m-1} \times_{W_{m-1}} W_m \arrow[d, "f_m"] \\ W_{m+1} \arrow[r, "\sigma_m"] & W_m \end{tikzcd}$$

where σ'_m is the local blowing up of V_m defined along the inverse image of the centre of σ_m, that is, $\sigma'_m = (f_m^{-1}(U_m), f_m^{-1}(E_m), \pi'_m)$. We choose $S(\mathfrak{e})$ subject to the following two conditions:

(1) If $\sigma^m = \sigma_0 \circ \cdots \circ \sigma_{m-1} : W_m \to W$ for all $m \geq 0$, then $\sigma^m \in \mathfrak{e}$;

Let $y_0 = y$, and for $m \geq 1$ define $y_m = p_{\sigma^m}(\mathfrak{e}) \in W_m$ (recall that we can associate to $\mathfrak{e}$ a unique étoile $j_{\sigma^m}^{-1}(\mathfrak{e}) = \mathfrak{e}_m$ over W_m which consists of all the local blowings up σ' over W_m such that $\sigma^m \circ \sigma' \in \mathfrak{e}$, and we define $p_{\sigma^m}(\mathfrak{e}) = p_{W_m}(\mathfrak{e}_m)$). Note that for $m \geq 0$ we have $\sigma_m(y_{m+1}) = y_m$. For example when $m = 0$, we have the equalities

$$p_W \circ j_{\sigma_0} = \sigma_0 \circ p_{W_1}, \qquad p_{\sigma_0} \circ j_{\sigma_0} = p_{W_1},$$

the first being proposition 3.2.10 and the second the definition of p_{σ_0}. Now

$$y_1 = p_{\sigma_0}(\mathfrak{e}) = p_{W_1} \circ j_{\sigma_0}^{-1}(\mathfrak{e}).$$

Thus

$$\sigma_0(y_1) = \sigma_0 \circ p_{W_1} \circ j_{\sigma_0}^{-1}(\mathfrak{e}) = p_W \circ j_{\sigma_0} \circ j_{\sigma_0}^{-1}(\mathfrak{e}) = p_W(\mathfrak{e}) = y_0.$$

The same argument shows that $\sigma_m(y_{m+1}) = y_m$ for all $m \geq 0$. Let $L_0 = L$, and for $m \geq 1$ set

$$L_m = L_0 \times_W \{y_m\} = L_{m-1} \times_{W_{m-1}} \{y_m\},$$

a non-empty compact subset of the fibre product $V \times_W W_m$.

(2) Suppose we have chosen $\sigma_0, \ldots, \sigma_{m-1}$. If $L_m \cap V_m = \emptyset$, let U_m be any open neighborhood of y_m in W_m, and let $E_m = \emptyset$, so $\sigma_m = \pi_m = \mathrm{id}_{|U_m}$.

If $L_m \cap V_m \neq \emptyset$, let (P_m, y_m) be the flatificator for $(f_m, L_m \cap V_m)$ and choose an open neighborhood U_m of y_m in W_m such that (P_m, y_m) has a representative P_m in U_m.

Now we want the centres of the σ_i to be nowhere dense, so we make the following modification to P_m. Let W_m^* be the strict transform of $U_m \backslash P_m$ by $\mathrm{id}_{W_m|U_m}$. That is, W_m^* is the unique smallest closed subspace of U_m such that $W_m^* \backslash P_m = U_m \backslash P_m$ and if W_m^{**} is a closed subspace of W_m^* that also has this property, then $W_m^{**} = W_m^*$. Now let $E_m = P_m \cap W_m^*$.

