

```

> restart :
> with(DifferentialGeometry) :
> with(Tools) : with(LinearAlgebra) :
> DGsetup([a, a1, w, x, y, z1, z], M, verbose);
FD := FrameData([da, da1, dw, dx, dy, dz1, dz], frame2) :
DGsetup(FD, [E], [alpha,  $\alpha^\#$ ,  $\sigma^\#$ , sigma, rho,  $\zeta^\#$ , zeta], verbose);

```

The following coordinates have been protected:

[a, a1, w, x, y, z1, z]

The following vector fields have been defined and protected:

[D_a, D_a1, D_w, D_x, D_y, D_z1, D_z]

The following differential 1-forms have been defined and protected:

[da, da1, dw, dx, dy, dz1, dz]

frame name: M

The following coordinates have been protected:

[a, a1, w, x, y, z1, z]

The following vector fields have been defined and protected:

[E1, E2, E3, E4, E5, E6, E7]

The following differential 1-forms have been defined and protected:

[α , $\alpha^\#$, $\sigma^\#$, σ , ρ , $\zeta^\#$, ζ]

frame name: frame2

(1)

```

> derive := proc(x, i) local y; y := op(1, x) : if (type(x, `+`) = true) then add(derive(op(j,
x), i), j = 1 .. nops(x)) elif
(type(x, `*`) = true) then expand(derive(y, i) *  $\frac{x}{y}$  + y * derive( $\frac{x}{y}$ , i))
elif
(type(x, `^`) = true) then op(2, x) * y(op(2, x) - 1) * derive(y, i) elif
(type(x, function) = true) then essai(x, i) elif (type(x, indexed) = true)
then essai(x, i) elif
(type(x, symbol) = true) then essai(x, i) else 0 fi end proc:
essai := proc(x, i); if i = 1 then x[alpha] elif i = 2 then x[ $\alpha^\#$ ] elif i = 3 then x[ $\sigma^\#$ ] elif i = 4
then x[sigma] elif i = 5 then x[rho] elif i = 6 then x[ $\zeta^\#$ ] elif i = 7 then x_zeta fi; end proc;

```

```
essai := proc(x, i)
```

```
if i = 1 then
```

```
x[ $\alpha$ ]
```

```
elif i = 2 then
```

```
x[ $\alpha^\#$ ]
```

```
elif i = 3 then
```

```
x[ $\sigma^\#$ ]
```

```
elif i = 4 then
```

```
x[ $\sigma$ ]
```

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elif $i=5$ **then**

$x[\rho]$

elif $i=6$ **then**

$x[\zeta^\#]$

elif $i=7$ **then**

$x[\zeta]$

end if

end proc

M > $Der := \text{proc}(x) ; \text{evalDG}(\text{add}(\text{derive}(x, i) \ \&\text{wedge} \ W[i], i=1..7)); \text{end proc};$

$Der := \text{proc}(x)$

$\text{DifferentialGeometry}:-\text{evalDG}(\text{add}(\text{DifferentialGeometry}:-\&\text{wedge}(\text{derive}(x, i), W[i]), i=1..7))$

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end proc

> $W := [\text{alpha}, \alpha^\#, \sigma^\#, \text{sigma}, \rho, \zeta^\#, \text{zeta}];$

$W := [\alpha, \alpha^\#, \sigma^\#, \sigma, \rho, \zeta^\#, \zeta]$

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frame2 > $List1 := W;$

frame2 > $List2 := \text{GenerateForms}(List1, 2);$

frame2 > **for** i **from** 1 **to** 6 **do for** j **from** $(i+1)$ **to** 7 **do**

$\text{tr1} \left[7 \cdot (i-1) - \frac{i \cdot (i-1)}{2} + (j-i) \right] := i; \text{tr2} \left[7 \cdot (i-1) - \frac{i \cdot (i-1)}{2} + (j-i) \right] := j; \text{od}; \text{od};$

M > $DF := \text{proc}(\text{omega}) \text{ local } T, Res; T := \text{GetComponents}(\text{omega}, List2);$

$Res[1] := \text{evalDG}(\text{add}(Der(T[i]) \ \&\text{wedge} \ List2[i], i=1..15));$

$Res[2] := \text{evalDG}(\text{add}(T[i] \ \&\text{wedge} \ dW[\text{tr1}[i]] \ \&\text{wedge} \ W[\text{tr2}[i]], i=1..21));$

