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## Travaux

## sur les symétries de Lie

## des équations aux dérivées partielles

## en relation avec la géométrie CR

Document 1, p. 2: Travail (publié) sur une caractérisation explicite des systèmes d'équations différentielles ordinaires qui sont localement équivalents au système décrivant une particule newtonienne dans un champ de forces nul. Calculs très difficiles d'élimination algébrico-différentielle; c'est avec cet article que j'ai appris à calculer, cf. les travaux sur Green-Griffiths.
Document 3, p. 93: Travail (publié) produisant des exemples de tubes CR non algébrisables en codimension quelconque.
Document 3, p. 144: Travail (publié) donnant des bornes pour la dimension du groupe des symétries de Lie de certains systèmes complets d'équations aux dérivées partielles.
Document 4, p. 185: Travail (récent et soumis) caractérisant explicitement les hypersurfaces analytiques réelles de $\mathbb{C}^{2}$ qui sont localement biholomorphes à la sphère $S^{3}$.
Document 5, p. 215: Travail (récent et soumis) caractérisant l'équivalence locale à la pseudo-sphère de Heisenberg dans $\mathbb{C}^{n}$ pour $n \geqslant 3$.
Document 6, p. 233: Travail de synthèse (à paraître) contenant de très nombreuses idées à développer en relation avec la modernisation des travaux de Lie.
[ 6 documents. En 2006, je me suis plus ou moins arrêté de travailler dans cette direction pour étudier les œuvres originales de Lie, plus puissantes que ce qui se fait actuellement dans cette direction.]

# Characterization of the Newtonian 

## free particle system

## in $m \geqslant 2$ dependent variables

Joël Merker

Abstract. We treat the problem of linearizability of a system of second order ordinary differential equations. The criterion we provide has applications to nonlinear Newtonian mechanics, especially in three-dimensional space.

Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, let $x \in \mathbb{K}$, let $m \geqslant 2$, let $y:=\left(y^{1}, \ldots, y^{m}\right) \in \mathbb{K}^{m}$ and let

$$
y_{x x}^{1}=F^{1}\left(x, y, y_{x}\right), \ldots \ldots, y_{x x}^{m}=F^{m}\left(x, y, y_{x}\right),
$$

be a collection of $m$ analytic second order ordinary differential equations, in general nonlinear. We obtain a new and applicable necessary and sufficient condition in order that this system is equivalent, under a point transformation

$$
\left(x, y^{1}, \ldots, y^{m}\right) \mapsto\left(X(x, y), Y^{1}(x, y), \ldots, Y^{m}(x, y)\right),
$$

to the Newtonian free particle system $Y_{X X}^{1}=\cdots=Y_{X X}^{m}=0$.
Strikingly, the explicit differential system that we obtain is of first order in the case $m \geqslant 2$, whereas according to a classical result due to Lie, it is of second order the case of a single equation $(m=1)$.

$$
\text { Acta Applicandæ Mathematicæ, } 92 \text { (2006), no. 2, 125-207 }
$$

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## §1. Introduction

Several physically meaningful systems of ordinary differential equations are of second order, as for instance the free particle in $m$-dimensional space, the damped or undamped harmonic oscillator, coupled or not, having constant time-dependent frequency or not, etc. Such systems are ubiquitous in Newtonian Mechanics, in Hamiltonian Dynamics and in General Relativity.

Two classical major problems are to classify these systems up to point or contact equivalence (Lie's Grail) and to recognize when they coincide with the Euler equations associated to a Lagrangian (inverse variational problem). In small dimensions, complete results hold (Lie, Tresse, Cartan; Darboux, Douglas). However, in arbitrary dimension, both tasks quickly exceed the human as well as the digital computer scale, due to the intrinsic complexity of the underlying symbolic computations (explosion, swelling) and to the exponentially increasing number of cases to be treated. We refer to Olver's monograph [Ol1995] for a panorama of problems, methods and results.

At least, as a first step in classification, with respect to applications, there are both a mathematical and a physical interest in determining concrete, explicit and applicable ("ready-made") criteria for a system of ordinary differential equation to be equivalent, via a local point transformation, to a linear equation.
1.1. Scalar equation. In this respect, we remind the celebrated linearizability criterion for a single equation, due to Lie. Let $\mathbb{K}=\mathbb{R}$ of $\mathbb{C}$. Let $x \in \mathbb{K}$ and $y \in \mathbb{K}$. Consider a local second order ordinary differential equation $y_{x x}=F\left(x, y, y_{x}\right)$, possibly nonlinear, with a locally $\mathbb{K}$-analytic right-hand side ${ }^{1}$.

Theorem 1.2. ([Lie1883], pp. 362-365; [GTW1989]; [Ol1995], p. 406) The following four conditions are equivalent:
(1) $y_{x x}=F\left(x, y, y_{x}\right)$ is equivalent under a local point transformation $(x, y) \mapsto(X, Y)$ to the free particle equation $Y_{X X}=0$;
(2) $y_{x x}=F\left(x, y, y_{x}\right)$ is equivalent to some linear equation $Y_{X X}=$ $G_{0}(X)+G_{1}(X) Y+H(X) Y_{X} ;$
(3) the local Lie symmetry group of $y_{x x}=F\left(x, y, y_{x}\right)$ is eightdimensional, locally isomorphic to the group $\operatorname{PGL}(3, \mathbb{K})$ of all projective transformations of $P_{2}(\mathbb{K})$.

[^0](4) $F_{y_{x} y_{x} y_{x} y_{x}}=0$, or equivalently $F=G+y_{x} H+\left(y_{x}\right)^{2} L+\left(y_{x}\right)^{3} M$, where $G, H, L, M$ are functions of $(x, y)$ that satisfy:
$0=-2 G_{y y}+\frac{4}{3} H_{x y}-\frac{1}{3} L_{x x}+2(G L) y-2 G_{x} M-4 G M_{x}+\frac{2}{3} H L_{x}-\frac{4}{3} H H_{y}$, $0=-\frac{2}{3} H_{y y}+\frac{4}{3} L_{x y}-2 M_{x x}+2 G M_{y}+4 G_{y} M-2(H M)_{x}-\frac{2}{3} H_{y} L+\frac{4}{3} L L_{x}$.

Section 2 of this paper is devoted to a detailed exposition of the original proof of the equivalence between (1) and (4), following [Lie1883]. In the contemporary literature, to the author's knowledge, there is no modern restitution of Lie's elegant proof, whereas the description of an alternative proof of Theorem 1.2 as a byproduct of É. Cartan's equivalence algorithm appears in the references [Tr1896], [Ca1924], [GTW1989], [Ol1995], [23]².

Lie's Grail would comprise:

- a complete classification of all Lie algebras of local vector fields; Lie achieved this task in dimension 2 over $\mathbb{C}$ (real case: [GKO1992]); however, as soon as the dimension is $\geqslant 3$, the complete classification is unknown, due to the intrinsic richness of imprimitive Lie algebras of vector fields;
- a list of all the possible Lie algebras that can be realized as infinitesimal Lie symmetry algebras of partial differential equations, together with their Levi-Malčev decomposition;
- an explicit Gröbner basis of the (noncommutative) algebra of all differential invariants of each equation in the list.
However, complete results hold only for the scalar second order ordinary differential equation. Some tables extracted from Lie's Gesammelte Abhandlungen and from Tresse's prized thesis [Tr1896] may be found in [Ol1995].
1.3. Systems. Let $x \in \mathbb{K}$, let $m \geqslant 2$, let $y:=\left(y^{1}, \ldots, y^{m}\right) \in \mathbb{K}^{m}$ and let
(1.4) $y_{x x}^{1}(x)=F^{1}\left(x, y(x), y_{x}(x)\right), \ldots \ldots, y_{x x}^{m}(x)=F^{m}\left(x, y(x), y_{x}(x)\right)$

[^1]be a collection of $m$ analytic second order ordinary differential equations, possibly nonlinear, of the most general form.

Motivated by physical applications and by geometrical questions, some authors (Leach [Le1980], Grissom-Thompson-Wilkens [GTW1989], González-López [GL1988], Fels [Fe 1995], Crampin-MartínezSarlet [CMS1996], Doubrov [Do2000], Grossman [Gr2000], MahomedSoh [MS2001], and others) have been interested in a least characterizing those that have the Lie symmetry group of maximal dimension. In [Le1980], based on the belief that the equivalence between (1), (2) and (3) of Theorem 1.2 would persist in the case $m \geqslant 2$, it was conjectured that the symmetry algebra of every linear system

$$
\begin{equation*}
y_{x x}^{j}=G_{0}^{j}(x)+\sum_{l=1}^{m} y^{l} G_{1, l}^{j}(x)+\sum_{l_{1}=1}^{m} y_{x}^{l_{1}} H_{l_{1}}^{j}(x), \quad j=1, \ldots, m, \tag{1.5}
\end{equation*}
$$

of $m \geqslant 2$ second order differential equations has a Lie symmetry group locally isomorphic to the full transformation group $\operatorname{PGL}(m+1, \mathbb{K})$ of the projective space $P_{m+1}(\mathbb{K})$. However, González-López [GL1988] infirmed this expectation, and produced a necessary and sufficient condition (see Corollary 1.8 below) for local equivalence of such linear systems to the free particle system

$$
\begin{equation*}
Y_{X X}^{j}=0, \quad j=1, \ldots, m \tag{1.6}
\end{equation*}
$$

Applying Lie's algorithm it is easily seen ([11]) that the free particle system has a local symmetry algebra of dimension $\leq m^{2}+4 m+3$, the bound being attained by $Y_{X X}^{j}=0, j=1, \ldots, m$, with group $\operatorname{PGL}(m+1, \mathbb{K})$.

In 1939, for fiber-preserving transformations only, Chern [Ch1939] conducted the É. Cartan algorithm through absorptions of torsion, normalizations and prolongations up to the reduction to an $\{e\}$-structure. Remarkably, in 1995, Fels [Fe1995] conducted the É. Cartan algorithm for general systems $y_{x x}^{j}=F^{j}\left(x, y, y_{x}\right)$ and for general point transformations. As a byproduct of the uniqueness of the obtained $\{e\}$-structure for which all invariant tensors vanish, Fels deduced in [Fe 1995] that the flat system $Y_{X X}^{j}=0$, $j=1, \ldots, m$, is, up to equivalence, the only system of second order possessing a symmetry group of maximal dimension. Alas, the $\{e\}$-structures obtained by Chern and by Fels are not parametric, so that the counterpart to (4) of Theorem 1.2 was lacking as soon as $m \geqslant 2$. The extraordinary heaviness of É. Cartan's method is a well known obstacle.

Recently, Neut [N2003] (see also [DNP2005]) wrote a Maple program to compute (among other $\{e\}$-structures) Fels' tensors in a parametric, explicit way. The digital computations succeeded in the case $m=2$, yielding Theorem 3 of [DNP2005], a statement that may be checked to be equivalent to

Theorem 1.7 (3) just below in the case $m=2$. In the physically most meaningful case $m=3$, a present-day computer is stuck ([N2003], [DNP2005]).

Extending Lie's less heavy computations ${ }^{3}$, we present here a complete solution to the characterization of $Y_{X X}^{j}=0$ for arbitrary $m \geqslant 2$.

