Université Paris-Sud
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Spécialité : Mathématiques

Présentée par :

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Sur la conjecture de Green-Griffiths logarithmique

(Dégénérescence algébrique effective des courbes holomorphes entières à valeurs dans des complémentaires d'hypersurfaces projectives de grand degré)

Soutenue le 03 Juillet 2014 devant la commission d'examen composée de :

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Résumé/Abstract

Résumé *L'objet d'étude de ce mémoire est la géométrie des courbes holomorphes entières à valeurs dans le complémentaire d'hypersurfaces génériques de l'espace projectif complexe. Les conjectures célèbres de Kobayashi et de Green-Griffiths énoncent que pour de telles hypersurfaces, de grand degré, les images de ces courbes entières doivent satisfaire certaines contraintes algébriques. En adaptant les techniques de jets développées notamment par Bloch, Green-Griffiths, Demailly, Siu, Diverio-Merker-Rousseau, pour les courbes à valeurs dans une hypersurface projective (cas dit compact), nous obtenons la dégénérescence algébrique des courbes entières* $f\colon \mathbb{C} \to \mathbb{P}^n \setminus X_d$ *(cas dit logarithmique), pour les hypersurfaces génériques* X_d *de* $\mathbb{P}^n$ *de degré* $d \geqslant (5n)^2\, n^n$*. Comme dans le cas compact, notre preuve repose essentiellement sur l'élimination algébrique de toutes les dérivées dans des équations différentielles qui sont vérifiées par toute courbe entière non constante. L'existence de telles équations différentielles est obtenue grâce aux inégalités de Morse holomorphes et à une variante simplifiée d'une formule de résidus originalement élaborée par Bérczi à partir de la formule de localisation équivariante d'Atiyah-Bott. La borne effective* $d \geqslant (5n)^2\, n^n$ *est obtenue par réduction radicale d'un calcul de résidus itérés de très grande ampleur. Ensuite, la déformation de ces équations différentielles par dérivation le long de champs de vecteurs obliques, dont l'existence est ici généralisée et clarifiée, nous permet d'engendrer suffisamment de nouvelles équations pour réaliser l'élimination algébrique finale évoquée ci-dessus.*

On the logarithmic Green-Griffiths conjecture

Abstract *The topic of this memoir is the geometry of holomorphic entire curves with values in the complement of generic hypersurfaces of the complex projective space. The well-known conjectures of Kobayashi and of Green-Griffiths assert that for such hypersurfaces, having large degree, the images of these curves shall fulfill algebraic constraints. By adapting the jet techniques developed notably by Bloch, Green-Griffiths, Demailly, Siu, Diverio-Merker-Rousseau, in the case of curves with values in projective hypersurfaces (so-called compact case), we obtain the algebraic degeneracy of entire curves* $f\colon \mathbb{C} \to \mathbb{P}^n \setminus X_d$ *(so called logarithmic case), for generic hypersurfaces of degree* $d \geqslant (5n)^2\, n^n$*. As in the compact case, our proof essentially relies on the algebraic elimination of all derivatives in differential equations that are satisfied by every nonconstant entire curve. The existence of such differential equations is obtained thanks to the holomorphic Morse inequalities and a simplified variant of a residue formula firstly developed by Bérczi from the Atiyah-Bott equivariant localization formula. The effective lower bound* $d \geqslant (5n)^2\, n^n$ *is obtained by radically simplifying a huge iterated residue computation. Next, the deformation of these differential equations by derivation along slanted vector fields, the existence of which is here generalized and clarified, allows us to generate sufficiently many new differential equations in order to realize the final algebraic elimination mentioned above.*

Mots-Clefs : hyperbolicité, positivité, conjecture de Green-Griffiths, hypersurface projective, jets logarithmiques, inégalités de Morse holomorphes, classes de Segre, résidus itérés, hypersurface universelle, champs de vecteurs obliques.

Keywords : hyperbolicity, positivity, Green-Griffiths conjecture, projective hypersurface, logarithmic jets, algebraic Morse inequalities, Segre classes, iterated residues, universal hypersurface, slanted vector fields.

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CHAPITRE I

Présentation des résultats

1. Motivations

Une variété complexe X est dite *hyperbolique* au sens de Brody lorsqu'il n'existe pas d'application holomorphe entière non constante $f : \mathbb{C} \to X$ à valeurs dans X. Le théorème de Picard (1879) donne un exemple de telle variété :

Théorème. *Il n'existe pas d'application holomorphe entière non constante :*

$$f : \mathbb{C} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

Le théorème de Green ([**21**]), qui généralise le théorème de Picard, fournit des exemples en dimension supérieure :

Théorème. *Il n'existe pas d'application holomorphe entière non constante :*

$$f : \mathbb{C} \to \mathbb{P}^n \setminus H_1 \cup \cdots \cup H_{2n+1},$$

à valeurs dans le complémentaire de la réunion de $2n + 1$ *hyperplans de* $\mathbb{P}^n$ *en position générale.*

La notion d'hyperbolicité peut être assouplie pour obtenir une notion plus faible, qui est la *dégénérescence algébrique* des courbes entières. Une courbe entière $\mathbb{C} \to X$ à valeurs dans une variété algébrique complexe X est dite *algébriquement dégénérée* s'il existe une sous-variété algébrique propre $Z \subsetneq X$ qui contient son image. Dans [**21**], Green prouve en fait le résultat plus général :

Théorème. *L'image d'une courbe entière* $f : \mathbb{C} \to \mathbb{P}^n$ *évitant* $n + k \geqslant n + 1$ *hyperplans en position générale est contenue dans un sous-espace linéaire projectif de dimension au plus la partie entière de* $\frac{n}{k}$. *Cette borne* $\lfloor \frac{n}{k} \rfloor$ *est optimale.*

On obtient donc que les courbes à valeurs dans le complémentaire de $n + 2$ hyperplans en position générale dans $\mathbb{P}^n$ sont algébriquement dégénérées.

On peut aussi considérer des hypersurfaces plus générales que des réunions d'hyperplans. L'une des motivations est la conjecture suivante ([**26**]), que nous choisissons de restreindre à l'hyperbolicité au sens de Brody, eu égard aux objectifs de ce mémoire :

Conjecture (Kobayashi). *Il n'existe pas d'application holomorphe entière non constante à valeurs dans le complémentaire* $\mathbb{P}^n \setminus X_d$ *d'une hypersurface générique* X_d *de degré* $d \geqslant 2n + 1$.

Ici *générique* signifie que les coefficients de l'équation définissante de X_d doivent se trouver en dehors d'une certaine sous-variété algébrique propre de l'espace projectif S de tous les coefficients des polynômes homogènes de degré d :

$$S := P\Big(\Gamma\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)\big)\Big),$$

lequel paramétrise les hypersurfaces algébriques de degré d dans $\mathbb{P}^n$. La borne $2n + 1$ peut à la fois être justifiée par le théorème de Green ci-dessus et par les travaux de

Zaĭdenberg ([**51**]), qui montrent que le complémentaire d'une hypersurface générique de degré $2n$ dans $\mathbb{P}^n$ n'est *pas* hyperbolique.

En ce qui concerne la dégénérescence algébrique, la conjecture de Green et Griffiths ([**22**]) énonce, dans un cadre très large, que toute variété algébrique lisse de type général X possède une sous-variété algébrique propre $Z \subsetneq X$, qui contient les images de toutes les courbes holomorphes entières non constantes $f\colon \mathbb{C} \to X$. Dans le cas considéré ici des hypersurfaces projectives, cette conjecture devient :

Conjecture (Green-Griffiths). *L'image d'une courbe entière $f\colon \mathbb{C} \to X_d$, à valeurs dans une hypersurface X_d de $\mathbb{P}^{n+1}$ de degré $d \geqslant n+3$ n'est pas Zariski-dense dans X_d.*

Conjecture (Green-Griffiths logarithmique). *L'image d'une courbe entière $f\colon \mathbb{C} \to \mathbb{P}^n \setminus X_d$, à valeurs dans le complémentaire dans $\mathbb{P}^n$ d'une hypersurface X_d de degré $d \geqslant n+2$ n'est pas Zariski-dense dans $\mathbb{P}^n$.*

La version forte de ces conjectures exige que la sous-variété algébrique propre Z de l'espace d'arrivée contenant l'image $f(\mathbb{C}) \subset Z$ ne dépende pas de la courbe entière non constante f.

Dans ce mémoire, une réponse positive est apportée à la conjecture de Green-Griffiths logarithmique forte, pour des hypersurfaces génériques de degré $d \geqslant (5n)^2\, n^n$.

2. L'approche des jets

À la suite de l'article de 1926 du mathématicien français André Bloch [**1**] sur les courbes entières à valeurs dans des variétés abéliennes, l'approche qui a dominé pour traiter les problèmes d'hyperbolicité est d'utiliser un grand nombre d'équations différentielles algébriques vérifiées par toute courbe entière. On se ramène ainsi à un problème d'élimination algébrique, et il suffit de montrer que le nombre d'inconnues est (beaucoup) plus grand que le nombre de contraintes.

Il est pratique de décrire ces équations différentielles en utilisant le formalisme des *jets*. Pour une application holomorphe $f\colon \mathbb{C} \to X$, à valeurs dans une variété complexe X et une trivialisation locale de $X \simeq \mathbb{C}^n$, le k-jet de f correspond à son développement de Taylor tronqué à l'ordre k. La notion locale de polynôme en les coefficients de Taylor $f_i', f_i'', \dots, f_i^{(k)}$ de f, de poids homogène m pour le nombre de "primes" a un sens global, comme le montrent les formules de changement de cartes de Faà di Bruno. On appelle *différentielle de jets* de degré k et de poids m le recollement de tels polynômes.

Dans son article, Bloch prouve le résultat d'hyperbolicité suivant :

Théorème. *Soit X un variété abélienne et soit $Y \subset X$ une sous-variété de X. Si Y ne contient le translaté d'aucune sous-variété abélienne de X, alors il n'existe pas de courbe entière non constante $f\colon \mathbb{C} \to Y$ à valeurs dans Y.*

Sa démonstration se déroule en trois étapes.

① La production explicite de *différentielles de jets holomorphes*, c'est-à-dire des polynômes dans les variables de jets $f_i^{(j)}$ à coefficients holomorphes, qui s'annulent sur un diviseur ample $A \subset X$.

② Le *théorème fondamental d'annulation* suivant, qui fait le lien entre différentielles de jets et hyperbolicité :

> **Théorème (**Lemme d'Ahlfors-Schwarz). *Si ω est une différentielle de jets holomorphe, s'annulant sur un diviseur ample $A \subset X$, et $f\colon \mathbb{C} \to X$ une courbe entière, alors le pullback $f^{\star}\omega \equiv 0$ s'annule identiquement sur $\mathbb{C}$.*

③ La détermination du lieu de dégénérescence : le translaté d'une sous-variété abélienne (qui doit être vide pour obtenir l'hyperbolicité).

Un demi siècle plus tard, Philip Green et Mark Griffiths ([**22**]) modernisent les concepts de Bloch et établissent plusieurs résultats fondamentaux sur la géométrie des courbes entières. Dans leur article de 1979, ils obtiennent le résultat :

Théorème. *Soit X une variété algébrique complexe vérifiant $q > \dim X$, où $q = \dim H^1(\overline{X}, \mathcal{O})$ pour n'importe quelle compactification $\overline{X}$ de X. Alors l'image de toute application holomorphe entière $\mathbb{C} \to X$ est contenue dans une sous-variété algébrique propre de X.*

Ce résultat avait déjà été énoncé par Bloch, et prouvé dans certains cas particuliers. Ochiai a également étudié certains cas particuliers. Comme Bloch et Ochiai, Green et Griffiths utilisent les *jets d'ordre supérieur*. Ils définissent les pseudo-métriques de jets et la notion de pseudo-métrique de jets à courbure sectionnelle holomorphe $\leqslant -1$. La démonstration du lemme d'Ahlfors-Schwarz implique que si la variété considérée X est munie d'une telle métrique sur ses k-jets, alors pour toute fonction holomorphe $f\colon \mathbb{C} \to X$, l'image $f(\mathbb{C})$ est contenue dans l'ensemble dégénéré de la métrique. Green et Griffiths choisissent ensuite la pseudo-métrique de façon à contrôler l'ensemble de dégénérescence.

En 1995 et ultérieurement aussi, Demailly([**7**]) a développé une technique de *jets invariants* prolongeant et améliorant l'approche de Green et Griffiths. Il introduit de nouveaux objets géométriques. Une *variété complexe dirigée* est par définition un sous-fibré vectoriel holomorphe $V \to X$ du fibré tangent $T_X \to X$ d'une variété complexe X. Pour étudier l'hyperbolicité de (X, V) on utilise la *méthode de la courbure négative* : par le lemme d'Ahlfors-Schwarz, l'existence d'une métrique hermitienne à courbure négative sur le fibré en droites tautologique $\mathcal{O}_{P(V)}(-1)$ implique la non existence de courbe entière non constante $f\colon \mathbb{C} \to X$ tangente à V. Le cas le plus intéressant est celui où le sous-fibré $V = T_X$ est le fibré tangent en entier, car on obtient alors l'hyperbolicité de X. Sous certaines hypothèses de positivité du fibré dual $V^\star$, il est possible de construire une telle métrique. Par exemple :

Théorème. *Soit (X, V) une variété complexe dirigée. Si $V^\star$ est ample, alors toute courbe entière $f\colon \mathbb{C} \to X$ tangente à V est constante.*

Pour généraliser la situation aux k-jets Demailly introduit une tour de fibrés projectifs. Cette construction a été généralisée au cas logarithmique par Dethloff et Lu ([**10**]).

Soit $\overline{X}$ une variété complexe et D un diviseur simple à croisements normaux sur $\overline{X}$, c'est-à-dire $D = \sum D_i$, où les composantes D_i sont des diviseurs lisses irréductibles qui s'intersectent transversalement. Une telle paire $(\overline{X}, D)$ est appelée une *paire logarithmique*. On note $T_{\overline{X}}(-\log D)$ le *fibré tangent logarithmique à $\overline{X}$ le long de D* ([**35**]). Pour tout sous-fibré V du fibré tangent logarithmique :

$$V \subset T_{\overline{X}}(-\log D) \subset T_{\overline{X}},$$

on construit ([**7, 10**]), pour tout ordre fixé $k \in \mathbb{N}$, la *tour de Demailly logarithmique* des fibrés de jets projectivisés :

$$\left(\overline{X}_k, D_k, V_k\right) \to \left(\overline{X}_{k-1}, D_{k-1}, V_{k-1}\right) \to \ldots \to \left(\overline{X}_1, D_1, V_1\right) \to \left(\overline{X}_0, D_0, V_0\right) := \left(\overline{X}, D, V\right),$$

ayant pour propriété principale que toute trajectoire tangente à V non constante $f\colon \Delta_R \to \overline{X} \setminus D$ se relève uniquement en une trajectoire $f_{[k]}\colon \Delta_R \to \overline{X}_k \setminus D_k$ tangente à V_k, qui ne dépend que du k-jet de f. Cette tour a pris une grande importance dans l'étude de la dégénérescence algébrique des courbes entières sur $\overline{X} \setminus D$ (*cf.* les articles de survol [**14, 38**]).

En effet, nous avons vu qu'une première étape dans la preuve de la dégénérescence algébrique des courbes entières est de montrer qu'il existe un polynôme P non nul sur $\overline{X}$, tel que toute courbe entière non constante $f\colon \mathbb{C} \to \overline{X} \setminus D$ satisfasse l'équation différentielle algébrique :

$$P\big(f, f', f'', \ldots, f^{(k)}\big) \equiv 0.$$

Étant par définition un fibré vectoriel projectif, la variété $\overline{X}_k$ est naturellement équipée d'un fibré en droites tautologique $\mathcal{O}_{\overline{X}_k}(-1)$. Soit $\pi_{k,0}$ la projection naturelle du k-ème niveau de la tour vers la base $\overline{X}_0 = \overline{X}$. Les images directes :

$$\mathcal{E}_{k,m} V^\star := (\pi_k)_\star \mathcal{O}_{\overline{X}_k}(m)$$

peuvent être vues comme des fibrés vectoriels d'opérateurs différentiels d'ordre k et de degré m agissant sur les courbes holomorphes dans X tangentes à V et invariantes par reparamétrage. Alors, J.-P. Demailly a démontré un théorème fondamental d'annulation ([**7, 10**]), qui stipule que pour toute section globale :

$$P \in H^0\big(\overline{X}_k, \mathcal{O}_{\overline{X}_k}(m) \otimes \pi_{k,0}^\star A^\vee\big) = H^0\big(\overline{X}, \mathcal{E}_{k,m} V^\star \otimes A^\vee\big),$$

à valeurs dans le dual $A^\vee$ d'un fibré en droites ample $A \to \overline{X}$, on a, pour tout courbe entière non constante $f\colon \mathbb{C} \to \overline{X} \setminus D$ tangente à V :

$$P\big(f, f', f'', \ldots, f^{(k)}\big) \equiv 0.$$

Ce théorème d'annulation (étape ② de la démonstration de Bloch) permet une approche alternative à la production explicite de différentielles de jets holomorphes (étape ① de la démonstration de Bloch). On peut se ramener à l'étude des sections globales du faisceau $\mathcal{O}_{\overline{X}_k}(m)$ qui s'annulent sur A. En contrepartie, le manque d'information sur les différentielles de jets obtenues complique l'étape ③ de la démonstration de Bloch. Dans certains cas, Diverio et Rousseau ([**16**]) ont même donné récemment des exemples où les équations qu'on obtient de cette façon ne suffisent pas pour obtenir des informations sur la base $\overline{X}_0$.

L'algèbre des jets invariants étudiée par Rousseau ([**41**]), par Merker ([**28**]), ainsi que par Bérczi et Kirwan ([**5**]) est compliquée.

3. La stratégie de Siu

En 1996 ([**48**]), en 2002 ([**45**]) et en 2004 ([**46**]), Siu a traité le cas des hypersurfaces projectives de dimension 2 (en degré non optimal) et il a proposé des idées générales adéquates, mais réputées difficiles à réaliser techniquement, afin de traiter la dimension n quelconque. Il utilise une stratégie en deux étapes :

①–② La construction explicite en coordonnées projectives de sections holomorphes globales du fibré des différentielles de jets, dans l'esprit des travaux de Bloch.

③ La déformation de ces sections pour obtenir suffisamment d'équations. Cette deuxième étape est inspirée des travaux de Ein-Clemens-Voisin ([**6, 18, 50**]) sur les courbes rationnelles, dans lesquels il est établit que :

Théorème. *Il n'existe pas de courbe rationnelle* $f\colon \mathbb{P}^1 \to X_d$ *à valeurs dans une hypersurface générique* X_d *de* $\mathbb{P}^n$ *de degré* $d \geqslant 2n - 1$.

Pour déformer les équations différentielles, on travaille en famille, sur l'hypersurface universelle de degré d :

$$\mathcal{H} := \Big\{ Z, P_s \colon P_s(Z) = 0 \Big\} \subset \mathbb{P}^n \times S.$$

Si ω est une famille de différentielles de k-jets sur les fibres de la deuxième projection, et si ω_s s'annule sur $A \subset H_s$, pour tout champs de vecteurs méromorphe V tangent à l'espace des k-jets verticaux de $\mathcal{H}$, dont l'ordre des pôles est plus petit que le degré d'annulation de ω, la dérivée de Lie $(\mathsf{V} \cdot \omega)_s$ vérifie toujours les hypothèses du lemme d'Ahlfors-Schwarz, et fournit donc une nouvelle équation différentielle.

Pour obtenir suffisamment d'équations indépendantes, on doit considérer les champs de vecteurs *obliques*, c'est-à-dire qui ont une composante non nulle dans la directions de l'espace des paramètres S et montrer que l'espace tangent aux jets verticaux possède un repère méromorphe avec des pôles d'ordre peu élevé.

La méthode de déformation de Siu, bien décrite dans [**13, 14, 38**], a été rendue effective par Păun ([**37**]) en dimension 2, et par Rousseau en dimension 3, à la fois dans le cas compact ([**43**]) et dans le cas logarithmique ([**42**]). Dans le cas compact, la technique a été généralisée en toute dimension par Merker ([**29**]), avec une amélioration importante de la détermination du lieu où l'énoncé d'engendrement global ③ est valide, qui a mené à une preuve de la dégénérescence algébrique *forte* des courbes entières à valeurs dans une hypersurface projective générique de grand degré, avec Diverio et Rousseau ([**13**]). Dans le contexte légèrement différent des familles d'hypersurfaces projectives, Mourougane ([**33**]) a mis la technique en œuvre en toute dimension et pour tout degré.

4. Existence de différentielles de jets

La construction explicite ① de différentielles de jets globales étant difficile, on utilise plutôt les méthodes cohomologiques développées par Demailly et Diverio ([**7, 10, 12, 19**]) pour obtenir l'existence d'équations différentielles (*malheureusement non explicites*). La stratégie adoptée pour obtenir des équations différentielles est de s'assurer de l'existence de sections globales non nulles du fibré en droite $\mathcal{O}_{\overline{X}_k}(m) \otimes \pi^\star_{k,0} A^\vee$, pour $m \gg 1$.

Une approche, avec les fibrés de Schur ([**7**]), consiste à contrôler les groupes de cohomologie paire supérieurs H^{2i} afin d'utiliser le théorème de Riemann-Roch. Dans [**40**], en dimension 3, Rousseau est capable de borner la dimension de H^2 en utilisant les inégalités de Morse holomorphes ([**7, 49**]). Ensuite dans [**32**], Merker traite le cas de la dimension arbitraire pour des différentielles de jets avec un grand ordre.

Avec une approche différente, dans [**8**], Demailly traite le cas de la dimension arbitraire en utilisant une version forte des inégalités de Morse holomorphes.

Une autre approche, développée dans des contextes variés ([**11, 12, 13, 3, 33, 2**]), consiste à appliquer les inégalités de Morse holomorphes pour prouver l'existence de sections d'un certain sous-faisceau plus simple du faisceau des différentielles de jets. Ceci conduit à établir la positivité d'un certain nombre d'intersection au k-ème niveau de la tour de Demailly :

$$I = \int_{\overline{X}_\kappa} p\Big(\pi^\star_{\kappa,1} u_1, \dots, \pi^\star_{\kappa,\kappa-1} u_{k-1}, u_k\Big),$$

où p est un polynôme, à coefficients dans l'anneau de cohomologie de la base, dans les premières classes de Chern :

$$u_i := c_1\Big(\mathcal{O}_{\overline{X}_i}(1)\Big).$$

5. Dégénérescence algébrique effective

La première confirmation de la conjecture de Green-Griffiths en toute dimension est un résultat récent de Diverio, Merker et Rousseau ([**13**]), qui établit l'énoncé avec une borne effective en utilisant les fibrés de Demailly et une version algébrique des inégalités de Morse holomorphes. Ils démontrent le théorème suivant :

Théorème. *Soit $X_d \subset \mathbb{P}^{n+1}$ une hypersurface projective lisse générique de* dimension arbitraire $n \geqslant 2$. *Si le degré de X_d est plus grand que l'entier* explicite 2^{n^5}, *alors il existe une sous-variété propre $Z \subset X_d$, telle que toute courbe holomorphe entière $f\colon \mathbb{C} \to X_d$ prend en fait ses valeurs dans $Z \supset f(\mathbb{C})$.*

Leur preuve est basée de façon essentielle sur la stratégie de déformation développée par Siu, combinée avec les techniques cohomologiques de Demailly. L'idée directrice est de produire un grand nombre d'équations différentielles algébriques que toute courbe entière doit satisfaire. Ensuite, l'énoncé de dégénérescence algébrique est obtenu (moralement) par élimination algébrique de toutes les dérivées. Le procédé qui produit de nombreuses équations différentielles se décline en deux étapes principales :

① – ② L'existence de différentielles de jets invariants s'annulant sur un diviseur ample dans les hypersurfaces projectives de grand degré, par des méthodes cohomologiques, en suivant les travaux de Demailly et de Diverio [**7, 11, 12**].

③ La seconde étape du raisonnement de Siu. L'engendrement global (modulo tensorisation) du fibré tangent à la variété des n-jets verticaux, en suivant les travaux de Siu et de Păun-Rousseau-Merker.

En 2010, Gergely Bérczi [**3**] a importé des techniques de la géométrie équivariante pour simplifier les calculs et améliorer la borne connue sur le degré. Il montre que le résultat est toujours valide lorsque $d \geqslant n^{8n}$.

En 2012, en utilisant une technique innovante basée sur des estimées probabilistes, Jean-Pierre Demailly ([**9**]) a amélioré la borne sur le degré des hypersurfaces projectives, et il a obtenu :

$$d \geqslant \frac{n^4}{3}\Big(n\log(n\log(24n))\Big)^n.$$

6. Principaux résultats de ce mémoire

Le premier but de cette thèse est d'exporter et d'améliorer les techniques de [**13**] pour étudier la dégénérescence algébrique des courbes entières $f\colon \mathbb{C} \to \mathbb{P}^n \setminus X_d$ à valeurs dans le complémentaire d'une hypersurface X_d de l'espace projectif $\mathbb{P}^n$.

Formule de résidus pour la première étape. Pour calculer le produit d'intersection :

$$I = \int_{\overline{X}_\kappa} p\Big(\pi_{\kappa,1}^\star u_1, \dots, \pi_{\kappa,\kappa-1}^\star u_{k-1}, u_k\Big),$$

la stratégie standard consiste à intégrer le long des fibres des projections :

$$\pi_{i,i-1}\colon \overline{X}_i \to \overline{X}_{i-1},$$

jusqu'à obtenir un produit d'intersection sur la base, $\overline{X}_0$, où l'intersection des classes de cohomologie devient plus simple.

Dans [**13**], Diverio-Merker-Rousseau procèdent à une élimination pas à pas des classes de Chern, et ils sont capables de dénouer l'interaction complexe entre les classes de cohomologie horizontales et les classes de cohomologie verticales par un tour de force technique. Ces calculs précis donnent un résultat *effectif*.

Dans [**33**] et dans [**2**], Mourougane et Brotbek utilisent intelligement les *classes de Segre* pour éviter une grande partie des calculs.

Dans [**3**], Bérczi utilise la géométrie équivariante pour prouver une *formule de résidus* en plusieurs variables, qui évite l'élimination pas à pas des classes de cohomologie horizontales et qui donne un résultat effectif.

Dans le chapitre II, les idées inspirées de ces auteurs sont combinées, pour prouver une formule de résidus en plusieurs variables similaire, qui est vraie dans de nombreux contextes géométriques, puisqu'elle est démontrée dans toute les situations où la tour de Demailly apparaît, *cf. e.g.* [**33, 2**].

La preuve emprunte la simplification technique de l'usage des classes de Segre, elle donne un résultat effectif, et elle utilise une formule dans l'esprit même de la formule de résidus de Bérczi.

On note $s(V_0)$ la classe de Segre totale du fibré vectoriel V_0 :

$$s_t(V_0) = 1 + t\, s_1(V_0) + t^2\, s_2(V_0) + \cdots,$$

qui est l'inverse de la classe de Chern totale $c_t(V_0)$. Un des points clefs, central dans le chapitre II, consiste à introduire de nouveaux générateurs de la cohomologie verticale :

$$v_i = \left(\pi_{i,1}^\star u_1 + \cdots + \pi_{i,i-1}^\star u_{i-1} + u_i\right) \in H^\bullet(\overline{X}_i),$$

lesquels apparaissent *naturellement* quand on considère les suites exactes qui définissent la tour des fibrés de jets projectivisés :

$$\overline{X}_k \to \overline{X}_{k-1} \to \cdots \to \overline{X}_1 \to \overline{X}_0.$$

Notons alors r le rang du fibré projectivisé $P(V_0)$. L'énoncé suivant, utile pour l'étape ①, est démontré dans le chapitre II :

Théorème principal (II**).** *Pour tout polynôme $f \in H^\bullet\left(\overline{X}_0\right)[t_1, \dots, t_k]$, en k variables $t_1, \dots, t_k$, à coefficients dans l'anneau de cohomologie de la base $\overline{X}_0$, la classe de cohomologie :*

$$f\left(\underline{v}\right) = f\left(v_1, \dots, v_k\right) \in H^\bullet\left(\overline{X}_k\right),$$

peut être intégrée le long des fibres du fibré projectif $\overline{X}_k \to \overline{X}_0$ en utilisant la formule :

$$\int_{\overline{X}_k} f\left(\underline{v}\right) = \underset{t_1^r \cdots t_k^r}{\mathrm{Coefficient}}\Big(I(t_1, \dots, t_k)\, \Phi_k(t_1, \dots, t_k)\Big),$$

où $I(\underline{t})$ est le produit d'intersection sur la base :

$$I(t_1, \dots, t_k) := \int_{\overline{X}_0} f(t_1, \dots, t_k)\, s_{1/t_1}(V_0) \cdots s_{1/t_k}(V_0),$$

et où $\Phi_k(\underline{t})$ est la fonction rationnelle universelle :

$$\Phi_k(t_1, \dots, t_k) := \prod_{1 \leqslant i < j \leqslant k} \frac{t_j - t_i}{t_j - 2\, t_i} \prod_{2 \leqslant i < j \leqslant k} \frac{t_j - 2\, t_i}{t_j - 2\, t_i + t_{i-1}}.$$

Clarification et généralisation de la stratégie des champs de vecteurs obliques. Ensuite, dans le chapitre III, les champs de vecteurs obliques de Siu-Merker sont développés en toute dimension dans le cadre logarithmique afin d'appliquer la méthode de déformation de Siu. L'énoncé suivant, central dans l'étape ③, y est démontré :

Théorème principal (III). *Si l'ordre des jets k est plus petit que le degré d, alors le 'twist' du fibré tangent aux k-jets verticaux de la variété logarithmique $\big(\mathbb{P}^n \times S, \mathcal{H}\big)$:*

$$T_{J_k^{vert}(\mathbb{P}^n\times S)(-\log \mathcal{H})} \otimes \Big(\mathcal{O}_{\mathbb{P}^n}\big(k^2+2k\big)\otimes \mathcal{O}_S\big(1\big)\Big)$$

est engendré par ses sections holomorphes globales en tout point du sous-espace des k-jets logarithmiques verticaux réguliers de courbes holomorphes évitant $\mathcal{H}$.

De plus, certains détails laissés au lecteur dans les travaux antérieurs sont complétés dans une approche simplifiée. La preuve fournit donc une clarification du lieu où l'énoncé d'engendrement global est valide, à la fois dans le cas compact, et dans le cas logarithmique.

Conjecture de Green-Griffiths logarithmique. Dans le chapitre IV, la dégénérescence algébrique effective des courbes entières non constantes à valeurs dans le complémentaire d'une hypersurface lisse générique de degré suffisamment grand est établie en utilisant la stratégie ① – ② – ③ :

Théorème principal (IV). *Si $X_d \subset \mathbb{P}^n$ est une hypersurface projective lisse générique de degré :*

$$d \geqslant (5n)^2\, n^n,$$

alors il existe une sous-variété propre $Z \subset \mathbb{P}^n$, de codimension au moins deux, telle que l'image de toute courbe entière non constante $f\colon \mathbb{C} \to \big(\mathbb{P}^n \setminus X_d\big)$ à valeurs dans le complémentaire de X_d, est en fait contenue dans $\big(Z \setminus X_d\big)$.

En dimensions $n = 2$ [**19, 44**] et $n = 3$ [**42**], ce théorème est déjà connu avec des estimations meilleures sur le degré.

La borne effective de ce théorème est également valable pour des courbes à valeurs dans une hypersurface lisse générique. C'est donc une amélioration significative des résultats de la littérature (*cf. supra*).

Les introductions des chapitres II,III, IV décrivent plus en détails ces résultats.

Overview of the main results of this memoir

Motivation

A complex manifold X is said *hyperbolic* in the sense of Brody when there is no nonconstant entire map $f\colon \mathbb{C} \to X$. The theorem of Picard (1879) gives an example of such a manifold:

Theorem. *There is no nonconstant entire map* $f\colon \mathbb{C} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

The theorem of Green ([**21**]), generalizes the theorem of Picard and gives examples in larger dimension:

Theorem. *There is no nonconstant entire map* $f\colon \mathbb{C} \to \mathbb{P}^n \setminus H_1 \cup \cdots \cup H_{2n+1}$ *with values in the complement of* $2n+1$ *hyperplanes of* $\mathbb{P}^n$ *in general position.*

The notion of hyperbolicity can be weaken to get a softer notion, that is the *algebraic degeneracy* of entire curves. An entire curve $\mathbb{C} \to X$ with values in a algebraic complex manifold X is said *algebraically degenerate* if there exists a proper algebraic subvariety $Z \subsetneq X$ containing its image. In [**21**], Green actually proves the more general statement:

Theorem. *The image of an entire curve* $f\colon \mathbb{C} \to \mathbb{P}^n$ *avoiding* $n+k \geqslant n+1$ *hyperplanes in general position is contained in a linear projective subspace having dimension at most the entire part in* $\frac{n}{k}$. *This bound* $\lfloor \frac{n}{k} \rfloor$ *is sharp.*

As a consequence the entire curves with values in the complement of $n+2$ hyperplanes in general position in $\mathbb{P}^n$ are algebraically degenerate.

It is interesting to consider more general hypersurfaces than unions of hyperplanes. One motivation is the following conjecture ([**26**]), that we deliberately restrain to the case of Brody hyperbolicity, considering the goals of this memoir:

Conjecture (Kobayashi). *There is no nonconstant entire map with values in the complement* $\mathbb{P}^n \setminus X_d$ *of a generic hypersurface* X_d *having degree* $d \geqslant 2n+1$.

Here, *generic* means that the coefficients of the defining equation of X_d must lie outside of a certain proper algebraic subvariety of the projective space S of all coefficients of homogeneous polynomial of degree d:

$$S := P\Big(\Gamma\Big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)\Big)\Big),$$

that parametrizes the algebraic hypersurface with degree d in $\mathbb{P}^n$. The bound $2n+1$ is justified both by the theorem of Green and by the work of Zaĭdenberg ([**51**]), where it is shown that the complement of a generic hypersurface of degree $2n$ in $\mathbb{P}^n$ is *not* hyperbolic.

Regarding algebraic degeneracy, the Green-Griffiths conjecture ([**22**]) assert, in a wide context, that any smooth algebraic variety X of general type contains a certain proper algebraic subvariety $Z \subsetneq X$, inside which all nonconstant entire map $f\colon \mathbb{C} \to X$

must necessarily lie. In the case of projective hypersurfaces, considered in this memoir, this conjecture becomes:

Conjecture (Green-Griffiths). *The image of an entire curve* $f\colon \mathbb{C} \to X_d$ *with values in a hypersurface* X_d *of* $\mathbb{P}^{n+1}$ *with degree* $d \geqslant n+3$ *is not Zariski-dense in* X_d.

Conjecture (logarithmic Green-Griffiths). *The image of an entire curve* $f\colon \mathbb{C} \to \mathbb{P}^n \setminus X_d$ *with values in the complement in* P^n *of a hypersurface* X_d *having degree* $d \geqslant n+2$ *is not Zariski-dense in* $\mathbb{P}^n$.

The strong version of these conjectures asserts that the proper algebraic subvariety Z of the goal space containing the image $f(\mathbb{C}) \subset Z$ does not depend on the nonconstant entire curve f.

In this memoir, a positive answer is given to the strong logarithmic Green-Griffiths conjecture, for generic hypersurfaces having degree $d \geqslant (5n)^2\, n^n$.

Jet Techniques

Following the seminal paper of 1926 by Bloch [**1**] on entire curves with values in abelian varieties, the main approach to treat hyperbolicity problems is the use of many differential equations satisfied by every entire curve. The problem is then reduced to an algebraic elimination, and it is sufficient to show that there are many more unknowns than constraints, in order to get a non zero analytic equation satisfied by any nonconstant entire curve.

The formalism of *jets* is a coordinate free description of these differential equations. For a holomorphic map $f\colon \mathbb{C} \to X$, with values in a complex manifold X and a coordinate system $X \simeq \mathbb{C}^n$, the k-jet of f corresponds to its truncated Taylor expansion at order k. The local notion of polynomial in the Taylor coefficients $f_i', f_i'', \dots, f_i^{(k)}$ of f, having homogeneous weight m for the number of "primes" has a global meaning, as it is shown by the Faà di Bruno formulae. The gluing of such polynomials is called a *jet differential* of degree k and of weight m.

The following hyperbolicity statement is proven by Bloch:

Theorem. *Let* X *be an abelian variety. If* $Y \subset X$ *is a subvariety that contains no translate of an abelian subvariety of* X*, then there is no nonconstant entire map* $f\colon \mathbb{C} \to Y$ *with values in* Y.

The proof of Bloch consists in three steps.

① The explicit construction of holomorphic jet differentials, that is polynomials in the jet variables $f_i^{(j)}$ with holomorphic coefficients, vanishing on an ample divisor $A \subset X$.

② The following *fundamental vanishing theorem*, that links jet differentials and hyperbolicity:

Theorem (Ahlfors-Schwarz lemma). *If* ω *is a holomorphic jet differential vanishing on an ample divisor* $A \subset X$*, and* $f\colon \mathbb{C} \to X$ *is a nonconstant entire curve, then the pullback* $f^\star\omega \equiv 0$ *vanishes identically on* $\mathbb{C}$.

③ The determining of the degeneracy locus: the translate of an abelian subvariety (that shall be empty in order to obtain hyperbolicity).

Fifty years later, Philip Green and Mark Griffiths ([**22**]) modernize the concepts of Bloch and establish several fundamental results on the geometry of entire curves. In 1979, they prove the following statement:

Theorem. *If X is an algebraic complex manifold such that $q > \dim X$, where $q = \dim H^1(\overline{X}, \mathcal{O})$, for any compactification $\overline{X}$ of X, then the image of any entire map $\mathbb{C} \to X$ is contained in a proper algebraic subvariety of X.*

This result had already been announced by Bloch, and proven in certain particular cases. Ochiai had already studied certain particular cases. Like Bloch and Ochiai, Green and Griffiths use *jet techniques*. They define jet pseudo metrics, and the notion of jet pseudo metrics with holomorphic sectional curvature $\leqslant -1$. The proof of the Ahlfors-Schwarz lemma implies that if X is equipped with such a metric on its k-jets, then every entire curve $f \colon \mathbb{C} \to X$ is algebraically degenerate in the degeneracy locus of the metric (step ②). Green and Griffiths choose carefully the pseudo metric in order to control its degeneracy locus (step ③).

In 1995 and also later, Demailly([**7**]) has developed a technique of *invariant jets*, pushing further and improving the approach of Green and Griffiths. New geometric objects are introduced. A *directed complex manifold* is by definition a holomorphic subbundle $V \to X$ of the holomorphic tangent bundle $T_X \to X$ to a complex manifold X. The hyperbolicy of (X, V) is treated by the method of negative curvature: by Ahlfors-Schwarz lemma, the existence of an hermitian metric with negative curvature on the tautological line bundle $\mathcal{O}_{P(V)}(-1)$ yields the non existence of a nonconstant entire curve $f \colon \mathbb{C} \to X$ tangent to V. The most interesting case is the case where the subbundle $V = T_X$ is the whole tangent bundle, because the hyperbolicity of X is obtained.

Under suitable positivity assumption on the dual $V^\star$, it is possible to construct such a metric, as an example the following statement holds:

Theorem. *Let (X, V) be a directed complex manifold. If $V^\star$ is ample then any entire curve $f \colon \mathbb{C} \to X$ tangent to V is constant.*

In order to generalize the results to k-jets, Demailly introduces a tower of projective bundles. This contruction and its properties have been generalized to the logarithmic setting by Dethloff and Lu ([**10**]).

Let $\overline{X}$ be a complex manifold and let D be a simple normal crossings divisor on $\overline{X}$, that is $D = \sum D_i$, where the components D_i are smooth irreducible divisors intersecting transversally. Such a pair (X, D) is called a logarithmic pair. Denote by $T_{\overline{X}}(-\log D)$ the *logarithmic tangent bundle to* $\overline{X}$ *along* D ([**35**]). For any subbundle V of the logarithmic tangent bundle:

$$V \subset T_{\overline{X}}(-\log D) \subset T_{\overline{X}},$$

and for any fixed order $k \in \mathbb{N}$, the *logarithmic Demailly tower* of projectivized jet bundles ([**7, 10**]):

$$\left(\overline{X}_k, D_k, V_k\right) \to \left(\overline{X}_{k-1}, D_{k-1}, V_{k-1}\right) \to \ldots \to \left(\overline{X}_1, D_1, V_1\right) \to \left(\overline{X}_0, D_0, V_0\right) := \left(\overline{X}, D, V\right),$$

is a construction with the main property that every nonconstant curve $f \colon \Delta_R \to \overline{X} \setminus D$ tangent to V lifts as a curve $f_{[k]} \colon \Delta_R \to \overline{X}_k \setminus D_k$ tangent to V_k, depending only on the k-jet of f. This tower is of great importance in the study of the algebraic degeneracy of entire curves on $\overline{X} \setminus D$ (*cf.* the enlightening surveys [**14, 38**]). Indeed, the first step toward the proof of algebraic degeneracy of entire curves is to prove the existence of a non zero polynomial P on $\overline{X}$ such that every nonconstant entire curve $f\colon \mathbb{C} \to \overline{X} \setminus D$ satisfies the algebraic differential equation:

$$P\Big(f, f', f'', \dots, f^{(k)}\Big) \equiv 0.$$

Being by definition a projective vector bundle, the manifold $\overline{X}_k$ comes naturally equipped with a tautological line bundle, $\mathcal{O}_{\overline{X}_k}(-1)$. Let $\pi_{k,0}$ denote the natural projection from the k-th level of the Demailly tower to the basis. The direct images:

$$\mathcal{E}_{k,m} V^\star := (\pi_k)_\star \mathcal{O}_{\overline{X}_k}(m)$$

may be seen as vector bundles of differential operators of order k and weighted degree m acting on germs of holomorphic curves in X tangent to V and invariant under reparametrization of the source. Then, J.-P. Demailly has established a fundamental vanishing theorem ([**7, 10**]), asserting that for any global holomorphic section:

$$P \in H^0\Big(\overline{X}_k, \mathcal{O}_{\overline{X}_k}(m) \otimes \pi_{k,0}^\star A^\vee\Big) = H^0\Big(\overline{X}, \mathcal{E}_{k,m} V^\star \otimes A^\vee\Big),$$

with values in the dual $A^\vee$ of an ample line bundle on $\overline{X}$, every nonconstant entire curve $f\colon \mathbb{C} \to \overline{X} \setminus D$ tangent to V satisfies the differential equation:

$$P\Big(f, f', f'', \dots, f^{(k)}\Big) \equiv 0.$$

This vanishing theorem (step ② in Bloch's proof) allows an alternative approach to the explicit construction of global holomorphic jet differentials (step ① in Bloch's proof). The problem reduces to the study of global holomorphic sections of the sheaf $\mathcal{O}_{\overline{X}_k}(m)$ vanishing on A. However, the lack of information on the obtained jet differentials make step ③ of Bloch's proof harder. In certain situation, Diverio and Rousseau ([**16**]) have even given examples where the obtained differential equations give no information on the basis X_0.

The algebra of invariant jets has been studied by Rousseau ([**41**]), by Merker ([**28**]), and by Bérczi and Kirwan ([**5**]) and is complicated.

Siu's Strategy

In 1996 ([**48**]), in 2002 ([**45**]) and in 2004 ([**46**]), Siu has treated the case of projective hypersurfaces of dimension 2 (for non optimal degrees) and has proposed adequate general ideas in order to treat the case of arbitrary dimension $n \geqslant 2$. Siu uses a strategy in two steps:

① – ② The explicit construction in projective coordinates of global holomorphic jet differentials vanishing on an ample divisor, in the spirit of Bloch.

③ The deformation of the obtained sections, in order to get sufficiently many differential equations. This second step is inspired by the works of Ein-Clemens-Voisin ([**6, 18, 50**]) on rational curves, in which it is established that:

Theorem. *There is no nonconstant rational curve $f\colon \mathbb{P}^1 \to X_d$ with values in a generic hypersurface X_d of P^n having degree $d \geqslant 2n-1$.*

In order to deform the differential equations, the universal hypersurface of degree d is considered:

$$\mathcal{H} := \Big\{Z, P_s\colon P_s(Z) = 0\Big\} \subset \mathbb{P}^n \times S.$$

When ω is a family of k-jet differentials on the fibers of the second projection, such that ω_s vanishes on an ample divisor $A \subset H_s$, and V is a meromorphic vector field tangent to the space of vertical k-jets of $\mathcal{H}$, with poles having order less than the vanishing order of ω, the Lie derivative $(V \cdot \omega)_s$ still satisfies the hypotheses of the Ahlfors-Schwarz lemma, and thus corresponds to a differential equation.

Sufficiently many algebraically independent equations are obtained by considering *slanted* vector fields, that is vector fields with a non zero component in the direction of the space of parameters S, if the tangent space to vertical jets has a meromorphic frame of low pole order.

The method of slanted vector fields introduced by Siu ([**46**]), nicely described in [**13, 14, 38**], has been pushed further by Păun ([**37**]) in dimension 2, and by Rousseau in dimension 3, both for the compact case ([**43**]) and for the logarithmic case ([**42**]). In the compact case, the technique has been generalized in any dimension by Merker ([**29**]), with a substantial improvement of the determination of the locus where the global generation statement ③ holds, leading to a proof of the *strong* algebraic degeneracy of entire curves with values in a generic projective hypersurface of large degree ([**13**]). In the slightly different context of projective hypersurfaces in families, Mourougane ([**33**]) has implemented the technique in any dimension and for any order.

Existence of Global Jet Differentials

The explicit construction of global jet differentials ① being a delicate problem, the cohomological methods developed by Demailly and Diverio ([**7, 10, 12, 19**]) are a nice alternative in order to obtain the existence of differential equations (*unfortunately non explicit ones*). One has thus to ensure the existence of global sections of the line bundle $\mathcal{O}_{\overline{X}_\kappa}(m) \otimes \pi^\star_{\kappa,0} A^\vee$, possibly with $m \gg 1$.

One approach, with Schur bundles ([**7**]), consists in bounding positive even cohomology groups H^{2i} in order to use the Riemann-Roch theorem. In [**40**], in dimension 3, Rousseau is able to bound the dimension of H^2 by use of the famous algebraic Morse inequalities ([**7, 49**]). Later in [**32**], Merker completes the case of arbitrary dimension, for high order jet differentials.

With a different approach, in [**8**], Demailly treats the case of arbitrary dimension by use of a stronger version of algebraic Morse inequalities.

