

```

> restart :
> with(DifferentialGeometry) :
> with(Tools) : with(LinearAlgebra) :
> DGsetup([l, w, x, y, z, z1], M, verbose);
FD := FrameData([dl, dw, dx, dy, dz, dz1], frame2) :
DGsetup(FD, [E], [Lambda, tau, sigma, rho, zeta, ζ#], verbose);

```

The following coordinates have been protected:

[l, w, x, y, z, z1]

The following vector fields have been defined and protected:

[D\_l, D\_w, D\_x, D\_y, D\_z, D\_z1]

The following differential 1-forms have been defined and protected:

[dl, dw, dx, dy, dz, dz1]

frame name: M

The following coordinates have been protected:

[l, w, x, y, z, z1]

The following vector fields have been defined and protected:

[E1, E2, E3, E4, E5, E6]

The following differential 1-forms have been defined and protected:

[Λ, τ, σ, ρ, ζ, ζ<sup>#</sup>]

frame name: frame2

(1)

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> derive := proc(x, i) local y; y := op(1, x) : if (type(x, `+`) = true) then add(derive(op(j,
x), i), j = 1 .. nops(x)) elif
(type(x, `*`) = true) then expand(derive(y, i) ·  $\frac{x}{y}$  + y · derive( $\frac{x}{y}$ , i))
elif
(type(x, `^`) = true) then op(2, x) · y(op(2, x) - 1) · derive(y, i) elif
(type(x, function) = true) then essai(x, i) elif (type(x, indexed) = true)
then essai(x, i) elif
(type(x, symbol) = true) then essai(x, i) else 0 fi end proc:

```

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M > Der := proc(x) ; evalDG( add(derive(x, i) &wedge W[i], i = 1 .. 6) ); end proc;
Der := proc(x)
DifferentialGeometry:-evalDG( add(DifferentialGeometry:-&wedge(derive(x, i), W[i]), i
= 1 .. 6) )
end proc

```

(2)

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> W := Vector([Lambda, tau, sigma, rho, zeta, ζ#]) :

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M > List1 := [Lambda, tau, sigma, rho, zeta, ζ#];
List1 := [Λ, τ, σ, ρ, ζ, ζ#]

```

(3)

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M > List2 := GenerateForms(List1, 2);

```

```

List2 := [Λ ∧ τ, Λ ∧ σ, Λ ∧ ρ, Λ ∧ ζ, Λ ∧ ζ#, τ ∧ σ, τ ∧ ρ, τ ∧ ζ, τ ∧ ζ#, σ ∧ ρ, σ ∧ ζ,
σ ∧ ζ#, ρ ∧ ζ, ρ ∧ ζ#, ζ ∧ ζ#]

```

(4)

```

M > for i from 1 to 5 do for j from (i + 1) to 6 do

```

$$tr1\left[6\cdot(i-1) - \frac{i\cdot(i-1)}{2} + (j-i)\right] := i : tr2\left[6\cdot(i-1) - \frac{i\cdot(i-1)}{2} + (j-i)\right] := j;$$

**od; od;**

**M** > *DF* := **proc**(omega) **local** *T, Res*; *T* := *GetComponents*(omega, *List2*);  
*Res*[1] := *evalDG*(*add*(*Der*(*T*[*i*]) &wedge *List2*[*i*], *i* = 1 ..15));  
*Res*[2] := *evalDG*(*add*(*T*[*i*] &wedge *dW*[*tr1*[*i*]] &wedge *W*[*tr2*[*i*]], *i* = 1 ..15));  
*Res*[3] := *evalDG*(*add*(*T*[*i*] &wedge *W*[*tr1*[*i*]] &wedge *dW*[*tr2*[*i*]], *i* = 1 ..15) );  
*evalDG*(*Res*[1] + *Res*[2] - *Res*[3]);  
**end proc**;

**M** > *List3* := *GenerateForms*(*List1*, 3) :

**M** > *BI* := **proc**(omega) **local** *R, i*; *R* := *GetComponents*(*DF*(omega), *List3*); **for** *i* **from** 1 **to** 20 **do** *print*(*R*[*i*] = 0); **od**; **end proc**;

