

> restart :
 > with(LinearAlgebra) :
 > with(DifferentialGeometry) :
 > DGsetup([c, e, cl, el, z[1], z[2], w[1], w[2], v, l], M2, verbose);
 The following coordinates have been protected:

$$[c, e, cl, el, z_1, z_2, w_1, w_2, v, l]$$

The following vector fields have been defined and protected:

$$[D_c, D_e, D_{cl}, D_{el}, D_{z_1}, D_{z_2}, D_{w_1}, D_{w_2}, D_v, D_l]$$

The following differential 1-forms have been defined and protected:

$$[dc, de, dcl, del, dz_1, dz_2, dw_1, dw_2, dv, dl]$$

frame name: M2

(1)

$$\begin{aligned} > q[1] := \frac{dc}{c} : q[2] := \frac{-Icl de - Ie dcl}{c cl} + \frac{I dce}{c^2} : q[3] := \frac{dcl}{cl} : q[4] \\ &:= \frac{Iel dc + Ic del}{c cl} - \frac{Idcl el}{cl^2} : \end{aligned}$$

$$\begin{aligned} > Q[1] := &-\frac{1}{2} I c cl (w_1 + w_2 z_1) dz_1 + \frac{\frac{1}{4} I c cl (w_1^2 + 2 w_2 z_1 w_1 + z_1^2 w_2^2) dz_2}{-1 + z_2 w_2} \\ &+ \frac{1}{2} I c cl (z_1 + w_1 z_2) dw_1 - \frac{\frac{1}{4} I c cl (z_1^2 + 2 z_2 z_1 w_1 + w_1^2 z_2^2) dw_2}{-1 + z_2 w_2} + \frac{1}{2} c cl (-1 \\ &+ z_2 w_2) dv : \end{aligned}$$

$$\begin{aligned} > Q[2] := &\left(c - \frac{1}{2} w_1 e cl - \frac{1}{2} e w_2 z_1 cl \right) dz_1 \\ &+ \frac{1}{4} \frac{(-4 c w_1 - 4 c w_2 z_1 + e w_1^2 cl + 2 w_1 e w_2 z_1 cl + z_1^2 w_2^2 e cl) dz_2}{-1 + z_2 w_2} + \frac{1}{2} e cl (z_1 \\ &+ w_1 z_2) dw_1 - \frac{1}{4} \frac{e cl (z_1^2 + 2 z_2 z_1 w_1 + w_1^2 z_2^2) dw_2}{-1 + z_2 w_2} - \frac{1}{2} I e cl (-1 + z_2 w_2) dv : \end{aligned}$$

$$\begin{aligned} > Q[3] := &\left(e - \frac{1}{4} \frac{e^2 cl w_1}{c} - \frac{1}{4} \frac{e^2 cl w_2 z_1}{c} \right) dz_1 \\ &+ \frac{1}{4} \frac{1}{cl (-1 + z_2 w_2)} \left(\left(\frac{1}{2} \frac{e^2 cl^2 z_1^2 w_2^2}{c} - 4 e w_2 z_1 cl + \frac{e^2 cl^2 w_2 z_1 w_1}{c} + 4 c \right. \right. \\ &\left. \left. - 4 w_1 e cl + \frac{1}{2} \frac{e^2 cl^2 w_1^2}{c} \right) dz_2 \right) + \frac{1}{4} \frac{e^2 cl (z_1 + w_1 z_2) dw_1}{c} \\ &- \frac{1}{8} \frac{e^2 cl (z_1^2 + 2 z_2 z_1 w_1 + w_1^2 z_2^2) dw_2}{c (-1 + z_2 w_2)} - \frac{1}{4} \frac{I e^2 cl (-1 + z_2 w_2) dv}{c} : \end{aligned}$$

$$> Q[4] := \frac{1}{2} el c (w_1 + w_2 z_1) dz_1 - \frac{1}{4} \frac{el c (w_1^2 + 2 w_2 z_1 w_1 + z_1^2 w_2^2) dz_2}{-1 + z_2 w_2} + \left(cl \right.$$

$$\begin{aligned}
& -\frac{1}{2} el z_1 c - \frac{1}{2} el w_1 z_2 c) dw_1 \\
& -\frac{1}{4} \frac{(4 cl z_1 + 4 cl w_1 z_2 - el c z_1^2 - 2 el c z_2 z_1 w_1 - el c w_1^2 z_2^2) dw_2}{-1 + z_2 w_2} + \frac{1}{2} I el c (-1 \\
& + z_2 w_2) dv :
\end{aligned}$$