We claim that $E_m = P_m \cap W_m^*$ is a closed, nowhere dense subspace of U_m. We know that P_m is an analytic subset of U_m, so at a given point $z \in U_m$ either P_m is nowhere dense in a neighborhood of z or there is a local component U'_m of U_m such that $P_m \cap U'_m = U'_m$. Thus W_m^* is the closure of the complement of P_m in the components of the U_m in which P_m is nowhere dense. In particular, W_m^* contains at most a proper analytic subset of any component U'_m such that $P_m \cap U'_m = U'_m$, so E_m is nowhere dense in U_m. Let us make this more precise. We have a finite number of irreducible components $U_m = \bigcup_{j=1}^r U_m^j$ (the global irreducible components of U_m, which may not be the local irreducible components of (U_m, y_m)) and we have the result (see e.g. [4, Ch. 2, §5]):

Proposition 4.3.1. *If A, B are analytic subsets of a reduced complex space then $\overline{A \backslash B}$ is the union of the irreducible components of A which are not contained in B.*

Here the closure always exists since reduced complex spaces are determined by the topological space. In fact we will see below that W_m is reduced, so this is the most general statement that we need. Returning to the nowhere-denseness of E_m, we have that the strict transform W_m^* is just $\overline{U_m \backslash P_m}$, so W_m^* is the union of all the U_m^j with $U_m^j \nsubseteq P_m$. We know that a proper analytic subset of an irreducible complex space is nowhere dense, so

$$E_m = P_m \cap W_m^* = \bigcup_{U_m^j \nsubseteq P_m} (U_m^j \cap P_m)$$

is nowhere dense. Next let us show that (2) implies the property (1), i.e. if $S(\mathfrak{e})$ is chosen satisfying (2) then for each $m \geq 0$ we have that

$$\sigma^m = \sigma_0 \circ \cdots \circ \sigma_{m-1} \in \mathfrak{e}.$$

To see this we recall corollary 3.3.3:

> Let X be a complex space and let $\tau : X' \to X$ be a finite sequence of local blowings up. Let $\tau' = (U', E', \pi')$ be a local blowing up over X', and let $\mathfrak{e}'$ be an étoile over X' such that $P_{W'}(\mathfrak{e}') \in U'$. If E' is nowhere dense in some neighborhood of $P_{W'}(\mathfrak{e}')$ in U' then $\tau \circ \tau' \in j_\tau(\mathfrak{e}')$.

In our situation we take $\mathfrak{e}' = j_{\sigma^m}^{-1}(\mathfrak{e})$, $E' = E_m$, $U' = U_m$ and $\tau = \sigma^m$. Then we have

$$p_{W_m}(\mathfrak{e}') = p_{W_m} \circ j_{\sigma^m}^{-1}(\mathfrak{e}) = p_{\sigma^m}(\mathfrak{e}) \in U_m$$

and E_m is nowhere dense in U_m, so corollary 3.3.3 states that

$$\sigma^{m+1} = \sigma^m \circ \sigma_m \in \mathfrak{e}.$$

This shows that $\sigma^m \in \mathfrak{e}$ for all m, provided we define $\sigma^0 = \mathrm{id}_W \in \mathfrak{e}$. Before we continue, we make a brief digression on reduced complex spaces. Let X be a complex space, and let

$$\mathcal{N}_X : U \mapsto \{\phi \in \mathcal{O}_X(U) : \phi^n = 0 \text{ for some } n \geq 1\}$$

be the nilpotent sheaf of $\mathcal{O}_X$. The *reduction* X_{red} of X is the closed subspace defined by the (coherent) sheaf of ideals $\mathcal{N}_X$. The projections

$$\mathcal{O}_X(U) \to \frac{\mathcal{O}_X(U)}{\mathcal{N}_X(U)}$$

define a canonical morphism $\mathrm{red}_X : X_{\mathrm{red}} \to X$. We say that X is *reduced* if $X_{\mathrm{red}} = X$. If X is reduced and $U \subseteq X$ then $\mathcal{O}_X(U)$ maps injectively into the continuous functions $\mathcal{C}_X(U)$ (it is easily verified using the nullstellensatz that the kernel of $\mathcal{O}_X \to \mathcal{C}_X$ is the nilpotent sheaf). In particular, the topological structure of X is sufficient to determine the sheaf $\mathcal{O}_X$ of holomorphic functions. We need the following fact:

Proposition 4.3.2. *Let $\pi : X' \to X$ be the blowing up of a complex space X along a closed subspace $Y \subseteq X$ with ideal sheaf $\mathcal{I}$. If X is reduced then so is X'.*

Proof. We know that X' is the strict transform of X by π. Let $E = \pi^{-1}(Y)$ be the exceptional divisor of π. Then if X'' is a closed subspace such that $X''\backslash E = X'\backslash E$ then $X'' = X'$ (this is the minimality condition on the strict transform X'). Since X is reduced and π is an isomorphism outside E, we have $X'_{\mathrm{red}}\backslash E = (X'\backslash E)_{\mathrm{red}} = (X'\backslash E)$, and X'_{red} is a closed subspace of X', hence $X'_{\mathrm{red}} = X'$. □

Let us now make some remarks on the sequence of local blowings up $S(\mathfrak{e}) = \{\sigma_m = (U_m, E_m, \pi_m : W_{m+1} \to U_m)\}$.

Remarks 4.3.3.

(i) We suppose that W is reduced, so by the above proposition, W_m is reduced for all $m \geq 0$. Suppose that P_m is nowhere dense in U_m, that is, there is no component U'_m of U_m such that P_m contains the whole of U'_m. Then we must have that $W^*_m = W_m$, since this is true topologically, and therefore $E_m = P_m$.

(ii) We know that (P_m, y_m) is unique, and the germ (W^*_m, y_m) is unique. Thus the only indeterminacy of $S(\mathfrak{e})$ for a fixed choice of f, L and $\mathfrak{e}$ is the the neighborhood U_m of y_m chosen. That is, the σ_m are uniquely determined up to a restriction of their domains.

(iii) If f_m is flat at each point of $L_m \cap V_m$ then by the definition of the flatificator (P_m, y_m) of $(f_m, L_m \cap V_m)$ we have that $(P_m, y_m) = (W_m, y_m)$. It then follows that in a sufficiently small neighborhood U_m of y_m that $E_m \subseteq W^*_m = \emptyset$, and for this choice of the neighborhood U_m we have $\sigma_m = \mathrm{id}_{W_m|U_m}$. In fact, we have that if f_m is flat at each point of $L_m \cap V_m$ then the embedding $V_{m+1} \hookrightarrow V_m \times_{W_m} W_{m+1}$ is locally an isomorphism at each point of $L_{m+1} \cap V_{m+1}$.

Let us show the last claim of remark (iii). To begin we show that a flat module is torsion free.

Definition 4.3.4. Let $A \neq \{0\}$ be a commutative ring, and let $S \subseteq A$ be a multiplicatively closed subset contained in the set of non-zerodivisors of A. Let M be an A-module and let $m \in M$ be any element. We say that m is *S-torsion* if there exists $a \in S$ such that $a \cdot m = 0$.

With the notation of this definition, if $S = \{1, x, x^2, \ldots\}$ we say that m is *x-torsion.* If S is the set of all non-zerodivisors of A, then we say that m is *torsion.* If no non-zero element of M is torsion, we say that M is *torsion free.*

Proposition 4.3.5. *Let A be a commutative ring and let $x \in A$ be a non-zerodivisor. Let M be an A-module. Then*

$$\mathrm{Tor}_1^A(A/(x), M) = \{m \in M : xm = 0\}.$$

Thus if M is flat, then

$$\mathrm{Tor}_1^A(A/(x), M) = \{m \in M : xm = 0\} = 0,$$

for each nonzerodivisor x, so M is torsion free.

