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> restart :
> with(DifferentialGeometry) :
> with(Tools) : with(LinearAlgebra) :
> DGsetup([x, y, z, z1], [a], M, verbose);
    The following coordinates have been protected:
                [x, y, z, z1, a]
    The following vector fields have been defined and protected:
                [D_x, D_y, D_z, D_z1, D_a]
    The following differential 1-forms have been defined and protected:
                [dx, dy, dz, dz1, da]
                frame name: M

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(1)

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> Der := proc(x) local y; y := op(1, x) : if (type(x, '+' ) = true) then add(Der(op(i, x)), i = 1
.. nops(x)) elif
    (type(x, '*' ) = true) then expand( (x/y) * Der(y) + y * Der(x/y) ) elif
    (type(x, '^' ) = true) then op(2, x) * y^(op(2, x) - 1) * Der(y) elif
    ( (type(x, function) = true) or (type(x, symbol) = true) ) then S(x) * W[1]
+ Tau(x) * W[2] + L(x) * W[3] + L#(x) * W[4]
    else 0 fi end proc:
derivation := proc(x) : collect( Der(x), [W[1], W[2], W[3], W[4]]) : end proc:

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Les fonctions L et L[#]:

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> L := proc(x) local y; y := op(1, x) : if (type(x, '+' ) = true) then add(L(op(i, x)), i = 1
.. nops(x)) elif
    (type(x, '*' ) = true) then expand( L(y) * x/y + y * L(x/y) ) elif
    (type(x, '^' ) = true) then op(2, x) * y^(op(2, x) - 1) * L(y) elif
    (type(x, function) = true) then 'L'(x) elif
    (type(x, symbol) = true) then 'L'(x) else 0 fi end proc:

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> L# := proc(x) local y; y := op(1, x) : if (type(x, '+' ) = true) then add(L#(op(i, x)), i = 1
.. nops(x)) elif
    (type(x, '*' ) = true) then expand( L#(y) * x/y + y * L#(x/y) ) elif
    (type(x, '^' ) = true) then op(2, x) * y^(op(2, x) - 1) * L#(y) elif
    (type(x, function) = true) then 'L#'(x) elif
    (type(x, symbol) = true) then 'L#'(x) else 0 fi end proc:

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> Tau := proc(x) I * (L(L#(x)) - L#(L(x))) end proc:
M > S := proc(x) simplify(L(Tau(x)) - Tau(L(x))) end proc:
M > S(x);

```

$$IL(L(L^\#(x))) - 2IL(L^\#(L(x))) + IL^\#(L(L(x)))$$

(2)

Les fonctions C₀ et B₀, impliquées dans les normalisations c:= C₀ * a² et B₀:= h*a.

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> C_0 := f:

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> $B_0 := h :$

Les fonctions d et k , impliquées dans les normalisations $d:=j*a$ et $e:=k*a$

Le nouveau coframe initial (tilde, désigné par X), s'exprime en fonction du précédent (chech, désigné par U) par la relation $X:=p.U$.
où p est la matrice:

M > $p := Matrix([[1, 0, 0, 0], [0, 1, 0, 0], [j, 0, 1, 0], [k, 0, 0, 1]]) :$

M > $pinv := MatrixInverse(p) :$

M > $U := pinv.X :$

Le coframe initial chech, (désigné par U), s'exprime en fonction du précédent (chapeau, désigné par V) par la relation $U:=n.V$.
où n est la matrice:

> $n := Matrix([[1, 0, 0, 0], [f, 1, 0, 0], [0, h, 1, 0], [0, hl, 0, 1]]) :$

> $ninv := MatrixInverse(n) :$

M > $V := ninv.Vector([U[1], U[2], U[3], U[4]]) :$

Le coframe chapeau, désigné par V, s'exprime en fonction du coframe initial, W, par la relation $V:=m.W$.

> $m := Matrix([[[B^{-\frac{1}{2}}, 0, 0, 0], [0, 1, 0, 0], [0, 0, B^{-\frac{1}{2}}, 0], [0, 0, 0, B^{\frac{1}{2}}]]]) :$

> $minv := MatrixInverse(m) :$

> $W := minv.Vector([V[1], V[2], V[3], V[4]]) :$

On donne ensuite la matrice de groupe:

M > $g := Matrix([[a^3, 0, 0, 0], [0, a^2, 0, 0], [0, 0, a, 0], [0, 0, 0, a]]) ;$

$$g := \begin{bmatrix} a^3 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$$

(3)

M > $ginv := MatrixInverse(g) :$

M > $Mat := map(evalDG, (ExteriorDerivative(g).ginv)) :$

Il y a une unique forme de MC:

M > $t[1] := \frac{da}{a} :$

M > $FD := FrameData([t[1], dx, dy, dz, dz1], N) :$

$DGsetup(FD, [E], [\alpha[1], \sigma, \rho, \zeta, \zeta^\#], verbose) :$

The following coordinates have been protected:

$[x, y, z, z1, a]$

The following vector fields have been defined and protected:

[E1, E2, E3, E4, E5]

The following differential 1-forms have been defined and protected:

$$[\alpha_1, \sigma, \rho, \zeta, \zeta^\#]$$

(4)

Le coframe 'relevé' est noté Y. Il est relié au coframe de base X par la relation $Y:=g.X$.

$$\mathbf{N} > Y := \text{Vector}([\sigma, \rho, \zeta, \zeta^\#]) :$$

$$\mathbf{N} > X := \text{ginv}.Y :$$

Les equations de courbure de coframe W sont connues:

$$\mathbf{M} > dW[1] := H \cdot (W[1] \wedge W[2]) + F \cdot (W[1] \wedge W[4]) + Q \cdot (W[1] \wedge W[3]) + B \cdot (W[2] \wedge W[4]) + (W[2] \wedge W[3]) :$$

$$\mathbf{M} > dW[2] := G \cdot (W[1] \wedge W[2]) + E \cdot (W[1] \wedge W[4]) + P \cdot (W[1] \wedge W[3]) + A \cdot (W[2] \wedge W[4]) + I \cdot (W[3] \wedge W[4]) :$$

Les equations de courbures du coframe V ont été calculées dans le fichier second step:

$$\mathbf{N} > dV[1] := \text{evalDG}\left(B^{\frac{-1}{2}} \cdot dW[1] + \left(\text{Der}\left(B^{\frac{-1}{2}}\right) \wedge W[1]\right)\right) :$$

