Non-vanishing of Modular L-functions *

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1 Introduction

For a real Dirichlet character χ of conductor $n > 1$ one expects

Conjecture 1 *$L(s;\chi) \neq 0$ for $Re(s) > 1 - \frac{c}{(\log q)^A}$, where c and A are absolute constants.*

Siegel showed that there is atmost one zero in this region and this zero is known as Landau-Siegel zero. Duke [Duk] established a connection between Dirichlet L-function and the modular L-functions through which one can reduce the above problem to one almost free of the character χ. Iwaniec and Saranak made a major breakthrough [IS] in this aspect. Any slight improvement in their results would prove the above conjectures with $A = 4$. In this note we shall look at the results and techniques of Iwaniec and Saranak and their connection with the Landau-Siegel zeros.

2 Modular Forms

All the proofs of the results mentioned in this and the next section can be found in [Kob], [Miy], [Ogg] and [Ser].

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Let $\Gamma_0 = SL_2(\mathbb{Z})$, the set of all 2×2 matrices, with integer entries, whose determinant is 1. For a natural number N, let

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0 : c \equiv 0 \pmod N, a \equiv d \equiv 1 \pmod N \right\}, \text{ and}$$
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0 : c \equiv 0 \pmod N \right\}.$$

A subgroup Γ of Γ_0 is called *congruence subgroup* if there exists $N \geq 1$ such that $\Gamma_1(N) \subset \Gamma$. Hence $\Gamma_0(N)$ is a congruence subgroup.

Remark $\Gamma_1(N)$ is a finite index subgroup of Γ_0 and so are the congruence subgroups. It is therefore natural to ask whether there is a finite index subgroup Γ of Γ_0 which is not a congruence subgroup. There are many such Γ's but is not straight forward to construct one (look at [New]). Incidentally, the *congruence subgroup problem* asserts that for $n \geq 3$, any finite index subgroup of $SL_n(\mathbb{Z})$ is a congruence subgroup (the definition being similar). This was proved by Baass-Milner-Serre [BMS].

Let $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0, z \in \mathbb{C} \cup \{\infty\}$, define

$$\gamma z = \frac{az+b}{cz+d}. \tag{1}$$

Note that, $Im(\gamma z) = \frac{Im(z)}{|cz+d|^2}$ and that $(\gamma_1\gamma_2)(z) = (\gamma_1(\gamma_2(z))$ for all $\gamma, \gamma_1, \gamma_2 \in \Gamma_0$. Therefore, the above defines an action of Γ_0 on $\mathbb{H}$.

Remark Γ_0 acts transitively on $\mathbb{Q} \cup \{\infty\}$.

Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function and k be an integer. For $\gamma \in \Gamma_0$ we define

$$f(z)|_k\gamma = (cz+d)^{-k} f(\gamma z). \tag{2}$$

Definition A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called *Modular form* (respectively *cusp form*) of weight k for Γ_0, if $f|_k\gamma = f$ for all $\gamma \in \Gamma_0$ and f is holomorphic (resp. vanishing) at ∞.

The last condition is explained in the following. We note that $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0$. The condition $f|_kT = f$ translates to $f(z+1) = f(z)$ for $z \in \mathbb{H}$. This would imply that f has a Laurent-series expansion in terms of $e^{2\pi iz} = q$, that is, $f(z) = \sum_{n=-\infty}^{+\infty} a(n)q^n$. This is called the Fourier expansion (or q-expansion) of f. By f holomorphic (resp. vanishing) we mean that $a(n) = 0$ for $n < 0$ (resp. $n \leq 0$).

Another view point is to consider the map $\phi : \mathbb{H} \to \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ defined by $z \mapsto e^{2\pi i z} = q$. Under this map $i\infty$ maps to 0. If f satisfies the condition $f(z+1) = f(z)$ then by the map ϕ we a get a holomorphic function $\tilde{f}$ (corresponding to f) on $\mathbb{D} - \{0\}$. f being holomorphic (resp. vanishing) at ∞ is equivalent to $\tilde{f}$ being the same at the origin.

Now, we shall define Modular forms for a general congruence subgroup Γ. By a *cusp* of Γ we mean a Γ-equivalence class of $\mathbb{Q} \cup \{\infty\}$.

Definition A Modular form (resp. cusp form) of weight k for Γ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying $f|_k\gamma = f$ for all $\gamma \in \Gamma$ and which is holomorphic (resp. vanishing) at all the cusps.

Again, the last condition needs some explaination. Let $\mathfrak{a}$ be a representative of a cusp of Γ. By an earlier remark there exists $\gamma \in \Gamma_0$ such that $\gamma\mathfrak{a} = \infty$. f is holomorphic (resp. vanishing) at the cusp $\mathfrak{a}$ if $f|_k\gamma^{-1}$ is holomorphic (resp. vanishing) at ∞ (in the same sense as explained earlier).

Remarks

1. If $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma$, then a modular form f of weight k should satisfy $f(z) = (-1)^k f(z)$. Therefore, if $-I \in \Gamma$ then, for odd values of k there exists no nontrivial modular forms of weight k for Γ.

2. The set of all modular forms of weight k for Γ is a $\mathbb{C}$-vector space. We shall denote this space by $M_k(\Gamma)$. The set of cusp forms of weight k for Γ is a vector subspace of $M_k(\Gamma)$ which we shall denote by $S_k(\Gamma)$. Moreover, if $f \in M_k(\Gamma)$ and $g \in M_l(\Gamma')$ then $fg \in M_{k+l}(\Gamma \cap \Gamma')$.