$Res[3] := \text{evalDG}(\text{add}(T[i] \ \&\text{wedge} \ W[\text{tr1}[i]] \ \&\text{wedge} \ dW[\text{tr2}[i]], i=1..21));$
 $\text{evalDG}(Res[1] + Res[2] - Res[3]);$

end proc;

M > $List3 := \text{GenerateForms}(List1, 3);$

M > $BI := \text{proc}(\text{omega}) \text{ local } R, i; R := \text{GetComponents}(DF(\text{omega}), List3); \text{for } i \text{ from } 1 \text{ to } 35 \text{ do } \text{print}(R[i]=0); \text{od}; \text{end proc};$

M > $dW[1] := \text{evalDG}(\text{add}(U[i] \cdot W[\text{tr1}[i]] \ \&\text{wedge} \ W[\text{tr2}[i]], i=1..21));$

$dW_1 := U_1 \alpha \wedge \alpha^\# + U_2 \alpha \wedge \sigma^\# + U_3 \alpha \wedge \sigma + U_4 \alpha \wedge \rho + U_5 \alpha \wedge \zeta^\# + U_6 \alpha \wedge \zeta + U_7 \alpha^\# \wedge \sigma^\#$
 $+ U_8 \alpha^\# \wedge \sigma + U_9 \alpha^\# \wedge \rho + U_{10} \alpha^\# \wedge \zeta^\# + U_{11} \alpha^\# \wedge \zeta + U_{12} \sigma^\# \wedge \sigma + U_{13} \sigma^\# \wedge \rho$
 $+ U_{14} \sigma^\# \wedge \zeta^\# + U_{15} \sigma^\# \wedge \zeta + U_{16} \sigma \wedge \rho + U_{17} \sigma \wedge \zeta^\# + U_{18} \sigma \wedge \zeta + U_{19} \rho \wedge \zeta^\#$
 $+ U_{20} \rho \wedge \zeta + U_{21} \zeta^\# \wedge \zeta$

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frame2 > $dW[2] := \text{evalDG}(\text{add}(V[i] \cdot W[\text{tr1}[i]] \ \&\text{wedge} \ W[\text{tr2}[i]], i=1..21));$

$dW_2 := V_1 \alpha \wedge \alpha^\# + V_2 \alpha \wedge \sigma^\# + V_3 \alpha \wedge \sigma + V_4 \alpha \wedge \rho + V_5 \alpha \wedge \zeta^\# + V_6 \alpha \wedge \zeta + V_7 \alpha^\# \wedge \sigma^\#$
 $+ V_8 \alpha^\# \wedge \sigma + V_9 \alpha^\# \wedge \rho + V_{10} \alpha^\# \wedge \zeta^\# + V_{11} \alpha^\# \wedge \zeta + V_{12} \sigma^\# \wedge \sigma + V_{13} \sigma^\# \wedge \rho$

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$$\begin{aligned}
& + V_{14} \sigma^\# \wedge \zeta^\# + V_{15} \sigma^\# \wedge \zeta + V_{16} \sigma \wedge \rho + V_{17} \sigma \wedge \zeta^\# + V_{18} \sigma \wedge \zeta + V_{19} \rho \wedge \zeta^\# \\
& + V_{20} \rho \wedge \zeta + V_{21} \zeta^\# \wedge \zeta
\end{aligned}$$

$$\begin{aligned}
\text{frame2} > dW[4] := \text{evalDG}((2 W[1] + W[2]) \&wedge W[4] + II \cdot (W[3] \\
& \&wedge W[6]) + W[5] \&wedge W[7]); \\
dW_4 & := 2 \alpha \wedge \sigma + \alpha^\# \wedge \sigma + II \sigma^\# \wedge \zeta^\# + \rho \wedge \zeta
\end{aligned} \tag{7}$$

$$\begin{aligned}
> dW[3] := \text{evalDG}((2 W[2] + W[1]) \&wedge W[3] + II^\# \cdot (W[4] \&wedge W[7]) + W[5] \\
& \&wedge W[6]); \\
dW_3 & := \alpha \wedge \sigma^\# + 2 \alpha^\# \wedge \sigma^\# + II^\# \sigma \wedge \zeta + \rho \wedge \zeta^\#
\end{aligned} \tag{8}$$