**M** > *essai* := **proc**(*x, i*); **if** *i* = 1 **then** *x*[lambda] **elif** *i* = 2 **then** *x*[tau] **elif** *i* = 3 **then** *x*[sigma] **elif** *i* = 4 **then** *x*[rho] **elif** *i* = 5 **then**  $x_{\zeta^{\#}}$  **elif** *i* = 6 **then**  $x_{\zeta^{\#}}$  **fi**; **end proc**;

*essai* := **proc**(*x, i*)

**if** *i* = 1 **then**

*x*[λ]

**elif** *i* = 2 **then**

*x*[τ]

**elif** *i* = 3 **then**

*x*[σ]

**elif** *i* = 4 **then**

*x*[ρ]

**elif** *i* = 5 **then**

*x*[ζ]

**elif** *i* = 6 **then**

$x_{\zeta^{\#}}$

**end if**

**end proc**

**M** > *dW*[1] := *evalDG*(*add*(*U*[*i*]·*W*[*tr1*[*i*]] &wedge *W*[*tr2*[*i*]], *i* = 1 .. 15));

$$dW_1 := U_1 \wedge \tau + U_2 \wedge \sigma + U_3 \wedge \rho + U_4 \wedge \zeta + U_5 \wedge \zeta^{\#} + U_6 \tau \wedge \sigma + U_7 \tau \wedge \rho$$

$$+ U_8 \tau \wedge \zeta + U_9 \tau \wedge \zeta^{\#} + U_{10} \sigma \wedge \rho + U_{11} \sigma \wedge \zeta + U_{12} \sigma \wedge \zeta^{\#} + U_{13} \rho \wedge \zeta + U_{14} \rho \wedge \zeta^{\#}$$

$$+ U_{15} \zeta \wedge \zeta^{\#}$$

**M** > *dW*[2] := *evalDG*(4 · *W*[1] &wedge *W*[2] + *I*<sub>1</sub> · (*W*[2] &wedge *W*[5]) - *I*<sub>1</sub> · (*W*[2] &wedge *W*[6]) + 3 · *I*<sub>1</sub> · (*W*[3] &wedge *W*[4]) + *W*[3] &wedge *W*[5] + *W*[3] &wedge *W*[6]);

$$dW_2 := 4 \wedge \tau + I_1 \tau \wedge \zeta - I_1 \tau \wedge \zeta^{\#} + 3 I_1 \sigma \wedge \rho + \sigma \wedge \zeta + \sigma \wedge \zeta^{\#}$$

**M** > *dW*[3] := *evalDG*(3 · *W*[1] &wedge *W*[3] + *I*<sub>2</sub> · (*W*[2] &wedge *W*[4]) + *I*<sub>3</sub> · (*W*[2] &wedge *W*[5]) +  $I_3^{\#}$  · (*W*[2] &wedge *W*[6]) + *I*<sub>4</sub> · (*W*[3] &wedge *W*[4])

(5)

(6)

(7)

$$- \frac{I_1}{2} \cdot (W[3] \wedge W[5]) + \frac{I_1}{2} \cdot (W[3] \wedge W[6]) + W[4] \wedge W[5] \\ + W[4] \wedge W[6]);$$

$$dW_3 := 3 \Lambda \wedge \sigma + I_2 \tau \wedge \rho + I_3 \tau \wedge \zeta + I_3^\# \tau \wedge \zeta^\# + I_4 \sigma \wedge \rho - \frac{1}{2} I_1 \sigma \wedge \zeta + \frac{1}{2} I_1 \sigma \wedge \zeta^\# \\ + \rho \wedge \zeta + \rho \wedge \zeta^\# \quad (8)$$

$$\begin{aligned} > dW[4] := \text{evalDG}(2 \cdot W[1] \wedge W[4] + I_5 \cdot (W[2] \wedge W[3]) + I_6 \cdot (W[2] \\ \wedge W[4]) + I_7 \cdot (W[2] \wedge W[5]) + I_7^\# \cdot (W[2] \wedge W[6]) \\ + I_8 \cdot (W[3] \wedge W[5]) + I_8^\# \cdot (W[3] \wedge W[6]) - \frac{I_1}{2} \cdot (W[4] \wedge W[5]) + \frac{I_1}{2} \\ \cdot (W[4] \wedge W[6]) + I \cdot (W[5] \wedge W[6])); \end{aligned}$$

