> restart :

> with(LinearAlgebra) :

> with(DifferentialGeometry) : with(Tools) :

> DGsetup([c, e, c1, e1, z[1], z[2], w[1], w[2], v], M2, verbose);

The following coordinates have been protected:

$[c, e, c1, e1, z_1, z_2, w_1, w_2, v]$

The following vector fields have been defined and protected:

$[D\_c, D\_e, D\_c1, D\_e1, D\_z_1, D\_z_2, D\_w_1, D\_w_2, D\_v]$

The following differential 1-forms have been defined and protected:

$[dc, de, dc1, de1, dz_1, dz_2, dw_1, dw_2, dv]$

frame name: M2 **(1)**

> $q[1] := \frac{dc}{c} : q[2] := \frac{-\mathrm{I}\, c1\, de - \mathrm{I}\, e\, dc1}{c\, c1} + \frac{\mathrm{I}\, dc\, e}{c^2} : q[3] := \frac{dc1}{c1} : q[4] := \frac{\mathrm{I}\, e1\, dc + \mathrm{I}\, c\, de1}{c\, c1} - \frac{\mathrm{I}\, dc1\, e1}{c1^2} :$

> $$Q[1] := -\frac{1}{2}\, \mathrm{I}\, c\, c1\, (w_1 + w_2 z_1)\, dz_1 + \frac{\frac{1}{4}\, \mathrm{I}\, c\, c1\, (w_1^2 + 2\, w_2 z_1 w_1 + z_1^2 w_2^2)\, dz_2}{-1 + z_2 w_2} + \frac{1}{2}\, \mathrm{I}\, c\, c1\, (z_1 + w_1 z_2)\, dw_1 - \frac{\frac{1}{4}\, \mathrm{I}\, c\, c1\, (z_1^2 + 2\, z_2 z_1 w_1 + w_1^2 z_2^2)\, dw_2}{-1 + z_2 w_2} + \frac{1}{2}\, c\, c1\, (-1 + z_2 w_2)\, dv :$$

> $$Q[2] := \left(c - \frac{1}{2}\, w_1\, e\, c1 - \frac{1}{2}\, e\, w_2 z_1\, c1\right) dz_1 + \frac{1}{4}\, \frac{(-4\, c\, w_1 - 4\, c\, w_2 z_1 + e\, w_1^2\, c1 + 2\, w_1\, e\, w_2 z_1\, c1 + z_1^2 w_2^2\, e\, c1)\, dz_2}{-1 + z_2 w_2} + \frac{1}{2}\, e\, c1\, (z_1 + w_1 z_2)\, dw_1 - \frac{1}{4}\, \frac{e\, c1\, (z_1^2 + 2\, z_2 z_1 w_1 + w_1^2 z_2^2)\, dw_2}{-1 + z_2 w_2} - \frac{1}{2}\, \mathrm{I}\, e\, c1\, (-1 + z_2 w_2)\, dv :$$

> $$Q[3] := \left(e - \frac{1}{4}\, \frac{e^2\, c1\, w_1}{c} - \frac{1}{4}\, \frac{e^2\, c1\, w_2 z_1}{c}\right) dz_1 + \frac{1}{4}\, \frac{1}{c1\, (-1 + z_2 w_2)} \left(\left(\frac{1}{2}\, \frac{e^2\, c1^2\, z_1^2 w_2^2}{c} - 4\, e\, w_2 z_1\, c1 + \frac{e^2\, c1^2\, w_2 z_1 w_1}{c} + 4\, c - 4\, w_1\, e\, c1 + \frac{1}{2}\, \frac{e^2\, c1^2\, w_1^2}{c}\right) dz_2\right) + \frac{1}{4}\, \frac{e^2\, c1\, (z_1 + w_1 z_2)\, dw_1}{c} - \frac{1}{8}\, \frac{e^2\, c1\, (z_1^2 + 2\, z_2 z_1 w_1 + w_1^2 z_2^2)\, dw_2}{c\, (-1 + z_2 w_2)} - \frac{\frac{1}{4}\, \mathrm{I}\, e^2\, c1\, (-1 + z_2 w_2)\, dv}{c} :$$