Theorem 1.7. Suppose $m \geqslant 2$. The following three conditions are equivalent:
(1) the system $y_{x x}^{j}=F^{j}\left(x, y, y_{x}\right), j=1, \ldots, m$, is equivalent, under a local point transformation $\left(x, y^{j}\right) \mapsto\left(X, Y^{j}\right)$ to the free particle equation $Y_{X X}^{j}=0$;
(2) the local Lie symmetry group of $y_{x x}^{j}=F^{j}\left(x, y, y_{x}\right)$ is $\left(m^{2}+4 m+3\right)$ dimensional and locally isomorphic to $\operatorname{PGL}(m+1, \mathbb{K})$;
(3) the right hand sides $F^{j}\left(x, y, y_{x}\right)$ are of a special form, described as follows.
(i) There exist local $\mathbb{K}$-analytic functions $G^{j}, H_{l_{1}}^{j}, L_{l_{1}, l_{2}}^{j}$ and $M_{l_{1}, l_{2}}$, where $j, l_{1}, l_{2}=1, \ldots$, , enjoying the symmetries $L_{l_{1}, l_{2}}^{j}=L_{l_{2}, l_{1}}^{j}$ and $M_{l_{1}, l_{2}}=M_{l_{2}, l_{1}}$ and depending only on ( $x, y$ ) such that $F^{j}\left(x, y, y_{x}\right)$ may be written as the following specific cubic polynomial with respect to $y_{x}$ :
$y_{x x}^{j}=G^{j}+\sum_{l_{1}=1}^{m} y_{x}^{l_{1}} H_{l_{1}}^{j}+\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} L_{l_{1}, l_{2}}^{j}+y_{x}^{j} \cdot \sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} M_{l_{1}, l_{2}}$.
(ii) The functions $G^{j}, H_{l_{1}}^{j}, L_{l_{1}, l_{2}}^{j}$ and $M_{l_{1}, l_{2}}$ satisfy the following explicit system of four families of first order partial differential equations:

$$
\left\{\begin{align*}
0= & -2 G_{y^{l_{1}}}^{j}+2 \delta_{l_{1}}^{j} G_{y^{l_{2}}}^{l_{2}}+H_{l_{1}, x}^{j}-\delta_{l_{1}}^{j} H_{l_{2}, x}^{l_{2}}+  \tag{I}\\
& +2 \sum_{k=1}^{m} G^{k} L_{l_{1}, k}^{j}-2 \delta_{l_{1}}^{j} \sum_{k=1}^{m} G^{k} L_{l_{2}, k}^{l_{2}}+ \\
& +\frac{1}{2} \delta_{l_{1}}^{j} \sum_{k=1}^{m} H_{l_{2}}^{k} H_{k}^{l_{2}}-\frac{1}{2} \sum_{k=1}^{m} H_{l_{1}}^{k} H_{k}^{j},
\end{align*}\right.
$$

[^2]where the indices $j, l_{1}$ vary in $\{1,2, \ldots, m\}$;
\[

\left\{$$
\begin{aligned}
0= & -\frac{1}{2} H_{l_{1}, y^{2}}^{j}+\frac{1}{6} \delta_{l_{1}}^{j} H_{l_{2}, y^{l_{2}}}^{l_{2}}+\frac{1}{3} \delta_{l_{2}}^{j} H_{l_{1}, y_{1}}^{l_{1}}+ \\
& +L_{l_{1}, l_{2}, x}^{j}-\frac{1}{3} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}, x}^{l_{2}}-\frac{2}{3} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}, x}^{l_{1}}+ \\
& +G^{j} M_{l_{1}, l_{2}}-\frac{1}{3} \delta_{l_{1}}^{j} G^{l_{2}} M_{l_{2}, l_{2}}-\frac{2}{3} \delta_{l_{2}}^{j} G^{l_{1}} M_{l_{1}, l_{1}}+ \\
& +\frac{1}{3} \delta_{l_{1}}^{j} \sum_{k=1}^{m} G^{k} M_{l_{2}, k}-\frac{1}{3} \delta_{l_{2}}^{j} \sum_{k=1}^{m} G^{k} M_{l_{1}, k}- \\
& -\frac{1}{2} \sum_{k=1}^{m} H_{k}^{j} L_{l_{1}, l_{2}}^{k}+\frac{1}{2} \sum_{k=1}^{m} H_{l_{1}}^{k} L_{l_{2}, k}^{j}+ \\
& +\delta_{l_{1}}^{j}\left(\frac{1}{6} \sum_{k=1}^{m} H_{k}^{l_{2}} L_{l_{2}, l_{2}}^{k}-\frac{1}{6} \sum_{k=1}^{m} H_{l_{2}}^{k} L_{l_{2}, k}^{l_{2}}\right)+ \\
& +\delta_{l_{2}}^{j}\left(\frac{1}{3} \sum_{k=1}^{m} H_{k}^{l_{1}} L_{l_{1}, l_{1}}^{k}-\frac{1}{3} \sum_{k=1}^{m} H_{l_{1}}^{k} L_{l_{1}, k}^{l_{1}}\right),
\end{aligned}
$$\right.
\]

where the indices $j, l_{1}, l_{2}$ vary in $\{1,2, \ldots, m\}$;

$$
\left\{\begin{aligned}
0= & L_{l_{1}, l_{2}, y^{l_{3}}}^{j}-L_{l_{1}, l_{3}, y^{l_{2}}}^{j}+\delta_{l_{3}}^{j} M_{l_{1}, l_{2}, x}-\delta_{l_{2}}^{j} M_{l_{1}, l_{3}, x}+ \\
& +\frac{1}{2} H_{l_{3}}^{j} M_{l_{1}, l_{2}}-\frac{1}{2} H_{l_{2}}^{j} M_{l_{1}, l_{3}}+ \\
& +\frac{1}{2} \delta_{l_{1}}^{j} \sum_{k=1}^{m} H_{l_{3}}^{k} M_{l_{2}, k}-\frac{1}{2} \delta_{l_{1}}^{j} \sum_{k=1}^{m} H_{l_{2}}^{k} M_{l_{3}, k}+ \\
& +\frac{1}{2} \delta_{l_{3}}^{j} \sum_{k=1}^{m} H_{l_{1}}^{k} M_{l_{2}, k}-\frac{1}{2} \delta_{l_{2}}^{j} \sum_{k=1}^{m} H_{l_{1}}^{k} M_{l_{3}, k}+ \\
& +\sum_{k=1}^{m} L_{l_{1}, l_{3}}^{k} L_{l_{2}, k}^{j}-\sum_{k=1}^{m} L_{l_{1}, l_{2}}^{k} L_{l_{3}, k}^{j},
\end{aligned}\right.
$$

where the indices $j, l_{1}, l_{2}, l_{3}$ vary in $\{1, \ldots m\}$; and

$$
\begin{equation*}
\left\{0=M_{l_{1}, l_{2}, y^{l_{3}}}-M_{l_{1}, l_{3}, y^{l_{2}}}-\sum_{k=1}^{m} L_{l_{1}, l_{2}}^{k} M_{l_{3}, k}+\sum_{k=1}^{m} L_{l_{1}, l_{3}}^{k} M_{l_{2}, k},\right. \tag{IV}
\end{equation*}
$$

where the indices $l_{1}, l_{2}, l_{3}$ vary in $\{1, \ldots, m\}$.
Let us provide commentaries and explanations. The form of the righthand side of (3)(i) is the analog of the form of the right-hand side $F$ in (4) of Lie's Theorem 1.2. However, we notice that the right-hand side of (3)(i)
is not the most general degree three polynomial in the variables $y_{x}^{j}, j=$ $1, \ldots, m$ : some coefficients of the cubic terms vanish.

Very strikingly, the differential system (I), (II), (III), (IV) satisfied by the functions $G^{j}, H_{l_{1}}^{j}, L_{l_{1}, l_{2}}^{j}, M_{l_{1}, l_{2}}$ is of first order for $m \geqslant 2$, whereas the system (4) satisfied by $G, H, L, M$ in Lie's Theorem 1.2 is of second order for $m=1$. This confirms the main theorem of González-López [GL1988], that is recovered here in the special linear case (1.5).

Corollary 1.8. ([GL1988]) For $m \geqslant 2$, a linear (nonhomogeneous) system $y_{x x}^{j}=G_{0}^{j}(x)+\sum_{l=1}^{m} y^{l} G_{1, l}^{j}(x)+\sum_{l_{1}=1}^{m} y_{x}^{l_{1}} H_{l_{1}}^{j}(x)$ is equivalent to $Y_{X X}^{j}=0$ if and only if there exists a function $B(x)$ such that the $m \times m$ matrix $G_{1}$ may be written under the specific form:

$$
\begin{equation*}
G_{1, l}^{j}=\frac{1}{2} H_{l}^{j}-\frac{1}{4} \sum_{k=1}^{m} H_{k}^{j} H_{l}^{k}+\delta_{l}^{j} B . \tag{1.9}
\end{equation*}
$$

We also mention that Fels obtained a characterization of equivalence to the free particle $Y_{X X}^{j}=0, j=1, \ldots, m$, by the vanishing of two (nonexplicit) tensors $\widetilde{S}_{i k l}^{j}$ and $\widetilde{P}_{i}^{j}$ (Corollary 5.1 in [Fe1995]). In this reference, some parametric computations are achieved after restricting the initial $G$ structure, together with its subsequent prolongations, to the identity element of the group; explicit expressions of $\left.\widetilde{S}_{i k l}^{j}\right|_{\text {Id }}$ and of $\left.\widetilde{P}_{i}^{j}\right|_{\text {Id }}$ are then obtained, through already hard computations. The vanishing of the two tensors $\widetilde{P}_{i}^{j}$ and $\widetilde{S}_{i k l}^{j}$ at the identity of the structure group, namely, as computed in Lemma 4.1 of [Fe1995], yields (translating into our notation)

$$
\left\{\begin{align*}
0=\left.\left(\widetilde{S}_{i k l}^{j}\right)\right|_{\mathrm{Id}}= & F_{y_{x}^{i} y_{x}^{k} y_{x}^{l}}^{j}-\frac{1}{n+2} \sum_{l_{1}=1}^{m} \sum_{\sigma \in \mathfrak{S}_{3}} \delta_{\sigma(l)}^{j} F_{y_{x}^{l_{x}} y_{x}^{\sigma(i)} y_{x}^{\sigma(k)}}^{l_{1}},  \tag{1.10}\\
0=\left.\left(\widetilde{P}_{i}^{j}\right)\right|_{\mathrm{Id}}= & \frac{1}{2} D\left(F_{y_{x}^{i}}^{j}\right)-F_{y^{i}}^{j}-\frac{1}{4} \sum_{k=1}^{m} F_{y_{x}^{k}}^{j} F_{y_{x}^{i}}^{k}- \\
& -\frac{1}{m} \delta_{i}^{j}\left[\frac{1}{2} D\left(\sum_{k=1}^{m} F_{y_{x}^{k}}^{k}\right)-\sum_{k=1}^{m} F_{y^{k}}^{k}-\frac{1}{4} \sum_{k=1}^{m} \sum_{l=1}^{m} F_{y_{x}^{k}}^{l} F_{y_{x}^{l}}^{k}\right]
\end{align*}\right.
$$

where $D$ is the total differentiation operator $\frac{\partial}{\partial x}+\sum_{l=1}^{m} y_{x}^{l} \frac{\partial}{\partial y^{l}}+\sum_{l=1}^{m} F^{l} \frac{\partial}{\partial y_{x}^{l}}$, and where $i, j, k, l=1, \ldots, m$. Strikingly, one may check that the first equation is equivalent to (3)(i) of Theorem 1.2 and then that the second equation yields the (complicated) four families of first order partial differential equations (I), (II), (III) and (IV). So, in Corollary 5.1 of [Fe1995], one may replace the vanishing of $\widetilde{S}_{i k l}^{j}$ and of $\widetilde{P}_{i}^{j}$ by the vanishing of $\left.\widetilde{S}_{i k l}^{j}\right|_{\text {Id }}$ and of $\left.\widetilde{P}_{i}^{j}\right|_{\text {Id }}$, which were explicitly computed there.

This phenomenon could be explained as follows: as soon as the tensors $\widetilde{S}_{i k l}^{j}$ vanish, the system enjoys a projective connection (appendix of [Fe1995]); with such a connection, the tensors $\widetilde{P}_{i}^{j}$ then transform according to a specific rule via tensorial rotation formulas and their general expression may be deduced from their expression at the identity ${ }^{4}$. We have checked this, but as we try to avoid the method of equivalence, details will not be reproduced here. Similar observations appear in Hachtroudi [Ha1937].

Even if the expressions (1.10) are more compact than the (equivalent) conditions in Theorem 1.7 (3), we prefer the complete expressions of Theorem 1.7 (3), since they are more explicit and ready-made for checking whether a physically given nonlinear system is equivalent to a free particle. If the reader prefers compact expressions and "short" theorems, (s)he may replace the conditions of Theorem 1.7 (3) by (1.10).
Open problem 1.11. Characterize explicitly the linearizability of a Newtonian system in $m \geqslant 2$ degrees of freedom, i.e. local equivalence to :

$$
\begin{equation*}
Y_{X X}^{j}=G_{0}^{j}(X)+\sum_{l=1}^{m} Y^{l} G_{1, l}^{j}(X)+\sum_{l_{1}=1}^{m} Y_{X}^{l_{1}} H_{l_{1}}^{j}(X) \tag{1.12}
\end{equation*}
$$

1.13. Organization, avertissement and acknowledgment. Section 2 is devoted to a thorough restitution of Lie's original proof of the equivalence between (1) and (4) in Theorem 1.2. Section 3 is devoted to the formulation of combinatorial formulas yielding the general form of a system equivalent to $Y_{X X}^{j}=0, j=1, \ldots, m$, under a local $\mathbb{K}$-analytic point transformation $\left(x, y^{j}\right) \mapsto\left(X, Y^{j}\right)$, for general $m \geqslant 2$; the proof of the main technical Lemma 3.32 is exposed in Section 5. Section 4 is devoted to the final proof of the equivalence between (1) and (3) in Theorem 1.7, the equivalence between (1) and (2) being already proved by Fels [Fe1995].