3. For a character χ modulo N, we define $M_k(N,\chi) = \{f : \mathbb{H} \to \mathbb{C} : f(\gamma z) = \chi(d)f(z)$, for all $\gamma \in \Gamma_0(N)\}$, where d is the right bottom entry of γ. Note that this is a subspace of $M_k(\Gamma_1(N))$. Moreover, for the principal character this set is nothing but $M_k(\Gamma_0(N))$. It is an easy exercise in representation theory to show that $M_k(\Gamma_1(N)) = \bigoplus_\chi M_k(N,\chi)$, where the sum runs over all the characters modulo N. We shall mainly look at $M_k(N,\chi)$ as they give most of the informations.

4. Γ_0 is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and T. Therefore, it is enough to check the first condition in the definition of Modular forms for $\Gamma_0(N)$ for these generators (one should also note that $\Gamma_0(N)$ has many generators).

5. One can explicitly write down the dimension of $M_k(\Gamma_0)$. For the proof of the following refer [Ser].

$$\dim_{\mathbb{C}}(M_k(\Gamma_0)) = \begin{cases} \left[\frac{k}{12}\right]+1 & \text{for } k \text{ even}, k \geq 0, k \not\equiv 2 \pmod{12} \\ \left[\frac{k}{12}\right] & \text{for } k \text{ even}, k \geq 0, k \equiv 2 \pmod{12} \\ 0 & \text{otherwise} \end{cases}$$

Also, for $k > 2, k$ even, one has $\dim_{\mathbb{C}}(M_k(\Gamma_0)) = \dim_{\mathbb{C}}(S_k(\Gamma_0)) + 1$.

6. One can also define Modular forms for half-integral weights. There is a connection between half-integral weight Modular forms and integral weight Modular forms given by *Shimura Correspondence*. For more details about these refer [Kob], [Sh2]

3 Hecke Operators and Modular L-functions

3.1 Hecke Operators for Γ_0

For a finite dimensional $\mathbb{R}$-vector space V, a lattice L is a discrete subset of V such that there is a $\mathbb{R}$-basis $\{e_1, e_2, \ldots, e_n\}$ of V which is a $\mathbb{Z}$-basis for L, i.e, $L = \mathbb{Z}e_1 \bigoplus \mathbb{Z}e_2 \bigoplus \cdots \bigoplus \mathbb{Z}e_n$. Considering $\mathbb{C}$ as a vector space over R, let $\mathcal{L}$ be the set of all lattices of $\mathbb{C}$. Let $\mathcal{R}$ denote the free abelian group generated by $\mathcal{L}$, that is, $\mathcal{R} = \{\sum_{L \in \mathcal{L}} a_L L : a_L = 0$ for all but finitely many $L \in \mathcal{L}\}$.

Let $f : \mathbb{H} \to \mathbb{C}$ be a map satisfying $f|_k\gamma = f$ for all $\gamma \in \Gamma_0$. We can then define $F : \mathcal{R} \to \mathbb{C}$ by $F(L) = \omega_2^{-k} f(\frac{\omega_1}{\omega_2})$, where $\{\omega_1, \omega_2\}$ is a $\mathbb{Z}$-basis for L with $Im(\frac{\omega_1}{\omega_2}) > 0$. This definition is unambiguous for if $\{\omega_1', \omega_2'\}$ is another $\mathbb{Z}$-basis with $Im(\frac{w_1'}{\omega_2'}) > 0$, then there is an element $\gamma \in \Gamma_0$ such that $\gamma \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1' \\ w_2' \end{pmatrix}$, which would imply that $w_2'^{-k} f(\frac{w_1'}{\omega_2'}) = \omega_2^{-k} f(\frac{\omega_1}{\omega_2})$. Notice that F has the property: $F(tL) = t^{-k}F(L)$, for $t \in \mathbb{C}^*, L \in \mathcal{L}$. A function satisfying this property is called *lattice function*. We also have, $f(z) = F(< z, 1 >)$ and hence we have a one-one correspondence between $\{f : \mathbb{H} \to \mathbb{C} : f|_k\gamma = f$ for all $\gamma \in \Gamma_0\}$ and $\{F : \mathcal{R} \to \mathbb{C} : F(tL) = t^{-k}F(L)$ for all $t \in \mathbb{C}, L \in \mathcal{L}\}$.

We now define correspondences $T(n), R_\lambda$ on $\mathcal{R}$ by defining their action on the lattices. For $n \geq 1, L \in \mathcal{L}$,

$$T(n)L = \sum_{[L:L']=n} L', \tag{3}$$

the sum of sublattices of L with index n. The sum is finite since $[L : L'] = n$ would imply that $L \supset L' \supset \frac{1}{n}L$. For $\lambda \in \mathbb{C}^*$, we define $R_\lambda L = \lambda L$. We then have

Proposition 1 *The correspondences defined above satisfy:*

$$R_\lambda R_\mu = R_{\lambda\mu} \qquad \lambda, \mu \in \mathbb{C}^* \tag{4}$$

$$R_\lambda T(n) = T(n)R_\lambda \qquad \lambda \in \mathbb{C}^*, n \geq 1 \tag{5}$$

$$T(n)T(m) = T(nm) \qquad (n,m) = 1 \tag{6}$$

$$T(p^n)T(p) = T(p^n + 1) + pT(p^{n-1})R_p \quad p \text{ prime}, n \geq 1 \tag{7}$$

Thus the algebra generated by $T(p), R_\lambda, \lambda \in C^*, p$ a prime, is commutative and contains all the $T(n)$'s.