$$\mathbf{N} > dV[2] := dW[2] :$$

$$\mathbf{N} > dV[3] := \text{Der}\left(B^{\frac{-1}{2}}\right) \wedge W[3] :$$

$$\mathbf{N} > dV[4] := \text{Der}\left(B^{\frac{1}{2}}\right) \wedge W[4] :$$

$\mathbf{N} >$

Les équations de courbures du coframe U ont été calculées au step 3

$$\mathbf{N} > dU[1] := dV[1] :$$

$$\mathbf{N} > dU[2] := dV[2] + (\text{Der}(f) \wedge V[1]) + f \cdot dV[1] :$$

$$\mathbf{N} > dU[3] := dV[3] + (\text{Der}(h) \wedge V[2]) + h \cdot dV[2] :$$

$$\mathbf{N} > dU[4] := dV[4] + (\text{Der}(h1) \wedge V[2]) + h1 \cdot dV[2] :$$

Les équations de courbures du coframe X sont donc:

$$\mathbf{N} > dX[1] := dU[1] :$$

$$\mathbf{N} > dX[2] := dU[2] :$$

$$\mathbf{N} > dX[3] := dU[3] + (\text{Der}(j) \wedge V[1]) + j \cdot dV[1] :$$

$$\mathbf{N} > dX[4] := dU[4] + (\text{Der}(k) \wedge V[1]) + k \cdot dV[1] :$$

On peut maintenant calculer les équations de courbure du coframe 'relevé':

$$\mathbf{N} > \text{Omega} := \text{map}(\text{evalDG}, g.\text{Vector}([dX[1], dX[2], dX[3], dX[4]])) :$$

$$\mathbf{N} > \text{Mat} := \text{map}(\text{evalDG}, (\text{ExteriorDerivative}(g).\text{ginv})) :$$

N > $Mat2 := Mat \&MatrixWedge Y :$

N > $SE := map(evalDG, (Mat2 \&MatrixPlus Omega)) :$

N > $List := GenerateForms([\alpha[1], \sigma, \rho, \zeta, \zeta^\#], 2) :$

N > $result := \mathbf{proc}(l) \mathbf{local} k, t, X; X := 0 : t := expand(GetComponents(l, List)) : \mathbf{for} k \mathbf{from} 1 \mathbf{to} 10 \mathbf{do} X := X + t[k] \cdot List[k] \mathbf{od}; X; \mathbf{end} \mathbf{proc} :$

$Torsion := \mathbf{proc}(S, i, j) \mathbf{local} k, X; k := 5 \cdot (i - 1) - \frac{i \cdot (i - 1)}{2} + j - i; X$

$:= map(expand, GetComponents(S, List)); X[k]; \mathbf{end} \mathbf{proc} :$

N > $result(SE[1]);$

$$3 \alpha_1 \wedge \sigma + \left(-\frac{1}{2} \frac{L^\#(B) h l}{a^2 B^{3/2}} - \frac{1}{2} \frac{L(B) h}{a^2 \sqrt{B}} + \frac{1}{2} \frac{IL(L^\#(B))}{a^2 B} - \frac{1}{2} \frac{IL^\#(L(B))}{a^2 B} + \frac{j}{a^2} \right. \\ \left. + \frac{k}{a^2} - \frac{\sqrt{B} Q h}{a^2} - \frac{F h l}{a^2 \sqrt{B}} + \frac{H}{a^2} \right) \sigma \wedge \rho + \left(\frac{1}{2} \frac{L(B)}{a \sqrt{B}} - \frac{f}{a} + \frac{\sqrt{B} Q}{a} \right) \sigma \wedge \zeta \\ + \left(\frac{1}{2} \frac{L^\#(B)}{a B^{3/2}} - \frac{f}{a} + \frac{F}{a \sqrt{B}} \right) \sigma \wedge \zeta^\# + \rho \wedge \zeta + \rho \wedge \zeta^\# \quad (5)$$

N > $result(SE[2]);$

$$2 \alpha_1 \wedge \rho + \left(-\frac{1}{2} \frac{L^\#(B) h l f}{a^3 B^{3/2}} - \frac{1}{2} \frac{L(B) h f}{a^3 \sqrt{B}} - \frac{1}{2} \frac{IfL^\#(L(B))}{a^3 B} + \frac{IL^\#(L(f))}{a^3} + \frac{jf}{a^3} \right. \\ \left. + \frac{kf}{a^3} - \frac{\sqrt{B} f Q h}{a^3} - \frac{f F h l}{a^3 \sqrt{B}} + \frac{f H}{a^3} + \frac{L^\#(f) h l}{a^3 \sqrt{B}} + \frac{\sqrt{B} L(f) h}{a^3} + \frac{1}{2} \frac{IfL(L^\#(B))}{a^3 B} \right. \\ \left. - \frac{IL(L^\#(f))}{a^3} + \frac{I h l j}{a^3} - \frac{I h k}{a^3} + \frac{A k}{a^3 \sqrt{B}} - \frac{B h P}{a^3} - \frac{h l E}{a^3} + \frac{\sqrt{B} G}{a^3} \right) \sigma \wedge \rho \\ + \left(\frac{1}{2} \frac{f L(B)}{a^2 \sqrt{B}} - \frac{f^2}{a^2} + \frac{\sqrt{B} f Q}{a^2} - \frac{\sqrt{B} L(f)}{a^2} + \frac{I k}{a^2} - \frac{I h l f}{a^2} + \frac{B P}{a^2} \right) \sigma \wedge \zeta \\ + \left(\frac{1}{2} \frac{f L^\#(B)}{a^2 B^{3/2}} - \frac{f^2}{a^2} + \frac{f F}{a^2 \sqrt{B}} - \frac{L^\#(f)}{a^2 \sqrt{B}} - \frac{I j}{a^2} + \frac{I h f}{a^2} - \frac{A f}{a^2 \sqrt{B}} + \frac{E}{a^2} \right) \sigma \wedge \zeta^\# \\ + \left(\frac{f}{a} + \frac{I h l}{a} \right) \rho \wedge \zeta + \left(\frac{f}{a} - \frac{I h}{a} + \frac{A}{a \sqrt{B}} \right) \rho \wedge \zeta^\# + I \zeta \wedge \zeta^\# \quad (6)$$

N > $result(SE[3]);$

$$\alpha_1 \wedge \zeta + \left(\frac{j^2}{a^4} - \frac{IL(L^\#(j))}{a^4} - \frac{\sqrt{B} j Q h}{a^4} - \frac{j F h l}{a^4 \sqrt{B}} - \frac{1}{2} \frac{j L(B) h}{a^4 \sqrt{B}} - \frac{1}{2} \frac{h L^\#(B) k}{a^4 B^{3/2}} \right. \\ \left. + \frac{h A k}{a^4 \sqrt{B}} - \frac{h h l E}{a^4} + \frac{1}{2} \frac{I h L(L(L^\#(B)))}{a^4 \sqrt{B}} + \frac{1}{2} \frac{I h L^\#(L(L(B)))}{a^4 \sqrt{B}} \right) \quad (7)$$

$$\begin{aligned}
& - \frac{IhL(L^\#(L(B)))}{a^4\sqrt{B}} + \frac{Ihhlj}{a^4} + \frac{IL^\#(L(j))}{a^4} + \frac{\sqrt{B}L(j)h}{a^4} - \frac{L^\#(h)k}{a^4\sqrt{B}} + \frac{L^\#(j)hl}{a^4\sqrt{B}} \\
& - \frac{\sqrt{B}L(h)j}{a^4} - \frac{Bh^2P}{a^4} + \frac{\sqrt{B}hG}{a^4} + \frac{jH}{a^4} + \frac{kj}{a^4} - \frac{Ih^2k}{a^4} + \frac{I\sqrt{B}L(L(L^\#(h)))}{a^4} \\
& - \frac{2I\sqrt{B}L(L^\#(L(h)))}{a^4} + \frac{I\sqrt{B}L^\#(L(L(h)))}{a^4} \Big) \sigma \wedge \rho + \left(\frac{1}{2} \frac{L(B)j}{a^3\sqrt{B}} - \frac{jf}{a^3} \right. \\
& + \frac{\sqrt{B}jQ}{a^3} - \frac{\sqrt{B}L(j)}{a^3} + \frac{Ihk}{a^3} - \frac{Ihlf}{a^3} + \frac{BhP}{a^3} + \frac{\sqrt{B}L(h)f}{a^3} + \frac{1}{2} \frac{L^\#(B)k}{a^3B^{3/2}} \\
& - \frac{1}{2} \frac{L^\#(B)hlf}{a^3B^{3/2}} + \frac{\frac{1}{2}IfL(L^\#(B))}{a^3B} - \frac{\frac{1}{2}IfL^\#(L(B))}{a^3B} - \frac{\frac{1}{2}IL(L(L^\#(B)))}{a^3\sqrt{B}} \\
