Derived categories of coherent sheaves on projective spaces

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Grothendick's key observation was that the constructions of homological algebra do not barely yield cohomology groups but in fact complexes with a certain indeterminacy.

The idea of the derived category is the following:
a) An object X of an abelian category should be identified with all its resolutions.
b) The main reason for such an identification is that some most important functors, such as Hom, tensor products, Γ, should be redefined. Their usual definitions should only be applied to some special objects, which are acyclic with respect to this functor.
c) To adopt this point of view one must consider from the very beginning not only objects and their resolutions, but arbitrary complexes. Hence, the relation that enables us to identify an object and its resolution should be generalized to arbitrary complexes. This is the notion of quasi-isomorphism.
d) The redefinition of the functors makes them exact in a special sense.

Derived categories seem to be the appropriate place to do homological algebra. One of their great advantages is that the important functors of homological algebra which are left or right exact (Hom, $N\otimes_k$, where N is a fixed k-module, the global section functor Γ, etc.) become exact on the level of derived functors (with an appropriately modified definition of exact).

Derived and Triangulated categories

Definition | Proposition:
Let A be an abelian category, Kom(A) the category of complexes over A. There exists a category $D(A)$ and a functor Q:Kom$(A) \to D(A)$ with the following properties:
a) Q(f) is an isomorphism for any quasi-isomorphism f.
b) Any functor F:Kom$(A)\to D$ transforming quasi-isomorphisms into isomorphisms can be factorized through $D(A)$.
The category $D(A)$ is called the derived category of the abelian category A.

Definitions:
a) Given any complex $K^\bullet = (K^i, d_K^i)$ and for a fixed integer n define a new complex $K[n]^\bullet$ by $(K[n])^i = K^{n+i}, d_{K[n]} = (-1)^n d_K$.
For a morphism of complexes $f : K^\bullet \to L^\bullet$ let $f[n] : K[n]^\bullet \to L[n]^\bullet$ coincide with f componentwise.
$T^n : Kom(A) \to Kom(A), T^n(K^\bullet) = K[n]^\bullet, T^n(f) = f[n]$ is called the *translation by n* functor.

b) Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes. The *cone of* f is the following complex:
$C(f)^i = K[1]^i \oplus L^i$, $d_{C(f)}(k^{i+1}, l^i) = (-d_K k^{i+1}, f(k^{i+1}) + d_L l^i)$
c) The *cylinder of* f $Cyl(f)$ is the complex:
$Cyl(f) = K^\bullet \oplus K[1]^\bullet \oplus L^\bullet, d^i_{Cyl(f)}(k^i, k^{i+1}, l^i) = (d_K k^i - k^{i+1}, -d_K k^{i+1}, f(k^{i+1}) + d_L l^i)$

Lemma 1 *For any morphism* $f : K^\bullet \to L^\bullet$ *there exists the following commutative diagram in* $Kom(A)$ *with exact rows:*

$$\begin{array}{ccccccccc}
 & & 0 & \to & L^\bullet & \stackrel{\bar{\pi}}{\to} & C(f) & \stackrel{\delta}{\to} & K[1]^\bullet & \to & 0 \\
 & & & & \alpha \downarrow & & \| & & & & \\
0 & \to & K^\bullet & \stackrel{\bar{f}}{\to} & Cyl(f) & \stackrel{\pi}{\to} & C(f) & \to & 0 & & \\
 & & \| & & \beta \downarrow & & & & & & \\
 & & K^\bullet & \stackrel{f}{\to} & L^\bullet & & & & & &
\end{array}$$

It is functorial in f *and has the following property:*
α *and* β *are quasi-isomorphisms; moreover* $\beta\alpha = id_L$ *and* $\alpha\beta$ *is homotopic to* $id_{Cyl(f)}$ *so that* $L^\bullet$ *and* $Cyl(f)$ *are canonically isomorphic in the derived category.*

Proof: [Gel] Chapter 3

Definition:
a) A triangle in some category of complexes($Kom, D, D^+, \ldots$) is a diagram of the form
$K^\bullet \stackrel{u}{\to} L^\bullet \stackrel{v}{\to} M^\bullet \stackrel{w}{\to} K[1]^\bullet$
b) A morphism of triangles is a commutative diagram of the form

$$\begin{array}{ccccccc}
K^\bullet & \stackrel{u}{\to} & L^\bullet & \stackrel{v}{\to} & M^\bullet & \stackrel{w}{\to} & K[1]^\bullet \\
\downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\
K^\bullet & \stackrel{u}{\to} & L^\bullet & \stackrel{v}{\to} & M^\bullet & \stackrel{w}{\to} & K[1]^\bullet
\end{array}$$