Another approach, developed in various contexts ([**11, 12, 13, 3, 33, 2**]), consists in applying the holomorphic Morse inequalities in order to prove the existence of sections of a certain more tractable subsheaf of the sheaf of jet differentials. One is led to establish the positivity of a certain intersection number on the k-th level of the Demailly tower:

$$I = \int_{\overline{X}_k} p\Big(\pi^\star_{\kappa,1} u_1, \ldots, \pi^\star_{\kappa,\kappa-1} u_{k-1}, u_k\Big),$$

where p is a polynomial with coefficients in the cohomology ring of the basis, in the first Chern classes:

$$u_i := c_1\Big(\mathcal{O}_{\overline{X}_i}(1)\Big).$$

Effective Algebraic Degeneracy

The first result confirming the Green-Griffiths conjecture *in any dimension* is a recent result of Diverio, Merker et Rousseau ([**13**]), confirming the statement for projective hypersurfaces of large degree with an effective sufficient lower bound on the degree.

Theorem. *Let $X_d \subset \mathbb{P}^{n+1}$ be a generic smooth hypersurface of dimension $n \geqslant 2$. If the degree of X_d is larger than the explicit integer 2^{n^5}, then there exists a proper subvariety $Z \subset X_d$, containing the image of every nonconstant entire curve $f\colon \mathbb{C} \to X_d$.*

Their proof is based in an essential way on the deformation method of Siu, combined with techniques of Demailly and Diverio. The overall idea is to produce sufficiently many differential algebraic equations, that all entire curves has to satisfy. Then, the degeneracy statement is obtained by an algebraic elimination of all derivatives. The process producing many differential equations is in two steps:

①–② The existence of invariant jet differentials vanishing on an ample divisor, following works by Demailly and by Diverio [**7, 11, 12**].

③ The second step in Siu's strategy: the global generation (modulo twist) of the tangent bundle to the vertical n-jet manifold, that allows to deform the differential equations, following works by Siu and Păun-Rousseau-Merker.

In 2010, Gergely Bérczi ([**3**]) used equivariant geometry to simplify the computations and to improve the sufficient lower bound. He shows that the result still holds for $d \geqslant n^{8n}$.

In 2012, using an innovative technique based on probabilistic estimates Jean-Pierre Demailly ([**9**]) improved the sufficient lower bound on the degree of projective hypersurfaces to:

$$d \geqslant \frac{n^4}{3}\Big(n \log(n \log(24n))\Big)^n.$$

The first goal of this memoir is to export and improve the techniques of [**13**] to study the algebraic degeneracy of holomorphic entire curves $f\colon \mathbb{C} \to \mathbb{P}^n \setminus X_d$ with values in the complement of an algebraic hypersurface X_d of the projective space $\mathbb{P}^n$.

Residue Formula

We first provide a residue formula that simplify step ①. When computing the intersection number:

$$I = \int_{\overline{X}_\kappa} p\Big(\pi^\star_{\kappa,1} u_1, \dots, \pi^\star_{\kappa,\kappa-1} u_{k-1}, u_k\Big),$$

the standard strategy is to integrate along the fibers of the projections $\pi_{i,i-1}\colon \overline{X}_i \to \overline{X}_{i-1}$, until one obtains an intersection product on the basis $\overline{X}_0$, where the intersection of cohomology classes becomes simpler.

In [**13**], Diverio-Merker-Rousseau use step-by-step elimination of Chern classes, and are able to disentangle the complex intrication between horizontal and vertical cohomology classes by a technical tour de force. These precise computations yield *effectivity*.

In [**33**] and in [**2**], Mourougane et Brotbek make a clever use of *Segre classes* in order to avoid a large part of the computations.

In [**3**], Bérczi uses equivariant geometry in order to prove a *residue formula* in several variables, that avoids step-by-step elimination and yields effectivity.

In Chapter II, we combine ideas coming from these authors, in order to prove a similar residue formula in several variables, that is valid in a versatile geometric context, since it holds in any situation where the Demailly tower appears, *cf. e.g.* [**33, 2**].

Our proof borrows the technical simplification of the use of Segre classes, it yields computational effectivity, and it is in the very spirit of the residue formula of Bérczi.

Let $s(V_0)$ denote the total Segre class of the vector bundle $V_0 \subset T_{X_0}$:

$$s_t(V_0) = 1 + t\, s_1(V_0) + t^2\, s_2(V_0) + \cdots,$$

that is the inverse of the total Chern class $c_t(V_0)$. One key point, central in chapter II, is the introduction of new generators for the vertical cohomology:

$$v_i = \Big(\pi_{i,1}^{\star} u_1 + \cdots + \pi_{i,i-1}^{\star} u_{i-1} + u_i\Big) \in H^{\bullet}(\overline{X}_i),$$

that appear *naturally* by considering the short exact sequences defining the tower of projectivized bundles:

$$\overline{X}_k \to \overline{X}_{k-1} \to \cdots \to \overline{X}_1 \to \overline{X}_0.$$

Denote by r the rank of the projective bundle of lines $P(V)$. The following statement, useful for step ①, is established in chapter II:

Main Theorem (II). *For any polynomial f in k variables $t_1, \dots, t_k$ having coefficients in the cohomology ring of the basis, the cohomology class:*

$$f\big(\underline{v}\big) = f\big(v_1, \dots, v_k\big) \in H^{\bullet}\big(\overline{X}_k\big),$$

can be integrated along the fibers of the projective bundle $\overline{X}_k \to \overline{X}_0$ using the formula:

$$\int_{\overline{X}_k} f\big(\underline{v}\big) = \operatorname*{Coefficient}_{t_1^r \cdots t_k^r}\Big(I(t_1, \dots, t_k)\, \Phi_k(t_1, \dots, t_k)\Big)$$

where $I(\underline{t})$ is the intersection product on the basis:

$$I(t_1, \dots, t_k) = \int_{\overline{X}_0} f(t_1, \dots, t_k)\, s_{1/t_1}(V_0) \cdots s_{1/t_k}(V_0),$$

and where $\Phi_k(\underline{t})$ is the universal rational function:

$$\Phi_k(t_1, \dots, t_k) = \prod_{1 \leqslant i < j \leqslant k} \frac{t_j - t_i}{t_j - 2\,t_i} \prod_{2 \leqslant i < j \leqslant k} \frac{t_j - 2\,t_i}{t_j - 2\,t_i + t_{i-1}}.$$

Slanted Vector Fields

Next, the strategy of slanted vector fields used in step ③ is generalized and clarified. *In Chapter III*, the Siu-Merker slanted vector fields are developed for any dimension in the logarithmic setting so as to apply the deformation method of Siu. The following statement, central in step ③, is established there:

Main Theorem (III). *Suppose that the order k of the jets is smaller than the degree d, then the twisted holomorphic tangent bundle to the vertical k-jets of the log-manifold $\big(\mathbb{P}^n \times S, \mathcal{H}\big)$:*

$$T_{J_k^{vert}(\mathbb{P}^n \times S)(-\log \mathcal{H})} \otimes \Big(\mathcal{O}_{\mathbb{P}^n}\big(k^2 + 2k\big) \otimes \mathcal{O}_S\big(1\big)\Big)$$

is generated by its global holomorphic sections at every point of the subspace of regular vertical logarithmic k-jets of holomorphic curves avoiding $\mathcal{H}$.

Moreover, some details left to the reader in the preceding works are treated with a simplified new approach. The proof provides thus a clarification of the locus where the global generation statement holds, both in the compact case, and in the logarithmic case.

Logarithmic Green-Griffiths Conjecture

In Chapter IV, the effective algebraic degeneracy of nonconstant entire curves with values in the complement of a generic smooth projective hypersurface having sufficiently large degree is established using the strategy ①–②–③:

Main Theorem (IV). *If $X_d \subset \mathbb{P}^n$ is a generic smooth projective hypersurface having degree:*

$$d \geqslant (5n)^2\, n^n,$$

then there exists a proper subvariety $Z \subset \mathbb{P}^n$, of codimension at least two, such that the image of every nonconstant entire curve $f : \mathbb{C} \to \big(\mathbb{P}^n \setminus X_d\big)$ having values in the complement of X_d, actually lies in $(Z \setminus X_d)$.

In dimensions $n = 2$ [**19, 44**] and $n = 3$ [**42**], this theorem is already known, with lower estimates on the degree.

The effective lower bound in this theorem also holds for curves with values in a generic projective hypersurface. Thus, this is a significant improvement of the results in the literature (*cf. supra*).

These results are described in more detail in the introductions of chapters II, III, IV.

CHAPTER II

Invariant Jets and Segre Classes

Contents

1. Introduction

Let $\overline{X}$ be a complex manifold and let D be a divisor on $\overline{X}$ with *normal crossings*, that is $D = \sum D_i$, where the components D_i are smooth irreducible divisors that meet transversally. For such a pair $\big(\overline{X}, D\big)$, one denotes by $T_{\overline{X}}\big(-\log D\big)$ the *logarithmic tangent bundle of* $\overline{X}$ *along* D ([**35**]).

Given a subbundle:

$$V \subset T_{\overline{X}}(-\log D) \subset T_{\overline{X}}$$

of the logarithmic tangent bundle, one constructs ([**7, 10**]), for any fixed order $\kappa \in \mathbb{N}$, the *logarithmic Demailly tower* of projectivized bundles:

$$\big(\overline{X}_\kappa, D_\kappa, V_\kappa\big) \to \big(\overline{X}_{\kappa-1}, D_{\kappa-1}, V_{\kappa-1}\big) \to \ldots \to \big(\overline{X}_1, D_1, V_1\big) \to \big(\overline{X}_0, D_0, V_0\big) := \big(\overline{X}, D, V\big),$$

having the main property that every holomorphic map $g\colon \mathbb{C} \to \overline{X} \setminus D$ lifts as maps:

$$g_{[i]}\colon \mathbb{C} \to \overline{X}_i \setminus D_i \qquad {\scriptstyle (i=0,1,\ldots,\kappa)},$$

which depends only on the corresponding i-jet of g. Later on in section §2, we will describe precisely this construction, central here.

For any two integers $j, k \in 0, 1, \ldots, \kappa$, the composition of the projections $\pi_i\colon \overline{X}_i \to \overline{X}_{i-1}$ yields a natural projection from the j-th level of the Demailly tower to the lower k-th level:

$$\pi_{j,k} := \pi_{k+1} \circ \cdots \circ \pi_j\colon \overline{X}_j \to \overline{X}_k.$$

The Demailly tower is of great importance in the study of the algebraic degeneracy of entire curves on $\overline{X} \setminus D$ (*cf.* the enlightening surveys [**14, 38**]). A first step towards the proof of algebraic degeneracy of entire curves is to prove the existence of a non zero polynomial P on $\overline{X}$ such that every non constant entire curve $g\colon \mathbb{C} \to \overline{X} \setminus D$ satisfies the algebraic differential equation:

$$P_{g(t)}\Big(g'(t), g''(t), \ldots, g^{(\kappa)}(t)\Big) = 0, \quad \text{for all } t \in \mathbb{C}.$$

Being by definition a projective vector bundle, the manifold $\overline{X}_i$ comes naturally equipped with a tautological line bundle, $\mathcal{O}_{\overline{X}_i}(-1)$, the multiples of which are usually denoted by $\mathcal{O}_{\overline{X}_i}(m) := (\mathcal{O}_{\overline{X}_i}(-1)^\vee)^{\otimes m}$. The direct image:

$$\mathcal{O}\Big(E_{\kappa,m}(V_0)^\star(\log D_0)\Big) := (\pi_{\kappa,0})_\star \mathcal{O}_{\overline{X}_\kappa}(m)$$

is the sheaf of sections of a holomorphic bundle $E_{\kappa,m}(V_0)^\star(\log D_0)$, called the *Demailly-Semple bundle of jet differentials*, and a fundamental vanishing theorem ([**7, 10**]) states that for every global section:

$$P \in H^0\Big(\overline{X}_\kappa, \mathcal{O}_{\overline{X}_\kappa}(m) \otimes \pi_{\kappa,0}^\star A^\vee\Big) \simeq H^0\Big(\overline{X}, E_{\kappa,m}(V_0)^\star(\log D_0) \otimes A^\vee\Big),$$

with values in the dual $A^\vee$ of an ample line bundle $A \to \overline{X}$, one has as desired, for any κ-jet $(g', g'', \dots, g^{(\kappa)})$ of non constant entire map $g\colon \mathbb{C} \to \overline{X} \setminus D$:

$$P_{g(t)}\Big(g'(t), g''(t), \dots, g^{(\kappa)}(t)\Big) = 0, \quad \text{for all } t \in \mathbb{C}.$$

One has thus to ensure the existence of global sections of the line bundle $\mathcal{O}_{\overline{X}_\kappa}(m) \otimes \pi_{\kappa,0}^\star A^\vee$, possibly with $m \gg 1$.

One approach, with Schur bundles ([**7**]), consists in bounding positive even cohomology groups H^{2i} in order to use the Riemann-Roch theorem. In [**40**], in dimension 3, the author is able to bound the dimension of H^2 by use of the famous algebraic Morse inequalities ([**7, 49**]). Later in [**32**], the case of arbitrary dimension is completed, for high order jet differentials.

With a different approach, in [**8**] the case of arbitrary dimension is completed by use of a stronger version of algebraic Morse inequalities.

Another approach, developed in various contexts ([**3, 2, 11, 12, 13**]), consists in applying the holomorphic Morse inequalities in order to prove the existence of sections of a certain more tractable subbundle of the bundle of jet differentials. One is led to establish the positivity of a certain intersection number on the κ-th level of the Demailly tower:

$$I = \int_{\overline{X}_\kappa} f\Big(c_1\big(\pi_{\kappa,1}^\star \mathcal{O}_{\overline{X}_1}(1)\big), \dots, c_1\big(\pi_{\kappa,\kappa-1}^\star \mathcal{O}_{\overline{X}_{\kappa-1}}(1)\big), c_1\big(\mathcal{O}_{\overline{X}_\kappa}(1)\big)\Big),$$

where f is a polynomial of large degree:

$$\deg(f) = n + \kappa\,(\mathrm{rk}\ P(V)) = \dim(\overline{X}_\kappa),$$

in the first Chern classes $c_1\big(\pi_{\kappa,i}^\star \mathcal{O}_{\overline{X}_i}(1)\big)$.

When computing this intersection number, the standard strategy is to integrate along the fibers of the projections $\pi_{i,i-1}\colon \overline{X}_i \to \overline{X}_{i-1}$, until one obtains an intersection product on the basis $\overline{X}_0$, where the intersection of cohomology classes becomes simpler.

In [**13**], the authors use step-by-step elimination of Chern classes, and are able to disentangle the complex intrication between horizontal and vertical cohomology classes by a technical tour de force. These precise computations yield *effectivity*.

In [**2**], the author makes a clever use of *Segre classes* in order to avoid a large part of the computations, but on the other hand, effectivity cannot be reached.

In [**3**], the author uses equivariant geometry in order to prove a *residue formula* in several variables, that avoids step-by-step elimination and yields effectivity.

In the present paper, we combine ideas coming from these authors, in order to prove a similar residue formula in several variables, that is valid in a versatile geometric context, since it holds in any situation where the Demailly tower appears, *cf. e.g.* [**33, 2**]. Our proof borrows from [**2**] the technical simplification of the use of Segre classes, it yields computational effectivity as in [**13**], and it is in the very spirit of the formula of [**3**].

To enter into the details, by the Leray-Hirsch theorem ([**24**]), the cohomology ring $H^\bullet\big(\overline{X}_\kappa\big)$ of $\overline{X}_\kappa$ is the free module generated by the first Chern classes $c_1\big(\pi^\star_{\kappa,i}\mathcal{O}_{\overline{X}_i}(-1)\big)$ over the cohomology ring $H^\bullet\big(\overline{X}_0\big)$ of the basis $\overline{X}_0$, but the implementation of the computation (*cf.* 3.2) suggests to naturally consider a different basis for the vertical cohomology by introducing the line bundles:

$$L_i := \mathcal{O}_{\overline{X}_i}(-1) \otimes \pi^\star_{i,i-1}\mathcal{O}_{\overline{X}_{i-1}}(-1) \otimes \cdots \otimes \pi^\star_{i,1}\mathcal{O}_{\overline{X}_0}(-1) \qquad {\scriptstyle (i=1,\ldots,\kappa)}.$$

We will use the notation v_i for the first Chern class of the dual of this line bundle L_i (dropping the pullbacks):

$$v_i := c_1\big(L_i^\vee\big) = c_1\big(\mathcal{O}_{\overline{X}_i}(1)\big) + \cdots + c_1\big(\mathcal{O}_{\overline{X}_1}(1)\big) \qquad {\scriptstyle (i=1,\ldots,\kappa)}.$$

Note that this formula looks like a plain change of variables having inverse:

$$c_1\big(\mathcal{O}_{\overline{X}_i}(1)\big) = v_i - v_{i-1} \qquad {\scriptstyle (i=2,\ldots,\kappa)},$$

thus, clearly, the polynomial f appearing in the intersection product I above has also a polynomial expression in terms of $v_1, \ldots, v_\kappa$. We will shortly provide a formula in order to integrate a polynomial under this new form, still denoted f.

Let K be a field. A *multivariate formal series* in κ variables with coefficients in K is a collection of coefficients in K, indexed by $\mathbb{Z}^\kappa$:

$$\Psi\colon \mathbb{Z}^\kappa \to K.$$

The space of formal series is naturally a K-vector space.

In analogy with polynomials, it is usual to denote, without convergence consideration:

$$\Psi(t_1, \ldots, t_\kappa) := \sum_{i_1,\ldots,i_\kappa\in\mathbb{Z}} \Psi(i_1, \ldots, i_\kappa)\, {t_1}^{i_1} \cdots {t_\kappa}^{i_\kappa},$$

hence, in order to avoid confusion, we will write:

$$\big[t_1^{i_1} \cdots t_\kappa^{i_\kappa}\big]\big(\Psi(t_1, \ldots, t_\kappa)\big) := \Psi(i_1, \ldots, i_\kappa),$$

to extract the coefficient indexed by $i_1, \ldots, i_\kappa$, that is the coefficient of the monomial $t_1^{i_1} \cdots t_\kappa^{i_\kappa}$ in the expansion of $\Psi(t_1, \ldots, t_\kappa)$. The *support* of the formal series Ψ is the subset of indices at which Ψ is non zero:

$$\operatorname{supp}(\Psi) := \big\{\underline{i} \in \mathbb{Z}^\kappa \colon [t_1^{i_1} \cdots t_\kappa^{i_\kappa}]\big(\Psi\big) \neq 0\big\}.$$

One defines the *Cauchy product* $\Psi_1\Psi_2$ of two formal power series:

$$\Psi_1\Psi_2\colon (i_1, \ldots, i_\kappa) \mapsto \sum_{\underline{j}+\underline{k}=\underline{i}} [t_1^{j_1} \cdots t_\kappa^{j_\kappa}]\big(\Psi_1\big)\, [t_1^{k_1} \cdots t_\kappa^{k_\kappa}]\big(\Psi_2\big), \tag{1.0.1}$$

whenever the displayed sum is a finite sum for each κ-tuple:

$$\underline{i} := i_1, \ldots, i_\kappa.$$

For a fixed partial ordering on $\mathbb{Z}^\kappa$, when considering only the series having well ordered support, the Cauchy product of two such series is always meaningful, since the computation of the coefficient of each monomial involves only finitely many terms. Moreover, for each choice of partial ordering, the set of formal series having well ordered support, equipped with the Cauchy product, forms a field ([**36**, Theorem 13.2.11]).

We give two examples of such fields. A *multivariate Laurent series* is a multivariate formal series, the support of which is well ordered for the standard product order on $\mathbb{Z}^\kappa$. An *iterated Laurent series* is a multivariate formal series, the support of which is well ordered for the lexicographic order on $\mathbb{Z}^\kappa$. The field of iterated Laurent series is an extension of the field of multivariate Laurent series.

After several Laurent expansions at the origin, any rational function becomes an iterated Laurent series (but not necessarily a multivariate Laurent series), as we will explain in more details later in §3.

We come back to the subbundle $V = V_0 \subset T_{\overline{X}_0}\big(-\log D_0\big)$. The *total Segre class* of this bundle $V_0 \to \overline{X}_0$:

$$s_\bullet(V_0) = 1 + s_1(V_0) + s_2(V_0) + \cdots + s_{\dim(\overline{X}_0)}(V_0)$$

is the inverse of the total Chern class of V_0 in $H^\bullet(\overline{X}_0)$. This notion is strongly related to integration along the fibers of a projective vector bundle ([**20**]). We will be more explicit about this relation below in §3.

We are now in position to state the main result of this chapter. Recall for $i = 1, \ldots, \kappa$ the notation $v_i := c_1(L_i^\vee)$, set:

$$r := \mathrm{rk}\ P(V) = \mathrm{rk}\ V - 1,$$

and introduce the (finite) generating series:

$$s_u(V_0) = 1 + u\, s_1(V_0) + u^2\, s_2(V_0) + \cdots + u^{\dim(\overline{X}_0)}\, s_{\dim(\overline{X}_0)}(V_0).$$

Main Theorem. *For any polynomial:*

$$f \in H^\bullet(\overline{X}_0, V_0)[t_1, \ldots, t_\kappa],$$

in κ variables $t_1, \ldots, t_\kappa$, with coefficients in the cohomology ring $H^\bullet bigl(\overline{X}_0, V_0\big)$, the intersection number:

$$I := \int_{\overline{X}_\kappa} f(v_1, \ldots, v_\kappa)$$

is equal to the Cauchy product coefficient:

$$I = \Big[t_1^r \cdots t_\kappa^r\Big]\Big(\Phi_\kappa\big(t_1, \ldots, t_\kappa\big)\, I(t_1, \ldots, t_\kappa)\Big) \qquad [r = \mathrm{rk}\ P(V_0) = \cdots = \mathrm{rk}\ P(V_\kappa)],$$

where $\Phi_\kappa(t_1, \ldots, t_\kappa)$ is the universal rational function:

$$\Phi_\kappa(t_1, \ldots, t_\kappa) = \prod_{1 \leqslant i < j \leqslant \kappa} \frac{t_j - t_i}{t_j - 2\,t_i} \prod_{2 \leqslant i < j \leqslant \kappa} \frac{t_j - 2\,t_i}{t_j - 2\,t_i + t_{i-1}},$$

and where $I(t_1, \ldots, t_\kappa)$ is the multivariate Laurent polynomial involving only explicit data of the base manifold:

$$I(t_1, \ldots, t_\kappa) = \int_{\overline{X}_0} f(t_1, \ldots, t_\kappa)\, s_{1/t_1}(V_0) \cdots s_{1/t_\kappa}(V_0).$$

Concretely, the computation of this intersection number I, on the κ-th level $\overline{X}_\kappa$, can be brought down to the basis $\overline{X}_0$ as follows:

□ **Step 1a:** Compute on the basis $\overline{X}_0$ the intersection number with parameters $t_1, \dots, t_\kappa$:

$$I(t_1, \dots, t_\kappa) = \int_{\overline{X}_0} f(t_1, \dots, t_\kappa)\, s_{1/t_1}(V_0) \cdots s_{1/t_\kappa}(V_0),$$

and obtain a multivariate Laurent polynomial in $t_1, \dots, t_\kappa$ over $\mathbb{Q}$.

□ **Step 1b :** Expand the universal rational function $\Phi_\kappa(t_1, \dots, t_\kappa)$ successively with respect to $t_1, t_2, \dots$ up to t_κ. Obtain, not a multivariate Laurent series, but what has been called an iterated Laurent series, similarly denoted by $\mathit{\Phi}_\kappa(t_1, \dots, t_\kappa)$ – notice the slanted $\mathit{\Phi}$.

□ **Step 2:** Compute the Cauchy product:

$$I(t_1, \dots, t_\kappa)\, \mathit{\Phi}_\kappa(t_1, \dots, t_\kappa)$$

of the multivariate Laurent polynomial $I(t_1, \dots, t_\kappa)$ and of the iterated Laurent series $\mathit{\Phi}_\kappa(t_1, \dots, t_\kappa)$ in the field of iterated Laurent series over $\mathbb{Q}$. Lastly, extract the coefficient of the monomial $t_1^r \cdots t_\kappa^r$ in the obtained multivariate formal series, and receive the sought element $I \in \mathbb{Q}$.

Really computing I proves to be quite delicate in practice. The first effective result in any dimension towards the Green-Griffiths conjecture was obtained in 2010 by Diverio, Merker and Rousseau (**[13]**), using step-by-step algebraic elimination, for entire curves $\mathbb{C} \to X_d \subset \mathbb{P}^{n+1}$ with values in generic hypersurfaces X_d of degree d in $\mathbb{P}^{n+1}$, with an estimated sufficient lower bound:

$$d \geqslant 2^{n^5}.$$

Some time after, in **[3]**, Gergely Bérczi made a substantial progress by replacing the elimination step of **[13]** by an iterated residue formula, and he reached the lower bound:

$$d \geqslant n^{8n}.$$

Using our result, the difficulty is that in general, Step 1b does not yield a single iterated Laurent series, but produces an involved product of several iterated Laurent series. Then, it is very difficult to determine even the sign of any individual coefficient of $\mathit{\Phi}_\kappa(t_1, \dots, t_\kappa)$, because this amounts to disentangle the large product:

$$\prod_{2 \leqslant i \leqslant j \leqslant \kappa} \frac{t_j - 2\,t_i}{t_j - 2\,t_i + t_{i-1}} = \prod_{2 \leqslant i \leqslant j \leqslant \kappa} \left(1 - \sum_{p=0}^{\infty} \frac{t_{i-1}(2\,t_i - t_{i-1})^p}{t_j^{p+1}} \right).$$

On the other hand, it is relatively easy to control the absolute value of these coefficients, using a convergent majorant series with positive coefficients, whence suppressing the problem of signs:

$$\begin{aligned} |\mathrm{coeff}| \left(\prod_{2 \leqslant i \leqslant j \leqslant \kappa} \frac{t_j - 2\,t_i}{t_j - 2\,t_i + t_{i-1}} \right) &\leqslant \mathrm{coeff} \left(\prod_{2 \leqslant i \leqslant j \leqslant \kappa} \frac{t_j - 2\,t_i}{t_j - 2\,t_i - t_{i-1}} \right) = \\ &= \mathrm{coeff} \left(\prod_{2 \leqslant i \leqslant j \leqslant \kappa} \left(1 + \sum_{p=0}^{\infty} \frac{t_{i-1}(2\,t_i + t_{i-1})^p}{t_j^{p+1}} \right) \right) \end{aligned}$$

– notice that $+t_{i-1}$ in the denominator becomes $-t_{i-1}$. And this allows us to use (plainly) the triangle inequality in chapter IV, to attain an effective lower bound on the degree d

of generic smooth hypersurfaces $X_d \subset \mathbb{P}^n$ such that all entire curves $\mathbb{C} \to \mathbb{P}^n \setminus X_d$ are algebraically degenerate:

$$d \geqslant (5n)^2\, n^n,$$

a lower bound which also holds for curves with values in a generic hypersurface $X_d \subset \mathbb{P}^{n+1}$.

2. Demailly Tower of (Logarithmic) Directed Manifolds

A *directed manifold* is defined to be a couple (X, V) where X is a complex manifold, equipped with a (not necessary integrable) holomorphic subbundle $V \subset T_X$ of its holomorphic tangent bundle. There is a natural generalization of this definition in the logarithmic setting. A *log-directed manifold* is by definition a triple $(\overline{X}, D, V)$ where $(\overline{X}, D)$ is a log-manifold and the distribution $V \subset T_{\overline{X}}(-\log D)$ is a (not necessary integrable) subbundle of the logarithmic tangent bundle.

Given a log-directed manifold $(\overline{X}, D, V)$, following Dethloff and Lu [**10**], we construct the *Demailly tower of projectivized bundles* $(\overline{X}_i, D_i, V_i)$ on $\overline{X}$ by induction on $i \geqslant 0$. This construction is formally the same as the construction [**7**] of the Demailly tower in the so-called *compact case*, *i.e.* where there is no divisor D. The only slight modification to keep in mind in the genuine logarithmic setting is that V is a holomorphic subbundle of the logarithmic tangent bundle $T_{\overline{X}}\big(-\log D\big)$.

Projectivization of a log directed manifold ([7, 10]). Recall that for a vector bundle $E \to \overline{X}$ on a smooth manifold $\overline{X}$ with projective bundle of lines $\pi\colon P(E) \to \overline{X}$, the *tautological line bundle*:

$$\mathcal{O}_{P(E)}(-1) \to P(E)$$

is defined as the subbundle of the pullback bundle $\pi^\star E \to P(E)$ with trivial fiber. In other words, the fiber of $\mathcal{O}_{P(E)}(-1)$ at a point (x, v) is the complex line $\mathbb{C}v \subset E_x$ spanned by v inside of the vector space E_x.

For a line bundle $L \to P(E)$, we use the standard notation for *twisted line bundles*:

$$L(k) := L \otimes \mathcal{O}_{P(E)}(-1)^{\otimes -k} \qquad (k \in \mathbb{Z}).$$

Accordingly, in the particular case where $L = \mathcal{O}_{P(E)}$, one has:

$$\mathcal{O}_{P(E)}(k) := \mathcal{O}_{P(E)}(-1)^{\otimes -k}.$$

With this notation, the tautological line bundle satisfies the following *Euler exact sequence* ([**20**, B.5.8][**23**, p. 408–409]):

$$0 \to \mathcal{O}_{P(E)} \to \pi^\star E \otimes \mathcal{O}_{P(E)}(1) \to T_\pi \to 0,$$

where $T_\pi := \ker(\pi_\star)$ stands for the *relative tangent bundle* of $P(E)$ over $\overline{X}$, that itself fits into the following short exact sequence:

$$0 \to T_\pi \hookrightarrow T_{P(E)} \xrightarrow{\pi_\star} \pi^\star T_{\overline{X}} \to 0.$$

2.1. Projectivization of a log directed manifold. ([**7, 10**]) Take therefore a log directed manifold $\big(\overline{X}, D, V\big) =: \big(\overline{X}_0, D_0, V_0\big)$. Reasoning by induction, we suppose that a directed manifold $\big(\overline{X}_{i-1}, D_{i-1}, V_{i-1}\big)$ is given, and we construct the directed manifold $\big(\overline{X}_i, D_i, V_i\big)$ at the next stage.

We now recall the inductive step $\big(\overline{X}', V', D'\big) \xrightarrow{\pi} \big(\overline{X}, V, D\big)$ of the construction of the Demailly tower. Keep in mind that V is a subbundle of $T_{\overline{X}}(-\log D)$ and that V' has to be a subbundle of the logarithmic tangent bundle $T_{\overline{X}'}(-\log D')$.

The formal construction of Demailly involves therefore differentials

$$\pi_\star\colon T_{\overline{X}'}(-\log D') \to \pi^\star T_{\overline{X}}(-\log D).$$

Consequently we have to define a divisor $D' \subset \overline{X}'$ at the upper level such that the projection $\overline{X}' \to \overline{X}$ becomes a log-morphism. There is a natural choice of this divisor D', for which we will still have:

$$V' \subset T_{\overline{X}'}(-\log D').$$

For $\overline{X}'$ we take the total space $P(V)$ of the projective bundle of lines of V:

$$\overline{X}' := P(V) \xrightarrow{\pi} \overline{X}.$$

One has thus:

$$\dim \overline{X}' = \dim \overline{X} + \operatorname{rk} P(V).$$

In order to make π a log-morphism it is natural to set:

$$D' := \pi^{-1}(D) \subset \overline{X}'.$$

Next, by definition of the relative tangent bundle $T_\pi := \ker(\pi_\star)$ of the log-morphism π one has the following short exact sequence:

$$0 \to T_\pi \hookrightarrow T_{\overline{X}'}(-\log D') \xrightarrow{\pi_\star} \pi^\star T_{\overline{X}}(-\log D) \to 0,$$

and since by assumption $V \subset T_{\overline{X}}(-\log D)$, the tautological line bundle of $\overline{X}' = P(V)$ is a subbundle of the bundle in the right-hand slot:

$$\mathcal{O}_{\overline{X}'}(-1) \subset \pi^\star V \subset \pi^\star T_{\overline{X}}(-\log D),$$

whence one can define a subbundle $V' \subset T_{\overline{X}'}(-\log D')$ by taking:

$$V' := (\pi_\star)^{-1}\mathcal{O}_{\overline{X}'}(-1).$$

Equivalently, V' is defined by the following short exact sequence:

$$0 \to T_\pi \hookrightarrow V' \xrightarrow{\pi_\star} \mathcal{O}_{\overline{X}'}(-1) \to 0.$$

It is profitable to compare this short exact sequence with:

$$0 \to T_\pi \hookrightarrow T_{\overline{X}'} \xrightarrow{\pi_\star} \pi^\star T_{\overline{X}} \to 0.$$

In the left, one keeps all the vertical directions whereas in the right, one keeps only the single "tautological" direction among all horizontal directions.

The only thing to verify in order to get a tower of log directed manifold is that V' is a holomorphic subbundle of $T_{\overline{X}'}(-\log D')$. Since $(\pi_\star)^{-1}$ has maximal rank everywhere, as it is a bundle projection, this is the case ([**10**]).

2.2. Local picture of $(\overline{X}', V') \to (\overline{X}, V)$**.** We use local coordinates x, v, x', v' on $T_{P(V)} = T_{\overline{X}}$, where $x \in \overline{X}$ is the coordinate on the basis, $v \in P(V_x)$ is the coordinate on the fiber, $x' \in \pi^\star T_{\overline{X},x}$ is the coordinate on the horizontal tangent bundle and $v' \in T_\pi$ is the coordinate on the relative tangent bundle. The tautological line bundle of the projective space $P(V)$ is the subbundle of the pullback bundle $\pi^\star V \to P(V)$ locally given by:

$$\mathcal{O}_{P(V)}(-1) = \{(x, v; x')\colon x' \in v \subset \pi^\star V_x\},$$

and thus, by definition:

$$V' = \{(x, v; x', v')\colon x' \in v \subset \pi^\star V_x\}.$$

In other words, the direction of the *horizontal derivative* x' is assigned by v whereas the *vertical derivative* v' can have an arbitrary direction in the relative tangent bundle.

Here, the data x' varies in a 1-dimensional complex subspace, the rank of V' is thus the same as the rank of V, because:

$$\mathrm{rk}(V') = \mathrm{rk}(P(V)) + 1 = \mathrm{rk}(V) - 1 + 1.$$

2.3. Logarithmic Demailly tower. Starting from a bundle V_0 having rank:

$$\mathrm{rk}\ V_0 =: r + 1,$$

by iterating the construction κ times, we get a tower of projectivized bundles

$$\left(\overline{X}_\kappa, V_\kappa, D_\kappa\right) \xrightarrow{\pi_\kappa} \cdots \xrightarrow{\pi_2} \left(\overline{X}_1, V_1, D_1\right) \xrightarrow{\pi_1} \left(\overline{X}_0, V_0, D_0\right)$$

with $\mathrm{rk}\, V_i = r + 1$ and $n_i := \dim\ \overline{X}_i = \dim(\overline{X}_0) + i\left(\mathrm{rk}\, P(V_0)\right) = n + i\, r$.

2.4. Existence of global jet differentials. The fibers of the Demailly-Semple bundle of jet differentials $E_{\kappa,m}(V_0)^\star(\log D_0)$ carries much complexity ([**28, 30**]). In order to prove the existence of global jet differentials of order $\kappa = \dim(\overline{X})$, one is led to consider a much more tractable line bundle, constructed in [**7**, 6.13] as a linear combination with non negative integer coefficients $(a_1, \ldots, a_\kappa)$:

$$\mathcal{O}_{\overline{X}_k}(a_1, a_2, \ldots, a_k) := (\pi_{k,1})^\star \mathcal{O}_{\overline{X}_1}(a_1) \otimes (\pi_{k,2})^\star \mathcal{O}_{\overline{X}_2}(a_2) \otimes \cdots \otimes \mathcal{O}_{\overline{X}_k}(a_k).$$

If $a_1 + \ldots + a_\kappa = m$, the direct image $(\pi_{\kappa,0})_\star \mathcal{O}_{\overline{X}_\kappa}(a_1, \ldots, a_\kappa)$ may be seen as a subbundle of the Demailly-Semple bundle of jet differentials ([**7, 10**]).

For a suitable choice of the parameters $a_1, \ldots, a_\kappa$, the line bundle $\mathcal{O}_{\overline{X}_\kappa}(a_1, \ldots, a_\kappa)$ has some positivity properties, that can be used together with the following Demailly-Trapani *algebraic Morse inequalities* ([**49, 7**]) in order to establish the existence of global jet differentials.

(2.4.1**) Theorem (**Weak algebraic Morse inequalities). *For any holomorphic line bundle L on a N-dimensional compact manifold $\overline{X}$, that can be written as the difference $L = F \otimes G^\vee$ of two nef line bundles F and G, one has:*

$$h^0\left(\overline{X}, L^{\otimes k}\right) \geqslant k^N\ \frac{\left(F^N\right) - k\left(F^{N-1} \cdot G\right)}{N!} - o(k^N).$$

For a choice of $a_1, \ldots, a_\kappa \in \mathbb{N}^\kappa$ such that:

$$a_{\kappa-1} > 2\, a_\kappa > 0 \quad ; \quad a_i \geqslant 3\, a_{i+1} \quad (i = 1, \ldots, \kappa - 2), \tag{2.4.2}$$

the line bundle $\mathcal{O}_{\overline{X}_\kappa}(a_1, \ldots, a_\kappa)$ is relatively ample along the fibers of $\overline{X}_\kappa$ over $\overline{X}$ (*cf.* [**7**]). It hence suffices to multiply it by a sufficiently positive power $\pi_{\kappa,0}^\star A^{\otimes l}$ of a given ample line bundle $A \to \overline{X}$, in order to get an ample (hence nef) line bundle (*cf.* [**27**]). On the other hand $\pi_{\kappa,0}^\star A^{\otimes l+1}$ is nef for it is the pullback of a nef line bundle.

It gives an expression of the line bundle $\mathcal{O}_{\overline{X}_\kappa}(a_1, \ldots, a_\kappa) \otimes (\pi_{\kappa,0})^\star A^\vee$ as the difference $F \otimes G^\vee$ of two nef line bundles:

$$F := \mathcal{O}_{\overline{X}_\kappa}(a_1, \ldots, a_\kappa) \otimes (\pi_{\kappa,0})^\star A^{\otimes l} \quad \text{and} \quad G := (\pi_{\kappa,0})^\star A^{\otimes l+1}$$

In order to prove the existence of global sections:

$$P \in H^0\Big(\overline{X}_\kappa, \mathcal{O}_{\overline{X}_\kappa}(a_1, \ldots, a_\kappa) \otimes (\pi_{\kappa,0})^\star A^\vee\Big),$$

it hence remains to show the positivity of the following intersection number:

$$I := \int_{\overline{X}_\kappa} c_1(F)^{n_\kappa} - n_\kappa\, c_1(F)^{n_\kappa - 1} c_1(G) \qquad {\scriptstyle (n_\kappa = \dim \overline{X}_\kappa)}.$$

We will give a formula for computing such an intersection product.

3. Fiber Integration on the Demailly Tower

It is convenient to bring down the computation to the basis and we will now provide a formula for this purpose. Noteworthy, the proof of this formula involves iterated Laurent series, in the same spirit as the residue formula of Bérczi [**3, 4**]. However, we will not use equivariant geometry like this author, but only basic lemmas of intersection theory, more precisely some of the properties of Segre classes exposed in the book of Fulton [**20**, Chap. 3]. We now first briefly recall these properties.

3.1. Segre classes on the Demailly tower. To go down one level, from $\overline{X}_{i+1} = P(V_i)$ to $\overline{X}_i$, we will use the very definition of the j-th *Segre class* of a vector bundle $E \to X$ (having rank $r+1$), namely the fiber integration formula:

(3.1.1)
$$\int_X s_j(E)\,\alpha = \int_{P(E)} u^{j+r}\, p^\star \alpha \qquad {\scriptstyle (j \geqslant 0,\ \alpha \in H^\bullet X)},$$

where:

$$p\colon P(E) \to X \quad \text{and} \quad u := c_1\Big(\mathcal{O}_{P(E)}(-1)^\vee\Big).$$

We want to apply this formula in order to eliminate the powers of the first Chern classes $v_1, \dots, v_\kappa$ of the vector bundles $L_i \to \overline{X}_i$. We will proceed by induction.

It is well known that the total Segre class of a vector bundle is the same as the inverse of its total Chern class. Thus, the total Segre class enjoys the Whitney formula.

Because we will obtain a result that is independent of the geometric context, we will deliberately be ambiguous about it. The only property of the Demailly construction that we use in what follows is the existence of the two short exact sequences:

$$0 \to T_\pi \to V' \to \mathcal{O}_{\overline{X}'}(-1) \to 0$$

and:

$$0 \to \mathcal{O}_{\overline{X}'} \to \pi^\star V \otimes \mathcal{O}_{\overline{X}'}(1) \to T_\pi \to 0,$$

where $(\overline{X}', V') \xrightarrow{\pi} (\overline{X}, V)$ is the inductive step of the Demailly construction.

Now, consider the following observation: the twist by a line bundle does not change the projective bundle of lines of V', but only the transition functions. Moreover, one can twist short exact sequences by line bundles.

We can hence chose a line bundle L' on $\overline{X}'$ that makes the induction more easy. We will twist both short exact sequences by the same line bundle, because we do not want T_π to appear in the final formula below. Also, we do not want anymore the central term of the second short exact sequence to be a product of line bundles with different base spaces $\overline{X}$ and $\overline{X}'$ but rather want it to be the pullback of a single bundle on the lower level $\overline{X}$, that is:

$$\pi^\star V \otimes \mathcal{O}_{\overline{X}'}(1) \otimes L' = \pi^\star\Big(V \otimes L\Big),$$

for a certain $L \to \overline{X}$ (in practice given by the preceding induction steps). Consequently, we have to take:

(3.1.2)
$$L' := \mathcal{O}_{\overline{X}'}(-1) \otimes \pi^\star L = \mathcal{O}_{P(V \otimes L)}(-1).$$

Notice that accordingly the term $\mathcal{O}_{\overline{X}'}(-1)$ can now be replaced by $\left(L' \otimes \pi^\star L^\vee\right)$ in the first exact sequence.

Once twisted by L', the above two short exact sequences become:

$$\begin{cases} 0 \to T_\pi \otimes L' \to V' \otimes L' \to \left(L' \otimes \pi^\star L^\vee\right) \otimes L' \to 0 \\ 0 \to L' \to \pi^\star\left(V \otimes L\right) \to T_\pi \otimes L' \to 0 \end{cases}.$$

By the Whitney formula, the first line yields:

$$s(V' \otimes L') = s(T_\pi \otimes L')\, s\left((L')^{\otimes 2} \otimes \pi^\star L^\vee\right),$$

while the second line yields:

$$\pi^\star s\left(V \otimes L\right) = s\left(T_\pi \otimes L'\right) s\left(L'\right).$$

Thus, we can eliminate T_π, as it was our intention, in order to get the induction formula:

$$s\left(V' \otimes L'\right) = \frac{s\left((L')^{\otimes 2} \otimes \pi^\star L^\vee\right)}{s\left(L'\right)}\, \pi^\star s\left(V \otimes L\right).$$

Now, for a line bundle $L \to X$, the total Segre class is the finite sum:

$$s_\bullet(L) = \left(1 - c_1(L^\vee)\right)^{-1} = 1 + c_1(L^\vee) + c_1(L^\vee)^2 + \cdots + c_1(L^\vee)^{\dim(X)}$$

– we use the first Chern class of the dual in order to have positive signs.

Let:

$$v := c_1\left(L^\vee\right) \quad \text{and} \quad v' := c_1\left((L')^\vee\right).$$

We get the induction formula:

$$s\left(V' \otimes L'\right) = \varphi\left(v', v\right)\, \pi^\star s\left(V \otimes L\right), \tag{3.1.3}$$

where:

$$\varphi\left(x, y\right) := (1 - x) \sum_{k=0}^{n_{\kappa-1}} (2\,x - y)^k.$$

is the truncated double Taylor expansion of the rational function:

$$(1 - x)\left(1 - 2x + y\right)^{-1}.$$

For $x, y \in H^1\left(\overline{X}_i\right)$, $i = 0, 1, \ldots, \kappa - 1$, one has indeed:

$$\varphi(x, y) = \frac{1 - x}{1 - 2x + y}.$$

Considering (3.1.2), we construct the *ad hoc* sequence of line bundles $L_i \to \overline{X}_i$ by taking first the tautological line bundle $L_1 := \mathcal{O}_{\overline{X}_1}(-1)$ of V_0 and then the tautological line bundle of the twisted vector bundle $V_{i-1} \otimes L_{i-1}$:

$$L_i := \mathcal{O}_{\overline{X}_i}(-1) \otimes (\pi_i)^\star L_{i-1} = \mathcal{O}_{P(V_{i-1} \otimes L_{i-1})}(-1) \qquad (i=2,\ldots,\kappa).$$

We will denote by v_i the first Chern class of the dual of this line bundle:

$$v_i := c_1\left(L_i^\vee\right) = c_1\left(\mathcal{O}_{P(V_{i-1} \otimes L_{i-1})}(1)\right).$$

Then, by (3.1.3), on has the following inductive formulas, where, for simplicity, we omit the pullbacks:

$$s\left(V_1 \otimes L_1\right) = \varphi(v_i, 0)\, s\left(V_0\right) \quad \text{and} \quad s\left(V_i \otimes L_i\right) = \varphi(v_i, v_{i-1})\, s\left(V_{i-1} \otimes L_{i-1}\right), \qquad (i=2,\ldots,\kappa).$$

Notice that we can reformulate the positivity property (2.4.2) by using the more explicit expression of the line bundles L_i:

$$L_i = \mathcal{O}_{\overline{X}_i}(-1) \otimes \cdots \otimes (\pi_{i,2})^\star \mathcal{O}_{\overline{X}_2}(-1) \otimes (\pi_{i,1})^\star \mathcal{O}_{\overline{X}_1}(-1) \qquad (i=1,\ldots,\kappa).$$

and the inversion of these formulas

$$\mathcal{O}_{\overline{X}_i}(1) = L_i^\vee \otimes L_{i-1};$$

in analogy with $\mathcal{O}_{\overline{X}_\kappa}(a_1, \ldots, a_\kappa) = a_1\, \mathcal{O}_{\overline{X}_1}(-1)^\vee + \cdots + a_\kappa\, \mathcal{O}_{\overline{X}_\kappa}(-1)^\vee$ consider, a linear combination:

$$L(a_1, \ldots, a_\kappa) := a_1\, L_1^\vee + \cdots + a_\kappa\, L_\kappa^\vee$$

of the line bundles $L_i^\vee$, with non negative coefficients a_i, such that:

$$a_1 + 2\, a_2 + \cdots + \kappa\, a_\kappa = m \in \mathbb{N},$$

then, the line bundle $\pi_{\kappa,0}^\star L(a_1, \ldots, a_\kappa)$ may be seen as a certain subbundle of $E_{\kappa,m} V_0^\star(\log D_0)$ and if:

$$a_{\kappa-1} > a_\kappa \geqslant 1 \quad ; \quad a_i \geqslant 2(a_{i+1} + \cdots + a_\kappa) \quad (i\leqslant\kappa-2).$$

the line bundle $L(a_1, \ldots, a_\kappa)$ is relatively ample along the fibers of $\overline{X}_\kappa \to \overline{X}_0$.