& \left. + \frac{IL(L^\#(L(B)))}{a^3\sqrt{B}} - \frac{\frac{1}{2}IL^\#(L(L(B)))}{a^3\sqrt{B}} \right) \sigma \wedge \zeta + \left(-\frac{jf}{a^3} + \frac{jF}{a^3\sqrt{B}} - \frac{L^\#(j)}{a^3\sqrt{B}} \right. \\
& - \frac{Ihj}{a^3} + \frac{Ih^2f}{a^3} - \frac{hAf}{a^3\sqrt{B}} + \frac{hE}{a^3} + \frac{L^\#(h)f}{a^3\sqrt{B}} + \frac{1}{2} \frac{L^\#(B)hf}{a^3B^{3/2}} \Big) \sigma \wedge \zeta^\# + \left(\frac{j}{a^2} \right. \\
& + \frac{Ihllh}{a^2} - \frac{\sqrt{B}L(h)}{a^2} + \frac{1}{2} \frac{L^\#(B)hl}{a^2B^{3/2}} - \frac{\frac{1}{2}IL(L^\#(B))}{a^2B} + \frac{\frac{1}{2}IL^\#(L(B))}{a^2B} \Big) \rho \wedge \zeta \\
& + \left(\frac{j}{a^2} - \frac{Ih^2}{a^2} + \frac{hA}{a^2\sqrt{B}} - \frac{L^\#(h)}{a^2\sqrt{B}} - \frac{1}{2} \frac{L^\#(B)h}{a^2B^{3/2}} \right) \rho \wedge \zeta^\# + \left(\frac{Ih}{a} \right. \\
& \left. + \frac{1}{2} \frac{L^\#(B)}{aB^{3/2}} \right) \zeta \wedge \zeta^\#
\end{aligned}$$

N > result(SE[4]);

$$\begin{aligned}
\alpha_1 \wedge \zeta^\# + & \left(\frac{k^2}{a^4} - \frac{IL(L^\#(k))}{a^4} + \frac{IL^\#(L(k))}{a^4} - \frac{\sqrt{B}kQh}{a^4} - \frac{kFhl}{a^4\sqrt{B}} - \frac{Ihllhk}{a^4} \right. \\
& + \frac{IL(L^\#(B))k}{a^4B} - \frac{IL^\#(L(B))k}{a^4B} - \frac{\frac{1}{2}Ihll^\#(L(L(B)))}{a^4\sqrt{B}} - \frac{\frac{1}{2}Ihll(L(L(L^\#(B))))}{a^4\sqrt{B}} \\
& \left. + \frac{Ihll(L(L^\#(L(B))))}{a^4\sqrt{B}} - \frac{1}{2} \frac{L^\#(B)hllk}{a^4B^{3/2}} - \frac{L(B)hk}{a^4\sqrt{B}} + \frac{1}{2} \frac{hll(L(B)j)}{a^4\sqrt{B}} - \frac{BhllPh}{a^4} \right)
\end{aligned} \tag{8}$$

$$\begin{aligned}
& + \frac{hl Ak}{a^4 \sqrt{B}} + \frac{kj}{a^4} - \frac{\sqrt{B} L(hl) j}{a^4} - \frac{L^\#(hl) k}{a^4 \sqrt{B}} + \frac{\sqrt{B} hl G}{a^4} - \frac{hl^2 E}{a^4} + \frac{\sqrt{B} L(k) h}{a^4} \\
& + \frac{L^\#(k) hl}{a^4 \sqrt{B}} + \frac{kH}{a^4} + \frac{Ihl^2 j}{a^4} + \frac{I\sqrt{B} L(L(L^\#(hl)))}{a^4} - \frac{2I\sqrt{B} L(L^\#(L(hl)))}{a^4} \\
& + \frac{I\sqrt{B} L^\#(L(L(hl)))}{a^4} \left. \right) \sigma \wedge \rho + \left(\frac{L(B) k}{a^3 \sqrt{B}} - \frac{kf}{a^3} + \frac{\sqrt{B} k Q}{a^3} - \frac{\sqrt{B} L(k)}{a^3} + \frac{Ihl k}{a^3} \right. \\
& - \frac{Ihl^2 f}{a^3} + \frac{Bhl P}{a^3} + \frac{\sqrt{B} L(hl) f}{a^3} - \frac{1}{2} \frac{L(B) hlf}{a^3 \sqrt{B}} \left. \right) \sigma \wedge \zeta + \left(\frac{1}{2} \frac{L^\#(B) k}{a^3 B^{3/2}} - \frac{kf}{a^3} \right. \\
& + \frac{kF}{a^3 \sqrt{B}} - \frac{L^\#(k)}{a^3 \sqrt{B}} + \frac{Ihl hf}{a^3} - \frac{Ihl j}{a^3} - \frac{hl Af}{a^3 \sqrt{B}} + \frac{hl E}{a^3} + \frac{L^\#(hl) f}{a^3 \sqrt{B}} \\
& - \frac{1}{2} \frac{L(B) j}{a^3 \sqrt{B}} + \frac{1}{2} \frac{L(B) hf}{a^3 \sqrt{B}} + \frac{\frac{1}{2} IL(L(L^\#(B)))}{a^3 \sqrt{B}} + \frac{\frac{1}{2} IL^\#(L(L(B)))}{a^3 \sqrt{B}} \\
& + \frac{\frac{1}{2} IfL^\#(L(B))}{a^3 B} - \frac{IL(L^\#(L(B)))}{a^3 \sqrt{B}} - \frac{\frac{1}{2} IfL(L^\#(B))}{a^3 B} \left. \right) \sigma \wedge \zeta^\# + \left(\frac{k}{a^2} + \frac{Ihl^2}{a^2} \right. \\
& - \frac{\sqrt{B} L(hl)}{a^2} + \frac{1}{2} \frac{L(B) hl}{a^2 \sqrt{B}} \left. \right) \rho \wedge \zeta + \left(\frac{k}{a^2} - \frac{Ihl h}{a^2} + \frac{hl A}{a^2 \sqrt{B}} - \frac{L^\#(hl)}{a^2 \sqrt{B}} \right. \\
& - \frac{1}{2} \frac{L(B) h}{a^2 \sqrt{B}} + \frac{\frac{1}{2} IL(L^\#(B))}{a^2 B} - \frac{\frac{1}{2} IL^\#(L(B))}{a^2 B} \left. \right) \rho \wedge \zeta^\# + \left(\frac{Ihl}{a} \right. \\
& + \frac{1}{2} \frac{L(B)}{a \sqrt{B}} \left. \right) \zeta \wedge \zeta^\#
\end{aligned}$$

$$\mathbf{N} > A := -\frac{1}{2} \frac{L^\#(B) - Q^\# B + B L(B) + B^2 Q}{B} : A^\# := -B^\# \cdot A : B^\# := \frac{1}{B} :$$

$$\mathbf{N} > F := L(B) + B \cdot Q + A :$$

$$\mathbf{N} > E := L(A) + B \cdot P :$$

$$\mathbf{N} > h := -\frac{I}{2} \cdot \left(L^\#(B^\#) \cdot B^{\frac{1}{2}} + \frac{1}{3} \cdot \frac{Q^\#}{B^{\frac{1}{2}}} \right) :$$

$$\mathbf{N} > hl := \frac{I}{2} \cdot \left(\frac{L(B)}{B^{\frac{1}{2}}} + \frac{1}{3} \cdot B^{\frac{1}{2}} \cdot Q \right) :$$

$$\mathbf{N} > f := \frac{1}{2} \left(\frac{L(B)}{B^{\frac{1}{2}}} + B^{\frac{1}{2}} \cdot Q \right);$$

$$\mathbf{N} > j$$

$$:= \text{expand} \left(\frac{1}{a} \left(I a h^2 + \frac{a L^{\#}(B) h}{B^{3/2}} - \frac{1}{2} \frac{a h Q^{\#}}{\sqrt{B}} + \frac{1}{2} \frac{a h L(B)}{\sqrt{B}} + \frac{1}{2} a \sqrt{B} h Q + \frac{a L^{\#}(h)}{\sqrt{B}} \right) \right);$$

$$\mathbf{N} > k := \text{expand} \left(\frac{\left(-I a h l^2 + a \sqrt{B} L(h l) - \frac{1}{2} \frac{a L(B) h l}{\sqrt{B}} \right)}{a} \right);$$

$$\mathbf{N} > \text{result}(SE[1]);$$

$$3 \alpha_1 \wedge \sigma + \left(-\frac{\frac{1}{4} I L(B) Q^{\#}}{B a^2} - \frac{\frac{1}{4} I Q L^{\#}(B)}{B a^2} - \frac{\frac{1}{6} I L^{\#}(B) Q^{\#}}{B^2 a^2} - \frac{\frac{1}{6} I L^{\#}(Q^{\#})}{B a^2} + \frac{\frac{1}{2} I L(L(B))}{a^2} + \frac{\frac{1}{2} I L^{\#}(L^{\#}(B))}{B^2 a^2} + \frac{\frac{1}{6} I B L(Q)}{a^2} + \frac{\frac{1}{2} I L(L^{\#}(B))}{a^2 B} + \frac{H}{a^2} - \frac{\frac{1}{6} I L(B) Q}{a^2} - \frac{\frac{1}{2} I L^{\#}(L(B))}{a^2 B} - \frac{\frac{1}{2} I L(B)^2}{B a^2} - \frac{\frac{1}{18} I B Q^2}{a^2} + \frac{\frac{1}{18} I Q^{\#2}}{B a^2} - \frac{\frac{1}{2} I L^{\#}(B)^2}{B^3 a^2} \right) \sigma \wedge \rho + \frac{1}{2} \frac{\sqrt{B} Q \sigma \wedge \zeta}{a} + \frac{1}{2} \frac{Q^{\#} \sigma \wedge \zeta^{\#}}{\sqrt{B} a} + \rho \wedge \zeta + \rho \wedge \zeta^{\#} \quad (9)$$