$$\begin{aligned}
\text{M2} > Q[5] := & \frac{1}{4} \frac{el^2 c (w_1 + w_2 z_1) dz_1}{cl} - \frac{1}{8} \frac{el^2 c (w_1^2 + 2 w_2 z_1 w_1 + z_1^2 w_2^2) dz_2}{cl (-1 + z_2 w_2)} + \left(el \right. \\
& \left. - \frac{1}{4} \frac{el^2 c z_1}{cl} - \frac{1}{4} \frac{el^2 c w_1 z_2}{cl} \right) dw_1 \\
& - \frac{1}{4} \frac{1}{c (-1 + z_2 w_2)} \left(\left(-\frac{1}{2} \frac{el^2 c^2 w_1^2 z_2^2}{cl} + 4 el w_1 z_2 c - \frac{el^2 c^2 z_2 z_1 w_1}{cl} - 4 cl \right. \right. \\
& \left. \left. + 4 el z_1 c - \frac{1}{2} \frac{el^2 c^2 z_1^2}{cl} \right) dw_2 \right) + \frac{1}{4} \frac{I el^2 c (-1 + z_2 w_2) dv}{cl} :
\end{aligned}$$

$$\begin{aligned}
\text{P} > r[1] := & evalDG \left(q[1] - \frac{\frac{1}{4} I (cl^4 e^2 w_2 - 2 el e c^2 cl^2 + c^4 el^2 z_2)}{c^3 cl^3} \cdot Q[1] \right. \\
& \left. + \frac{(-e cl^2 w_2 + el c^2)}{cl c^2} \cdot Q[2] + \frac{w[2] \cdot cl}{c} \cdot Q[3] \right) :
\end{aligned}$$

$$\begin{aligned}
\text{M2} > r[2] := & evalDG \left(q[2] - I \cdot \frac{\frac{1}{2} I e el (-e cl^2 + el c^2 z_2)}{c^2 cl^3} \cdot Q[1] - \left(-\frac{\frac{1}{2} I e^2 cl w_2}{c^3} \right. \right. \\
& \left. \left. + \frac{\frac{1}{4} I (cl^4 e^2 w_2 - 2 el e c^2 cl^2 + c^4 el^2 z_2)}{c^3 cl^3} \right) \cdot Q[2] - \frac{I \cdot el}{cl} \cdot Q[3] \right. \\
& \left. + \frac{\frac{1}{2} I e (-e cl^2 + 2 el c^2 z_2)}{c^2 cl^2} \cdot Q[4] - \frac{I e z_2}{cl} \cdot Q[5] \right) :
\end{aligned}$$

$$\begin{aligned}
\text{M2} > r[3] := & evalDG \left(q[3] + \frac{\frac{1}{4} I (cl^4 e^2 w_2 - 2 el e c^2 cl^2 + c^4 el^2 z_2)}{c^3 cl^3} \cdot Q[1] \right. \\
& \left. - \frac{-e cl^2 + el c^2 z_2}{c cl^2} \cdot Q[4] + \frac{z_2 c}{cl} \cdot Q[5] \right) :
\end{aligned}$$

$$\begin{aligned}
\text{M2} > r[4] := & evalDG \left(q[4] + I \cdot \frac{\frac{1}{2} I e el (-e cl^2 w_2 + el c^2)}{cl^2 c^3} \cdot Q[1] - \left(\frac{\frac{1}{2} I el^2 c z_2}{cl^3} \right. \right. \\
& \left. \left. - \frac{\frac{1}{4} I (cl^4 e^2 w_2 - 2 el e c^2 cl^2 + c^4 el^2 z_2)}{c^3 cl^3} \right) \cdot Q[4] + \frac{I e}{c} \cdot Q[5] \right) :
\end{aligned}$$

$$+ \frac{\frac{1}{2} \text{I} e l (-2 e c l^2 w_2 + e l c^2)}{c l^2 c^2} \cdot Q[2] + \frac{\text{I} e l w_2}{c} \cdot Q[3] \Bigg) :$$

> $s[1] := \text{evalDG}(r[1] + l \cdot Q[1]) : s[2] := \text{evalDG}(r[2] + l \cdot Q[2]) : s[3] := \text{evalDG}(r[3] + l \cdot Q[1]) : s[4] := \text{evalDG}(r[4] + l \cdot Q[4]) :$

M2 > $Fr := \text{FrameData}([s[1], s[2], s[3], s[4], Q[1], Q[2], Q[3], Q[4], Q[5], dl], P) :$