(3.1.4) **Proposition.** *For any polynomial in the first Chern classes $v_1, \ldots, v_{i+1}$ having coefficients in (the pullback of) the cohomology of $\overline{X}_0$:*

$$f \in H^\bullet\big(\overline{X}_0\big)[v_1, \ldots, v_i, v_{i+1}],$$

the following formula of integration along the fibers of $\overline{X}_{i+1} \to \overline{X}_i$ holds:

$$\int_{\overline{X}_{i+1}} f(v_1, \ldots, v_i, v_{i+1}) = \Big[t_{i+1}^r\Big]\bigg(\int_{\overline{X}_i} f(v_1, \ldots, v_i, t_{i+1})\, s_{1/t_{i+1}}\Big(V_i \otimes L_i\Big)\bigg).$$

Proof. Firstly, by the Leray-Hirsch theorem:

$$H^\bullet\big(\overline{X}_0\big)[v_1, \ldots, v_i, v_{i+1}] = H^\bullet\big(\overline{X}_{i+1}\big)$$

Thus, f has values in the cohomology ring of $\overline{X}_{i+1}$.

The polynomial f is of the form:

$$f(v_1, \ldots, v_i, v_{i+1}) = \sum_{j=0}^{n_{i+1}} (v_{i+1})^j\, (\pi_{i+1})^\star f_j(v_1, \ldots, v_i).$$

By linearity, the formula will hold for any such sum, if it holds for every monomial:

$$v_{i+1}^j\, (\pi_{i+1})^\star f_j(v_1, \ldots, v_i).$$

Recall that the line bundles L_i are constructed by the inductive formula:

$$L_{i+1} = \mathcal{O}_{P(V_i)}(-1) \otimes \pi_{i+1}^\star L_i = \mathcal{O}_{P(V_i \otimes L_i)}(-1).$$

Thus, it can be thought of as the tautological line bundle of the projective bundle:

$$P(V_i \otimes L_i) \simeq P(V_i) =: \overline{X}_{i+1}.$$

Then the above fiber integration formula (3.1.1) yields at once:

$$(*) \qquad \int_{\overline{X}_{i+1}} v_{i+1}^j\, (\pi_{i+1})^\star f_j(v_1, \ldots, v_i) = \int_{\overline{X}_i} s_{j-r}\Big(V_i \otimes L_i\Big)\, f_j(v_1, \ldots, v_i).$$

In particular, this integral is zero for indices j smaller than r.

The problem is now to obtain the individual Segre classes from the total Segre class. In that aim, we will use the formalism of generating functions. Recall that, in analogy with Chern polynomial, for a vector bundle $E \to X$ over a N dimensional manifold X, we have introduced the generating function $s_u(E)$ of the Segre classes of E, that is:

$$s_u(E) := s_0(E) + u\, s_1(E) + u^2\, s_2(E) + \cdots + u^N\, s_N(E).$$

Then, by taking $t = 1/u$, we obtain a Laurent polynomial:

$$s_{1/t}(E) := \frac{s_0(E)}{t^0} + \frac{s_1(E)}{t^1} + \frac{s_2(E)}{t^2} + \cdots + \frac{s_N(E)}{t^N},$$

in which the $\big(j-r\big)$-th Segre class involved in the fiber integration appears as the coefficient:

$$s_{j-r}(E) = \big[1/t^{j-r}\big] s_{1/t}(E) = \big[t^r\big]\big(t^j\, s_{1/t}(E)\big).$$

Therefore, by replacing in the integration formula $(*)$:

$$\begin{aligned}\int_{\overline{X}_{i+1}} v_{i+1}^j\, (\pi_{i+1})^\star f_j(v_1,\dots,v_i) &= \int_{\overline{X}_i} \big[{t_{i+1}}^r\big]\Big(t_{i+1}^j\, s_{1/t_{i+1}}\big(V_i \otimes L_i\big)\Big)\, f_j(v_1,\dots,v_i).\\ &= \big[{t_{i+1}}^r\big]\bigg(\int_{\overline{X}_i} t_{i+1}^j\, f_j(v_i,\dots,v_1)\, s_{1/t_{i+1}}\big(V_i \otimes L_i\big)\bigg).\end{aligned}$$

Notice that inside of the parenthesis there is the product of a monomial by a Laurent polynomial. Thus, only a finite number of terms are involved and there is no objection to switching the integral and the coefficient extraction.

The obtained formula is exactly the sought formula for the considered monomial:

$$v_{i+1}^j\, (\pi_{i+1})^\star f_j(v_1,\dots,v_i),$$

and this ends the proof. □

3.2. Iteration of the integration formula. In order to iterate the fiber integration, we introduce the following formalism: for $i = 0, 1, \dots, \kappa$, we denote by $\underline{vt}_{\,i}$ the κ-tuple obtained from:

$$\underline{v} := (v_1, \dots, v_\kappa)$$

by replacing the last i components $v_{\kappa-i+1}$, …, v_κ by the corresponding parameters $t_{\kappa-i+1}, \dots, t_\kappa$, *i.e.*:

$$\underline{vt}_{\,i} := (v_1, \dots, v_{\kappa-i}, t_{\kappa-i+1}, \dots, t_\kappa) \qquad {\scriptstyle (i=0,1,\dots,\kappa)}.$$

With this notation the fiber integration formula (3.1.4) just above yields directly, that for any polynomial in the first Chern classes $v_1, \dots, v_{i+1}$ having coefficients in (the pullback of) the cohomology of $\overline{X}_0$, being a Laurent polynomial in the formal parameters $t_{i+2}, \dots, t_\kappa$:

$$f \in H^\bullet\big(\overline{X}_0\big)[v_1, \dots, v_i, v_{i+1}][t_{i+2}, t_{i+2}^{-1}, \dots, t_\kappa, t_\kappa^{-1}],$$

the following formula of integration along the fibers of $\overline{X}_{i+1} \to \overline{X}_i$ holds:

$$\int_{\overline{X}_i} f\big(\underline{vt}_{\,\kappa-i}\big) = \big[t_i^r\big] \int_{\overline{X}_{i-1}} f\big(\underline{vt}_{\,\kappa-i+1}\big)\, s_{1/t_i}\big(V_{i-1} \otimes L_{i-1}\big). \tag{3.2.1}$$

Notice that in the above formula the form of the polynomial appearing in the integrand:

$$f\big(\underline{vt}_{\,\kappa-i+1}\big)\, s_{1/t_i}\big(V_{i-1} \otimes L_{i-1}\big) \in H^\bullet\big(\overline{X}_0\big)[v_1, \dots, v_i][t_{i+1}, t_{i+1}^{-1}, \dots, t_\kappa, t_\kappa^{-1}],$$

allows to iterate this formula in order to integrate along the fibers of $\overline{X}_{i-1} \to \overline{X}_{i-2}$. For short, we denote the appearing polynomial rings by:

$$\Lambda[\underline{vt}_{\kappa-i}] := H^\bullet\big(\overline{X}_0\big)[v_1, \dots, v_i][t_{i+1}, t_{i+1}^{-1}, \dots, t_\kappa, t_\kappa^{-1}] \quad {\scriptstyle (i=0,1,\dots,\kappa)}.$$

One has thus:

$$\Lambda[\underline{vt}_0] = H^\bullet\big(\overline{X}_0\big)[\underline{v}] = H^\bullet\big(\overline{X}_\kappa\big) \quad \text{and} \quad \Lambda[\underline{vt}_\kappa] = H^\bullet\big(\overline{X}_0\big)[\underline{t}, \underline{t}^{-1}].$$

We have first to investigate the dependence with respect to v_i of the appearing power series $s_{1/t_{i+1}}\big(V_i \otimes L_i\big)$. The induction formula (3.1.1) precisely provides us with this information. Thanks to it, we can split the power series $s_{1/t}\big(V_i \otimes L_i\big)$ in two parts:

$$s_{1/t_j}\big(V_i \otimes L_i\big) = \underbrace{\varphi\Big(\frac{v_i}{t_j}, \frac{v_{i-1}}{t_j}\Big)}_{\in \Lambda[\underline{vt}_{\kappa-i}]} \; \underbrace{s_{1/t_j}\big(V_{i-1} \otimes L_{i-1}\big)}_{\in \Lambda[\underline{vt}_{\kappa-i+1}]},$$

or for $i = 1$:

$$s_{1/t_j}\big(V_1 \otimes L_1\big) = \underbrace{\varphi\Big(\frac{v_1}{t_j}, 0\Big)}_{\in \Lambda[\underline{vt}_{\kappa-1}]} \; \underbrace{s_{1/t_j}\big(V_0 \otimes L_0\big)}_{\in H^\bullet(\overline{X}_0)[\underline{t}, \underline{t}^{-1}]},$$

the first of which depends on v_i whereas the second does not.

Write for short:

$$\varPhi_{k,l}(t_1, \dots, t_\kappa) := \varphi\Big(\frac{t_k}{t_l}, \frac{t_{k-1}}{t_l}\Big) \qquad {\scriptstyle (k=2,\dots,\kappa-1,\ k+1 \leqslant l \leqslant \kappa)},$$

and:

$$\varPhi_{1,l}(t_1, \dots, t_\kappa) := \varphi\Big(\frac{t_1}{t_l}, 0\Big) \qquad {\scriptstyle (2 \leqslant l \leqslant \kappa)},$$

in such way that, for any two positive integers $k < l$:

$$s_{1/t_l}\big(V_k \otimes L_k\big) = \varPhi_{k,l}\big(\underline{vt}_k\big)\, s_{1/t_l}\big(V_{k-1} \otimes L_{k-1}\big) \quad {\scriptstyle (1 \leqslant k < l \leqslant \kappa)}. \tag{3.2.2}$$

Let $\varPhi_i$ be the product of the terms in the i last lines of the array:

$$\begin{pmatrix} 1 & \varPhi_{1,2} & \cdots & \varPhi_{1,\kappa} \\ \vdots & \ddots \; \varPhi_{2,3} & \cdots & \varPhi_{2,\kappa} \\ & & \ddots & \vdots \\ \vdots & & \ddots & \varPhi_{\kappa-1,\kappa} \\ 1 & \cdots & \cdots & 1 \end{pmatrix},$$

that is the product of $\big(i(i-1)/2\big)$ terms:

$$\varPhi_i(t_1, \dots, t_\kappa) := \prod_{\kappa-i+1 \leqslant k < l \leqslant \kappa} \varPhi_{k,l}(t_1, \dots, t_\kappa).$$

As an example, $\varPhi_1(t_1, \dots, t_\kappa) = 1$.

The following lemma will be used in order to isolate the variable $v_{\kappa-i}$, that is to eliminate after the i-th step of the fiber integration.

(3.2.3) Lemma (Isolation of $v_{\kappa-i}$). *For any $i = 0, 1, \ldots, \kappa-1$ one has the following relation between Φ_i and Φ_{i+1}:*

$$\Phi_i\big(\underline{vt}_i\big) \prod_{j=\kappa-i+1}^{\kappa} s_{1/t_j}\big(V_{\kappa-i} \otimes L_{\kappa-i}\big) = \Phi_{i+1}\big(\underline{vt}_i\big) \prod_{j=\kappa-i+1}^{\kappa} s_{1/t_j}\big(V_{\kappa-(i+1)} \otimes L_{\kappa-(i+1)}\big).$$

Proof. Recall the induction formula displayed above:

$$\frac{s_{1/t_j}\big(V_i \otimes L_i\big)}{s_{1/t_j}\big(V_{i-1} \otimes L_{i-1}\big)} = \Phi_{i,j}\big(\underline{vt}_{\kappa-i}\big).$$

Thus, one has:

$$\frac{\prod_{j=\kappa-i+1}^{\kappa} s_{1/t_j}\big(V_{\kappa-i} \otimes L_{\kappa-i}\big)}{\prod_{j=\kappa-i+1}^{\kappa} s_{1/t_j}\big(V_{\kappa-(i+1)} \otimes L_{\kappa-(i+1)}\big)} = \prod_{j=\kappa-i+1}^{\kappa} \Phi_{\kappa-i,j}\big(\underline{vt}_{\kappa-i}\big).$$

Now, by definition of Φ_i and Φ_{i+1}:

$$\frac{\Phi_{i+1}}{\Phi_i}\big(\underline{vt}_{\kappa-i}\big) = \frac{\prod_{\kappa-i\leqslant k<l\leqslant\kappa} \Phi_{k,l}}{\prod_{\kappa-i+1\leqslant k<l\leqslant\kappa} \Phi_{k,l}}\big(\underline{vt}_{\kappa-i}\big) = \prod_{l=\kappa-i+1}^{\kappa} \Phi_{\kappa-i,l}\big(\underline{vt}_{\kappa-i}\big).$$

Hence, we get the announced result. □

Notice that in the right hand side of the obtained formula, only the first factor depends on $v_{\kappa-i}$.

This result is given by anticipation of the proof of main theorem (3.2.4). However we can already notice that, *e.g.*:

$$\Phi_1\big(\underline{vt}_1\big) \prod_{j=\kappa-1+1}^{\kappa} s_{1/t_j}\big(V_{\kappa-i} \otimes L_{\kappa-i}\big) = s_{1/t_\kappa}\big(V_{\kappa-1} \otimes L_{\kappa-1}\big),$$

is the term appearing in the first step of the fiber integration.

Main result.

(3.2.4) Theorem (Fiber Integration on the Demailly tower). *Any polynomial:*

$$f \in H^\bullet(\overline{X}_0, V_0)[t_1, \ldots, t_\kappa],$$

in κ variables $t_1, \ldots, t_\kappa$, with coefficients in the cohomology ring $H^\bullet(\overline{X}_0, V_0)$, yields a cohomology class:

$$f\big(\underline{v}\big) = f\big(v_1, \ldots, v_\kappa\big) \in H^\bullet\big(\overline{X}_\kappa\big),$$

that can be integrated along the fibers of the projective bundle $\overline{X}_\kappa \to \overline{X}_0$ according to the formula:

$$\int_{\overline{X}_\kappa} f\big(\underline{v}\big) = \big[t_1^r \cdots t_\kappa^r\big]\Big(\Phi_\kappa\big(\underline{t}\big) \int_{\overline{X}_0} f\big(\underline{t}\big)\, s_{1/t_1}(V_0) \cdots s_{1/t_\kappa}(V_0)\Big).$$

Proof. We will prove by induction that for $i = 0, 1, \ldots, \kappa$, one has:

$$\int_{\overline{X}_\kappa} f\big(\underline{v}\big) = \big[t_{\kappa-i+1}^r \cdots t_\kappa^r\big] \int_{\overline{X}_{\kappa-i}} f_i\big(\underline{vt}_i\big),$$

with:

$$f_i\big(\underline{vt}_i\big) := f\big(\underline{vt}_i\big)\, \Phi_i\big(\underline{vt}_i\big) \prod_{k=\kappa-i+1}^{\kappa} s_{1/t_k}(V_{\kappa-i} \otimes L_{\kappa-i}).$$

Then, for $i = \kappa$:

$$\int_{\overline{X}_\kappa} f\big(\underline{v}\big) = \big[t_1^r \cdots t_\kappa^r\big] \int_{\overline{X}_0} f_\kappa\big(\underline{t}\big),$$

with:

$$f_\kappa\big(\underline{t}\big) := f\big(\underline{t}\big)\, \Phi_\kappa\big(\underline{t}\big) \prod_{k=1}^{\kappa} s_{1/t_k}(V_0 \otimes L_0).$$

That is the desired formula because $L_0 = \mathcal{O}_{\overline{X}_0}$.

For $i = 0$, this is tautological. Now, assume that the formula holds for the index i, that is to say:

$$(*) \qquad \int_{\overline{X}_\kappa} f\big(\underline{v}\big) = \big[t_{\kappa-i+1}^r \cdots t_\kappa^r\big] \int_{\overline{X}_{\kappa-i}} f_i\big(\underline{vt}_i\big),$$

According to lemma (3.2.3), f_i can also be written:

$$f_i\big(\underline{vt}_i\big) = f\big(\underline{vt}_i\big)\, \Phi_{i+1}\big(\underline{vt}_i\big) \underbrace{\prod_{k=\kappa-i+1}^{\kappa} s_{1/t_k}(V_{\kappa-(i+1)} \otimes L_{\kappa-(i+1)})}_{\in \Lambda[\underline{vt}_{i+1}]}.$$

Now applying lemma (3.2.1):

$$\int_{\overline{X}_{\kappa-i}} f_i\big(\underline{vt}_i\big) = \big[t_{\kappa-i}^r\big] \int_{\overline{X}_{\kappa-i-1}} f_i\big(\underline{vt}_{i+1}\big) s_{1/t_{\kappa-i}}(V_{\kappa-(i+1)} \otimes L_{\kappa-(i+1)}).$$

It remains to state that:

$$f_i\big(\underline{vt}_{i+1}\big) s_{1/t_{\kappa-i}}(V_{\kappa-(i+1)} \otimes L_{\kappa-(i+1)}) = \\ f\big(\underline{vt}_{i+1}\big)\, \Phi_{i+1}\big(\underline{vt}_{i+1}\big) \prod_{k=\kappa-i+1}^{\kappa} s_{1/t_k}(V_{\kappa-(i+1)} \otimes L_{\kappa-(i+1)}) \\ s_{1/t_{\kappa-i}}(V_{\kappa-(i+1)} \otimes L_{\kappa-(i+1)}).$$

Here, we recognize the expression:

$$f_{i+1}\big(\underline{vt}_{i+1}\big) = f\big(\underline{vt}_{i+1}\big)\, \Phi_{i+1}\big(\underline{vt}_{i+1}\big) \prod_{k=\kappa-i}^{\kappa} s_{1/t_k}(V_{\kappa-(i+1)} \otimes L_{\kappa-(i+1)}).$$

Thus, we can replace the integrand in order to get:

$$\int_{\overline{X}_{\kappa-i}} f_i\big(\underline{vt}_i\big) = \big[t_{\kappa-i}^r\big] \int_{\overline{X}_{\kappa-i-1}} f_{i+1}\big(\underline{vt}_{i+1}\big).$$

Using the induction hypothesis $(*)$, one finally gets the desired formula, for the index $i+1$:

$$\int_{\overline{X}_\kappa} f\big(\underline{v}\big) = \big[t_{\kappa-i+1}^r \cdots t_\kappa^r\big] \int_{\overline{X}_{\kappa-i}} f_i\big(\underline{vt}_i\big) = \big[t_{\kappa-i}^r \cdots t_\kappa^r\big] \int_{\overline{X}_{\kappa-(i+1)}} f_{i+1}\big(\underline{vt}_{i+1}\big).$$

This complete the proof. □

3.3. Iterated Laurent series. Next, we introduce the formalism of Laurent series expansion in several variables, in order to write Φ_κ more compactly.

Univariate case. In the univariate case, let K be a field, let $K[[t]]$ denote the *ring of formal power series in the indeterminate t over K* and let $K[[t, t^{-1}]]$ denote the *space of formal series in the indeterminate t over K*.

An element Ψ of $K[[t, t^{-1}]]$ is a formal combination:

$$\Psi := \sum_{i \in \mathbb{Z}} \Psi_i \, t^i,$$

with coefficients $\Psi_i \in K$. Recall the notation:

$$\big[t^i\big]\Big(\sum \Psi_i \, t^i\Big) := \Psi_i.$$

The *support* of a formal series Ψ is the subset of $\mathbb{Z}$ consisting of exponents i for which the corresponding coefficient in the expansion of Ψ is not zero:

$$\operatorname{supp} \Psi := \Big\{ i \colon [t^i]\big(\Psi\big) \neq 0 \Big\} \subset \mathbb{Z}.$$

The *Cauchy product* $\Psi_1 \, \Psi_2$ of two formal power series Ψ_1 and Ψ_2 in $K[[t]]$ is the series with coefficients:

$$[t^i]\big(\Psi_1 \, \Psi_2\big) = \sum_{j+k=i} [t^j]\big(\Psi_1\big) \, [t^k]\big(\Psi_2\big). \tag{3.3.1}$$

It is well defined, because the computation of the coefficient of the monomial t^i involves only finitely many terms. Equipped with this law, $K[[t]]$ becomes a ring.

A formal power series Ψ in $K[[t]]$ is invertible with respect to the Cauchy product if and only if has a non zero constant term $[t^0](\Psi)$. In particular, the following *geometric series formula* is valid in $K[[t]]$:

$$(1 - t)^{-1} = \sum_{i \geqslant 0} t^i. \tag{3.3.2}$$

It is not possible to extend the law (3.3.1) to the full space of formal series $K[[t, t^{-1}]]$: in general, the Cauchy product of two formal series $\Psi_1, \Psi_2 \in K[[t, t^{-1}]]$ is not defined, because some appearing formal coefficients

$$[t^i]\big(\Psi_1 \, \Psi_2\big) = \sum_{j \in \mathbb{Z}} [t^j]\big(\Psi_1\big) \, [t^{i-j}]\big(\Psi_2\big)$$

could be *infinite* diverging series.

Classically, a *Laurent series* in the indeterminate t is a formal series $\Psi \in K[[t, t^{-1}]]$ whose support is bounded from below by a certain constant N, possibly negative, depending on Ψ:

$$\Psi = \sum_{i \geqslant N} \Psi_i \, t^i.$$

We will denote by $K(\!(t)\!)$ the space of Laurent series. One may check that Laurent series in $K(\!(t)\!)$ are nothing but polynomials in t^{-1} with coefficients in the ring $K[[t]]$:

$$K(\!(t)\!) = K[[t]][t^{-1}].$$

Hence the law of $K[[t]]$ induces a law on $K(\!(t)\!)$, for which $K(\!(t)\!)$ becomes a ring.

More concretely, the problem of convergence that we mentioned just above fortunately disappears for the formal product of two Laurent series, since each obtained coefficient is a *finite* sum:

$$[t^i]\big(\Psi_1 \, \Psi_2\big) = \sum_{j=N_1}^{i-N_2} [t^j]\big(\Psi_1\big) \, [t^{i-j}]\big(\Psi_2\big), \tag{3.3.3}$$

where $N_1 := \min \operatorname{supp} \Psi_1$, $N_2 := \min \operatorname{supp} \Psi_2$.

Also, thanks to the formula (3.3.2), we can define the formal inverse (for the Cauchy product) of any Laurent series of the form:

$$\Psi = \sum_{i \geqslant N} \Psi_i \, t^i$$

with initial coefficient $\Psi_N \neq 0$, as follows:

$$\Psi^{-1} = \frac{1}{\Psi_N \, t^N} \sum_{k \geqslant 0} \Big(- \sum_{j \geqslant 1} \frac{\Psi_{N+j}}{\Psi_N} \, t^j \Big)^k, \tag{3.3.4}$$

because the computation of the coefficient of any power of t in the later expression involves only a finite number of appearing k-th powers. The result is indeed a Laurent series, because its support is visibly bounded from below. To sum up, with these natural product structure, $K(\!(t)\!)$ becomes a field.

Some important spaces are naturally embedded in $K(\!(t)\!)$, namely the ring of Laurent polynomials $K[t, t^{-1}]$ and notably, also the field of rational functions:

$$K(t) := \operatorname{Frac} K[t].$$

Indeed, the support of a polynomial $Q \in K[t]$, considered as a formal series, is finite. Consequently, it is naturally a Laurent series. Then, by formula (3.3.4) above, we can construct a formal inverse of Q in the field of Laurent series. Now, any rational function of the form:

$$\frac{P(t)}{Q(t)} = P(t) \, Q^{-1}(t),$$

with also $P \in R[t]$, can be expanded as a Laurent series: it suffices to use the multiplication rule (3.3.3) in order to compute the product (in the field of Laurent series) of the numerator P by the formal inverse "Q^{-1}" obtained after using the expansion rule (3.3.4). Thus, any rational function enjoys a natural Laurent expansions.

When a Laurent series expansion is done by applying the rule (3.3.4), we will mention that it is done "*under the assumption* $t \ll 1$". This yields an injective morphism of fields:

$$\Psi^0 \colon K(t) \hookrightarrow K(\!(t)\!),$$

that we call *Laurent expansion of rational functions at the origin*.

Multivariate case. We now want to generalize this setting to the multivariate case. Some objects have natural generalizations, as *e.g.* the vector space of formal series in the indeterminates $t_1, \dots, t_\kappa$:

$$K[[t_1, t_1^{-1}, \dots, t_\kappa, t_\kappa^{-1}]],$$

the ring of Laurent polynomials:

$$K[t_1, t_1^{-1}, \dots, t_\kappa, t_\kappa^{-1}]$$

and the field of rational functions as well:

$$K(t_1, \dots, t_\kappa).$$

However, in order to expand a rational function of $t_1, \dots, t_\kappa$ under the form of a generalized Laurent series, it is necessary to assign at first a total ordering to the variables t_i.

As an example, for the simplest case $\kappa = 2$, consider a rational function $Q \in K(t_1, t_2)$. Through the two isomorphisms:

$$K(t_1, t_2) \simeq K(t_1)(t_2) \simeq K(t_2)(t_1),$$

the function $Q \in K(t_1, t_2)$ can be seen either as a rational function $Q \in K(t_1)(t_2)$ of the variable t_2 with coefficients in the field $K' = K(t_1)$, or as a rational function $Q \in K(t_2)(t_1)$ of the variable t_1 with coefficients in the field $K' = K(t_2)$. Thus, a direct application of the procedure (3.3.4) of Laurent expansion at the origin provides *two* ways of expanding Q at the origin: either we take $Q \in K(t_1)(t_2)$, then expand it with respect to t_2 in the field $K(t_1)\langle\!\langle t_2 \rangle\!\rangle$ of Laurent series at the origin with coefficients in $K(t_1)$:

$$Q(t_1, t_2) \mapsto \sum_{i \geqslant N} \Psi_i(t_1)\, t_2^i,$$

and lastly we expand all coefficients $\Psi_i(t_1) \in K(t_1)$ in the field of Laurent series with coefficients in K, or we reverse the roles of t_2, t_1 — *i.e.* we consider Q as an element of $K(t_2)(t_1)$ — and we perform the corresponding two successive Laurent expansions.

Observe that once we have written $Q \in K(t_2)(t_1)$ — explicitly: $Q \in K'(t_1)$ with the field of coefficients being $K' = K(t_2)$ — we have a *unequivocal* Laurent expansion of Q at the origin, since here t_1 *is* the variable, while $t_2 \in K'$ is a plain coefficient. It is only in the second step, when we consider the coefficients $\Psi_i(t_2)$, that the symbol t_2 becomes truly a variable. In summary, when we write $Q \in K(t_2)(t_1)$, we work step by step in the univariate setting: *firstly* with the formal variable t_1, *secondly* with the formal variable t_2.

The more precise example of:

$$Q\colon (t_1, t_2) \mapsto (t_1 - t_2)^{-1}$$

can be further analyzed as follows. We want to define two new fields $K\langle\!\langle t_1, t_2 \rangle\!\rangle$ and $K\langle\!\langle t_2, t_1 \rangle\!\rangle$ — taking account of the order in which t_1 and t_2 appear — such that: when working in $K(t_1)(t_2)$, the result of the successive series expansions of Q at the origin:

$$\frac{1}{t_1} \cdot \left(1 + \frac{t_2}{t_1} + \left(\frac{t_2}{t_1}\right)^2 + \cdots\right) \in K\langle\!\langle t_2, t_1 \rangle\!\rangle,$$

will be an element of $K\langle\!\langle t_2, t_1 \rangle\!\rangle$ while, when working in $K(t_2)(t_1)$, the result of the successive series expansions of Q at the origin:

$$-\frac{1}{t_2} \cdot \left(1 + \frac{t_1}{t_2} + \left(\frac{t_1}{t_2}\right)^2 + \cdots\right) \in K\langle\!\langle t_1, t_2 \rangle\!\rangle,$$

will be an element of $K\langle\!\langle t_1, t_2 \rangle\!\rangle$.

Notice that this last expression is a Laurent series in t_1 but is *not* a Laurent series in t_2, because they are infinitely many negative exponents i_2 such that the monomial ${t_2}^{i_2}$ has a non-zero coefficient. As a consequence, it is *not* an element of the usual space of multivariate Laurent series:

$$K(\!(t_1, t_2)\!) := K[[t_1, t_2]][t_1^{-1}, t_2^{-1}].$$

Nevertheless, for any (fixed) integer i_1, the coefficient of ${t_1}^{i_1}$ is a Laurent series in the variable t_2. This heuristic justifies that we set as a definition:

$$K\langle\!\langle t_1, t_2 \rangle\!\rangle := K(\!(t_2)\!)(\!(t_1)\!),$$

that is to say the field of Laurent series in the variable t_1 with coefficients that are Laurent series in t_2. It is indeed a field for the sum and the Cauchy product (with the variable t_1), since we have already seen that, when K' is a field, $K'(\!(t)\!)$ is a field. We can thus inductively define a vector subspace of the space of formal power series $K[[t_1, t_1^{-1}, \dots, t_{k+1}, t_{k+1}^{-1}]]$ by:

$$K\langle\!\langle t_1 \rangle\!\rangle := K(\!(t_1)\!),$$
$$K\langle\!\langle t_1, \dots, t_{k+1} \rangle\!\rangle := K\langle\!\langle t_2, \dots, t_{k+1} \rangle\!\rangle(\!(t_1)\!),$$

that is clearly a field in itself.

One easily convinces oneself that the successive series expansions at zero (taking account of the ordering of the variables) yield an injective morphism of fields:

$$\Psi^0\colon K(t_\kappa)(t_{\kappa-1})\dots(t_1) \hookrightarrow K\langle\langle t_1, t_2, \dots, t_\kappa\rangle\rangle,$$

that we call *Laurent expansion of rational functions at the origin, under the assumption* $t_1 \ll \cdots \ll t_{n-1} \ll t_n \ll 1$. The map Ψ^0 is indeed injective because its image contains only summable series, therefore its left inverse is the successive summation for $t_\kappa, t_{\kappa-1} \dots, t_1$.

Here, the notation $t_1 \ll t_2 \ll \cdots \ll t_\kappa \ll 1$ is an abbreviation for the κ univariate assumptions:

$$\text{"}t_k \ll 1\text{"} \qquad {\scriptstyle (k=1\dots\kappa)},$$

formulated in the field of coefficients $K'_k = K(t_\kappa)\dots(t_{k+1})$ — *i.e.* where 1 stands for any rational expression involving $t_{k+1}, \dots, t_\kappa$ — when making the $(n-k+1)$-th step of expansion of all coefficients.

The idea is that for two integers $k < k'$ the variable t_k is infinitely smaller than any (positive or negative) power of the variable $t_{k'}$. The first hypothesis means that we first expand Q at the origin as a rational function of t_1, formally considering any rational expression made of constants of K, and variables $t_2, \dots, t_\kappa$ as elements of the field of coefficients. Then, when expanding the coefficients of the resulting series, we forget t_1 and we have: $t_2 \ll t_3 \ll \dots t_\kappa \ll 1$, what accordingly means $t_2 \ll 1$, where the field of coefficients is $K(t_\kappa)\dots(t_3)$. We iterate the procedure until we get $t_\kappa \ll 1$, that is the one-dimensional case.

Thus, an element of $K\langle\langle t_1, \dots, t_\kappa\rangle\rangle$ should be seen as a Laurent series in t_1 whose coefficients are Laurent series in t_2 whose coefficients are Laurent series in t_3 and so on... Accordingly we call such an element an *iterated Laurent series*.

It is a bigger space than the space of multivariate Laurent series. A formal series Ψ is an element of $K\langle\langle t_1, \dots, t_\kappa\rangle\rangle$ if and only if its support is well ordered for the lexicographic order. This condition is clearly weaker than to be bounded from below for the standard product order on $\mathbb{Z}^\kappa$ (see the example of $(t_1 - t_2)^{-1}$ just above).

We can extend the coefficient extraction operator to the field of rational functions $K(t_1, \dots, t_\kappa)$ by using the injection Ψ^0. For a rational function $Q \in K(t_1, \dots, t_\kappa)$, we always imply the assumption $t_1 \ll \cdots \ll t_\kappa \ll 1$ and we define the coefficient extraction operator:

$$\big[t_1^{i_1} \cdots t_\kappa^{i_\kappa}\big]\big(Q\big) := \big[t_1^{i_1} \cdots t_\kappa^{i_\kappa}\big]\big(\Psi^0(Q)\big).$$

This convention in turn allows us to define the Cauchy product of a rational function by an iterated Laurent series, by using the same formalism as in (3.3.1).

Integration formula. We are now in position to state a more tractable version of formula (3.2.4):

(3.3.5) **Theorem** (Fiber Integration on the Demailly tower). *For any polynomial:*

$$f \in H^\bullet(\overline{X}_0, V_0)[t_1, \dots, t_\kappa],$$

in κ variables $t_1, \dots, t_\kappa$, with coefficients in the cohomology ring $H^\bullet(\overline{X}_0, V_0)$, having total degree at most n_κ, the cohomology class:

$$f\big(\underline{v}\big) = f\big(v_1, \dots, v_\kappa\big) \in H^\bullet\big(\overline{X}_\kappa\big),$$

can be integrated along the fibers of the projective bundle $\overline{X}_\kappa \to \overline{X}_0$ according to the formula:

$$\int_{\overline{X}_\kappa} f\big(\underline{v}\big) = \big[t_1^r \cdots t_\kappa^r\big]\Big(\Phi_\kappa\big(\underline{t}\big) \int_{\overline{X}_0} f\big(\underline{t}\big) s_{1/t_1}(V_0) \cdots s_{1/t_\kappa}(V_0)\Big),$$

where Φ_κ is the universal rational function:

$$\Phi_\kappa(t_1,\dots,t_\kappa)=\prod_{1\leqslant i<j\leqslant\kappa}\frac{t_j-t_i}{t_j-2\,t_i}\prod_{2\leqslant i<j\leqslant\kappa}\frac{t_j-2\,t_i}{t_j-2\,t_i+t_{i-1}}.$$

Proof. The product Φ_κ can be reshaped as follows:

$$\Phi_\kappa(t_1,\dots,t_\kappa)=\prod_{j=2}^{\kappa-1}\underbrace{\frac{t_j-t_1}{t_j-2\,t_1}}_{\Phi_{1,j}(\underline{t})}\prod_{i=2}^{j-1}\underbrace{\frac{t_j-t_i}{t_j-2\,t_i+t_{i-1}}}_{\Phi_{i,j}(\underline{t})}.$$

For $1\leqslant i<j\leqslant\kappa-1$, let:

$$R_{i,j}(\underline{t}):=\Psi^0(\Phi_{i,j}(\underline{t}))-\Phi_{i,j}(\underline{t}).$$

By coming back to the definitions, it is immediate to see that these remainders are:

$$R_{1,j}(\underline{t})=\left(1-\frac{t_i}{t_j}\right)\sum_{k>n_{\kappa-1}}\frac{(2t_i)^k}{t_j^k}\quad\text{and}\quad R_{i,j}(\underline{t})=\left(1-\frac{t_i}{t_j}\right)\sum_{k>n_{\kappa-1}}\frac{(2t_i-t_{i-1})^k}{t_j^k}\qquad{\scriptstyle(2\leqslant i<j\leqslant\kappa-1)}.$$

Noteworthy, the supports of these remainders satisfy:

$$\operatorname{supp}R_{i,j}\subset\Big\{(i_1,\dots,i_\kappa)\colon i_j<-n_{\kappa-1}\Big\}\cap\bigcap_{k>j}\Big\{(i_1,\dots,i_\kappa)\colon i_k=0\Big\}.$$

One can write:

$$\Psi^0(\Phi(\underline{t}))=\prod_{\text{entries}}\begin{pmatrix}1 & \Phi_{1,2}+R_{1,2} & \cdots & \Phi_{1,\kappa}+R_{1,\kappa}\\ \vdots & \ddots & \ddots & \vdots\\ & & & \Phi_{\kappa-1,\kappa}+R_{\kappa-1,\kappa}\\ 1 & \cdots & & 1\end{pmatrix}.$$

We clean inductively this array of the remainder in the k-th column. Let array_j be the array deduced from the above array by dropping the remainders in the j last columns. We will show that:

$$\Big[t_1^r\cdots t_\kappa^r\Big]\Big(\prod\mathsf{array}_j\,I(t_1,\dots,t_\kappa)\Big)=\Big[t_1^r\cdots t_\kappa^r\Big]\Big(\prod\mathsf{array}_{j+1}\,I(t_1,\dots,t_\kappa)\Big)$$

where:

$$I(t_1,\dots,t_\kappa):=\int_{\overline{X}_0}f(t_1,\dots,t_\kappa)\,s_{1/t_1}(V_0)\cdots s_{1/t_\kappa}(V_0).$$

We generalize the notation $I(t_1,\dots,t_\kappa)$ by setting:

$$I(t_1,\dots,t_i):=\Big[t_{i+1}^r\cdots t_\kappa^r\Big]\Big(I(\underline{t})\prod_{k=i+1}^{\kappa-1}\prod_{j=1}^{k-1}\Phi_{j,k}(\underline{t})\Big)\qquad{\scriptstyle(i=0,1,\dots,\kappa)}.$$

Then we claim (find the proof below):

$$\operatorname{supp}I(t_1,\dots,t_{\kappa-j})\subset\Big\{(i_1,\dots,i_{\kappa-j})\colon i_{\kappa-j}\leqslant n_\kappa\Big\}.$$

On the other hand, one is easily convinced that:

$$\prod\mathsf{array}_j=\prod\mathsf{array}_{j+1}+R_j\prod_{l=\kappa-j+1}^{\kappa-1}\prod_{k=1}^{l-1}\Phi_{k,l}(\underline{t}),$$

where R_j is a given series such that:

$$\operatorname{supp} R_j(t_1,\dots,t_{\kappa-j}) \subset \Big\{(i_1,\dots,i_{\kappa-j})\colon i_{\kappa-j} < -n_{\kappa-1}\Big\}.$$

It is now clear that the second part cannot contribute to the coefficient of $t_{\kappa-j}^r$ in:

$$[t_{\kappa-j+1}^r \cdots t_\kappa^r]\Big(\prod \mathsf{array}_j\, I(\underline{t})\Big),$$

because the degree of $t_{\kappa-j}$ in the corresponding term:

$$[t_{\kappa-j+1}^r \cdots t_\kappa^r]\Big(R_j(t_1,\dots,t_{\kappa-j}) \prod_{l=\kappa-j+1}^{\kappa-1} \prod_{k=1}^{l-1} \Phi_{k,l}(\underline{t})\, I(\underline{t})\Big) = R_j(t_1,\dots,t_{\kappa-j})\, I(t_1,\dots,t_{\kappa-j})$$

is *strictly* less than $-n_{\kappa-1} + n_\kappa = r$.

Consequently:

$$\Big[t_{\kappa-j}^r \cdots t_\kappa^r\Big]\Big(\prod \mathsf{array}_j\, I(t_1,\dots,t_\kappa)\Big) = \Big[t_{\kappa-j}^r \cdots t_\kappa^r\Big]\Big(\prod \mathsf{array}_{j+1}\, I(t_1,\dots,t_\kappa)\Big),$$

and by extraction of the coefficient of the monomial $t_1^r \cdots t_{\kappa-j-1}^r$, as announced:

$$\Big[t_1^r \cdots t_\kappa^r\Big]\Big(\prod \mathsf{array}_j\, I(t_1,\dots,t_\kappa)\Big) = \Big[t_1^r \cdots t_\kappa^r\Big]\Big(\prod \mathsf{array}_{j+1}\, I(t_1,\dots,t_\kappa)\Big).$$

An induction finishes the proof because:

$$\Psi^0(\Phi_\kappa) = \prod \mathsf{array}_0 \quad \text{and} \quad \Phi_\kappa = \prod \mathsf{array}_\kappa. \qquad \square$$

We have added a lot of non contributive terms, however in practice (*cf.* chap. IV), the above reformulation of (3.2.4) is more efficient, because it takes account of the convergence of the series at stake.

Finally, we prove our claim above in the proof, that:

$$\operatorname{supp} I(t_1,\dots,t_{\kappa-j}) \subset \Big\{(i_1,\dots,i_{\kappa-j})\colon i_{\kappa-j} \leqslant n_\kappa\Big\}.$$

Actually we will be more precise and show that for $j = 1,\dots,\kappa$:

$$\operatorname{supp} I(t_1,\dots,t_j) \subset \Big\{(i_1,\dots,i_j)\colon i_j \leqslant n_j\Big\}.$$

Proof. In order to prove this statement, it is easier to work with genuine polynomials, and not Laurent polynomials. An important remark is that for any two integers $k < l$, the Laurent polynomial $t_l^{n_{\kappa-1}} \Phi_{k,l}(\underline{t})$ is in fact a genuine polynomial, having degree $n_{\kappa-1}$. Thus, we rather consider:

$$t_j^n I(t_1,\dots,t_j) = \Big[t_{j+1}^{m_{j+1}} \cdots t_{\kappa-1}^{m_{\kappa-1}}\, t_\kappa^{n+r}\Big]\Big(\Big(t_j^n \cdots t_\kappa^n\, I(\underline{t})\Big) \prod_{l=j+1}^{\kappa-1} \prod_{k=1}^{l-1} \Big(t_l^{n_{\kappa-1}} \Phi_{k,l}(\underline{t})\Big)\Big),$$

where

$$m_l = n + r + (l-1)(n_{\kappa-1}).$$

The appearing polynomial in $t_j, t_{j+1},\dots,t_\kappa$ has degree:

$$\deg_{t_j,\dots,t_\kappa}\Big(\Big(t_j^n \cdots t_\kappa^n\, I(\underline{t})\Big) \prod_{l=j+1}^{\kappa-1} \prod_{k=1}^{l-1} \Big(t_l^{n_{\kappa-1}} \Phi_{k,l}(\underline{t})\Big)\Big) \leqslant n_\kappa + n + \sum_{l=j+1}^{\kappa-1} (m_l - r) + n.$$

Extracting the coefficient of $\Big[t_{j+1}^{m_{j+1}} \cdots t_{\kappa-1}^{m_{\kappa-1}}\, t_\kappa^{n+r}\Big]$ decrease the degree by at least:

$$\sum_{l=j+1}^{\kappa-1} m_l + (n+r).$$

Finally we get a polynomial in t_j having degree:

$$\deg_{t_j}\Big(t_j^n I(t_1,\dots,t_j)\Big) \leqslant n + n_\kappa - r - \sum_{l=j+1}^{\kappa-1} r = n + n_j.$$

Thus, as announced:

$$\deg_{t_j}\Big(I(t_1,\dots,t_j)\Big) \leqslant n_j. \qquad \square$$

Notice that our theorem holds as well without the (natural) technical assumption on the degree of f.

CHAPTER III

Slanted Vector Fields for Logarithmic Jets Spaces

Contents

1. Introduction

The formalism of *jets* is a coordinate-free description of the differential equations that holomorphic curve may satisfy. For a map $f\colon \mathbb{C} \to X$, valued in a complex projective manifold X, the k-jet map $f_{[k]}\colon \mathbb{C} \to J_kX$ valued in the k-jet bundle J_kX corresponds to the truncated Taylor expansion of f at order k in some local coordinates system. In J_kX, each *jet-coordinate* $f_i^{(j)}$ shall be considered as an independant coordinate, as a consequence, each algebraic differential equation (with holomorphic coefficients) of order k shall be thought of as a polynomial equation in J_kX:

$$P(f', f'', \dots, f^{(k)}) \equiv 0.$$

Similarly, if $D \subset X$ is a normal crossings divisor, the submanifold $J_kX(-\log D) \subset J_kX$ of *logarithmic k-jets* on X along D can be defined by considering the logarithmic derivatives in the direction of D (see below).

Schwarz lemma. A (logarithmic) *k-jet differential* is locally a polynomial in the (logarithmic) jet-coordinates $f_i^{(j)}$ having constant homogeneous weight, when the weight of $f_i^{(j)}$ is the number of "primes" j. The jet differentials enjoy the following *fundamental vanishing theorem*:

If σ is a holomorphic jet differential on X with logarithmic poles along D, vanishing on an ample divisor, and f is a holomorphic map $\mathbb{C} \to X \setminus D$, then the pullback $f^\star\sigma = P(f', \dots, f^{(k)})$ vanishes identically on $\mathbb{C}$.

When the canonical divisor $K_X + D$ is big, an interesting question, motivated by the longstanding Green-Griffiths conjecture ([**22**]), is the algebraic degeneracy of such holomorphic maps $\mathbb{C} \to X \setminus D$. Starting with a lot of differential equations as above, the overall idea is to decrease the degree of the differential equations by algebraic elimination until the obtainment of a differential equation of degree 0, that is an analytic equation satisfied by every entire curve $f\colon \mathbb{C} \to X \setminus D$. We will briefly recall the key points of this strategy, successfully implemented both in the compact setting ($X \setminus D = X_d \subset \mathbb{P}^n$, [**13**]) and in the logarithmic setting ($X \setminus D = \mathbb{P}^n \setminus X_d$, chapter IV).For more details, the reader is referred to the comprehensive recent article [**47**] by Yum-Tong Siu.

Siu's strategy. The general idea is that the vector fields $\mathsf{V} \in T_{J_k X}$ applied to P produce new differential equations. However, the obtained equation is not necessarily satisfied. Indeed, if the pole order of a vector field V is bigger than the vanishing order of σ, then the hypothesis of the above vanishing theorem are not satisfied by $\mathsf{V} \cdot \sigma$ anymore! It is thus important to control the pole order of the vector fields. On a regular hypersurface of high degree, there cannot be sufficiently nonzero meromorphic vector fields having low pole order, but one can use the positivity of the moduli space of degree d hypersurfaces in $\mathbb{P}^n$ in order to get a lot of low pole order vector fields.

Let $S = \mathbb{P}^{\binom{n+d}{d}-1}$ be the moduli space of all degree d hypersurfaces in $\mathbb{P}^n$, that is the projective space of homogeneous polynomials of degree d on $\mathbb{P}^n$:

$$S := \mathbb{P}\left\{\sum_{|\alpha|=d} A_\alpha Z^\alpha \colon A_\alpha \in \mathbb{C}\right\}.$$

The *universal family of degree d hypersurfaces in* $\mathbb{P}^n$ is the subspace $\mathcal{H} \subset \mathbb{P}^n \times S$ defined by:

$$\mathcal{H} := \left\{[Z], [A]\colon \sum_{|\alpha|=d} A_\alpha Z^\alpha = 0\right\}.$$

The space of vertical logarithmic jets is the subspace $J_k^{\text{vert}}(\mathbb{P}^n \times S)(-\log \mathcal{H}) \subset J_k(\mathbb{P}^n \times S)(-\log \mathcal{H})$, consisting of jets tangent to the fibers of the second projection $\mathbb{P}^n \times S \to S$. These jets are introduced in order to use the Schwarz lemma fiberwise.