Proof of proposition 4.3.5. The exact sequence

$$0 \to (x) \to A \to A/(x) \to 0$$

is in fact a free resolution of $A/(x)$. Thus tensoring with M, we have that $\mathrm{Tor}_1^A(A/(x), M)$ is the cohomology of the sequence

$$0 \to M \xrightarrow{\cdot x} M$$

where $\cdot x$ means multiplication by x. But this cohomology group is just the kernel of $\cdot x$, which is by definition $\{m \in M : xm = 0\}$. $\square$

Since flatness is preserved by base change, we have that $V_m \times_{W_m} W_{m+1} \to W_{m+1}$ is flat at each point of L_{m+1}. It follows from the fact that a flat module is torsion free that the same holds for the restriction of this map to the closed subspace V_{m+1}:

$$\begin{tikzcd}
V_{m+1} \arrow[dr, "\iota"] \arrow[drr, "\sigma'_m"] \arrow[ddr, "f_{m+1}"'] & & \\
& V_m \times_{W_m} W_{m+1} \arrow[r, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] & V_m \arrow[d, "f_m"] \\
& W_{m+1} \arrow[r, "\sigma_m"'] & W_m
\end{tikzcd}$$

where ι is the closed embedding of V_{m+1} in the fibre product. Indeed, since a flat module is torsion free, and since flatness is an open property, for any point $z_0 \in L_{m+1}$ we have that the local ring $\mathcal{O}_{V_m \times_{W_m} W_{m+1}, z}$ is a torsion free $\mathcal{O}_{W_{m+1}, y_{m+1}}$-module for all z in a neighborhood of z_0. Let $\mathcal{I}_m$ be the ideal sheaf of E_m

in $\mathcal{O}_{U_m}$. Then $\mathcal{I}_m\mathcal{O}_{W_{m+1}}$ is invertible, since σ_m is the blowing up with centre E_m. Now $\mathcal{I}_m\mathcal{O}_{V_m\times_{W_m}W_{m+1}}$ is locally principal in a neighborhood of z_0, and it is torsion free over $\mathcal{O}_{W_{m+1},y_{m+1}}$, hence generated by a non-zerodivisor. It follows that $\mathcal{I}_m\mathcal{O}_{V_m\times_{W_m}W_{m+1}}$ is an invertible $\mathcal{O}_{V_m\times_{W_m}W_{m+1}}$-module in a neighborhood of L_{m+1}. Now by the universal property for $\sigma'_m = (f_m^{-1}(U_m), f_m^{-1}(E_m), \pi'_m)$ there exists locally in a neighborhood of L_{m+1}, a unique $\tau : V_m \times_{W_m} W_{m+1} \to V_m$ such that $\sigma'_m \circ \tau = \mathrm{pr}_1$. We also have $\mathrm{pr}_1 \circ \iota = \sigma'_m$, so

$$\sigma'_m \circ (\tau \circ \iota) = \sigma'_m.$$

Since σ'_m is strict, the map $(\tau \circ \iota)$ satisfying this equality is unique, so $\tau \circ \iota = \mathrm{id}$ (note that this holds only locally in a neighborhood of L_{m+1}). Next we show that near L_{m+1}, $\iota \circ \tau = \mathrm{id}$. To make the diagrams easier to draw, let $Z_m = V_m \times_{W_m} W_{m+1}$. Take a diagram as follows:

$$\begin{tikzcd}
Z_m \arrow[dr, "\tau"] \arrow[drr, bend left, "\mathrm{pr}_1"] \arrow[ddr, bend right, "\mathrm{pr}_2"'] & & \\
& V_{m+1} \arrow[dr, "\iota"] \arrow[drr, "\sigma'_m"] \arrow[ddr, "f_{m+1}"'] & \\
& & Z_m \arrow[r, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] & V_m \arrow[d, "f_m"] \\
& & W_{m+1} \arrow[r, "\sigma_m"'] & W_m
\end{tikzcd}$$

Now we have

$$\mathrm{pr}_1 \circ \iota \circ \tau = \sigma'_m \circ \tau = \mathrm{pr}_1 \tag{4.3.1}$$