For a lattice function F we define,

$$(R_\lambda F)(L) = F(R_\lambda L) = \lambda^{-k} F(L),$$

$$(T(n)F)(L) = F(T(n)L) = \sum_{[L:L']=n} F(L').$$

Therefore, the equations (6) and (7) translate to

$$(T(n)T(m))F = T(nm)F \qquad (n,m) = 1 \tag{8}$$

$$(T(p^n)T(p))F = T(p^{n+1}) + p^{1-k}T(p^{n-1})F \quad n \geq 1, p \text{ prime} \tag{9}$$

The correspondence between lattice functions and functions $f : \mathbb{H} \to \mathbb{C}$, satisfying $f|_k\gamma = f$ for all $\gamma \in \Gamma_0$, allows us to extend these actions to the set of Modular forms.

For $f \in M_k(\Gamma_0)$, let F be the corresponding lattice function. We then define

$$T(n)f(z) = n^{k-1}T(n)F(< z, 1 >) \tag{10}$$

(the multiplication by n^{k-1} is to clear the denominators in the later equations). By manipulating the matrices we get

$$T(n)f(z) = n^{k-1} \sum_{\substack{ad=n \\ 0 \leq b < d}} d^{-k} f(\frac{az+b}{d}) \tag{11}$$

The equations (8) and (9) now translate to

$$T(n)T(m)f = T(nm)f \qquad (n,m)=1 \tag{12}$$
$$T(p^n)T(p)f = T(p^{n+1})f + p^{k-1}T(p^{n-1})f \quad n \geq 1, p \text{ prime} \tag{13}$$

These $T(n)$'s are called the *Hecke operators*. The importance of these operators comes from their eigenvectors and eigenvalues. To see this, we shall first look at the Fourier coefficients of $T(n)f$. Let $f(z) = \sum_{m\geq 0} a(m)q^m$ and $(T(n)f)(z) = \sum_{m\geq 0} b(m)q^m$. Then we have

Proposition 2

$$b(m) = \sum_{d|(n,m)} d^{k-1} a(\frac{mn}{d^2}). \tag{14}$$

Corollary 1 *With the same notations as above we have,*

1. $b(0) = (\sum_{d|n} d^{k-1})a(0), b(1) = a(n)$.
2. *If* $n = p$ *a prime then*

$$b(m) = a(pm), \quad \text{for } m \not\equiv 0 \pmod{p},$$

$$b(m) = a(pm) + p^{k-1}a(m/p), \quad \text{for } m \equiv 0 \pmod{p}.$$

3. *If* $f \in M_k(\Gamma_0)$ *(resp.* $S_k(\Gamma_0)$*) then so does* $T(n)f$.

Now, suppose that $f(z) = \sum_{m\geq 0} a(m)q^m \in M_k(\Gamma_0)$ is a non-constant eigen vector for all the $T(n)$'s. Then we have $a(1) \neq 0$. Moreover, if $a(1) = 1$, then $T(n)f = a(n)f$ for all $n \geq 1$. This follows easily from the above results. Thus we have from (12) and (13),

$$a(n)a(m) = a(nm) \qquad (n,m)=1 \tag{15}$$
$$a(p^n)a(p) = a(p^{n+1}) + p^{k-1}a(p^{n-1}) \quad n \geq 1, p \text{ prime} \tag{16}$$

3.2 L-functions and twisted forms

For $f(z) = \sum_{n\geq 0} a(n)q^n$, we let $L(s;f) = \sum_{n\geq 1} a(n)n^{-s}$. Therefore, for a common eigenform f, the L-function admits an Euler product,

$$L(s,f) = \sum_{n\geq 1} a(n)n^{-s} = \prod_{p \text{ prime}} (1 - a(p)p^{-s} + p^{k-1-2s})^{-1} \tag{17}$$

With little more effort one can define Hecke operators for the general modular forms. For $f \in M_k(N, \chi)$ we will obtain

$$T(p^n)T(p)f = T(p^{n+1})f + \chi(p)p^{k-1}T(p^{n-1})f, p \not| N, p \text{ prime}, n \geq 1, \quad (18)$$

$$T(p^{n+1})f = T(p)^n, p|N, p \text{ prime}. \quad (19)$$

Therefore the L-function of a common eigenfunction takes the Euler product

$$L(s, f) = \prod_{p \text{ prime}} (1 - a(p)p^{-s} + \chi(p)p^{k-1-2s})^{-1}. \quad (20)$$

Note that $\chi(p) = 0$ whenever $p|N$.
Remark A modular form being a common eigenfunction is equivalent to the corresponding L-function having an Euler product.

Introducing Dirichlet character into the Fourier expansion, we get what are called *twisted forms*. Let $f(z) = \sum_{n \geq 0} a(n)q^n \in M_k(N, \chi)$ and let d_1 be the conductor of χ. Let ψ be a primitive Dirichlet character modulo d_2. We define $f_\psi(z) = \sum_{n \geq 0} \psi(n)a(n)q^n$.