We call *slanted vector field* a vector field on $J_k^{\text{vert}}(\mathbb{P}^n \times S)(-\log \mathcal{H})$ that is not tangential to the space of k-jets of the vertical fiber at a generic point of $J_k^{\text{vert}}(\mathbb{P}^n \times S)(-\log \mathcal{H})$. When low pole order meromorphic slanted vector fields are used, we will show that it is possible to produce sufficiently many differential equations in order to eliminate the derivatives $f', \dots, f^{(k)}$ in the differential algebraic equations.

Background. The method of slanted vector fields has been introduced by Siu ([**46**]), and is motivated by the work of Clemens-Ein-Voisin ([**6, 18, 50**]) on rational curves. It has been pushed further by Păun ([**37**]) in dimension 2, and by Rousseau in dimension 3, both for the compact case ([**43**]) and for the logarithmic case ([**42**]). In the compact case, the technique has been generalized in any dimension by Merker ([**29**]), with a substantial improvement of the determination of the locus where the global generation statement fails, leading to a proof of the *strong* algebraic degeneracy of entire curves with values in a generic projective hypersurface of large degree ([**13**]). In the slightly different context of projective hypersurfaces in families, Mourougane ([**33**]) has implemented the technique in any dimension and for any order.

In the present work, these Siu-Merker slanted vector fields are developed in the *logarithmic setting*. Moreover, some details left to the reader in the previous works are completed in a new approach. The proof provides thus an important clarification of the locus where the global generation statement fails, in both compact an logarithmic cases.

Main result. Low pole order meromorphic frames of slanted vector fields are constructed on the space of vertical logarithmic k-jets along the universal family $\mathcal{H}$ of degree d projective hypersurfaces in $\mathbb{P}^n$. Let $\eta\colon f_{[\kappa]} \mapsto f(0)$ denotes the evaluation of the jets. The following statement is established:

Main Theorem. *Suppose that the order k of the jets is smaller than the degree d, then the twisted holomorphic tangent bundle to the vertical k-jets of the log-manifold* $\big(\mathbb{P}^n \times S, \mathcal{H}\big)$*:*

$$T_{J_k^{vert}(\mathbb{P}^n\times S)(-\log\mathcal{H})} \otimes \eta^\star\Big(\mathcal{O}_{\mathbb{P}^n}\big(k^2+2k\big)\otimes\mathcal{O}_S\big(1\big)\Big)$$

is generated by its holomorphic global sections at every point of the subspace of regular vertical logarithmic k-jets of holomorphic curves avoiding $\mathcal{H}$*.*

Organization of the chapter. The chapter is organized as follows. Firstly, in §1, we recall the formalism of jets. Next, in §2, in order to study the logarithmic vector fields, we generalize the strategy implemented by Erwan Rousseau in [**42**] for complements of hypersurfaces in $\mathbb{P}^3$, by locally straightening out the universal hypersurface $\mathcal{H}$. Then we compare two natural systems of jet-coordinates, namely the standard jet-coordinates and the logarithmic jet-coordinates. The first system of coordinates is useful for defining vertical jets, but not adapted to describe logarithmic jets. The second system of coordinates is well adapted to the formalism of logarithmic jets, but the equations to satisfy for a tangent vector field to be vertical are slightly more complicated. Lastly, in §3, we prove the main theorem on global generation of the tangent bundle. In the directions tangent to the space of parameters S, the arguments introduced by Merker for the compact case ([**29**]) directly apply, and the space of vertical vector fields has the right codimension $k+1$, provided one works outside of the set $\{z'_j \neq 0\}$ where the derivative in the j-th direction does not vanish. In the remaining directions, tangent to the space spanned by the jet coordinates, the matrix approach presented in the work of Păun ([**37**]), Rousseau ([**43, 42**]) and Merker ([**29**]) is generalized and reformulated with new arguments. Some new vector fields that generate all the remaining tangent directions outside the set $\{z'_j \neq 0\}$ are introduced, for which it is easy to find a slanted counterpart by solving a simple linear system.

Typographical conventions and notation.

— Throughout this text, lower case greek letters (like α, β, γ, …) will be used to denote *multi-indices* in $\mathbb{N}^n$. The entries of such a multi-index α are written $\alpha_1, \dots, \alpha_n$. As a special case, δ^j will denote the multi-index with entries δ_i^j, where δ is the usual Kronecker symbol. In other words:

$$\alpha = \sum_{i=1}^{n} \alpha_i\,\delta^i.$$

— In order to shrink the notation, $\alpha!$ will denote the product of factorials:

$$\alpha! := \alpha_1! \cdots \alpha_n!,$$

and for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ the notation z^α will be used for the monomial:

$$z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

— The relation $\leqslant$ will stand for the standard *product order* on $\mathbb{N}^n$, namely:

$$\lambda \leqslant \lambda' \quad\Leftrightarrow\quad \lambda_i \leqslant \lambda'_i \quad {\scriptstyle (i=1,\dots,n)}.$$

— The *length* of a multi-indice $\lambda \in \mathbb{N}^n$ is the sum:

$$|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_k,$$

and its *weight* is the weighted sum:

$$\|\lambda\| := \lambda_1 + 2\lambda_2 + \cdots + k\lambda_k.$$

Clearly, for any multi-indice $\lambda \in \mathbb{N}^n$, $|\lambda| \leqslant \|\lambda\|$. The equality case corresponds to the multi-indices of the form $\lambda = (\lambda_1, 0 \ldots, 0)$.

Important Notation.

— We will rather use the notation $f^{(j)}$ for the univariate j-th Taylor coefficient than for the univariate j-th derivative, *i.e.*:

$$f^{(j)} := \frac{1}{j!}\frac{\partial^j f}{\partial t^j}.$$

A reason is that this choice simplifies the coefficients of the Faà di Bruno formulae.

Faà di Bruno formulae. Let $f\colon \mathbb{C} \to X$ be a regular mapping of class $\mathscr{C}^k$ and let $h\colon \mathbb{C} \to \mathbb{C}$ be a parametrization of the source of class $\mathscr{C}^k$. Then for any integer $j = 0, 1, \ldots, k$, the j-th Taylor coefficient of the composition $f \circ h$ is given by:

$$\big(f \circ h\big)^{(j)} = \sum_{i \leqslant j} \mathscr{B}_{i,j}(h)\, \frac{\partial_z^i}{i!} f \circ h, \tag{1.0.6}$$

where $\mathscr{B}_{i,j}$ is the Bell's polynomial:

$$\mathscr{B}_{i,j}(h) := \sum_{|\mu|=i, \|\mu\|=j} \frac{|\mu|!}{\mu!}\, h^{(1)\mu_1} \cdots h^{(k)\mu_k}. \tag{1.0.7}$$

Let $f\colon \mathbb{C}^n \to X$ be a regular mapping of class $\mathscr{C}^k$ and let $h\colon \mathbb{C} \to \mathbb{C}^n$ be a parametrization of the source of class $\mathscr{C}^k$. Then for any integer $j = 0, 1, \ldots, k$, the j-th Taylor coefficient of the composition $f \circ h$ is given by:

$$\big(f \circ h\big)^{(j)} = \sum_{|\lambda| \leqslant j} \mathscr{B}_{\lambda,j}(h)\, \frac{\partial_{z_1}^{\lambda_1}}{\lambda_1!} \cdots \frac{\partial_{z_n}^{\lambda_n}}{\lambda_n!} f \circ h, \tag{1.0.8}$$

where $\mathscr{B}_{\lambda,j}$ is the multivariate Bell's polynomial:

$$\mathscr{B}_{\lambda,j}\big(h_1, \ldots, h_n\big) := \sum_{|\mu|=j} \mathscr{B}_{\lambda_1,\mu_1}(h_1) \cdots \mathscr{B}_{\lambda_n,\mu_n}(h_n). \tag{1.0.9}$$

2. Logarithmic Jet Bundles.

2.1. Jet manifold. Let X be a n-dimensional complex manifold and $k \in \mathbb{N}$ an integer. At any point $x \in X$, the set $J_k X_x$ of *k-jets of germs of parametrized curves* at x is by definition the quotient space of the set of local holomorphic curves $f\colon (\mathbb{C}, 0) \to (X, x)$ defined on an open neighborhood Ω_f of the origin by the equivalence relation:

$$f \overset{k}{\sim} g \Leftrightarrow \big(f^{(j)}(0) = g^{(j)}(0) \text{ for } j = 0, 1, \cdots, k\big),$$

that is f is equivalent to g if and only if the Taylor coefficients of f and g at the origin are equal up to order k for some local coordinate system, on an open neighborhood $\Omega_{f,g} \subset \Omega_f \cap \Omega_g$ of the origin. This relation is independent of the choice of a system of coordinates around x.

The example of polynomials shows that every k-jet may be represented by a local holomorphic map. The space $J_k X_x$ therefore identifies, by Taylor formula, to the vector space $\mathbb{C}^{nk}$.

Moreover, we claim that the collection of these vector spaces gives rise to a holomorphic fiber bundle:

$$J_kX := \bigsqcup_{x\in X}\big(x, J_kX_x\big),$$

the projection on the basis X being the evaluation map η of germs at the origin. This bundle is called the *k-jet manifold of X*.

An illustrative example is the bundle of 1-jets $J_1X \simeq T_X$, that is isomorphic to the holomorphic tangent bundle to X. However this picture is not very representative, because for $k > 1$, it is well known that J_kX is *not* a vector bundle.

Let Ω be an open subset of $\mathbb{C}$ and $f\colon \Omega \to X$ a local holomorphic mapping, then f lifts canonically to a map:

$$f_{[k]}\colon \Omega \to J_kX,$$

called the *k-jet of f*, such that $\eta \circ f_{[k]} \equiv f$. When Ω and X are equipped with coordinates, this map $f_{[k]}$ is determined by the truncation at order k of the Taylor expansion of f.

2.2. Local triviality in coordinates. To get a local trivialization of J_kX around a point $x \in X$, the first naive idea is to consider the derivatives in some local coordinates $(z_1, \dots, z_n)$ over an open subset $U \subset X$ around x. A construction due to Noguchi **[35]** allows more general "derivatives", that are potentially more adapted to the geometric situation that will be dealt with below. Here "derivative"means pullback of local meromorphic 1-forms, as by the very definition of the pullback of a 1-form, for every local holomorphic map $f\colon \Omega \to U$, one has:

$$f^\star \mathrm{d}z_i|_t = \mathrm{d}f_i|_t = f_i'(t)\,\mathrm{d}t,$$

where we equip the complex plane $\mathbb{C}$ with the standard coordinate t.

We will use more general local meromorphic 1-forms that only $\mathrm{d}z_i$, and we even do not demand them to be locally exact. Let thus $\omega \in T_U^\star$ be a local meromorphic 1-form over an open subset $U \subset X$ and let $f\colon \Omega \to X$ be a local holomorphic mapping over an open subset $\Omega \subset \mathbb{C}$. This mapping f induces a meromorphic function $A'\colon \Omega \to \mathbb{C}$ by the formula:

$$f^\star\omega|_t =: A'(t)\mathrm{d}t.$$

By definition of the pullback of a differential form, $A'(t)$ is the coefficient of the linear map $\mathbb{C} \to \mathbb{C}$:

$$f^\star\omega|_t := v \mapsto \omega|_{f(t)}\big(\mathrm{d}f_t v\big).$$

Thus, the obtained map A' depends only on the 1-jet $f_{[1]}$ of f. More generally, the derivatives:

$$\partial_t^j A' := \frac{\mathrm{d}^j}{\mathrm{d}t^j}\big(A'\big) \qquad {\scriptstyle (j=1,\dots,k-1)}$$

up to order $k-1$ of A' are well-defined and depend only on the lifting $f_{[k]}$. One gets a meromorphic mapping $\tilde\omega\colon J_kX|_U \to \mathbb{C}^k$ such that:

$$\tilde\omega\big(f_{[k]}\big) := \Big(A', \frac{\partial_t A'}{2!}, \dots, \frac{\partial_t^{k-1}A'}{k!}\Big)(0).$$

If ω is taken holomorphic, then $\tilde\omega$ becomes also holomorphic. In the simplest particular case where $\omega = \mathrm{d}z_i$ is locally exact, if f_i denotes the component of f in the direction z_i, one gets the definition of the derivative of f_i at x. Note that any holomorphic 1-forms aren't locally exact, as it is shown by the theorem of normalization of Darboux.

In the basic particular case were locally $\omega = \frac{\mathrm{d}z_i}{z_i}$, we get the logarithmic derivative of f_i.

For a complex manifold X of dimension n, we say that the n-tuple of holomorphic 1-forms $\left(\omega^1, \dots, \omega^n\right)$ is a *holomorphic frame* of $T^\star_X$ if $\omega^1 \wedge \cdots \wedge \omega^n$ is a volume form. Recall that η denotes the evaluation of germs at the origin.

(2.2.1) *Given a holomorphic frame $\left(\omega^1, \dots, \omega^n\right)$ for the holomorphic cotangent bundle $T^\star_X$, the biholomorphic map:*

$$\eta \times \left(\tilde{\omega}^1, \dots, \tilde{\omega}^n\right)\colon J_kX|_U \longrightarrow U \times (\mathbb{C}^k)^n,$$

is called the local trivialization *of J_kX associated to the holomorphic frame $(\omega^1, \dots, \omega^n)$.*

2.3. Noguchi's logarithmic jet bundles. Now we set the *logarithmic setting*. Let again X be a manifold of dimension n. A *reduced divisor* on the manifold X is an effective divisor with all multiplicities equal to 1. At a point $x \in X$, a reduced divisor D is said to *have only simple normal crossings* if locally it looks like the union of coordinate hyperplanes: there exists a coordinate neighborhood U at x with a local holomorphic coordinate system $(z_1, \dots, z_n)$ such that $U \cap D$ is the zero-locus $\left\{(z_1 \cdots z_\ell) = 0\right\}$ for some integer $\ell \leqslant n$. Such a local coordinate system is said to be a *logarithmic coordinate system along D at x*. A reduced divisor is a *simple normal crossing* divisor if it has only simple normal crossing at every point. In this case such a pair (X, D) is called a *log-manifold*. Notice that ℓ depends on the point, *e.g.* if $x \in X \setminus D$, then $\ell = 0$.

Let $\mathcal{O}_X$ denote the structure sheaf of X. Let D be a simple normal crossing divisor on X and $\mathcal{O}_X(D)$ denote its Cartier sheaf. The *$\mathcal{O}_X$-module of logarithmic 1-forms of X along D* is defined in [**25**] as the $\mathcal{O}_X$-submodule $\mathcal{T}^\star_X(\log D) \subset \mathcal{T}^\star_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$, such that:

— If $\ell = 0$, that is x is a point of the open part of $\left(X, D\right)$, the stalk at x is defined as:

$$\mathcal{T}^\star_X(\log D)_x := \mathcal{T}^\star_{X,x}$$

— If $x \in D$ is a point of the divisor, take a neighborhood U of x equipped with a logarithmic coordinate system $(z_1, \dots, z_n)$ along D at x such that: $U \cap D = \left\{z_1 \cdots z_\ell = 0\right\}$, then the stalk is defined as:

$$\mathcal{T}^\star_X(\log D)_x := \sum_{i=1}^{\ell} \mathcal{O}_{X,x} \frac{\mathrm{d}z_i}{z_i} + \sum_{i=\ell+1}^{n} \mathcal{O}_{X,x}\, \mathrm{d}z_i.$$

Clearly $\mathcal{T}^\star_X(\log D)$ is locally free of rank n. The associated vector bundle, denoted by $T^\star_X(\log D)$, is called the *logarithmic cotangent bundle*. Its dual $T_X(-\log D)$ is called the *logarithmic tangent bundle* to X along D.

Recall, after [**35**], that a holomorphic section $s \in H^0\left(U, J_kX\right)$ on an open subset $U \subset X$ is said to be a *logarithmic jet field along D* of order k if the maps:

$$\tilde{\omega} \circ s|_V\colon V \to \mathbb{C}^k$$

are holomorphic, for all logarithmic 1-form $\omega \in H^0\left(V, T^\star_X(\log D)\right)$, where $V \subset U$ is an arbitrary open subset of U.

The sets of logarithmic jet fields along D over open subsets of X form a complete presheaf that defines a sheaf $\mathcal{J}_kX(-\log D)$ over X, called the sheaf of germs of logarithmic jet fields along D over X. This sheaf is isomorphic to the sheaf of sections of a holomorphic subbundle of J_kX. This subbundle, that we denote by:

$$J_kX(-\log D) \subset J_kX$$

is called the *logarithmic jet bundle* of order k along D.

2.4. Local triviality in coordinate. Let U be an open chart on (X, D) with logarithmic coordinates $z_1, \dots, z_n$ along D such that the normal crossing divisor D is the zero-set locus $\{z_1 z_2 \cdots z_\ell = 0\}$ for some $\ell < n$. Let $\omega^1 := \frac{\mathrm{d}z_1}{z_1}, \dots, \omega^\ell := \frac{\mathrm{d}z_\ell}{z_\ell}$ and $\omega^{\ell+1} := \mathrm{d}z_{\ell+1}, \dots, \omega^n := \mathrm{d}z_n$. Recall we have a biholomorphic map:

$$\eta \times (\tilde{\omega}_1, \dots, \tilde{\omega}_n) \colon J_k X(-\log D)|_U \to U \times \mathbb{C}^{kn}.$$

Let $s \in H^0\big(U, J_k X(-\log D)\big)$ be a logarithmic section. Then in this trivialization, $s(z) = \big(f_i^{(j)}(z)\big)_{j=1,\dots,k}^{i=1,\dots,n}$ has holomorphic jet coordinates $f_i^{(j)}$ given in the system of local coordinates $(z_1, \dots, z_n)$ on X around x by:

$$\begin{cases} f_i^{(j)} = (\log \circ s_i)^{(j)} & 0 \leqslant i \leqslant \ell \\ f_i^{(j)} = s_i^{(j)} & \ell + 1 \leqslant i \leqslant n \end{cases}$$

Recall the basic inclusion:

$$s \in J_k X(-\log D) \subset J_k X$$

Thus the standard trivialization on $U \subset \mathbb{C}$ in which $s(z) = \big(s_i^{(j)}(z)\big)_{j=1,\dots,k}^{i=1,\dots,n}$, associated with the holomorphic frame $\mathrm{d}z_1, \dots, \mathrm{d}z_n$, is also available.

These two coordinates systems are related by the univariate Faà di Bruno formula (1.0.6) applied on $s_i = \exp \circ \log(s_i)$ and evaluated at $0 \in \mathbb{C}$ that states, for $i = 0, 1, \dots, \ell$:

$$s_i^{(j)} = s_i \sum_{\|\mu\| = j} \frac{f_i^{(\mu)}}{\mu!}. \tag{2.4.1}$$

In the remaining directions $i = \ell + 1, \dots, n$, the jet coordinates $s_i^{(j)} = f_i^{(j)}$ stay unchanged.

2.5. Regular jets. A jet field $j \in J_k X$ is termed *singular* if it is the lift of a stationary curve, *i.e.* $j = f_{[k]}$ with $f' = 0$. The subset of singular jets will be denoted by $J_k^{\mathrm{sing}} X$. A logarithmic jet field is said *singular* if it is in the Zariski closure:

$$J_k^{\mathrm{sing}} X(-\log D) := \overline{J_k^{\mathrm{sing}} X \cap J_k X(-\log D)}^{\mathrm{Zar}}.$$

A (logarithmic) jet field that is not singular is termed *regular*.

2.6. Log-morphism. Let (X, D) and (X', D') be log-manifolds. A holomorphic map $\pi \colon X' \to X$ such that $\pi^{-1}(D) \subset D'$ is called a *log-morphism* $X' \to X$. Such a map enjoys good functorial properties ([**25**]):

(2.6.1**)** *Given a log-morphism $\pi \colon (X', D') \to (X, D)$, one can define a pullback morphism:*

$$\pi^\star \colon \pi^{-1} T_X^\star(\log D) \to T_{X'}^\star(\log D')$$

from the vector bundle of logarithmic 1*-forms of X along D to the vector bundle of logarithmic* 1*-form of X' along D'.*

Recall that the logarithmic 1-forms play a fundamental role in the definition of logarithmic jet fields. Consequently, the above proposition (2.6.1) allows to define further morphisms ([**10**]):

(2.6.2) *A log-morphism* $\pi\colon (X', D') \to (X, D)$ *induces vector bundle morphisms (for any* $k > 0$*):*

$$\pi_{[k]}\colon J_kX'(-\log D') \to \pi^{-1}J_kX(-\log D),$$

by the formula:

$$\pi_{[k]}\big(f_{[k]}\big) := (\pi \circ f)_{[k]}.$$

In the important case $k = 1$ *this morphism is called the* differential of π*:*

$$\pi_\star := \pi_{[1]}\colon T_{X'}(-\log D') \to \pi^{-1}T_X(-\log D).$$

For a log-morphism $\pi\colon (X', D') \to (X, D)$ that is moreover a fiber bundle, the *vertical bundle* T_π is defined as the kernel of the differential $\pi_\star$. One has thus, by definition, the following short exact sequence of vector bundles over X':

$$0 \to T_\pi \to T_{X'}(-\log D') \xrightarrow{\pi_\star} T_X(-\log D) \to 0.$$

In local coordinates $(x, v) \in \pi^{-1}(U)$ over an open subset $U \subset X$ the fiber of the bundle T_π is identified at a point (x, v) with the tangent space to the fiber $\pi^{-1}(x)$ at v.

3. Vertical Logarithmic Jets

In this section, we compute the algebraic equations of the tangent space to the vertical logarithmic jets.

3.1. Straightening out the universal hypersurface. By its very definition, the universal family $\mathcal{H}$ of hypersurfaces of degree d in $\mathbb{P}^n$ is given in the system of coordinates $\big([Z], [A]\big)$ on $\mathbb{P}^n \times S$ as the following zero set of the universal homogeneous polynomial of degrees $(d, 1)$:

$$\mathcal{H} := \Big\{0 = \sum_{|\alpha|=d} A_\alpha\, Z^\alpha\Big\} \subset \mathbb{P}^n \times S = \mathbb{P}^n \times \mathbb{P}^{\binom{d+n}{d}-1}.$$

This zero locus can be straightened out in the following way. Introduce a new homogeneous "Z-coordinate" $W \in \mathbb{C}$ associated with a new homogeneous "A-coordinate" $A_0 \in \mathbb{C}$, thought of as the coefficient of the monomial W^d. Accordingly consider the new homogeneous polynomial equation of total degree d:

$$A_0\, W^d = \sum_{|\alpha|=d} A_\alpha\, Z^\alpha,$$

and define $\mathcal{X}$ as the zero set:

$$\mathcal{X} := \Big\{A_0\, W^d = \sum_{|\alpha|=d} A_\alpha\, Z^\alpha\Big\} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{\binom{d+n}{d}}.$$

There is a natural *forgetful map*:

$$\pi\colon \mathbb{P}^{n+1} \times \mathbb{P}^{\binom{d+n}{d}} \setminus (\{\forall Z_i = 0\} \cup \{\forall A_\alpha = 0\}) \to \mathbb{P}^n \times \mathbb{P}^{\binom{d+n}{d}-1},$$

that consists in erasing both W and A_0.

Notice that:

$$\mathcal{X} \cap (\{\forall Z_i = 0\} \cup \{\forall A_\alpha = 0\}) \subset \mathcal{X} \cap (\{A_0 = 0\} \cup \{\forall A_\alpha = 0\})$$

Indeed, if $Z = 0$, the equation of $\mathcal{X}$ becomes: $A_0\, W^d = 0$. This implies that either A_0 or W must be zero. But W cannot be zero, because the homogeneous coordinates of the point $[W : 0 : \ldots : 0]$ in the projective space $\mathbb{P}^{n+1}$ cannot be all simultaneously zero. Thus A_0 must be zero.

Let $\mathcal{X}^*$ be the restriction of $\mathcal{X}$ to the affine chart, pointed at the origin:

$$\{A_0 \neq 0\} \setminus \{[1:0:\ldots:0]\} \simeq \mathbb{C}^{n+1} \setminus \{0\}.$$

Then, the projection $\pi|_{\mathcal{X}^*}\colon \mathcal{X}^* \to \mathbb{P}^n \times S$ is well defined and moreover, it is a branched cover of degree d that ramifies exactly over $\mathcal{H}$. Since $A_0 \neq 0$, the inverse image of the universal family $\mathcal{H}$ under this projection identifies with the (straight) hyperplane

$$D := (\pi|_{\mathcal{X}^*})^{-1}(\mathcal{H}) = \{W = 0\}.$$

The map $\pi\colon (\mathcal{X}^*, D) \to (\mathbb{P}^n \times S, \mathcal{H})$ is therefore a log-morphism, that induces a canonical holomorphic map on the spaces of jets of logarithmic curves:

$$\pi_{[k]}\colon J_k\mathcal{X}^*(-\log D) \to \pi^\star J_k(\mathbb{P}^n \times S)(-\log \mathcal{H}),$$

that is clearly dominant, as $d\pi_{[k]}$ is of maximal rank. This projection $\pi_{[k]}$ also send vertical jets on vertical jets. We will thus study the vertical logarithmic jets upstairs, where it is more easy to use logarithmic jet-coordinates.

3.2. Vertical jets in coordinates. Equip the affine chart:

$$\{Z_0 \neq 0\} \subset \mathbb{P}^{n+1}$$

with the standard inhomogeneous coordinates:

$$[W : Z_0 : Z_1 : \ldots : Z_n] \mapsto (w, z_1, \ldots, z_n)$$

(where $z_j = Z_j/Z_0$ and $w = W/Z_0$), and equip also the affine chart, pointed at the origin:

$$\{A_0 \neq 0\} \setminus \{[1:0:\ldots:0]\} \subset \mathbb{P}^{\binom{d+n}{d}}$$

with the standard inhomogeneous coordinates:

$$[A_0 : A_\alpha] \mapsto (a_\alpha)_{\{\alpha \in \mathbb{N}^n \colon |\alpha| \leqslant d\}},$$

where (notice that α_0 does not appear anymore in the indices of a_α):

$$a_{\alpha_1,\ldots,\alpha_n} := A_{(d-\alpha_1-\cdots-\alpha_n,\alpha_1,\ldots,\alpha_n)}/A_0.$$

In these coordinates, the restriction $\mathcal{X}_0$ of $\mathcal{X}^*$ to the chart:

$$\{Z_0 \neq 0\} \times \{A_0 \neq 0\} \setminus \{[1:0:\ldots:0]\}$$

is the zero-set:

$$\mathcal{X}_0 := \Big\{w^d = \sum_{|\alpha| \leqslant d} a_\alpha\, z^\alpha\Big\},$$

and the restriction of D is the hyperplane:

$$D_0 := \Big\{w = 0\Big\}.$$

One can use two meromorphic coframes to trivialize the jets. By using the holomorphic coframe:

$$\mathrm{d}w, \mathrm{d}z_1, \ldots, \mathrm{d}z_n, \mathrm{d}a_\alpha,$$

one gets the standard jet-coordinates:

$$\Big(w^{(j)}, z_1^{(j)}, \ldots, z_n^{(j)}, a_\alpha^{(j)}\Big)_{0 \leqslant j \leqslant k} \in \mathbb{C}^{\left(1+n+\binom{d+n}{d}\right)(k+1)}.$$

On the other hand, by choosing the adapted meromorphic coframe:

$$\mathrm{d}(\log w), \mathrm{d}z_1, \ldots, \mathrm{d}z_n, \mathrm{d}a_\alpha,$$

one gets the logarithmic jet-coordinates along $D_0 = \{w = 0\}$:

$$\Big((\log w)^{(j)}, z_1^{(j)}, \ldots, z_n^{(j)}, a_\alpha^{(j)}\Big)_{0 \leqslant j \leqslant k} \in \mathbb{C}^{\left(1+n+\binom{d+n}{d}\right)(k+1)}.$$

Whereas the z-jet-coordinates and the a-jet-coordinates stay unchanged, the Faà di Bruno formula yields the following relations between the standard jet coordinates $w^{(j)}$ and the logarithmic jet coordinates $(\log w)^{(j)}$:

$$w^{(j)} = w \sum_{\|\lambda\|=j} \frac{1}{\lambda!} (\log w)^{(\lambda)}.$$

Throughout the end of this text, we will use the short notation $\partial_{\bullet} = \frac{\partial}{\partial \bullet}$, for the vector fields on $\mathbb{C}^{\left(1+n+\binom{d+n}{d}\right)(k+1)}$ in the above two trivializations. In this notation, the *formal differentiation of jets* D is the following vector field, that mimic the action of the derivation in standard coordinates:

$$\text{(3.2.1)} \qquad \mathsf{D} := \sum_{j=0}^{k-1} (j+1) z_1^{(j+1)} \, \partial_{z_1^{(j)}} + \cdots + \sum_{j=0}^{k-1} (j+1) z_n^{(j+1)} \, \partial_{z_n^{(j)}} + \sum_{j=0}^{k-1} (j+1) w^{(j+1)} \, \partial_{w^{(j)}}.$$

Here, the coefficient $j+1$ appears only because we use the Taylor coefficients.

Recall that a jet was termed vertical if it is tangent to the projection $\mathbb{P}^{n+1} \times \mathbb{P}^{\binom{d+n}{d}} \to \mathbb{P}^{\binom{d+n}{d}}$. By definition, a formal differentiation in the $(n+1)$ variables $z_1, \dots, z_n, w$ of the defining equation:

$$w^d \equiv F(a, z) := \sum_{|\alpha| \leqslant d} a_\alpha \, z^\alpha,$$

gives the $(k+1)$ defining equations of the submanifold of vertical jets $J_k^{\text{vert}} \mathcal{X}_0$:

$$\text{(3.2.2)} \qquad J_k^{\text{vert}} \mathcal{X}_0 = \left\{ 0 = w^d - F = \mathsf{D}\left(w^d - F\right) = \frac{\mathsf{D}^2}{2!}\left(w^d - F\right) = \cdots = \frac{\mathsf{D}^k}{k!}\left(w^d - F\right) \right\}.$$

The submanifold of *vertical logarithmic k-jets* along D_0:

$$J_k^{\text{vert}} \mathcal{X}_0(-\log D_0) = J_k^{\text{vert}} \mathcal{X}_0 \cap J_k \mathcal{X}_0(-\log D_0).$$

consists of logarithmic k-jets tangent to the fibers of the natural projection over the parameter space.

(3.2.3) *The submanifold of logarithmic vertical k-jets is defined by the $(k+1)$ equations:*

$$w^d \sum_{\|\lambda\|=j} d^{|\lambda|} \, \frac{1}{\lambda!} \, (\log w)^{(\lambda)} = \sum_{|\alpha| \leqslant d} a_\alpha \, \frac{\mathsf{D}^j}{j!}\left(z^\alpha\right) \qquad {\scriptstyle (j=0,1,\dots,k)}.$$

Proof. Here, notice the following subtlety: the formal differentitation of jets D is naturally defined in standard coordinates, whereas the logarithmic jets are naturally defined in logarithmic coordinates. Hence, to get the local equations of the space of logarithmic vertical k-jets, one shall substitute the log-variables $(\log w)^{(j)}$ to the standard variables $w^{(j)}$ in the $(k+1)$ equations of the space of (standard) vertical k-jets $J_k^{\text{vert}}(Y_0)$:

$$\frac{\mathsf{D}^j}{j!}(w^d) = \frac{\mathsf{D}^j}{j!}\left(\sum_{|\alpha| \leqslant d} a_\alpha \, z^\alpha \right) \qquad {\scriptstyle (j=0,1,\dots,k)}.$$

In other words, one has to express the iterations D^j of the map D in terms of the log-variables.

Considering that $w^d = \exp\circ(d\log)w$ and using Faà di Bruno formula:

$$\frac{\mathsf{D}^j}{j!}\big(w^d\big) = \sum_{||\lambda||=j}\frac{1}{\lambda!}w^d\,(d\log w)^{(\lambda)} = w^d\sum_{||\lambda||=j} d^{|\lambda|}\,\frac{1}{\lambda!}\,(\log w)^{(\lambda)},$$

one gets the $(k+1)$ explicit equations of $J_k^{\text{vert}}\mathcal{X}_0$ in logarithmic coordinates:

$$w^d\sum_{||\lambda||=j} d^{|\lambda|}\,\frac{1}{\lambda!}\,(\log w)^{(\lambda)} = \sum_{|\alpha|\leqslant d} a_\alpha\,\frac{\mathsf{D}^j}{j!}\big(z^\alpha\big) \qquad {\scriptstyle (j=0,1,\dots,k)},$$

where notably the w-coordinate and the w-log-derivatives are now separated. □

3.3. Algebraic equations of the tangent space. Now that the equations of the subspace of logarithmic vertical k-jets are expressed in log-jet-coordinates, it is straightforward to derive the algebraic equation of its tangent space.

Consider a vector field on the tangent space $\big(\mathbb{C}^{1+k}\big)^n\times\big(\mathbb{C}^{1+k}\big)\times\big(\mathbb{C}^{1+k}\big)^{\binom{d+n}{d}}$ under the general form:

$$\mathsf{V} := \mathsf{V}_z + \mathsf{V}_w + \mathsf{V}_\alpha,$$

where:

— the vector field V_z is tangent to the space of z-jet-coordinates and has coefficients v_i^j:

$$\mathsf{V}_z := \sum_{1\leqslant i\leqslant n}\sum_{0\leqslant j\leqslant k} v_i^j\,\partial_{z_i^{(j)}},$$

— the vector field V_w is a logarithmic vector field tangent to the space of w-log-jet-coordinates and has coefficients v_w^j:

$$\mathsf{V}_w := \sum_{0\leqslant j\leqslant k} v_w^j\,\partial_{(\log w)^{(j)}},$$

— the vector field V_α is tangent to the space of a-jet-coordinates and has coefficients v_α^j:

$$\mathsf{V}_\alpha := \sum_{|\alpha|\leqslant d}\sum_{0\leqslant j\leqslant k} v_\alpha^j\,\partial_{a_\alpha^{(j)}}.$$

(3.3.1**)** *The $(k+1)$ algebraic differential equations to be satisfied by the vector field:*

$$\mathsf{V} = \mathsf{V}_z + \mathsf{V}_w + \mathsf{V}_\alpha$$

to be tangent to $J_k^{vert}\mathcal{X}_0(-\log D_0)$ are the following, for orders of derivation $j = 0, 1, \dots, k$:

$$\sum_{|\alpha|\leqslant d} d\,a_\alpha\,\sum_{r=0}^{j} v_w^r\,\frac{\mathsf{D}^{j-r}}{(j-r)!}(z^\alpha) = \sum_{i=1}^{n}\sum_{|\alpha|\leqslant d} a_\alpha\,\sum_{r=0}^{j} v_i^r\,\frac{\mathsf{D}^{j-r}}{(j-r)!}\partial_{z_i}(z^\alpha) + \sum_{|\alpha|\leqslant d} v_\alpha^0\,\frac{\mathsf{D}^j}{j!}(z^\alpha).$$

Proof. A vector field:

$$\mathsf{V} = \mathsf{V}_z + \mathsf{V}_w + \mathsf{V}_\alpha$$

on $\left(\mathbb{C}^{k+1}\right)\times\left(\mathbb{C}^{k+1}\right)^n\times\left(\mathbb{C}^{k+1}\right)^{\binom{d+n}{d}}$ is tangent to $J_k^{\text{vert}}\mathcal{X}_0(-\log D_0)$ if it vanishes over the equations (3.2.3):

$$\mathsf{V}\cdot\left(w^d\sum_{\|\lambda\|=j} d^{|\lambda|}\,\frac{1}{\lambda!}\,(\log w)^{(\lambda)}\right)=\mathsf{V}\cdot\left(\sum_{|\alpha|\leqslant d} a_\alpha\,\frac{\mathsf{D}^j}{j!}(z^\alpha)\right)\qquad {\scriptstyle (j=0,1,\dots,k)}.$$

Some parts of V trivially vanish on one or the other side of this equations. It is more judicious to expand V in the above equations:

$$\mathsf{V}_w\left(w^d\sum_{\|\lambda\|=j} d^{|\lambda|}\,\frac{1}{\lambda!}\,(\log w)^{(\lambda)}\right)=\sum_{|\alpha|\leqslant d} a_\alpha\;\mathsf{V}_z\frac{\mathsf{D}^j}{j!}(z^\alpha)+\sum_{|\alpha|\leqslant d}\mathsf{V}_\alpha(a_\alpha)\,\frac{\mathsf{D}^j}{j!}(z^\alpha).$$

Here, the left hand side involves logarithmic jet-coordinates, whereas the right hand side involves standard jet-coordinates. We will now compute the three appearing terms separately.

1. Recall that:

$$\mathsf{V}_w:=\sum_{0\leqslant r\leqslant k} v_w^r\,\partial_{(\log w)^{(r)}},$$

thus by substituting this full expression for the vector field V_w, we get the following expression for the first term:

$$\begin{aligned}\mathsf{V}_w\left(w^d\sum_{\|\lambda\|=j} d^{|\lambda|}\,\frac{1}{\lambda!}\,(\log w)^{(\lambda)}\right)&=\\ &v_w^0\,d\,w^d\sum_{\|\lambda\|=j} d^{|\lambda|}\,\frac{1}{\lambda!}\,(\log w)^{(\lambda)}+w^d\sum_{1\leqslant r\leqslant j} v_w^r\sum_{\|\lambda\|=j} d^{|\lambda|}\,\frac{1}{\lambda!}\,\partial_{(\log w)^{(r)}}\left((\log w)^{(\lambda)}\right),\end{aligned}$$

where, in the indices of the sum, we take account that the appearing logarithmic jet-coordinate have order $r\leqslant j$.

The partial derivatives appearing in the last summand can be replaced by:

$$\partial_{(\log w)^{(r)}}\left((\log w)^{(\lambda)}\right)=\lambda_r(\log w)^{(\lambda-\delta^r)}.$$

By the change of variable $\lambda\leftarrow\lambda-\delta^r$ in the last sum — notice that the terms with $\lambda_r=0$ vanish —, one obtains:

$$\sum_{\|\lambda\|=j} d^{|\lambda|}\,\frac{1}{\lambda!}\,\partial_{(\log w)^{(r)}}\left((\log w)^{(\lambda)}\right)=\sum_{\|\lambda\|=j} d^{|\lambda|}\,\frac{\lambda_r}{\lambda!}\,(\log w)^{(\lambda-\delta^r)}=\sum_{\|\lambda\|=j-r} d^{|\lambda|+1}\,\frac{1}{\lambda!}(\log w)^{(\lambda)}.$$

The first term is now written under the form:

$$d\,w^d\left(v_w^0\sum_{\|\lambda\|=j} d^{|\lambda|}\,\frac{1}{\lambda!}\,(\log w)^{(\lambda)}+\sum_{1\leqslant r\leqslant j} v_w^r\sum_{\|\lambda\|=j-r} d^{|\lambda|}\,\frac{1}{\lambda!}\,(\log w)^{(\lambda)}\right).$$

Actually, it was rather artificial to separate the case $r=0$.

It remains to replace the w-jet-coordinates with z-jet-coordinates by considering the defining equations of $J_k^{\text{vert}}\mathcal{X}_0(-\log D_0)$:

$$w^d\sum_{\|\lambda\|=j-r} d^{|\lambda|}\,\frac{1}{\lambda!}\,(\log w)^{(\lambda)}=\sum_{|\alpha|\leqslant d} a_\alpha\,\frac{\mathsf{D}^{j-r}}{(j-r)!}(z^\alpha).$$

Hence, at a point of $J_k^{\mathrm{vert}}\mathcal{X}_0(-\log D_0)$ the jet-coordinates satisfy:

$$\boxed{\mathsf{V}_w\left(w^d \sum_{\|\lambda\|=j} d^{|\lambda|}\,\frac{1}{\lambda!}\,(\log w)^{(\lambda)}\right) = d\sum_{0\leqslant r\leqslant j} v_w^r \sum_{|\alpha|\leqslant d} a_\alpha\,\frac{\mathsf{D}^{j-r}}{(j-r)!}(z^\alpha).}$$

2. Recall that:

$$\mathsf{V}_z := \sum_{1\leqslant i\leqslant n}\sum_{0\leqslant r\leqslant k} v_i^j\,\partial_{z_i^{(j)}}$$

thus by substituting this full expression for the vector field V_z, we get the following expression for the second term:

$$\sum_{|\alpha|\leqslant d} a_\alpha\,\mathsf{V}_z\frac{\mathsf{D}^j}{j!}(z^\alpha) = \sum_{|\alpha|\leqslant d} a_\alpha \sum_{1\leqslant i\leqslant n}\sum_{0\leqslant r\leqslant j} v_i^r\left(\partial_{z_i^{(r)}}\frac{\mathsf{D}^j}{j!}\right)(z^\alpha),$$

where, in the indices of the sum, we take account that the term $\mathsf{D}^j(z^\alpha)$ involves only jet coordinates of order at most j.

The next step to get tractable equations is to eliminate the partial derivatives. In that aim, use the following elementary fact, for $r\leqslant j$:

$$\frac{\partial}{\partial z_i^{(r)}}\,\frac{\mathsf{D}^j}{j!}(z^\alpha) = \frac{\mathsf{D}^{j-r}}{(j-r)!}\frac{\partial}{\partial z_i}(z^\alpha).$$

In conclusion:

$$\boxed{\sum_{|\alpha|\leqslant d} a_\alpha\,\mathsf{V}_z\frac{\mathsf{D}^j}{j!}(z^\alpha) = \sum_{|\alpha|\leqslant d} a_\alpha \sum_{1\leqslant i\leqslant n}\sum_{0\leqslant r\leqslant j} v_i^r\,\frac{\mathsf{D}^{j-r}}{(j-r)!}\frac{\partial}{\partial z_i}z^\alpha.}$$

3. Recall that:

$$\mathsf{V}_\alpha := \sum_{|\alpha|\leqslant d}\sum_{0\leqslant j\leqslant k} v_\alpha^j\,\partial_{a_\alpha^{(j)}},$$

thus by substituting this full expression for the vector field V_α, we get the following expression for the third term:

$$\boxed{\sum_{|\alpha|\leqslant d}\mathsf{V}(a_\alpha)\,\frac{\mathsf{D}^j}{j!}(z^\alpha) = \sum_{|\alpha|\leqslant d} v_\alpha^0\,\frac{\mathsf{D}^j}{j!}(z^\alpha).}$$

Notice that the a-derivatives with order $r\geqslant 1$ are not involved in the tangency of V_α. The vector field:

$$\sum_{|a|\leqslant d}\sum_{1\leqslant j\leqslant k} v_\alpha^j\partial_{a_\alpha^{(j)}}$$

is automaticaly tangent to the space of vertical jets.

Gathering the partial results **1**, **2**, **3**, the equations for V to be tangent to the vertical jets can be formulated as:

$$\sum_{|\alpha|\leqslant d}\sum_{0\leqslant r\leqslant j} v_w^r\,d\,a_\alpha\,\frac{\mathsf{D}^{j-r}}{(j-r)!}(z^\alpha) = \sum_{|\alpha|\leqslant d}\sum_{1\leqslant i\leqslant n}\sum_{0\leqslant r\leqslant j} v_i^r\,a_\alpha\,\frac{\mathsf{D}^{j-r}}{(j-r)!}(\partial_{z_i}z^\alpha) + \sum_{|\alpha|\leqslant d} v_\alpha^0\,\frac{\mathsf{D}^j}{j!}(z^\alpha).$$

□

Remark that these "differential" equations are strictly speaking algebraic equations, in the space of jets. Noticing that these equations form a linearly free system of rank $(k+1)$, one infers:

(3.3.2) *The algebraic subspace of logarithmic vertical jets $J_k^{vert}\mathcal{X}_0(-\log D_0)$ is of pure codimension $(k+1)$ in the ambient space $\mathbb{C}^{(k+1)\left(n+1+\binom{d+n}{d}\right)}$.*

4. Low Pole Order Slanted Vector Fields

We prove the following result:

(4.0.3) *Suppose that the order of jets k is smaller than the degree d, then the (twisted) holomorphic tangent bundle to vertical jets of the log-manifold $(\mathbb{P}^n \times S, \mathcal{H})$:*

$$T_{J_k^{vert}(\mathbb{P}^n\times S)(-\log \mathcal{H})} \otimes \eta^\star \left(\mathcal{O}_{\mathbb{P}^n}\big(k(k+2)\big) \otimes \mathcal{O}_S(1)\right)$$

is generated by its holomorphic global sections at every point of the subspace:

$$J_k^{vert,reg}(\mathbb{P}^n \times S)(-\log \mathcal{H}) \setminus \eta^{-1}(\mathcal{H})$$

of regular vertical logarithmic k-jets of holomorphic curves avoiding $\mathcal{H}$.

In order to prove theorem (4.0.3) it hence suffices to prove the analog theorem for $(\mathcal{X}, D)$, namely:

(4.0.4) *Suppose that the order of jets k is smaller than d, then the (twisted) holomorphic tangent bundle to vertical jets of the log-manifold $(\mathcal{X}, D)$:*

$$T_{J_k^{vert}\mathcal{X}(-\log D)} \otimes \eta^\star \left(\mathcal{O}_{\mathbb{P}^{n+1}}\big(k(k+2)\big) \otimes \mathcal{O}_{\mathbb{P}\binom{d+n}{d}}(1)\right)$$

is globally generated over the subspace $J_k^{vert,reg}\mathcal{X}(-\log D)\setminus\eta^{-1}(D)$ of regular vertical logarithmic jets of holomorphic curves avoiding D.

We distinguish two packages of vector fields. In a first package — see 4.1 —, we put the vector fields tangent to the space of parameters. In a second package — see 4.2 —, we put the *slanted* vector fields that are the sum of a vector field tangent to the vertical fiber and of a slanted part.

Any point of $J_k^{\mathrm{vert,reg}}\mathcal{X}_0(-\log D_0)$ lies on a least one chart $\{z_i^{(1)} \neq 0\}$. Recall that $\delta^i \in \mathbb{N}^n$ is the multi-index $(0,\dots,0,1,0,\dots,0)$ with a 1 at the i-th column and 0 elsewhere.