Some word about style and intentions. We wanted the proof of the equivalence between (1) and (3) be totally complete, every tiny detail being rigorously and patiently checked. This is why we decided to carefully detail each intermediate computational step, seeking first the combinatorics of the formal calculations in the case $m=1$ and devising then the underlying combinatorics for the case $m \geqslant 2$. Actually, the size of differential expressions is relatively impressive, as will become soon evident. Thus, no intermediate symbolic computation will be hidden, hence essentially no checking work is left to the reader, as would have been the case if we did not have typed all the computations.

[^3]We also would like to point out that except in every specific standard situations, as with the much studied Riemann and Ricci tensors, presentday computer programs are not yet powerful enough to apply the É. Cartan equivalence method when the number of some collection of variables is a general integer. All the formulas obtained in Sections 2, 3 and 4 were first treated completely by hand and then, some of them were confirmed afterwards with the help of MAPLE release 6 in the cases $m=2$ and $m=3$. The author is indebted to Sylvain Neut and to Michel Petitot, from the University of Lille 1 , for their help in computer machine confirmations.

## §2. Proof of LIE's THEOREM

2.1. Argument. This preliminary section contains a detailed exposition of Lie's original proof of the equivalence between (1) and (4) in Theorem 1.2. Since our goal is to guess the combinatorics of computations in several variables, it will be a crucial point for us to explain thoroughly and patiently each step of Lie's computation. Without such an intuitive control, it would be hopeless to conduct any generalization to several variables. Hence we shall respect a fundamental principle: always explain clearly and completely what sort of computation is achieved at each step. Also, we shall many times introduce some appropriate new notation.
2.2. Combinatorics of the second order prolongation of a point transformation. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Let $(x, y) \mapsto(X(x, y), Y(x, y))$ be a local $\mathbb{K}$-analytic invertible transformation, defined in a neighborhood of the origin in $\mathbb{K}^{2}$, which maps the second order differential equation $y_{x x}=F\left(x, y, y_{x}\right)$ to the flat equation $Y_{X X}=0$. By assumption, the Jacobian determinant

$$
\Delta(x \mid y):=\left|\begin{array}{cc}
X_{x} & X_{y}  \tag{2.3}\\
Y_{x} & Y_{y}
\end{array}\right|
$$

is nowhere vanishing. Since the equation $Y_{X X}=0$ is left unchanged by any affine transformation in the $(X, Y)$ space, we can (and we shall) assume that the transformation is tangent to the identity at the origin, namely the above Jacobian matrix equals the identity matrix at $(x, y)=(0,0)$.

The computation how the differential equation in the $(X, Y)$ coordinates is related to the differential equation in the $(x, y)$-coordinates is classical, cf. [Lie1883], [Tr1896], [BK1989], [Ib1992]: let us remind it. A local graph $\{y=y(x)\}$ being transformed to a local graph $\{Y=Y(X)\}$, we have a direct formula for the first derivative $Y_{X}$ :

$$
\begin{equation*}
Y_{X}:=\frac{d Y}{d X}=\frac{d x \cdot \partial Y(x, y(x)) / \partial x}{d x \cdot \partial X(x, y(x)) / \partial x}=\frac{Y_{x}+y_{x} Y_{y}}{X_{x}+y_{x} X_{y}} \tag{2.4}
\end{equation*}
$$

This yields the prolongation of the transformation to the first order jet space. For the second order prolongation, introducing the second order total differentiation operator (which geometrically corresponds to differentiation along graphs $\{(x, y(x))\})$ defined by

$$
\begin{equation*}
D:=\frac{\partial}{\partial x}+y_{x} \frac{\partial}{\partial y}+y_{x x} \frac{\partial}{\partial y_{x}}, \tag{2.5}
\end{equation*}
$$

we may compute, simplify and reorder the expression of the second order derivative in the $(X, Y)$-coordinates:
(2.6)

$$
\left\{\begin{aligned}
Y_{X X}:=\frac{d^{2} Y}{d X^{2}} \equiv \frac{D Y_{X}}{D X} & =\frac{D\left[\left(Y_{x}+y_{x} Y_{y}\right)\left(X_{x}+y_{x} X_{x}\right)^{-1}\right]}{X_{x}+y_{x} X_{y}}= \\
& =\frac{1}{\left[X_{x}+y_{x} X_{y}\right]^{3}}\left\{y_{x x}\left[X_{x} Y_{y}-Y_{x} X_{y}\right]+X_{x} Y_{x x}-Y_{x} X_{x x}+\right. \\
& +y_{x}\left[2\left(X_{x} Y_{x y}-Y_{x} X_{x y}\right)-\left(X_{x x} Y_{y}-Y_{x x} X_{y}\right)\right]+ \\
& +y_{x} y_{x}\left[X_{x} Y_{y y}-Y_{x} X_{y y}-2\left(X_{x y} Y_{y}-Y_{x y} X_{y}\right)\right]+ \\
& \left.+y_{x} y_{x} y_{x}\left[-\left(X_{y y} Y_{y}-Y_{y y} X_{y}\right)\right]\right\} .
\end{aligned}\right.
$$

Even if not too complicated, the internal combinatorics of this expression has to be analyzed and expressed thoroughly. First of all, as $Y_{X X}=0$ by assumption, we may erase the cubic factor $\left[X_{x}+y_{x} X_{y}\right]^{-3}$. Next, as the factor of $y_{x x}$ in the right-hand side of (2.6), we just recognize the Jacobian $\Delta(x \mid y)$ expressed in (2.3) above. Also, all the other factors are modifications of the Jacobian $\Delta(x \mid y)$, whose combinatorics may be understood as follows.

There exist exactly three possible distinct second order derivatives: $x x, x y$ and $y y$. There are also exactly two columns in (2.3). By replacing each of the two columns of first order derivative in $\Delta(x \mid y)$ by any column of second order derivative (leaving $X$ and $Y$ unchanged), we may build exactly six new determinants

$$
\left\{\begin{array}{lll}
\Delta(x x \mid y) & \Delta(x y \mid y) & \Delta(y y \mid y)  \tag{2.7}\\
\Delta(x \mid x x) & \Delta(x \mid x y) & \Delta(x \mid y y)
\end{array}\right.
$$

where for instance

$$
\left\{\Delta(\underline{x x} \mid y):=\left|\begin{array}{cc}
X_{\underline{x x}} & X_{y}  \tag{2.8}\\
Y_{\underline{x x}} & Y_{y}
\end{array}\right| \quad \text { and } \quad \Delta(x \mid \underline{x y}):=\left|\begin{array}{cc}
X_{x} & X_{\underline{x y}} \\
Y_{x} & Y_{\underline{x y}}
\end{array}\right| .\right.
$$

Hence, by rewriting (2.6), we see that the equation $y_{x x}=F\left(x, y, y_{x}\right)$ equivalent to $Y_{X X}=0$ may be written under the general explicit form, involving
determinants

$$
\left\{\begin{array}{c}
0=y_{x x} \cdot\left|\begin{array}{cc}
X_{x} & X_{y} \\
Y_{x} & Y_{y}
\end{array}\right|+\left|\begin{array}{cc}
X_{x} & X_{x x} \\
Y_{x} & Y_{x x}
\end{array}\right|+y_{x} \cdot\left\{2\left|\begin{array}{cc}
X_{x} & X_{x y} \\
Y_{x} & Y_{x y}
\end{array}\right|-\left|\begin{array}{cc}
X_{x x} & X_{y} \\
Y_{x x} & Y_{y}
\end{array}\right|\right\}+  \tag{2.9}\\
\quad+y_{x} y_{x} \cdot\left\{\left|\begin{array}{cc}
X_{x} & X_{y y} \\
Y_{x} & Y_{y y}
\end{array}\right|-2\left|\begin{array}{cc}
X_{x y} & X_{y} \\
Y_{x y} & Y_{y}
\end{array}\right|\right\}+y_{x} y_{x} y_{x} \cdot\left\{-\left|\begin{array}{cc}
X_{y y} & X_{y} \\
Y_{y y} & Y_{y}
\end{array}\right|\right\}
\end{array}\right.
$$

or equivalently, after solving in $y_{x x}$, i.e. after dividing by the Jacobian $\Delta(x \mid y)$ :
(2.10)

$$
\left\{\begin{aligned}
y_{x x}= & -\frac{\Delta(x \mid x x)}{\Delta(x \mid y)}+y_{x} \cdot\left\{-2 \frac{\Delta(x \mid x y)}{\Delta(x \mid y)}+\frac{\Delta(x x \mid y)}{\Delta(x \mid y)}\right\}+ \\
& +\left(y_{x}\right)^{2} \cdot\left\{-\frac{\Delta(x \mid y y)}{\Delta(x \mid y)}+2 \frac{\Delta(x y \mid y)}{\Delta(x \mid y)}\right\}+\left(y_{x}\right)^{3} \cdot\left\{\frac{\Delta(y y \mid y)}{\Delta(x \mid y)}\right\}
\end{aligned}\right.
$$

At this point, it will be convenient to slightly contract the notation by introducing a new family of square functions as follows. We first index the coordinates $(x, y)$ as $\left(y^{0}, y^{1}\right)$, namely we introduce the two notational equivalences

$$
\begin{equation*}
y^{0} \equiv x, \quad y^{1} \equiv y \tag{2.11}
\end{equation*}
$$

which will be very convenient in the sequel, especially to write down general combinatorial formulas anticipating our treatment of the case of $m \geqslant 2$ dependent variables $\left(y^{1}, \ldots, y^{m}\right)$, to be achieved in Sections 3, 4 and 5 below. With this convention at hand, our six square functions $\square_{y^{j_{1}} y^{j_{2}}}^{k_{1}}$, symmetric with respect to the lower indices, where $0 \leqslant j_{1}, j_{2}, k_{1} \leqslant 1$, are defined by

$$
\left\{\begin{array}{lll}
\square_{x x}^{0}:=\frac{\Delta(x x \mid y)}{\Delta(x \mid y)}, & \square_{x y}^{0}:=\frac{\Delta(x y \mid y)}{\Delta(x \mid y)}, & \square_{y y}^{0}:=\frac{\Delta(y y \mid y)}{\Delta(x \mid y)}  \tag{2.12}\\
\square_{x x}^{1}:=\frac{\Delta(x \mid x x)}{\Delta(x \mid y)}, & \square_{x y}^{1}:=\frac{\Delta(x \mid x y)}{\Delta(x \mid y)}, & \square_{y y}^{1}:=\frac{\Delta(x \mid y y)}{\Delta(x \mid y)} .
\end{array}\right.
$$

Here of course, the upper index designates the column upon which the second order derivative appears, itself being encoded by the two lower indices. Even if this is hidden in the notation, we shall remember that the square functions are explicit rational expressions in terms of the second order jet of the transformation $(x, y) \mapsto(X(x, y), Y(x, y))$. However, we shall be aware of not confusing the index in the square functions with a second order partial derivative of some function " $\square^{j}$ ", denoted by the square symbol: indeed, the partial derivatives are hidden in some determinant.