Lemma 1 *$f_\psi \in M_k(M, \psi^2\chi)$, where M is the least common multiple of $N, d_1 d_2$ and d_2^2. If f is a cusp form then so is f_ψ.*

3.3 Functional Equation

Lemma 2 *A modular form f satisfies the conditions (C):*

f converges absolutely and uniformly on any compact subset of $\mathbb{H}$,
$f(z) = O(Im(z)^{-v}), v \in \mathbb{R}$, as $z \to 0$ uniformly.

For $f(z) = \sum_{n \geq 0} a(n)q^n$ and a character ψ of conductor m, we define

$$\Lambda_N(s; f, \psi) = (\frac{2\pi}{m\sqrt{N}})^{-s}\Gamma(X)L(s; f, \psi). \quad (21)$$

By $\Lambda_N(s; f)$ we shall mean that ψ is trivial and hence $m = 1$.

Theorem 1 (Hecke) *Let $f(z) = \sum_{n \geq 0} a(n)q^n$ and $g(z) = \sum_{n \geq 0} b(n)q^n$ satisfy C. For positive integers k and N, the following are equivalent:*
(A) $g(z) = (-i\sqrt{N}z)^{-k} f(\frac{-1}{Nz})$

(B) $\Lambda_N(s;f)$ *and* $\Lambda_N(s;g)$ *can be analytically continued to the whole s-plane, satisfy the functional equation*

$$\Lambda_N(s;f) = \Lambda_N(k-s;g), and$$

$\Lambda_N(s,f) + \frac{a(0)}{s} + \frac{b(0)}{k-s}$ *is holomorphic on the whole s-plane and bounded on any vertical strip.*

As a immediate corollary we have,

Corollary 2 *Let* $f(z) = \sum_{n\geq 0} a(n)q^n$ *satisfy condition C. Then* $f \in M_k(\Gamma_0)$ *if and only if* $\Lambda(S,f) = (2\pi)^{-s}\Gamma(s)L(s,f)$ *has an analytic continuation,* $\Lambda(s;f) + \frac{a(0)}{s} + \frac{(-1)^{k/2}a(0)}{k-s}$ *is holomorphic on whole of* $\mathbb{H}$*, bounded on any vertical strip,* $\Lambda(s;f) = (-1)^{k/2}\Lambda(k-s;f)$*.*

We shall for the present time being fix N. For a Dirichlet character ψ of conductor m let

$$\begin{aligned} C_\psi &= \chi(m)\psi(-N)W(\psi)/W(\overline{\psi}) \\ &= \chi(m)\psi(N)W(\psi)^2/m, \end{aligned}$$

where

$$W(\phi) = \sum_{n \pmod n} \phi(n)e^{2\pi i n/m}$$

is the *Gauss sum* of ϕ. Let $\omega_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Then if $f \in M_k(N,\chi)$ then $f|_k\omega_N \in M_k(N,\overline{\chi})$, thus giving an isomorphism between $M_k(N,\chi) and M_k(N,\overline{\chi})$. By Hecke's theorem we will get

Theorem 2 *Let* $f \in S_k(N,\chi)$ *and* ψ *a primitive Dirichlet character of conductor* $m, (m,N) = 1$*. Then* $\Lambda_N(s;f,\psi)$ *can be holomorphically continued to the s-plane, bounded on any vertical strip and satisfies*

$$\Lambda_N(s;f,\psi) = i^k C_\psi \Lambda_N(k-s;f|_k\omega_N,\overline{\psi}). \tag{22}$$

We have the converse of this theorem but is not as straightforward as expected.

Theorem 3 (A. Weil) *Suppose that* $f(z) = \sum_{n\geq 0} a(n)q^n$ *satisfy the condition C. Let* $\mathcal{M}$ *be a set of primes meeting every primitive arithmetic progression* $a+nb, (a,b)=1$ *with* $(m,N)=1$ *for* $m \in \mathcal{M}$. *Let* $\epsilon = \pm 1$. *Assume that* $\Lambda_N(s;f) + N^{-s/2}(\frac{a(0)}{s} + \epsilon\frac{a(0)}{(k-s)})$ *is bounded in every vertical strip and that* $\Lambda_N(s;f) = \epsilon N^{k/2-s}\Lambda(k-s)$. *Further, for all characters* χ *of conductor* $m \in \mathcal{M}$, *assume that* $\Lambda_N(s;f,\chi)$ *is bounded in every vertical strip and that* $\Lambda_N(s;f,\chi) = \epsilon C_\chi N^{k/2-s}\Lambda(k-s;f,\overline{\chi})$, *where* $C_\chi = \psi(m)\chi(-n)W(\chi)/W(\overline{\chi})$, *with* ψ *being a character modulo* N. *Then* $f \in M_k(N,\psi)$ *and* f *satisfies the functional equation* $f = \epsilon i^k f|_k\omega_N$. *Moreover if* $L(s;f)$ *converges absolutely at* $s = k-\delta$ *for some* $\delta > 0$, *then* f *is a cusp form.*

Remark Loosely speaking, $\Lambda_N(s;f,\chi)$ is symmetric with respect to the *critical point* $s = k/2$. So, it is interesting to know what happens at the critical point. In case of weight 2, the order of vanishing of L-function associated with a newform for $\Gamma_0(N)$ at the critical point is conjecturally equal to the rank of the group of rational points on an elliptic curve.