Moreover by the universal property for the blowing up σ_m there is a unique mapping $\phi : Z_m \to W_{m+1}$ such that

$$\sigma_m \circ \phi = \mathrm{pr}_1 \circ f_m.$$

Therefore we have

$$\mathrm{pr}_2 \circ \iota \circ \tau = \mathrm{pr}_2,$$

since both sides of this equation satisfy the property of ϕ. Now by the universal property for the fibre product Z_m, the mapping $\iota \circ \tau$ satisfying this last equality and (4.3.1) is unique, and id_{Z_m} has this property, so $\iota \circ \tau = \mathrm{id}$ (again we recall that this is only true in a neighborhood of L_{m+1}). We conclude that the closed embedding ι is locally an isomorphism in a neighborhood of each point of $L_{m+1} \cap V_{m+1}$. In particular, since the projection $Z_m = V_m \times_{W_m} W_{m+1} \to W_{m+1}$ is flat at each point of L_{m+1}, it follows that f_{m+1} is flat at each point of $L_{m+1} \cap V_{m+1}$.

4.4 Proof of the local flattening theorem

To establish the local flattening theorem we show that there exists a certain rung $m_0 \geq 0$ such that for all $m \geq m_0$ each σ_m is a restriction of the identity mapping of W_m, after possibly shrinking the open sets U_m. This is the main content of the proof of the local flattening theorem; it is elementary to show that taking σ^{m_0} as one of the mappings σ_α of the statement of this theorem for each étoile $\mathfrak{e}$ gives us a (possibly infinite) set containing the required sequences of local blowings up.

Proposition 4.4.1 (Main proposition)**.** *Let $f : V \to W$ be a morphism of complex spaces, $y \in W$, $L \subset\subset f^{-1}(y)$ and $S(\mathfrak{e})$ satisfying (1) and (2) above. Then there exists $m_0 \geq 0$ such that for all $m \geq m_0$ the strict transform f_m of f by σ^m is flat at every point of $L_m \cap V_m$, and E_m is empty in some neighborhood of y_m in W_m.*

Note that by the remark (ii) above, if f_m is flat at every point of $L_m \cap V_m$, then E_m is empty in some neighborhood of y_m in W_m, so it suffices to prove only the first part of the statement. Let us prove proposition 4.4.1. The main fact we want to prove is that if f_{m+1} is not flat at some point of $L_{m+1} \cap V_{m+1}$ then there exists at least one $z_{m+1} \in L_{m+1}$ such that:

$$f_{m+1}^{-1}(y_{m+1}) \hookrightarrow \mathrm{pr}_2^{-1}(y_{m+1}) \xrightarrow{\sim} f_m^{-1}(y_m)$$

is a strict inclusion at z_{m+1}. Since we cannot have an infinite sequence of such strict inclusions, the main proposition must then follow. In order to prove the existence of z_{m+1} we will show that the hypotheses of proposition 4.2.8 hold for the restricted diagram

$$\begin{tikzcd}
W_m^{*\prime} \times_{W_m^*} f_m^{-1}(W_m^*) \arrow[r, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] & f_m^{-1}(W_m^*) \arrow[d, "f_m"] \\
W_m^{*\prime} \arrow[r, "\sigma_m"'] & W_m^*
\end{tikzcd}$$

where $W_m^{*\prime}$ is the strict transform of W_m^* by σ_m. Now we need a quick:

Lemma 4.4.2. *Let X be a complex space, let $U \subseteq X$ be an open subspace. Let F_1, F_2 be closed subspaces of U such that $F_1 \cup F_2 = U$. Let $E = F_1 \cap F_2 \subseteq U$, and let $\sigma = (U, E, \pi : X' \to U)$ be the local blowing up with centre E. Then if F_i' is the strict transform of F_i by σ, then $F_1' \cap F_2' = \emptyset$ and $F_1' \cup F_2' = X'$.*