At this point, we may summarize what we have established so far.
Lemma 2.13. The equation $y_{x x}=F\left(x, y, y_{x}\right)$ is equivalent to the flat equation $Y_{X X}=0$ if and only if there exist two local $\mathbb{K}$-analytic functions
$X(x, y)$ and $Y(x, y)$ such that it may be written under the form
$y_{x x}=-\square_{x x}^{1}+y_{x} \cdot\left(-2 \square_{x y}^{1}+\square_{x x}^{0}\right)+\left(y_{x}\right)^{2} \cdot\left(-\square_{y y}^{1}+2 \square_{x y}^{0}\right)+\left(y_{x}\right)^{3} \cdot \square_{y y}^{0}$.
At this point, for heuristic reasons, it may be useful to compare the righthand side of (2.14) with the classical expression of the prolongation to the second order jet space of a general vector field of the form $L:=X(x, y) \frac{\partial}{\partial x}+$ $Y(x, y) \frac{\partial}{\partial y}$, which is given, according to [Lie1883], [Ol1986], [BK1989], by

$$
\left\{\begin{align*}
L^{(2)} & =X \frac{\partial}{\partial x}+Y \frac{\partial}{\partial y}+\left[Y_{x}+y_{x} \cdot\left(Y_{y}-X_{x}\right)+\left(y_{x}\right)^{2} \cdot\left(-X_{y}\right)\right] \frac{\partial}{\partial y_{x}}+  \tag{2.15}\\
& +\left[Y_{x x}+y_{x} \cdot\left(2 Y_{x y}-X_{x x}\right)+\left(y_{x}\right)^{2} \cdot\left(Y_{y y}-2 X_{x y}\right)+\left(y_{x}\right)^{3} \cdot\left(-X_{y y}\right)+\right. \\
& \left.+y_{x x} \cdot\left(Y_{y}-2 X_{x}\right)+y_{x} y_{x x} \cdot\left(-3 X_{y}\right)\right] \frac{\partial}{\partial y_{x x}} .
\end{align*}\right.
$$

We immediately see that (up to an overall minus sign) the right-hand side of (2.14) is formally analogous to the second line of (2.15) : the letter $X$ corresponds to the symbol $\square^{0}$ and the letter $Y$ corresponds to the symbol $\square^{1}$. This analogy is no mystery, just because the formula for $L^{(2)}$ is classically obtained by differentiating at $\varepsilon=0$ the second order prolongation $[\exp (\varepsilon L)(\cdot)]^{(2)}$ of the flow of $L!$

In fact, as we assumed that the transformation $(x, y) \mapsto$ $(X(x, y), Y(x, y))$ is tangent to the identity at the origin, we may think that $X_{x} \cong 1, X_{y} \cong 0, Y_{x} \cong 0$ and $Y_{y} \cong 1$, whence the Jacobian $\Delta(x \mid y) \cong 1$ and moreover

$$
\left\{\begin{array}{rlr}
\square_{x x}^{0} \cong X_{x x}, & \square_{x y}^{0} \cong X_{x y}, & \square_{y y}^{0} \cong X_{y y}  \tag{2.16}\\
\square_{x x}^{1} \cong Y_{x x}, & \square_{x y}^{1} \cong Y_{x y}, & \square_{y y}^{1} \cong Y_{y y}
\end{array}\right.
$$

By means of this (abusive) notational correspondence, we see that, up to an overall minus sign, the right-hand side of (2.14) transforms precisely to the second line of (2.15). This analogy will be useful in devising combinatorial formulas for the generalization of Lemma 2.13 to the case of $m \geqslant 2$ variables $\left(y^{1}, \ldots, y^{m}\right)$, see Lemmas 3.22 and 3.32 below.
2.17. Continuation. Clearly, since the right-hand side of (2.14) is a polynomial of degree three in $y_{x}$, the first condition of Theorem 1.2 (4) immediately holds. We are therefore led to establish that the second condition is necessary and sufficient in order that there exist two local $\mathbb{K}$-analytic functions $X(x, y)$ and $Y(x, y)$ which solve the following system of nonlinear second order partial differential equations (remind that the second order jet
of $(X, Y)$ is hidden in the square functions):

$$
\left\{\begin{align*}
G & =-\square_{x x}^{1},  \tag{2.18}\\
H & =-2 \square_{x y}^{1}+\square_{x x}^{0}, \\
L & =-\square_{y y}^{1}+2 \square_{x y}^{0}, \\
M & =\square_{y y}^{0} .
\end{align*}\right.
$$

In the remainder of this section, following [Lie1883], p. 364, we shall study this second order system by introducing two auxiliary systems of partial differential equations which are complete, and we shall see in $\S 2.38$ below that the compatibility conditions (insuring involutivity, hence complete integrability) of the second auxiliary system exactly provide the two partial differential equations appearing in Theorem 1.2 (4).
2.19. First auxiliary system. We notice that in (2.18), there are two more square functions $\square_{x x}^{0}, \square_{x y}^{0}, \square_{y y}^{0}, \square_{x x}^{1}, \square_{x y}^{1}, \square_{y y}^{1}$, than functions $G, H, L$ and $M$. Hence, as a trick, let us introduce six new independent functions $\Pi_{j_{1}, j_{2}}^{k_{1}}$ of $(x, y)$, symmetric with respect to the lower indices, for $0 \leqslant j_{1}, j_{2}, k_{1} \leqslant 1$ and let us seek necessary and sufficient conditions in order that there exist solutions $(X, Y)$ to the first auxiliary system:

$$
\left\{\begin{array}{lll}
\square_{x x}^{0}=\Pi_{0,0}^{0}, & \square_{x y}^{0}=\Pi_{0,1}^{0}, & \square_{y y}^{0}=\Pi_{1,1}^{0}  \tag{2.20}\\
\square_{x x}^{1}=\Pi_{0,0}^{1}, & \square_{x y}^{1}=\Pi_{0,1}^{1}, & \square_{y y}^{1}=\Pi_{1,1}^{1} .
\end{array}\right.
$$

According to the (aprooximate) identities (2.16), this system looks like a complete second order system of partial differential equations in two variables $(x, y)$ and in two unknowns $(X, Y)$. More rigorously, by means of elementary algebraic operations, taking account of the fact that $X_{x} \cong 1$, $X_{y} \cong 0, Y_{x} \cong 0$ and $Y_{y} \cong 1$, one may transform this sytem in a true second order complete system, solved with respect to the top order derivatives, namely of the form

$$
\left\{\begin{align*}
X_{x x} & =\Lambda_{0,0}^{0}, & X_{x y}=\Lambda_{0,1}^{0}, & X_{y y}=\Lambda_{1,1}^{0}  \tag{2.21}\\
Y_{x x} & =\Lambda_{0,0}^{1}, & Y_{x y}=\Lambda_{0,1}^{1}, & Y_{y y}=\Lambda_{1,1}^{1}
\end{align*}\right.
$$

where the $\Lambda_{j_{1}, j_{2}}^{k_{1}}$ are local $\mathbb{K}$-analytic functions of $\left(x, y, X, Y, X_{x}, X_{y}, Y_{x}, Y_{y}\right)$. For such a system, the compatibility conditions [which are necessary and sufficient for the existence of a solution $(X, Y)$ ] are easily formulated:

$$
\begin{cases}\left(\Lambda_{0,0}^{0}\right)_{y}=\left(\Lambda_{0,1}^{0}\right)_{x}, & \left(\Lambda_{0,1}^{0}\right)_{y}=\left(\Lambda_{1,1}^{0}\right)_{x}  \tag{2.22}\\ \left(\Lambda_{0,0}^{1}\right)_{y}=\left(\Lambda_{0,1}^{1}\right)_{x}, & \left(\Lambda_{0,1}^{1}\right)_{y}=\left(\Lambda_{1,1}^{1}\right)_{x}\end{cases}
$$

Equivalently, we may express the compatibility conditions directly with the system (2.20), without transforming it to the form (2.21). This direct strategy will be more appropriate.
2.23. Compatibility conditions for the first auxiliary system. Indeed, to begin with, let us remind that the $\Delta(\cdot \mid \cdot)$ are determinant, hence we have the skew-symmetry relation $\Delta\left(x^{a} y^{b} \mid x^{c} y^{d}\right)=-\Delta\left(x^{c} y^{d} \mid x^{a} y^{b}\right)$ and the following two formulas for partial differentiation

$$
\left\{\begin{array}{l}
{\left[\Delta\left(x^{a} y^{b} \mid x^{c} y^{d}\right)\right]_{x}=\Delta\left(x^{a+1} y^{b} \mid x^{c} y^{d}\right)+\Delta\left(x^{a} y^{b} \mid x^{c+1} y^{d}\right)}  \tag{2.24}\\
{\left[\Delta\left(x^{a} y^{b} \mid x^{c} y^{d}\right)\right]_{y}=\Delta\left(x^{a} y^{b+1} \mid x^{c} y^{d}\right)+\Delta\left(x^{a} y^{b} \mid x^{c} y^{d+1}\right)}
\end{array}\right.
$$

With these formal rules at hand, as an exercise, let us compute for instance the following cross differentiation (remember that the lower index in the square functions is not a partial derivative):

$$
\left\{\begin{align*}
&\left(\square_{x x}^{0}\right)_{y}-\left(\square_{x y}^{0}\right)_{x}=\frac{\partial}{\partial y}\left(\frac{\Delta(x x \mid y)}{\Delta(x \mid y)}\right)-\frac{\partial}{\partial x}\left(\frac{\Delta(x y \mid y)}{\Delta(x \mid y)}\right)=  \tag{2.25}\\
&=\frac{1}{[\Delta(x \mid y)]^{2}}\left\{\frac{\Delta(x x y \mid y) \cdot \Delta(x \mid y)^{(a)}}{}+\Delta(x x \mid y y) \cdot \Delta(x \mid y)-\right. \\
&-\frac{\Delta(x y \mid y) \cdot \Delta(x x \mid y)}{(\mathrm{b}}-\Delta(x \mid y y) \cdot \Delta(x x \mid y)- \\
&-\underline{\Delta(x x y \mid y) \cdot \Delta(x \mid y)}(\text { a }) \\
&+\underline{\Delta(x y \mid x y) \cdot \Delta(x \mid y)^{(c)}}+ \\
&=\frac{1}{[\Delta(x \mid y)]^{2}}\{\Delta(x x \mid y y) \cdot \Delta(x \mid y)-\Delta(x \mid y y) \cdot \Delta(x x \mid y)+ \\
&+\Delta(x y \mid y) \cdot \Delta(x \mid x y)\} .
\end{align*}\right.
$$

Crucially, we observe that the third order derivatives kill each other and disappear, see the underlined terms with (a) appended. Also, two products of two determinants $\Delta(\cdot \mid \cdot)$ involving a second order derivative upon one column of each determinant kill each other: they are underlined with (b) appended. Finally, by antisymmetry of determinants, the term $\Delta(x y \mid x y)$. $\Delta(x \mid y)$ vanishes gratuitously: it is underlined with (c)appended.

However, there still remains one term involving second order derivatives upon the two columns of a determinant: it is $\Delta(x x \mid y y)$.

We must transform this unpleasant term $\Delta(x x \mid y y) \cdot \Delta(x \mid y)$ and express it as a product of two determinants, each involving a second order derivative only in one column. To this aim, we have:

Lemma 2.26. The following three relations between the differential determinants $\Delta(\cdot \mid \cdot)$ hold true:
(2.27)

$$
\left\{\begin{array}{l}
\Delta(x x \mid x y) \cdot \Delta(x \mid y)=\Delta(x x \mid y) \cdot \Delta(x \mid x y)-\Delta(x y \mid y) \cdot \Delta(x \mid x x), \\
\Delta(x x \mid y y) \cdot \Delta(x \mid y)=\Delta(x x \mid y) \cdot \Delta(x \mid y y)-\Delta(y y \mid y) \cdot \Delta(x \mid x x), \\
\Delta(x y \mid y y) \cdot \Delta(x \mid y)=\Delta(x y \mid y) \cdot \Delta(x \mid y y)-\Delta(y y \mid y) \cdot \Delta(x \mid x y) .
\end{array}\right.
$$

Proof. Each of these three formal identities is an immediate direct consequence of the following Plücker type identity, easily verified by developing all the determinants :
(2.28)
$\left.\begin{array}{ll}A_{1} & B_{1} \\ A_{2} & B_{2}\end{array}|\cdot| \begin{array}{ll}C_{1} & D_{1} \\ C_{2} & D_{2}\end{array}\left|=\left|\begin{array}{ll}A_{1} & D_{1} \\ A_{2} & D_{2}\end{array}\right| \cdot\right| \begin{array}{ll}C_{1} & B_{1} \\ C_{2} & B_{2}\end{array}\left|-\left|\begin{array}{ll}B_{1} & D_{1} \\ B_{2} & D_{2}\end{array}\right| \cdot\right| \begin{array}{ll}C_{1} & A_{1} \\ C_{2} & A_{2}\end{array} \right\rvert\,$,
where the variables $A_{1}, A_{2}, B_{1}, B_{2}, C_{1}, C_{2}, D_{1}, D_{2} \in \mathbb{K}$ are arbitrary.
Thanks to the second identity (2.27), we may therefore transform the result left above in the last two lines of (2.25); as desired, it will remain determinants having only one second order derivative per column, so that after division by $[\Delta(x \mid y)]^{2}$, we discover a quadratic expression involving only the square functions themselves:
(2.29)

$$
\left\{\begin{aligned}
\left(\square_{x x}^{0}\right)_{y}-\left(\square_{x y}^{0}\right)_{x}= & \frac{1}{[\Delta(x \mid y)]^{2}}\{\Delta(x x \mid y) \cdot \Delta(x \mid y y)-\Delta(y y \mid y) \cdot \Delta(x \mid x x)- \\
& -\Delta(x \mid y y) \cdot \Delta(x x \mid y)+\Delta(x y \mid y) \cdot \Delta(x \mid x y)\} \\
= & \frac{1}{[\Delta(x \mid y)]^{2}}\{-\Delta(y y \mid y) \cdot \Delta(x \mid x x)+\Delta(x y \mid y) \cdot \Delta(x \mid x y)\} \\
& =-\square_{y y}^{0} \cdot \square_{x x}^{1}+\square_{x y}^{0} \cdot \square_{x y}^{1} .
\end{aligned}\right.
$$