3.4 Petersson inner product and New forms

For cusp forms $f, g \in M_k(\Gamma)$ we define

$$< f, g >= \int_{\Gamma\setminus\mathbb{H}} f(z)\overline{g(z)}y^{k-2}dz \tag{23}$$

This converges, hence defines an inner product on $S_k(N,\chi)$ and is called *Petersson inner product.* One of the main properties of the inner product is the following

Theorem 4 (Petersson) *For* $f, g \in S_k(\Gamma_0(N))$ *we have* $< T(n)f, g >=< f, T(n)g >$, *if* $(n,N) = 1$.

As a consequence we have

Theorem 5 *The Fourier coefficients of an eigenform are real and algebraic.*

and

Theorem 6 $S_k(\Gamma_0(N))$ *has an orthonormal basis each of whose forms is an eigenfunction for all* $T(n)$ *with* $(n,N) = 1$.

The reason why we need to force the condition $(n, N) = 1$ is that if $f \in S_k(\Gamma_0(d))$ for some $d|N, then f \in S_k(\Gamma_0(N))$. Also, if $N = dd'$ and if $g \in S_k(\Gamma_0(d'))$ then $g(dz) \in S_k(\Gamma_0(N))$. The forms that arise in these two fashions are called *old forms* and one cannot expect these forms to be common eigenfunction for all the $T(n)$'s. However, the subspace orthogonal to the space spanned by old forms has an orthonormal basis whose forms are eigenfunctions for all the $T(n)$'s.
Definition A cusp form $f \in S_k(N, \chi)$ is called a *new form* if it is a common eigenfunction for all $T(n)$'s, $n \geq 1$ and is orthogonal to all the old forms.

4 Duke's result

4.1 Statements

For $N \geq 1$, let $\mathcal{F}$ denote the set of all newforms of weight 2 for $\Gamma_0(N)$. Note that these are precisely the ones which correspond to the elliptic curves.

Theorem 7 (Duke) *Suppose that χ is a fixed primitive Dirichlet character modulo d. Then there is a positive absolute constant C and a constant C_d depending only on d such that for prime $N > C_d$ there are at least $CN\log^{-2}N$ forms $f \in \mathcal{F}_N$ for which $L(1; f, \chi) \neq 0$.*

Given two real distinct characters χ_1 and χ_2, let $P_f(s) = L(s; f, \chi_1)L(s; f, \chi_2)$.

Theorem 8 (Duke) *Suppose that $\chi_1 \pmod{d}_1$ and $\chi_2 \pmod{d}_2$ are fixed distinct primitive real Dirichlet character. Then there are positive constants C_1 and C_2 such that there are at least $C_2N\log^{-10}N$ forms $f \in \mathcal{F}_N$ with*

$$\mathrm{ord}_{s=1}P_f(s) = \begin{cases} 0 & \text{if } \chi_1\chi_2(-N) = 1 \\ 1 & \text{if } \chi_1\chi_2(-N) = -1 \end{cases}$$

provided $N > C_1$ is prime.

4.2 Sketch of Proof

For a modular form f we denote by $a_f(n)$ its nth Fourier coefficient.

Lemma 3 *Let $N \geq 1, k \geq 1$. Let $\mathcal{F}$ be an orthonormal basis for the set of cusp forms of weight k for $\Gamma_o(N)$. Then for m and n positive integers*

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}} \sum_{f\in\mathcal{F}} \frac{a_f(m)a_f(n)}{< f, f >} = \delta_{mn} + 2\pi i^k \sum_{c\equiv 0(N)} c^{-1} S(m, n; c) J_{k-1}(\frac{4\pi\sqrt{mn}}{c}), \tag{24}$$

where

$$S(m, n; c) = \sum_{ad\equiv 1(c)} e^{2\pi i(\frac{ma+nd}{c})}$$

is the classical Kloosterman sum and $J_n(x) = \sum_{l\geq 0} \frac{(-1)^l}{l!\Gamma(l+n+1)}(\frac{x}{2})^{2l+n+1}$ is the Bessel's J-function.

As a consequence of Riemann Hypothesis for curves, which was proved by A. Weil, we get the inequality

$$S(m, n; c) \leq (m, n, c)^{\frac{1}{2}} c^{\frac{1}{2}} d(c) \tag{25}$$

where $d(c)$ is the divisor function. For $x \geq 0$ we have

$$J_n(x) \leq \min\{x^{-1/2}, x^n\}. \tag{26}$$

Using these bounds we get

Corollary 3 *For m and n positive integers and N prime*

$$|\sum_{f\in\mathcal{F}_N} \frac{1}{4\pi < f, f >} \frac{a_f(m)}{\sqrt{m}} \frac{a_f(n)}{\sqrt{n}} - \delta_{mn}| \leq 539 N^{-3/2} (m, n)^{1/2} \sqrt{mn} \tag{27}$$

The L-function associated with $f \in \mathcal{F}_N$ can be broken into two parts one of which is a rapidly decreasing function and hence will not contribute much to the value of L-function at the critical value. We recall the functional equation (21) for weight 2

$$(\frac{2\pi}{d\sqrt{N}})^{-s} \Gamma(s) L(s; f, \chi) = -C_\chi (\frac{2\pi}{d\sqrt{N}})^{s-2} \Gamma(2-s) L(2-s; f, \chi). \tag{28}$$

Lemma 4 *For any $x > 0$, let $A(x) = \sum_{n\geq 1} \chi(n) a_f(n) n^{-1} e^{-2\pi n/x}$. Then we have $L(1; f, \chi) = A(x) - C_\chi \overline{A}(\frac{Nq^2}{x})$.*

Choosing $x = d^2 N \log N$ in the above lemma and using (27) we get

Proposition 3 *Let χ be a fixed primitive Dirichlet character modulo d. Then we have*

$$\sum_{f \in \mathcal{F}_N} \frac{L(1; f, \chi)}{4\pi < f, f >} = 1 + O(N^{-1/2} \log N)$$

for N prime, the implied constant depending only on d.