4.1. First package of vectors fields. We start by looking for vector fields tangent to the space of parameters, *i.e.* under the specific form:

$$\mathsf{V} = \sum_{|\alpha|\leqslant d} v_\alpha\, \partial_{a_\alpha}.$$

In this setting, the system (3.3.1) reduces to:

$$0 = \sum_{|\alpha|\leqslant d} v_\alpha\, \frac{\mathsf{D}^j}{j!}(z^\alpha) \qquad {\scriptstyle (j=0,\dots,k)}.$$

Lower lengths. Let $\alpha \in \mathbb{N}^n$ be a multi-index whose length verifies $|\alpha| \leqslant k < d$ and with moreover $\alpha \neq 0, \delta^i, \dots, k\delta^i$. Following **[29]**, we look for a vector field of the specific form:

$$\mathsf{T}_\alpha := -\Delta\, \partial_{a_\alpha} + v_0\, \partial_{a_0} + v_{\delta^i}\, \partial_{a_{\delta^i}} + \cdots + v_{k\delta^i}\, \partial_{a_{k\delta^i}}$$

where all coefficients are assumed to be holomorphic functions of the variables $w, z_1, \dots, z_n$ on Ω_0.

(4.1.1) *Suppose that the order of jets k is smaller than the degree d. For every multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant k$, that is distinct from the $k+1$ multi-indices $0, \delta^i, \dots, k\delta^i$, there exist a coefficient Δ and coefficients $v_0, v_{\delta^i}, \dots, v_{k\delta^i}$, that are holomorphic functions of the variables $(w, z_1, \dots, z_n)$ on Ω_0 , such that the vector field:*

$$\mathsf{T}_\alpha := -\Delta\,\partial_{a_\alpha} + v_0\,\partial_{a_0} + v_{\delta^i}\,\partial_{a_{\delta^i}} + \cdots + v_{k\delta^i}\,\partial_{a_{k\delta^i}}$$

is tangent to the space of logarithmic vertical jets $J_k^{vert}(\mathcal{X}_0)(-\log D_0)$.

Proof. The system of $k+1$ equations (3.3.1) has the matrix form:

$$\begin{bmatrix} 1 & z_i & \dots & z_i^k \\ 0 & \mathsf{D}(z_i) & \dots & \mathsf{D}(z_i^k) \\ \vdots & \vdots & & \vdots \\ 0 & \frac{\mathsf{D}^k}{k!}(z_i) & \dots & \frac{\mathsf{D}^k}{k!}(z_i^k) \end{bmatrix} \begin{bmatrix} v_0 \\ v_{\delta^i} \\ \vdots \\ v_{k\delta^i} \end{bmatrix} = \Delta \begin{bmatrix} z^\alpha \\ \mathsf{D}(z^\alpha) \\ \vdots \\ \frac{\mathsf{D}^k}{k!}(z^\alpha) \end{bmatrix}.$$

The first column contains only one non-zero coefficient, at the first position. Hence, the first unknown is involved in only the first equation of this linear system. The system made of the remaining k last lines :

$$\begin{bmatrix} \mathsf{D}(z_i) & \dots & \mathsf{D}(z_i^k) \\ \vdots & & \vdots \\ \frac{\mathsf{D}^k}{k!}(z_i) & \dots & \frac{\mathsf{D}^k}{k!}(z_i^k) \end{bmatrix} \begin{bmatrix} v_{\delta^i} \\ \vdots \\ v_{k\delta^i} \end{bmatrix} = \Delta \begin{bmatrix} \mathsf{D}(z^\alpha) \\ \vdots \\ \frac{\mathsf{D}^k}{k!}(z^\alpha) \end{bmatrix}$$

only involves the k unknowns $v_{\delta^i}, \dots, v_{k\delta^i}$ and classically, it can be (uniquely) solved using Cramer's rule provided its determinant is not zero. In this view, set Δ to be the determinant of this linear system, the value of which was calculated by Merker [**29**]:

$$\Delta := \det \begin{bmatrix} \mathsf{D}(z_i) & \cdots & \mathsf{D}(z_i^k) \\ \vdots & & \vdots \\ \frac{\mathsf{D}^k}{k!}(z_i) & \cdots & \frac{\mathsf{D}^k}{k!}(z_i^k) \end{bmatrix} = \left(z_i'\right)^{\frac{k(k+1)}{2}}.$$

The crucial point is that it is non-zero because we made the *very simple* assumption $z_i' \neq 0$. Hence by Cramer's rule the system has a (unique) solution and more precisely we have:

$$v_{j\varepsilon_i} = \det \begin{bmatrix} \mathsf{D}(z_i) & \dots & \mathsf{D}(z_i^{j-1}) & \mathsf{D}(z^\alpha) & \mathsf{D}(z_i^{j+1}) & \dots & \mathsf{D}(z_i^k) \\ \vdots & & \vdots & & \vdots & & \\ \frac{\mathsf{D}^k}{k!}(z_i) & \dots & \frac{\mathsf{D}^k}{k!}(z_i^{k-1}) & \frac{\mathsf{D}^k}{k!}(z^\alpha) & \frac{\mathsf{D}^k}{k!}(z_i^{j+1}) & \dots & \frac{\mathsf{D}^k}{k!}(z_i^k) \end{bmatrix},$$

where we changed the k-th column in the determinant. The last unknown is finally determined by the remaining first equation:

$$v_0 = \Delta\, z^\alpha - v_{\delta^i}\, z_i - \cdots - v_{k\delta^i}\, z_i^k.$$

□

Higher lengths.

(4.1.2) *Fix a multi-index β with length $|\beta| = k+1$ and consider the multi-indices $\alpha \geqslant \beta$ (having length $k+1 \leqslant |\alpha| \leqslant d$). The vector field defined by:*

$$\mathsf{T}_{\alpha,\beta} := \sum_{\gamma \leqslant \beta} (-1)^{|\gamma|} \frac{\beta!}{(\beta-\gamma)!\gamma!}\, z^\gamma\, \partial_{a_{\alpha-\gamma}}.$$

is tangent to the space of vertical logarithmic jets $J_\kappa^{vert}(\mathcal{X}_0)(-\log D_0)$.

Proof [**29**]. Recall that the successive formal derivations of z^α take the form:

$$\frac{\mathsf{D}^j}{j!}\big(z^\alpha\big) = \sum_{|\lambda|\leqslant j} \mathscr{B}_{\lambda,j}(z)\, \frac{\partial_{z_1}^{\lambda_1}\cdots\partial_{z_n}^{\lambda_n}}{\lambda_1!\cdots\lambda_n!}\big(z^\alpha\big).$$

Hence the system of equations (3.3.1) that a vector field tangent to the space of parameters $\{a_\alpha\}$ has to satisfy writes:

$$0 \equiv \sum_{|\lambda|\leqslant j} \mathscr{B}_{\lambda,j}(z) \sum_{|\alpha|\leqslant d} v_\alpha\, \frac{\partial_{z_1}^{\lambda_1}\cdots\partial_{z_n}^{\lambda_n}}{\lambda_1!\cdots\lambda_n!}\big(z^\alpha\big).$$

We observe that it hence suffices that the coefficients v_α fulfill the following stronger system of equations:

$$(*) \qquad 0 \equiv \sum_{|\alpha|\leqslant d} v_\alpha\, \partial_{z_{i_1}}\cdots\partial_{z_{i_j}}\big(z^\alpha\big),$$

for all $j = 0, 1, \dots, k$ and all $i_1, \dots, i_j = 1, \dots, n$, because all the above coefficients of $z^{(\Gamma)}$ are clearly of this form.

Now, for any multi-index β with length $|\beta| = k+1$ and any multi-index $\alpha \geqslant \beta$, we claim that the considered vector field:

$$\mathsf{T}_{\alpha,\beta} := \sum_{\gamma\leqslant\beta} (-1)^{|\gamma|}\, \frac{\beta!}{(\beta-\gamma)!\gamma!}\, z^\gamma\, \partial_{a_{\alpha-\gamma}}.$$

satisfies the stronger system $(*)$ just above. Let us sketch the main idea of the proof.

On one hand, the desired cancelations write:

$$(**) \qquad 0 \equiv \sum_{\gamma\leqslant\beta} (-1)^{|\gamma|}\, \frac{\beta!}{(\beta-\gamma)!\gamma!}\, z^\gamma\, \partial_{z_{i_1}}\cdots\partial_{z_{i_j}}\big(z^{\alpha-\gamma}\big).$$

On the other hand, introducing an auxiliary complex variable $y \in \mathbb{C}^n$ and applying multinomial expansion, one gets the general basic formula,:

$$y^{\alpha-\beta}\, (y-z)^\beta \equiv \sum_{\lambda'\leqslant\lambda} (-1)^{|\gamma|}\, \frac{\beta!}{(\beta-\gamma)!\gamma!}\, y^{\alpha-\beta} y^{\beta-\gamma}\, z^\gamma.$$

While $j < |\beta| = k+1$ the derivative by $\partial_{z_{i_1}}\cdots\partial_{z_{i_j}}$ of the left hand side vanishes on the diagonal $y = z$. This remark leads to the formal identity:

$$(***) \qquad 0 \equiv \sum_{\gamma\leqslant\beta} (-1)^{|\gamma|}\, \frac{\beta!}{(\beta-\gamma)!\gamma!}\, z^{\alpha-\gamma}\, \partial_{z_{i_1}}\cdots\partial_{z_{i_j}}\big(z^\gamma\big).$$

Comparing $(**)$ and $(***)$, it looks like we have to transfer the derivatives from the right monomial z^γ to the left monomial $z^{\alpha-\gamma}$. This can be done by induction on j using Leibniz' rule. □

4.2. Second package of vector fields.

(4.2.1) *For $i = 1, \dots, n$, the slanted vector field:*

$$\mathsf{T}_i := \partial_{z_i} - \sum_{|\alpha|\leqslant d-1} (\alpha_i + 1)\, a_{\alpha+\delta^i}\, \partial_{a_\alpha}$$

is tangent to the space of vertical logarithmic jets $J_k^{vert}(\mathcal{X}_0)(-\log D_0)$.

Proof. For $\mathsf{V}_z = \partial_{z_i}$ and $\mathsf{V}_w = 0$ the $k+1$ defining equations (3.3.1) become:

$$\begin{aligned}0 &= \sum_{|\alpha|\leqslant d} \alpha_i\, a_\alpha\, \mathsf{D}^j(z^{\alpha-\delta^i}) + \sum_{|\alpha|\leqslant d} v^0_\alpha\, \mathsf{D}^j(z^\alpha)\\ &= \mathsf{D}^j\Big(\sum_{|\alpha|\leqslant d} \alpha_i\, a_\alpha\, z^{\alpha-\delta^i} + \sum_{|\alpha|\leqslant d} v^0_\alpha\, z^\alpha\Big)\\ &= \mathsf{D}^j\Big(\sum_{|\alpha|\leqslant d-1} (\alpha_i+1)\, a_{\alpha+\delta^i}\, z^\alpha + \sum_{|\alpha|\leqslant d} v^0_\alpha\, z^\alpha\Big)\end{aligned}$$

The choice $v_\alpha = (\alpha_i+1)\, a_{\alpha+\delta^i}$ for $|\alpha| \leqslant (d-1)$ and $v_\alpha = 0$ for $|\alpha| = d$ solves the problem. □

(4.2.2**)** *The slanted vector field:*

$$\mathsf{T}_w := \partial_{\log w} + \sum_{|\alpha|\leqslant d} d\, a_\alpha\, \partial_{a_\alpha}$$

is tangent to the space of vertical logarithmic jets $J_k^{vert}(\mathcal{X}_0)(-\log D_0)$.

Proof. For $\mathsf{V}_z = 0$ and $\mathsf{V}_w = \partial_{\log w}$, the $(k+1)$ defining equations (3.3.1) become:

$$0 = \sum_{|\alpha|\leqslant d} (v_\alpha - d\, a_\alpha)\, \mathsf{D}^j z^\alpha \qquad (j=0,1,\dots,k).$$

So it suffices to take $v_\alpha = d\, a_\alpha$ in order to cancel all the equations. □

Next, in the remaining directions, tangent to the space spanned by the jet derivatives, we can improve the matrix approach presented in the work of Păun, Rousseau and Merker by introducing vector fields that generate all the remaining tangent directions outside the set $\{z'_{i_0} \neq 0\}$ where the derivative in the fixed i_0-th direction vanishes.

Change of coordinates. Consider the non singular jets, for which at least one first derivative z'_{i_0} is not zero. Without loss of generality, assume $i_0 = 1$.

Next, introduce the differential operator of total degree zero:

$$\Delta = \frac{\mathsf{D}}{\mathsf{D}(z_1)}.$$

(4.2.3**)** *The powers of the operators D and Δ satisfy the triangular system:*

$$\frac{\mathsf{D}^q}{q!} = \sum_{p=0}^{q} \mathscr{B}_{p,q}(z_1)\, \frac{\Delta^p}{p!} \qquad (q=0,1,\dots,k),$$

that is invertible outside the set $\{z'_1 = 0\}$ where the derivative in the direction z_1 vanishes.

Proof. Starting with the univariate Faà di Bruno formula:

$$\big(g\circ h\big)^{(q)} = \sum_{p\leqslant q} \mathscr{B}_{p,q}(h)\, g^{(p)}\circ h,$$

one states the formal identity:

$$\begin{aligned}\big(g\circ h\big)^{(q+1)} &= \sum_{p\leqslant q+1} \mathscr{B}_{p,q+1}(h)\, g^{(p)}\circ h\\ &= \sum_{p\leqslant q} \frac{\mathsf{D}}{q+1}\mathscr{B}_{p,q}(h)\, g^{(p)}\circ h + \sum_{p\leqslant q} \frac{p+1}{q+1}\mathscr{B}_{p,q}(h)\, h'\, g^{(p+1)}\circ h,\end{aligned}$$

that yields:

$$\mathscr{B}_{p,q+1}(h) = \frac{\mathsf{D}\big(\mathscr{B}_{p,q}(h)\big) + p\,h'\,\mathscr{B}_{p-1,q}(h)}{q+1}.$$

Applying this result to $h = z_1$, it is now easy to prove the stated formula by induction.
Recall that by definition:

$$\mathscr{B}_{p,q}(h) = \sum_{|\mu|=p, \|\mu\|=q} \frac{|\mu|!}{\mu!} (h^{(1)})^{\mu_1} \cdots (h^{(k)})^{\mu_k}.$$

The system is clearly triangular, because there is no multi-integer with $|\mu| > \|\mu\|$, thus if $p > q$: $\mathscr{B}_{p,q} = 0$. Moreover, the only multi-integer such that $\|\mu\| = |\mu| = p$ is $\mu = (p, 0, \ldots, 0)$, whence:

$$\mathscr{B}_{p,p}(z_1) = (z_1')^p.$$

The system has thus determinant:

$$\det(\mathscr{B}_{p,q}) = (z_1')^{0+1+\cdots+k}. \qquad \square$$

As a consequence, we can reformulate the defining equations, outside of the set $\{z_1' = 0\}$:

$$0 \equiv \mathsf{D}^j \left(w^d - f\right) \quad {\scriptstyle (j=0,1,\ldots,k)} \qquad \overset{z_1'\neq 0}{\Leftrightarrow} \qquad 0 \equiv \Delta^j \left(w^d - f\right) \quad {\scriptstyle (j=0,1,\ldots,k)}.$$

In the direction of z_1, define:

$$x_1^{(j)} := \frac{1}{j!}\mathsf{D}^j(z_i) = z_1^{(j)} \qquad {\scriptstyle (j=0,1,\ldots,k)},$$

and for all other directions $z_i, i \geqslant 2$, define:

$$x_i^{(j)} := \frac{1}{j!}\Delta^j(z_i) \qquad {\scriptstyle (j=0,1,\ldots,k)},$$

lastly define:

$$x_w^{(j)} := \frac{1}{j!}\Delta^j(\log w) \qquad {\scriptstyle (j=0,1,\ldots,k)}.$$

Euler type vector fields. In the new coordinates x:

$$\Delta = \underbrace{\left(\frac{\partial}{\partial x_1} + \frac{x_1''}{x_1'}\frac{\partial}{\partial x_1'} + \cdots + \frac{x_1^{(k)}}{x_1'}\frac{\partial}{\partial x_1^{(k-1)}}\right)}_{\Delta_r} + \underbrace{\sum_{i,w}\left(x_i'\frac{\partial}{\partial x_i} + \cdots + x_i^{(k)}\frac{\partial}{\partial x_i^{(k-1)}}\right)}_{\Delta_\theta}$$

On $\mathbb{C}[z] = \mathbb{C}[x]$, the iterates of Δ will never produce derivatives in the direction of x_1, because $\Delta(x_1) = 1$. Thus we can use the simpler expression of Δ:

$$\Delta^j = \left(\Delta^j\right)|_{\mathbb{C}[x]} = \left(\frac{\partial}{\partial x_1} + \Delta_\theta\right)^j.$$

Another consequence of this fact is that for $j \geqslant 1$, the vector field $\partial/\partial x_1^{(j)}$ is automatically vertical. This is a consequence of the homogeneity of the defining equations. These vector fields are thus of Euler type. Their expressions in the old coordinates are rather involved.

Generators of the vector space. For fixed directions x_1 and x_i, for integers $p = 0, 1, \ldots, k$, let us consider the following vector fields:

$$\frac{\partial}{\partial x_i{}^{(p)}} = \sum_{q=0}^{k} \mathscr{B}_{p,q}(x_1) \frac{\partial}{\partial z_i^{(q)}} = \sum_{|\mu|=p} \frac{|\mu|!}{\mu!}\, x_1^{(\mu)} \frac{\partial}{\partial z_i^{(\|\mu\|)}}.$$

The simpler examples of such vector fields are constructed in the case $p = 0$, where the vector field has the plain expression:

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial z_i},$$

and in the case $p = 1$, where the vector field $\partial/\partial x_i'$ is the same as the vector field implicitly used in the matrix approach presented in the work of Păun ([**37**]):

$$\frac{\partial}{\partial x_i'} = \sum_{q=0}^{k} z_1^{(q)} \frac{\partial}{\partial z_i^{(q)}}.$$

In the general case, $\partial/\partial x_i^{(p)}$ is under the form:

$$\mathsf{W}_p := \frac{\partial}{\partial x_i^{(p)}} = \sum_{q=0}^{k} \mathscr{B}_{p,q}(z_1) \frac{\partial}{\partial z_i^{(q)}} = \sum_{q=0}^{k} \left(\sum_{|\mu|=p, \|\mu\|=q} \frac{|\mu|!}{\mu!}\, z_1^{(\mu)} \right) \frac{\partial}{\partial z_i^{(q)}}.$$

As $\|\mu\| \geqslant |\mu|$, the terms for $q = 0, 1, \ldots, p-1$ are always 0. Moreover, the only multi-integer such that $\|\mu\| = |\mu| = p$ is $\mu = (p, 0, \ldots, 0)$. On thus obtains:

$$\mathsf{W}_p = \left(z_1'\right)^p \frac{\partial}{\partial z_i^{(p)}} + \sum_{q=p+1}^{k} \mathscr{B}_{p,q}(z_1) \frac{\partial}{\partial z_i^{(q)}}.$$

Now, take a linear combination with complex coefficients of $\mathsf{W}_0, \mathsf{W}_1, \ldots, \mathsf{W}_k$:

$$\alpha_0\, \mathsf{W}_0 + \alpha_1\, \mathsf{W}_1 + \cdots + \alpha_k\, \mathsf{W}_k = \beta_0 \frac{\partial}{\partial z_i} + \beta_1 \frac{\partial}{\partial z_i^{(1)}} + \cdots + \beta_k \frac{\partial}{\partial z_i^{(k)}}.$$

Then, the coordinates β are obtained from the coefficients α by the matrix product:

$$\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_k \end{bmatrix} \begin{bmatrix} 1 & * & * & * & * \\ 0 & z_1' & * & * & * \\ \vdots & & (z_1')^2 & * & * \\ & & & \ddots & * \\ 0 & \cdots & & 0 & (z_1')^k \end{bmatrix} = \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_k \end{bmatrix}.$$

The appearing $(k+1) \times (k+1)$ matrix has coefficients:

$$\mathscr{B}_{p,q}(z_1) = \sum_{|\mu|=p, \|\mu\|=q} \frac{|\mu|!}{\mu!}\, z_1^{(\mu)}$$

and is thus upper triangular. Its determinant is the plain monomial $\left(z_1'\right)^{0+1+2+\cdots+k}$, whence it is very clear that the vector fields $\mathsf{W}_0, \mathsf{W}_1, \ldots, \mathsf{W}_k$ span the tangent vector space in the i-th direction, at points where $z_1' \neq 0$.

Computation of the slanted part. We will next investigate how to add a slanted part $-\mathsf{V}_\alpha$ to these generators W_p in order that the sum $\widetilde{\mathsf{W}}_p$ of the two vector fields is tangent to the space of vertical jets. Accordingly we consider the slanted vector fields of the form:

$$\widetilde{\mathsf{W}}_p := \mathsf{W}_p - \mathsf{V}_\alpha,$$

where the slanted part V_α depends on the coordinates a and *only* on the coordinate x_1 in the vertical direction:

$$\mathsf{V}_\alpha := \sum_{|\alpha|\leqslant d} v_\alpha(a, x_1)\frac{\partial}{\partial a_\alpha}.$$

The coefficients v_α will be set *a posteriori*.

(4.2.4) *The $k+1$ equations to be satisfied by the slanted part for $\widetilde{\mathsf{W}_p}$ to be tangent to the vector space of vertical jets are:*

$$\mathsf{V}_\alpha \cdot \partial^j_{x_1}\big(f\big) \equiv p!\binom{j}{p}\partial^{j-p}_{x_1}\partial_{x_i}\big(F\big) \qquad (j=0,1,\dots,k).$$

Proof. This simplification of the equations to be satisfied relies on the basic identity:

$$\Delta^j = \big(\partial_{x_1} + \Delta_\theta\big)^j = \sum_{r=0}^{j}\binom{j}{r}\Delta_\theta^{j-r}\,\partial^r_{x_1} = \sum_{r=0}^{j}\binom{j}{r}\partial^{j-r}_{x_1}\,\Delta^r_\theta.$$

The p first equations are:

$$(*) \qquad \widetilde{\mathsf{W}}_p \cdot \Delta^j(F) = 0 \quad \Leftrightarrow \quad \partial_{x_i^{(p)}}\Delta^j(F) \equiv \mathsf{V}_\alpha\Delta^j(F) \qquad (j=0,1,\dots,p-1).$$

In the reformulation of each equation, the left hand side is obviously 0, because no p-th derivative can appear after less than p derivations! For $p=0$, one obtains $\mathsf{V}_\alpha F \equiv 0$. We will show by induction that eventually:

$$\mathsf{V}_\alpha \cdot \partial^j_{x_1}\big(F\big) \equiv 0 \qquad (j=0,1,\dots,p-1).$$

Accordingly, assume that:

$$\mathsf{V}_\alpha \cdot \partial^r_{x_1}\big(F\big) \equiv 0 \qquad (r=0,1,\dots,j-1).$$

The j-th equation in $(*)$ yields:

$$0 \equiv \mathsf{V}_\alpha\Delta^j(F) \equiv \mathsf{V}_\alpha\sum_{r=0}^{j}\binom{j}{r}\Delta_\theta^{j-r}\,\partial^r_{x_1}(F).$$

Now V_α depends only on the coordinates a_α and x_1, thus it commutes with Δ_θ:

$$0 \equiv \sum_{r=0}^{j}\binom{j}{r}\Delta_\theta^{j-r}\,\mathsf{V}_\alpha\partial^r_{x_1}(F).$$

Taking account of the induction hypothesis, the terms for $r<j$ cancel and it remains only:

$$0 \equiv \mathsf{V}_\alpha\partial^j_{x_1}F.$$

This concludes the proof for the first p equations.

For the remaining equations, with $j \geqslant p$, we use the elementary fact (easily proven by induction),that for $i \neq 1$ and $r \geqslant p$:

$$\partial_{x_i^{(p)}} \cdot \Delta_\theta^r\Big|_{\mathbb{C}[x]} = p!\binom{r}{p}\Delta_\theta^{r-p} \cdot \partial_{x_i}\Big|_{\mathbb{C}[x]}.$$

As a consequence:

$$\begin{aligned}\partial_{x_i^{(p)}}\Delta^j(F) &= \partial_{x_i^{(p)}} \sum_{r=0}^{j} \binom{j}{r}\partial_{x_1}^{j-r}\ \Delta_\theta^r(F)\\ &= \sum_{r=p}^{j} \frac{j!}{(j-r)!(r-p)!}\partial_{x_1}^{j-r}\ \Delta_\theta^{r-p}\partial_{x_i}(F).\end{aligned}$$

On the other hand:

$$\mathsf{V}_\alpha\Delta^j(F) = \sum_{r=0}^{j} \binom{j}{r}\Delta_\theta^{j-r}\ \mathsf{V}_\alpha\partial_{x_1}^r(F)$$

Assuming by induction that:

$$\mathsf{V}_\alpha \cdot \partial_{x_1}^r\big(F\big) \equiv p!\binom{r}{p}\partial_{x_1}^{r-p}\partial_{x_i}\big(F\big) \qquad (r=0,1,\ldots,j-1),$$

one computes:

$$\begin{aligned}\mathsf{V}_\alpha\Delta^j(F) &= \mathsf{V}_\alpha\partial_{x_1}^j(F) + \sum_{r=0}^{j-1} \binom{j}{r}\Delta_\theta^{j-r}\ p!\binom{r}{p}\partial_{x_1}^{r-p}\partial_{x_i}\big(F\big)\\ &= \mathsf{V}_\alpha\partial_{x_1}^j(F) + \sum_{r=p}^{j-1} \frac{j!}{(j-r)!(r-p)!}\Delta_\theta^{j-r}\ \partial_{x_1}^{r-p}\partial_{x_i}\big(F\big).\end{aligned}$$

Thus:

$$\begin{aligned}0 &\equiv \partial_{x_i^{(p)}}\Delta^j(F) - \mathsf{V}_\alpha\Delta^j(F)\\ &\equiv \sum_{r=p}^{j} \frac{j!}{(j-r)!(r-p)!}\partial_{x_1}^{j-r}\ \Delta_\theta^{r-p}\partial_{x_i}(F) - \mathsf{V}_\alpha\partial_{x_1}^j(F) - \sum_{r=p}^{j-1} \frac{j!}{(j-r)!(r-p)!}\Delta_\theta^{j-r}\ \partial_{x_1}^{r-p}\partial_{x_i}\big(F\big)\end{aligned}$$

It remains to make the change of variables: $r - p \leftrightarrow j - r$ in the second sum, and to simplify all terms appearing with both signs in order to obtain as announced:

$$0 \equiv \frac{p!}{(j-p)!}\partial_{x_1}^{j-p}\partial_{x_i}(F) - \mathsf{V}_\alpha\partial_{x_1}^j(F).$$

This concludes the proof. □

For the supplementary variable w, we can follow the same strategy.

(4.2.5**)** *The equation to be satisfied by the vector field:*

$$\partial_{x_w^{(p)}} + \mathsf{V}_\alpha$$

to be tangent to the space of vertical jets are:

$$\mathsf{V}_\alpha \cdot \partial_{x_1}^j F = d\ p!\binom{j}{p}\partial_{x_1}^{j-p}F. \qquad (j=0,1,\ldots,k).$$

Proof. The proof is the same. For $j = 0, 1, \ldots, p-1$:

$$\mathsf{V}_\alpha \cdot \partial^j_{x_1}\big(F\big) \equiv 0.$$

For $j \geqslant p$:

$$\partial_{x_w^{(p)}} \Delta^j(w^d) = dp!\binom{j}{p}\Delta^{j-p}F = d\sum_{r=0}^{j-p} p!\binom{j}{p}\binom{j-p}{r}\partial^{j-p-r}_{x_1}\Delta^r_\theta(F)$$
$$= d\sum_{r=0}^{j-p}\frac{j!}{(j-p-r)!r!}\partial^{j-p-r}_{x_1}\Delta^r_\theta(F).$$

On the other hand:

$$\mathsf{V}_\alpha\Delta^j(F) = \sum_{r=0}^{j}\binom{j}{r}\Delta^{j-r}_\theta\ \mathsf{V}_\alpha\partial^r_{x_1}(F)$$

Assuming by induction that:

$$\mathsf{V}_\alpha \cdot \partial^r_{x_1}F = p!\binom{r}{p}d\,\partial^{r-p}_{x_1}F. \qquad (r=0,1,\ldots,j-1).$$

one computes:

$$\mathsf{V}_\alpha\Delta^j(F) = \mathsf{V}_\alpha\partial^j_{x_1}(F) + d\sum_{r=1}^{j}\binom{j}{r}\Delta^{r-j}_\theta\ p!\binom{r}{p}\partial^{r-p}_{x_1}\big(F\big)$$
$$= \mathsf{V}_\alpha\partial^j_{x_1}(F) + d\sum_{r=p}^{j-1}\frac{j!}{(r-p)!(j-r)!}\Delta^{r-j}_\theta\ \partial^{r-p}_{x_1}\big(F\big).$$

It remains to make the change of variables: $r \leftrightarrow j-r$ in the sum, and to simplify all terms appearing in both expressions in order to obtain as announced:

$$d\ \frac{j!}{(j-p)!}\partial^{j-p}_{x_1}(F) \equiv \mathsf{V}_\alpha\partial^j_{x_1}(F).$$

This concludes the proof. □

It remains to solve these linear systems. In both cases, the right hand side is a polynomial of total degree less than d in x and linear in a, not depending on v_α and the left hand side is $\mathsf{V}_\alpha \cdot \partial^j_{x_1}F$. Assume that $v_\alpha(a, x_1)$ is written under the form:

$$v_\alpha = \sum_{s=0}^{k} v^s_\alpha(a)\,x^s_1,$$

Then the left hand side becomes:

$$\mathsf{V}_\alpha \cdot \partial^j_{x_1}F = \sum_{|\alpha|\leqslant d}\sum_{s=0}^{k}\Big(\partial^j_{x_1}x^{\alpha_1}_1\Big)_{x_1=1}\ v^s_\alpha(a)\ x_1^{\alpha_1-j+s}x_2^{\alpha_2}\cdots x_n^{\alpha_n}.$$

For $|\beta| \leqslant d$, the coefficient of $x^{\beta-j\delta^1}$ is thus:

$$\sum_{s=0}^{k}\Big(\partial^j_{x_1}x_1^{\beta_1-s}\Big)_{x_1=1}\ v^s_{\beta-s\delta^1}(a).$$

Notably the system $\{v^s_\alpha, \alpha + s\delta^1 = \beta\}$ can be solved independently, because these variables appear only in the $k+1$ equations corresponding to the coefficient of $x^{\beta-j\delta^1}$ in the j-th equation.

One needs the following lemma:

(4.2.6**)** *For any fixed integer β_1, the following square matrix is invertible:*

$$L_{\beta_1} := \left[\left(\frac{1}{j!}\partial_{x_1}^j x_1^{\beta_1-s}\right)_{x_1=1}\right]_{\substack{j=0,1,\dots,k\\ s=0,1,\dots,k}}.$$

Proof. If not, then the system made of its columns is linearly dependent. A column of the matrix L_{β_1} contains all the Taylor coefficients of $x_1^{\beta_1-s}$ at $x_1 = 1$ up to order k. As a result, we get a *non zero* linear combination:

$$u(x_1) := \sum_{s=0}^{k} u_s\, x_1^{\beta_1-s}.$$

all the Taylor coefficients of which up to order k are zero at $x_1 = 1$. The same property then visibly holds for the polynomial:

$$x_1^{\beta_1}\, u(1/x_1) = \sum_{s=0}^{k} u_s\, x_1^s.$$

Thus, all the coefficients u_s are zero, in contradiction with the preceding assumption that $u \not\equiv 0$. □

As a consequence, it is possible to solve the systems (4.2.4) and (4.2.5) by taking:

$$v_\alpha = \sum_{s=0}^{k} v_\alpha^s(a)\, x_1^s,$$

where, by Cramer's rule, one may assume that v_α^s is linear in a.

Pole order of meromorphic prolongations. By considering the successive derivations of $z_i = Z_i/Z_0$, it is easy to see that the pole order of the meromorphic prolongation of $z_i^{(j)}$ is $j + 1$. By additivity, the pole order of T_α is $k(k + 2)$ in the vertical direction and 0 in the horizontal direction, the pole order of $\mathsf{T}_{\alpha,\beta}$ is $k + 2$ in the vertical direction and 0 in the horizontal direction, the pole order of T_i and T_w is 0 in the vertical direction and 1 in the horizontal direction, lastly the pole order of $\widetilde{\mathsf{W}}_p$ is $p \leqslant k$ in the vertical direction, and 1 in the horizontal direction. More details are available in [**29**].

These observations yield the constants $k^2 + 2k$ and 1 in the main theorem.

CHAPTER IV

The Logarithmic Green-Griffiths Conjecture

Contents

1. Introduction

According to a longstanding conjecture of Shoshichi Kobayashi [**26**], there should exist no nonconstant entire curves with values in the complements of generic projective hypersurfaces of sufficiently large degree in $\mathbb{P}^n$. Regarding algebraic degeneracy of these curves, the Green-Griffiths conjecture [**22**] asserts, in a wide context, that a nonconstant entire curve with values in a smooth algebraic variety of general type is never Zariski-dense. The strong version of this conjecture asserts moreover that the proper algebraic subvariety Z of the target space containing the image $f(\mathbb{C}) \subset Z$ does not depend on the nonconstant entire curve f. In the case of complements of projective hypersurfaces, considered in this work, this conjecture becomes the following. If $H \subset \mathbb{P}^n$ is a hypersurface having degree $d \geqslant n+2$, then there should exist a proper subvariety $Z \subset \mathbb{P}^n$, such that the image of every nonconstant entire curve $f\colon \mathbb{C} \to \big(\mathbb{P}^n \setminus H\big)$, actually lies in $(Z \setminus H)$.

In this chapter, a positive answer is given to the strong logarithmic Green-Griffiths conjecture, for generic hypersurfaces having degree $d \geqslant (5n)^2\, n^n$. Precisely, we prove the following.

Main Theorem. *If $H \subset \mathbb{P}^n$ is a generic hypersurface having degree*

$$d \,\geqslant\, (5n)^2\, n^n,$$

then there exists a proper subvariety $Z \subset \mathbb{P}^n$, of codimension at least two, such that the image of every nonconstant entire curve $f\colon \mathbb{C} \to (\mathbb{P}^n \setminus H)$, having values in the complement of H, actually lies in $(Z \setminus H)$.

In dimensions $n = 2$ [**19, 44**] and $n = 3$ [**42**], this theorem is already known with more precise lower bounds on the degree. In [**44, 39**], Erwan Rousseau also treated the case of hypersurfaces with several smooth components, in dimension 2.

In order to prove the main theorem, we use the general strategy described in [**13, 15, 38**], already implemented in [**37, 42, 43, 13, 3**], that combines the *extrinsic approach* of Siu [**6, 18, 50, 46, 29, 31**] with the *intrinsic approach* of Demailly [**7, 10, 12, 19**].

The chapter is organized as follows.

§2. In the first section, the main result on degeneracy of entire curves on complements of generic smooth projective hypersurfaces is proved. The main idea is to produce *many* algebraic differential equations satisfied by every nonconstant entire curve. Admitting for the moment the existence of *one* such differential equation, and working in the universal family of complements of projective hypersurfaces of degree d, lots of *algebraically independent* differential equations are produced by using *low pole order slanted vector fields* ([**46, 29**]). These vector fields have been studied in our context in the chapter III, which generalizes the dimension 2 and dimension 3 cases due to Erwan Rousseau ([**42, 44**]).

§3. The second section is devoted to proving the existence of one differential equation (result admitted in the first section). The intrinsic strategy outlined in [**12**] is followed. It relies on the construction of *invariant jet differentials* of Demailly ([**7**]), generalized to the logarithmic setting by Dethloff and Lu ([**10**]). The use of weak algebraic Morse inequalities ([**49, 7**]) provides a control of the cohomology, sufficient for our goal. It reduces the problem to the positivity of a certain intersection product on the κ-th level of the *Demailly tower* of projectivized jet bundles. This intersection product is a polynomial $I(d)$ in the degree d of the hypersurface depending on parameters $\underline{a}$ and δ, that can be adjusted later. A variant of the multivariate residue formula of Bérczi ([**3**]), presented in chapter II, allows us to integrate along the fibers of the Demailly tower. Then, it remains to estimate one coefficient in the complicated Cauchy product of many (convergent) multivariate formal series.

§4. In the third section, the computation of this coefficient is implemented. The above-mentioned multivariate residue formula of chapter II shows that the intersection number $I(d)$ on the projectivized κ-jet bundle can be computed as a coefficient of the Cauchy product of a universal rational term by a simpler term involving only explicit data of the base manifold. To evaluate $I(d)$, it is necessary to tame the intricate combinatorics of this universal term. The overall approach, already adopted by Bérczi in [**3**], is to identify some central terms among the numerous combinations contributing to the coefficient of each power d^p. This identification leads to a modified version $\widetilde{I}(d)$ of the polynomial $I(d)$ having much simpler coefficients. It is easy to compute its largest root $\widetilde{\lambda}$. Then, under suitable numerical hypotheses on the parameters $\underline{a}$ and δ — that yield the effective degree bound $d \geqslant (5n)^2 n^n$ of the main theorem —, the estimation of the largest root of the polynomial $I(d)$ is reduced to the much easier computation of the largest root $\widetilde{\lambda}$ of the simplified polynomial $\widetilde{I}(d)$. Many technical results are postponed to appendices 5 and 6.

§5,6. The last two sections: appendices 5 and 6, form the *technical core* of this chapter. In the first appendix, the leading coefficient of the intersection product $I(d)$ is studied, and its positivity for a suitable (explicit) choice of the parameters $\underline{a}$ and δ is stated. In the second appendix, the remaining coefficients of $I(d)$ are studied and an upper bound λ for the largest root of this polynomial is derived.

Although most of the reasonings are valid for general jet-orders $n \leqslant \kappa \leqslant d$, we will soon restrict ourselves to the case $\kappa = n$, in order to avoid going into additional technical details, specially to lighten the appendices.

2. Effective Algebraic Degeneracy of Entire Curves

The formalism of jets, that will now be recalled, allows a coordinate-free description of differential equations and as a result more compact statements, although coordinates stay an essential tool in the machinery of proofs.

2.1. Logarithmic jet bundles. In what follows, Δ denotes a complex disk of any radius, that can vary.

Jet manifold. Let $\overline{X}$ be a complex manifold. Classically, the *κ-jet manifold* of $\overline{X}$ is a coordinate-free construction with the same information as the κ-th-order Taylor polynomial of germs of holomorphic maps $\Delta \to \overline{X}$. For a germ of holomorphic map $f\colon \Delta \to \overline{X}$, the *$\kappa$-jet* of f is the equivalence class of germs $g\colon \Delta \to \overline{X}$ that osculate with f to order κ at the origin of Δ. Then, for a target point $x \in \overline{X}$, denote by $J_\kappa \overline{X}_x$ the space of all κ-jets $f_{[\kappa]}$ of germs $f\colon (\Delta, 0) \to (\overline{X}, x)$. The collection $J_\kappa \overline{X} \xrightarrow{\eta} \overline{X}$ of these spaces is a fiber bundle, called the *κ-jet manifold* of $\overline{X}$. The map η is the evaluation of the jets at the origin.

For a meromorphic differential 1-form $\omega \in \Gamma_{\mathrm{loc}}\big(T^\star_{\overline{X}}\big)$ and a germ of holomorphic map $f\colon \Delta \to \overline{X}$ the pullback $f^\star\omega$ is necessarily of the form $f^\star\omega = A(t)dt$ for a meromorphic function $A\colon \mathbb{C} \to \mathbb{C}$. Thus, each such 1-form ω induces a meromorphic map

$$\widetilde{\omega}\colon f_{[\kappa]} \mapsto \Big(A(t), A'(t), \dots, A^{(\kappa-1)}(t)\Big) \qquad [J_\kappa X \to \mathbb{C}^\kappa].$$

On an open set $U \subset \overline{X}$, the trivialization $\Gamma_U\big(J_\kappa \overline{X}\big) \to U \times \big(\mathbb{C}^\kappa\big)^n$ associated to a meromorphic local coframe $\omega_1 \wedge \cdots \wedge \omega_n \neq 0$ is

$$\sigma \mapsto (\eta \circ \sigma\,;\, \widetilde{\omega}_1 \circ \sigma, \dots, \widetilde{\omega}_n \circ \sigma)\,.$$

The components $A_i^{(j)}$ of $\widetilde{\omega}_i \circ \sigma$ are called *jet coordinates* and correspond to the derivatives of the germs $\sigma_x\colon t \mapsto \sigma_x(t)$ with respect to the complex variable $t \in \Delta$. Note that all of these objects are holomorphic if ω is.

Logarithmic jet manifold along a normal crossing divisor. A divisor $D \subset \overline{X}$ has only *normal crossings* if at each point $x \in \overline{X}$, there is an integer $\ell = \ell(x)$ and a centered coordinate system $z_1, \dots, z_\ell, z_{\ell+1}, \dots, z_n$ on $\overline{X}$ around x such that $D \simeq \mathrm{div}(z_1 \cdots z_\ell) \subset \mathbb{C}^n$. For such a normal crossing divisor D, one defines

$$\mathcal{T}^\star_{\overline{X}}(\log D)_x := \mathcal{O}_{\overline{X}}\,\frac{dz_1}{z_1} + \cdots + \mathcal{O}_{\overline{X}}\,\frac{dz_\ell}{z_\ell} + \mathcal{O}_{\overline{X}}\,dz_{\ell+1} + \cdots + \mathcal{O}_{\overline{X}}\,dz_n.$$

It is a locally free $\mathcal{O}_{\overline{X}}$-module of rank n that is the sheaf of sections of a vector bundle $T^\star_{\overline{X}}(\log D)$, called the *logarithmic cotangent bundle of $\overline{X}$ along D* ([**25**]).

A local section $\sigma \in \Gamma_U(J_\kappa \overline{X})$ over an open set $U \in \overline{X}$ is termed *logarithmic* if for any logarithmic cotangent vector field $\omega \in \Gamma_V(T^\star_{\overline{X}}(\log D))$, defined on a smaller open set $V \subset U$, the meromorphic function $\widetilde{\omega} \circ \sigma$ is actually holomorphic, or, in other words, if σ has holomorphic jet coordinates in the adapted logarithmic coframe generating $\mathcal{T}^\star_{\overline{X}}(\log D)$. These sections define a subsheaf of sections of $J_\kappa \overline{X}$ and this subsheaf is itself the sheaf of sections of a holomorphic affine bundle $J_\kappa \overline{X}(-\log D)$, called the *logarithmic κ-jet manifold of $\overline{X}$ along D* ([**35**]).

Invertible jets. A jet field $j \in J_\kappa \overline{X}$ is termed *singular* at a point $x \in \overline{X}$ if it is the lift of a stationary curve, *i.e.* $j = f_{[\kappa]}$ with $f(0) = x$ and $f_1'(0) = \cdots = f_n'(0) = 0$. The subset of singular jets will be denoted by $J_\kappa^{sing} \overline{X}$. Note that for a logarithmic pair $(\overline{X}, D)$, the logarithmic jet bundle $J_\kappa \overline{X}(-\log D)|_{X\setminus D}$ and the holomorphic jet bundle $J_\kappa(\overline{X} \setminus D)$

coincide on the open part $\overline{X} \setminus D$. This observation allows to define singular logarithmic jet fields, as follows. A logarithmic jet field is said *singular* if it is in the topological closure of the subset $J_\kappa^{sing}(\overline{X} \setminus D)$ in $J_\kappa \overline{X}(-\log D)$. A (logarithmic) jet field that is not singular is termed *invertible* (or *regular*).

Jet differentials. Recall the concept of jet differentials, after **[22, 7, 10]**. "It is a coordinate-free description of the holomorphic differential equations that a germ of curve may satisfy".

One has a $\mathbb{C}^\star$-action on $J_\kappa \overline{X}(-\log D)$ by rescaling of the source. Indeed if h_λ is the homothety with ratio $\lambda \in \mathbb{C}^\star$, and f is a logarithmic jet along $D = \mathrm{div}(z_1 \cdots z_\ell)$, the jet coordinates change as follows:

$$\begin{cases} \big(\log \circ f_i \circ h_\lambda\big)^{(j)} = \lambda^j \,\big(\log \circ f_i\big)^{(j)} \circ h_\lambda & {\scriptstyle [i=1,\dots,\ell]} \\ \big(f_i \circ h_\lambda\big)^{(j)} = \lambda^j\, f_i^{(j)} \circ h_\lambda & {\scriptstyle [i=\ell+1,\dots,n]}. \end{cases}$$

The Faà di Bruno formulas (at the end of the introduction to chapter III) show that the concept of polynomial on the fibers of $J_\kappa \overline{X}(-\log D)$ makes sense. One can thus consider the *Green-Griffiths jet bundle* $\mathcal{E}_{\kappa,m} T^\star_{\overline{X}}(\log D\big) \to \overline{X}$ of differential operators of *order* κ and *weighted degree* m, the fiber of which consists of the complex valued polynomials $Q(f', \dots, f^{(\kappa)})$ on the fibers of $J_\kappa \overline{X}(-\log D)$ of weighted degree m with respect to the $\mathbb{C}^\star$-action by rescaling of the source, that is:

$$Q\Big((f \circ h_\lambda)_{[\kappa]}\Big) = \lambda^m\, Q\Big(f_{[\kappa]}\Big).$$

2.2. Fundamental vanishing theorem. The link between algebraic degeneracy and jet bundles is nowadays classical. One has indeed the following *fundamental vanishing theorem* (**[7, 10]**), here stated for $(\overline{X}, D) = (\mathbb{P}^n, H)$, where H is a smooth hypersurface in $\mathbb{P}^n$.

(2.2.1**)** *Let P be a non zero global jet differential of order κ and weighted degree m, vanishing with order $\varepsilon > 0$ on $\mathcal{O}_{\mathbb{P}^n}(1)$:*

$$P \in H^0\Big(\mathbb{P}^n, \mathcal{E}_{\kappa,m} T^\star_{\mathbb{P}^n}(\log H) \otimes \mathcal{O}_{\mathbb{P}^n}(\varepsilon)^\vee\Big) \setminus \Big\{0\Big\}.$$

Then every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to \Big(\mathbb{P}^n \setminus H\Big)$ must satisfy the corresponding algebraic differential equation of degree κ:

$$P_{f(t)}\Big(f^{(1)}(t), \dots, f^{(\kappa)}(t)\Big) = 0 \qquad \textit{for all } t \in \mathbb{C}.$$

A strategy **[46, 37, 43, 42, 13, 15]** for proving the *algebraic degeneracy* of entire curves $f\colon \mathbb{C} \to \mathbb{P}^n \setminus H$, *i.e.* for obtaining an algebraic differential equation of degree 0 that every entire curve has to satisfy, is then to generate *many* global jet differentials. Using the above fundamental vanishing theorem, this gives algebraic differential equations that every entire curve must satisfy. Next, one can (morally) get rid of the differentials $f^{(1)}, \dots, f^{(\kappa)}$ by algebraic elimination.