> $$Q[4] := \frac{1}{2}\, e1\, c\, (w_1 + w_2 z_1)\, dz_1 - \frac{1}{4}\, \frac{e1\, c\, (w_1^2 + 2\, w_2 z_1 w_1 + z_1^2 w_2^2)\, dz_2}{-1 + z_2 w_2} + \Big(c1$$

$$- \frac{1}{2}\, e1\, z_1\, c - \frac{1}{2}\, e1\, w_1\, z_2\, c \Big)\, dw_1$$

$$- \frac{1}{4}\, \frac{\left(4\, c1\, z_1 + 4\, c1\, w_1\, z_2 - e1\, c\, z_1^2 - 2\, e1\, c\, z_2\, z_1\, w_1 - e1\, c\, w_1^2\, z_2^2\right) dw_2}{-1 + z_2\, w_2} + \frac{1}{2}\, \mathrm{I}\, e1\, c\, \big(-1$$

$$+ z_2\, w_2\big)\, dv :$$

> $Q[5] := \frac{1}{4}\, \frac{e1^2\, c\, (w_1 + w_2\, z_1)\, dz_1}{c1} - \frac{1}{8}\, \frac{e1^2\, c\, \left(w_1^2 + 2\, w_2\, z_1\, w_1 + z_1^2\, w_2^2\right) dz_2}{c1\, (-1 + z_2\, w_2)} + \Big(e1$

$$- \frac{1}{4}\, \frac{e1^2\, c\, z_1}{c1} - \frac{1}{4}\, \frac{e1^2\, c\, w_1\, z_2}{c1} \Big)\, dw_1$$

$$- \frac{1}{4}\, \frac{1}{c\, (-1 + z_2\, w_2)} \Bigg(\bigg(-\frac{1}{2}\, \frac{e1^2\, c^2\, w_1^2\, z_2^2}{c1} + 4\, e1\, w_1\, z_2\, c - \frac{e1^2\, c^2\, z_2\, z_1\, w_1}{c1} - 4\, c1$$

$$+ 4\, e1\, z_1\, c - \frac{1}{2}\, \frac{e1^2\, c^2\, z_1^2}{c1} \bigg)\, dw_2 \Bigg) + \frac{\frac{1}{4}\, \mathrm{I}\, e1^2\, c\, (-1 + z_2\, w_2)\, dv}{c1} :$$

> $Fr := FrameData([q[1], q[2], q[3], q[4], Q[1], Q[2], Q[3], Q[4], Q[5]], P) :$

M2 > $DGsetup\big(Fr, [E], [\mathrm{delta}[1], \mathrm{delta}[2], \mathrm{delta}[3], \mathrm{delta}[4], \mathrm{rho}, \mathrm{kappa}, \mathrm{zeta}, \kappa^{\#}, \zeta^{\#}]\big);$

frame name: P **(2)**

M2 > $visualisation := \mathbf{proc}\ (l);\ subs\big(\{w[1] = conjugate(z[1]), w[2] = conjugate(z[2]), c1 = conjugate(c), e1 = conjugate(e), \kappa^{\#} = conjugate(\mathrm{kappa}), \zeta^{\#} = conjugate(\mathrm{zeta})\}, l\big);\ \mathbf{end\ proc}:$

M2 > $visualisation(ExteriorDerivative(\mathrm{rho}));$

$$\delta_1 \wedge \rho + \delta_3 \wedge \rho - \frac{\left(-e\, \bar{c}^2\, \bar{z}_2 + \bar{e}\, c^2\right) \rho \wedge \kappa}{\bar{c}\, c^2} - \frac{\bar{z}_2\, \bar{c}\, \rho \wedge \zeta}{c} + \frac{\left(-e\, \bar{c}^2 + \bar{e}\, c^2\, z_2\right) \rho \wedge \kappa^{\#}}{\bar{c}^2\, c} - \frac{z_2\, c\, \rho \wedge \zeta^{\#}}{\bar{c}} + \mathrm{I}\, \kappa \wedge \kappa^{\#} \quad (3)$$

M2 > $visualisation(ExteriorDerivative(\mathrm{kappa}));$

$$\delta_1 \wedge \kappa + \delta_2 \wedge \rho - \frac{\frac{1}{2}\, \mathrm{I}\, e^2\, \bar{c}\, \bar{z}_2\, \rho \wedge \kappa}{c^3} + \frac{\mathrm{I}\, \bar{e}\, \rho \wedge \zeta}{\bar{c}} - \frac{\frac{1}{2}\, \mathrm{I}\, e\, \left(-e\, \bar{c}^2 + 2\, \bar{e}\, c^2\, z_2\right) \rho \wedge \kappa^{\#}}{\bar{c}^2\, c^2} + \frac{\mathrm{I}\, e\, z_2\, \rho \wedge \zeta^{\#}}{\bar{c}} - \frac{\bar{z}_2\, \bar{c}\, \kappa \wedge \zeta}{c} + \zeta \wedge \kappa^{\#} \quad (4)$$