It is an isomorphism if f, g, h are isomorphisms in the corresponding category.
c) A triangle is said to be distinguished if it is isomorphic to the middle row
$K^\bullet \stackrel{\bar{f}}{\to} Cyl(f) \stackrel{\pi}{\to} C(f) \stackrel{\delta}{\to} K[1]^\bullet$
of some diagram of the form as in the preceding lemma.

Definition:
A functor from $D(A) \to D(B)$ is said to be exact if it takes distinguished triangles to distinguished triangles.

The derived functor of an additive left exact functor F:$A \to B$ is a pair consisting of an exact functor RF:$D^+(A) \to D^+(B)$ and a morphism of functors $\epsilon_F : Q_B \circ K^+(F) \to$RF$\circ Q_A$

$$\begin{array}{ccccc}
 & & D^+(A) & & \\
 & Q_A \nearrow & & \searrow RF & \\
K^+(A) & & & & D^+(B) \\
 & K^+(F) \searrow & & \nearrow Q_B & \\
 & & K^+(B) & &
\end{array}$$

satisfying the following universal property:

for any exact functor G:$D^+(A) \to D^+(B)$ and any morphism of functors $\epsilon : Q_B \circ K^+(F) \to G \circ Q_A$ there exists a unique morphism of functors $\eta : RF \to G$ making the following diagram commutative.

$$\begin{array}{ccc} & Q_B \circ K^+(F) & \\ \epsilon_F \swarrow & & \searrow \epsilon \\ RF \circ Q_A & \overset{\eta \circ Q_A}{\to} & G \circ Q_A \end{array}$$

Similar theorem holds for right exact functor.

Definition:
Let D be an additive category. The structure of a triangulated category on D is given by the following data a,b that must satisfy the axioms TR1 to TR4 below:
a) An additive automorphism T:$D \to D$ called the translation functor.
We write X[n] = T^n(X)   f[n] = T^n(f)
b) The class of distinguished triangles.

TR1:
(a) For any X∈ObD, the triangle
$X \xrightarrow{id} X \longrightarrow 0 \longrightarrow X[1]$
is distinguished.
(b) A triangle isomorphic to a distinguished one is itself distinguished.
(c) Any morphism $X \xrightarrow{u} Y$ can be completed to distinguished triangle
$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$
TR2: A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished iff the triangle
$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is distinguished.
TR3: A diagram of the following form can be completed:

$$\begin{array}{ccccccc} X & \to & Y & \to & Z & \to & X[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ X' & \to & Y' & \to & Z' & \to & X'[1] \end{array}$$

TR4: Any diagram of the form 'upper cap' can be completed to an octahedron diagram.(for diagram see [Gel])

Theorem: A Derived category is triangulated.
Proof: [Gel] Chapter 3

Let C be a triangulated category. We shall say that a family of its objects $\{x_i\}$ generates C, if the smallest full triangulated subcategory containing them is equivalent with C.

Remark 1 *If $\{x_i\}$ generate C then the objects in C are upto translations, cones of morphisms*
$u : x_i \to x_j,\ u \in Hom_C(x_i, x_j)$

Lemma 2 *Let C and D be triangulated categories, F:C $\rightarrow$ D be an exact functor, $\{x_i\}$ be a family of objects of C. Let us assume that $\{x_i\}$ generates C, $\{F(x_i)\}$ generates D, and for any pair x_i,x_j from the family,*
F:Hom(x_i,x_j) $\rightarrow$ Hom($F(x_i)$,$F(x_j)$)
is an isomorphism. Then F is an equivalence of categories.