$$\mathbf{N} > \text{result}(SE[2]);$$

$$2 \alpha_1 \wedge \rho + \left(-\frac{\frac{1}{18} I Q L^{\#}(B) Q^{\#}}{B^{3/2} a^3} - \frac{\frac{1}{18} I L(B) Q Q^{\#}}{\sqrt{B} a^3} - \frac{\frac{1}{6} I Q L^{\#}(B) L(B)}{B^{3/2} a^3} + \frac{\sqrt{B} G}{a^3} + \frac{\frac{1}{12} I \sqrt{B} Q L(L(B))}{a^3} - \frac{\frac{1}{8} I \sqrt{B} L(B) Q^2}{a^3} + \frac{\frac{1}{4} I L(B) L(L(B))}{\sqrt{B} a^3} - \frac{\frac{1}{18} I Q L^{\#}(Q^{\#})}{\sqrt{B} a^3} - \frac{\frac{1}{4} I L(B) L(Q^{\#})}{\sqrt{B} a^3} + \frac{\frac{1}{12} I L^{\#}(L(B)) Q}{\sqrt{B} a^3} + \frac{\frac{1}{4} I L(B) L^{\#}(Q)}{\sqrt{B} a^3} - \frac{\frac{1}{2} I P L^{\#}(B)}{\sqrt{B} a^3} + \frac{\frac{1}{6} I \sqrt{B} P Q^{\#}}{a^3} - \frac{\frac{1}{8} I L(B)^2 Q^{\#}}{B^{3/2} a^3} + \frac{\frac{1}{54} I Q Q^{\#2}}{\sqrt{B} a^3} + \frac{\frac{1}{6} I Q L^{\#}(L^{\#}(B))}{B^{3/2} a^3} - \frac{\frac{1}{4} I L(B) L^{\#}(L(B))}{B^{3/2} a^3} - \frac{\frac{1}{2} I \sqrt{B} L(B) P}{a^3} \right) \quad (10)$$

$$\begin{aligned}
& + \frac{\frac{1}{12} \text{IQL}(L^\#(B))}{\sqrt{B} a^3} - \frac{\frac{1}{12} \text{I}\sqrt{B} \text{QL}(Q^\#)}{a^3} - \frac{\frac{1}{6} \text{IB}^{3/2} \text{QP}}{a^3} - \frac{\frac{1}{36} \text{I}\sqrt{B} \text{Q}^\# \text{L}(Q)}{a^3} \\
& + \frac{\frac{1}{4} \text{IL}^\#(B) \text{L}(L(B))}{B^{3/2} a^3} + \frac{\frac{1}{4} \text{IL}^\#(B) \text{L}(Q)}{\sqrt{B} a^3} + \frac{\frac{1}{12} \text{I}\sqrt{B} \text{L}^\#(Q) \text{Q}}{a^3} \\
& + \frac{\frac{1}{12} \text{IQ}^\# \text{L}(L(B))}{\sqrt{B} a^3} + \frac{\frac{5}{216} \text{I}\sqrt{B} \text{Q}^2 \text{Q}^\#}{a^3} + \frac{1}{2} \frac{\sqrt{B} \text{HQ}}{a^3} + \frac{1}{2} \frac{\text{HL}(B)}{\sqrt{B} a^3} \\
& - \frac{\frac{1}{8} \text{IL}(B)^3}{B^{3/2} a^3} - \frac{\frac{1}{24} \text{IB}^{3/2} \text{Q}^3}{a^3} - \frac{\frac{1}{2} \text{IL}(L^\#(L(B)))}{\sqrt{B} a^3} - \frac{\frac{1}{2} \text{I}\sqrt{B} \text{L}(L^\#(Q))}{a^3} \\
& + \frac{\frac{1}{2} \text{I}\sqrt{B} \text{L}^\#(L(Q))}{a^3} - \frac{\frac{5}{8} \text{IL}^\#(B) \text{L}(B)^2}{B^{5/2} a^3} - \frac{\frac{1}{6} \text{IQL}^\#(B)^2}{B^{5/2} a^3} + \frac{\frac{1}{4} \text{I}\sqrt{B} \text{L}(B) \text{L}(Q)}{a^3} \\
& + \frac{\frac{1}{12} \text{IB}^{3/2} \text{QL}(Q)}{a^3} - \frac{\frac{1}{24} \text{IQL}(B)^2}{\sqrt{B} a^3} + \frac{\frac{3}{4} \text{IL}(L^\#(B)) \text{L}(B)}{B^{3/2} a^3} - \frac{\frac{1}{8} \text{IQ}^2 \text{L}^\#(B)}{\sqrt{B} a^3} \\
& + \left. \frac{\frac{1}{2} \text{IL}^\#(L(L(B)))}{a^3 \sqrt{B}} \right) \sigma \wedge \rho + \left(\frac{3}{4} \frac{\text{L}(B)^2}{B a^2} + \frac{1}{6} \frac{\text{L}(B) \text{Q}}{a^2} + \frac{11}{36} \frac{B \text{Q}^2}{a^2} - \frac{\text{L}(L(B))}{a^2} \right. \\
& - \left. \frac{2}{3} \frac{B \text{L}(Q)}{a^2} + \frac{B \text{P}}{a^2} \right) \sigma \wedge \zeta + \left(\frac{1}{18} \frac{\text{Q}^{\#2}}{B a^2} + \frac{1}{4} \frac{B \text{Q}^2}{a^2} + \frac{1}{4} \frac{\text{L}(B)^2}{B a^2} - \frac{1}{2} \frac{B \text{L}(Q)}{a^2} \right. \\
& - \frac{1}{2} \frac{\text{L}(L(B))}{a^2} + \frac{\text{L}(B) \text{L}^\#(B)}{B^2 a^2} - \frac{1}{2} \frac{\text{L}^\#(B)^2}{B^3 a^2} - \frac{1}{2} \frac{\text{L}(L^\#(B))}{B a^2} - \frac{1}{2} \frac{\text{L}^\#(L(B))}{B a^2} \\
& - \left. \frac{1}{6} \frac{\text{L}^\#(Q^\#)}{B a^2} + \frac{1}{2} \frac{\text{L}^\#(L^\#(B))}{B^2 a^2} - \frac{1}{6} \frac{\text{L}^\#(B) \text{Q}^\#}{B^2 a^2} - \frac{1}{2} \frac{\text{L}^\#(Q)}{a^2} + \frac{1}{2} \frac{\text{L}(Q^\#)}{a^2} + \frac{B \text{P}}{a^2} \right) \\
& \sigma \wedge \zeta^\# + \frac{1}{3} \frac{\sqrt{B} \text{Q} \rho \wedge \zeta}{a} + \frac{1}{3} \frac{\text{Q}^\# \rho \wedge \zeta^\#}{\sqrt{B} a} + \text{I} \zeta \wedge \zeta^\#
\end{aligned}$$

N > result(SE[3]);

$$\begin{aligned}
\alpha_1 \wedge \zeta + & \left(-\frac{1}{4} \frac{\text{L}^\#(B)^4}{B^6 a^4} - \frac{1}{324} \frac{\text{Q}^{\#4}}{B^2 a^4} - \frac{1}{4} \frac{\text{L}^\#(L^\#(B))^2}{B^4 a^4} - \frac{1}{36} \frac{\text{L}^\#(Q^\#)^2}{B^2 a^4} \right. \\
& - \frac{1}{3} \frac{\text{L}(L^\#(L(Q^\#)))}{a^4} - \frac{1}{2} \frac{\text{L}^\#(L(L(L^\#(B))))}{B a^4} + \frac{1}{6} \frac{\text{L}^\#(L(L(Q^\#)))}{a^4} \\
& \left. - \frac{1}{2} \frac{\text{L}(L(L^\#(L^\#(B))))}{B a^4} + \frac{1}{6} \frac{\text{L}(L(L^\#(Q^\#)))}{a^4} + \frac{1}{2} \frac{\text{L}(L^\#(L^\#(L^\#(B))))}{B^2 a^4} \right)
\end{aligned} \tag{11}$$

$$\begin{aligned}
& -\frac{1}{6} \frac{L(L^\#(L^\#(Q^\#)))}{B a^4} + \frac{1}{6} \frac{L^\#(L(L^\#(Q^\#)))}{B a^4} - \frac{1}{2} \frac{L^\#(L(L^\#(L^\#(B))))}{B^2 a^4} \\
& + \frac{L(L^\#(L(L^\#(B))))}{B a^4} - \frac{1}{24} \frac{L^\#(B) Q L(Q)}{a^4} + \frac{1}{12} \frac{L^\#(B) Q P}{a^4} \\
& + \frac{1}{8} \frac{Q^\# L(B) L(L^\#(B))}{B^2 a^4} - \frac{1}{6} \frac{P L^\#(B) Q^\#}{B a^4} + \frac{1}{12} \frac{L^\#(B) Q^\# L(Q)}{B a^4} \\