$$dW_4 := 2 \Lambda \wedge \rho + I_5 \tau \wedge \sigma + I_6 \tau \wedge \rho + I_7 \tau \wedge \zeta + I_7^\# \tau \wedge \zeta^\# + I_8 \sigma \wedge \zeta + I_8^\# \sigma \wedge \zeta^\# \\ - \frac{1}{2} I_1 \rho \wedge \zeta + \frac{1}{2} I_1 \rho \wedge \zeta^\# + I \zeta \wedge \zeta^\# \quad (9)$$

$$\begin{aligned} \mathbf{M} > dW[5] := \text{evalDG}(W[1] \wedge W[5] + I_9 \cdot (W[2] \wedge W[3]) + I_{10} \cdot (W[2] \\ \wedge W[4]) + I_{11} \cdot (W[2] \wedge W[5]) + I_{12} \cdot (W[2] \wedge W[6]) \\ + I_{13} \cdot (W[3] \wedge W[4]) + I_{14} \cdot (W[3] \wedge W[5])); \end{aligned}$$

$$dW_5 := \Lambda \wedge \zeta + I_9 \tau \wedge \sigma + I_{10} \tau \wedge \rho + I_{11} \tau \wedge \zeta + I_{12} \tau \wedge \zeta^\# + I_{13} \sigma \wedge \rho + I_{14} \sigma \wedge \zeta \quad (10)$$

$$\begin{aligned} \mathbf{M} > dW[6] := \text{evalDG}(W[1] \wedge W[6] + I_9^\# \cdot (W[2] \wedge W[3]) + I_{10}^\# \cdot (W[2] \\ \wedge W[4]) + I_{11}^\# \cdot (W[2] \wedge W[6]) + I_{12}^\# \cdot (W[2] \wedge W[5]) \\ + I_{13}^\# \cdot (W[3] \wedge W[4]) + I_{14}^\# \cdot (W[3] \wedge W[6])); \end{aligned}$$

$$dW_6 := \Lambda \wedge \zeta^\# + I_9^\# \tau \wedge \sigma + I_{10}^\# \tau \wedge \rho + I_{12}^\# \tau \wedge \zeta + I_{11}^\# \tau \wedge \zeta^\# + I_{13}^\# \sigma \wedge \rho + I_{14}^\# \sigma \wedge \zeta^\# \quad (11)$$

**M** >

**frame2** > BI(dW[2]);

$$-4 U_2 = 0$$

$$-4 U_3 = 0$$

$$I_1 - 4 U_4 + I_{1\lambda} = 0$$

$$-I_1 - 4 U_5 - I_{1\lambda} = 0$$

$$3 II_1 + 3 II_{1\lambda} = 0$$

$$0 = 0$$

$$0 = 0$$

$$0 = 0$$

$$0 = 0$$

$$0 = 0$$

$$\begin{aligned}
\dot{I}_{10} + I_{10} + 3 II_1 I_6 + I_1 \dot{I}_{13} - I_1 I_{13} + 4 U_{10} + 3 II_1 &= 0 \\
\dot{I}_{12} + I_{11} + 3 II_1 I_7 - I_1 I_{14} + 4 U_{11} - I_{1\sigma} &= 0 \\
\dot{I}_{11} + I_{12} + 3 II_1 \dot{I}_7 + I_1 \dot{I}_{14} + 4 U_{12} + I_{1\sigma} &= 0 \\
I_2 - 3 II_1 I_3 + 4 U_{13} - I_{1\rho} &= 0 \\
I_2 - 3 II_1 \dot{I}_3 + 4 U_{14} + I_{1\rho} &= 0 \\
I_3 - \dot{I}_3 + 4 U_{15} + I_{1\zeta} + I_{1\zeta}^\# &= 0 \\
6 II_1^2 + I_4 + 3 II_1 I_{1\zeta} &= 0 \\
-6 II_1^2 + I_4 + 3 II_1 I_{1\zeta}^\# &= 0 \\
0 &= 0 \\
0 &= 0
\end{aligned}$$