In sum, the result of the cross differentiation $\left(\square_{x x}^{0}\right)_{y}-\left(\square_{x y}^{0}\right)_{x}$ is a quadratic expression in terms of the square functions themselves! Following the same recipe (with no surprise), one may establish the following relations, listing all the compatibility conditions (the first one is nothing else than (2.29)):

$$
\left\{\begin{array}{l}
\left(\square_{x x}^{0}\right)_{y}-\left(\square_{x y}^{0}\right)_{x}=-\square_{x x}^{1} \cdot \square_{y y}^{0}+\square_{x y}^{1} \cdot \square_{x y}^{0},  \tag{2.30}\\
\left(\square_{x y}^{0}\right)_{y}-\left(\square_{y y}^{0}\right)_{x}=-\square_{x y}^{0} \cdot \square_{y x}^{0}-\square_{x y}^{1} \cdot \square_{y y}^{0}+\square_{y y}^{0} \cdot \square_{x x}^{0}+\square_{y y}^{1} \cdot \square_{x y}^{0}, \\
\left(\square_{x x}^{1}\right)_{y}-\left(\square_{x y}^{1}\right)_{x}=-\square_{x x}^{0} \cdot \square_{y x}^{1}-\square_{x x}^{1} \cdot \square_{y y}^{1}+\square_{x y}^{0} \cdot \square_{x x}^{1}+\square_{x y}^{1} \cdot \square_{x y}^{1}, \\
\left(\square_{x y}^{1}\right)_{y}-\left(\square_{y y}^{1}\right)_{x}=-\square_{x y}^{0} \cdot \square_{y x}^{1}+\square_{y y}^{0} \cdot \square_{x x}^{1} .
\end{array}\right.
$$

Instead of checking patiently each of the remaining three cross above cross differentiation identities, it is better to establish directly the following general relation.

Lemma 2.31. Remind from (2.11) that we identify $y^{0}$ with $x$ and $y^{1}$ with $y$ and let $0 \leqslant j_{1}, j_{2}, j_{3}, k_{1} \leqslant 1$. Then

$$
\begin{equation*}
\left(\square_{y^{j_{1}} y^{j_{2}}}^{k_{1}}\right)_{y^{j_{3}}}-\left(\square_{y^{j_{1}} j^{j_{3}}}^{k_{1}}\right)_{y^{j_{2}}}=-\sum_{k_{2}=0}^{1} \square_{y^{j_{1}} y^{j_{2}}}^{k_{2}} \cdot \square_{y^{j_{3}} y^{k_{2}}}^{k_{1}}+\sum_{k_{2}=0}^{1} \square_{y^{j_{1}} y^{j_{3}}}^{k_{2}} \cdot \square_{y^{j_{2}} y^{k_{2}}}^{k_{1}} . \tag{2.32}
\end{equation*}
$$

This lemma is left to the reader; anyway, we shall complete the proof of a generalization of Lemma 2.31 to the case of $m \geqslant 1$ dependent variables $\left(y^{1}, \ldots, y^{m}\right)$ in Section 2 below (Lemma 3.40).

Coming back to the first auxiliary system (2.20), we therefore have obtained a necessary and sufficient condition for the existence of $(X, Y)$ : the functions $\Pi_{j^{1}, j^{2}}^{k_{1}}$ should satisfy the following system of first order partial differential equations, just obtained from (2.30) by replacing the square functions by the Pi functions:
(2.33)

$$
\left\{\begin{array}{l}
\left(\Pi_{0,0}^{0}\right)_{y}-\left(\Pi_{0,1}^{0}\right)_{x}=-\Pi_{0,0}^{1} \cdot \Pi_{1,1}^{0}+\Pi_{0,1}^{1} \cdot \Pi_{0,1}^{0} \\
\left(\Pi_{0,1}^{0}\right)_{y}-\left(\Pi_{1,1}^{0}\right)_{x}=-\Pi_{0,1}^{0} \cdot \Pi_{0,1}^{0}-\Pi_{0,1}^{1} \cdot \Pi_{1,1}^{0}+\Pi_{1,1}^{0} \cdot \Pi_{0,0}^{0}+\Pi_{1,1}^{1} \cdot \Pi_{0,1}^{0} \\
\left(\Pi_{0,0}^{1}\right)_{y}-\left(\Pi_{0,1}^{1}\right)_{x}=-\Pi_{0,0}^{0} \cdot \Pi_{0,1}^{1}-\Pi_{0,0}^{1} \cdot \Pi_{1,1}^{1}+\Pi_{0,1}^{0} \cdot \Pi_{1,1}^{1}+\Pi_{0,1}^{1} \cdot \Pi_{0,1}^{1}, \\
\left(\Pi_{0,1}^{1}\right)_{y}-\left(\Pi_{1,1}^{1}\right)_{x}=-\Pi_{0,1}^{0} \cdot \Pi_{0,1}^{1}+\Pi_{1,1}^{0} \cdot \Pi_{0,0}^{1} .
\end{array}\right.
$$

2.34. Second auxiliary system. It is now time to come back to the functions $G, H, L$ and $M$ and to get rid of the auxiliary "Pi" functions. Unfortunately, we cannot invert directly the linear system (2.18), hence we must choose two specific square functions as principal unknowns, and the best, from a combinatorial point of view, is to choose $\square_{x x}^{0}$ and $\square_{y y}^{1}$. Remind that by (2.20), we have $\square_{x x}^{0}=\Pi_{0,0}^{0}$ and $\square_{y y}^{1}=\Pi_{1,1}^{1}$. For clarity, it will be useful to adopt the notational equivalences

$$
\begin{equation*}
\Theta^{0} \equiv \Pi_{0,0}^{0} \quad \text { and } \quad \Theta^{1} \equiv \Pi_{1,1}^{1} . \tag{2.35}
\end{equation*}
$$

We may therefore quasi-inverse the linear system (2.18), obtaining that the four functions $\Pi_{0,0}^{1}, \Pi_{0,1}^{1}, \Pi_{0,1}^{0}$ and $\Pi_{1,1}^{0}$ may be expressed in terms of the functions $G, H, L$ and $M$ and in terms of the remaining two principal unknowns (2.35), which yields:

$$
\left\{\begin{array}{l}
\Pi_{0,0}^{1}=\square_{x x}^{1}=-G  \tag{2.36}\\
\Pi_{0,1}^{1}=\square_{x y}^{1}=-\frac{1}{2} H+\frac{1}{2} \Theta^{0}, \\
\Pi_{0,1}^{0}=\square_{x y}^{0}=\frac{1}{2} L+\frac{1}{2} \Theta^{1} \\
\Pi_{1,1}^{0}=\square_{y y}^{0}=M
\end{array}\right.
$$

Replacing now each of these four expressions in the compatibility conditions of the first auxiliary system (2.33), solving the four equations with respect to $\Theta_{y}^{1}, \Theta_{x}^{0}, \Theta_{x}^{1}$ and $\Theta_{y}^{0}$, we get after hygienic simplifications what we shall call the second auxiliary system, which is a complete system of first order
partial derivatives in the remaining two principal unknowns $\Theta^{0}$ and $\Theta^{1}$ :

$$
\left\{\begin{array}{l}
\Theta_{y}^{1}=-L_{y}+2 M_{x}+H M-\frac{1}{2} L^{2}+M \Theta^{0}+\frac{1}{2}\left(\Theta^{1}\right)^{2},  \tag{2.37}\\
\Theta_{x}^{0}=-2 G_{y}+H_{x}+G L-\frac{1}{2} H^{2}-G \Theta^{1}+\frac{1}{2}\left(\Theta^{0}\right)^{2}, \\
\Theta_{x}^{1}=-\frac{2}{3} H_{y}+\frac{1}{3} L_{x}+2 G M-\frac{1}{2} H L-\frac{1}{2} H \Theta^{1}+\frac{1}{2} L \Theta^{0}+\frac{1}{2} \Theta^{0} \Theta^{1}, \\
\Theta_{y}^{0}=-\frac{1}{3} H_{y}+\frac{2}{3} L_{x}+2 G M-\frac{1}{2} H L-\frac{1}{2} H \Theta^{1}+\frac{1}{2} L \Theta^{0}+\frac{1}{2} \Theta^{0} \Theta^{1} .
\end{array}\right.
$$

We do not comment the intermediate computations, since they offer no new combinatorial discovery.
2.38. Precise lexicographic rules. We group first order derivatives before zeroth order derivatives; in each group, we respect the lexicographic order of appearance given by the sequence $G, H, L, M, \Theta^{0}, \Theta^{1}$; we always put rational coefficient of every differential monomial in its left; consequently, we accept a minus sign just after an equality sign, as for instance in $(2.36)_{1}$ and in $(2.37)_{2}$; for clarity, we prefer to write a complicated differential equation as $0=\Phi$, with 0 on the left, instead of $\Phi=0$, since $\Phi$ may incoporate 10 , 20 and up to 150 monomials, as will happen for instance in the next sections below.
2.39. Compatibility conditions for the second auxiliary system. Clearly, the necessary and sufficient condition for the existence of solutions $\left(\Theta^{0}, \Theta^{1}\right)$ to the second auxiliary system (2.37) is that the two cross differentiations vanish:

$$
\left\{\begin{array}{l}
0=\left(\Theta_{x}^{0}\right)_{y}-\left(\Theta_{y}^{0}\right)_{x},  \tag{2.40}\\
0=\left(\Theta_{x}^{1}\right)_{y}-\left(\Theta_{y}^{1}\right)_{x} .
\end{array}\right.
$$

Using (2.37), we shall see that we exactly obtain the two second order partial differential equations written in Theorem 1.2 (4). For completeness, we shall perform completely the computation of the first compatibility condition (2.40) and leave the second as an (easy) exercise.

First of all, inserting (2.37) and using the rule of Leibniz for the differentiation of a product, let us write the crude result, performing neither any
simplification nor any reordering:
(2.41)

$$
\left\{\begin{aligned}
0= & \left(\Theta_{x}^{0}\right)_{y}-\left(\Theta_{y}^{0}\right)_{x} \\
= & -2 G_{y y}+H_{x y}+G_{y} L+G L_{y}-H H_{y}-G_{y} \Theta^{1}-G \Theta_{y}^{1}+\Theta^{0} \Theta_{y}^{0}+ \\
& +\frac{1}{3} H_{x y}-\frac{2}{3} L_{x x}-2 G_{x} M-2 G M_{x}+\frac{1}{2} H_{x} L+\frac{1}{2} H L_{x}+ \\
& +\frac{1}{2} H_{x} \Theta^{1}+\frac{1}{2} H \Theta_{x}^{1}-\frac{1}{2} L_{x} \Theta^{0}-\frac{1}{2} L \Theta_{x}^{0}-\frac{1}{2} \Theta_{x}^{0} \Theta^{1}-\frac{1}{2} \Theta^{0} \Theta_{x}^{1} .
\end{aligned}\right.
$$