Fix any primitive characters χ_1 modulo d_1 and χ_2 modulo d_2. Let $P_f(s) = L(s; f, \chi_1)L(s; f, \chi_2) = \sum_{n \geq 1} b_f(n) n^{-s}$. For $r = 0, 1$ let

$$g_r(x) = \frac{1}{2\pi i} \int_{Re(s)=3/4} (2\pi)^{-2s} \Gamma(s) \Gamma(s - r + 1) x^{-s} ds \tag{29}$$

Further, let $B_r(x) = \sum_{n \geq 1} b_f(n) n^{-1} g_r(n/x)$. Then we have

Lemma 5 *Let $f \in \mathcal{F}_N$ for $N \geq 1$ and suppose that χ_1 and χ_2 are primitive with $(d_1 d_1, N) = 1$. For any $x > 0$ we have*

$$P_f(1) = B_0(x) + \hat{\epsilon} \overline{B_0}((N d_1 d_2)^2 / x) \tag{30}$$

while if $P_f(1) = 0$ then for any $x > 0$ we have

$$P_f'(1) = B_1(x) - \hat{\epsilon} \overline{B_1}((N d_1 d_2)^2 / x) \tag{31}$$

where $\hat{\epsilon} = \chi_1 \chi_2(N)(W(\chi_1) W(\chi_2))^2 (d_1 d_2)^{-1}$.

And we derive using this lemma

Proposition 4 *Let χ_1 modulo d_1 and χ_1 modulo d_2 be primitive Dirichlet characters such that either $\chi_1 = \overline{\chi_2}$ or χ_1 and χ_2 are real distinct. In the first case we have*

$$\sum_{f \in \mathcal{F}_N} \frac{P_f(1)}{4\pi < f, f >} = \prod_{p | d_1} (1 - p^{-1}) log N + c_1 + O(N^{-1/2} \log N)$$

for N prime with $(d_1, N) = 1$ where c_1 and the implied constant depend only on d_1. Otherwise

$$\sum_{f \in \mathcal{F}_N} \frac{P_f(1)}{4\pi < f, f >} = 2L(1, \chi_1 \chi_2) + O(N^{-1/2} \log N)$$

for N prime with $\chi_1\chi_2(-N) = 1$ while

$$\sum_{f\in\mathcal{F}_N} \frac{P_f'(1)}{4\pi < f, f >} = 2L(1, \chi_1\chi_2)\log N + c_2 + O(N^{-1/2}\log N)$$

for N prime with $\chi_1\chi_2(-N) = -1$, where c_2 and the implied constant depend only on $d_1 d_2$.

Calculation using Cauchy-Schwartz ineqaulity will yeild

Proposition 5 *Let χ be a primitive Dirichlet character modulo d. Then there is a constant C_d depending only on d such that for prime$N > C_d$*

$$\sum_{f\in\mathcal{F}_N; L(1,f,\chi)\neq 0} \frac{1}{4\pi < f, f >} >> \log^{-1} N$$

the implied constant being absolute. Further if χ_1 and χ_2 are distince real primitive Dirichlet character modulo d_1 and d_2 respectively, then for $r = 0, 1$,

$$\sum_{f\in\mathcal{F}_N; P_f^{(r)}>0} \frac{1}{4\pi < f, f >} >> \log^{-9} N$$

for N, a sufficiently large prime with $\chi_1\chi_2(-N) = (-1)^r$, the implied constants depending only on $d_1 d_2$.

The Theorems 7 and 8 follow from the above Proposition along with the fact that $\frac{1}{4\pi<f,f>} << N^{-1}\log N$.

5 Results of Iwaniec and Sarnak

5.1 Results

We shall first set up few notations that we use in this section. $\mathcal{H}_k(N)$ will denote the set of all new forms of weight k for $\Gamma_0(N)$. $\mathcal{H}_k^{\pm}(N) = \{f \in \mathcal{H}_k(N) : \Lambda(s, f) = \pm\Lambda(k - s, f)\}$. For simplicity we let $S_k(N) = S_k(\Gamma_0(N))$. Let h be a smooth test function and let $H = \int_0^\infty h(t)dt$. For $f \in S_k(N)$ we let $\omega_f = \frac{\Gamma(k)}{(4\pi)^k}\frac{1}{<f,f>}$.

Let K be a large parameter. We define

$$\mathcal{A}_K[X_f] = \sum_{k \text{ even}} \frac{h(k/K)}{|\mathcal{H}_k(1)|} \sum_{f\in\mathcal{H}_k(1)} \omega_f X_f \tag{32}$$

and

$$\mathcal{B}_N[X_f] = \sum_{f\in\mathcal{H}_k(N)} \omega_f X_f. \tag{33}$$

X_f is a variable function of f which we shall choose later. Note that $\mathcal{H}_k(1) = \dim S_k(1)$ since all the common eigenforms are new. Thus $\mathcal{H}_k(1) \sim k/12$ as $k \to \infty$. Setting $n = m = 1$ in (24) and using the inequalities (25) and (26) one gets

$$\sum_{f\in\mathcal{H}_k(1)} \omega_f \sim k/12$$

as $k \to \infty$. Thus we have $\mathcal{A}_K[1] \sim \frac{1}{2}HK$ as $k \to \infty$. In a similar fashion we get $\mathcal{B}_N[1] \sim N$ as $N \to \infty$. Let χ be a real primitive Dirichlet character modulo d. Then one has

Proposition 6 *Let δ be a small constant. Then we have*

$$\mathcal{A}_K[L(k/2; f, \chi)] \sim HK, \tag{34}$$

the asymptotic being uniform for $d \leq K^\delta$ as $K \to \infty$. Also,

$$\mathcal{B}_N[L(k/2; f, \chi)] \sim N, \tag{35}$$

uniformly for $d \leq N^\delta$.