Proof. Let X_0 be the strict transform of $F_1 \cup F_2$ by σ, so $X_0 = F_1' \cup F_2' = X'$. As in the proof of lemma 3.3.6, if $\mathcal{I}_i$ is the ideal sheaf of F_i then we can write $\mathcal{I}_i\mathcal{O}_{X'} = \mathcal{Q}_i(\mathcal{I}_1 + \mathcal{I}_2)\mathcal{O}_{X'}$ for coherent ideal sheaves $\mathcal{Q}_i$ in $\mathcal{O}_V$ such that $\mathcal{Q}_1 + \mathcal{Q}_2 = \mathcal{O}_{X'}$. Thus the closed subspaces F_i'' defined by the $\mathcal{Q}_i$ are disjoint. Since $\mathcal{Q}_i \supseteq \mathcal{I}_i\mathcal{O}_{X'}$ we have $F_i' \subseteq F_i''$ and therefore $F_1' \cap F_2' = \emptyset$. $\square$

In our situation, we take $F_1 = P_m$, $F_2 = W_m^*$ and let P_m' and $W_m^{*\prime}$ be their respective strict transforms by

$$\sigma_m = (U_m, E_m = P_m \cap W_m^*, \pi_m)$$

We then have that $W_{m+1} = P_m' \sqcup W_m^{*\prime}$ is a disjoint union of closed subspaces.

Claim: $y_{m+1} \in W_{m+1}^{*\prime}$

Proof: Suppose that $y_{m+1} \in P'_m$. Since $W_{m+1} = P'_m \sqcup W^{*\prime}_m$ is disconnected, there is an open neighborhood U of y_{m+1} in W_{m+1} such that $U \subseteq P'_m$. We therefore have a diagram:

$$\begin{tikzcd} V_m \times_{W_m} U \arrow[r, "\sigma'_m"] \arrow[d, "\mathrm{pr}_2"'] & V_m \arrow[d, "f_m"] \\ U \arrow[r, "\sigma_m"] & W_m \end{tikzcd}$$

where all the mappings are restricted to the indicated domains, so σ_m maps into P_m. Now by the definition of the flatificator (P_m, y_m) for $(f_m, L_m \cap V_m)$ we have that the base extension of f_m by σ_m is flat at each point of $L_{m+1} = L \times_W \{y_{m+1}\} \subseteq \mathrm{pr}_2^{-1}(y_{m+1})$, so by the same argument we used in the discussion of remark (iii), the closed embedding of V_{m+1} in $V_m \times_{W_m} W_{m+1}$ is an isomorphism in a neighborhood of each point of $L_{m+1} \cap V_{m+1}$ so f_{m+1} is flat at each point of $L_{m+1} \cap V_{m+1}$, which contradicts our assumption. Therefore we must have $y_{m+1} \in W^{*\prime}_m$.

Now, we know that $W^{*\prime}_m \to W^*_m$ is the blowing up with centre $E_m \cap W^*_m = E_m$, and by lemma 4.2.3, (E_m, y_m) is the flatificator for $(f_{m|f_m^{-1}(W^*_m)}, L_m \cap f_m^{-1}(W^*_m))$. Now we must have $V_{m+1} = f_{m+1}^{-1}(W^{*\prime}_m) \sqcup f_{m+1}^{-1}(P'_m)$, and the map $f_{m+1}^{-1}(W^{*\prime}_m) \to W^{*\prime}_m$ is the strict transform of $f_m^{-1}(W^*_m) \to W^*_m$ by $\sigma_{m|W^{*\prime}_m}$. Therefore we can apply proposition 4.2.8, which states that there exists at least one $z_m \in L_m \cap f_m^{-1}(W^*_m)$ such that if $z_{m+1} = z_m \times y_{m+1}$, the inclusion

$$f_{m+1}^{-1}(y_{m+1}) \hookrightarrow f_m^{-1}(y_m)$$

is strict at z_{m+1}. Now if there is no m_0 as claimed in proposition 4.4.1 then there is an infinite sequence of embeddings

$$\cdots \hookrightarrow f_{m+1}^{-1}(y_{m+1}) \hookrightarrow f_m^{-1}(y_m) \hookrightarrow \cdots$$