In sections 3 and 4, using algebraic Morse inequalities, the existence of *one* such differential equation of order $\kappa = n$ is proved:

(2.2.2**)** *Let $H \subset \mathbb{P}^n$ be a smooth hypersurface of degree d and let δ be a positive rational number such that:*

$$d \geqslant (52\, n^n) \textit{ and } (35\, n^n)\, \delta \leqslant 1.$$

Then, for jet order $k = n$ and weighted degrees $m \gg d$ large enough, the vector space of logarithmic global jet differentials along H vanishing on the ample line bundle $\mathcal{K}_{\mathbb{P}^n}(H)^{2\delta m}$ has positive dimension:

$$\dim H^0\Big(\mathbb{P}^n, \mathcal{E}_{n,m} T^\star_{\mathbb{P}^n}\big(\log H\big) \otimes \big(\mathcal{K}_{\mathbb{P}^n}(H)^\vee\big)^{2\delta m}\Big) \geqslant 1.$$

The next step is to use the variational method of Voisin-Siu [**50, 46**], namely to deform the differential equations just obtained, by considering the projective hypersurfaces in family, in order to get sufficiently many algebraically independent differential equations.

2.3. Universal family of complements of projective hypersurfaces. Let S be the space of parameters of degree d hypersurfaces of $\mathbb{P}^n$:

$$S := |\mathcal{O}_{\mathbb{P}^n}(d)| = \mathbb{P}\, H^0\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)\big).$$

Consider the universal family of degree d hypersurfaces in $\mathbb{P}^n$

$$\mathcal{H} := \{(x, P) \in \mathbb{P}^n \times S \colon P(x) = 0\},$$

and the complement $\mathcal{X}$ of this universal family:

$$\mathcal{X} := \big(\mathbb{P}^n \times S\big) \setminus \mathcal{H}.$$

Denote the two natural projections to the factors of $\mathbb{P}^n \times S$ by

$$\begin{array}{ccc} & \mathbb{P}^n \times S & \\ {}^{\mathrm{pr}_1}\swarrow & & \searrow^{\mathrm{pr}_2} \\ \mathbb{P}^n & & S \end{array}.$$

If $s \in S$ is a point of the parameter space of $\mathcal{H}$, then $\mathcal{X}_s := (\mathrm{pr}_2|_{\mathcal{X}})^{-1}(s) = \big(\mathbb{P}^n \setminus H_s\big) \times \{s\}$, where $H_s \subset \mathbb{P}^n$ is the projective hypersurface of degree d parametrized by s:

$$H_s := \mathrm{pr}_1\big((\mathrm{pr}_2|_{\mathcal{H}})^{-1}(s)\big).$$

Moreover, for a generic $s \in S$, the hypersurface H_s is smooth.

Slanted vector fields. The universal family $\mathcal{H}$ is a normal crossing divisor of degree $(d, 1)$ in $\mathbb{P}^n \times S$. The space of the logarithmic κ-jets along $\mathcal{H}$ of the log-manifold $\big(\mathbb{P}^n \times S, \mathcal{H}\big)$ will be denoted hereafter by:

$$\mathcal{J}_\kappa := J_\kappa(\mathbb{P}^n \times S)(-\log \mathcal{H}),$$

and we will denote by $\mathcal{V}_\kappa$ the submanifold of *vertical logarithmic jet fields* of order k:

$$\mathcal{V}_\kappa := J_k^{\mathrm{vert}}(\mathbb{P}^n \times S)(-\log \mathcal{H}) \subset \mathcal{J}_\kappa,$$

consisting of κ-jets tangent to the fibers of the second projection $\mathrm{pr}_2 \colon \mathbb{P}^n \times S \to S$. Lastly we will denote by $\mathcal{V}_\kappa^*$ the submanifold of invertible regular jets.

Let η denote the evaluation of the jets $\mathcal{J}_\kappa \to \mathbb{P}^n \times S$. In the chapter III, we prove the following statement:

(2.3.1) *For degrees $d \geqslant \kappa$, the twisted holomorphic tangent sheaf to vertical jets of the log manifold* $(\mathbb{P}^n \times S, \mathcal{H})$

$$\mathcal{T}_{\mathcal{V}_\kappa} \otimes \eta^\star \Big(\mathcal{O}_{\mathbb{P}^n}\big(\kappa(\kappa+2)\big) \otimes \mathcal{O}_S(1)\Big)$$

is generated by its global holomorphic sections over the subspace $\mathcal{V}_\kappa^* \setminus \eta^{-1}\mathcal{H}$ of logarithmic κ-jets of non stationary holomorphic curves $\mathbb{C} \to \mathcal{X}$ tangent to the fibers of $\mathbb{P}^n \times S \to S$.

2.4. Proof of the main theorem. In this proof, since we use numerical estimates of sections 3 and 4, we will take $\kappa = n$.

Let $f\colon \mathbb{C} \to \mathcal{X}$ be a nonconstant entire curve having values in the complement $\mathcal{X} \subset \mathbb{P}^n \times S$ of the universal family $\mathcal{H}$ of projective hypersurfaces of degree d. Assume f is tangent to the logarithmic relative tangent bundle of pr_2:

$$\mathcal{V}_1 := \ker\big(\mathrm{pr}_{2\star}\big).$$

This assumption means that $\mathrm{pr}_2 \circ f \equiv s_0$ is constant. In other words, f maps $\mathbb{C}$ to the slice:

$$\mathcal{X}_{s_0} := \big(\mathbb{P}^n \setminus H_{s_0}\big) \times \big\{s_0\big\} \subset \mathcal{X}.$$

Recall that H_{s_0} is smooth for a generic $s_0 \in S$, which we assume.

By the very definition of the relative tangent bundle, one has:

$$\mathcal{V}_1|_{\mathcal{X}_s} = T_{\mathcal{X}_s}(-\log \mathcal{H}_s) = (\mathrm{pr}_2)^\star\, T_{\mathbb{P}^n}(-\log H_s).$$

By the above statement 2.2.2, for a suitable choice of $0 < \delta \ll 1$ independently of s_0, noticing as Mourougane in [**33**] that the semicontinuity theorem ([**34**, II.§5 p.50]) yields that there is a suitable choice of $m \gg d \gg 1$ valid for general hypersurfaces, that we fix, there exists a non zero jet differential having positive vanishing order:

$$\sigma_{s_0} \in H^0\Big(\mathbb{P}^n \times \{s_0\}, \mathcal{E}_{n,m}\mathcal{V}_1^\star \otimes \eta^\star \mathcal{O}_{\mathbb{P}^n}(2\delta m(d-n-1))^\vee\big|_{\mathcal{X}_{s_0}}\Big),$$

with zero locus: $Z_{s_0} := \{x \in \mathbb{P}^n \colon \sigma_{s_0}(x) = 0\} \subsetneq \mathbb{P}^n \times \{s_0\}$.

Because the vanishing order of the section σ_{s_0} is positive, one can apply the fundamental vanishing theorem and obtain an algebraic equation satisfied by the n-jet of f in $\overline{X}_n$:

$$\sigma_{s_0}\left(f^{(1)}, \ldots, f^{(n)}\right) = \sigma_{pr_2 \circ f}\left(f^{(1)}, \ldots, f^{(n)}\right) \equiv 0.$$

In order to obtain information on the image of f itself, one obvious problem is that the projection to the base $\overline{X}_0$ of the algebraic subset cut off by this equation in $\overline{X}_n$ might be onto ; but in fact, we will shortly establish that the image of f lies in the much smaller common zero set $Z_{s_0} \in \overline{X}$ of all coefficients of this polynomial,*i.e.*:

$$\sigma_{pr_2 \circ f} \equiv 0.$$

By the semicontinuity theorem, there exists a Zariski closed subset $\Sigma \subset S$ such that if s_0 lies in $S \setminus \Sigma$, then σ_{s_0} can be extended to a Zariski dense open set containing s_0 as a holomorphic family of non zero jet differentials:

$$\sigma := \Big\{\sigma_s \in H^0\Big(\mathbb{P}^n \times \{s\}, \mathcal{E}_{n,m} T^\star_{\mathbb{P}^n}(\log H_s) \otimes \mathcal{O}_{\mathbb{P}^n}(2\delta m(d-n-1))^\vee\Big)\Big\}.$$

Make the two generic assumptions that $s_0 \notin \Sigma$ and that $H_{s_0} \subset \mathbb{P}^n$ is smooth and, in order to get rid of the term $\eta^\star \mathcal{O}_S(1)$ in (2.3.1), remove a supplementary hyperplane of S, if necessary.

The sections of $\mathcal{E}_{n,m} T^\star_{\mathbb{P}^n}(\log H) \otimes \eta^\star \mathcal{K}_{\mathbb{P}^n}(H)^{-2\delta m}$ can be interpreted as invariant maps

$$J_n \mathbb{P}^n(-\log H) \to \eta^\star \mathcal{K}_{\mathbb{P}^n}(H)^{-2\delta m}.$$

Recall that:

$$\mathcal{K}_{\mathbb{P}^n}(H) = \mathcal{O}_{\mathbb{P}^n}(d-n-1).$$

Working locally on a neighbourhood U of one such s_0, consider the section:

$$\sigma\colon \mathcal{V}_n|_{\mathrm{pr}_2^{-1} U} \to \eta^\star \mathcal{O}_{\mathbb{P}^n}\big(-2\delta m(d-n-1)\big)\big|_{\mathrm{pr}_2^{-1} U},$$

and p slanted vector fields provided by (2.3.1):

$$\mathsf{v}_1, \ldots, \mathsf{v}_p \colon \mathrm{pr}_2^{-1} U \to T_{\mathcal{V}_n} \otimes \eta^\star \mathcal{O}_{\mathbb{P}^n}(n(n+2))|_{\mathrm{pr}_2^{-1} U}.$$

These slanted vector fields are useful in order to generate lots of new vector fields by Lie derivative of the given section σ. Indeed, by induction, the Lie derivative $\big(\mathsf{v}_p \cdots \mathsf{v}_1 \cdot \sigma\big)_{s_0}$ is a non zero section of the vector space:

$$H^0\Big(\mathbb{P}^n \times \{s_0\}, \mathcal{E}_{n,m} T^\star_{\mathbb{P}^n}(\log H_{s_0}) \otimes \mathcal{O}_{\mathbb{P}^n}\big(-2m\delta(d-n-1) + p\, n(n+2)\big)\Big).$$

Each derivative decreases the vanishing order by at most $n(n+2)$.

While the vanishing order of $\big(\mathsf{v}_p \cdots \mathsf{v}_1 \cdot \sigma\big)_{s_0}$ is still positive, *i.e.* while:

$$2\delta m(d-n-1) > p\, n\, (n+2),$$

the above argument remains valid, and one infers that:

$$\big(\mathsf{v}_p \cdots \mathsf{v}_1 \cdot \sigma\big)_{s_0}\big(f_{[n]}\big) \equiv 0.$$

In this way, *lots of algebraic differential equations* can be obtained, as soon as the vanishing order of the initial section σ is large enough.

Sufficient lower bound on the degree. If $f(\mathbb{C}) \not\subset Z_{s_0}$, the element of contradiction is the following. Pick $t_0 \in \mathbb{C}$ such that $f(t_0) \notin Z_{s_0}$. Up to moving a bit t_0, one may assume that $f'(t_0) \neq 0$, because $f_{[n]}(\mathbb{C}) \not\subset \mathcal{V}_n^{sing}$. Now, the result of global generation yields:

(2.4.1**)** **(**[**13**, pp.176–178, (ii)]) *If $f(t_0) \notin Z_s$ and $f'(t_0) \neq 0$, then there exist $q \leqslant m$ vector fields*

$$\mathsf{v}_1, \ldots, \mathsf{v}_q \colon \mathrm{pr}_2^{-1} U \to T_{\mathcal{V}_n} \otimes \eta^\star \mathcal{O}_{\mathbb{P}^n}(n(n+2))|_{\mathrm{pr}_2^{-1} U}.$$

such that:

$$\big(\mathsf{v}_q \cdots \mathsf{v}_1 \cdot \sigma\big)_{s_0}\big(f_{[n]}(t_0)\big) \neq 0.$$

Sketch of proof. The idea of the proof is that locally around s_0 the section σ can be viewed as a polynomial in $z_1', \ldots, z_n', \ldots, z_1^{(n)}, \ldots, z_n^{(n)}$ of degree at most q, with coefficients depending holomorphically on s (but not on the jet coordinates). This polynomial is not zero only if it has at least one non zero coefficient, but, by Taylor formula, these coefficients are all of the form:

$$\Big(\frac{\partial}{\partial z_{i_q}^{(k_q)}} \cdots \frac{\partial}{\partial z_{i_1}^{(k_1)}} \cdot \sigma\Big)_{s_0},$$

for $q \leqslant m$. It suffices to chose vector fields such that

$$\mathsf{v}_k|_{s_0, f_{[n]}(t_0)} = \frac{\partial}{\partial z_{i_k}^{(j_k)}} \qquad {\scriptstyle (k=1,\ldots,q)},$$

which is always possible, by the result of global generation. □

Thus, if for any $p \leqslant m$ and any vector fields $\mathsf{v}_1, \ldots, \mathsf{v}_p$, one has the vanishing:

$$0 \equiv \big(\mathsf{v}_p \cdots \mathsf{v}_1 \cdot \sigma\big)_{s_0}\big(f_{[n]}\big),$$

then the sought contradiction is obtained. In particular, by the fundamental vanishing theorem, it is the case if the vanishing order is positive for all $p \leqslant m$. In other words, if the degree d satisfies:

$$2\delta m(d-n-1) > mn(n+2),$$

then, the entire curves with values in the complement of a generic hypersurface of degree d are algebraically degenerate. After dividing by m, the condition on the degree becomes:

$$d > (n+1) + \frac{n(n+2)}{2\delta}.$$

Now, recall that by assumption $35n^n\,\delta \leqslant 1$. After fixing: $35n^n\,\delta = 1$, in order to minimize the lower bound, one finally obtains the stronger condition (for $n \geqslant 5$):

$$d \geqslant (5n)^2\,n^n > n^{n+2}\left(\frac{n+1}{n^{n+2}} + \frac{35(n+2)}{2n}\right).$$

Here, to obtain a better looking lower bound, we assume $n \geqslant 6$, but one can drop the hypothesis since for $n \leqslant 5$, this lower bound on the degree also holds, by explicit computation (see Table 1 below, page 77).

Notice that this bound $(5n)^2 n^n > 52n^n$ is sufficiently large in order to prove the existence of one global jet differential.

Codimension of the degeneracy locus. In [**17**], the authors show that the degeneracy locus of entire curves has no divisorial components. As a result, its codimension is at least two.

3. Existence of Global Logarithmic Jet Differentials

In this section, the intrinsic strategy outlined in [**12**] is followed in order to prove the existence theorem 2.2.2, *viz.*:

(3.0.1) *For any smooth hypersurface $H \subset \mathbb{P}^n$ of degree d and any positive rational number δ such that:*

$$d \geqslant (52\,n^n) \text{ and } (35\,n^n)\,\delta \leqslant 1,$$

the vector space of logarithmic global jet differentials along H of order $k = n$ and weighted degree $m \gg n^n$ vanishing on $\mathcal{K}_{\mathbb{P}^n}(H)^{2\delta m}$ has positive dimension:

$$\dim H^0\Big(\mathbb{P}^n, \mathcal{E}_{n,m} T^\star_{\mathbb{P}^n}(\log H) \otimes \mathcal{K}_{\mathbb{P}^n}(H)^{-2\delta m}\Big) \geqslant 1.$$

We incorporate a new key ingredient to this strategy, namely a multivariate residue formula, in order to obtain the above effective result in any dimension. In what follows, we explain how all the arguments link together and, strictly speaking, the proof of the existence theorem itself will be achieved in the next section §3.

A general intrinsic method for obtaining global jet differentials relies on the construction of Demailly ([**7**]), generalized to the logarithmic setting by Dethloff and Lu ([**10**]), that we will now recall.

3.1. Demailly tower of logarithmic directed manifolds. A *log-directed manifold* is a triple $\left(\overline{X}, D, V\right)$, where $\left(\overline{X}, D\right)$ is a log manifold and V is a holomorphic subbundle of the logarithmic tangent bundle $T_{\overline{X}}(-\log D)$. Given such a log-directed manifold $(\overline{X}_0, D_0, V_0)$, one constructs on $\overline{X}_0$ the *Demailly tower of log directed manifolds*:

$$(\overline{X}_\kappa, D_\kappa, V_\kappa) \xrightarrow{\pi_\kappa} (\overline{X}_{\kappa-1}, D_{\kappa-1}, V_{\kappa-1}) \xrightarrow{\pi_{\kappa-1}} \ldots \xrightarrow{\pi_1} (\overline{X}_0, D_0, V_0),$$

by induction on $\kappa \geqslant 0$. This construction has the same formal properties as in the so-called *compact case*, *i.e.* where there is no divisor D_0. Here, in the genuine logarithmic setting, V_k is a holomorphic subbundle of the logarithmic tangent bundle $T_{\overline{X}_k}(-\log D_k)$.

Recall quickly the inductive step $\left(\overline{X}', D', V'\right) \xrightarrow{\pi} \left(\overline{X}, D, V\right)$ of the construction of the log Demailly tower ([**10**]). The space $\overline{X}'$ is the total space $P(V)$ of the projective bundle of lines of V:

$$\overline{X}' := P(V) \xrightarrow{\pi} \overline{X},$$

and in order to make π a log-morphism it is natural to set:

$$D' := \pi^{-1}(D) \subset \overline{X}'.$$

The *relative tangent bundle* T_π of the log-morphism $\pi\colon (\overline{X}, D) \to (\overline{X}', D')$ is defined as the kernel of the differential $\pi_\star$:

$$0 \to T_\pi \hookrightarrow T_{\overline{X}'}(-\log D') \xrightarrow{\pi_\star} \pi^\star T_{\overline{X}}(-\log D) \to 0. \tag{3.1.1}$$

Keep in mind that V is a subbundle of the logarithmic tangent bundle $T_{\overline{X}}(-\log D)$, displayed in the right of this short exact sequence, and that V' has to be a subbundle of the logarithmic tangent bundle $T_{\overline{X}'}(-\log D')$, displayed in the center of this short exact sequence.

The tautological line bundle of $\overline{X}' = P(V)$ is a subbundle of $\pi^\star T_{\overline{X}}(-\log D)$ because:

$$\mathcal{O}_{\overline{X}'}(-1) \subset \pi^\star V \subset \pi^\star T_{\overline{X}}(-\log D).$$

One can thus define a subbundle $V' \subset T_{X'}(-\log D')$, locally isomorphic to $\pi^\star V$, by taking:

$$V' := (\pi_\star)^{-1}\mathcal{O}_{\overline{X}'}(-1).$$

Equivalently, V' is defined by the following short exact sequence:

$$0 \to T_\pi \hookrightarrow V' \xrightarrow{\pi^\star} \mathcal{O}_{\overline{X}'}(-1) \to 0. \tag{3.1.2}$$

Comparing the two short exact sequences (3.1.1) and (3.1.2), notice that in the left, one keeps all the vertical directions whereas in the right, one keeps only the single "tautological" direction among all horizontal directions.

3.2. The direct image formula. For any integer $i = 1, \dots, \kappa - 1$, the composition of the projections $\pi_j\colon \overline{X}_j \to \overline{X}_{j-1}$ yields a projection:

$$\pi_{\kappa,i} := \pi_{i+1} \circ \cdots \circ \pi_\kappa\colon \overline{X}_\kappa \to \overline{X}_i.$$

One can pullback the tautological line bundle of X_i to the κ-th level, thereby obtaining κ line bundles:

$$(\pi_{\kappa,1})^\star\mathcal{O}_{\overline{X}_1}(-1),\ (\pi_{\kappa,2})^\star\mathcal{O}_{\overline{X}_2}(-1),\ \dots\ ,\ (\pi_{\kappa,\kappa-1})^\star\mathcal{O}_{\overline{X}_{\kappa-1}}(-1),\ \mathcal{O}_{\overline{X}_\kappa}(-1).$$

The linear combinations of these line bundles with integer coefficients $(a_1, \dots, a_\kappa) \in \mathbb{N}^\kappa$:

$$\mathcal{O}_{\overline{X}_\kappa}(a_1, \dots, a_\kappa) := \mathcal{O}_{\overline{X}_\kappa}(a_\kappa) \otimes (\pi_{\kappa,\kappa-1})^\star\mathcal{O}_{\overline{X}_{\kappa-1}}(a_{\kappa-1}) \otimes \cdots \otimes (\pi_{\kappa,1})^\star\mathcal{O}_{\overline{X}_1}(a_1).$$

enjoy positivity properties ([**7, 12**]):

(3.2.1) ([12, proposition §3.2]) *If $a_1 \geqslant 3\,a_2, \dots, a_{\kappa-2} \geqslant 3\,a_{\kappa-1}$, and $a_{\kappa-1} \geqslant 2\,a_\kappa > 0$, then the line bundle*

$$\mathcal{F} = \mathcal{O}_{\overline{X}_\kappa}(a_1, \dots, a_\kappa) \otimes \pi_{\kappa,0}^\star\mathcal{O}_{\overline{X}}(\ell)$$

is nef provided that $\ell \geqslant 2\,(a_1 + \cdots + a_\kappa)$.

Recall [**27**, 1.4.1] that a line bundle $\mathcal{L} \to X$ over X is numerically effective (*nef*) if for every irreducible curve $C \subset X$, one has $\int_C c_1(\mathcal{L}) \geqslant 0$.

Here the first factor $\mathcal{O}_{\overline{X}_\kappa}(a_1, \dots, a_\kappa)$ of $\mathcal{F}$ is positive on curves lying in the fibers, and the second factor $\mathcal{O}_{\overline{X}}(\ell)$ compensate the negativity than could occur in the horizontal directions.

In chapter II, we show that it is natural to work with the κ line bundles:

$$\mathcal{L}_i := \mathcal{O}_{\overline{X}_i}(-1, \dots, -1) \to \overline{X}_i \qquad {\scriptstyle (i=1,\dots,\kappa)},$$

that can be viewed as the tautological line bundles of certain twists of the distribution V_i. For integers coefficients $(a_1, \dots, a_\kappa) \in \mathbb{N}^\kappa$, in analogy with the notation

$$\mathcal{O}_{\overline{X}_k}(a_1, \dots, a_\kappa) = (\pi_{\kappa,1})^\star (\mathcal{O}_{\overline{X}_i}(-1)^\vee)^{a_1} \otimes \cdots \otimes (\pi_{\kappa,\kappa})^\star (\mathcal{O}_{\overline{X}_k}(-1)^\vee)^{a_1},$$

let us use the notation:

$$\mathcal{L}(a_1, \dots, a_\kappa) := (\pi_{\kappa,1})^\star (\mathcal{L}_1^\vee)^{a_1} \otimes \cdots \otimes (\pi_{\kappa,\kappa})^\star (\mathcal{L}_\kappa^\vee)^{a_\kappa}.$$

By additivity, the translation formula is:

$$\mathcal{L}(a_1, a_2, \dots, a_\kappa) = \mathcal{O}_{\overline{X}_\kappa}(a_1 + a_2 + \cdots + a_\kappa, a_2 + \cdots + a_\kappa, \dots, a_\kappa). \tag{3.2.2}$$

Considering this formula (3.2.2) —that can be thought of as a plain change of variables—, one can easily transpose existing results on the line bundles $\mathcal{O}_{\overline{X}_\kappa}(\underline{a})$ ([**7, 10**]) into results on the line bundles $\mathcal{L}(\underline{a})$. One has the following *direct image formula*:

(3.2.3) *For any κ-tuple $(a_1, \dots, a_\kappa) \in \mathbb{Z}^\kappa$ with:*

$$a_i + \cdots + a_\kappa \geqslant 0 \qquad {\scriptstyle [i=1,\dots,\kappa]},$$

the direct image $(\pi_{\kappa,0})_\star \mathcal{L}(a_1, \dots, a_\kappa)$ may be seen as a subsheaf of the Green-Griffiths bundle $\mathcal{E}_{\kappa,\mu} V_0^\star(\log D_0)$, where:

$$\mu = \mu(\underline{a}) := 1\, a_1 + 2\, a_2 \cdots + \kappa\, a_\kappa \in \mathbb{N}.$$

And using again (3.2.2) one has also the analog of Proposition 3.2.1:

(3.2.4) *In the logarithmic absolute case, if $a_1, \dots, a_\kappa$ are κ positive integers having weighted sum $\mu(\underline{a}) = 1\, a_1 + \cdots + \kappa\, a_\kappa$, and satisfying the inequalities:*

$$a_i \geqslant 3\, a_{i+1} > 0 \qquad {\scriptstyle [i=1,\dots,\kappa-1]},$$

then the line bundle $\mathcal{F} := \mathcal{L}\big(a_1, \dots, a_\kappa\big) \otimes (\pi_{\kappa,0})^\star \mathcal{O}_{\mathbb{P}^n}\big(2\mu(\underline{a})\big) \to \overline{X}_\kappa$ is nef.

Proof. The plain translation of Proposition (3.2.1) yields the result for integers $a_1, \dots, a_\kappa$ satisfying:

$$(a_1 + \cdots + a_\kappa) \geqslant 3\,(a_2 + \cdots + a_\kappa), \dots, (a_{\kappa-2} + \cdots + a_\kappa) \geqslant 3\,(a_{\kappa-1} + a_\kappa),$$
$$\text{and} \qquad (a_{\kappa-1} + a_\kappa) \geqslant 2\, a_\kappa > 0,$$

but we make a stronger assumption. Indeed, if $a_i \geqslant 3\, a_{i+1}$, then:

$$(a_i + \cdots + a_{\kappa-1} + a_\kappa) \geqslant 3\,(a_{i+1} + \cdots + a_\kappa) + a_\kappa. \qquad \square$$

In order to prove the existence of sections of the Green-Griffiths logarithmic jet bundle, it is sufficient to prove the existence of sections of $\mathcal{L}(a_1, \dots, a_\kappa)$ for a certain suitable choice of the parameters $a_1, \dots, a_\kappa$. More generally, for any ample line bundle $\mathcal{A}$:

$$\dim H^0\Big(\overline{X}_0, \mathcal{E}_{\kappa,\mu(\underline{a})} V_0^\star(\log D_0) \otimes \mathcal{A}^\vee\Big) \geqslant \dim H^0\Big(\overline{X}_\kappa, \mathcal{L}(\underline{a}) \otimes \pi_{\kappa,0}^\star \mathcal{A}^\vee\Big).$$

3.3. Morse inequalities. Starting with a hypersurface $H \subset \mathbb{P}^n$ of degree d, fix the log pair $(\overline{X}_0, D_0) := (\mathbb{P}^n, H)$, and also fix the distribution V_0 by taking the whole logarithmic tangent space $V_0 := T_{\mathbb{P}^n}(-\log H)$. Now, let as above denote the Demailly tower constructed on the log directed manifold $\big(\mathbb{P}^n, H, T_{\mathbb{P}^n}(-\log H)\big)$ by:

$$\big(\overline{X}_\kappa, D_\kappa\big) \to \big(\overline{X}_{\kappa-1}, D_{\kappa-1}\big) \to \ldots \to \big(\overline{X}_1, D_1\big) \to \big(\overline{X}_0, D_0\big).$$

This context will be called the *logarithmic absolute case*. Now, we consider the line bundle:

$$\mathcal{L}(a_1, \ldots, a_\kappa) \otimes (\pi_{\kappa,0})^\star \mathcal{A}^\vee = \mathcal{L}(a_1, \ldots, a_\kappa) \otimes (\pi_{\kappa,0})^\star \mathcal{K}_{\mathbb{P}^n}(H)^{-2\mu\delta}$$

It is the difference of two nef line bundles $\mathcal{F}, \mathcal{G}$ because one has the trivial identity (where the pullbacks are now omitted):

$$\mathcal{L}(a_1, \ldots, a_\kappa) \otimes \mathcal{A}^\vee = \underbrace{\Big(\mathcal{L}(a_1, \ldots, a_\kappa) \otimes \mathcal{O}_{\mathbb{P}^n}(2\mu)\Big)}_{=:\mathcal{F}} \otimes \underbrace{\Big(\mathcal{O}_{\mathbb{P}^n}\big(2\mu + 2\mu\delta(d-n-1)\big)\Big)^\vee}_{=:\mathcal{G}}.$$

Indeed, it has precisely be said just above that $\mathcal{F}$ is nef, and $\mathcal{G}$ is also nef because it is the pullback of a nef line bundle.

Together with the following Demailly-Trapani *Morse inequalities* this fact provides with a (rough) control the cohomology of the line bundle $\mathcal{L}(a_1, \ldots, a_\kappa) \otimes \mathcal{A}^\vee \to \overline{X}_\kappa$.

(3.3.1**) ([49, 7])** *For any holomorphic line bundle $\mathcal{L}$ on a N-dimensional compact manifold X, that can be written as the difference $\mathcal{L} = \mathcal{F} \otimes \mathcal{G}^\vee$ of two nef line bundles $\mathcal{F}$ and $\mathcal{G}$, one has:*

$$\dim H^0\big(X, \mathcal{L}^{\otimes k}\big) \geqslant k^N\, \frac{\big(\mathcal{F}^N\big) - k\big(\mathcal{F}^{N-1}\cdot \mathcal{G}\big)}{N!} - o(k^N).$$

Recall that by definition [**27**], the intersection number $\big(L_1 \cdots L_k\big)$ of k line bundles on a k dimensional variety X denotes:

$$\big(L_1 \cdots L_k\big) := \int_X c_1(L_1) \cdots c_1(L_k).$$

These algebraic Morse inequalities (3.3.1) can now be applied to the constructed line bundle $\mathcal{L}(\underline{a}) \otimes \mathcal{A}^\vee \to \overline{X}_\kappa$. By induction the dimension $n_\kappa := \dim(\overline{X}_\kappa)$ is simple to compute:

$$n_\kappa = \dim(\overline{X}_0) + \kappa\,\big(\operatorname{rk} P(V_0)\big) = n + \kappa\,(n-1).$$

(3.3.2**)** *If the integers $a_1, \ldots, a_\kappa$ satisfy the inequalities:*

$$a_i \geqslant 3\, a_{i+1} > 0 \qquad {\scriptstyle [i=1,\ldots,\kappa-1]},$$

and if $\delta \in \mathbb{Q}$ is a fixed positive rational number, it is sufficient to establish the positivity of the intersection number:

$$I := \int_{\overline{X}_\kappa} c_1(\mathcal{F})^{n_\kappa} - n_\kappa\, c_1(\mathcal{F})^{n_\kappa - 1} c_1(\mathcal{G}),$$

where:

$$\begin{cases} c_1(\mathcal{F}) = 2\mu(\underline{a})\, h + a_1\, c_1(\mathcal{L}_1^\vee) + \cdots + a_\kappa\, c_1(\mathcal{L}_\kappa^\vee) \\ c_1(\mathcal{G}) = \big(1 + \delta(d-n-1)\big) 2\mu(\underline{a})\, h, \end{cases}$$

in order to guarantee the existence of non zero global sections in the vector space:

$$H^0\Big(\overline{X}_\kappa, \mathcal{E}_{\kappa,m}T^\star_{\mathbb{P}^n}\big(\log H\big) \otimes (\pi_{\kappa,0})^\star\big(\mathcal{K}_{\mathbb{P}^n}(H)^\vee\big)^{2\delta m}\Big),$$

for asymptotic weighted degree $m \gg 1$.

Proof. Assume that one can find $\underline{a} = (a_1, \dots, a_\kappa)$, such that $I > 0$ for $d \geqslant \lambda(\underline{a})$. By the above algebraic Morse inequality, one has, for large integers k:

$$\dim H^0\Big(\overline{X}_\kappa, \big(\mathcal{L}(\underline{a}) \otimes \mathcal{K}_{\mathbb{P}^n}(H)^{-2\delta\mu(\underline{a})}\big)^{\otimes k}\Big) \geqslant k^{n_\kappa}\,\frac{I}{n_\kappa!} - o(k^{n_\kappa}).$$

Now, both $\mathcal{L}(\underline{a})$ and $\mu(\underline{a})$ are linear in $\underline{a}$. Thus:

$$\big(\mathcal{L}(\underline{a}) \otimes \mathcal{K}_{\mathbb{P}^n}(H)^{-2\delta\mu(\underline{a})}\big)^{\otimes k} = \mathcal{L}(k\underline{a}) \otimes \mathcal{K}_{\mathbb{P}^n}(H)^{-2\delta\mu(k\underline{a})}.$$

Since $k\underline{a}$ still satisfies the inequalities $k\,a_i \geqslant 3\,k\,a_{i+1} > 0$, for $i = 1, \dots, \kappa - 1$, the sheaf $\mathcal{L}(k\underline{a})$ is a subsheaf of the Green-Griffiths logarithmic jet bundle $\mathcal{E}_{\kappa,m}T^\star_{\mathbb{P}^n}(\log H)$ for the weighted degree

$$m := \mu(k\,\underline{a}) = k\,\mu(\underline{a}).$$

By taking k large enough, one has:

$$\dim H^0\Big(\overline{X}_\kappa, \mathcal{L}(k\underline{a}) \otimes \mathcal{K}_{\mathbb{P}^n}(H)^{-2\delta\mu(k\underline{a})}\Big) \geqslant \frac{1}{2}\,k^{n_\kappa}\,\frac{I}{n_\kappa!} > 0.$$

Thus, for $m \gg 1$ large enough, *not* effective:

$$\dim H^0\Big(\overline{X}_\kappa, \mathcal{E}_{\kappa,m}T^\star_{\mathbb{P}^n}(\log H) \otimes \mathcal{K}_{\mathbb{P}^n}(H)^{-2\delta m}\Big) \geqslant 1. \qquad \square$$

Denote the first Chern classes of the vertical line bundles $\mathcal{L}_i$ by:

$$v_i := c_1\big(\mathcal{L}_i^\vee\big) \qquad [i=1,\dots,\kappa].$$

then the integrand of the above intersection product:

$$f(v_1, \dots, v_\kappa) := c_1(\mathcal{F})^{n_\kappa} - n_\kappa\, c_1(\mathcal{F})^{n_\kappa-1} c_1(\mathcal{G})$$

can be written in this notation as:

$$f(v_1, \dots, v_\kappa) := \big(2\mu h + a_1 v_1 + \cdots + a_\kappa v_\kappa\big)^{n_\kappa} - \\ \big(1 + \delta(d - n - 1)\big)\, n_\kappa\, 2\mu h\big(2\mu h + a_1 v_1 + \cdots + a_\kappa v_\kappa\big)^{n_\kappa - 1}.$$

We will next give a formula for integrating f under this form.

3.4. Formal computation on the Demailly tower. It is convenient to bring down the computation to the base $\overline{X}_0 = \overline{X}$. In chapter II, we provide a formula in that aim. This formula expresses intersection products on $\overline{X}_\kappa$ as coefficients of an *iterated Laurent series*, in the very spirit of the predating residue formula of Bérczi ([**3**]).

Iterated Laurent series. An iterated Laurent series is a multivariate formal series having well ordered support with respect to the lexicographic order on $\mathbb{Z}^n$. The main advantage of using the subspace of iterated Laurent series over using the whole space of multivariate formal series is that the Cauchy product is well defined. There is an injection of the field of rational functions in the field of iterated Laurent series, called the iterated Laurent series expansion at the origin. The reader is referred to the section II.3.3 for a precise description of this process. By convention, in the product of an iterated Laurent series and a rational function, the rational function will always be replaced by its expansion. Accordingly, such a product is in fact the Cauchy product of two iterated Laurent series. Lastly, the expression $[M]\Phi$ will denote the coefficient of a monomial M in the multivariate formal series Φ.

Integration formula. We can now recall the integration formula (II.3.3.5) of chapter II. In analogy with Chern polynomial, for a vector bundle $E \to X$ over a N dimensional manifold X, define $s_u(E)$ to be the generating function of the Segre classes of E, that is:

$$s_u(E) := s_0(E) + u\, s_1(E) + u^2\, s_2(E) + \cdots + u^n\, s_N(E).$$

(3.4.1**)** *For any polynomial $f \in H^\bullet(\overline{X}_0, V_0)[t_1, \ldots, t_\kappa]$, in κ variables $t_1, \ldots, t_\kappa$, with coefficients in the cohomology ring of the base, the intersection number:*

$$I := \int_{\overline{X}_\kappa} f(v_1, \ldots, v_\kappa)$$

is equal to the Cauchy product coefficient:

$$I = \big[t_1^{n-1} \cdots t_\kappa^{n-1}\big]\bigg(\int_{\mathbb{P}^n} f(t_1, \ldots, t_\kappa) \prod_{j=1}^{\kappa} s_{1/t_j}\big(T^\star_{\mathbb{P}^n}(\log D)\big) \prod_{1\leqslant i<j\leqslant\kappa} \bigg(\frac{t_j - t_i}{t_j - 2t_i}\bigg) \prod_{2\leqslant i<j\leqslant\kappa} \bigg(\frac{t_j - 2\,t_i}{t_j - 2\,t_i + t_{i-1}}\bigg)\bigg).$$

This formula allows to eliminate simultaneously all the vertical first Chern classes v_i.

(3.4.2**)** *In the "logarithmic absolute case", for any homogeneous polynomial:*

$$f \in \mathbb{C}[h][t_1, \ldots, t_\kappa] = \mathbb{C}[h, t_1, \ldots, t_\kappa]$$

having total degree $n_\kappa = \dim\, \overline{X}_\kappa$ (with respect to the variables $h, t_1, \ldots, t_\kappa$), the corresponding cohomology class $f(v_1, \ldots, v_\kappa) \in H^\bullet(\overline{X}_\kappa)$ can be integrated on $\overline{X}_\kappa$ using the formula:

$$\int_{\overline{X}_\kappa} f(v_1, \ldots, v_\kappa) = \big[h^n t_1^n \cdots t_\kappa^n\big]\big(\mathsf{A}(t_1, \ldots, t_\kappa)\, \mathsf{B}(t_1, \ldots, t_\kappa)\, \mathsf{C}(t_1, \ldots, t_\kappa)\big),$$

where A is the polynomial with coefficients in $\mathbb{C}[h]$ obtained from f by:

$$\mathsf{A}(t_1, \ldots, t_\kappa) := (dh + t_1) \cdots (dh + t_\kappa)\, f(t_1, \ldots, t_\kappa),$$

where B is the Laurent polynomial with coefficients in $\mathbb{C}[h]$:

$$\mathsf{B}(t_1, \ldots, t_\kappa) := \sum_{j_1, \ldots, j_\kappa \geqslant 0} \binom{n + j_1}{n} \cdots \binom{n + j_\kappa}{n} \frac{(-h)^{j_1 + \cdots + j_\kappa}}{t_1^{j_1} \cdots t_\kappa^{j_\kappa}},$$

and where C is the (unequivocal) iterated Laurent series expansion of the universal rational function:

$$\mathsf{C}(t_1, \ldots, t_\kappa) := \prod_{1\leqslant i<j\leqslant\kappa} \bigg(\frac{t_j - t_i}{t_j - 2t_i}\bigg) \prod_{2\leqslant i<j\leqslant\kappa} \bigg(\frac{t_j - 2\,t_i}{t_j - 2\,t_i + t_{i-1}}\bigg).$$

with respect to the order $t_1 \ll \cdots \ll t_\kappa \ll 1$.

Proof. One has:

$$\int_{\overline{X}_\kappa} f(v_1,\dots,v_\kappa) = \Big[t_1^{n-1}\cdots t_\kappa^{n-1}\Big]\bigg(\int_{\mathbb{P}^n} f(t_1,\dots,t_\kappa)\prod_{j=1}^{\kappa} s_{1/t_j}\Big(T^\star_{\mathbb{P}^n}(\log D)\Big)$$
$$\prod_{1\leqslant i<j\leqslant\kappa}\left(\frac{t_j-t_i}{t_j-2t_i}\right)\prod_{2\leqslant i<j\leqslant\kappa}\left(\frac{t_j-2\,t_i}{t_j-2\,t_i+t_{i-1}}\right)\bigg).$$

In the second line, we recognize the universal series C:

(∗)
$$\int_{\overline{X}_\kappa} f(v_1,\dots,v_\kappa) = \Big[t_1^{n-1}\cdots t_\kappa^{n-1}\Big]\bigg(\int_{\mathbb{P}^n} f(t_1,\dots,t_\kappa)\prod_{j=1}^{\kappa} s_{1/t_j}\Big(T^\star_{\mathbb{P}^n}(\log D)\Big)\,\mathsf{C}(t_1,\dots,t_\kappa)\bigg).$$

By a classic computation, the total Segre class of the base is:

$$s_\bullet(V_0) = s_\bullet\Big(T_{\mathbb{P}^n}(-\log H)\Big) = \frac{(1+d\,h)}{(1+h)^{n+1}}.$$

Thus the appearing Laurent polynomials $s_{1/t_j}\Big(T^\star_{\mathbb{P}^n}(\log D)\Big)$ have the expression:

$$s_{1/t_i}(V_0) = \frac{t_i+dh}{t_i}\sum_{j_i=0}^{n}\binom{n+j_i}{j_i}\left(\frac{-h}{t_i}\right)^{j_i},$$

and the product of these expressions for $i=1,\dots,\kappa$ is:

$$\prod_{i=1}^{\kappa} s_{1/t_i}\Big(T^\star_{\mathbb{P}^n}(\log D)\Big) = \frac{t_1+dh}{t_1}\cdots\frac{t_\kappa+dh}{t_\kappa}\,\mathsf{B}(t_1,\dots,t_\kappa).$$

Then (∗) becomes:

$$\int_{\overline{X}_\kappa} f(v_1,\dots,v_\kappa) =$$
$$\Big[t_1^{n-1}\cdots t_\kappa^{n-1}\Big]\bigg(\frac{1}{t_1\cdots t_\kappa}\int_{\mathbb{P}^n} f(t_1,\dots,t_\kappa)\prod_{i=1}^{\kappa}(dh+t_i)\,\mathsf{B}(t_1,\dots,t_\kappa)\,\mathsf{C}(t_1,\dots,t_\kappa)\bigg),$$

where $\mathsf{A}(t_1,\dots,t_k) = \prod_{i=1}^{\kappa}(dh+t_i)\,f(t_1,\dots,t_\kappa)$ now appears in the integrand. So we obtain:

$$\int_{\overline{X}_\kappa} f(v_1,\dots,v_\kappa) = \Big[t_1^{n-1}\cdots t_\kappa^{n-1}\Big]\bigg(\frac{1}{t_1\cdots t_\kappa}\int_{\mathbb{P}^n}\mathsf{A}(t_1,\dots,t_\kappa)\,\mathsf{B}(t_1,\dots,t_\kappa)\,\mathsf{C}(t_1,\dots,t_\kappa)\bigg)$$
$$= \Big[t_1^{n}\cdots t_\kappa^{n}\Big]\bigg(\int_{\mathbb{P}^n}\mathsf{A}(t_1,\dots,t_\kappa)\,\mathsf{B}(t_1,\dots,t_\kappa)\,\mathsf{C}(t_1,\dots,t_\kappa)\bigg).$$

Now, it remains to integrate polynomials in the hyperplane class h on $\mathbb{P}^n$. For degree reason, the integrand is a multiple of h^n. The coefficient of this monomial is the sought intersection product, because it is known that: $\int_{\mathbb{P}^n} h^n = 1$.

Finally, our computation becomes the very concrete combinatorial problem:

$$\int_{X_\kappa} f(v_1,\dots,v_\kappa) = \Big[h^n t_1^n\cdots t_\kappa^n\Big]\bigg(\mathsf{A}(t_1,\dots,t_\kappa)\,\mathsf{B}(t_1,\dots,t_\kappa)\,\mathsf{C}(t_1,\dots,t_\kappa)\bigg). \qquad \square$$

Moreover, this formula stays true without assumption on the degree of f, because the only power of h that does not vanish by integration on the base is the n-th power.

4. Implementation of the Computation

In this section and until the end of the chapter, we will restrict ourselves to the case $\kappa = n$. In particular,

$$n_\kappa = n + n(n-1) = n^2.$$

We also assume $n \geqslant 6$ when necessary, because for $n \leqslant 5$ the value of the largest root of $I(d)$ obtained by explicit computation with a computer algebra system — for $\underline{a} = (2\,3^{n-2}, \dots, 6, 2, 1)$ — is much less than $(5n)^2\, n^n$ (see Table 1 below).

	$\lceil\lambda\rceil$			τ (ms)
	$c = 0$	$c = (n+1)^2$	$(5n)^2 n^n$	
$n = 2$	15	231	400	12
$n = 3$	75	2 075	6 075	60
$n = 4$	306	13 229	102 400	1 156
$n = 5$	1 154	71 420	1 953 125	107 366

Table 1. Small dimensions: Largest roots for $I(d)$ and indicative time of computation τ

We will now establish that if $n \geqslant 6$, $m \gg d \geqslant 52\, n^n$ and if $0 < 35\, n^n\, \delta \leqslant 1$ then

$$\dim H^0\Big(\mathbb{P}^n, \mathcal{E}_{n,m} T^\star_{\mathbb{P}^n}\big(\log H\big) \otimes \mathcal{K}_{\mathbb{P}^n}(H)^{-2m\delta}\Big) \geqslant 1.$$

For this, according to the previous section, it is sufficient to prove — under the same hypotheses — the positivity of the intersection product:

$$I = \Big[h^n t_1^n \cdots t_n^n\Big]\Big(\mathsf{A}(\underline{t})\, \mathsf{B}(\underline{t})\, \mathsf{C}(\underline{t})\Big),$$

for a fixed choice of weights $a_1, a_2, \dots, a_n$, where $\underline{t}$ stands for the n-tuple $t_1, \dots, t_n$.

4.1. Preliminary expansion of A. The first appearing term is the polynomial:

$$\mathsf{A}(t_1, \dots, t_n) = (d\,h + t_1) \cdots (d\,h + t_n)\, f(t_1, \dots, t_n).$$

It is the only term involving d.