Proof:
a) By hypothesis F is a fully faithful functor.
b) F is exact $\Rightarrow$ it commutes with the translation functor and takes distinguished triangles to distinguished triangles.
Further for a morphism u,
F(C(u)) $\cong$ C(F(u))
Follows from Axiom TR3 applied to the following diagram:

$$\begin{array}{ccccccc} F(x_i) & \overset{F(u)}{\rightarrow} & F(x_j) & \rightarrow & F(C(u)) & \rightarrow & F(x_i[1]) \\ \downarrow id & & \downarrow id & & \downarrow & & \downarrow iso \\ F(x_i) & \overset{F(u)}{\rightarrow} & F(x_j) & \rightarrow & C(F(u)) & \rightarrow & F(x_i)[1] \end{array}$$

$\Rightarrow$ By preceding remark every object $y \in ObD$ is isomorphic to an object of the form F(x) for some $x \in ObC$
$\Rightarrow$ F is an equivalence of categories.

Sheaf Cohomology

Proposition 1 *If A is a ring, then every A-module is isomorphic to a submodule of an injective A-module.*

Proposition 2 *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category $\mathcal{M}od(X)$ of sheaves of $\mathcal{O}_X$-modules has enough injectives.*

Sketch of Proof: For any sheaf of $\mathcal{O}_X$-modules, $\mathcal{F}$ consider $I = \Pi_{x \in X} j_*(I_x)$ where I_x is an injective module containing $\mathcal{F}_x$ and j:{ x } $\rightarrow$X is the inclusion map.
Then I is an injective $\mathcal{O}_X$ -module and the natural map from $\mathcal{F}$ to I is injective

Proposition 3 *If $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is a short exact sequence in $\mathcal{M}od(X)$, then for any $\mathcal{G}$ we have long exact sequences*
$0 \rightarrow Hom(\mathcal{G}, \mathcal{F}') \rightarrow Hom(\mathcal{G}, \mathcal{F}) \rightarrow Hom(\mathcal{G}, \mathcal{F}'') \rightarrow Ext^1(\mathcal{G}, \mathcal{F}') \rightarrow Ext^1(\mathcal{G}, \mathcal{F}) \rightarrow Ext^1(\mathcal{G}, \mathcal{F}'') \rightarrow ...$, *and*
$0 \rightarrow Hom(\mathcal{F}'', \mathcal{G}) \rightarrow Hom(\mathcal{F}, \mathcal{G}) \rightarrow Hom(\mathcal{F}', \mathcal{G}) \rightarrow Ext^1(\mathcal{F}'', \mathcal{G}) \rightarrow Ext^1(\mathcal{F}, \mathcal{G}) \rightarrow Ext^1(\mathcal{F}', \mathcal{G}) \rightarrow ...$,

Sketch of Proof:
The first exact sequence is obtained by applying $Hom(\mathcal{G}, .)$ to injective resolutions of $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ and taking the long exact sequence of cohomology.
The second sequence is obtained by taking an injective resolution $\mathcal{I}^\bullet$ of $\mathcal{G}$ and applying $Hom(., \mathcal{I}^\bullet)$ on $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ and taking the long exact sequence of cohomology.

Cohomology on Projective Spaces

Let A be a noetherian ring, let S = A$[x_0, ...x_r]$ and let X = Proj S be the projective space P^r_A over A. Let $\mathcal{O}_X(1)$ be the twisting sheaf of Serre. For any sheaf of $\mathcal{O}_X$-module $\mathcal{F}$ we denote by by $\Gamma_*(\mathcal{F})$ the graded S-module $\bigoplus_{n\in Z}\Gamma(X, \mathcal{F}(r))$.

Theorem 1 *Let A be a noetherian ring , and let X= P^r_A, with r≥1. Then :*
(a) the rational map $S \to \Gamma_*(\mathcal{O}_X) = \bigoplus_{n\in Z} H^0(X, \mathcal{O}_X(n))$ is an isomorphism of graded S-modules;
(b) $H^i(X, \mathcal{O}_X(n)) = 0$ for $0<i<r$ and all $n \in Z$;
(c) $H^r(X, \mathcal{O}_X(-r-1)) \cong A$;
(d) the natural map $H^0(X, \mathcal{O}_X(r)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \longrightarrow H^r(X, \mathcal{O}_X(-r-1)) \cong A$ is a perfect pairing of finitely generated free A-modules, for each n∈ Z.

Proof: [Har1] Chapter 3.