$$\begin{aligned}
\text{frame2} > dW[5] := \text{evalDG}((W[1] + W[2]) \&wedge W[5] + I2 \cdot (W[3] \\
& \&wedge W[4]) + I3 \cdot (W[3] \&wedge W[6]) + I3^\# \cdot (W[4] \\
& \&wedge W[7]) + I \cdot W[7] \&wedge W[6]); \\
dW_5 & := \alpha \wedge \rho + \alpha^\# \wedge \rho + I2 \sigma^\# \wedge \sigma + I3 \sigma^\# \wedge \zeta^\# + I3^\# \sigma \wedge \zeta - I \zeta^\# \wedge \zeta
\end{aligned} \tag{9}$$

$$\begin{aligned}
\text{frame2} > dW[7] := \text{evalDG}(W[1] \&wedge W[7] + \text{evalDG}(\text{add}(T[i - 11] \\
& \cdot W[\text{tr}I[i]] \&wedge W[\text{tr}2[i]], i = 12 .. 21))); \\
dW_7 & := \alpha \wedge \zeta + T_1 \sigma^\# \wedge \sigma + T_2 \sigma^\# \wedge \rho + T_3 \sigma^\# \wedge \zeta^\# + T_4 \sigma^\# \wedge \zeta + T_5 \sigma \wedge \rho + T_6 \sigma \wedge \zeta^\# \\
& + T_7 \sigma \wedge \zeta + T_8 \rho \wedge \zeta^\# + T_9 \rho \wedge \zeta + T_{10} \zeta^\# \wedge \zeta
\end{aligned} \tag{10}$$

$$\begin{aligned}
\text{frame2} > dW[6] := \text{evalDG}(W[2] \&wedge W[6] + \text{evalDG}(\text{add}(S[i - 11] \\
& \cdot W[\text{tr}I[i]] \&wedge W[\text{tr}2[i]], i = 12 .. 21))); \\
dW_6 & := \alpha^\# \wedge \zeta^\# + S_1 \sigma^\# \wedge \sigma + S_2 \sigma^\# \wedge \rho + S_3 \sigma^\# \wedge \zeta^\# + S_4 \sigma^\# \wedge \zeta + S_5 \sigma \wedge \rho + S_6 \sigma \wedge \zeta^\# \\
& + S_7 \sigma \wedge \zeta + S_8 \rho \wedge \zeta^\# + S_9 \rho \wedge \zeta + S_{10} \zeta^\# \wedge \zeta
\end{aligned} \tag{11}$$

$$\begin{aligned}
\text{frame2} > BI(dW[4]); \\
& 0 = 0 \\
& V_1 + 2 U_1 = 0 \\
& 0 = 0 \\
& 0 = 0 \\
& 0 = 0 \\
& V_2 + 2 U_2 = 0 \\
& 0 = 0 \\
& -II + II_\alpha = 0 \\
& 0 = 0 \\
& -V_4 - 2 U_4 = 0 \\
& -V_5 - 2 U_5 = 0 \\
& -V_6 - 2 U_6 = 0 \\
& 0 = 0 \\
& 0 = 0 \\
& 0 = 0
\end{aligned}$$

$$\begin{aligned}
V_7 + 2 U_7 &= 0 \\
0 &= 0 \\
2 II + II_{\alpha^{\#}} &= 0 \\
0 &= 0 \\
-V_9 - 2 U_9 &= 0 \\
-V_{10} - 2 U_{10} &= 0 \\
-V_{11} - 2 U_{11} &= 0 \\
0 &= 0 \\
0 &= 0 \\
0 &= 0 \\
-T_1 - II S_5 - V_{13} - 2 U_{13} &= 0 \\
-II S_6 - V_{14} - 2 U_{14} - II_{\sigma} &= 0 \\
-II S_7 + I_2 - V_{15} - 2 U_{15} &= 0 \\
T_3 - II S_8 - II_{\rho} &= 0 \\
T_4 - II S_9 &= 0 \\
-II S_{10} + I_3 + II_{\zeta} &= 0 \\
T_6 + V_{19} + 2 U_{19} &= 0 \\
T_7 + V_{20} + 2 U_{20} &= 0 \\
-II II^{\#} + V_{21} + 2 U_{21} &= 0 \\
-T_{10} &= 0
\end{aligned}$$