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**frame2** >  $BI(dW[3]);$

$$\begin{aligned}
3 U_1 &= 0 \\
3 I_2 + I_{2\lambda} &= 0 \\
2 I_3 + I_{3\lambda} &= 0 \\
2 \dot{I}_3 + \dot{I}_{3\lambda} &= 0 \\
2 I_4 - 3 U_3 + I_{4\lambda} &= 0 \\
-\frac{1}{2} I_1 - 3 U_4 - \frac{1}{2} I_{1\lambda} &= 0 \\
\frac{1}{2} I_1 - 3 U_5 + \frac{1}{2} I_{1\lambda} &= 0 \\
0 &= 0 \\
0 &= 0 \\
0 &= 0 \\
-\dot{I}_9 - I_9 + \frac{1}{2} I_1 \dot{I}_{10} - \frac{1}{2} I_1 I_{10} + I_4 I_6 - \dot{I}_3 \dot{I}_{13} - I_3 I_{13} - 3 U_7 + I_{4\tau} - I_{2\sigma} &= 0 \\
\frac{1}{2} I_1 \dot{I}_{12} - \frac{1}{2} I_1 I_{11} + I_4 I_7 - I_3 I_{14} - I_2 I_8 + I_5 - 3 U_8 - \frac{1}{2} I_{1\tau} - I_{3\sigma} &= 0 \\
\frac{1}{2} I_1 \dot{I}_{11} - \frac{1}{2} I_1 I_{12} + I_4 \dot{I}_7 - \dot{I}_3 \dot{I}_{14} - I_2 \dot{I}_8 + I_5 - 3 U_9 + \frac{1}{2} I_{1\tau} - \dot{I}_{3\sigma} &= 0 \\
\dot{I}_{12} + I_{11} - I_1 I_2 + I_6 - I_4 I_3 - I_{3\rho} + I_{2\zeta} &= 0 \\
\dot{I}_{11} + I_{12} + I_1 I_2 + I_6 - I_4 \dot{I}_3 - \dot{I}_{3\rho} + I_{2\zeta}^\# &= 0
\end{aligned}$$

$$\begin{aligned}
& -II_2 + I_7 - I_7^\# + \frac{3}{2} I_1 I_3 + \frac{3}{2} I_1 I_3^\# - I_{3\zeta}^\# + I_{3\zeta^\#} = 0 \\
& I_{14} + \frac{1}{2} I_1 I_4 + 3 II_1 I_3 - I_2 + 3 U_{13} + \frac{1}{2} I_{1\rho} + I_{4\zeta} = 0 \\
& I_{14}^\# - \frac{1}{2} I_1 I_4 + 3 II_1 I_3^\# - I_2 + 3 U_{14} - \frac{1}{2} I_{1\rho} + I_{4\zeta^\#} = 0 \\
& -II_4 + I_8 - I_8^\# + I_3^\# - I_3 + 3 U_{15} - \frac{1}{2} I_{1\zeta} - \frac{1}{2} I_{1\zeta^\#} = 0 \\
& 0 = 0
\end{aligned}$$