M2 > $visualisation(ExteriorDerivative(\mathrm{zeta}));$

$$\delta_1 \wedge \zeta + \mathrm{I}\, \delta_2 \wedge \kappa - \delta_3 \wedge \zeta + \frac{\frac{1}{2}\, \mathrm{I}\, e\, \bar{e}\, \left(-e\, \bar{c}^2 + \bar{e}\, c^2\, z_2\right) \rho \wedge \kappa}{c^2\, \bar{c}^3} - \frac{\frac{1}{2}\, \mathrm{I}\, \left(\bar{e}^2\, z_2\, c^4 + e^2\, \bar{z}_2\, \bar{c}^4 - 2\, \bar{e}\, e\, c^2\, \bar{c}^2\right) \rho \wedge \zeta}{c^3\, \bar{c}^3} - \frac{\bar{c}\, e\, \bar{z}_2\, \kappa \wedge \zeta}{c^2} \quad (5)$$

$$+\frac{1}{2}\frac{e\left(-e\,\bar{c}^2+2\,\bar{e}\,c^2 z_2\right)\kappa\wedge\kappa^{\#}}{c^2\,\bar{c}^2}-\frac{e\,z_2\,\kappa\wedge\zeta^{\#}}{\bar{c}}-\frac{\left(-e\,\bar{c}^2+\bar{e}\,c^2 z_2\right)\zeta\wedge\kappa^{\#}}{\bar{c}^2\,c}+\frac{z_2\,c\,\zeta\wedge\zeta^{\#}}{\bar{c}}$$

P > *visualisation*(*ExteriorDerivative*($\kappa^{\#}$));

$$\delta_3\wedge\kappa^{\#}+\delta_4\wedge\rho-\frac{\frac{1}{2}\,\mathrm{I}\,\bar{e}\left(-2\,e\,\bar{c}^2\,\bar{z}_2+\bar{e}\,c^2\right)\rho\wedge\kappa}{c^2\,\bar{c}^2}-\frac{\mathrm{I}\,\bar{e}\,\bar{z}_2\,\rho\wedge\zeta}{c}+\frac{\frac{1}{2}\,\mathrm{I}\,\bar{e}^2\,z_2\,c\,\rho\wedge\kappa^{\#}}{\bar{c}^3}-\frac{\mathrm{I}\,e\,\rho\wedge\zeta^{\#}}{c}-\kappa\wedge\zeta^{\#}-\frac{z_2\,c\,\kappa^{\#}\wedge\zeta^{\#}}{\bar{c}} \qquad (6)$$

> *visualisation*(*ExteriorDerivative*($\zeta^{\#}$));

$$-\delta_1\wedge\zeta^{\#}+\delta_3\wedge\zeta^{\#}-\mathrm{I}\,\delta_4\wedge\kappa^{\#}+\frac{\frac{1}{2}\,\mathrm{I}\,\bar{e}\,e\left(-e\,\bar{c}^2\,\bar{z}_2+\bar{e}\,c^2\right)\rho\wedge\kappa^{\#}}{\bar{c}^2\,c^3}+\frac{\frac{1}{2}\,\mathrm{I}\left(\bar{e}^2 z_2\,c^4+e^2\,\bar{z}_2\,\bar{c}^4-2\,\bar{e}\,e\,c^2\,\bar{c}^2\right)\rho\wedge\zeta^{\#}}{c^3\,\bar{c}^3}+\frac{1}{2}\frac{\bar{e}\left(-2\,e\,\bar{c}^2\,\bar{z}_2+\bar{e}\,c^2\right)\kappa\wedge\kappa^{\#}}{c^2\,\bar{c}^2}-\frac{\left(-e\,\bar{c}^2\,\bar{z}_2+\bar{e}\,c^2\right)\kappa\wedge\zeta^{\#}}{\bar{c}\,c^2}+\frac{\bar{e}\,\bar{z}_2\,\zeta\wedge\kappa^{\#}}{c}-\frac{\bar{z}_2\,\bar{c}\,\zeta\wedge\zeta^{\#}}{c}-\frac{\bar{e}\,z_2\,c\,\kappa^{\#}\wedge\zeta^{\#}}{\bar{c}^2} \qquad (7)$$

P > $r[1] := evalDG\left(q[1]-\frac{\frac{1}{4}\,\mathrm{I}\left(c1^4\,e^2\,w_2-2\,e1\,e\,c^2\,c1^2+c^4\,e1^2\,z_2\right)}{c^3\,c1^3}\cdot Q[1]+\frac{\left(-e\,c1^2\,w_2+e1\,c^2\right)}{c1\,c^2}\cdot Q[2]+\frac{w[2]\cdot c1}{c}\cdot Q[3]\right):$