M2 > $\text{DGsetup}(Fr, [E], [\text{pi}[1], \text{pi}[2], \text{pi}[3], \text{pi}[4], \text{rho}, \text{kappa}, \text{zeta}, \kappa^\#, \zeta^\#, \text{Lambda}]);$
frame name: P (2)

M2 > $\text{visualisation} := \text{proc}(l); \text{subs}(\{w[1] = \text{conjugate}(z[1]), w[2] = \text{conjugate}(z[2]),$
 $c l = \text{conjugate}(c), e l = \text{conjugate}(e), \kappa^\# = \text{conjugate}(\text{kappa}), \zeta^\#$
 $= \text{conjugate}(\text{zeta})\}, l); \text{end proc}:$

P > $\text{ExteriorDerivative}(\text{rho});$

$$\pi_1 \wedge \rho + \pi_3 \wedge \rho + \text{I} \kappa \wedge \kappa^\# \quad (3)$$

P > $\text{ExteriorDerivative}(\text{kappa});$

$$\pi_1 \wedge \kappa + \pi_2 \wedge \rho + \zeta \wedge \kappa^\# \quad (4)$$

P > $\text{ExteriorDerivative}(\text{zeta});$

$$\pi_1 \wedge \zeta + \text{I} \pi_2 \wedge \kappa - \pi_3 \wedge \zeta \quad (5)$$

P > $\text{ExteriorDerivative}(\kappa^\#);$

$$\pi_3 \wedge \kappa^\# + \pi_4 \wedge \rho - \kappa \wedge \zeta^\# \quad (6)$$

P > $\text{ExteriorDerivative}(\zeta^\#);$

$$-\pi_1 \wedge \zeta^\# + \pi_3 \wedge \zeta^\# - \text{I} \pi_4 \wedge \kappa^\# \quad (7)$$

P > $\text{ExteriorDerivative}(\text{pi}[1]);$

$$l \pi_1 \wedge \rho + \frac{1}{2} \frac{(-e c l^2 w_2 + e l c^2) \pi_2 \wedge \rho}{c^2 c l} + l \pi_3 \wedge \rho - \frac{1}{2} \frac{(-e c l^2 + e l c^2 z_2) \pi_4 \wedge \rho}{c c l^2} \quad (8)$$

$$- \text{I} \pi_4 \wedge \kappa - \frac{1}{8} \frac{1}{c^5 c l^4} \left((-4 l c^5 c l^3 e l + 4 l c^3 c l^5 e w_2 - 2 \text{I} e l^2 e c^4 c l^2 - \text{I} e^3 c l^6 w_2^2 \right.$$

$$\left. + 3 \text{I} e l e^2 c^2 c l^4 w_2 + \text{I} e l^3 c^6 z_2 - \text{I} e l^2 c^4 z_2 e c l^2 w_2 \right) \rho \wedge \kappa$$

$$- \frac{\frac{1}{4} \text{I} (e^2 c l^4 w_2^2 - 2 w_2 e l e c^2 c l^2 + c^4 e l^2) \rho \wedge \zeta}{c l^2 c^4} - \frac{1}{8} \frac{1}{c l^5 c^4} \left((2 \text{I} e l e^2 c^2 c l^4 \right.$$

$$\left. + \text{I} e l^3 c^6 z_2^2 - 3 \text{I} e l^2 e c^4 c l^2 z_2 - 4 l c^3 c l^5 e - \text{I} e^3 c l^6 w_2 + 4 l c^5 c l^3 e l z_2 \right.$$

$$\left. + \text{I} e^2 c l^4 w_2 e l c^2 z_2 \right) \rho \wedge \kappa^\# + \frac{\frac{1}{4} \text{I} (-2 z_2 e l e c^2 c l^2 + c l^4 e^2 + c^4 e l^2 z_2^2) \rho \wedge \zeta^\#}{c l^4 c^2}$$

$$- \rho \wedge \Lambda + \zeta \wedge \zeta^\#$$

P > $\text{ExteriorDerivative}(\text{pi}[2]);$

$$l\pi_1 \wedge \kappa + \pi_2 \wedge \pi_3 + \frac{1}{2} \frac{(-ecl^2 w_2 + elc^2) \pi_2 \wedge \kappa}{c^2 cl} + l\pi_3 \wedge \kappa \quad (9)$$

$$- \frac{1}{2} \frac{(-ecl^2 + elc^2 z_2) \pi_4 \wedge \kappa}{ccl^2} - \pi_4 \wedge \zeta + \frac{1}{16} \frac{1}{c^6 cl^6} \left((c^8 el^4 z_2^2 + cl^8 e^4 w_2^2 \right.$$