Theorem 2 *Let A be a ring, let Y = Spec A, and let X = P^n_A. Then there is an exact sequence of sheaves on X,*
$0 \longrightarrow \Omega_{X/Y} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \to 0$

Theorem 3 *Let k be a field, and let X = P^n_k. Then there is an exact sequence of sheaves on X,*
$0 \longrightarrow \Omega^i(i) \longrightarrow \bigwedge^i V \otimes \mathcal{O} \longrightarrow \Omega^{i-1}(i) \longrightarrow 0$, $\forall i \geq 1$ *where* $V = H^0(P, \mathcal{O}(1))$

Proof Consider the map $\psi : \mathcal{O}_X(-1)^{\wedge^i V} \longrightarrow \Omega^{i-1}(i-1)$
given by ,
$\psi|_{U_j} : \mathcal{O}_X(-1)^{\wedge^i V}|_{U_j} \longrightarrow \Omega^{i-1}(i-1)|_{U_j}$
let $v_0, ..., v_n$ be a basis for V (fix it).
$\psi|_{U_j} : e_{v_{k_1}\wedge v_{k_2}\wedge ...\wedge v_{k_n}}|_{U_j} \longmapsto x_j^{i-1}(\Sigma(-1)^{l+1}x_{k_l}d(\frac{x_{k_1}}{x_j}) \wedge ... \wedge d(\hat{\frac{x_{k_l}}{x_j}}) \wedge ... \wedge d(\frac{x_{k_i}}{x_j}))$ (where $k_i = j$ take $d(\frac{x_{k_i}}{x_j}) = 0$).
Glues on $U_j \cap U_{j'}$. Hence a map on the whole sheaf.
fix $k_1...k_{i+1}$ then
$(x_{k_1}, \ldots, x_{k_{i+1}}) = \Sigma(-1)^{l+1}x_{k_l}.e_{v_{k_1}\wedge v_{k_2}\wedge ...\wedge \hat{v_{k_l}}\wedge ...\wedge v_{k_{i+1}}} \longmapsto 0$.
Now, $\mathcal{O}_X(-1)^{\wedge^i V}|_{U_j} \longrightarrow^{\psi|_{U_j}} \Omega^{i-1}(i-1)|_{U_j}$ is a surjective homomorphism of free f-g-modules.
$\Rightarrow$ Rank$(\mathcal{O}_X(-1)^{\wedge^i V}|_{U_j}) = Rank(ker(\psi|_{U_j})) + Rank(\Omega^{i-1}(i-1)|_{U_j})$.
$\Rightarrow$ Rank$(ker(\psi|_{U_j})) = \binom{n+1}{i}$ - $\binom{n}{i-1} = \binom{n}{i}$
Consider the map $\Omega^i(i-1)|_{U_j} \longrightarrow ker(\psi)|_{U_j}$
$d(\frac{x_{k_1}}{x_j}) \wedge ... \wedge d(\frac{x_{k_i}}{x_j}) \longmapsto \frac{1}{x_j^{i+1}}(x_j, x_{k_1}, ..., x_{k_i})$
This map is an isomorphism and it glues on intersections. Hence, it is a map on the whole space.
So, we have the exact sequence,
$0 \longrightarrow \Omega^i(i-1) \longrightarrow \mathcal{O}(-1)^{\wedge^i V} \longrightarrow \Omega^{i-1}(i-1) \longrightarrow 0$

i.e. $0 \longrightarrow \Omega^i(i-1) \longrightarrow \wedge^i V \otimes \mathcal{O}(-1) \longrightarrow \Omega^{i-1}(i-1) \longrightarrow 0$
i.e. $0 \longrightarrow \Omega^i(i) \longrightarrow \wedge^i V \otimes \mathcal{O} \longrightarrow \Omega^{i-1}(i) \longrightarrow 0$
Another proof:

Lemma 3 *If* $0 \longrightarrow M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \longrightarrow 0$ *is exact as A-modules and* $\bigwedge^2 M'' = 0$. *Then* $0 \longrightarrow \bigwedge^i M' \longrightarrow \bigwedge^i M \longrightarrow \bigwedge^{i-1} M' \otimes M'' \longrightarrow 0$ *is exact.*