M2 > $r[2] := evalDG\left(q[2]-I\cdot\frac{\frac{1}{2}\,\mathrm{I}\,e\,e1\left(-e\,c1^2+e1\,c^2\,z_2\right)}{c^2\,c1^3}\cdot Q[1]-\left(-\frac{\frac{1}{2}\,\mathrm{I}\,e^2\,c1\,w_2}{c^3}+\frac{\frac{1}{4}\,\mathrm{I}\left(c1^4\,e^2\,w_2-2\,e1\,e\,c^2\,c1^2+c^4\,e1^2\,z_2\right)}{c^3\,c1^3}\right)\cdot Q[2]-\frac{I\cdot e1}{c1}\cdot Q[3]+\frac{\frac{1}{2}\,\mathrm{I}\,e\left(-e\,c1^2+2\,e1\,c^2\,z_2\right)}{c^2\,c1^2}\cdot Q[4]-\frac{\mathrm{I}\,e\,z_2}{c1}\cdot Q[5]\right):$

M2 > $r[3] := evalDG\left(q[3]+\frac{\frac{1}{4}\,\mathrm{I}\left(c1^4\,e^2\,w_2-2\,e1\,e\,c^2\,c1^2+c^4\,e1^2\,z_2\right)}{c^3\,c1^3}\cdot Q[1]-\frac{-e\,c1^2+e1\,c^2\,z_2}{c\,c1^2}\cdot Q[4]+\frac{z_2\,c}{c1}\cdot Q[5]\right):$

M2 > $r[4] := evalDG\left(q[4] + I\cdot \frac{\frac{1}{2}\, \mathrm{I}\, e1\, e\left(-e\, c1^2 w_2 + e1\, c^2\right)}{c1^2 c^3}\cdot Q[1] - \left(\frac{\frac{1}{2}\, \mathrm{I}\, e1^2\, c\, z_2}{c1^3} - \frac{\frac{1}{4}\, \mathrm{I}\left(c1^4 e^2 w_2 - 2\, e1\, e\, c^2 c1^2 + c^4 e1^2 z_2\right)}{c^3 c1^3}\right)\cdot Q[4] + \frac{\mathrm{I}\, e}{c}\cdot Q[5] + \frac{\frac{1}{2}\, \mathrm{I}\, e1\left(-2\, e\, c1^2 w_2 + e1\, c^2\right)}{c1^2 c^2}\cdot Q[2] + \frac{\mathrm{I}\, e1\, w_2}{c}\cdot Q[3]\right):$

M2 > *antivisualisation* := **proc**(l); *subs*({ *conjugate*($z[1]$) = $w[1]$, *conjugate*($z[2]$) = $w[2]$, *conjugate*(c) = $c1$, *conjugate*(e) = $e1$, }, l); **end proc**:

List := *GenerateForms*([delta[1], delta[2], delta[3], delta[4], rho, kappa, zeta, $\kappa^\#$, $\zeta^\#$], 2);

$$\left[\delta_1 \wedge \delta_2, \delta_1 \wedge \delta_3, \delta_1 \wedge \delta_4, \delta_1 \wedge \rho, \delta_1 \wedge \kappa, \delta_1 \wedge \zeta, \delta_1 \wedge \kappa^\#, \delta_1 \wedge \zeta^\#, \delta_2 \wedge \delta_3, \delta_2 \wedge \delta_4, \delta_2 \wedge \rho, \delta_2 \wedge \kappa, \delta_2 \wedge \zeta, \delta_2 \wedge \kappa^\#, \delta_2 \wedge \zeta^\#, \delta_3 \wedge \delta_4, \delta_3 \wedge \rho, \delta_3 \wedge \kappa, \delta_3 \wedge \zeta, \delta_3 \wedge \kappa^\#, \delta_3 \wedge \zeta^\#, \delta_4 \wedge \rho, \delta_4 \wedge \kappa, \delta_4 \wedge \zeta, \delta_4 \wedge \kappa^\#, \delta_4 \wedge \zeta^\#, \rho \wedge \kappa, \rho \wedge \zeta, \rho \wedge \kappa^\#, \rho \wedge \zeta^\#, \kappa \wedge \zeta, \kappa \wedge \kappa^\#, \kappa \wedge \zeta^\#, \zeta \wedge \kappa^\#, \zeta \wedge \zeta^\#, \kappa^\# \wedge \zeta^\#\right] \quad \mathbf{(8)}$$