Next, replacing each first order derivative $\Theta_{x}^{0}, \Theta_{y}^{0}, \Theta_{x}^{1}$ and $\Theta_{y}^{1}$ occuring in (2.41) by its expression given in (2.37), we obtain (suffering a little) as a brute result, before any simplification (except that we put all second order derivatives in the beginning):

$$
\left\{\begin{aligned}
0= & -2 G_{y y}+\frac{4}{3} H_{x y}-\frac{2}{3} L_{x x}+ \\
& +G_{y} L+G L_{y}-H H_{y}-G_{y} \Theta^{1}+G L_{y}-2 G M_{x}-G H M+ \\
& +\frac{1}{2} G(L)^{2}-G M \Theta^{0}-\frac{1}{2} G\left(\Theta^{1}\right)^{2}-\frac{1}{3} H_{y} \Theta^{0}+\frac{2}{3} L_{x} \Theta^{0}+ \\
& +2 G M \Theta^{0}-\frac{1}{2} H L \Theta^{0}-\frac{1}{2} H \Theta^{0} \Theta^{1}+\frac{1}{2} L\left(\Theta^{0}\right)^{2}+\frac{1}{2}\left(\Theta^{0}\right)^{2} \Theta^{1}- \\
& -2 G_{x} M-2 G M_{x}+\frac{1}{2} H_{x} L+\frac{1}{2} H L_{x}+\frac{1}{2} H_{x} \Theta^{1}+ \\
& +\frac{1}{6} H L_{x}-\frac{1}{3} H H_{y}+G H M-\frac{1}{4}(H)^{2} L-\frac{1}{4}(H)^{2} \Theta^{1}+ \\
& +\frac{1}{4} H L \Theta^{0}+\frac{1}{4} H \Theta^{0} \Theta^{1}-\frac{1}{2} L_{x} \Theta^{0}+G_{y} L-\frac{1}{2} H_{x} L-\frac{1}{2} G(L)^{2}+ \\
& +\frac{1}{4} H^{2} L+\frac{1}{2} G L \Theta^{1}-\frac{1}{4} L\left(\Theta^{0}\right)^{2}+G_{y} \Theta^{1}-\frac{1}{2} H_{x} \Theta^{1}- \\
& -\frac{1}{2} G L \Theta^{1}+\frac{1}{4} H^{2} \Theta^{1}+\frac{1}{2} G\left(\Theta^{1}\right)^{2}-\frac{1}{4}\left(\Theta^{0}\right)^{2} \Theta^{1}-\frac{1}{6} L_{x} \Theta^{0}+ \\
& +\frac{1}{3} H_{y} \Theta^{0}-G M \Theta^{0}+\frac{1}{4} H L \Theta^{0}+\frac{1}{4} H \Theta^{0} \Theta^{1}-\frac{1}{4} L\left(\Theta^{0}\right)^{2}- \\
& -\frac{1}{4}\left(\Theta^{0}\right)^{2} \Theta^{1} .
\end{aligned}\right.
$$

Now, we can simplify this brute expression by chasing every couple (or triple, or quadruple) of terms killing each other. After (patient) simplification and lexicographic ordering, we obtain the equation

$$
\left\{\begin{align*}
0=-2 G_{y y}+ & \frac{4}{3} H_{x y}-\frac{2}{3} L_{x x}+  \tag{2.43}\\
& +2(G L)_{y}-2 G_{x} M-4 G M_{x}+\frac{2}{3} H L_{x}-\frac{4}{3} H H_{y}
\end{align*}\right.
$$

which is exactly the first equation of (4) of Theorem 1.2. The treatment of the second one is totally similar. This completes the proof of the equivalence between (1) and (4) in Theorem 1.2.
2.44. Interlude: about hand-computed formulas. In Section 4 below, when dealing with several dependent variables $y^{1}, \ldots, y^{m}$, many simplifications of identities which are much more massive than (2.42) will occur several times. It is therefore welcome to explain how we manage to achieve such computations, without mistakes at the end and strictly by hand. One of the trick is to use colors, which, unfortunately, cannot be restituted in this printed document. Another trick is to underline and to number the terms which disappear together, by pair, by triple, by quadruple, etc. This trick is illustrated in the detailed identity (2.45) below, extracted from our manuscript, which is a copy of (2.42) together with the designation of all the terms which vanish together. Hence, we keep a written track of each intermediate step of every computation and of every simplification. Checking the correctness of a computation simply by reading is then the easiest way, both for the writer and for the reader, although of course, it takes time, anyway.

On the contrary, when relying upon a digital computer, most intermediate steps are invisible; the chase of mistakes is by reading the program and by testing it on several instances. Alas, all the finest intuitions which may awake in the extreme inside of a long computation are essentially absent, the mind believing that the machine is stronger for such tasks. This last belief is in part true, in the case where straightforward known computations are concerned, and in part untrue, in the case where some new hidden mathematical reality is concerned.

For us, the challenge is to control everything in a sea of signs. Computations are to be organized like a living giant coral tree, all part of which should be clearly visible in a transparent fluid of thought, and permanently subject to corrections.

Indeed, it often happens that going through a problem involving massive formal computations, some disharmony or some incoherency is discovered. Then one has to inspect every living atom in the preceding branches of the
growing coral tree of computations until some very tiny or ridiculous mistake is found. In addition to making easy the reading, a perfectly rigorous way of writing the formal identities which respects a large amount of virtual conventions facilitates to reorganize rapidly the coral tree after a mistake has been found. The accumulation of new virtual conventions, all of which we cannot speak, constitutes another coral meta-tree and another profound collection of trick. Finally, we use a blank fluid corrector to avoid copying to much.

Extracted from our manuscript, here is the identity (2.42) with the underlining-numbering of all the vanishing terms (without the original colours) until we get the final equation (2.43):

$$
\begin{align*}
& 0=-2 G_{y y}+\frac{4}{3} H_{x y}-\frac{2}{3} L_{x x}+ \\
& +G_{y} L+G L_{y}-H H_{y}-{\underline{G y} \Theta^{1}}_{(0)}+G L_{y}-2 G M_{x}-\underline{G H M}_{(\mathbb{k})}+ \\
& +\underline{\frac{1}{2}}^{\underline{1}(L)^{2}} \text { (a) }-\underline{G M \Theta^{0}} \text { (b) }-\underline{\frac{1}{2} G\left(\Theta^{1}\right)^{2}} \text { (c) }-\underline{\frac{1}{3} H_{y} \Theta^{0}}+\text { (d) }^{\frac{2}{3} L_{x} \Theta^{0}}+ \\
& +{\underline{2 G M \Theta^{0}}(\text { (b) }}_{-\frac{1}{2} H L \Theta^{0}}^{(f}-\underline{\frac{1}{2} H \Theta^{0} \Theta^{1}}\left(\text { (g) }+\underline{\frac{1}{2} L\left(\Theta^{0}\right)^{2}}(\text { h }) \underline{\frac{1}{2}\left(\Theta^{0}\right)^{2} \Theta^{1}}\right. \text { (i) } \\
& -2 G_{x} M-2 G M_{x}+\frac{1}{2} H_{x} L+\frac{1}{2} H L_{x}+\underline{\underline{1}}_{\frac{1}{2} H_{x} \Theta^{1}}^{(j}+ \\
& +\frac{1}{6} H L_{x}-\frac{1}{3} H H_{y}+\underline{G H M_{(k)}}-\underline{\frac{1}{4}(H)^{2} L}-(1) \underline{\frac{1}{4}(H)^{2} \Theta^{1}}( \tag{2.45}
\end{align*}
$$

$$
\begin{aligned}
& -\underline{\frac{1}{2} G L \Theta^{1}}+\underline{(\mathrm{n})}+\underline{\frac{1}{4} H^{2} \Theta^{1}}+\underline{(\mathrm{m})} \underline{\frac{1}{2} G\left(\Theta^{1}\right)^{2}} \text { (c) }-\underline{\frac{1}{4}\left(\Theta^{0}\right)^{2} \Theta^{1}}-\underline{(\mathrm{i})}-\underline{\frac{1}{6} L_{x} \Theta^{0}}+
\end{aligned}
$$

$$
\begin{aligned}
& \left.-\underline{\frac{1}{4}}\left(\Theta^{0}\right)^{2} \Theta^{1} \dot{i}\right)
\end{aligned}
$$

As may be observed, the order in which we discover the terms which vanish is governed by chance. After some terms are underlined, they are automatically disregarded by the eyes, which lightens the chasing of other terms to be simplified. To collect the remaining terms in order to obtain the final expression (2.43), our method is similar: we underline the terms which may be
summed together. However, whereas we use the red pencil to underline the vanishing terms, we use the green pencil to underline the remaining terms. This small trick is to avoid as much as possible to copy several times some long formal expressions. Finally, we reorder everything lexicographically, so as to get the conclusion (2.43). In order to obtain the final equation (2.43) as efficiently as possible, we read the remaining terms, picking them directly in lexicographic order. If, by lack of luck, one or two terms are forgotten by the eyes and not written in the right place, we copy once more the very final result in the right order, or we use the blank corrector.

Of course, such a refined methodology could seem to be essentially superfluous for such relatively accessible computations. However, when passing to several dependent variables, the current expressions will be approximatively five times more massive. We may really ascertain that a clever methodology of hand computations is helpful in this category.

## §3. Systems of second order ordinary differential EQUATIONS EQUIVALENT TO FREE PARTICLES

3.1. Combinatorics of the second order prolongation of a point transformation. In this section, we endeavour to explain how Lie's theorem and proof may be generalized to the case of several dependent variables. As in the statement of Theorem 1.7 (1), let us assume that the system $y_{x x}^{j}=F^{j}\left(x, y, y_{x}\right), j=1, \ldots, m$, is equivalent under an invertible point transformation $(x, y) \mapsto(X(x, y), Y(x, y))$ to the free particle system $Y_{X X}^{j}=0, j=1, \ldots, m$. By assumption, the Jacobian determinant

$$
\Delta\left(x\left|y^{1}\right| \ldots \mid y^{m}\right):=\left|\begin{array}{cccc}
X_{x} & X_{y^{1}} & \ldots & X_{y^{m}}  \tag{3.2}\\
Y_{x}^{1} & Y_{y^{1}}^{1} & \ldots & Y_{y^{m}}^{1} \\
\ldots & \ldots & \ldots & \ldots \\
Y_{x}^{m} & Y_{y^{1}}^{m} & \ldots & Y_{y^{m}}^{m}
\end{array}\right|
$$

does not vanish at the origin. As in the case $m=1$, since the flat system $Y_{X X}^{j}=0$ is left unchanged by any affine transformation, we can (and we shall) assume that the transformation is tangent to the identity at the origin, so that the above Jacobian matrix equals the identity matrix at $(x, y)=$ $(0,0)$, whence in a neighborhood of the origin it is close to the identity matrix, namely

$$
\begin{equation*}
X_{x} \cong 1, \quad X_{y^{j}} \cong 0, \quad Y_{x}^{j} \cong 0, \quad Y_{y^{j_{1}}}^{j_{2}} \cong \delta_{j_{1}}^{j_{2}} \tag{3.3}
\end{equation*}
$$

Inductive formulas for the computation how the differential equation in the $(X, Y)$ coordinates is related to the differential equation in the $(x, y)$ coordinates may be found in [BK1989], [Ol1995]; the explicit formulas are not achieved in these references. Let us recall the inductive formulas, just on the
computational level (differential-geometric conceptional background about graph transformations may be found in [Ol1986], Ch. 2).

First of all, we seek how the $Y_{X}^{j}:=\frac{d Y^{j}}{d X}$ are explicitely related to the $y_{x}^{l}$. It suffices to replace, in the identity

$$
\begin{equation*}
Y_{X}^{j} \cdot\left(X_{x} d x+\sum_{l=1}^{m} X_{y^{l}} d y^{l}\right)=Y_{X}^{j} d X=d Y^{j}=Y_{x}^{j} d x+\sum_{l=1}^{m} Y_{y^{l}}^{j} d y^{l} \tag{3.4}
\end{equation*}
$$

the differentials $d y^{l}$ by $y_{x}^{l} d x$ and then to identify the coefficient of $d x$ on both sides, which rapidly yields the formulas

$$
\begin{equation*}
Y_{X}^{j}=\frac{Y_{x}^{j}+\sum_{l=1}^{m} y_{x}^{l} Y_{y^{l}}^{j}}{X_{x}+\sum_{l=1}^{m} y_{x}^{l} X_{y^{l}}}, \tag{3.5}
\end{equation*}
$$

for $j=1, \ldots, m$.
Next, we seek how the $Y_{X X}^{j}:=\frac{d^{2} Y^{j}}{d X^{2}}=\frac{d Y_{X}^{j}}{d X}$ are related to the $y_{x}^{l_{1}}, y_{x x}^{l_{2}}$. It suffices to again replace each $d y_{l}$ by $y_{x}^{l} d x$ and each $d y_{x}^{l}$ by $y_{x x}^{l} d x$ in the identity

$$
\left\{\begin{align*}
Y_{X X}^{j} \cdot\left(X_{x} d x+\sum_{l=1}^{m} X_{y^{l}} d y^{l}\right) & =Y_{X X}^{j} \cdot d X=d Y_{X}^{j}  \tag{3.6}\\
& =\frac{\partial Y_{X}^{j}}{\partial x} d x+\sum_{l=1}^{m} \frac{\partial Y_{X}^{j}}{\partial y^{l}} d y^{l}+\sum_{l=1}^{m} \frac{\partial Y_{X}^{j}}{\partial y_{x}^{l}} d y_{x}^{l} \\
& =\left(\frac{\partial Y_{X}^{j}}{\partial x}+\sum_{l=1}^{m} \frac{\partial Y_{X}^{j}}{\partial y^{l}} y_{x}^{l}+\sum_{l=1}^{m} \frac{\partial Y_{X}^{j}}{\partial y_{x}^{l}} y_{x x}^{l}\right) \cdot d x
\end{align*}\right.
$$

Before entering the precise combinatorics of the explicit expression of $Y_{X X}^{j}$, let us observe that the last term of (3.6) simply writes $D\left(Y_{X}^{j}\right) d x$, where $D$ denotes the total differentiation operator (of order two) defined by

$$
\begin{equation*}
D:=\frac{\partial}{\partial x}+\sum_{l=1}^{m} y_{x}^{l} \frac{\partial}{\partial y^{l}}+\sum_{l=1}^{m} y_{x x}^{l} \frac{\partial}{\partial y_{x}^{l}} . \tag{3.7}
\end{equation*}
$$

Since $d X \equiv D X$ after replacing each $d y^{l}$ by $y_{x}^{l} d x$, it follows that we may compactly rewrite (3.6) as

$$
\begin{equation*}
Y_{X X}^{j} D X \cdot d x=D\left(Y_{X}^{j}\right) \cdot d x \tag{3.8}
\end{equation*}
$$

Consequently, the expressions of $Y_{X}^{j}$ (obtained in (3.5)) and of $Y_{X X}^{j}$ are (3.9)
$Y_{X}^{j}=\frac{D Y^{j}}{D X} \quad$ and $\quad Y_{X X}^{j}=\frac{D\left(Y_{X}^{j}\right)}{D X}=\frac{D D Y^{j} \cdot D X-D D X \cdot D Y^{j}}{[D X]^{3}}$.