Recall that in our situation, the polynomial f to be integrated on the Demailly tower is:

$$f(t_1, \dots, t_n) := \Big(2\mu h + a_1 t_1 + \cdots + a_n t_n\Big)^{n^2} - n^2\Big(\delta\, d + (1 - \delta\, n - \delta)\Big) 2\mu h \Big(2\mu h + a_1 t_1 + \cdots + a_n t_n\Big)^{n^2-1},$$

where μ is the weighted sum of the coefficients a_i:

$$\mu = \mu(\underline{a}) = 1\, a_1 + \cdots + n\, a_n,$$

and let us introduce the short notation:

$$f_i(t_1, \dots, t_n) := \frac{n^2!}{(n^2 - i)!} (2\mu)^i \Big(a_1 t_1 + \ldots + a_n t_n\Big)^{n^2 - i} \qquad [i=0,1,\dots,n]$$

for the terms (not depending on d) that appear in the expansion of f with respect to the hyperplane class h:

$$f(t_1, \dots, t_n) = \sum_{i=0}^{n} h^i\, (\alpha_i - d\beta_i)\, f_i(t_1, \dots, t_n),$$

where the rational coefficients α_i and β_i have the respective expressions:

$$\alpha_i := \frac{1-(1-\delta\, n-\delta)i}{i!} \quad \text{and} \quad \beta_i := \frac{\delta i}{i!} \qquad {\scriptstyle [i=0,1,\ldots,n]}.$$

In particular:

$$\alpha_0 = 1,\ \alpha_1 = \delta(n+1),\ \beta_0 = 0,\ \beta_1 = \beta_2 = \delta.$$

Also one has, by using the notation e_j for the j-th elementary symmetric function of n variables:

$$(d\,h + t_1)\cdots(d\,h+t_n) = \sum_{j=0}^{n} (d\,h)^{n-j} e_j(t_1,\ldots,t_n).$$

Thus A is the following product of polynomials in the variables $d, h, t_1, \ldots, t_n$:

$$\mathsf{A}(t_1,\ldots,t_n) = \Big(\sum_{j=0}^{n} d^{n-j}\, h^{n-j}\, e_j(\underline{t})\Big)\Big(\sum_{i=0}^{n} h^i\, (\alpha_i - d\,\beta_i)\, f_i(\underline{t})\Big).$$

Recall that $\beta_0 = 0$, thus A has degree at most n with respect to d, because in its expression d appears only as a factor of the product dh, and $h^{n+1} = 0$.

(4.1.1) *For $p = 0, 1, \ldots, n$, the coefficient A_p of d^{n-p} in A is:*

$$\mathsf{A}_p(\underline{t}) := \big[d^{n-p}\big]\mathsf{A}(\underline{t}) = h^{n-p} \sum_{q=0}^{p} h^q\Big(\alpha_q\, e_p(\underline{t})\, f_q(\underline{t}) - \beta_{q+1}\, e_{p+1}(\underline{t})\, f_{q+1}(\underline{t})\Big).$$

Proof. Recall:

$$\mathsf{A}(t_1,\ldots,t_n) = \sum_{i=0}^{n}\Big((\alpha_i - d\,\beta_i)\, h^i\, f_i(\underline{t})\Big)\sum_{j=0}^{n}\Big(d^{n-j}\, h^{n-j}\, e_j(\underline{t})\Big).$$

There are two ways to obtain d^{n-p}; indeed the first polynomial is linear in d. Selecting the constant coefficient (with respect to d) in the first polynomial and the coefficient of d^{n-p} in the second polynomial, one obtains the first contribution:

$$(*) \qquad \sum_{i=0}^{p} \alpha_i\, h^i\, f_i(\underline{t}) \cdot h^{n-p}\, e_p(\underline{t}),$$

and alternatively, selecting the slope (with respect to d) in the first polynomial and the coefficient of d^{n-p-1} in the second polynomial, one obtains the second contribution:

$$(**) \qquad \sum_{i=0}^{p+1} -\beta_i\, h^i\, f_i(\underline{t}) \cdot h^{n-(p+1)}\, e_{p+1}(\underline{t}).$$

Here the sums stop at index p and $p+1$, because $h^{n+1} = 0$.

For any integer $q = 0, 1, \ldots, p$, the coefficient of h^{n-p+q} in the first contribution $(*)$ is: $\alpha_q\, f_q(\underline{t})\, e_p(\underline{t})$, and the coefficient of h^{n-p+q} in the second contribution $(**)$ is: $-\beta_{q+1}\, f_{q+1}(\underline{t})\, e_{p+1}(\underline{t})$. One can easily deduce the expansion of A_p:

$$\mathsf{A}_p = h^{n-p} \sum_{q=0}^{p} h^q\,\Big(\alpha_q\, e_p(\underline{t})\, f_q(\underline{t}) - \beta_{q+1}\, e_{p+1}(\underline{t})\, f_{q+1}(\underline{t})\Big).$$

That is the announced result. □

4.2. Examination of the terms B and C. The term B appearing in Proposition 3.4.2 is the Laurent polynomial:

$$\mathsf{B}(t_1,\dots,t_\kappa) := \sum_{j_1,\dots,j_\kappa\geqslant 0} \binom{n+j_1}{n}\cdots\binom{n+j_\kappa}{n}\frac{(-h)^{j_1+\cdots+j_\kappa}}{t_1^{j_1}\cdots t_\kappa^{j_\kappa}}.$$

For multi-indices $j\in\mathbb{Z}^n$, define the coefficients $\mathsf{B}_j := (-1)^{j_1+\cdots+j_n}\binom{n+j_1}{n}\cdots\binom{n+j_n}{n}$ in such way that:

$$\mathsf{B} = \sum_{j\in\mathbb{N}^n}\mathsf{B}_j\left(\frac{h}{t_1}\right)^{j_1}\cdots\left(\frac{h}{t_n}\right)^{j_n}.$$

The coefficients B_j are simple. On the contrary, it is highly challenging to understand the Laurent series expansion of the rational expression:

$$\mathsf{C}(t_1,\dots,t_n) := \prod_{1\leqslant i<j\leqslant n}\left(\frac{t_j-t_i}{t_j-2t_i}\right)\prod_{2\leqslant i<j\leqslant n}\left(\frac{t_j-2\,t_i}{t_j-2\,t_i+t_{i-1}}\right).$$

Nevertheless, the shape of C allows to extract some more information on the support of its iterated Laurent series expansion. Let us use the notation

$$\mathsf{C}_{\boldsymbol{k}} := \left[t_1^{k_1}\cdots t_n^{k_n}\right]\mathsf{C}(t_1,\dots,t_n),$$

for the (complicated) coefficient of the monomial $t_1^{k_1}\cdots t_n^{k_n}$ in the iterated Laurent series expansion of C.

(4.2.1) *The coefficient* $\mathsf{C}_{\boldsymbol{k}}$ *is zero unless* $k_1+\cdots+k_n=0$ *and* $k_i+\cdots+k_n\leqslant 0$ *, for each* $i=2,\dots,n$.

Proof. At each step of the iterated Laurent series expansion of C, one manipulates only products of the homogeneous monomials t_i/t_j for indices $1\leqslant i<j\leqslant n$. □

We will still use the notation C for the Laurent series expansion:

$$\mathsf{C}(t_1,\dots,t_n) = \sum_{k_1+\cdots+k_n=0}\mathsf{C}_{\boldsymbol{k}}\,t_1^{k_1}\cdots t_n^{k_n}.$$

Notice that one can use the rational expression of C whenever $\underline{t}$ is in the domain of convergence of C. We will also use the transparent notation:

$$|\mathsf{C}| = \sum_{k_1+\cdots+k_n=0}|\mathsf{C}_{\boldsymbol{k}}|\,t_1^{k_1}\cdots t_n^{k_n}.$$

4.3. Guide to the computation. For the moment, we will assume that — to a first approximation — both terms B and C can be replaced by 1 and also assume $\delta=0$. Later, we will come back in more details to this point. We introduce the auxiliary (positive) constants:

$$\widetilde{I_p} := \left[t_1^n\cdots t_n^n\right]\Big(e_p(t_1,\dots,t_n)\,f_p(t_1,\dots,t_n)\Big)\qquad{\scriptstyle [p=0,1,\dots,n]}.\tag{4.3.1}$$

Compare this with the actual coefficients of d^{n-p} in I:

$$I_p := \left[h^pt_1^n\cdots t_n^n\right]\left(\sum_{q=0}^{p}h^q\left(\alpha_q\,e_p(\underline{t})\,f_q(\underline{t})-\beta_{q+1}\,e_{p+1}(\underline{t})\,f_{q+1}(\underline{t})\right)\mathsf{B}(\underline{t})\;\mathsf{C}(\underline{t})\right)\qquad{\scriptstyle [p=0,1,\dots,n]}.$$

Although this seems to be a radical simplification, It will be proved later that for all $p=0,1,\dots,n$, the coefficient I_p has nearly the same size as the simplified coefficient

$\widetilde{I}_p$, provided the parameters $a_1, \dots, a_n$ are *suitably adjusted* ($\mathcal{H}_1$) and that $\delta \ll 1$ is taken *small enough* ($\mathcal{H}_2$).

Here is the guide to our computation:

(1) *Firstly*, we will compute a sufficient lower bound $\widetilde{\lambda}(\underline{a})$ on the degree d, such that the approximated polynomial:

$$\widetilde{I}(d) := \widetilde{I}_0\, d^n - \Big(\widetilde{I}_1\, d^{n-1} + \cdots + \widetilde{I}_n\Big)$$

has positive values for degrees $d \geqslant \widetilde{\lambda}(\underline{a})$.

The goal of all the technicalities that will follow is to bring the computation back to this *much easier* computation.

(2) *Secondly*, we will study the leading coefficient, and prove that under *ad hoc* hypotheses ($\mathcal{H}_1$) and ($\mathcal{H}_2$), it has a positive value, that is at least two thirds of $\widetilde{I}_0$:

$$I_0 = \underbrace{\Big[h^n t_1^n \cdots t_n^n\Big]\Big(\mathsf{A}_0(\underline{t})|_{\delta=0}\Big)}_{=\widetilde{I}_0} + \underbrace{\Big[h^n t_1^n \cdots t_n^n\Big]\Big(\mathsf{A}_0(\underline{t})|_{\delta=0}\,\big(\mathsf{C}(\underline{t}) - 1\big)\Big)}_{\text{negligible for } a_1 \gg \cdots \gg a_n\ (\mathcal{H}_1)} + \underbrace{\delta\, I_0'}_{\text{negligible for } \delta \ll 1\ (\mathcal{H}_2)}.$$

(3) *Thirdly*, we will study the remaining coefficients and derive a sufficient lower bound $\lambda(\underline{a})$ on the degree d — similar to $\widetilde{\lambda}(\underline{a})$ — from the relative sizes of the coefficients. Again, we will see that:

$$I_p = \underbrace{\Big[h^n t_1^n \cdots t_n^n\Big]\Big(\mathsf{A}_p(\underline{t})|_{\delta=0}\,\mathsf{B}(\underline{t})\Big)}_{\sim\text{multiple of } \widetilde{I}_p} + \underbrace{\Big[h^n t_1^n \cdots t_n^n\Big]\Big(\mathsf{A}_p(\underline{t})|_{\delta=0}\,\mathsf{B}(\underline{t})\,\big(\mathsf{C}(\underline{t}) - 1\big)\Big)}_{\text{negligible for } a_1 \gg \cdots \gg a_n\ (\mathcal{H}_1)} + \underbrace{\delta\, I_p'}_{\text{negligible for } \delta \ll 1\ (\mathcal{H}_2)}.$$

4.4. Quick justification of the approximations. Recall that I_p is the Cauchy product coefficient:

$$I_p = \Big[h^n t_1^n \cdots t_n^n\Big]\Big(\mathsf{A}_p\big(\underline{t}\big)\ \mathsf{B}\big(\underline{t}\big)\ \mathsf{C}\big(\underline{t}\big)\Big)$$

where A_p is the polynomial:

$$\mathsf{A}_p = h^{n-p} \sum_{q=0}^{p} h^q\Big(\alpha_q\, e_p(\underline{t})\, f_q(\underline{t}) - \beta_{q+1}\, e_{p+1}(\underline{t})\, f_{q+1}(\underline{t})\Big),$$

and where B and C are the iterated Laurent series:

$$\mathsf{B} = \sum_{j} \mathsf{B}_j \left(\frac{h}{t_1}\right)^{j_1} \cdots \left(\frac{h}{t_n}\right)^{j_n} \quad \text{and} \quad \mathsf{C} = \sum_{|\boldsymbol{k}|=0} \mathsf{C}_{\boldsymbol{k}}\, t_1^{k_1} \cdots t_n^{k_n}.$$

As a consequence, I_p can be written as:

$$I_p = \sum_{i=j-\boldsymbol{k}} \sum_{q=0}^{p} \alpha_q \Big[t_1^{n+i_1} \cdots t_n^{n+i_n}\Big]\Big(e_p(\underline{t})\, f_q(\underline{t})\Big)\, \mathsf{B}_j\, \mathsf{C}_{\boldsymbol{k}} - \sum_{i=j-\boldsymbol{k}} \sum_{q=0}^{p} \beta_{q+1} \Big[t_1^{n+i_1} \cdots t_n^{n+i_n}\Big]\Big(e_{p+1}(\underline{t})\, f_{q+1}(\underline{t})\Big)\, \mathsf{B}_j\, \mathsf{C}_{\boldsymbol{k}}.$$

Notice that since both polynomials $e_p\, f_q$ and $e_{p+1}\, f_{q+1}$ have homogeneous degree $n^2 + p - q$, the summands are zero unless:

$$i_1 + \cdots + i_n = j_1 + \cdots + j_n = p - q.$$

The only freedom that we have in order to make the estimation of this term easier is the choice of the parameters $a_1 > \cdots > a_n > 0$ that appear in:

$$f_i(t_1,\dots,t_n) = \frac{n^2!}{(n^2-i)!}\big(2\mu(\underline{a})\big)^i\big(a_1t_1+\cdots+a_nt_n\big)^{n^2-i}.$$

This latitude will become clearer after we explain how to simplify the coefficients

$$\big[t_1^{n+i_1}\cdots t_n^{n+i_n}\big]\big(e_p(t_1,\dots,t_n)\ f_q(t_1,\dots,t_n)\big)$$

in order to deal with more tractable plain $\underline{a}$-monomials.

Recall the auxiliary constant introduced in (4.3.1):

$$\widetilde{I_p} = \big[t_1^n\cdots t_n^n\big]\big(e_p(t_1,\dots,t_n)\, f_p(t_1,\dots,t_n)\big).$$

It stands for the absolute value of the coefficients of d^{n-p} in the simplified polynomial $\widetilde{I}$, appearing in the preceding guide. Each coefficient appearing in the computation of the coefficient I_p of d^{n-p} in I can be compared with the unique coefficient $\widetilde{I_p}$ appearing in the computation of the coefficient of d^{n-p} in $\widetilde{I}$:

(4.4.1) *For any integer $p = 0, 1, \dots, n$, one has:*

$$\widetilde{I_p} = \frac{(n^2)!}{(n!)^n}\,(a_1\cdots a_n)^n\,(2n\mu)^p\, e_p\Big(\frac{1}{a_1},\dots,\frac{1}{a_n}\Big),$$

an for any integer $q \leqslant p$ and any multi-index $\big(i_1,\dots,i_n\big)\in\mathbb{Z}^n$, such that $i_1+\cdots+i_n = p-q$, one has:

$$\big[t_1^{n+i_1}\cdots t_n^{n+i_n}\big]\big(e_p(t_1,\dots,t_n)\ f_q(t_1,\dots,t_n)\big) \leqslant a_1^{i_1}\cdots a_n^{i_n}\ \frac{\widetilde{I_p}}{(2n\mu)^{p-q}}.$$

Before we prove this key lemma, let us explain in a few words how to use this latitude. We will set $a_1 \gg \cdots \gg a_n$ in order to make the annoying contributions negligible with respect to the "central" term $\frac{\alpha_p}{p!}\widetilde{I_p}$, obtained for $\boldsymbol{i} = (0,\dots,0)\in\mathbb{Z}^n$.

Indeed, since $i_1+\cdots+i_n = p-q$, using the telescoping products:

$$\bigg(\frac{a_j}{2n\mu(\underline{a})}\bigg) = \bigg(\frac{a_j}{a_{j-1}}\bigg)\cdots\bigg(\frac{a_2}{a_1}\bigg)\bigg(\frac{a_1}{2n\mu(\underline{a})}\bigg)$$

the formula stated just above can be written:

$$\big[t_1^{n+i_1}\cdots t_n^{n+i_n}\big]\big(e_p(\underline{t})\ f_q(\underline{t})\big) \leqslant \frac{1}{q!}\bigg(\frac{a_1}{2n\mu(\underline{a})}\bigg)^{i_1+\cdots+i_n}\bigg(\frac{a_2}{a_1}\bigg)^{i_2+\cdots+i_n}\cdots\bigg(\frac{a_n}{a_{n-1}}\bigg)^{i_n}\widetilde{I_p}.$$

Moreover, since $\boldsymbol{i} = \boldsymbol{j}-\boldsymbol{k}$, all multi-indices $\boldsymbol{i}$ involved in the computation have entries such that:

$$i_j+\cdots+i_n \geqslant 0 \qquad (j=1,\dots,n),$$

because the entries of $\boldsymbol{j}$ have to be non negative and we have seen that $\mathsf{C}_{\boldsymbol{k}}$ is not zero only for

$$k_i+\cdots+k_n \leqslant 0 \qquad (i=1,\dots,n).$$

Thus, whereas the coefficient of $\boldsymbol{i} = (0,\dots,0)$ will always be 1, for other indices, taking parameters $a_i \gg a_{i+1} > 0$ — that clearly fulfil the condition (3.2.4) —, these coefficients can be made as small as wanted.

It follows that the technical simplification $a_1 \gg \cdots \gg a_n$ yields the desired approximation $I_p \simeq \frac{\alpha_p}{p!}\widetilde{I_p}$, at least asymptotically. *However*, if one is interested in effective computation, the parameters $a_1, \dots, a_n$ have to be chosen with care. Indeed, a very quickly decreasing sequence $\underline{a}$ will guarantee a very large positive leading coefficient, but will also increase the size of the largest positive root of the polynomial $I(d)$.

Proof of the lemma. By definition:

$$f_q(t_1, \dots, t_n) = \frac{(n^2)!}{(n^2-q)!} (2\mu)^q \Big(a_1\, t_1 + \cdots + a_n\, t_n\Big)^{n^2-q}.$$

We will expand this (n^2-q)-power. In this view, recall that for any integer $k, \ell \in \mathbb{N}$ the standard binomial formula yield by induction on k the following *multinomial formula*:

$$\Big(x_1 + \cdots + x_k\Big)^{\ell} = \sum_{j_1+\cdots+j_k=\ell} \frac{\ell!}{j_1! \cdots j_k!}\, x_1^{j_1} \cdots x_k^{j_k}.$$

Thus f_q becomes:

$$f_q(t_1, \dots, t_n) = (2\mu)^q \sum_{j_1+\cdots+j_n=n^2-q} \frac{n^2!}{(n^2-q)!} \frac{(n^2-q)!}{j_1! \cdots j_n!}\, a_1^{j_1} \cdots a_n^{j_n}\, t_1^{j_1} \cdots t_n^{j_n}.$$

On the other hand, by definition, the p-th elementary symmetric function of n variables $x_1, \dots, x_{\cdot}$ is the following homogeneous sum of monomials:

$$(*) \qquad e_p(x_1, \dots, x_n) = \sum_{\substack{(\varepsilon_1,\dots,\varepsilon_n)\in\{0,1\}^n \\ \varepsilon_1+\cdots+\varepsilon_n=p}} x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n},$$

the total degree of which is p and in which the exponent of each variable $x_1, \dots, x_n$ is at most 1.

The product of the two polynomials $e_p(\underline{t})$ and $f_q(\underline{t})$ is thus:

$$e_p(\underline{t})\, f_q(\underline{t}) = (2\mu)^q \sum_{\substack{j_1+\cdots+j_n=n^2-q \\ \varepsilon_1+\cdots+\varepsilon_n=p}} \frac{n^2!}{j_1! \cdots j_n!}\, a_1^{j_1} \cdots a_n^{j_n}\, t_1^{j_1+\varepsilon_1} \cdots t_n^{j_n+\varepsilon_n},$$

where as above $(j_1, \dots, j_n) \in \mathbb{N}^n$ and $(\varepsilon_1, \dots, \varepsilon_n) \in \{0,1\}^n$.

Computation of $\widetilde{I_p}$. Now, in the case where $p = q$ and $i_1 = \cdots = i_n = 0$, for $j_1 + \varepsilon_1 = n, \dots, j_n + \varepsilon_n = n$, all j_k are non negative and the appearing multinomial coefficient has always the same value:

$$\frac{n^2!}{j_1! \cdots j_n!} = \frac{n^2!}{(n-\varepsilon_1)! \cdots (n-\varepsilon_n)!} = \frac{n^2!}{(n!)^{n-p}\, (n-1)!^p} = n^p\, \frac{(n^2)!}{(n!)^n},$$

because p of the ε_k's are 1 and the $n-p$ remaining ε_k's are 0.

As a consequence:

$$\Big[t_1^n \cdots t_n^n\Big]\Big(e_p(\underline{t})\, f_p(\underline{t})\Big) = (2n\mu)^p \sum_{\varepsilon_1+\cdots+\varepsilon_n=p} \frac{(n^2)!}{(n!)^n}\, a_1^{n-\varepsilon_1} \cdots a_n^{n-\varepsilon_n}.$$

Next, one can factorize the multinomial coefficient and $a_1^n \cdots a_n^n$ and use the above definition $(*)$ of the elementary symmetric function from the right to the left, in order to obtain:

$$\begin{aligned}
\Big[t_1^n \cdots t_n^n\Big]\Big(e_p(\underline{t})\, f_p(\underline{t})\Big) &= (2n\mu)^p\, \frac{(n^2)!}{(n!)^n}\, a_1^n \cdots a_n^n \sum_{\varepsilon_1+\cdots+\varepsilon_n=p} \Big(\frac{1}{a_1}\Big)^{\varepsilon_1} \cdots \Big(\frac{1}{a_n}\Big)^{\varepsilon_n} \\
&= (2n\mu)^p\, \frac{(n^2)!}{(n!)^n}\, a_1^n \cdots a_n^n\; e_p\Big(\frac{1}{a_1}, \dots, \frac{1}{a_n}\Big).
\end{aligned}$$

Proof of the estimations. We come back to the general formula

$$e_p(\underline{t})\, f_q(\underline{t}) = \frac{(2\mu)^q}{q!} \sum_{\substack{j_1+\cdots+j_n=n^2-q \\ \varepsilon_1+\cdots+\varepsilon_n=p}} \frac{n^2!}{j_1!\cdots j_n!}\, a_1^{j_1}\cdots a_n^{j_n}\, t_1^{j_1+\varepsilon_1}\cdots t_n^{j_n+\varepsilon_n}.$$

Of course, one has:

$$\left[t_1^{n+i_1}\cdots t_n^{n+i_n}\right]\Big(e_p(\underline{t})\, f_q(\underline{t})\Big) = \frac{(2\mu)^q}{q!} \sum_{\substack{\varepsilon_1+\cdots+\varepsilon_n=p \\ \forall k:\; n+i_k-\varepsilon_k\geqslant 0}} \frac{n^2!}{(n+i_1-\varepsilon_1)!\cdots(n+i_n-\varepsilon_n)!}\, a_1^{n+i_1-\varepsilon_1}\cdots a_n^{n+i_n-\varepsilon_n}.$$

By distinguishing the case ($i<0$) and the case ($i>0$) it is easy to show that in both cases: $\frac{n!}{(n+i)!}\leqslant n^{-i}$. Thus, by summing the powers of n:

$$\frac{1}{(n+i_1-\varepsilon_1)!\cdots(n+i_n-\varepsilon_n)!} \leqslant \frac{n^{p-(p-q)}}{(n!)^n} = \frac{n^q}{(n!)^n}.$$

In the right hand side, the dependence on $\varepsilon_1,\dots,\varepsilon_n$ disappears. Then, as above, one can factorize the expression and use the above definition (∗) of the elementary symmetric function from the right to the left, in order to obtain:

$$\left[t_1^{n+i_1}\cdots t_n^{n+i_n}\right]\Big(e_p(\underline{t})\, f_q(\underline{t})\Big) \leqslant (2n\mu)^q\, \frac{(n^2)!}{(n!)^n}\, a_1^{n+i_1}\cdots a_n^{n+i_n}\, e_p\Big(\frac{1}{a_1},\dots,\frac{1}{a_n}\Big).$$

It remains only to compare with the expression of the auxiliary constant $\widetilde{I}_p$ in order to conclude. In intermediary computation, its explicit value was computed to be:

$$\widetilde{I}_p = (2n\mu)^p\, \frac{(n^2)!}{(n!)^n}\, a_1^n\cdots a_n^n\;\, e_p\Big(\frac{1}{a_1},\dots,\frac{1}{a_n}\Big),$$

whence:

$$\left[t_1^{n-i_1}\cdots t_n^{n-i_n}\right]\Big(e_p(\underline{t})\; f_q(\underline{t})\Big) \leqslant a_1^{i_1}\cdots a_n^{i_n}\, \frac{\widetilde{I}_p}{(2n\mu)^{p-q}}. \qquad \square$$

4.5. Simplified computation. Now, we achieve our program. As a first step, let us simplify the problem and let us consider the polynomial inequality:

$$\widetilde{I}_0\, d^n \geqslant \widetilde{I}_1\, d^{n-1} + \cdots + \widetilde{I}_{n-1}\, d + \widetilde{I}_n.$$

Because $\widetilde{I}_0>0$, this inequality is checked as soon as d is larger than the (largest) positive root of the polynomial:

$$\widetilde{I}(d) = \widetilde{I}_0\, d^n - \widetilde{I}_1\, d^{n-1} - \cdots - \widetilde{I}_{n-1}\, d - \widetilde{I}_n.$$

Dividing the polynomial $\widetilde{I}$ by the appearing (huge) positive constant $\widetilde{I}_0$, already computed to be:

$$\widetilde{I}_0 = \frac{(n^2)!}{(n!)^n}\,(a_1\cdots a_n)^n,$$

does not change the values for which $\widetilde{I}$ is positive. In order to bound from above the absolute values of the roots of a polynomial, it suffices to estimate the relative size of its coefficients, according to the following lemma:

(4.5.1**)** *All complex roots of a nonconstant complex algebraic equation of degree n:*

$$c_0x^n = c_1x^{n-1} + \cdots + c_{n-1}x + c_n$$

have an absolute value less than or equal to the Fujiwara's bound*:*

$$\lambda = 2 \max_{p=1,\dots,n} \left|\frac{c_p}{c_0}\right|^{1/p}.$$

Following this strategy of bounding $\widetilde{I}_p/\widetilde{I}_0$, we will now compute the relative sizes of the coefficients of $\widetilde{I}$ and after that compare each non-leading coefficient with the leading coefficient:

(4.5.2) *The Fujiwara's bound for the polynomial $\widetilde{I}$:*

$$\widetilde{\lambda}(a_1,\dots,a_n) := 2 \max_{1\leqslant p\leqslant n} \left|\frac{\widetilde{I}_p}{\widetilde{I}_0}\right|^{1/p}$$

has the value

$$\widetilde{\lambda}(\underline{a}) = 4n\,\mu(\underline{a})\,\big(1/a_1 + \cdots + 1/a_n\big).$$

Proof. For $p = 0, 1, \dots, n$, we have computed:

$$\widetilde{I}_p = (2n\mu)^p\,\frac{(n^2)!}{(n!)^n}\,a_1^n \cdots a_n^n\;\, e_p\Big(\frac{1}{a_1},\dots,\frac{1}{a_n}\Big),$$

in particular the leading term of $\widetilde{I}(d)$ is:

$$\widetilde{I}_0 = \frac{(n^2)!}{(n!)^n}\,a_1^n \cdots a_n^n.$$

Thus, first dividing by $\widetilde{I}_0$ and then taking account of the homogeneous degree of e_p:

$$\frac{\widetilde{I}_p}{\widetilde{I}_0} = (2n\mu)^p\;e_p\Big(\frac{1}{a_1},\dots,\frac{1}{a_n}\Big) = e_p\Big(\frac{2n\mu}{a_1},\dots,\frac{2n\mu}{a_n}\Big),$$

whence the sought Fujiwara bound is:

$$\widetilde{\lambda} = 2 \max_{1\leqslant p\leqslant n} \bigg(e_p\Big(\frac{2n\mu}{a_1},\dots,\frac{2n\mu}{a_n}\Big)\bigg)^{1/p}.$$

Now, the classical *Maclaurin's inequality* precisely states that for $n \in \mathbb{N}$ and for a set of positive reals $x_1, \dots, x_n$:

$$e_1(\underline{x}) \geqslant e_2(\underline{x})^{1/2} \geqslant \cdots \geqslant e_n(\underline{x})^{1/n}.$$

One can directly deduce from this that:

$$\widetilde{\lambda} = 2 \max_{1\leqslant p\leqslant n} \bigg(e_p\Big(\frac{2n\mu}{a_1},\dots,\frac{2n\mu}{a_n}\Big)\bigg)^{1/p} = 2\,e_1\Big(\frac{2n\mu}{a_1},\dots,\frac{2n\mu}{a_n}\Big) = 4n\mu\,\Big(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\Big),$$

which ends the proof. □

Anticipating hypothesis $(\mathcal{H}_1)$ by a couple of pages, we will immediately estimate the final value of $\widetilde{\lambda}$.

(4.5.3) *If $n \geqslant 6$ and $\underline{a} = \big(n^n, n^{n-1}, \dots, n, 1\big)$, the approximated Fujiwara's bound $\widetilde{\lambda}(a_1,\dots,a_n)$ satisfies:*

$$4\,n^n \leqslant \widetilde{\lambda}(a_1,\dots,a_n) \leqslant 7\,n^n.$$

Proof. Recall that:

$$\widetilde{\lambda}(\underline{a}) = 4\,n\,\mu(\underline{a})\Big(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\Big).$$

After computing the partial sum

$$\sum_{i=1}^{n} i\,x^{i-1} = \frac{d}{dx}\Big(\frac{x^{n+1}-1}{x-1}\Big) = \frac{n\,x^{n+1} - (n+1)\,x^n + 1}{(x-1)^2},$$

one can express explicitely the weighted sum

$$\mu(\underline{a}) = n^{n-1}\sum_{i=1}^{n} i\Big(\frac{1}{n}\Big)^{i-1} = \frac{n^{n+1}-n^2}{(n-1)^2},$$

and of course:

$$\sum_{i=1}^{n}\frac{1}{a_i} = \sum_{i=1}^{n}\frac{n^i}{n^n} = \frac{n^{n+1}-1}{(n-1)n^n}.$$

Thus:

$$\widetilde{\lambda}(\underline{a}) = \frac{n^3\,(n^{n-1}-1)\,(n^{n+1}-1)}{(n-1)^3\,n^{n-1}\,n^{n+1}}\,4\,n^n;$$

for $n \geqslant 6$, we see that $\widetilde{\lambda}(\underline{a}) \leqslant 7\,n^n$, which is the stated bounds on $\widetilde{\lambda}$. □

Using the above Fujiwara's bound, and taking account that the leading coefficient is positive, we obtain that, under hypothesis $(\mathcal{H}_1)$, the approximated polynomial $\widetilde{I}$ is positive for degrees:

$$d \geqslant 7\,n^n$$

Next we will explain how to follow the same strategy with the (more complicated) actual computation.

4.6. Estimation of the leading coefficient. Actually, the leading coefficient I_0 of the polynomial:

$$I(d) = I_0\,d^n + I_1\,d^{n-1} + \cdots + I_{n-1}\,d + I_n$$

is the Cauchy product coefficient:

$$I_0 = \Big[h^n t_1^n \cdots t_n^n\Big]\Big(\mathsf{A}_0(t_1,\cdots,t_n)\,\mathsf{B}(t_1,\cdots,t_n)\,\mathsf{C}(t_1,\cdots,t_n)\Big),$$

and we want to prove that it is *positive* and to justify that it is comparable to the simpler coefficient:

$$\widetilde{I}_0 = \Big[t_1^n \cdots t_n^n\Big]\Big(f_0(t_1,\ldots,t_n)\Big).$$

There are indeed some similarities between the computations of these two coefficients (*cf. supra* for the earlier computation of $\widetilde{I}_0$).

Recall that A_0 is the coefficient of d^n in A and it has the expression:

$$\mathsf{A}_0(t_1,\ldots,t_n) = h^n\Big(f_0\big(\underline{t}\big) - \delta\,e_1\big(\underline{t}\big)f_1\big(\underline{t}\big)\Big),$$

Thus $[h^n]\mathsf{A}_0(t_1,\ldots,t_n)$ and $f_0(t_1,\ldots,t_n)$ coincide for $\delta = 0$.

Moreover, since in the multivariate polynomial A, the exponent of h is already the same as the exponent of h in the sought-after monomial $h^n t_1^n \cdots t_n^n$, the other term depending on h, namely

$$\mathsf{B}(t_1, \dots, t_n) = \sum_{i_1,\dots,i_n \geqslant 0} (-1)^{i_1+\cdots+i_n} \binom{n+i_1}{n} \cdots \binom{n+i_n}{n} \frac{h^{i_1+\cdots+i_n}}{t_1^{i_1}\cdots t_n^{i_n}},$$

can be replaced by its truncation $\mathsf{B}(\underline{t}) = 1 + O(h)$ — here $O(h)$ denotes a polynomial multiple of h in $\mathbb{C}[h, t_1, \dots, t_n]$ —, and the resulting simplified computation is:

$$I_0 = \Big[h^n t_1^n \cdots t_n^n\Big]\Big(\mathsf{A}_0(h, t_1, \cdots, t_n)\, \mathsf{C}(t_1, \cdots, t_n)\Big),$$

that is the sum:

$$I_0 = \sum_{k_1+\cdots+k_n=0} \Big[h^n t_1^{n-k_1} \cdots t_n^{n-k_n}\Big]\Big(\mathsf{A}_0(h, t_1, \cdots, t_n)\Big)\Big[t_1^{k_1}\cdots t_n^{k_n}\Big]\Big(\mathsf{C}(t_1, \cdots, t_n)\Big).$$

That can be written as the difference of two sums:

$$I_0 = \sum_{k_1+\cdots+k_n=0} \Big[t_1^{n-k_1}\cdots t_n^{n-k_n}\Big] f_0\big(\underline{t}\big)\, \Big[t_1^{k_1}\cdots t_n^{k_n}\Big]\mathsf{C}\big(\underline{t}\big) - \delta \sum_{k_1+\cdots+k_n=0} \Big[t_1^{n-k_1}\cdots t_n^{n-k_n}\Big]\Big(e_1\big(\underline{t}\big) f_1\big(\underline{t}\big)\Big)\Big[t_1^{k_1}\cdots t_n^{k_n}\Big]\mathsf{C}\big(\underline{t}\big).$$

Of course, by taking $\delta \ll 1$ small enough, the second of these sums becomes negligible.

It remains to study the first sum above. Here, there is a delicate pairing between the numerous coefficients of the polynomial f_0 and the corresponding coefficients of C. The combinatorics of the first family of coefficients is well understood, whereas the combinatorics of the second family of coefficients is *very intricate*.

Notice that the coefficient $\widetilde{I_0}$ corresponds to the single term of this sum indexed by $\boldsymbol{i} = (0, \dots, 0)$. Thus, comparing I_0 and $\widetilde{I_0}$ amounts to replace C by its constant term 1 (and also take $\delta = 0$).

According to Lemma 4.4.1, it would suffice to take a sequence $a_1, \dots, a_n$ that decreases quickly enough, in order to make all terms appearing in the sum:

$$\sum_{\substack{k_1+\cdots+k_n=0\\ \boldsymbol{i}\neq(0,\dots,0)}} \Big[t_1^{n-k_1}\cdots t_n^{n-k_n}\Big] f_0\big(\underline{t}\big)\, \Big[t_1^{k_1}\cdots t_n^{k_n}\Big]\mathsf{C}\big(\underline{t}\big)$$

negligible with respect to:

$$\widetilde{I_0} = \Big[t_1^{n-0}\cdots t_n^{n-0}\Big] f_0\big(\underline{t}\big)\, \Big[t_1^{0}\cdots t_n^{i_0}\Big]\mathsf{C}\big(\underline{t}\big).$$

Actually, below in appendix 5 we show the following.

(4.6.1) *In the logarithmic absolute case, where $V = T_{\mathbb{P}^n}(-\log H)$, for $n \geqslant 6$, fix the parameter $\underline{a}$ by taking:*

$$(\mathcal{H}_1) \qquad \underline{a} = \Big(n^n, n^{n-1}, \dots, n, 1\Big),$$

and a posteriori take $\delta = \delta(\underline{a})$ small enough:

$$(\mathcal{H}_2) \qquad 5\,\delta(\underline{a})\,\widetilde{\lambda}(\underline{a}) \leqslant 1.$$

Then the leading coefficient I_0 of the polynomial $I(d)$ is positive and it is at least two thirds of the coefficient $\widetilde{I_0}$:

$$I_0 \geqslant \frac{2}{3}\widetilde{I_0} > 0.$$

4.7. Computation of the Fujiwara's bound. Next, in order to compute the Fujiwara's bound of the polynomial $I(d)$, we estimate the other coefficients. Considering the exponent $1/p$ in the definition of the Fujiwara's bound, one is easily convinced that it is more important to have a good estimate on the coefficient I_1 of the monomial d^{n-1}. Thus we treat it separately. Below in appendix 6.1 we show that for any $a_1, \ldots, a_n \in \mathbb{N}$, the absolute value of the coefficient I_1 of d^{n-1} in the polynomial I is bounded from above by:

$$|I_1| \leqslant \left(\frac{1 + 3\,\delta\widetilde{\lambda}}{2} + \frac{1 + \delta\widetilde{\lambda}}{2n} + \delta(n+1) \right) |\mathsf{C}|\Big(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\Big)\,\widetilde{I_1}.$$

In particular, under hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, we show that:

$$|I_1| \leqslant 5\,\widetilde{I_1}. \tag{4.7.1}$$

For the remaining coefficients, we do not expand the term B, thus we get a slightly larger upper bound. Below in appendix 6.2 we show that for any $a_1, \ldots, a_n \in \mathbb{N}$, for $p = 0, 1, \ldots, n$, the absolute value of the coefficient I_p of d^{n-p} in the polynomial I is bounded from above by:

$$|I_p| \leqslant \left(\frac{2 + \delta\widetilde{\lambda}}{2} \left(\frac{2\,n}{2\,n-1}\right)^{n+1} \right) |\mathsf{C}|\left(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\right)\widetilde{I_p}.$$

In particular, for $n \geqslant 6$, under hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$:

$$|I_p| \leqslant 12\,\widetilde{I_p}. \tag{4.7.2}$$

One can now compare the actual Fujiwara's bound λ with the approximated Fujiwara's bound $\widetilde{\lambda}$. Recall that, by definition, $\widetilde{\lambda}$ is the Fujiwara's bound of the simplified polynomial $\widetilde{I}$:

$$\widetilde{\lambda}(\underline{a}) = 2 \max_{1 \leqslant p \leqslant n} \left(\frac{\widetilde{I_p}}{\widetilde{I_0}} \right)^{1/p}.$$

In (4.6.1) above, we have proved that, under hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, one has:

$$I_0 \geqslant \frac{2}{3}\widetilde{I_0}$$

and in (4.7.1) just above that, under the same hypotheses:

$$|I_1| \leqslant 5\widetilde{I_1}.$$

Thus, by combining these two inequalities, we obtain:

$$\frac{|I_1|}{|I_0|} \leqslant \frac{15}{2} \frac{\widetilde{I_1}}{\widetilde{I_0}}$$

Similarly for $p = 2, \ldots, n$, the inequality corresponding to (4.7.2) is:

$$\left(\frac{|I_p|}{|I_0|}\right)^{1/p} \leqslant 18^{1/p} \left(\frac{\widetilde{I_p}}{\widetilde{I_0}}\right)^{1/p} \leqslant 3\sqrt{2} \left(\frac{\widetilde{I_p}}{\widetilde{I_0}}\right)^{1/p}.$$

Thus, under hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, the Fujiwara's bound of the polynomial $I(d)$ satisfies:

$$\lambda(\underline{a}) := 2 \max_{p=1,\ldots,n} \left| \frac{I_p}{I_0} \right|^{1/p} \leqslant \frac{15}{2}\,\widetilde{\lambda}(\underline{a}) \leqslant \frac{15}{2}\,7\,n^n.$$

Then, we simplify the bounds on d and δ by using that for $n \geqslant 6$:

$$\begin{cases} \lambda \leqslant \frac{15}{2}\, 4 \left(\frac{n}{n-1}\right)^3 n^n \leqslant 52\, n^n \\ 1/\delta \geqslant 35\, n^n \geqslant 5\, 4 \left(\frac{n}{n-1}\right)^3 n^n \Rightarrow 5\delta\widetilde{\lambda} \leqslant 1 \end{cases}.$$

Appendix of chapter IV

5. Positivity of the Leading Coefficient

In this appendix, we establish how the technical hypotheses ($\mathcal{H}_1$) and ($\mathcal{H}_2$) yield the positivity of the leading coefficient I_0 of $I(d)$.

5.1. Proof of proposition (4.6.1)**.** The leading coefficient of $I(d)$ has the expression:

$$(*)\quad I_0 = \sum_{k_1+\cdots+k_n=0} \Big[t_1^{n-k_1}\cdots t_n^{n-k_n}\Big] f_0\big(\underline{t}\big)\,\Big[t_1^{k_1}\cdots t_n^{k_n}\Big]\mathsf{C}\big(\underline{t}\big) - \delta \sum_{k_1+\cdots+k_n=0} \Big[t_1^{n-k_1}\cdots t_n^{n-k_n}\Big]\Big(f_1\big(\underline{t}\big)e_1\big(\underline{t}\big)\Big)\Big[t_1^{k_1}\cdots t_n^{k_n}\Big]\mathsf{C}\big(\underline{t}\big).$$

We introduce some notation for the positive and negative contributions to the first sum, as follows:

$$I_0^+ := \sum_{\substack{k_1+\cdots+k_n=0\\ [\underline{t}^k]\mathsf{C}(\underline{t})>0}} \Big[t_1^{n-k_1}\cdots t_n^{n-k_n}\Big] f_0\big(\underline{t}\big)\,\Big[t_1^{k_1}\cdots t_n^{k_n}\Big]\mathsf{C}\big(\underline{t}\big),$$

$$I_0^- := -\sum_{\substack{k_1+\cdots+k_n=0\\ [\underline{t}^k]\mathsf{C}(\underline{t})<0}} \Big[t_1^{n-k_1}\cdots t_n^{n-k_n}\Big] f_0\big(\underline{t}\big)\,\Big[t_1^{k_1}\cdots t_n^{k_n}\Big]\mathsf{C}\big(\underline{t}\big).$$

and also for the slope of I_0 with respect to δ:

$$I_0' := \sum_{k_1+\cdots+k_n=0} \Big[t_1^{n-k_1}\cdots t_n^{n-k_n}\Big]\Big(f_1(\underline{t})\,e_1(\underline{t})\Big)\Big[t_1^{k_1}\cdots t_n^{k_n}\Big]\mathsf{C}\big(\underline{t}\big).$$

Then the above equation $(*)$ becomes: $I_0 = I_0^+ - I_0^- - \delta\, I_0'$. It is proved below in sections (5.2), (5.3), (5.4), that under the hypothesis ($\mathcal{H}_1$), one has the following estimates.

— *Firstly*, the sum of positive contributions is bounded from below by:

$$I_0^+ \geqslant 2\,\widetilde{I}_0.$$

— *Secondly*, the sum of negative contributions is bounded from above by:

$$I_0^- \leqslant \big(5/6\big)\widetilde{I}_0.$$

— *Thirdly*, the slope with respect to δ is bounded from above by:

$$|I_0'| \leqslant \big(5\widetilde{\lambda}(\underline{a})/2\big)\,\widetilde{I}_0.$$

Putting these three estimates together, finishes the proof of (4.6.1). Indeed, one obtains:

$$I_0 \geqslant I_0^+ - I_0^- - \delta\,|I_0'| \geqslant \widetilde{I}_0\left(2 - \frac{5}{6} - \frac{5\,\delta\widetilde{\lambda}}{2}\right) \geqslant \widetilde{I}_0\left(\frac{7 - 15\,\delta\widetilde{\lambda}}{6}\right).$$

It suffices to take, as in ($\mathcal{H}_2$):

$$5\,\delta\big(\underline{a}\big)\widetilde{\lambda}\big(\underline{a}\big) \leqslant 1,$$

in order to obtain as announced:

$$I_0 \geqslant \frac{2}{3}\widetilde{I_0} > 0.$$

5.2. Partial expansion of C and positive contributions. In (4.2.1), we have seen that the coefficient C_k is zero unless for each $i = 1, \ldots, n$:

$$k_i + \cdots + k_n \leqslant 0.$$

Taking account that the coefficient C_k is zero unless:

$$k_1 + \cdots + k_n = 0.$$

It can be reformulated in saying that the coefficient C_k is zero unless for each $i = 1, \ldots, n$:

$$k_1 + \cdots + k_i \geqslant 0.$$

This suggests to make a change of variables in order to deal with formal series.

In this chapter, we will not do this change of variables, but it is useful to introduce the length:

$$\ell(k_1, k_2, \ldots, k_n) = (k_1) + (k_1 + k_2) \cdots + (k_1 + k_2 + \cdots + k_n) = \sum_{i=1}^{n} (n-i)\, k_i,$$

that would correspond to the sum of the exponents in the new variables.

We will also use the notation:

$$\ell(t_1^{k_1} \cdots t_n^{k_n}) := \ell(k_1, \ldots, k_n),$$

for the weighted degree of a monomial, and for $l \in \mathbb{N}$, we will also write $O(l)$ for a series that involves only monomials with weighted degree at least l. Using this notation, observe that:

$$\ell(t_i/t_j) = j - i.$$

This basic observation will allow us to easily expand up to terms of degree 3 the factors appearing in the product:

$$\mathsf{C}(\underline{t}) = \prod_{1 \leqslant i < j \leqslant n} \frac{t_j - t_i}{t_j - 2t_i} \prod_{2 \leqslant i < j \leqslant n} \frac{t_j - 2\,t_i}{t_j - 2t_i + t_{i-1}}.$$

(5.2.1) *For every integers i, j such that $1 \leqslant i < j \leqslant n$:*

$$\frac{t_j - t_i}{t_j - 2\,t_i} = 1 + \frac{t_i}{t_j} + 2\left(\frac{t_i}{t_j}\right)^2 + O(3).$$

and for every integers i, j such that $2 \leqslant i < j \leqslant n$:

$$\frac{t_j - 2\,t_i}{t_j - 2\,t_i + t_{i-1}} = 1 - \frac{t_{i-1}}{t_j} + O(3).$$

Proof. For every two integers i, j such that $1 \leqslant i < j \leqslant n$, the series expansion:

$$\frac{t_j - t_i}{t_j - 2\,t_i} = 1 + \frac{t_i}{t_j}\frac{1}{1 - 2\,\frac{t_i}{t_j}} = 1 + \frac{t_i}{t_j}\sum_{k \geqslant 0}\left(2\,\frac{t_i}{t_j}\right)^k$$

yields:

$$\frac{t_j - t_i}{t_j - 2\,t_i} = 1 + \frac{t_i}{t_j} + 2\left(\frac{t_i}{t_j}\right)^2 + O\Big(3\,(j-i)\Big).$$

Next, for every two integers i, j such that $2 \leqslant i < j \leqslant n$, the series expansion:

$$\frac{t_j - 2\,t_i}{t_j - 2\,t_i + t_{i-1}} = 1 + \frac{-t_{i-1}}{t_j}\frac{1}{1 - \frac{2\,t_i - t_{i-1}}{t_j}} = 1 - \frac{t_{i-1}}{t_j}\sum_{k \geqslant 0}\left(\frac{2\,t_i - t_{i-1}}{t_j}\right)^k$$

yields:

$$\frac{t_j - 2\,t_i}{t_j - 2\,t_i + t_{i-1}} = 1 - \frac{t_{i-1}}{t_j} + O\Big((j - i + 1) + (j - i)\Big). \qquad \square$$

These elementary observations yield the following.