proof
Consider $\bigwedge^i M' \longrightarrow^{\wedge^i \phi} \bigwedge^i M$
$m_1' \wedge ... \wedge m_i' \longrightarrow \phi(m_1') \wedge ... \wedge \phi(m_i')$
This is clearly injective.
Consider the map $\bigwedge^i M \xrightarrow{\psi_i} \bigwedge^{i-1} M \otimes M''$
$m_1 \wedge m_2 \wedge ... \wedge m_i \longmapsto \Sigma_j (-1)^j \psi(m_j) m_1 \wedge ... \wedge \hat{m}_j \wedge ... \wedge m_i$
Clearly $\mathrm{Ker}(\psi_i) = \bigwedge^i \phi(M') = \bigwedge^i \phi(\bigwedge M')$
$\psi_{i-1}\psi_i = 0 \Rightarrow \mathrm{Im}(\psi_i) \subset \bigwedge^{i-1} M' \otimes M''$
Infact $\mathrm{Im}\psi_i = \bigwedge^{i-1} M' \otimes M''$.
(Consider $\psi_i(m_1, ..., m_i)$ where $m_1 \notin M'$ and $m_j \in M' \; \forall j \neq 1$).
Hence, the lemma.

Applying it to ,
$0 \longrightarrow \Omega^1(1) \longrightarrow V \otimes \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0$
Since $\bigwedge^2 \mathcal{O}(1) = 0$ (locally free of rank 1) we get
$0 \longrightarrow \Omega^i(i) \longrightarrow \bigwedge^i V \otimes \mathcal{O} \longrightarrow \Omega^{i-1}(i) \longrightarrow 0$
Hence proved

Theorem 4 $Hom(\mathcal{O}, \Omega^j(j+1)) = H^0(P^n, \Omega^j(j+1)) = \bigwedge^{j+1} V$

Proof
In $U_0 \cap U_1$, if $k_1, ..., k_j \neq 0$
$x_1^{j+1} d(\frac{x_{k_1}}{x_1}) \wedge ... \wedge d(\frac{x_{k_j}}{x_1}) = x_0^j . x_1 . d(\frac{x_{k_1}}{x_0}) \wedge ... \wedge d(\frac{x_{k_j}}{x_0}) +$
$x_0^j \sum (-1)^l x_{k_l} d(\frac{x_1}{x_0}) \wedge d(\frac{x_{k_1}}{x_0}) \wedge ... \wedge d(\frac{\hat{x}_{k_l}}{x_0}) ... \wedge d(\frac{x_{k_j}}{x_0})$
If $k_1 = 0$ (say),
$x_1^{j+1} d(\frac{x_0}{x_1}) \wedge ... \wedge d(\frac{x_{k_j}}{x_1}) = -x_0^{j+1} d(\frac{x_1}{x_0}) \wedge ... \wedge d(\frac{x_{k_j}}{x_0})$
Now, suppose $f_0, ..., f_n$ are such that $f_i \in \Omega^j(j+1))_{U_i}$ and they glue on intersection.
say $f_i = x_i^{j+1} \sum f^i_{k_1,...,k_j} d(\frac{x_{k_1}}{x_i}) \wedge ... \wedge d(\frac{x_{k_j}}{x_i})$
In $U_0 \cap U_1$, $f_0 = f_1$
$\Longrightarrow$ 1)$x_0 f^0_{k_1,...,k_j} = x_1 f^1_{k_1,...,k_j}$ if $k_1, ..., k_j \neq 0, 1$
2)$x_0 f^0_{1,k_2,...,k_j} = -x_1 f^1_{0,k_2,...,k_j} + \sum_{l \neq 1, k_2 ... k_j} \pm x_l f^1_{k_2,...,l,...,k_j}$
Similar conditon for f_i, f_j.
Now, consider, for a fixed $k_1, ..., k_{j+1}$
$f_l^{k_1,...,k_{j+1}} = x_l^j \sum x_{k_i} d(\frac{x_{k_1}}{x_l}) \wedge ... \wedge d(\frac{\hat{x}_{k_i}}{x_l}) ... \wedge d(\frac{x_{k_{j+1}}}{x_l})$
then $f_l^{k_1,...,k_{j+1}}$'s satisfy the condition 1,2 (for varying l).
$\Longrightarrow$ they extend to a global section.

Moreover, $f_l^{k_1,...,k_{j+1}}$'s form a basis for the space of global sections as can be checked easily (for varying $k_1, ..., k_{j+1}$).
$\Longrightarrow$ the map.
$(f_0^{k_1,...,k_{j+1}}, ..., f_n^{k_1,...,k_{j+1}}) \longmapsto v_{k_1} \wedge ... \wedge v_{k_{j+1}}$
from
$H^0(P^n, \Omega^s(j+1)) \longrightarrow \bigwedge^{j+1} V$ is an isomorphism.