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**frame2** >  $BI(dW[4]);$

$$\begin{aligned}
& 5 I_5 + I_{5\lambda} = 0 \\
& 4 I_6 + 2 U_1 + I_{6\lambda} = 0 \\
& 3 I_7 + I_{7\lambda} = 0 \\
& 3 I_7^\# + I_{7\lambda}^\# = 0 \\
& 2 U_2 = 0 \\
& 2 I_8 + I_{8\lambda} = 0 \\
& 2 I_8^\# + I_{8\lambda}^\# = 0 \\
& -\frac{1}{2} I_1 - 2 U_4 - \frac{1}{2} I_{1\lambda} = 0 \\
& \frac{1}{2} I_1 - 2 U_5 + \frac{1}{2} I_{1\lambda} = 0 \\
& 0 = 0 \\
& -\frac{1}{2} I_1 I_9^\# + \frac{1}{2} I_1 I_9 + I_8^\# I_{10}^\# + I_8 I_{10} - I_7^\# I_{13}^\# - I_7 I_{13} - I_5 I_4 + 2 U_6 - I_{6\sigma} + I_{5\rho} = 0 \\
& -I_9^\# + I_8^\# I_{12}^\# + I_8 I_{11} - I_7 I_{14} - I_6 I_8 - I_5 I_1 + I_{8\tau} - I_{7\sigma} + I_{5\zeta} = 0 \\
& I_8^\# I_{11}^\# + I_8 I_{12} - I_7^\# I_{14}^\# - I_6 I_8^\# + I_5 I_1 + II_9 + I_{8\tau}^\# - I_{7\sigma}^\# + I_{5\zeta^\#} = 0 \\
& -I_{10}^\# + \frac{1}{2} I_1 I_{12}^\# - \frac{1}{2} I_1 I_{11} - I_1 I_6 - I_5 + I_2 I_8 - 2 U_8 - \frac{1}{2} I_{1\tau} - I_{7\rho} + I_{6\zeta} = 0 \\
& \frac{1}{2} I_1 I_{11}^\# - \frac{1}{2} I_1 I_{12} + I_1 I_6 - I_5 + II_{10} + I_2 I_8^\# - 2 U_9 + \frac{1}{2} I_{1\tau} - I_{7\rho}^\# + I_{6\zeta^\#} = 0 \\
& II_{11} - II_6 + II_{11} + \frac{3}{2} I_1 I_7 + \frac{3}{2} I_1 I_7^\# + I_8^\# I_3 - I_8 I_3^\# - I_{7\zeta}^\# + I_{7\zeta^\#} = 0 \\
& -I_{13}^\# - \frac{1}{2} I_1 I_{14} + I_8 I_4 + 3 II_1 I_7 - I_6 - 2 U_{11} - \frac{1}{2} I_{1\sigma} - I_{8\rho} = 0 \\
& \frac{1}{2} I_1 I_{14}^\# + II_{13} + I_8^\# I_4 + 3 II_1 I_7^\# - I_6 - 2 U_{12} + \frac{1}{2} I_{1\sigma} - I_{8\rho}^\# = 0
\end{aligned}$$

$$\begin{aligned}
I\dot{I}_{14} + II_{14} + \dot{I}_7 - I_7 - \dot{I}_{8\zeta} + I_{8\zeta} &= 0 \\
\dot{I}_8 - I_8 + 2 U_{15} - \frac{1}{2} I_{1\zeta} - \frac{1}{2} I_{1\zeta} &= 0
\end{aligned}
\tag{14}$$

**frame2** > *BI(dW[5]);*

$$\begin{aligned}
6 I_9 + I_{9\lambda} &= 0 \\
5 I_{10} + I_{10\lambda} &= 0 \\
4 I_{11} + U_1 + I_{11\lambda} &= 0 \\
4 I_{12} + I_{12\lambda} &= 0 \\
4 I_{13} + I_{13\lambda} &= 0 \\
3 I_{14} + U_2 + I_{14\lambda} &= 0 \\
0 &= 0 \\
U_3 &= 0 \\
0 &= 0 \\
-U_5 &= 0 \\
I_{14} I_{10} + I_{13} I_6 - I_{12} \dot{I}_{13} - I_{11} I_{13} - I_9 I_4 + I_{13\tau} - I_{10\sigma} + I_{9\rho} &= 0 \\
I_7 I_{13} - I_8 I_{10} - \frac{1}{2} I_1 I_9 + U_6 + I_{14\tau} - I_{11\sigma} + I_{9\zeta} &= 0 \\
I_{14} I_{12} + I_{13} \dot{I}_7 - I_{12} \dot{I}_{14} - I_{10} \dot{I}_8 + \frac{1}{2} I_1 I_9 - I_{12\sigma} + I_{9\zeta} &= 0 \\
-\frac{1}{2} I_{10} I_1 - I_9 + I_{14} I_2 - I_{13} I_3 + U_7 - I_{11\rho} + I_{10\zeta} &= 0 \\
\frac{1}{2} I_{10} I_1 - I_9 - I_{13} \dot{I}_3 - I_{12\rho} + I_{10\zeta} &= 0 \\
-I I_{10} - I_{14} \dot{I}_3 + I_1 I_{12} + I_1 I_{11} - U_9 - I_{12\zeta} + I_{11\zeta} &= 0 \\
I_{13} I_1 + I_{14} I_4 + 3 I I_{11} I_1 - I_{10} + U_{10} - I_{14\rho} + I_{13\zeta} &= 0 \\
-I_{13} I_1 + 3 I I_{12} I_1 - I_{10} + I_{13\zeta} &= 0 \\
-I I_{13} - \frac{1}{2} I_1 I_{14} + I_{12} - I_{11} - U_{12} + I_{14\zeta} &= 0 \\
-I_{14} - U_{14} &= 0
\end{aligned}
\tag{15}$$