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frame2 > BI(dW[3]);

$$\begin{aligned}
2 V_1 + U_1 &= 0 \\
0 &= 0 \\
0 &= 0 \\
0 &= 0 \\
0 &= 0 \\
-2 V_3 - U_3 &= 0 \\
-2 V_4 - U_4 &= 0 \\
-2 V_5 - U_5 &= 0 \\
-2 V_6 - U_6 &= 0 \\
0 &= 0 \\
0 &= 0 \\
2 II^{\#} &= 0 \\
0 &= 0 \\
0 &= 0 \\
0 &= 0
\end{aligned}$$

$$\begin{aligned}
& -2 V_8 - U_8 = 0 \\
& -2 V_9 - U_9 = 0 \\
& -2 V_{10} - U_{10} = 0 \\
& -2 V_{11} - U_{11} = 0 \\
& 0 = 0 \\
& 0 = 0 \\
& -II^\# = 0 \\
& 0 = 0 \\
& 0 = 0 \\
& 0 = 0 \\
& -S_1 + II^\# T_2 + 2 V_{16} + U_{16} = 0 \\
& II^\# T_3 + I_2 + 2 V_{17} + U_{17} = 0 \\
& II^\# T_4 + 2 V_{18} + U_{18} = 0 \\
& S_3 + 2 V_{19} + U_{19} = 0 \\
& S_4 + 2 V_{20} + U_{20} = 0 \\
& II II^\# + 2 V_{21} + U_{21} = 0 \\
& S_6 - II^\# T_8 = 0 \\
& S_7 - II^\# T_9 = 0 \\
& -II^\# T_{10} - I_3^\# = 0 \\
& -S_{10} = 0
\end{aligned} \tag{13}$$

frame2 > $dW[1] := evalDG(U[2] \cdot (W[1] \wedge W[3]) + U[3] \cdot (W[1] \wedge W[4]) + U[7] \cdot (W[2] \wedge W[3]) + U[8] \cdot (W[2] \wedge W[4]) + add(U[i] \cdot W[tr1[i]] \wedge W[tr2[i]], i = 13 .. 21));$

$$\begin{aligned}
dW_1 := & U_2 \alpha \wedge \sigma^\# + U_3 \alpha \wedge \sigma + U_7 \alpha^\# \wedge \sigma^\# + U_8 \alpha^\# \wedge \sigma + U_{13} \sigma^\# \wedge \rho + U_{14} \sigma^\# \wedge \zeta^\# \\
& + U_{15} \sigma^\# \wedge \zeta + U_{16} \sigma \wedge \rho + U_{17} \sigma \wedge \zeta^\# + U_{18} \sigma \wedge \zeta + U_{19} \rho \wedge \zeta^\# + U_{20} \rho \wedge \zeta \\
& + U_{21} \zeta^\# \wedge \zeta
\end{aligned} \tag{14}$$

frame2 > $dW[2] := evalDG(conjugate(U[8]) \cdot (W[1] \wedge W[3]) + conjugate(U[7]) \cdot (W[1] \wedge W[4]) + conjugate(U[3]) \cdot (W[2] \wedge W[3]) + conjugate(U[2]) \cdot (W[2] \wedge W[4]) + add(V[i] \cdot W[tr1[i]] \wedge W[tr2[i]], i = 13 .. 21));$

$$\begin{aligned}
dW_2 := & \bar{U}_8 \alpha \wedge \sigma^\# + \bar{U}_7 \alpha \wedge \sigma + \bar{U}_3 \alpha^\# \wedge \sigma^\# + \bar{U}_2 \alpha^\# \wedge \sigma + V_{13} \sigma^\# \wedge \rho + V_{14} \sigma^\# \wedge \zeta^\# \\
& + V_{15} \sigma^\# \wedge \zeta + V_{16} \sigma \wedge \rho + V_{17} \sigma \wedge \zeta^\# + V_{18} \sigma \wedge \zeta + V_{19} \rho \wedge \zeta^\# + V_{20} \rho \wedge \zeta \\
& + V_{21} \zeta^\# \wedge \zeta
\end{aligned} \tag{15}$$

frame2 > $BI(dW[4]);$

$$\begin{aligned}
&0 = 0 \\
&0 = 0 \\
&0 = 0 \\
&0 = 0 \\
&0 = 0 \\
&\overline{U}_8 + 2 U_2 = 0 \\
&0 = 0 \\
&-II + II_\alpha = 0 \\
&0 = 0 \\
&0 = 0 \\
&0 = 0 \\
&0 = 0 \\
&0 = 0 \\
&0 = 0 \\
&0 = 0 \\
&\overline{U}_3 + 2 U_7 = 0 \\
&0 = 0 \\
&2 II + II_{\alpha^\#} = 0 \\
&0 = 0 \\
&0 = 0 \\
&0 = 0 \\
&0 = 0 \\
&0 = 0 \\
&0 = 0 \\
&0 = 0 \\
&0 = 0 \\
&-T_1 - II S_5 - V_{13} - 2 U_{13} = 0 \\
&-II S_6 - V_{14} - 2 U_{14} - II_\sigma = 0 \\
&-II S_7 + I_2 - V_{15} - 2 U_{15} = 0 \\
&T_3 - II S_8 - II_\rho = 0 \\
&T_4 - II S_9 = 0 \\
&-II S_{10} + I_3 + II_\zeta = 0 \\
&T_6 + V_{19} + 2 U_{19} = 0 \\
&T_7 + V_{20} + 2 U_{20} = 0 \\
&-II I^\# + V_{21} + 2 U_{21} = 0 \\
&-T_{10} = 0
\end{aligned}$$