& - \frac{1}{18} \frac{Q^\# L(Q^\#) L^\#(B)}{B^2 a^4} + \frac{1}{12} \frac{L(L(B)) L^\#(B) Q^\#}{B^2 a^4} + \frac{1}{6} \frac{L(L^\#(B)) L^\#(B) Q^\#}{B^3 a^4} \\
& + \frac{1}{24} \frac{L^\#(Q) Q^\# L(B)}{B a^4} - \frac{1}{8} \frac{L(B) L^\#(B) L^\#(Q)}{B^2 a^4} - \frac{1}{24} \frac{L^\#(L(B)) L(B) Q^\#}{B^2 a^4} \\
& + \frac{3}{8} \frac{L(B) L^\#(B) L^\#(L(B))}{B^3 a^4} - \frac{1}{24} \frac{L^\#(B) L^\#(Q) Q}{B a^4} + \frac{1}{72} \frac{L^\#(L(B)) Q^\# Q}{B a^4} \\
& + \frac{1}{72} \frac{L^\#(Q) Q^\# Q}{a^4} + \frac{5}{24} \frac{L^\#(B) L^\#(L(B)) Q}{B^2 a^4} - \frac{1}{3} \frac{L^\#(B) L^\#(L(B)) Q^\#}{B^3 a^4} \\
& - \frac{1}{24} \frac{Q^\# L(B) L(Q^\#)}{B a^4} + \frac{1}{24} \frac{Q^\# L(B) L(L(B))}{B a^4} - \frac{1}{144} \frac{Q^\# L(B)^2 Q}{B a^4} \\
& + \frac{1}{24} \frac{Q^\# L(B) L(Q)}{a^4} - \frac{1}{12} \frac{Q^\# L(B) P}{a^4} + \frac{1}{72} \frac{Q^\# Q L(L^\#(B))}{B a^4} - \frac{1}{72} \frac{Q^\# Q L(Q^\#)}{a^4} \\
& + \frac{1}{72} \frac{Q^\# Q L(L(B))}{a^4} - \frac{1}{48} \frac{Q^\# Q^2 L(B)}{a^4} + \frac{1}{72} \frac{B Q^\# Q L(Q)}{a^4} + \frac{1}{48} \frac{L(B)^2 Q L^\#(B)}{B^2 a^4} \\
& - \frac{1}{8} \frac{L(L(B)) L(B) L^\#(B)}{B^2 a^4} - \frac{1}{8} \frac{L(Q) L(B) L^\#(B)}{B a^4} - \frac{5}{8} \frac{L^\#(B) L(B) L(L^\#(B))}{B^3 a^4} \\
& + \frac{1}{8} \frac{L^\#(B) L(B) L(Q^\#)}{B^2 a^4} + \frac{1}{4} \frac{L^\#(B) L(B) P}{B a^4} - \frac{7}{24} \frac{L^\#(B) Q L(L^\#(B))}{B^2 a^4} \\
& - \frac{1}{2} \frac{I H L^\#(B)^2}{B^3 a^4} + \frac{1}{18} \frac{I H Q^\#^2}{B a^4} - \frac{1}{6} \frac{I H L^\#(Q^\#)}{B a^4} + \frac{1}{2} \frac{I G L^\#(B)}{B a^4} \\
& + \frac{1}{2} \frac{I H L^\#(L^\#(B))}{B^2 a^4} - \frac{1}{12} \frac{I H Q Q^\#}{a^4} - \frac{1}{36} \frac{L(B) L^\#(B) Q Q^\#}{B^2 a^4} - \frac{1}{6} \frac{I H L^\#(B) Q^\#}{B^2 a^4}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \frac{IHL(B) L^\#(B)}{B^2 a^4} - \frac{1}{12} \frac{IHL(B) Q^\#}{B a^4} + \frac{1}{4} \frac{IHQL^\#(B)}{B a^4} + \frac{1}{324} \frac{Q^{\#3} Q}{B a^4} \\
& + \frac{1}{144} \frac{Q^2 Q^{\#2}}{a^4} + \frac{1}{36} \frac{P Q^{\#2}}{a^4} - \frac{1}{144} \frac{B Q^3 Q^\#}{a^4} + \frac{1}{36} \frac{L^\#(B)^2 Q^{\#2}}{B^4 a^4} - \frac{1}{6} \frac{L^\#(B)^3 Q^\#}{B^5 a^4} \\
& - \frac{L^\#(B)^3 L(B)}{B^5 a^4} - \frac{1}{4} \frac{L^\#(B)^3 Q}{B^4 a^4} + \frac{1}{2} \frac{L^\#(B)^2 L^\#(L^\#(B))}{B^5 a^4} - \frac{1}{6} \frac{L^\#(B)^2 L^\#(Q^\#)}{B^4 a^4} \\
& + \frac{1}{54} \frac{Q^{\#3} L^\#(B)}{B^3 a^4} + \frac{1}{108} \frac{Q^{\#3} L(B)}{B^2 a^4} - \frac{1}{18} \frac{Q^{\#2} L^\#(L^\#(B))}{B^3 a^4} + \frac{1}{54} \frac{Q^{\#2} L^\#(Q^\#)}{B^2 a^4} \\
& + \frac{5}{16} \frac{L(B)^2 L^\#(B)^2}{B^4 a^4} + \frac{1}{144} \frac{L(B)^2 Q^{\#2}}{B^2 a^4} + \frac{1}{16} \frac{Q^2 L^\#(B)^2}{B^2 a^4} + \frac{1}{6} \frac{L^\#(L^\#(B)) L^\#(Q^\#)}{B^3 a^4} \\
& + \frac{1}{12} \frac{L(B) L^\#(L(Q^\#))}{B a^4} - \frac{1}{4} \frac{L^\#(L(L^\#(B))) Q}{B a^4} + \frac{1}{18} \frac{Q^{\#2} L^\#(L(B))}{B^2 a^4} \\
& + \frac{L^\#(B) L^\#(L(L^\#(B)))}{B^3 a^4} - \frac{1}{4} \frac{L^\#(L(L^\#(B))) L(B)}{B^2 a^4} - \frac{1}{4} \frac{L^\#(B) L^\#(L(Q))}{B a^4} \\
& + \frac{1}{2} \frac{L(L(L^\#(B))) L^\#(B)}{B^2 a^4} + \frac{1}{4} \frac{L^\#(L(L(B))) L^\#(B)}{B^2 a^4} + \frac{1}{12} \frac{L^\#(L(L(B))) Q^\#}{B a^4} \\
& - \frac{3}{4} \frac{L(L^\#(L(B))) L^\#(B)}{B^2 a^4} - \frac{1}{12} \frac{L(L^\#(L(B))) Q^\#}{B a^4} + \frac{1}{4} \frac{P L^\#(B)^2}{B^2 a^4} \\
& - \frac{1}{18} \frac{L(L^\#(B)) Q^{\#2}}{B^2 a^4} - \frac{3}{4} \frac{L(L^\#(B)) L^\#(L^\#(B))}{B^3 a^4} + \frac{1}{12} \frac{L(L^\#(B)) L^\#(Q^\#)}{B^2 a^4} \\
& + \frac{1}{108} \frac{L(Q^\#) Q^{\#2}}{B a^4} - \frac{1}{12} \frac{L(Q^\#) L^\#(L^\#(B))}{B^2 a^4} + \frac{1}{36} \frac{L(Q^\#) L^\#(Q^\#)}{B a^4} \\
& - \frac{1}{8} \frac{L(L(B)) L^\#(B)^2}{B^3 a^4} - \frac{1}{8} \frac{L(Q) L^\#(B)^2}{B^2 a^4} - \frac{1}{72} \frac{L(Q) Q^{\#2}}{a^4} + \frac{1}{16} \frac{L(B)^3 L^\#(B)}{B^3 a^4} \\
& - \frac{1}{48} \frac{L(B)^3 Q^\#}{B^2 a^4} + \frac{1}{48} \frac{Q^3 L^\#(B)}{a^4} - \frac{1}{72} \frac{L(L(B)) Q^{\#2}}{B a^4} - \frac{1}{12} \frac{L(L^\#(L^\#(B))) Q^\#}{B^2 a^4} \\
& + \frac{1}{12} \frac{L(L^\#(Q^\#)) Q^\#}{B a^4} - \frac{1}{12} \frac{L^\#(B) L(L^\#(Q^\#))}{B^2 a^4} + \frac{1}{6} \frac{L^\#(B)^2 L(Q^\#)}{B^3 a^4}
\end{aligned}$$

$$\begin{aligned}
& -\frac{5}{4} \frac{L^\#(B) L(L^\#(L^\#(B)))}{B^3 a^4} + \frac{7}{4} \frac{L(L^\#(B)) L^\#(B)^2}{B^4 a^4} - \frac{1}{6} \frac{L^\#(L(B)) L^\#(Q^\#)}{B^2 a^4} \\
& + \frac{L^\#(L^\#(B)) L^\#(L(B))}{B^3 a^4} - \frac{1}{9} \frac{Q^\# L^\#(L(Q^\#))}{B a^4} + \frac{1}{12} \frac{L^\#(L(Q)) Q^\#}{a^4} \\
& + \frac{1}{4} \frac{L(L^\#(L^\#(B))) Q}{B a^4} + \frac{1}{4} \frac{L(L^\#(L^\#(B))) L(B)}{B^2 a^4} - \frac{1}{12} \frac{L(L^\#(Q)) Q^\#}{a^4} \\
& - \frac{1}{12} \frac{Q L(L^\#(Q^\#))}{a^4} + \frac{1}{6} \frac{L^\#(B) L^\#(L(Q^\#))}{B^2 a^4} - \frac{3}{2} \frac{L^\#(B)^2 L^\#(L(B))}{B^4 a^4} \\
& + \frac{1}{6} \frac{L^\#(L(L^\#(B))) Q^\#}{B^2 a^4} + \frac{1}{36} \frac{L^\#(L^\#(Q^\#)) Q}{B a^4} - \frac{1}{12} \frac{L^\#(L^\#(L^\#(B))) Q}{B^2 a^4} \\