(5.2.2) *The expansion of* C *with respect to the order:*

$$t_1 \ll t_2 \ll \cdots \ll t_n \ll 1$$

up to terms of weighted length ℓ *at least* 3 *is the following sum of terms having positive coefficients:*

$$\mathsf{C}(t_1, \ldots, t_n) = 1 + \sum_{i=1}^{n-1}\frac{t_i}{t_{i+1}} + 2\sum_{i=1}^{n-1}\frac{t_i^2}{t_{i+1}^2} + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\frac{t_i}{t_{i+1}}\frac{t_j}{t_{j+1}} + O(3).$$

Proof. Recall that $\mathsf{C}(t_1, \ldots, t_n)$ is by definition (the expansion of) the product:

$$\mathsf{C}(t_1, \ldots, t_n) = \prod_{1 \leqslant i < j \leqslant n}\left(\frac{t_j - t_i}{t_j - 2t_i}\right)\prod_{2 \leqslant i < j \leqslant n}\left(\frac{t_j - 2\,t_i}{t_j - 2t_i + t_{i-1}}\right).$$

By the lemma just above, the second product has the partial expansion:

$$\prod_{2 \leqslant i < j \leqslant n}\left(\frac{t_j - 2\,t_i}{t_j - 2t_i + t_{i-1}}\right) = \prod_{1 \leqslant i \leqslant n-2}\left(1 - \frac{t_i}{t_{i+2}}\right) + O\big(3\big).$$

because for $j \geqslant i + 2$ the length $j - i + 1 \geqslant 3$. Thus:

$$\prod_{2 \leqslant i < j \leqslant n}\left(\frac{t_j - 2\,t_i}{t_j - 2t_i + t_{i-1}}\right) = 1 - \sum_{i=1}^{n-2}\frac{t_i}{t_{i+2}} + O\big(3\big).$$

One has also:

$$\begin{aligned}\prod_{1 \leqslant i < j \leqslant n}\left(\frac{t_j - t_i}{t_j - 2t_i}\right) &= \prod_{1 \leqslant i < j \leqslant n}\left(1 + \frac{t_i}{t_j} + 2\left(\frac{t_i}{t_j}\right)^2\right) + O(3).\\ &= \prod_{1 \leqslant i \leqslant n-1}\left(1 + \frac{t_i}{t_{i+1}} + 2\left(\frac{t_i}{t_{i+1}}\right)^2\right)\prod_{1 \leqslant i \leqslant n-2}\left(1 + \frac{t_i}{t_{i+2}}\right) + O(3).\\ &= 1 + \sum_{i=1}^{n-1}\frac{t_i}{t_{i+1}} + 2\sum_{i=1}^{n-1}\frac{t_i^2}{t_{i+1}^2} + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\frac{t_i}{t_{i+1}}\frac{t_j}{t_{j+1}} + \sum_{i=1}^{n-2}\frac{t_i}{t_{i+2}} + O(3).\end{aligned}$$

We will consider separately $j = 2$, $j = 3$ and $j \geqslant 4$ and show that in all cases:

$$\frac{t_j - t_1}{t_j - 2t_1}\prod_{i=2}^{j-1}\frac{t_j - t_i}{t_j - 2t_i + t_{i-1}} = 1 + \frac{t_{j-1}}{t_j} + 2\left(\frac{t_{j-1}}{t_j}\right)^2 + O(3).$$

For the first case $j = 2$, there is only one term, and by the lemma just above:

$$\frac{t_2 - t_1}{t_2 - 2t_1} = 1 + \frac{t_1}{t_2} + 2\left(\frac{t_1}{t_2}\right)^2 + O(3).$$

Hence, we are done with this case.

For the second case $j = 3$, note that $\ell\big((t_1/t_3)^2\big) = 4$, thus:

$$\frac{t_3 - t_1}{t_3 - 2t_1} = 1 + \frac{t_1}{t_3} + O(3).$$

There is only one supplementary term in the product, namely:

$$\frac{t_3 - t_2}{t_3 - 2\,t_2 + t_1} = 1 + \frac{t_2}{t_3} - \frac{t_1}{t_3} + 2\left(\frac{t_2}{t_3}\right)^2 + O(3).$$

By doing the product of these two terms, the coefficient of $\frac{t_1}{t_3}$ cancels and all cross products of fractions are of weighted degree at least 3, hence:

$$\frac{t_3 - t_1}{t_3 - 2t_1}\,\frac{t_3 - t_2}{t_3 - 2\,t_2 + t_1} = 1 + \frac{t_2}{t_3} + 2\left(\frac{t_2}{t_3}\right)^2 + O(3).$$

For the remaining case $j \geqslant 4$, note that $\ell(t_j/t_1) \geqslant 3$ thus:

$$\frac{t_j - t_1}{t_j - 2t_1} = 1 + O(3).$$

In the same spirit, if $j - i \geqslant 3$:

$$\frac{t_j - t_i}{t_j - 2t_i + t_{i-1}} = 1 + O(3).$$

The remaining factors of the product are obtained for $i = j - 1$:

$$\frac{t_j - t_{j-1}}{t_j - 2t_{j-1} + t_{j-2}} = 1 + \frac{t_{j-1}}{t_j} - \frac{t_{j-2}}{t_j} + 2\left(\frac{t_{j-1}}{t_j}\right)^2 + O(3),$$

and for $i = j - 2$:

$$\frac{t_j - t_{j-2}}{t_j - 2t_{j-2} + t_{j-3}} = 1 + \frac{t_{j-2}}{t_j} + O(3).$$

In the exact same way as above for $j = 3$, we obtain:

$$\frac{t_j - t_1}{t_j - 2t_1}\;\frac{t_j - t_{j-1}}{t_j - 2t_{j-1} + t_{j-2}}\;\frac{t_j - t_{j-2}}{t_j - 2t_{j-2} + t_{j-3}}\;\prod_{i=2}^{j-3}\frac{t_j - t_i}{t_j - 2t_i + t_{i-1}} = 1 + \frac{t_{j-1}}{t_j} + 2\left(\frac{t_{j-1}}{t_j}\right)^2 + O(3).$$

It remains to state that the product of the obtained expressions for $j = 2, 3, \ldots, n$ is:

$$\prod_{j=2}^{n}\Bigg(1 + \underbrace{\left(\frac{t_{j-1}}{t_j}\right)}_{\ell=1} + 2\underbrace{\left(\frac{t_{j-1}}{t_j}\right)^2}_{\ell=2} + O(3)\Bigg) = 1 + \left(\sum_{2\leqslant j\leqslant n}\left(\frac{t_{j-1}}{t_j} + 2\,\frac{t_{j-1}^2}{t_j^2}\right)\right) + \left(\sum_{2\leqslant j_1<j_2\leqslant n}\frac{t_{j_1-1}}{t_{j_1}}\,\frac{t_{j_2-1}}{t_{j_2}}\right) + O(3).$$

This is because in the second parenthesis, that contains the cross products of fractions, the only terms that have degree least than 3 are the product of fractions of degree 1. By shifting all indices by -1 one obtains the truncated expansion:

$$\mathsf{C}(t_1, \ldots, t_n) = 1 + \sum_{i=1}^{n-1}\frac{t_i}{t_{i+1}} + 2\sum_{i=1}^{n-1}\frac{t_i^2}{t_{i+1}^2} + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\frac{t_i}{t_{i+1}}\,\frac{t_j}{t_{j+1}} + O(3),$$

as announced. □

Notice that in the new variables, this iterated Laurent series expansion of C coincide with the usual multivariate Taylor expansion of the expression obtained from C.

We state that all terms of order less than 3 are non negative in such way that this "Taylor" series expansion of C allows to give a lower bound for the sum of positive contributions I_0^+:

(5.2.3) *Under hypotheses ($\mathcal{H}_1$) and ($\mathcal{H}_2$), the leading term of the polyomial $\widetilde{I}$ enjoys*

$$I_0^+ \geqslant 2\widetilde{I}_0.$$

Proof. Recall that by definition, I_0^+ is the following sum of positive contributions to the leading coefficient of I:

$$I_0^+ = \sum_{\substack{k_1+\cdots+k_n=0 \\ [\underline{t}^k]\mathsf{C}(\underline{t})>0}} \Big[t_1^{n-k_1}\cdots t_n^{n-k_n}\Big]\Big(f_0\big(\underline{t}\big)\Big)\Big[t_1^{k_1}\cdots t_n^{k_n}\Big]\Big(\mathsf{C}\big(\underline{t}\big)\Big).$$

Thanks to the multinomial formual, it is easy to compute that the appearing coefficients of f_0 are:

$$\Big[t_1^{n-k_1}\cdots t_n^{n-k_n}\Big]\Big(f_0(t_1,\dots,t_n)\Big) = \frac{(a_1\cdots a_n)^n}{a_1^{k_1}\cdots a_n^{k_n}}\frac{(n^2)!}{(n-k_1)!\cdots(n-k_n)!}.$$

The coefficient $\widetilde{I}_0$ is the same coefficient for $\boldsymbol{k}=(0,\dots,0)$:

$$\Big[t_1^{n}\cdots t_n^{n}\Big]\Big(f_0(t_1,\dots,t_n)\Big) = (a_1\cdots a_n)^n\frac{(n^2)!}{(n!)^n}.$$

Thus:

$$\Big[t_1^{n-k_1}\cdots t_n^{n-k_n}\Big]\Big(f_0(t_1,\dots,t_n)\Big) = \frac{\widetilde{I}_0}{a_1^{k_1}\cdots a_n^{k_n}}\frac{(n)!}{(n-k_1)!}\cdots\frac{(n)!}{(n-k_n)!}.$$

On the other hand, the above "Taylor" expansion of C provide us with some points in the set

$$\Big\{k_1+\cdots+k_n=0\colon [\underline{t}^k]\mathsf{C}(\underline{t})>0\Big\}.$$

Here is the list of the corresponding coefficients:

t^i	1	t_i/t_{i+1}	t_i^2/t_{i+1}^2	t_i/t_{i+2}	$t_it_j/t_{i+1}t_{j+1}$
coeff/$\widetilde{I}_0$	1	$\frac{n\,a_{i+1}}{(n+1)\,a_i}$	$\frac{n(n-1)\,a_{i+1}^2}{(n+1)(n+2)\,a_i^2}$	$\frac{n\,a_{i+2}}{(n+1)\,a_i}$	$\frac{n^2\,a_{i+1}a_{j+1}}{(n+1)^2\,a_ia_j}$

In conclusion:

$$I_0^+ \geqslant \widetilde{I}_0\Bigg(1+\sum_{i=1}^{n-1}\frac{n\,a_{i+1}}{(n+1)\,a_i}+2\sum_{i=1}^{n-1}\frac{n(n-1)\,a_{i+1}^2}{(n+1)(n+2)\,a_i^2}+\sum_{i=1}^{n-2}\frac{n\,a_{i+2}}{(n+1)\,a_i}+\sum_{i=1}^{n-3}\sum_{j=i+2}^{n-1}\frac{n^2\,a_{i+1}a_{j+1}}{(n+1)^2\,a_ia_j}\Bigg).$$

For $\underline{a}=\big(n^n,\dots,n,1\big)$, one gets the sum:

$$\frac{I_0^+}{\widetilde{I}_0}\geqslant\Bigg(1+\frac{(n-1)}{(n+1)}\frac{n}{n}+\frac{(n-1)^2}{(n+1)(n+2)}\frac{2\,n}{n^2}+\frac{(n-2)}{(n+1)}\frac{n}{n^2}+\frac{(n-3)(n-2)}{(n+1)^2}\frac{n^2}{2\,n^2}\Bigg),$$

and we see that, for $n\geqslant 5$, this quantity is more than 2. □

5.3. Evaluation of $|\mathsf{C}|$ and negative contributions. Next, we control the negative contributions that could appear by multiplying A by C:

$$I_0^- = -\sum_{\substack{k_1+\cdots+k_n=0 \\ [\underline{t}^k]\mathsf{C}(\underline{t})<0}} \Big[t_1^{n-k_1}\cdots t_n^{n-k_n}\Big]\Big(f_0\big(\underline{t}\big)\Big)\Big[t_1^{k_1}\cdots t_n^{k_n}\Big]\Big(\mathsf{C}\big(\underline{t}\big)\Big).$$

We will use the transparent notation:

$$\mathsf{C}_{k_1,\dots,k_n} := \Big[t_1^{k_1}\cdots t_n^{k_n}\Big]\mathsf{C}(t_1,\dots,t_n).$$

By the triangle inequality, taking account of lemma (4.4.1) (for $p = q = 0$), we get:

$$I_0^- \leqslant \sum_{\substack{k_1+\cdots+k_n=0\\ \mathsf{C}_{k_1,\ldots,k_n}<0}} \left(\frac{a_1}{2n\mu(\underline{a})}\right)^{-k_1} \cdots \left(\frac{a_n}{2n\mu(\underline{a})}\right)^{-k_n} \widetilde{I_0}\, |\mathsf{C}_{k_1,\ldots,k_n}|.$$

Now, $k_1 + \cdots + k_n = 0$ thus:

$$I_0^- \leqslant \widetilde{I_0} \sum_{\substack{k_1+\cdots+k_n=0\\ \mathsf{C}_{k_1,\ldots,k_n}<0}} |\mathsf{C}_{k_1,\ldots,k_n}| \left(\frac{1}{a_1}\right)^{k_1} \cdots \left(\frac{1}{a_n}\right)^{k_n}. \tag{5.3.1}$$

Recall that $|\mathsf{C}|(\underline{t})$ denotes the (convergent) iterated Laurent series generated by the absolute value of the coefficients of the series C:

$$|\mathsf{C}|(t_1,\ldots,t_n) := \sum_{i\in\mathbb{Z}^n} |\mathsf{C}_{i_1,\ldots,i_n}|\, t_1^{i_1}\cdots t_n^{i_n}.$$

We will use the basic result:

$$\sum_{\substack{k_1+\cdots+k_n=0\\ \mathsf{C}_{k_1,\ldots,k_n}<0}} |\mathsf{C}_{k_1,\ldots,k_n}| \left(\frac{1}{a_1}\right)^{k_1} \cdots \left(\frac{1}{a_n}\right)^{k_n} = \frac{1}{2}\left(|\mathsf{C}|\left(\frac{1}{a_1},\ldots,\frac{1}{a_n}\right) - \mathsf{C}\left(\frac{1}{a_1},\ldots,\frac{1}{a_n}\right)\right).$$

Now, we fix $a_1,\ldots,a_n$, in order to establish the estimates:

(5.3.2) *Let $(a_i)_{i=1,\ldots,n}$ be the decreasing geometric sequence*

$$a_i := (n)^{n-i} \qquad (i=1,\ldots,n),$$

then for $n \geqslant 6$, the following inequalities hold:

$$\frac{2}{3}\,|\mathsf{C}|\left(\frac{1}{a_1},\ldots,\frac{1}{a_n}\right) \leqslant \mathsf{C}\left(\frac{1}{a_1},\ldots,\frac{1}{a_n}\right) \leqslant |\mathsf{C}|\left(\frac{1}{a_1},\ldots,\frac{1}{a_n}\right) \leqslant 5.$$

Coming back to (5.3.1), we get at once:

$$I_0^- \leqslant \widetilde{I_0}\,\frac{1}{2}\left(1-\frac{2}{3}\right)|\mathsf{C}|\left(\frac{1}{a_1},\ldots,\frac{1}{a_n}\right)\widetilde{I_0} \leqslant \frac{5}{6}\,\widetilde{I_0}.$$

Proof of the inequalities. Observe that the coefficient of C are very complicated. In order to bypass this difficulty, we will use a *majorant series* $\widehat{\mathsf{C}}$ for C, in the sense that the Taylor series expansion of $\widehat{\mathsf{C}}$ has only non negative coefficients, that are furthermore upper bounds for the absolute value of the corresponding Taylor coefficients of C:

$$|\mathsf{C}_{\boldsymbol{i}}| \leqslant \widehat{\mathsf{C}}_{\boldsymbol{i}} \qquad (\boldsymbol{i} \in \mathbb{N}^n).$$

Moreover, we will work in the domain of convergence of the series, in order to use their rational expressions.

For each factor of

$$\mathsf{C}(t_1,\ldots,t_n) = \prod_{1\leqslant i<j\leqslant n} \frac{t_j - t_i}{t_j - 2t_i} \prod_{2\leqslant i<j\leqslant n} \frac{t_j - 2\,t_i}{t_j - 2t_i + t_{i-1}},$$

it is easy to obtain such a majorant series, by replacing the series coefficients by their absolute values. The first kind of fractions has only positive coefficients. Indeed:

$$\frac{t_j - t_i}{t_j - 2t_i} = 1 + \sum_{k\geqslant 0} 2^k \left(\frac{t_i}{t_j}\right)^{k+1}.$$

The second kind of fractions has coefficients which sign is determined by the exponent of t_{i-1}, indeed:

$$\frac{t_j - 2\,t_i}{t_j - 2t_i + t_{i-1}} = 1 + \frac{(-t_{i-1})}{t_j} \sum_{k\geqslant l\geqslant 0} (-1)^l\, 2^{k-l}\binom{k}{l}\, t_{i-1}^l t_i^{k-l} t_j^k$$

whereas, by changing the sign of t_{i-1}:

$$\frac{t_j - 2\,t_i}{t_j - 2t_i - t_{i-1}} = 1 + \frac{t_{i-1}}{t_j} \sum_{k\geqslant l\geqslant 0} 2^{k-l}\binom{k}{l}\, t_{i-1}^l t_i^{k-l} t_j^k.$$

The majorant series is then obtained by taking the product of these pieces, that is:

$$\widehat{\mathsf{C}}(t_1,\ldots,t_n) = \prod_{1\leqslant i<j\leqslant n} \frac{t_j - t_i}{t_j - 2t_i} \prod_{2\leqslant i<j\leqslant n} \frac{t_j - 2\,t_i}{t_j - 2t_i - t_{i-1}},$$

The detail of this fact is elementary and left to the reader.

Then:

$$\widehat{\mathsf{C}}\left(\frac{1}{a_1},\ldots,\frac{1}{a_n}\right) = \prod_{1\leqslant i<j\leqslant n} \frac{a_i/a_j - 1}{a_i/a_j - 2} \prod_{2\leqslant i<j\leqslant n} \frac{a_i/a_j - 2}{a_i/a_j - (2 + a_i/a_{i-1})} = \prod_{1\leqslant i<j\leqslant n} \frac{n^{j-i} - 1}{n^{j-i} - 2} \prod_{2\leqslant i<j\leqslant n} \frac{n^{j-i} - 2}{n^{j-i} - (2 + 1/n)}$$

$$= \prod_{k=1}^{n-1} \left(\frac{n^k - 1}{n^k - 2}\right)^{n-k} \prod_{k=1}^{n-1} \left(\frac{n^k - 2}{n^k - (2 + 1/n)}\right)^{n-1-k} = \prod_{k=1}^{n-1} \left(\frac{\left(n^k - 1\right)^{n-k}}{(n^k - 2)\,(n^k - (2 + 1/n))^{n-1-k}}\right)$$

and similarly;

$$\mathsf{C}\left(\frac{1}{a_1},\ldots,\frac{1}{a_n}\right) = \prod_{k=1}^{n-1} \left(\frac{\left(n^k - 1\right)^{n-k}}{(n^k - 2)\,(n^k - (2 - 1/n))^{n-1-k}}\right).$$

This two products have many common terms and their difference is:

$$\frac{\widehat{\mathsf{C}}\left(\frac{1}{a_1},\ldots,\frac{1}{a_n}\right)}{\mathsf{C}\left(\frac{1}{a_1},\ldots,\frac{1}{a_n}\right)} = \prod_{k=1}^{n-1} \frac{\left(n^k - 2 + 1/n\right)^{n-1-k}}{\left(n^k - 2 - 1/n\right)^{n-1-k}} = \prod_{k=1}^{n-1} \left(1 + \frac{2}{n^{k+1} - 2n - 1)}\right)^{n-(k+1)}$$

That is, after a shift of k by 1:

$$\frac{\widehat{\mathsf{C}}\left(\frac{1}{a_1},\ldots,\frac{1}{a_n}\right)}{\mathsf{C}\left(\frac{1}{a_1},\ldots,\frac{1}{a_n}\right)} = \prod_{k=2}^{n} \left(1 + \frac{2}{n^k - 2n - 1}\right)^{n-k}.$$

In order to prove that for $n \geqslant 6$, this quantity is less than $\frac{3}{2}$, consider the two first cases $n = 6, 7$ with a computer algebra system, and then use a rough estimation for large enough cases, *e.g.*:

$$\ln \frac{\widehat{\mathsf{C}}}{\mathsf{C}} \leqslant \sum_{k=2}^{n} \frac{2(n-k)}{n^k - 2n - 1} \leqslant 2\sum_{k=2}^{n} \frac{(n-2)}{(n-2)^k} \leqslant \frac{2}{(n-3)},$$

that yields the upper bound by exponentiation, for $n \geqslant 8$.

We estimate $\widehat{\mathsf{C}}$ in the same way. We consider the cases $7 \leqslant n \leqslant 11$ with a computer algebra system and then one has:

$$\widehat{\mathsf{C}}\left(\frac{1}{a_1},\ldots,\frac{1}{a_n}\right) = \prod_{k=1}^{n-1} \left(\frac{\left(n^k - 1\right)^{n-k}}{(n^k - 2)\,(n^k - (2 + 1/n))^{n-1-k}}\right) \leqslant \prod_{k=1}^{n-1} \left(\frac{n^k - 1}{n^k - (2 + 1/n)}\right)^{n-k}$$

That is:

$$\widehat{\mathsf{C}}\Big(\frac{1}{a_1},\dots,\frac{1}{a_n}\Big) \leqslant \prod_{k=1}^{n-1}\Big(1+\frac{n+1}{n^{k+1}-2n-1}\Big)^{n-k} = \prod_{k=2}^{n}\Big(1+\frac{n+1}{n^{k}-2n-1}\Big)^{n+1-k}$$

We infer that:

$$\ln(\widehat{\mathsf{C}}) \leqslant \sum_{k=2}^{n}\frac{(n+1-k)(n+1)}{n^k-2n-1} \leqslant n^2\sum_{k\geqslant 2}\Big(\frac{1}{n-2}\Big)^k \leqslant \frac{n^2}{(n-2)(n-3)}.$$

For $n \geqslant 12$, by exponentiation:

$$\widehat{\mathsf{C}}\Big(\frac{1}{a_1},\dots,\frac{1}{a_n}\Big) \leqslant \exp\Big(\frac{n^2}{(n-2)(n-3)}\Big) \leqslant 5. \qquad \square$$

5.4. Estimation of the slope. Lastly, we study the slope of I_0 with respect to δ:

$$I_0' = \sum_{k_1+\cdots+k_n=0}\Big[t_1^{n-k_1}\cdots t_n^{n-k_n}\Big]\Big(f_1(t_1,\dots,t_n)\,e_1(t_1,\dots,t_n)\Big)\,\Big[t_1^{k_1}\cdots t_n^{k_n}\Big]\Big(\mathsf{C}(t_1,\dots,t_n)\Big).$$

By the triangle inequality, taking account of lemma (4.4.1) (for $p=1$ and $q=0$), we get:

$$I_0^- \leqslant \widetilde{I_1}\sum_{k_1+\cdots+k_n=0}\Big(\frac{a_1}{2n\mu(\underline{a})}\Big)^{-k_1}\cdots\Big(\frac{a_n}{2n\mu(\underline{a})}\Big)^{-k_n}\;|\mathsf{C}_{k_1,\dots,k_n}| = \widetilde{I_1}\;|\mathsf{C}|\Big(\frac{1}{a_1},\dots,\frac{1}{a_n}\Big).$$

On the other hand, in (4.5.2), we have established that $\widetilde{\lambda} = 2\widetilde{I_1}/\widetilde{I_0}$, hence:

$$|I_0'| \leqslant \frac{\widetilde{\lambda}(\underline{a})}{2}\;|\mathsf{C}|\Big(\frac{1}{a_1},\dots,\frac{1}{a_n}\Big)\,\widetilde{I_0} \leqslant \frac{5\widetilde{\lambda}(\underline{a})}{2}\widetilde{I_0}.$$

6. Estimation of the Non-leading Coefficients

In this appendix, we estimate the other coefficients. We also show that the term B *is always negligible.*

6.1. Estimation of the coefficient of d^{n-1} in I. Next, we consider the coefficient of d^{n-1} in I, that is the Cauchy product coefficient:

$$I_1 = \Big[h^n t_1^n\cdots t_n^n\Big]\Big(\mathsf{A}_1(t_1,\dots,t_n)\;\mathsf{B}(t_1,\dots,t_n)\;\mathsf{C}(t_1,\dots,t_n)\Big).$$

(6.1.1) *For any integer $n\in\mathbb{N}$, and for any choice of the parameters $(a_1,\dots,a_n)\in\mathbb{N}^n$:*

$$|I_1| \leqslant |\mathsf{C}|\Big(\frac{1}{a_1},\dots,\frac{1}{a_n}\Big)\widetilde{I_1}.$$

Proof. The term $\mathsf{C}(t_1,\dots,t_n)$ does not involve the variable h, thus:

$$I_1 = \Big[t_1^n\cdots t_n^n\Big]\Big(\Big[h^n\Big]\Big(\mathsf{A}_1(t_1,\dots,t_n)\,\mathsf{B}(t_1,\dots,t_n)\Big)\mathsf{C}(t_1,\dots,t_n)\Big).$$

The expansion formula (4.1.1) above, page 78, shows that the first factor of this product:

$$\mathsf{A}_1(t_1,\dots,t_n) = \Big[d^{n-1}\Big]\mathsf{A}(t_1,\dots,t_n)$$

is the following polynomial multiple of h^{n-1}:

$$\mathsf{A}_1(t_1,\dots,t_n) = h^n\Big(\delta\,(n+1)\,e_1(\underline{t})\,f_1(\underline{t}) - \delta\,e_2(\underline{t})\,f_2(\underline{t})\Big) + h^{n-1}\Big(e_1(\underline{t})\,f_0(\underline{t}) - \delta\,e_2(\underline{t})\,f_1(\underline{t})\Big),$$

with the notation of (4.1.1), taking account of

$$\alpha_0 = 1,\;\; \alpha_1 = \delta(n+1),\;\;\text{and}\;\; \beta_1=\beta_2=\delta.$$

Hence, in the computation of the coefficient

$$\big[h^n\big]\Big(\mathsf{A}_1(t_1,\dots,t_n)\,\mathsf{B}(t_1,\dots,t_n)\Big) = \big[h\big]\Big(\frac{\mathsf{A}_1(t_1,\dots,t_n)}{h^{n-1}}\,\mathsf{B}(t_1,\dots,t_n)\Big),$$

one can replace the iterated Laurent series:

$$\mathsf{B}(t_1,\dots,t_n) = \sum_{j_1,\dots,j_n\geqslant 0}\binom{n+j_1}{n}\cdots\binom{n+j_n}{n}\frac{(-h)^{j_1+\cdots+j_n}}{t_1^{j_1}\cdots t_n^{j_n}},$$

by its truncation up to polynomial multiples of h^2:

$$\mathsf{B}(t_1,\dots,t_n) = 1-\sum_{i=1}^{n}(n+1)\frac{h}{t_i}+O(h^2).$$

The coefficient of h^n becomes:

$$\begin{aligned}\big[h^n\big]\Big(\mathsf{A}_1(t_1,\dots,t_n)\,\mathsf{B}(t_1,\dots,t_n)\Big) = {}&\\ &\delta\,(n+1)\,e_1(\underline{t})\,f_1(\underline{t})-\delta\,e_2(\underline{t})\,f_2(\underline{t})\\ &-(n+1)\Big(e_1(\underline{t})\,f_0(\underline{t})-\delta\,e_2(\underline{t})\,f_1(\underline{t})\Big)\sum_{i=1}^{n}\frac{1}{t_i}.\end{aligned}$$

Recall the convention:

$$\mathsf{C}_{k_1,\dots,k_n} = \big[t_1^{k_1}\cdots t_n^{k_n}\big]\Big(\mathsf{C}(t_1,\dots,t_n)\Big).$$

Then:

$$\big[t_1^{k_1}\cdots t_n^{k_n}\big]\Big(\frac{1}{t_i}\mathsf{C}(t_1,\dots,t_n)\Big) = \mathsf{C}_{k_1,\dots,k_{i-1},k_i+1,k_{i+1},\dots,k_n}.$$

Hence:

$$\begin{aligned}I_1 = \delta&\sum_{k_1+\cdots+k_n=0}\big[t_1^{n-k_1}\cdots t_n^{n-k_n}\big]\Big((n+1)\,e_1(\underline{t})\,f_1(\underline{t})-e_2(\underline{t})\,f_2(\underline{t})\Big)\,\mathsf{C}_{k_1,\dots,k_n}\\ -(n+1)&\sum_{k_1+\cdots+k_n=-1}\big[t_1^{n-k_1}\cdots t_n^{n-k_n}\big]\Big(e_1(\underline{t})\,f_0(\underline{t})-\delta\,e_2(\underline{t})\,f_1(\underline{t})\Big)\,\Big(\mathsf{C}_{k_1+1,\dots,k_n}+\cdots+\mathsf{C}_{k_1,\dots,k_n+1}\Big).\end{aligned}$$

By the triangle inequality, and applying four times lemma (4.4.1):

$$\begin{aligned}|I_1| \leqslant \delta\,\Big((n+1)\,\widetilde{I_1}+\widetilde{I_2}\Big)&\sum_{k_1+\cdots+k_n=0}\Big(\frac{1}{a_1}\Big)^{k_1}\cdots\Big(\frac{1}{a_n}\Big)^{k_n}\;|\mathsf{C}|_{k_1,\dots,k_n}\\ +(n+1)(\widetilde{I_1}+\delta\,\widetilde{I_2})&\sum_{k_1+\cdots+k_n=-1}\frac{1}{2n\mu(\underline{a})}\Big(\frac{1}{a_1}\Big)^{k_1}\cdots\Big(\frac{1}{a_n}\Big)^{k_n}\;\Big(|\mathsf{C}|_{k_1+1,\dots,k_n}+\cdots+|\mathsf{C}|_{k_1,\dots,k_n+1}\Big).\end{aligned}$$

Now the sum in the second line expands as follows in n sums:

$$\begin{aligned}\sum_{k_1+\cdots+k_n=-1}\Big(\frac{1}{a_1}\Big)^{k_1}\cdots\Big(\frac{1}{a_n}\Big)^{k_n}\;\Big(|\mathsf{C}|_{k_1+1,\dots,k_n}+\cdots+|\mathsf{C}|_{k_1,\dots,k_n+1}\Big) = {}&\\ \sum_{(k_1+1)+\cdots+k_n=0}\Bigg[\Big(\frac{1}{a_1}\Big)^{(k_1+1)-1}\cdots\Big(\frac{1}{a_n}\Big)^{k_n}\;|\mathsf{C}|_{k_1+1,\dots,k_n}\Bigg]&+\\ \cdots&\\ +\sum_{k_1+\cdots+(k_n+1)=0}\Bigg[\Big(\frac{1}{a_1}\Big)^{k_1}\cdots\Big(\frac{1}{a_n}\Big)^{(k_n+1)-1}\;|\mathsf{C}|_{k_1,\dots,k_n+1}\Bigg]&,\end{aligned}$$

and it suffices to make the appropriate shift of indices in each of these sums in order to obtain:

$$\sum_{k_1+\cdots+k_n=-1}\Big(\frac{1}{a_1}\Big)^{k_1}\cdots\Big(\frac{1}{a_n}\Big)^{k_n}\Big(|\mathsf{C}|_{k_1+1,\ldots,k_n}+\cdots+|\mathsf{C}|_{k_1,\ldots,k_n+1}\Big)=$$

$$a_1\,|\mathsf{C}|\Big(\frac{1}{a_1},\ldots,\frac{1}{a_n}\Big)+\cdots+a_n\,|\mathsf{C}|\Big(\frac{1}{a_1},\ldots,\frac{1}{a_n}\Big).$$

The later sum is smaller than $\mu(\underline{a})|\mathsf{C}|\big(\frac{1}{a_1},\ldots,\frac{1}{a_n}\big)$ because by definition:

$$\mu(a_1,\ldots,a_n)=1\,a_1+\cdots+n\,a_n.$$

One can now obtain, as desired, a constant multiple of $\widetilde{I_1}$ bounding the coefficient $|I_1|$ from above:

$$\begin{aligned}|I_1|&\leqslant\Big(\delta\,\big((n+1)\,\widetilde{I_1}+\widetilde{I_2}\big)+\frac{n+1}{2n}(\widetilde{I_1}+\delta\,\widetilde{I_2})\Big)\,|\mathsf{C}|\Big(\frac{1}{a_1},\ldots,\frac{1}{a_n}\Big)\\&\leqslant\widetilde{I_1}\Big(\delta\,(n+1+\widetilde{\lambda})+\frac{n+1}{2n}(1+\delta\,\widetilde{\lambda})\Big)\,|\mathsf{C}|\,\Big(\frac{1}{a_1},\ldots,\frac{1}{a_n}\Big)\end{aligned}$$

Notice that this estimate are true without assumption $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$. Now, taking account of $5\delta\widetilde{\lambda}\leqslant 1$ and $n\geqslant 6$, one has:

$$\Big(\delta\,(n+1+\widetilde{\lambda})+\frac{n+1}{2n}(1+\delta\,\widetilde{\lambda})\Big)=\left(\frac{1+3\,\delta\widetilde{\lambda}}{2}+\frac{1+\delta\widetilde{\lambda}}{2n}+\delta(n+1)\right)\leqslant\Big(\frac{8}{10}+\frac{1}{10}+\frac{n+1}{5\widetilde{\lambda}}\Big)\leqslant 1,$$

because $\widetilde{\lambda}\geqslant 2\,(n+1)$. One has finally (without using hypothesis $(\mathcal{H}_1)$):

$$|I_1|\leqslant|\mathsf{C}|\Big(\frac{1}{a_1},\ldots,\frac{1}{a_n}\Big)\widetilde{I_1}.\qquad\square$$

6.2. Estimation of the other coefficients. We consider the coefficients of the remaining monomials $d^{n-p}=d^{n-2},\ldots,d^1,d^0$. As we have already seen twice above, the coefficient of d^{n-p} in A is a polynomial multiple of h^{n-p}. Because we seek the coefficient of h^n, this allowed us to simplify the computation by replacing the iterated Laurent series:

$$\mathsf{B}(t_1,\ldots,t_n)=\sum_{j_1,\ldots,j_n\geqslant 0}\binom{n+j_1}{n}\cdots\binom{n+j_n}{n}\frac{(-h)^{j_1+\cdots+j_n}}{t_1^{j_1}\cdots t_n^{j_n}}$$

by its truncations $B\simeq 1+O(h)$ for $p=0$ and $B\simeq 1-\sum_{i=1}^n(n+1)\frac{h}{t_i}+O(h^2)$ for $p=1$. Of course in general this allows to replace the iterated Laurent series B by its truncation up to polynomial multiples of h^{p+1}. As p goes from 0 to n, this strategy reaches soon its limits, since the number of terms remaining in the above-mentioned truncated expansion of B increases dramatically. Consequently, the precise estimation of the coefficient:

$$I_p=\Big[d^{n-p}h^n t_1^n\cdots t_n^n\Big]\Big(\mathsf{A}(t_1,\ldots,t_n)\,\mathsf{B}(t_1,\ldots,t_n)\,\mathsf{C}(t_1,\ldots,t_n)\Big)$$

is technically more and more involved.

This consideration quickly justify that for $p\geqslant 2$ we will not use anymore the expanded expression:

$$\mathsf{B}(t_1,\ldots,t_n)=\sum_{j_1,\ldots,j_n\geqslant 0}\binom{n+j_1}{n}\cdots\binom{n+j_n}{n}\frac{(-h)^{j_1+\cdots+j_n}}{t_1^{j_1}\cdots t_n^{j_n}}$$

as in the preceding estimations of I_0 and I_1, but that we will use a simpler expression instead. Namely, in order to use the triangle inequality, we will forget about the signs of the coefficients of B and use the corresponding series with non negative coefficients:

$$|\mathsf{B}|(t_1,\ldots,t_n) = \sum_{j_1,\ldots,j_n\geqslant 0} \binom{n+j_1}{n}\cdots\binom{n+j_n}{n}\frac{(+h)^{j_1+\cdots+j_n}}{t_1^{j_1}\cdots t_n^{j_n}}$$

and we will even manage to use the sum of this series:

$$|\mathsf{B}|(t_1,\ldots,t_n) = \prod_{i=1}^{n}\frac{1}{\left(1-\frac{h}{t_1}\right)^{n+1}},$$

the estimation of which is easier. Actually, we will no longer need to know anything about the coefficients of B.

This level of precision will be sufficient in view of the application of Fujiwara's bound:

$$d \geqslant 2 \max_{1\leqslant p\leqslant n}\left|\frac{I_p}{I_0}\right|^{\frac{1}{p}},$$

because we see that for $p \geqslant 2$, the obtained ratio will be decreased by an exponent $\frac{1}{p}\leqslant\frac{1}{2}$.

(6.2.1) *For $p = 0, 1, \ldots, n$ and any choice of the parameters $a_1,\ldots,a_n$, the absolute value of the coefficient of d^{n-p} in the polynomial I is bounded from above by:*

$$|I_p| \leqslant \left(\widetilde{I_p}+\delta\widetilde{I_{p+1}}\right)|\mathsf{B}|\Big(\frac{2n\mu h}{a_1},\ldots,\frac{2n\mu h}{a_n}\Big)|\mathsf{C}|\Big(\frac{1}{a_1},\ldots,\frac{1}{a_n}\Big).$$

Proof. Recall that $I(d)$ is the Cauchy product coefficient:

$$I = \left[h^n t_1^n\cdots t_n^n\right]\Big(\mathsf{A}(t_1,\ldots,t_n)\,\mathsf{B}(t_1,\ldots,t_n)\,\mathsf{C}(t_1,\ldots,t_n)\Big).$$

Recall that:

$$\mathsf{B}(t_1,\ldots,t_n) = \sum_{j\in\mathbb{N}^n}\mathsf{B}_j\left(\frac{h}{t_1}\right)^{j_1}\cdots\left(\frac{h}{t_n}\right)^{j_n}$$

and:

$$\mathsf{C}(t_1,\ldots,t_n) = \sum_{k_1+\cdots+k_n=0}\mathsf{C}_{\boldsymbol{k}}\,t_1^{k_1}\cdots t_n^{k_n}.$$

Recall lastly that $\mathsf{A}_p = [d^{n-p}]\mathsf{A}$. The coefficient $I_p = [d^{n-p}]I$ is thus:

$$\begin{aligned} I_p &= \sum_{\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}=0}\left[h^{n-|j|}t_1^{n+i_1}\cdots t_n^{n+i_n}\right]\!\left(\mathsf{A}_p\right)\left[h^{|j|}t_1^{-j_1}\cdots t_n^{-j_n}\right]\!\left(\mathsf{B}\right)\left[t_1^{k_1}\cdots t_n^{k_n}\right]\!\left(\mathsf{C}\right)\\ &= \sum_{\boldsymbol{j}-\boldsymbol{j}+\boldsymbol{k}=0}\left[h^{n-|\boldsymbol{i}|}\,t_1^{n+i_1}\cdots t_n^{n+i_n}\right]\!\left(\mathsf{A}_p\right)\mathsf{B}_{\boldsymbol{j}}\,\mathsf{C}_{\boldsymbol{k}}.\end{aligned}$$

Next, we extract the coefficient of $h^{n-|j|}$ by using the expansion formula (4.1.1), and we get:

$$\begin{aligned} I_p = \sum_{\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}=0}\alpha_{p-|j|}\left[t_1^{n+i_1}\cdots t_n^{n+i_n}\right]\!\Big(e_p(\underline{t})\,f_{p-|j|}(\underline{t})\Big)\,\mathsf{B}_{\boldsymbol{j}}\,\mathsf{C}_{\boldsymbol{k}}\\ + \sum_{\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}=0}\beta_{p-|j|+1}\left[t_1^{n+i_1}\cdots t_n^{n+i_n}\right]\!\Big(e_{p+1}(\underline{t})\,f_{p-|j|+1}(\underline{t})\Big)\,\mathsf{B}_{\boldsymbol{j}}\mathsf{C}_{\boldsymbol{k}}.\end{aligned}$$

Recall that $|\alpha_{p-q}| \leqslant 1$ and $|\beta_{p-q+1}| \leqslant \delta$. By the triangle inequality and using (4.4.1):

$$|I_p| \leqslant \sum_{i-j+k=0} \widetilde{I}_p \left(\frac{1}{2n\mu}\right)^{|j|} \frac{a_1^{j_1}\cdots a_n^{j_n}}{a_1^{k_1}\cdots a_n^{k_n}} |\mathsf{B}_j|\,|\mathsf{C}_k| + \widetilde{I}_{p+1} \sum_{i-j+k=0} \delta \left(\frac{1}{2n\mu}\right)^{|j|} \frac{a_1^{j_1}\cdots a_n^{j_n}}{a_1^{k_1}\cdots a_n^{k_n}} |\mathsf{B}_j|\,|\mathsf{C}_k|.$$

We factorize $\widetilde{I}_p + \delta\,\widetilde{I}_{p+1}$ and separate $\boldsymbol{j}$ and $\boldsymbol{k}$ and obtain:

$$|I_p| \leqslant \left(\widetilde{I}_p + \delta\,\widetilde{I}_{p+1}\right) \sum_{j_1+\cdots+j_n=q} \left(\frac{1}{2n\mu}\right)^q a_1^{j_1}\cdots a_n^{j_n} |\mathsf{B}_j| \sum_{k_1+\cdots+k_n=0} \frac{1}{a_1^{k_1}\cdots a_n^{k_n}} |\mathsf{C}_k|.$$

The announced result is obtained by identifying the two appearing sums:

$$|I_p| \leqslant \left(\widetilde{I}_p + \delta\widetilde{I}_{p+1}\right) |\mathsf{B}| \left(\frac{2n\mu h}{a_1}, \ldots, \frac{2n\mu h}{a_n}\right) |\mathsf{C}| \left(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\right). \qquad \square$$

We have already estimated $|\mathsf{C}|\left(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\right)$. In order to apply Fujiwara's bound,it remains to estimate $|\mathsf{B}|\left(\frac{2n\mu h}{a_1}, \ldots, \frac{2n\mu h}{a_n}\right)$. We will now see that — without using hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ — this term is bounded from above by the term of a converging (thus bounded) sequence.

(6.2.2) *For any integer $n \in \mathbb{N}$, for any choice of the parameters $(a_1, \ldots, a_n) \in \mathbb{N}^n$ and for $\mu = (1\,a_1 + \cdots + na_n)$:*

$$|\mathsf{B}| \left(\frac{2n\mu h}{a_1}, \ldots, \frac{2n\mu h}{a_n}\right) \leqslant \left(\frac{2\,n}{2\,n-1}\right)^{n+1} \qquad (\to e^{1/2}),$$

Proof. Recall that:

$$|\mathsf{B}|(t_1, \ldots, t_n) = \sum_{j_1,\ldots,j_n \geqslant 0} \binom{n+j_1}{n} \cdots \binom{n+j_n}{n} \frac{h^{j_1+\cdots+j_n}}{t_1^{j_1}\cdots t_n^{j_n}}.$$

Thus:

$$|\mathsf{B}| \left(\frac{2n\mu h}{a_1}, \ldots, \frac{2n\mu h}{a_n}\right) = \sum_{j_1,\ldots,j_n \geqslant 0} \binom{n+j_1}{n} \cdots \binom{n+j_n}{n} \left(\frac{a_1}{2n\mu}\right)^{j_1} \cdots \left(\frac{a_n}{2n\mu}\right)^{j_n}$$

Now $2n\mu > a_i$ for $i = 1, \ldots, n$, hence the left hand side expression is:

$$|\mathsf{B}| \left(\frac{2n\mu h}{a_1}, \ldots, \frac{2n\mu h}{a_n}\right) = \prod_{i=1}^n \left(1 - \frac{a_i}{\mu}\frac{1}{2n}\right)^{-(n+1)},$$

and by taking the logarithm:

$$\ln |\mathsf{B}| \left(\frac{2n\mu h}{a_1}, \ldots, \frac{2n\mu h}{a_n}\right) = (n+1) \sum_{i=1}^n \left|\ln\left(1 - \frac{a_i}{\mu}\frac{1}{2n}\right)\right|.$$

Next, use the concavity of the logarithm function and the fact that:

$$0 \leqslant a_i \leqslant \mu(\underline{a}) = 1\,a_1 + \cdots + n\,a_n \qquad (i=1,\ldots,n)$$

in order to obtain:

$$\left|\ln\left(1 - \frac{a_i}{\mu}\frac{1}{2n}\right)\right| \leqslant \frac{a_i}{\mu}\left|\ln\left(1 - \frac{1}{2\,n}\right)\right| = \frac{a_i}{\mu}\ln\left(\frac{2\,n}{2\,n-1}\right) \qquad [i=1,\ldots,n],$$

and sum these inequalities in order to get the upper bound:

$$\begin{aligned}\ln|\mathsf{B}|\left(\frac{2n\mu h}{a_1},\dots,\frac{2n\mu h}{a_n}\right) &\leqslant \frac{(a_1+\cdots+a_n)}{\mu(\underline{a})}\,(n+1)\ln\left(\frac{2\,n}{2\,n-1}\right)\\ &\leqslant (n+1)\ln\left(\frac{2\,n}{2\,n-1}\right).\end{aligned}$$

Finally, the increasingness of the exponential function yields the announced result. □

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