Torsion := **proc**(S, i, j) **local** k, X; $k := 9\cdot(i-1) - \frac{i\cdot(i-1)}{2} + j - i$; $X :=$ *GetComponents*(S, *List*); $X[k]$; **end proc**:

result := **proc**(l) **local** k, t, X; $X := 0 : t :=$ (*expand*(*simplify*(*GetComponents*(l, *List*)))) : **for** k **from** 1 **to** 36 **do** $X := X + t[k]\cdot List[k]$ **od**; X; **end proc**:

M2 > *ExteriorDerivative*(rho);

$$\delta_1 \wedge \rho + \delta_3 \wedge \rho - \frac{\left(-e\, c1^2 w_2 + e1\, c^2\right)\rho \wedge \kappa}{c1\, c^2} - \frac{w_2\, c1\, \rho \wedge \zeta}{c} + \frac{\left(-e\, c1^2 + e1\, c^2 z_2\right)\rho \wedge \kappa^\#}{c1^2\, c} - \frac{z_2\, c\, \rho \wedge \zeta^\#}{c1} + \mathrm{I}\, \kappa \wedge \kappa^\# \quad \mathbf{(9)}$$

P > *visualisation*(*result*(*ExteriorDerivative*(rho)));

$$\delta_1 \wedge \rho + \delta_3 \wedge \rho + \left(\frac{\overline{c}\, e\, \overline{z_2}}{c^2} - \frac{\overline{e}}{\overline{c}}\right)\rho \wedge \kappa - \frac{\overline{z_2}\, \overline{c}\, \rho \wedge \zeta}{c} + \left(-\frac{e}{c} + \frac{\overline{e}\, z_2\, c}{\overline{c}^2}\right)\rho \wedge \kappa^\# - \frac{z_2\, c\, \rho \wedge \zeta^\#}{\overline{c}} + \mathrm{I}\, \kappa \wedge \kappa^\# \quad \mathbf{(10)}$$

P > *visualisation*(*result*(*ExteriorDerivative*(kappa)));

$$\delta_1 \wedge \kappa + \delta_2 \wedge \rho - \frac{\frac{1}{2}\, \mathrm{I}\, e^2\, \overline{c}\, \overline{z_2}\, \rho \wedge \kappa}{c^3} + \frac{\mathrm{I}\, \overline{e}\, \rho \wedge \zeta}{\overline{c}} + \left(\frac{\frac{1}{2}\, \mathrm{I}\, e^2}{c^2} - \frac{\mathrm{I}\, e\, \overline{e}\, z_2}{\overline{c}^2}\right)\rho \wedge \kappa^\# + \frac{\mathrm{I}\, e\, z_2\, \rho \wedge \zeta^\#}{\overline{c}} - \frac{\overline{z_2}\, \overline{c}\, \kappa \wedge \zeta}{c} + \zeta \wedge \kappa^\# \quad \mathbf{(11)}$$

P > *visualisation*(*result*(*ExteriorDerivative*(zeta)));

$$\delta_1 \wedge \zeta + I\,\delta_2 \wedge \kappa - \delta_3 \wedge \zeta + \left(-\frac{\frac{1}{2}\, I\, e^2\, \bar{e}}{c^2\, \bar{c}} + \frac{\frac{1}{2}\, I\, e\, \bar{e}^2\, z_2}{\bar{c}^3} \right) \rho \wedge \kappa + \left(-\frac{\frac{1}{2}\, I\, \bar{e}^2\, z_2\, c}{\bar{c}^3} - \frac{\frac{1}{2}\, I\, e^2\, \bar{c}\, \bar{z}_2}{c^3} + \frac{I\, \bar{e}\, e}{c\, \bar{c}} \right) \rho \wedge \zeta - \frac{\bar{c}\, e\, \bar{z}_2\, \kappa \wedge \zeta}{c^2} + \left(-\frac{1}{2}\, \frac{e^2}{c^2} + \frac{e\, \bar{e}\, z_2}{\bar{c}^2} \right) \kappa \wedge \kappa^{\#} - \frac{e\, z_2\, \kappa \wedge \zeta^{\#}}{\bar{c}} + \left(\frac{e}{c} - \frac{\bar{e}\, z_2\, c}{\bar{c}^2} \right) \zeta \wedge \kappa^{\#} + \frac{z_2\, c\, \zeta \wedge \zeta^{\#}}{\bar{c}} \tag{12}$$

P >