Theorem 5 $H^n(P, \Omega^i(i-n-1)) = \bigwedge^i V$

Proof We compute using Cech Cohomology.
$C^n = \Gamma(v_0 \wedge v_1 \wedge ... \wedge v_n, \Omega^i(i-n-1))$
$C^{n-1} = \Gamma(v_1 \wedge v_2 \wedge ... \wedge v_n) \times \Gamma(v_0 \wedge v_2 \wedge ... \wedge v_n) \times ... \times \Gamma(v_0 \wedge v_1 \wedge ... \wedge v_{n-1})$
Let $(f_0, f_1, ..., f_n) \in C^{n-1}$, $f_i \in \Gamma(v_0 \wedge ... \wedge \hat{v_i} \wedge ... \wedge v_n)$
then, $f_0 = x_1^{i-n-1} f^0_{k_1...k_i} d(\frac{x_{k_1}}{x_1}) \wedge ... \wedge d(\frac{x_{k_i}}{x_i})$
$f_j = x_0^{i-n-1} f^0_{k_1...k_i} d(\frac{x_{k_1}}{x_0}) \wedge ... \wedge d(\frac{x_{k_i}}{x_0})$
$f^0_{k_1,...,k_i} \in k[x_0, ...x_n](\frac{1}{x_1}, ..., \frac{1}{x_n})$
$f^j_{k_1,...,k_i} \in k[x_0, ...x_n](\frac{1}{x_1}, ..., \hat{\frac{1}{x_j}}, ..., \frac{1}{x_n})$
Image of $(f_0, f_1, ..., f_n)$ in $C^n = \sum_{j=0}^n (-1)^j f_j|_{v_0 \wedge v_1 \wedge ... \wedge v_n}$
$= f_0|_{v_0 \wedge v_1 \wedge ... \wedge v_n} + ((\frac{x_0}{x_1})^{n+1} f^o_{k_1,...,k_i} + \sum_{j=1}^n (-1)^j f^j_{k_1,...,k_i})$
The image of C^{n-1} is the submodule of C^n generated by the elements of the form,
$(x_o^{i-n-1}).f.d(\frac{x_k}{x_0}) \wedge ... \wedge d(\frac{x_{k_i}}{x_0})$
s.t. f $\neq$ constant $\times \frac{x_o^n}{x_1,...,x_n} \Longrightarrow H^n(P^n, \Omega^i(i-n-1)) \cong \bigwedge^i V$
the map being,
$(x_0^{i-n-1})(\frac{x_o^n}{x_1,...,x_n}) d(\frac{x_{k_1}}{x_0}) \wedge ... \wedge d(\frac{x_{k_i}}{x_0}) \longmapsto v_{k_1} \wedge v_{k_2} \wedge ... \wedge v_{k_i}$

Theorem 6 *Serre duality for P_k^n:*
Let $X = P_k^n$ over a field K. Then:
(a) $H^n(X, \omega_X) \cong k$. Fix ane such isomorphism.
(b) For any coherent sheaf $\mathcal{F}$ on X, the natural pairing
$Hom(\mathcal{F}, \omega) \times H^n(X, \mathcal{F}) \to H^n(X, \omega) \cong k$ is a perfect pairing of f.d.v.s over K.
(c) For every i$\geq$ 0 there is a natural funtorial isomorphism,
$Ext^i(\mathcal{F}, \omega) \cong H^{n-i}(X, \mathcal{F})^*$.

Proposition 4 $Hom(\Omega^i(i), \mathcal{O}) \cong \bigwedge^i V^*$

Proof
$Hom(\Omega^i(i), \mathcal{O}) = Ext^0(\Omega^i(i-n-1), \omega)$ (by Serre duality)
$= H^n(\mathcal{O}, \Omega^i(i-n-1))*$
$= (\bigwedge^i V)^* = \bigwedge^i V^*$

The Main Theorem

Let P be n-dimensional projective space over k, V = H^0(P,$\mathcal{O}(1)$)