As, by assumption, the system $y_{x x}^{j}=F^{j}\left(x, y, y_{x}\right)$ transforms to the flat system $Y_{X X}^{j}=0$, after erasing the denominator of (3.7), we come to the equations

$$
\begin{equation*}
0=D D Y^{j} \cdot D X-D D X \cdot D Y^{j} \tag{3.10}
\end{equation*}
$$

for $j=1, \ldots, m$. However, this too simple and too compact expression of the system $y_{x x}^{j}=F^{j}\left(x, y, y_{x}\right)$ is of no use and we must develope (patiently!) the explicit expressions of $D D Y^{j}$, of $D X$, of $D D X$ and of $D Y^{j}$, using the complete expression of $D$ defined in (3.6).

At this point, we would like to stress that it constitutes already a nontrivial computational and combinatorial task to obtain a complete explicit formula for the system $y_{x x}^{j}=F^{j}\left(x, y, y_{x}\right)$ hidden in the compact form (3.10), which would be the generalization of the nice formula (2.9) involving modifications of the Jacobian determinant. For general $m \geqslant 2$, the complete proofs are postponed to Section 5 below.

Since it would be intuitively unsatisfactory to provide directly the final simplified expression of the development of (3.10) in the general case $m \geqslant 2$, let us firstly describe step by step how one may guess what is the generalization of (2.9).

For instance, in the case $m=2$, by a direct and relatively short computation which consists in developing plainly (3.10), we obtain for $j=1,2$ :

$$
\left\{\begin{align*}
0= & -X_{x x} Y_{x}^{j}+Y_{x x}^{j} X_{x}+  \tag{3.11}\\
& +y_{x}^{1} \cdot\left[-X_{x x} Y_{y^{1}}^{j}+Y_{x x}^{j} X_{y^{1}}-2 X_{x y^{1}} Y_{x}^{j}+2 Y_{x y^{1}}^{j} X_{x}\right]+ \\
& +y_{x}^{2} \cdot\left[-X_{x x} Y_{y^{2}}^{j}+Y_{x x}^{j} X_{y^{2}}-2 X_{x y^{2}} Y_{x}^{j}+2 Y_{x y^{2}}^{j} X_{x}\right]+ \\
& +y_{x}^{1} y_{x}^{1} \cdot\left[-2 X_{x y^{1}} Y_{y^{1}}^{j}+2 Y_{x y^{1}}^{j} X_{y^{1}}-X_{y^{1} y^{1}} Y_{x}^{j}+Y_{y^{1} y^{1}}^{j} X_{x}\right]+ \\
& +y_{x}^{1} y_{x}^{2} \cdot\left[-2 X_{x y^{1}} Y_{y^{2}}^{j}+2 Y_{x y^{1}}^{j} X_{y^{2}}-2 X_{x y^{2}} Y_{y^{1}}^{j}+2 Y_{x y^{2}}^{j} X_{y^{1}}-\right. \\
& \left.-2 X_{y^{1} y^{2}} Y_{x}^{j}+2 Y_{y^{1} y^{2}}^{j} X_{x}\right]+ \\
& +y_{x}^{2} y_{x}^{2} \cdot\left[-2 X_{x y^{2}} Y_{y^{2}}^{j}+2 Y_{x y^{2}}^{j} X_{y^{2}}-X_{y^{2} y^{2}} Y_{x}^{j}+Y_{y^{2} y^{2}}^{j} X_{x}\right]+ \\
& +y_{x}^{1} y_{x}^{1} y_{x}^{1} \cdot\left[-X_{y^{1} y^{1}} Y_{y^{1}}^{j}+Y_{y^{1} y^{1}}^{j} X_{y^{1}}\right]+ \\
& +y_{x}^{1} y_{x}^{1} y_{x}^{2} \cdot\left[-X_{y^{1} y^{1}} Y_{y^{2}}^{j}+Y_{y^{1} y^{1}}^{j} X_{y^{2}}-2 X_{y^{1} y^{2}} Y_{y^{1}}^{j}+2 Y_{y^{1} y^{2}}^{j} X_{y^{1}}\right]+ \\
& +y_{x}^{1} y_{x}^{2} y_{x}^{2} \cdot\left[-X_{y^{2} y^{2}} Y_{y^{1}}^{j}+Y_{y^{2} y^{2}}^{j} X_{y^{1}}-2 X_{y^{1} y^{2}} Y_{y^{2}}^{j}+2 Y_{y^{1} y^{2}}^{j} X_{y^{2}}\right]+ \\
& +y_{x}^{2} y_{x}^{2} y_{x}^{2} \cdot\left[-X_{y^{2} y^{2}} Y_{y^{2}}^{j}+Y_{y^{2} y^{2}}^{j} X_{y^{2}}\right]+ \\
& +y_{x x}^{1} \cdot\left[-X_{y^{1}} Y_{x}^{j}+Y_{y^{1}}^{j} X_{x}+y_{x}^{2} \cdot\left\{-X_{y^{1}} Y_{y^{2}}^{j}+Y_{y^{1}}^{j} X_{y^{2}}\right\}\right]+ \\
& +y_{x x}^{2} \cdot\left[-X_{y^{2}} Y_{x}^{j}+Y_{y^{2}}^{j} X_{x}+y_{x}^{1} \cdot\left\{-X_{y^{2}} Y_{y^{1}}^{j}+Y_{y^{2}}^{j} X_{y^{1}}\right\}\right] .
\end{align*}\right.
$$

Unfortunately, the above two equations are not solved with respect to $y_{x x}^{1}$ and to $y_{x x}^{2}$. Consequently, if we abbreviate them as a linear system of the form

$$
\left\{\begin{array}{l}
0=A^{1}+y_{x x}^{1} \cdot B_{1}^{1}+y_{x x}^{2} \cdot B_{2}^{1}  \tag{3.12}\\
0=A^{2}+y_{x x}^{1} \cdot B_{1}^{2}+y_{x x}^{2} \cdot B_{2}^{2}
\end{array}\right.
$$

we have to solve for $y_{x x}^{1}$ and for $y_{x x}^{2}$ by means of the classical rule of Cramer. Here, it is rather quick to check manually that the determinant of this system has the following nice expression:

$$
\left\{\begin{align*}
\left|\begin{array}{ll}
B_{1}^{1} & B_{2}^{1} \\
B_{1}^{2} & B_{2}^{2}
\end{array}\right| & =\Delta\left(x\left|y^{1}\right| y^{2}\right) \cdot\left\{X_{x}+y_{x}^{1} X_{y^{1}}+y_{x}^{2} X_{y^{2}}\right\}  \tag{3.13}\\
& =\Delta\left(x\left|y^{1}\right| y^{2}\right) \cdot D X
\end{align*}\right.
$$

However, the complete solving for $y_{x x}^{1}$ and for $y_{x x}^{2}$ requires some more time. After a direct and rather long hand computation (or alternately, using Maple
or Mathematica) one obtains formulas involving hidden $3 \times 3$ determinants, which have to be guessed by the intuition; the first equation that we obtain, namely for $y_{x x}^{1}$ is as follows:

$$
\begin{align*}
0= & y_{x x}^{1} \cdot\left|\begin{array}{ccc}
X_{x} & X_{y^{1}} & X_{y^{2}} \\
Y_{x}^{1} & Y_{y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{x}^{2} & Y_{y^{1}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|+\left|\begin{array}{cccc}
X_{x} & X_{x x} & X_{y^{2}} \\
Y_{x}^{1} & Y_{x x}^{1} & Y_{y^{2}}^{1} \\
Y_{x}^{2} & Y_{x x}^{2} & Y_{y^{2}}^{2}
\end{array}\right|+ \\
& +y_{x}^{1} \cdot\left\{2\left|\begin{array}{ccc}
X_{x} & X_{x y^{1}} & X_{y^{2}} \\
Y_{x}^{1} & Y_{x y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{x}^{2} & Y_{x y^{1}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|-\left|\begin{array}{ccc}
X_{x x} & X_{y^{1}} & X_{y^{2}} \\
Y_{x x}^{1} & Y_{y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{x x}^{2} & Y_{y^{1}} & Y_{y^{2}}^{2}
\end{array}\right|\right\}+  \tag{3.14}\\
& +y_{x}^{2} \cdot\left\{2\left|\begin{array}{ccc}
X_{x} & X_{x y^{2}} & X_{y^{2}}^{1} \\
Y_{x}^{1} & Y_{x y^{2}}^{1} & Y_{y^{2}}^{1} \\
Y_{x}^{2} & Y_{x y^{2}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|\right\}+
\end{align*}
$$

$$
+y_{x}^{1} y_{x}^{1} \cdot\left\{\left\{\left.\begin{array}{ccc}
X_{x} & X_{y^{1} y^{1}} & X_{y^{2}} \\
Y_{x}^{1} & Y_{y^{1} y^{1}} & Y_{y^{2}}^{1} \\
Y_{x}^{2} & Y_{y^{1} y^{1}}^{2} & Y_{y^{2}}^{2}
\end{array}|-2| \begin{array}{ccc}
X_{x y^{1}} & X_{y^{1}} & X_{y^{2}} \\
Y_{x y^{1}}^{1} & Y_{y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{x y^{1}}^{2} & Y_{y^{1}}^{2} & Y_{y^{2}}^{2}
\end{array} \right\rvert\,\right\}+\right.
$$

$$
+y_{x}^{1} y_{x}^{2}\left\{2\left|\begin{array}{ccc}
X_{x} & X_{y^{1}} y^{2} & X_{y^{2}} \\
Y_{x}^{1} & Y_{y^{1} y^{2}} & Y_{y^{2}}^{1} \\
Y_{x}^{2} & Y_{y^{1} y^{2}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|-2\left|\begin{array}{ccc}
X_{x y^{2}} & X_{y^{1}} & X_{y^{2}} \\
Y_{x y^{2}}^{1} & Y_{y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{x y^{2}}^{2} & Y_{y^{1}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|\right\}+
$$

$$
+y_{x}^{2} y_{x}^{2} \cdot\left\{\left|\begin{array}{ccc}
X_{x} & X_{y^{2} y^{2}} & X_{y^{2}} \\
Y_{x}^{1} & Y_{y^{2}}^{1} y^{2} & Y_{y^{2}}^{1} \\
Y_{x}^{2} & Y_{y^{2} y^{2}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|\right\}+
$$

$$
+y_{x}^{1} y_{x}^{1} y_{x}^{1} \cdot\left\{-\left|\begin{array}{ccc}
X_{y^{1}} y^{1} & X_{y^{1}} & X_{y^{2}} \\
Y_{y^{1} y^{1}} & Y_{y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{y^{1} y^{1}}^{2} & Y_{y^{1}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|\right\}+y_{x}^{1} y_{x}^{1} y_{x}^{2} \cdot\left\{-2\left|\begin{array}{ccc}
X_{y^{1} y^{2}} & X_{y^{1}} & X_{y^{2}} \\
Y_{y^{1} y^{2}}^{1} & Y_{y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{y^{1} y^{2}}^{2} & Y_{y^{2}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|\right\}+
$$

$$
+y_{x}^{1} y_{x}^{2} y_{x}^{2} \cdot\left\{-\left|\begin{array}{ccc}
X_{y^{2} y^{2}} & X_{y^{1}} & X_{y^{2}} \\
Y_{y^{2}}^{1} y^{2} & Y_{y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{y^{2} y^{2}}^{2} & Y_{y^{1}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|\right\}
$$