For $f \in S_k(N)$ we define

$$M(f, \chi) = \sum_{m\leq M} x_m a_f(m)\chi(m)m^{-k/2} \tag{36}$$

with real variables x_m, which we shall choose later. These are called *mollifiers*. The idea is to minimize $\mathcal{A}_K[L^2(k/2; f, \chi)M^2(f, \chi)]$ subject to the linear constraint $\mathcal{A}_K[L(k/2; f, \chi)M(f, \chi)] = HK$. Asymptotically the optimal choice of x_m is

$$x_m \sim \frac{\mu(m)m(\log M/m)^2}{2\sigma(m)\zeta(2)\log M}. \tag{37}$$

where μ is the Möbius function and σ is the divisor function. We can then approximate the mollifier to

$$M(f,\chi) \sim \sum_{m\le M} \mu_f(m)\chi(m)m^{-k/2}\left(1-\frac{\log m}{\log M}\right), \tag{38}$$

where $\mu_f(m) = \mu(m)a_f(m)$ if $m = ab^2$ is cubefree with $(b, N) = 1$, and $\mu_f(m) = 0$ otherwise.

Theorem 9 *For the mollifier as above of length $M \le D^{-1}K(\log K)^{-20}$, we have*

$$\mathcal{A}_K[L(k/2; f,\chi)M(f,\chi)] \sim HK, \tag{39}$$

$$\mathcal{A}_K[L^2(k/2; f,\chi)M^2(f,\chi)] \sim 2HK\left(1+\frac{\log DK}{\log M}\right) \tag{40}$$

uniformly for $d \le K^\delta$, as $K \to \infty$.

Similarly for $\mathcal{B}_N$ we have,

Theorem 10 *Fix $k \ge 2, k$ even. Let N be a large, squarefree number with $(d, N) = 1$. For a mollifier of length $M \le D^{-1}\sqrt{(N)}(\log N)^{-20}$, we have*

$$\mathcal{B}_N[L(k/2; f,\chi)M(f,\chi)] \sim N, \tag{41}$$

$$\mathcal{B}_N[L^2(k/2; f,\chi)M^2(f,\chi)] \sim 2N\left(1+\frac{\log D\sqrt{N}}{\log M}\right) \tag{42}$$

uniformly for $d \le N^\delta$, as $N \to \infty$ satisfying $\phi(n) \sim N$.

The summation operators $\mathcal{A}$ and $\mathcal{B}$ are over all the new forms irrespective of their sign in the functional equation. However, we know that if the sign in the functional equation is negative then the L-function vanishes. Manipulating the summation operators with the sign of the functional equation we can get rid of the forms with the negative sign and Cauchy-Schwartz inequality will yeild

Theorem 11 *Let d be fixed and $b(k)\log k \to 0$ as $k \to \infty$. Then*

$$\lim_{K\to\infty} \frac{\mathcal{A}_K\{f \in \mathcal{H}_k^+(1) | L(k/2; f,\chi) \ge b(k)\}}{\mathcal{A}_K\{f \in \mathcal{H}_k^+(1)\}} \ge \frac{1}{2}. \tag{43}$$

An analogous result can be proved for the summation operator $\mathcal{B}_N$. We can remove the wieght ω_f easily as it is related to the L-function $\sum_{n\geq 1} a_f(n)^2 n^{-s}$. Thus we get

Theorem 12 *Let $\mathcal{H}_k(1)$ be the set of normalized eigenforms for $S_k(\Gamma_0(1))$. Fix a positive integer d and let χ be a real Dirichlet character of conductor d. Then*

$$\lim_{K\to\infty} \sum_{k\leq K, i^k=\chi(-1)} \frac{\#\{f \in \mathcal{H}_k(1) | L(k/2; f, \chi) \geq (\log k)^{-2}\}}{\#\{f \in \mathcal{H}_k(1)\}} \geq \frac{1}{2}. \tag{44}$$

And,

Theorem 13 *Let $\mathcal{H}_k(N)$ be the set of newforms for $\Gamma_0(N)$. Fix positive integers d and k. Let χ be a real primitive Dirichlet character of conductor d. Then*

$$\lim_{N\to\infty} \frac{\#\{f \in \mathcal{H}_k(N) | \epsilon(f,\chi) = +1, L(k/2; f, \chi) \geq (\log N)^{-2}\}}{\#\{f \in \mathcal{H}_k(N) | \epsilon(f,\chi) = +1\}} \geq \frac{1}{2} \tag{45}$$

where N runs over squarefree numbers coprime with d and such that $\phi(N) \sim N$.

Similar to the theorems of Duke mentioned in the previous section, we can estimate the sums involving product of two L-functions arising from different characters. For more details refer [IS]. Also, the following theorem can be derived in case of $d = 1$.