Lemma 4 *For any pair i,j such that* $0 \leq i, j \leq n$, *and* $l \geq 1$
(a) $Hom(\mathcal{O}(-i), \mathcal{O}(-j)) = S^{i-j}(V) \quad Ext^l(\mathcal{O}(-i), \mathcal{O}(-j)) = 0$
(b) $Hom(\Omega^i(i), \Omega^j(j)) = \Lambda^{i-j}(V^*) \quad Ext^l(\Omega^i(i), \Omega^j(j)) = 0$
where composition of morphisms coincides with multiplication in S(V) and $\Lambda(V^*)$*, respectively.*

Proof:
(a) Follows from preceding theorem. $Hom(\mathcal{O}(-i), \mathcal{O}(-j)) = H^0(P, \mathcal{O}(i-j)) = S^{i-j}(V)$
$Ext^l(\mathcal{O}(-i), \mathcal{O}(-j)) = H^l(P, \mathcal{O}(i-j)) = 0$
(b) Applying $Hom(\bullet, \Omega^{j-1}(j))$ and $Hom(\Omega^i(i), \bullet)$ to the exact sequence in theorem 2 for i and j respectively, we get
$0 \longrightarrow Hom(\Omega^{i-1}(i), \Omega^{j-1}(j)) \longrightarrow Hom(\bigwedge^i V \otimes \mathcal{O}, \Omega^{j-1}(j))...$
and
$0 \longrightarrow Hom(\Omega^i(i), \Omega^j(j)) \longrightarrow Hom(\Omega^i(i), \bigwedge^j V \otimes \mathcal{O}) \longrightarrow Hom(\Omega^i(i), |Omega^{j-1}(j))...$
By five lemma, we get
$Hom(\Omega^{i-1}(i-1), \Omega^{j-1}(j-1)) = Hom(\Omega^i(i), \Omega^j(j))$ which by induction =
$Hom(\Omega^{i-j}(i-j), \mathcal{O}) = \bigwedge^{i-j} V^*$.

2. Let A be a graded algebra.
Notation:
A[i] is the free one-dimensional graded A-module with distinguished generator of degree i; $M_{[0,n]}$(A) is the full subcategory of the category of graded A-modules and morphisms of degree 0, whose objects are isomorphic with finite direct sums of A[i], where $0 \leq i \leq n$; $K^b_{[0,n]}$(A) is the category whose objects are finite complexes over $M_{[0,n]}$(A), and whose morphisms are morphisms of complexes modulo null-homotopic ones.

3. With an (n+1)-dimensional vector space V over the field k are associated two graded algebras S(V) and $\Lambda(V^*)$.
We set $K_\Lambda = K^b_{[0,n]}(\Lambda(V^*)), K_S = K^b_{[0,n]}(S(V))$.

Let $\mathcal{M}od$(P) be the category of coherent sheaves on P, and $D^b(P)$ be its derived category. It follows from Lemma 4 that there exist natural additive functors,
$F_1' : M_{[0,n]}(\Lambda(V^*)) \to \mathcal{M}od(P)$ and $F_2' : M_{[0,n]}(S(V)) \to \mathcal{M}od(P)$ such that
$F_1'(\Lambda(V^*)[i]) = \Omega^i(i), \quad F_2'(S(V)[i]) = \mathcal{O}(-i)$

They extend canonically to exact functors
$F_1 : K_\Lambda \to D^b(P), \quad F_2 : K_S \to D^b(P)$

Theorem 7 *F_1 and F_2 are equivalence of categories*

Proof
We verify that F_i satify Lemma 2

Proposition 5 *$\Omega^i(i)$ generate $D^b(P)$.*

Proof
We have the exact sequence,
$0 \longrightarrow \Omega \longrightarrow \mathcal{O}(-1)^{n+1} \longrightarrow \mathcal{O} \longrightarrow 0$
$\Longrightarrow$ the exact sequence,
$0 \longrightarrow \Omega^1(-1) \longrightarrow \mathcal{O}^{n+1} \longrightarrow \mathcal{O}(-1) \longrightarrow 0$
$\Longrightarrow$ the sequence, (taking dual)
$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}^{n+1} \longrightarrow T(-1) \longrightarrow 0$
Thus, the stalk of $T(-1)$ at a point $x_0 \in P^n$ is, $T(-1)_{x_0} \cong C^{n+1}/Cv$, where $x_0 = [v]$