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frame2 > *BI*(*dW*[3]);

$$\begin{aligned}
&0 = 0 \\
&0 = 0
\end{aligned}$$

$$\begin{aligned}
0 &= 0 \\
0 &= 0 \\
0 &= 0 \\
-2 \overline{U}_7 - U_3 &= 0
\end{aligned}$$

$$\begin{aligned}
0 &= 0 \\
0 &= 0 \\
0 &= 0 \\
0 &= 0 \\
0 &= 0
\end{aligned}$$

$$2 II^\# = 0$$

$$\begin{aligned}
0 &= 0 \\
0 &= 0 \\
0 &= 0
\end{aligned}$$

$$-2 \overline{U}_2 - U_8 = 0$$

$$\begin{aligned}
0 &= 0 \\
0 &= 0 \\
0 &= 0 \\
0 &= 0 \\
0 &= 0
\end{aligned}$$

$$-II^\# = 0$$

$$\begin{aligned}
0 &= 0 \\
0 &= 0 \\
0 &= 0
\end{aligned}$$

$$-S_1 + II^\# T_2 + 2 V_{16} + U_{16} = 0$$

$$II^\# T_3 + I_2 + 2 V_{17} + U_{17} = 0$$

$$II^\# T_4 + 2 V_{18} + U_{18} = 0$$

$$S_3 + 2 V_{19} + U_{19} = 0$$

$$S_4 + 2 V_{20} + U_{20} = 0$$

$$II II^\# + 2 V_{21} + U_{21} = 0$$

$$S_6 - II^\# T_8 = 0$$

$$S_7 - II^\# T_9 = 0$$

$$-II^\# T_{10} - I_3^\# = 0$$

$$-S_{10} = 0$$

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frame2 > *BI(dW[5]);*

$$\begin{aligned}
0 &= 0 \\
0 &= 0 \\
0 &= 0 \\
0 &= 0
\end{aligned}$$

$$\begin{aligned}
& 0 = 0 \\
& 2 I_2 + I_{2_\alpha} = 0 \\
& \overline{U}_8 + U_2 = 0 \\
& I_{3_\alpha} = 0 \\
& 0 = 0 \\
& \overline{U}_7 + U_3 = 0 \\
& 0 = 0 \\
& 2 I_3^\# = 0 \\
& 0 = 0 \\
& 0 = 0 \\
& 0 = 0 \\
& 2 I_2 + I_{2_\alpha^\#} = 0 \\
& \overline{U}_3 + U_7 = 0 \\
& 2 I_3 + I_{3_\alpha^\#} = 0 \\
& 0 = 0 \\
& \overline{U}_2 + U_8 = 0 \\
& 0 = 0 \\
& 0 = 0 \\
& 0 = 0 \\
& 0 = 0 \\
& 0 = 0 \\
& I_3^\# T_2 - I_3 S_5 + I_{2_\rho} = 0 \\
& I T_1 + I_3^\# T_3 - I_3 S_6 - I_{3_\sigma} + I_{2_\zeta^\#} = 0 \\
& I_3^\# T_4 - I_3 S_7 - I S_1 + I_{2_\zeta} = 0 \\
& I T_2 - I_3 S_8 - V_{14} - U_{14} - I_{3_\rho} = 0 \\
& -I_3 S_9 - I_2 - I S_2 - V_{15} - U_{15} = 0 \\
& -I T_4 - I_3 S_{10} - I S_3 + I_3^\# I I + I_{3_\zeta} = 0 \\
& I T_5 - I_3^\# T_8 + I_2 - V_{17} - U_{17} = 0 \\
& -I_3^\# T_9 - I S_5 - V_{18} - U_{18} = 0 \\
& -I T_7 - I_3^\# T_{10} - I S_6 - I_3 I I^\# = 0 \\
& -I T_9 - I S_8 + V_{21} + U_{21} = 0
\end{aligned}$$

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