Theorem 14 *Fix an integer $k \geq 2$ and a small number $\delta > 0$. Let $\sum'$ denote the summation over squarefree N satisfying $\phi(N) \geq (1-\delta)N$. Let $X(\delta)$ denote the number of such N's with $X \leq N \leq 2X$. Then for all sufficiently large X*

$$\frac{1}{X(\delta)} \sum_{X\leq N\leq 2X}{}' \frac{\#\{f \in \mathcal{H}_k^+(N) | L(k/2; f) \geq (\log N)^{-2}\}}{\#\{f \in \mathcal{H}_k^+(N)\}} \geq c$$

where $c > 1/2$.

The main idea is to use the formula (24) with which we can express the average sums in terms of sums of Kloosterman sums. Therefore, the problem reduces to estimating the sums of Kloosterman sums. The introduction of mollifiers is a known idea, for example, used by Selberg to show that a positive proportion of the zeros of the zeta-function are on the critical line (refer [Sel]). However, the estimation of the variables is not a straight forward one. It is similar to the method of Selberg, but is subtler.

5.2 Landau-Siegel zeros

The motivation for the above work came from the Landau-Siegel zero problem. The connection betweem modular L-functions and the Dirichlet L-functions comes from the following (χ is assumed to be a quadratic character of conducter d):

Proposition 7 *If $\chi(-N) = 1$, then*

$$\mathcal{B}_N^+ [L(k/2; f)L(k/2; f, \chi)] \sim NL(1, \chi) \tag{46}$$

uniformly for $d \leq N^\delta$ as $N \to \infty$ with $\phi(N) \sim N, N$ squarefree. ($\mathcal{B}_N^+$ denotes the summation operator with the condition that the sign in the functional equation of f is positive).

To show that $L(s, \chi) \neq 0$ for $s > 1 - c/(\log d)^a$ it is enough to show that $L(1, \chi)$ is large. Results of Waldsuprger [Wal] say that $L(k/2; f, \chi) \geq 0$. Hence, by the above asymptotic formula it is enough for one to show that $L(k/2; f)L(k/2; f, \chi)$ is large for many f's. However, trying to show that the product is large is as hard as solving the Landau-Siegel zero problem. One can consider the sets $A = \{f \in \mathcal{H}_k^+(N) | L(k/2; f) \text{ is large}\}$ and $B = \{f \in \mathcal{H}_k^+(N) | L(k/2; f, \chi) \text{ is large}\}$. The idea is to establish results of form $|A| = c|H_k^+(N)|$ and $|B| = c'|H_k^+(N)|$. If $c + c' > 1$ then we have that $|A \cap B| \geq (c + c' - 1)|H_k^+(N)|$ which gives the desired result.

If ϵ and ϵ_χ are the signs of $L(s; f)$ and $L(s; f, \chi)$ respectively in their functional equation, then we have $\epsilon\epsilon_\chi = \chi(-N)$. Therefore, if $\chi(-N) = -1$ then either $L(k/2; f) = 0$ or $L(k/2; f, \chi) = 0$. Hence, the contribution of their product to (46) is zero. Thus we have to consider only those N's for which $\chi(-N) = 1$. What Iwaniec and Sarnak could get is $c = c' = 1/2$, the limit case. If one can improve the value of c then the Landau-Siegel zero problem is solved. Thus they have reduced a problem involving a quadratic character χ to a problem that is almost free of χ. The only place where the characater occurs is in the condition $\chi(-N) = 1$. All the methods of Iwaniec-Sarnak fail to extend the value of c, mainly because of this condition.

In fact, Iwaniec and Sarnak conjecture the following:

Conjecture 2 *Let $0 < \beta \le 1$. For any $\epsilon > 0$, there is an effective constant $K(\beta, \epsilon)$ such that for every $K \ge K(\beta, \epsilon)$*

$$\mathcal{A}_K^+\{f \in \mathcal{H}_k(1) | L(k/2, f) \ge (\log k)^{-2}\} \ge (\beta - \epsilon)\mathcal{A}_K^+[1] \tag{47}$$

for $K \ge K(\beta, \epsilon)$.

Conjecture 3 *Let $k \ge 2$ be even and $0 < \beta \le 1$. For any $\epsilon > 0$, there is an effective constant $N(\beta, \epsilon)$ such that*

$$\mathcal{B}_N^+\{f \in \mathcal{H}_k(N) | L(k/2, f) \ge (\log N)^{-2}\} \ge (\beta - \epsilon)\mathcal{B}_N^+[1] \tag{48}$$

for every $N \ge N(\beta, \epsilon), N$ squarefree.

The Theorem 11 and its counterpart establish the conjecture for $\beta \le 1/2$. One expects the above conjecture to be true for $\beta = 1$. The Landau-Siegel zero problem will be solved if one proves any of the above two conjectures for some $\beta > 1/2$. Note that there is a difference between Theorem 14 and Conjecture 3. In the Theorem 14 we sum over all the N's between X and $2X$ whereas in Conjecture 3 we require the result for all N sufficiently large.

5.3 Conclusion

The importance of these results also comes from their connection with the Elliptic curves. The non-vanishing of the L-functions associated with the elliptic curves is conjecturally equivalent to the group of rational points on the elliptic curve being finite. Also, the results have certain applications to the rank of quotients of $J_0(N)$ and Brumer's conjecture. One should note that all the ideas involved are analytic. As mentioned in [IS] it might be possible to improve the results using algebraic methods.

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