Consider the map,
$s' : p_1^*(\mathcal{O}(-1)) \otimes p_2^*(\mathcal{O}) \longrightarrow p_1^*(\mathcal{O}) \otimes T(-1)$
(locally) given by,
$s'(x,y)(v) = v \ modCw,\ v, w \in P^n,\ [v] = x, [w] = y$
This gives a global section, $s \in H^0(P_n \times P_n, p_1^*(\mathcal{O}(1)) \otimes p_2^*(T(-1)))$.
Now $s(x,y) = 0 \Leftrightarrow$ v $modCw = 0 \Leftrightarrow v = \lambda w \Leftrightarrow x = y$ i.e. the zero locus of s is the diagonal $\Delta \subset P_n \times P_n$
Taking the Koszul resolution w.r.t s we get the exact sequence,
$0 \longrightarrow \bigwedge^n(p_1^*(\mathcal{O}(-1)) \otimes p_2^*(\Omega^1(1))) \longrightarrow \bigwedge^{n-1}(p_1^*(\mathcal{O}(-1)) \otimes p_2^*(\Omega^1(1))) \longrightarrow ...$
$\longrightarrow p_1^*(\mathcal{O}(-1)) \otimes p_2^*(\Omega^1(1)) \longrightarrow p_1^*(\mathcal{O}) \otimes p_2^*((O)) \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$

i.e. the exact sequence,
$0 \longrightarrow p_1^*(\mathcal{O}(-n)) \otimes p_2^*(\Omega^n(n)) \longrightarrow p_1^*(\mathcal{O}(-n+1)) \otimes p_2^*(\Omega^{n-1}(n-1)) \longrightarrow ... \longrightarrow$
$p_1^*(\mathcal{O}(-1)) \otimes p_2^*(\Omega^1(1)) \longrightarrow \mathcal{O}_{P^n \times P^n} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0.$
Denote it by $C^\bullet$. Now, for any $X \in \mathcal{O}bD^b(P)$.
Consider $C^\bullet \otimes^L Lp_1^*(X)$
$0 \longrightarrow p_2^*(\Omega^n(n) \otimes p_1^*(X(-n)) \longrightarrow ... \longrightarrow p_2^*(\Omega^1(1)) \otimes p_1^*(X(-1)) \longrightarrow \mathcal{O}_\Delta \otimes p_1^*(X) \longrightarrow 0$

$\Longrightarrow 0 \longrightarrow p_2^*(\Omega^n(n) \otimes p_1^*(\Omega^n(X(-n))) \longrightarrow ... \longrightarrow p_2^*(\Omega^1(1)) \otimes p_1^*(X(-1)) \longrightarrow 0$
is quasi-isomorphic to $\mathcal{O}_\Delta \otimes p_1^*(X)$.
In the language of derived categories, $\mathcal{O}_\Delta \otimes^L Lp_1^*(X)$ belongs to the full triangulated subcategory of $D^b(P^n \times P^n)$ generated by sheaves of the form, $p_2^*(\Omega^i(i)) \otimes p_1^*(Y)$ for $Y \in Ob(D^b(P))$
Applying Rp_{2*} we see that X belongs to the full triangulated subcategory generated by $\Omega^i(i)$.

Note: $Rp_{2*}(\mathcal{O}_\Delta \otimes^L Lp_1^*(X)) \cong X$
and $Rp_{2*}(p_2^*(\Omega^i(i)) \otimes p_1^*(Y)) \cong \Omega^i(i) \otimes R\Gamma(Y)$ which is a direct sum of $\Omega^i(i)$.

Proposition 6 *$\mathcal{O}(-i)$ generate $D^b(P)$.*

Proof Analogous to that of $\Omega^i(i)$.

Now by Definition $\Lambda(V^*)[i]$ generate $K_\Lambda(resp, S(V)[i] generate\ K_S)$.
F_i's are exact and by Lemma 4,
$Hom(\Lambda(V^*)[i], \Lambda(V^*)[j]) = \Lambda^{i-j}(V^*) = Hom(\Omega^i(i), \Omega^j(j)) =$
$Hom(F_1(\Lambda(V^*)[i]), F_1(\Lambda(V^*)[j])$
and $Hom(S(V)[i], S(V)[j]) = S^{i-j}(V) =$
$Hom(\mathcal{O}(-i), \mathcal{O}(-j)) = Hom(F_1(S(V)[i]), F_2(S(V)[j]))$
$\Longrightarrow$ By Lemma 2, the theorem is proved.

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