each of which is strict at some point varying with m, which is impossible. Indeed the lemma 2.3.4 states that the union of the ideal sheaves (corresponding to the intersection of the fibres) must be locally isomorphic to the ideal sheaf of one of the fibres in a neighborhood of each point. Since $L \subset\subset f^{-1}(y)$ is compact we need choose only a finite number of such local isomorphisms, and we cannot have an infinite descending sequence. This concludes the proof of the main proposition. We are now ready to prove the local flattening theorem; we retain the notation used above. For each étoile $\mathfrak{e}$ with $p_W(\mathfrak{e}) = y$ we choose a sequence

$$S(\mathfrak{e}) = \{\sigma_m = (U_m, E_m, \pi_m) : m = 0, 1, \ldots\}.$$

Then by proposition 4.4.1 there exists $m = m(\mathfrak{e}) \geq 0$ such that the strict transform f_m of f by $\sigma^m = \sigma_0 \circ \cdots \circ \sigma_{m-1} : V_m \to W_m$ is flat at each point of $L_m \cap V_m$. Flatness being an open property, there exists a neighborhood N_m of $L_m \cap V_m$ in V_m such that f_m is flat in N_m. Recall that we defined the local blowing up

$$\sigma'_i = (f_i^{-1}(U_i), f_i^{-1}(E_i), \pi'_i : V_{i+1} \to f^{-1}(U_i)), \qquad i \geq 0$$

as in the following diagram:

$$\begin{tikzcd} V_{i+1} \arrow[r, "\sigma_i'"] \arrow[d, "f_{i+1}"] & V_i \subseteq V \times_W W_i = V_{i-1} \times_{W_{i-1}} W_i \arrow[d, "f_i"] \\ W_{i+1} \arrow[r, "\sigma_i"] & W_i \end{tikzcd}$$

Let $\sigma^{m\prime} = \sigma_0' \circ \cdots \circ \sigma_{m-1}' : V_m \to V$. Then we have

$$(\sigma^{m\prime})^{-1}(L) \subseteq L \times_W W_m \subseteq V \times_W W_m$$

via the closed embedding $V_m \to V \times_W W_m$. Therefore the restriction of f_m to $(\sigma^{m\prime})^{-1}(L)$ is proper. Then for a sufficiently small neighborhood M_m of y_m in W_m we have

$$f_m^{-1}(M_m) \cap (\sigma^{m\prime})^{-1}(L) \subseteq N_m.$$

We write $f_\alpha : V_\alpha \to W_\alpha$ for the morphism $f_m^{-1}(M_m) \to M_m$ induced by f_m. Then f_α is flat at each point corresponding to $L \subseteq V$. In order to write f_α as the strict transform by a finite sequence of local blowings up we make the following modification. By the remarks above, after a possibly shrinking the set U_m, we have that σ_m is a restriction of the identity mapping of W_m. Thus we may replace σ_m by

$$\sigma_m = (U_m, E_m, \pi_m) = (M_m, \emptyset, \mathrm{id}_{W_m|M_m}).$$

and now f_α is the strict transform of f by $\sigma_\alpha = \sigma^{m+1}$, since the additional mapping σ_m in the composition has only the effect of replacing f_m with its restriction to $f_m^{-1}(M_m)$. Thus statements (i) and (iii) of the local flattening theorem are verified for σ_α. Now we list the étoiles $\{\mathfrak{e}_\alpha : \alpha \in A\}$ such that $p_W(\mathfrak{e}_\alpha) = y$, and let $\sigma_\alpha \in \mathfrak{e}_\alpha$ be the finite sequence of local blowings up just obtained. Then

$$p_W^{-1}(y) = \{\mathfrak{e}_\alpha : \alpha \in A\} \subseteq \bigcup_{\alpha \in A} \mathcal{E}_{\sigma_\alpha}.$$

Since p_W is proper, we may replace A with a finite subset that also has this property. Now by proposition 3.3.10, (ii) of the local flattening theorem is also satisfied.

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