& + \frac{1}{12} \frac{L(B) L^\#(L^\#(Q^\#))}{B^2 a^4} - \frac{1}{4} \frac{L(B) L^\#(L^\#(L^\#(B)))}{B^3 a^4} - \frac{1}{12} \frac{L(B) L(L^\#(Q^\#))}{B a^4} \\
& + \frac{1}{4} \frac{L^\#(B) L(L^\#(Q))}{B a^4} + \frac{1}{12} \frac{Q L^\#(L(Q^\#))}{a^4} - \frac{1}{6} \frac{IG Q^\#}{a^4} - \frac{1}{24} \frac{L^\#(B) Q L(L(B))}{B a^4} \\
& + \frac{1}{16} \frac{L^\#(B) Q^2 L(B)}{B a^4} - \frac{1}{36} \frac{Q Q^\# L^\#(Q^\#)}{B a^4} + \frac{1}{24} \frac{L^\#(B) Q L(Q^\#)}{B a^4} - \frac{1}{36} \frac{B Q^\# Q P}{a^4} \\
& - \frac{1}{12} \frac{L^\#(B)^2 L(B) Q^\#}{B^4 a^4} - \frac{1}{12} \frac{L^\#(B)^2 Q Q^\#}{B^3 a^4} + \frac{1}{6} \frac{L^\#(B) Q^\# L^\#(L^\#(B))}{B^4 a^4} \\
& - \frac{1}{18} \frac{L^\#(B) Q^\# L^\#(Q^\#)}{B^3 a^4} - \frac{1}{8} \frac{L(B)^2 L^\#(B) Q^\#}{B^3 a^4} + \frac{1}{12} \frac{L(B) L^\#(B)^2 Q}{B^3 a^4} \\
& + \frac{L(B) L^\#(B) L^\#(L^\#(B))}{B^4 a^4} + \frac{1}{12} \frac{L(B) L^\#(B) L^\#(Q^\#)}{B^3 a^4} + \frac{1}{12} \frac{L(B) Q^\# L^\#(L^\#(B))}{B^3 a^4} \\
& - \frac{1}{12} \frac{L(B) Q^\# L^\#(Q^\#)}{B^2 a^4} - \frac{1}{24} \frac{Q^2 L^\#(B) Q^\#}{B a^4} + \frac{1}{3} \frac{Q L^\#(B) L^\#(L^\#(B))}{B^3 a^4} \\
& + \frac{1}{18} \frac{Q Q^\# L^\#(L^\#(B))}{B^2 a^4} \Big) \sigma \wedge \rho + \left(\frac{1}{6} \frac{IL(L^\#(Q^\#))}{\sqrt{B} a^3} - \frac{3}{2} \frac{IL^\#(B)^2 L(B)}{B^{7/2} a^3} \right)
\end{aligned}$$

$$\begin{aligned}
& - \frac{11}{216} \frac{I\sqrt{B} Q^2 Q^\#}{a^3} - \frac{1}{6} \frac{I\sqrt{B} P Q^\#}{a^3} + \frac{1}{36} \frac{IQQ^\#^2}{\sqrt{B} a^3} + \frac{1}{18} \frac{IL(B) Q^\#^2}{B^{3/2} a^3} \\
& + \frac{1}{8} \frac{IL(B)^2 L^\#(B)}{B^{5/2} a^3} + \frac{IL(B) L^\#(L^\#(B))}{B^{5/2} a^3} - \frac{1}{6} \frac{IL(B) L^\#(Q^\#)}{B^{3/2} a^3} + \frac{1}{4} \frac{IQ L^\#(L^\#(B))}{B^{3/2} a^3} \\
& - \frac{1}{12} \frac{IQ L^\#(Q^\#)}{\sqrt{B} a^3} - \frac{1}{4} \frac{IL^\#(L(B)) Q}{\sqrt{B} a^3} - \frac{1}{4} \frac{IQ L^\#(B)^2}{B^{5/2} a^3} + \frac{1}{8} \frac{IQ^2 L^\#(B)}{\sqrt{B} a^3} \\
& + \frac{1}{2} \frac{IPL^\#(B)}{\sqrt{B} a^3} - \frac{1}{9} \frac{IQ^\# L(Q^\#)}{\sqrt{B} a^3} + \frac{1}{9} \frac{I\sqrt{B} L(Q) Q^\#}{a^3} - \frac{1}{4} \frac{IL^\#(B) L(L(B))}{B^{3/2} a^3} \\
& + \frac{1}{4} \frac{IL(L^\#(B)) Q}{\sqrt{B} a^3} + \frac{IL^\#(B) L(L^\#(B))}{B^{5/2} a^3} + \frac{1}{4} \frac{IL(L^\#(B)) L(B)}{B^{3/2} a^3} - \frac{1}{4} \frac{IL^\#(B) L(Q)}{\sqrt{B} a^3} \\
& + \frac{1}{6} \frac{IL(L(B)) Q^\#}{\sqrt{B} a^3} + \frac{1}{6} \frac{IL^\#(B) L(Q^\#)}{B^{3/2} a^3} + \frac{1}{6} \frac{IL(L^\#(B)) Q^\#}{B^{3/2} a^3} - \frac{1}{3} \frac{IL(B) L^\#(B) Q^\#}{B^{5/2} a^3} \\
& - \frac{1}{36} \frac{IL(B) Q Q^\#}{\sqrt{B} a^3} - \frac{1}{12} \frac{IQ L^\#(B) Q^\#}{B^{3/2} a^3} - \frac{1}{8} \frac{IL(B)^2 Q^\#}{B^{3/2} a^3} - \frac{1}{4} \frac{IL(B) L^\#(L(B))}{B^{3/2} a^3} \\
& - \frac{1}{2} \frac{IL(L^\#(L^\#(B)))}{B^{3/2} a^3} + \frac{IL(L^\#(L(B)))}{\sqrt{B} a^3} - \frac{1}{2} \frac{IL^\#(L(L(B)))}{\sqrt{B} a^3} \\
& - \frac{1}{2} \frac{IL(L(L^\#(B)))}{\sqrt{B} a^3} \Big) \sigma \wedge \zeta + \left(- \frac{1}{6} \frac{I\sqrt{B} P Q^\#}{a^3} + \frac{1}{8} \frac{IL(B)^2 L^\#(B)}{B^{5/2} a^3} \right. \\
& + \frac{1}{8} \frac{IQ^2 L^\#(B)}{\sqrt{B} a^3} + \frac{1}{2} \frac{IPL^\#(B)}{\sqrt{B} a^3} - \frac{1}{4} \frac{IL^\#(B) L(L(B))}{B^{3/2} a^3} - \frac{1}{4} \frac{IL^\#(B) L(Q)}{\sqrt{B} a^3} \\
& - \frac{1}{24} \frac{I\sqrt{B} Q^2 Q^\#}{a^3} - \frac{1}{24} \frac{IL(B)^2 Q^\#}{B^{3/2} a^3} - \frac{1}{4} \frac{IL^\#(B) L^\#(Q)}{B^{3/2} a^3} - \frac{1}{2} \frac{IL^\#(B)^2 Q^\#}{B^{7/2} a^3} \\
& + \frac{1}{2} \frac{IL^\#(B)^2 L(B)}{B^{7/2} a^3} + \frac{1}{12} \frac{IL^\#(L(B)) Q^\#}{B^{3/2} a^3} - \frac{1}{6} \frac{IQ^\# L^\#(Q^\#)}{B^{3/2} a^3} + \frac{1}{12} \frac{IL^\#(Q) Q^\#}{\sqrt{B} a^3} \\
& + \frac{2IL^\#(B) L^\#(L^\#(B))}{B^{7/2} a^3} - \frac{1}{4} \frac{IL^\#(B) L^\#(L(B))}{B^{5/2} a^3} + \frac{1}{3} \frac{IL^\#(L^\#(B)) Q^\#}{B^{5/2} a^3}
\end{aligned}$$

$$\begin{aligned}
& - \frac{\frac{1}{12} \text{I} Q^\# L(Q^\#)}{\sqrt{B} a^3} - \frac{\frac{1}{4} \text{I} L^\#(B) L(L^\#(B))}{B^{5/2} a^3} + \frac{\frac{1}{4} \text{I} L^\#(B) L(Q^\#)}{B^{3/2} a^3} \\
& + \frac{\frac{1}{12} \text{I} L(L^\#(B)) Q^\#}{B^{3/2} a^3} + \frac{\frac{1}{12} \text{I} \sqrt{B} L(Q) Q^\#}{a^3} - \frac{\frac{1}{6} \text{I} L(B) L^\#(B) Q^\#}{B^{5/2} a^3} \\
& + \frac{\frac{1}{12} \text{I} Q^\# L(L(B))}{\sqrt{B} a^3} - \frac{\frac{3}{2} \text{I} L^\#(B)^3}{B^{9/2} a^3} + \frac{\frac{1}{6} \text{I} L^\#(L^\#(Q^\#))}{B^{3/2} a^3} - \frac{\frac{1}{2} \text{I} L^\#(L^\#(L^\#(B)))}{B^{5/2} a^3} \\
& + \frac{\frac{1}{54} \text{I} Q^{\#3}}{B^{3/2} a^3} \left. \right) \sigma \wedge \zeta^\# + \left(- \frac{\frac{1}{2} \text{I} L^\#(B)^2}{B^3 a^2} + \frac{\frac{1}{18} \text{I} Q^{\#2}}{B a^2} - \frac{\frac{1}{6} \text{I} L^\#(B) Q^\#}{B^2 a^2} \right. \\
& + \frac{\text{I} L(B) L^\#(B)}{B^2 a^2} - \frac{\frac{1}{12} \text{I} L(B) Q^\#}{B a^2} + \frac{\frac{1}{4} \text{I} Q L^\#(B)}{B a^2} - \frac{\frac{1}{18} \text{I} Q Q^\#}{a^2} + \frac{\frac{1}{2} \text{I} L^\#(L^\#(B))}{B^2 a^2} \\