**frame2** >

*EqSet := proc(omega) local R; R  
:= convert(GetComponents(DF(omega), List3), set); end proc;*

**frame2** > *SolBianchi := proc(t) local i, S; S := { }; for i from 1 to nops(t) do S  
:= EqSet(dW[t[i]]) union S od; solve(S, Variables); end proc;*

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*SolBianchi* := **proc**(*t*) (16)

**local** *i, S*;

*S* := { }; **for** *i* **to** *nops*(*t*) **do** *S* := *union*(*EqSet*(*dW*[*t*[*i*]]), *S*) **end do**; *solve*(*S*, *Variables*)

**end proc**

**frame2** > *Variables* := {*seq*(*U<sub>p</sub>*, *i* = 1 ..15)};

*Variables* := {*U<sub>1</sub>*, *U<sub>2</sub>*, *U<sub>3</sub>*, *U<sub>4</sub>*, *U<sub>5</sub>*, *U<sub>6</sub>*, *U<sub>7</sub>*, *U<sub>8</sub>*, *U<sub>9</sub>*, *U<sub>10</sub>*, *U<sub>11</sub>*, *U<sub>12</sub>*, *U<sub>13</sub>*, *U<sub>14</sub>*, *U<sub>15</sub>*} (17)

**frame2** > *SolBianchi*([2]);

**frame2** > *EqSet*(*dW*[2]);

{0, -4 *U<sub>2</sub>*, -4 *U<sub>3</sub>*, 3 *II<sub>1</sub>* + 3 *II<sub>1λ</sub>*, -*I<sub>1</sub>* - 4 *U<sub>5</sub>* - *I<sub>1λ</sub>*, *I<sub>1</sub>* - 4 *U<sub>4</sub>* + *I<sub>1λ</sub>*, -6 *II<sub>1</sub><sup>2</sup>* + *I<sub>4</sub>* + 3 *II<sub>1ζ<sup>#</sup></sub>*, 6 *II<sub>1</sub><sup>2</sup>* + *I<sub>4</sub>* (18)

+ 3 *II<sub>1ζ</sub>*, *I<sub>2</sub>* - 3 *II<sub>1</sub>* *I<sub>3</sub>* + 4 *U<sub>13</sub>* - *I<sub>1ρ</sub>*, *I<sub>2</sub>* - 3 *II<sub>1</sub>* *I<sub>3</sub><sup>#</sup>* + 4 *U<sub>14</sub>* + *I<sub>1ρ</sub>*, *I<sub>3</sub>* - *I<sub>3</sub><sup>#</sup>* + 4 *U<sub>15</sub>* + *I<sub>1ζ</sub>* + *I<sub>1ζ<sup>#</sup></sub>*,

*I<sub>11</sub><sup>#</sup>* + *I<sub>12</sub>* + 3 *II<sub>1</sub>* *I<sub>7</sub><sup>#</sup>* + *I<sub>1</sub>* *I<sub>14</sub><sup>#</sup>* + 4 *U<sub>12</sub>* + *I<sub>1σ</sub>*, *I<sub>12</sub><sup>#</sup>* + *I<sub>11</sub>* + 3 *II<sub>1</sub>* *I<sub>7</sub>* - *I<sub>1</sub>* *I<sub>14</sub>* + 4 *U<sub>11</sub>* - *I<sub>1σ</sub>*, *I<sub>10</sub><sup>#</sup>*

+ *I<sub>10</sub>* + 3 *II<sub>1</sub>* *I<sub>6</sub>* + *I<sub>1</sub>* *I<sub>13</sub><sup>#</sup>* - *I<sub>13</sub>* *I<sub>1</sub>* + 4 *U<sub>10</sub>* + 3 *II<sub>1τ</sub>*}

**frame2** >