$$\left. - 16l^2 c^6 cl^6 - 4el^3 c^6 z_2 ecl^2 - 4cl^6 e^3 w_2 elc^2 + 4c^4 el^2 cl^4 e^2 + 2el^2 c^4 z_2 cl^4 e^2 w_2) \right)$$

$$\rho \wedge \kappa) - \frac{\frac{1}{4} I(e^2 cl^4 w_2^2 - 2w_2 el e c^2 cl^2 + c^4 el^2)}{cl^2 c^4} \kappa \wedge \zeta$$

$$- \frac{1}{8} \frac{1}{cl^5 c^4} \left((2Iel e^2 c^2 cl^4 + Iel^3 c^6 z_2^2 - 3Iel^2 e c^4 cl^2 z_2 - 4lc^3 cl^5 e - Ie^3 cl^6 w_2 \right.$$

$$\left. + 4lc^5 cl^3 el z_2 + Ie^2 cl^4 w_2 el c^2 z_2) \kappa \wedge \kappa^\# \right)$$

$$+ \frac{\frac{1}{4} I(-2z_2 el e c^2 cl^2 + cl^4 e^2 + c^4 el^2 z_2^2) \kappa \wedge \zeta^\#}{cl^4 c^2} - \kappa \wedge \Lambda$$

P > Sigma := evalDG $\left(dl - l \cdot s[1] - \frac{1}{2} \frac{(-ecl^2 w_2 + elc^2)}{c^2 cl} s[2] - l \cdot s[3] \right.$

$$+ \frac{1}{2} \frac{(-ecl^2 + elc^2 z_2)}{ccl^2} s[4] - \frac{1}{16} \frac{1}{c^6 cl^6} \left((c^8 el^4 z_2^2 + cl^8 e^4 w_2^2 \right.$$

$$\left. - 16l^2 c^6 cl^6 - 4el^3 c^6 z_2 ecl^2 - 4cl^6 e^3 w_2 elc^2 + 4c^4 el^2 cl^4 e^2 \right.$$

$$\left. + 2el^2 c^4 z_2 cl^4 e^2 w_2) \right) \cdot Q[1] - \frac{1}{8} \frac{1}{c^5 cl^4} \left((-4lc^5 cl^3 el + 4lc^3 cl^5 e w_2 \right.$$

$$\left. - 2Iel^2 e c^4 cl^2 - Ie^3 cl^6 w_2^2 + 3Iel e^2 c^2 cl^4 w_2 + Iel^3 c^6 z_2 - Iel^2 c^4 z_2 ecl^2 w_2) \right.$$

$$\left. \right) \cdot Q[2] - \frac{\frac{1}{4} I(e^2 cl^4 w_2^2 - 2w_2 el e c^2 cl^2 + c^4 el^2)}{cl^2 c^4} Q[3]$$

$$- \frac{1}{8} \frac{1}{cl^5 c^4} \left(2Iel e^2 c^2 cl^4 + Iel^3 c^6 z_2^2 - 3Iel^2 e c^4 cl^2 z_2 - 4lc^3 cl^5 e \right.$$

$$\left. - Ie^3 cl^6 w_2 + 4lc^5 cl^3 el z_2 + Ie^2 cl^4 w_2 el c^2 z_2) \cdot Q[4] \right.$$

$$\left. + \frac{\frac{1}{4} I(-2z_2 el e c^2 cl^2 + cl^4 e^2 + c^4 el^2 z_2^2)}{cl^4 c^2} \cdot Q[5] \right) :$$

$$\begin{aligned}
\mathbf{P} &> \text{expand}\left(\frac{1}{8 \cdot c^5 c l^4} \left((-4 l c^5 c l^3 e l + 4 l c^3 c l^5 e w_2 - 2 I e l^2 e c^4 c l^2 - I e^3 c l^6 w_2^2 \right. \right. \\
&\quad \left. \left. + 3 I e l e^2 c^2 c l^4 w_2 + I e l^3 c^6 z_2 - I e l^2 c^4 z_2 e c l^2 w_2) \right) \right); \\
&-\frac{1}{2} \frac{l e l}{c l} + \frac{1}{2} \frac{c l l e w_2}{c^2} - \frac{\frac{1}{4} I e l^2 e}{c c l^2} - \frac{\frac{1}{8} I c l^2 e^3 w_2^2}{c^5} + \frac{\frac{3}{8} I e l e^2 w_2}{c^3} + \frac{\frac{1}{8} I c e l^3 z_2}{c l^4} \\
&\quad - \frac{\frac{1}{8} I e l^2 z_2 e w_2}{c c l^2}
\end{aligned} \tag{10}$$