& \left. - \frac{\frac{1}{6} \text{I} L^\#(Q^\#)}{B a^2} - \frac{\text{I} L(L^\#(B))}{B a^2} + \frac{\frac{1}{6} \text{I} L(Q^\#)}{a^2} + \frac{\frac{1}{2} \text{I} L^\#(L(B))}{B a^2} \right) \rho \wedge \zeta \\
& + \frac{1}{6} \frac{Q^\# \zeta \wedge \zeta^\#}{\sqrt{B} a}
\end{aligned}$$

N > result(SE[4]);

$$\begin{aligned}
\alpha_1 \wedge \zeta^\# + & \left(\frac{1}{24} \frac{L^\#(B) Q L(Q)}{a^4} + \frac{1}{12} \frac{L^\#(B) Q P}{a^4} - \frac{1}{8} \frac{Q^\# L(B) L(L(B))}{B a^4} \right. \\
& + \frac{1}{48} \frac{Q^\# L(B)^2 Q}{B a^4} + \frac{1}{24} \frac{Q^\# L(B) L(Q)}{a^4} - \frac{1}{12} \frac{Q^\# L(B) P}{a^4} - \frac{1}{24} \frac{Q^\# Q L(L(B))}{a^4} \\
& - \frac{1}{48} \frac{Q^\# Q^2 L(B)}{a^4} - \frac{1}{72} \frac{B Q^\# Q L(Q)}{a^4} - \frac{1}{16} \frac{L(B)^2 Q L^\#(B)}{B^2 a^4} \\
& + \frac{3}{8} \frac{L(L(B)) L(B) L^\#(B)}{B^2 a^4} - \frac{5}{24} \frac{L(Q) L(B) L^\#(B)}{B a^4} + \frac{1}{4} \frac{L^\#(B) L(B) P}{B a^4} \\
& - \frac{5}{1296} \frac{B Q^3 Q^\#}{a^4} - \frac{3}{16} \frac{L(B)^3 L^\#(B)}{B^3 a^4} + \frac{1}{16} \frac{L(B)^3 Q^\#}{B^2 a^4} + \frac{5}{432} \frac{Q^3 L^\#(B)}{a^4} \\
& \left. + \frac{5}{1296} \frac{B^2 Q^4}{a^4} + \frac{3}{2} \frac{L(L^\#(L(L(B))))}{a^4} + \frac{1}{2} \frac{B L(L^\#(L(Q)))}{a^4} - \frac{L^\#(L(L(L(B))))}{a^4} \right)
\end{aligned} \tag{12}$$

$$\begin{aligned}
& - \frac{1}{3} \frac{BL^\#(L(L(Q)))}{a^4} - \frac{3}{16} \frac{L(B)^4}{B^2 a^4} - \frac{1}{2} \frac{L(L(L^\#(L(B))))}{a^4} - \frac{1}{6} \frac{BL(L(L^\#(Q)))}{a^4} \\
& - \frac{1}{4} \frac{L(L(B))^2}{a^4} - \frac{1}{36} \frac{B^2 L(Q)^2}{a^4} - \frac{1}{4} \frac{L(L^\#(B)) L(B) Q}{B a^4} + \frac{1}{12} \frac{QL(B) L(L(B))}{a^4} \\
& - \frac{1}{36} \frac{BQL(B) L(Q)}{a^4} + \frac{1}{4} \frac{L^\#(L(B)) QL(B)}{B a^4} - \frac{1}{36} \frac{L(B) QL^\#(Q)}{a^4} \\
& + \frac{1}{12} \frac{QL(B) L(Q^\#)}{a^4} + \frac{1}{6} \frac{BQL(B) P}{a^4} + \frac{1}{6} \frac{IBGQ}{a^4} + \frac{1}{36} \frac{IBHQ^2}{a^4} \\
& + \frac{1}{6} \frac{IBHL(Q)}{a^4} - \frac{1}{4} \frac{IHL(B)^2}{B a^4} + \frac{1}{6} \frac{IHL(B) Q}{a^4} + \frac{5}{24} \frac{L^\#(B) QL(L(B))}{B a^4} \\
& + \frac{13}{144} \frac{L^\#(B) Q^2 L(B)}{B a^4} - \frac{1}{36} \frac{BQ^\#QP}{a^4} + \frac{1}{36} \frac{B^2 Q^2 P}{a^4} - \frac{1}{8} \frac{L(B)^3 Q}{B a^4} \\
& - \frac{1}{72} \frac{L(B)^2 Q^2}{a^4} + \frac{3}{8} \frac{L(B)^2 L(L(B))}{B a^4} + \frac{1}{24} \frac{L(B)^2 L(Q)}{a^4} + \frac{5}{216} \frac{BQ^3 L(B)}{a^4} \\
& - \frac{1}{8} \frac{L(B)^2 L^\#(Q)}{B a^4} - \frac{1}{4} \frac{L(L(L(B))) L^\#(B)}{B a^4} + \frac{1}{8} \frac{L(B)^2 L(Q^\#)}{B a^4} + \frac{1}{4} \frac{L(B)^2 P}{a^4} \\
& + \frac{1}{12} \frac{L(L(L(B))) Q^\#}{a^4} + \frac{1}{36} \frac{BL(L(Q)) Q^\#}{a^4} - \frac{1}{216} \frac{BL^\#(Q) Q^2}{a^4} \\
& - \frac{1}{6} \frac{BL(L(B)) L(Q)}{a^4} - \frac{1}{2} \frac{L(L^\#(B)) L(L(B))}{B a^4} + \frac{1}{2} \frac{L(L(L^\#(B))) L(B)}{B a^4} \\
& + \frac{1}{72} \frac{BQ^2 L(Q^\#)}{a^4} + \frac{1}{24} \frac{BQ^2 L(L(B))}{a^4} + \frac{1}{216} \frac{B^2 Q^2 L(Q)}{a^4} - \frac{1}{4} \frac{L(B) L^\#(L(Q))}{a^4} \\
& + \frac{3}{8} \frac{L(L^\#(B)) L(B)^2}{B^2 a^4} - \frac{3}{2} \frac{L(B) L(L^\#(L(B)))}{B a^4} + \frac{1}{12} \frac{L(L(B)) L^\#(Q)}{a^4} \\
& + \frac{1}{36} \frac{BL(Q) L^\#(Q)}{a^4} + \frac{1}{6} \frac{L(L^\#(L(B))) Q}{a^4} - \frac{1}{72} \frac{L(L^\#(B)) Q^2}{a^4} \\
& + \frac{1}{18} \frac{BQL(L^\#(Q))}{a^4} - \frac{3}{8} \frac{L(B)^2 L^\#(L(B))}{B^2 a^4} - \frac{1}{12} \frac{L^\#(B) L(L(Q))}{a^4}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{12} \frac{L^\#(L(B)) L(Q)}{a^4} + \frac{1}{6} \frac{L(B) L(L^\#(Q))}{a^4} + \frac{3}{4} \frac{L(L(B)) L^\#(L(B))}{B a^4} \\
& - \frac{1}{72} \frac{L^\#(L(B)) Q^2}{a^4} - \frac{1}{12} \frac{B Q L^\#(L(Q))}{a^4} + \frac{3}{4} \frac{L(B) L^\#(L(L(B)))}{B a^4} \\
& - \frac{1}{4} \frac{L^\#(L(L(B))) Q}{a^4} + \left. \frac{\frac{1}{2} IHL(L(B))}{a^4} + \frac{\frac{1}{2} IGL(B)}{a^4} \right) \sigma \wedge \rho + \left(-\frac{\frac{1}{8} IL(B)^2 Q}{\sqrt{B} a^3} \right. \\
& + \frac{\frac{1}{8} I\sqrt{B} Q^2 L(B)}{a^3} + \frac{\frac{3}{4} IL(L(B)) L(B)}{\sqrt{B} a^3} - \frac{\frac{1}{4} I\sqrt{B} L(Q) L(B)}{a^3} \\
& + \frac{\frac{1}{4} I\sqrt{B} L(L(B)) Q}{a^3} + \frac{\frac{1}{12} IB^{3/2} QL(Q)}{a^3} + \frac{\frac{1}{2} I\sqrt{B} PL(B)}{a^3} + \frac{\frac{1}{6} IB^{3/2} PQ}{a^3} \\
& \left. - \frac{\frac{3}{8} IL(B)^3}{B^{3/2} a^3} + \frac{\frac{5}{216} IB^{3/2} Q^3}{a^3} - \frac{\frac{1}{2} I\sqrt{B} L(L(L(B)))}{a^3} - \frac{\frac{1}{6} IB^{3/2} L(L(Q))}{a^3} \right) \\
& \sigma \wedge \zeta + \left(\frac{\frac{1}{12} I\sqrt{B} L(Q) Q^\#}{a^3} + \frac{\frac{1}{8} I\sqrt{B} Q^2 L(B)}{a^3} - \frac{\frac{1}{4} I\sqrt{B} L(Q) L(B)}{a^3} \right. \\
& + \frac{\frac{1}{2} I\sqrt{B} PL(B)}{a^3} + \frac{\frac{1}{6} IB^{3/2} PQ}{a^3} - \frac{\frac{1}{12} IQL^\#(B)^2}{B^{5/2} a^3} - \frac{\frac{1}{36} I\sqrt{B} Q^\# Q^2}{a^3} \\
& - \frac{\frac{1}{6} IL^\#(B) L(Q)}{\sqrt{B} a^3} - \frac{\frac{1}{4} IL(B)^2 Q^\#}{B^{3/2} a^3} + \frac{\frac{1}{108} IQQ^\#^2}{\sqrt{B} a^3} + \frac{\frac{1}{12} IQL^\#(L^\#(B))}{B^{3/2} a^3} \\
& - \frac{\frac{1}{36} IQL^\#(Q^\#)}{\sqrt{B} a^3} + \frac{\frac{1}{4} IL(B) L(Q^\#)}{\sqrt{B} a^3} + \frac{\frac{1}{3} IL^\#(L(B)) Q}{\sqrt{B} a^3} - \frac{\frac{1}{12} IL(B) L^\#(Q)}{\sqrt{B} a^3} \\
& + \frac{\frac{1}{36} I\sqrt{B} L^\#(Q) Q}{a^3} + \frac{\frac{1}{4} IQ^\# L(L(B))}{\sqrt{B} a^3} - \frac{\frac{1}{4} IL(B) L(L(B))}{\sqrt{B} a^3} \\
& - \frac{\frac{1}{12} I\sqrt{B} QL(L(B))}{a^3} - \frac{\frac{1}{12} IB^{3/2} QL(Q)}{a^3} + \frac{\frac{1}{24} IQL(B)^2}{\sqrt{B} a^3} \\
& \left. - \frac{\frac{1}{2} IL(L^\#(B)) L(B)}{B^{3/2} a^3} + \frac{\frac{1}{18} IQ^2 L^\#(B)}{\sqrt{B} a^3} + \frac{IL(B) L^\#(L(B))}{B^{3/2} a^3} - \frac{\frac{1}{3} IQL(L^\#(B))}{\sqrt{B} a^3} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{\frac{1}{12} I\sqrt{B} QL(Q^\#)}{a^3} + \frac{\frac{1}{6} IQL^\#(B) L(B)}{B^{3/2} a^3} - \frac{\frac{1}{36} IQL^\#(B) Q^\#}{B^{3/2} a^3} - \frac{\frac{1}{12} IL(B) QQ^\#}{\sqrt{B} a^3} \\
& + \frac{\frac{1}{8} IL(B)^3}{B^{3/2} a^3} + \frac{\frac{1}{24} IB^{3/2} Q^3}{a^3} - \frac{\frac{1}{6} I\sqrt{B} L^\#(L(Q))}{a^3} + \frac{\frac{1}{2} IL(L(L^\#(B)))}{a^3 \sqrt{B}} \\
& - \frac{IL(L^\#(L(B)))}{a^3 \sqrt{B}} \left. \right) \sigma \wedge \zeta^\# + \left(-\frac{\frac{1}{2} IL(B)^2}{B a^2} + \frac{\frac{1}{2} IL(L^\#(B))}{a^2 B} - \frac{IL^\#(L(B))}{B a^2} \right. \\
& - \frac{\frac{1}{6} IL(B) Q}{a^2} + \frac{\frac{1}{4} IL(B) Q^\#}{B a^2} + \frac{\frac{1}{18} IQQ^\#}{a^2} - \frac{\frac{1}{12} IQL^\#(B)}{B a^2} - \frac{\frac{1}{18} IBQ^2}{a^2} \\
& \left. + \frac{\frac{1}{2} IL(L(B))}{a^2} + \frac{\frac{1}{6} IBL(Q)}{a^2} - \frac{\frac{1}{6} IL^\#(Q)}{a^2} \right) \rho \wedge \zeta^\# - \frac{1}{6} \frac{\sqrt{B} Q\zeta \wedge \zeta^\#}{a}
\end{aligned}$$

N > `exp1 := expand(a^2·Torsion(SE[4], 3, 5));`

$$\begin{aligned}
\text{exp1} := & -\frac{\frac{1}{2} IL(B)^2}{B} + \frac{\frac{1}{2} IL(L^\#(B))}{B} - \frac{IL^\#(L(B))}{B} - \frac{1}{6} IL(B) Q + \frac{\frac{1}{4} IL(B) Q^\#}{B} \\
& + \frac{1}{18} IQQ^\# - \frac{\frac{1}{12} IQL^\#(B)}{B} - \frac{1}{18} IBQ^2 + \frac{1}{2} IL(L(B)) + \frac{1}{6} IBL(Q) \\
& - \frac{1}{6} IL^\#(Q)
\end{aligned} \tag{13}$$

N > `exp2 := expand(a^2·Torsion(SE[3], 3, 4));`

$$\begin{aligned}
\text{exp2} := & -\frac{\frac{1}{2} IL^\#(B)^2}{B^3} + \frac{\frac{1}{18} IQ^\#}{B} - \frac{\frac{1}{6} IL^\#(B) Q^\#}{B^2} + \frac{IL(B) L^\#(B)}{B^2} - \frac{\frac{1}{12} IL(B) Q^\#}{B} \\
& + \frac{\frac{1}{4} IQL^\#(B)}{B} - \frac{1}{18} IQQ^\# + \frac{\frac{1}{2} IL^\#(L^\#(B))}{B^2} - \frac{\frac{1}{6} IL^\#(Q^\#)}{B} - \frac{IL(L^\#(B))}{B} \\
& + \frac{1}{6} IL(Q^\#) + \frac{\frac{1}{2} IL^\#(L(B))}{B}
\end{aligned} \tag{14}$$

N > `conj := proc(x) local y; y := op(1, x) : if (type(x, '+') = true) then add(conj(op(i, x)), i = 1 .. nops(x)) elif`

`(type(x, '*') = true) then simplify(conj(y) · conj($\left(\frac{x}{y}\right)$)) elif`

`(type(x, '^') = true) then conj(y)op(2, x) elif (type(x, 'complex') = true) then conjugate(x) elif`

`x = Q then Q# elif x = P then P# elif x = B then B# elif x = A then A#`

`elif x = Q# then Q elif x = P# then P elif x = A# then A elif x = B#`

`then B elif x = a then a elif (type(x, function) = true) then if op(0, x) = L`

then ($L^\#(\text{conj}(y))$) **else** ($L(\text{conj}(y))$) **end if end if;end proc:**

N > *conjugue* := **proc**(x) ; *expand*($\text{Sub}(\text{conj}(x))$) **end proc:**

N > *substitution* := **proc**(s) ; *subs*($\left(\text{sqrt}\left(\frac{1}{B}\right) = \frac{1}{\text{sqrt}(B)}, s\right)$) **end proc;**

substitution := **proc**(s) *subs*($\text{sqrt}(1/B) = 1/\text{sqrt}(B), s$) **end proc**

(15)

N > *Sub* := **proc**(s) ; **if** ($\text{type}(s, '+') = \text{true}$) **then** *add*($\text{Sub}(\text{op}(i, s))$), $i = 1 .. \text{nops}(s)$)
elif ($\text{type}(s, '^')$ **and** $\text{op}(2, s) < 0$) = *true*

then $\frac{1}{\text{substitution}(\text{op}(1, s)^{-\text{op}(2, s)})}$ **elif** ($\text{type}(s, '*') = \text{true}$) **then** *Sub*($\text{op}(1, s)$)

Sub($\frac{s}{\text{op}(1, s)})$) **else** *substitution*(s) **fi end proc:**

N > *conjugue*(h);

$$\frac{\frac{1}{2} IL(B)}{\sqrt{B}} + \frac{1}{6} I\sqrt{B} Q$$

(16)

N > *conjugue*(*exp2*) - *exp1*;

0

(17)

N > j ;

$$-\frac{\frac{1}{2} IL^\#(B)^2}{B^3} + \frac{\frac{1}{18} IQ^\#2}{B} - \frac{\frac{1}{6} IL^\#(B) Q^\#}{B^2} + \frac{\frac{1}{4} IL(B) L^\#(B)}{B^2} - \frac{\frac{1}{12} IL(B) Q^\#}{B}$$

(18)

$$+ \frac{\frac{1}{4} IQ L^\#(B)}{B} - \frac{\frac{1}{12} IQ Q^\#}{B} + \frac{\frac{1}{2} IL^\#(L^\#(B))}{B^2} - \frac{\frac{1}{6} IL^\#(Q^\#)}{B}$$

N > k ;

$$\frac{1}{6} IL(B) Q + \frac{1}{36} IB Q^2 + \frac{1}{2} IL(L(B)) + \frac{1}{6} IBL(Q) - \frac{\frac{1}{4} IL(B)^2}{B}$$

(19)

N > *conjugue*(j);

$$-\frac{\frac{1}{2} IL(B)^2}{B} - \frac{1}{18} IB Q^2 - \frac{1}{6} IL(B) Q - \frac{\frac{1}{4} IL(B) L^\#(B)}{B^2} - \frac{\frac{1}{12} IQ L^\#(B)}{B}$$

(20)

$$+ \frac{\frac{1}{4} IL(B) Q^\#}{B} + \frac{1}{12} IQ Q^\# + \frac{1}{2} IL(L(B)) + \frac{1}{6} IBL(Q)$$

N > $S(x)$;

$$IL(L(L^\#(x))) - 2IL(L^\#(L(x))) + IL^\#(L(L(x)))$$

(21)

> *simplify*($L^\#(\text{Tau}(x)) - \text{Tau}(L^\#(x))$);

$$2IL^\#(L(L^\#(x))) - IL^\#(L^\#(L(x))) - IL(L^\#(L^\#(x)))$$

(22)