This formula and the next have been checked by Sylvain Neut and Michel Petitot with the help of Maple. We notice that the size is not negligible, but fortunately, there appears some combinatorics, much more visible than
in (3.11). The second equation that we obtain, namely for $y_{x x}^{2}$, is as follows:

$$
\begin{aligned}
& 0=y_{x x}^{2} \cdot\left|\begin{array}{ccc}
X_{x} & X_{y^{1}} & X_{y^{2}} \\
Y_{x}^{1} & Y_{y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{x}^{2} & Y_{y^{1}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|+\left|\begin{array}{ccc}
X_{x} & X_{y^{1}} & X_{x x} \\
Y_{x}^{1} & Y_{y^{1}}^{1} & Y_{x x}^{1} \\
Y_{x}^{2} & Y_{y^{2}}^{2} & Y_{x x}^{2}
\end{array}\right|+ \\
& +y_{x}^{1} \cdot\left\{2\left|\begin{array}{ccc}
X_{x} & X_{y^{1}} & X_{x y^{1}} \\
Y_{x}^{1} & Y_{y^{1}}^{1} & Y_{x y^{1}}^{1} \\
Y_{x}^{2} & Y_{y^{1}}^{2} & Y_{x y^{1}}^{2}
\end{array}\right|\right\}+ \\
& +y_{x}^{2} \cdot\left\{2\left|\begin{array}{ccc}
X_{x} & X_{y^{1}} & X_{x y^{2}} \\
Y_{x}^{1} & Y_{y^{1}}^{1} & Y_{x y^{2}}^{1} \\
Y_{x}^{2} & Y_{y^{1}}^{2} & Y_{x y^{2}}^{2}
\end{array}\right|-\left|\begin{array}{ccc}
X_{x x} & X_{y^{1}} & X_{y^{2}} \\
Y_{x x}^{1} & Y_{y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{x x}^{2} & Y_{y^{1}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|\right\}+ \\
& +y_{x}^{1} y_{x}^{1} \cdot\left\{\left|\begin{array}{ccc}
X_{x} & X_{y^{1}} & X_{y^{1} y^{1}} \\
Y_{x}^{1} & Y_{y^{1}}^{1} & Y_{y^{1}}^{1} y^{1} \\
Y_{x}^{2} & Y_{y^{1}}^{2} & Y_{y^{1} y^{1}}^{2}
\end{array}\right|\right\}+ \\
& +y_{x}^{1} y_{x}^{2}\left\{2\left|\begin{array}{ccc}
X_{x} & X_{y^{1}} & X_{y^{1} y^{2}} \\
Y_{x}^{1} & Y_{y^{1}}^{1} & Y_{y^{1} y^{2}}^{1} \\
Y_{x}^{2} & Y_{y^{2}}^{2} & Y_{y^{1} y^{2}}^{2}
\end{array}\right|-2\left|\begin{array}{ccc}
X_{x y^{1}} & X_{y^{1}} & X_{y^{2}} \\
Y_{x y^{1}}^{1} & Y_{y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{x y^{1}}^{2} & Y_{y^{2}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|\right\}+ \\
& +y_{x}^{2} y_{x}^{2} \cdot\left\{\left|\begin{array}{ccc}
X_{x} & X_{y^{1}} & X_{y^{2} y^{2}} \\
Y_{x}^{1} & Y_{y^{1}}^{1} & Y_{y^{2} y^{2}} \\
Y_{x}^{2} & Y_{y^{1}}^{2} & Y_{y^{2} y^{2}}^{2}
\end{array}\right|-2\left|\begin{array}{ccc}
X_{x y^{2}} & X_{y^{1}} & X_{y^{2}} \\
Y_{x y^{2}}^{1} & Y_{y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{x y^{2}}^{2} & Y_{y^{1}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|\right\}+ \\
& +y_{x}^{1} y_{x}^{1} y_{x}^{2} \cdot\left\{-\left|\begin{array}{ccc}
X_{y^{1} y^{1}} & X_{y^{1}} & X_{y^{2}} \\
Y_{y^{1} y^{1}}^{1} & Y_{y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{y^{1} y^{1}}^{2} & Y_{y^{1}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|\right\}+y_{x}^{1} y_{x}^{2} y_{x}^{2} \cdot\left\{-2\left|\begin{array}{ccc}
X_{y^{1} y^{2}} & X_{y^{1}} & X_{y^{2}} \\
Y_{y^{1} y^{2}}^{1} & Y_{y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{y^{1} y^{2}}^{2} & Y_{y^{2}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|\right\}+ \\
& +y_{x}^{2} y_{x}^{2} y_{x}^{2} \cdot\left\{-\left|\begin{array}{ccc}
X_{y^{2}} y^{2} & X_{y^{1}} & X_{y^{2}} \\
Y_{y^{2}}^{1} y^{2} & Y_{y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{y^{2} y^{2}}^{2} & Y_{y^{1}}^{2} & Y_{y^{2}}^{2}
\end{array}\right|\right\} .
\end{aligned}
$$

Importantly, the obtained formulas seem to be analogous to the formula (2.9), since we observe that the coefficients of the degree three polynomial in the $y_{x}^{l}$ are modifications of the Jacobian determinant $\Delta\left(x\left|y^{1}\right| y^{2}\right)$.

To describe the underlying combinatorics, let us observe that there exist exactly six possible distinct second order derivatives: $x x, x y^{1}, x y^{2}, y^{1} y^{1}$, $y^{1} y^{2}$ and $y^{2} y^{2}$. There are also exactly three columns in the Jacobian determinant (3.2). By replacing each of the three columns of first order derivatives by a column of second order detivatives (leaving $X, Y^{1}$ and $Y^{2}$ unchanged),
we may build exactly eighteen new determinants

$$
\left\{\begin{array}{lrl}
\Delta\left(x x\left|y^{1}\right| y^{2}\right) & \Delta\left(x|x x| y^{2}\right) & \Delta\left(x\left|y^{1}\right| x x\right)  \tag{3.16}\\
\Delta\left(x y^{1}\left|y^{1}\right| y^{2}\right) & \Delta\left(x\left|x y^{1}\right| y^{2}\right) & \Delta\left(x\left|y^{1}\right| x y^{1}\right) \\
\Delta\left(x y^{2}\left|y^{1}\right| y^{2}\right) & \Delta\left(x\left|x y^{2}\right| y^{2}\right) & \Delta\left(x\left|y^{1}\right| x y^{2}\right) \\
\Delta\left(y^{1} y^{1}\left|y^{1}\right| y^{2}\right) & \Delta\left(x\left|y^{1} y^{1}\right| y^{2}\right) & \Delta\left(x\left|y^{1}\right| y^{1} y^{1}\right) \\
\Delta\left(y^{1} y^{2}\left|y^{1}\right| y^{2}\right) & \Delta\left(x\left|y^{1} y^{2}\right| y^{2}\right) & \Delta\left(x\left|y^{1}\right| y^{1} y^{2}\right) \\
\Delta\left(y^{2} y^{2}\left|y^{1}\right| y^{2}\right) & \Delta\left(x\left|y^{2} y^{2}\right| y^{2}\right) & \Delta\left(x\left|y^{1}\right| y^{2} y^{2}\right)
\end{array}\right.
$$

where for instance

$$
\left\{\begin{align*}
& \Delta\left(\underline{y^{1} y^{2}}\left|y^{1}\right| y^{2}\right):=\left|\begin{array}{ccc}
X_{y^{1} y^{2}} & X_{y^{1}} & X_{y^{2}} \\
Y_{y^{1} y^{2}}^{1} & Y_{y^{1}}^{1} & Y_{y^{2}}^{1} \\
Y_{\underline{y^{1} y^{2}}}^{2} & Y_{y^{1}}^{2} & Y_{y^{2}}^{2}
\end{array}\right| \quad \text { and }  \tag{3.17}\\
& \Delta\left(x\left|y^{1}\right| \underline{x y^{2}}\right):=\left|\begin{array}{ccc}
X_{x} & X_{y^{1}} & X_{x y^{2}} \\
Y_{x}^{1} & Y_{y^{1}}^{1} & Y_{x y^{2}}^{1} \\
Y_{x}^{2} & Y_{y^{1}}^{2} & Y_{x y^{2}}^{2}
\end{array}\right|
\end{align*}\right.
$$

Hence, using the $\Delta$-notation, we may rewrite the two equation (3.14) and (3.15) under a more compact form; after division by the Jacobian determinant $\Delta\left(x\left|y^{1}\right| y^{2}\right)$, the first equation becomes:

$$
\left\{\begin{align*}
0= & y_{x x}^{1}+\frac{\Delta\left(x|x x| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}+y_{x}^{1} \cdot\left\{2 \frac{\Delta\left(x\left|x y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}-\frac{\Delta\left(x x\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\}+  \tag{3.18}\\
& +y_{x}^{2} \cdot\left\{2 \frac{\Delta\left(x\left|x y^{2}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\}+y_{x}^{1} y_{x}^{1} \cdot\left\{\frac{\Delta\left(x\left|y^{1} y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}-2 \frac{\Delta\left(x y^{1}\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\}+ \\
& +y_{x}^{1} y_{x}^{2} \cdot\left\{2 \frac{\Delta\left(x\left|y^{1} y^{2}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}-2 \frac{\Delta\left(x y^{2}\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\}+y_{x}^{2} y_{x}^{2} \cdot\left\{\frac{\Delta\left(x\left|y^{2} y^{2}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\}+ \\
& +y_{x}^{1} y_{x}^{1} y_{x}^{1} \cdot\left\{-\frac{\Delta\left(y^{1} y^{1}\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\}+y_{x}^{1} y_{x}^{1} y_{x}^{2} \cdot\left\{-2 \frac{\Delta\left(y^{1} y^{2}\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\}+ \\
& +y_{x}^{1} y_{x}^{2} y_{x}^{2} \cdot\left\{-\frac{\Delta\left(y^{2} y^{2}\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\} .
\end{align*}\right.
$$

Similarly, the second equation takes the form:
(3.19)

$$
\left\{\begin{aligned}
0= & y_{x x}^{2}+\frac{\Delta\left(x\left|y^{1}\right| x x\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}+y_{x}^{1} \cdot\left\{2 \frac{\Delta\left(x\left|y^{1}\right| x y^{1}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\}+ \\
& +y_{x}^{2} \cdot\left\{2 \frac{\Delta\left(x\left|y^{1}\right| x y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}-\frac{\Delta\left(x x\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\}+y_{x}^{1} y_{x}^{1} \cdot\left\{\frac{\Delta\left(x\left|y^{1}\right| y^{1} y^{1}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\}+ \\
& +y_{x}^{1} y_{x}^{2} \cdot\left\{2 \frac{\Delta\left(x\left|y^{1}\right| y^{1} y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}-2 \frac{\Delta\left(x y^{1}\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\}+ \\
& +y_{x}^{2} y_{x}^{2} \cdot\left\{\frac{\Delta\left(x\left|y^{2}\right| y^{2} y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}-2 \frac{\Delta\left(x y^{2}\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\}+ \\
& +y_{x}^{1} y_{x}^{1} y_{x}^{2} \cdot\left\{-\frac{\Delta\left(y^{1} y^{1}\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\}+y_{x}^{1} y_{x}^{2} y_{x}^{2} \cdot\left\{-2 \frac{\Delta\left(y^{1} y^{2}\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\}+ \\
& +y_{x}^{2} y_{x}^{2} y_{x}^{2} \cdot\left\{-\frac{\Delta\left(y^{2} y^{2}\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}\right\} .
\end{aligned}\right.
$$

Since the formulas are still of a consequent size, analogously to what was achieved in Section 2, we shall introduce a new family of square functions as follows. We first index the coordinates $\left(x, y^{1}, \ldots, y^{m}\right)$ as $\left(y^{0}, y^{1}, \ldots, y^{m}\right)$, namely we introduce the notational equivalence

$$
\begin{equation*}
y^{0} \equiv x \tag{3.20}
\end{equation*}
$$

which will be very convenient in the sequel, especially in order to write general formulas. With this convention at hand, our eighteen square functions $\square_{y^{l_{1}} y_{2}{ }^{l_{2}}}^{k_{1}}$, defined for $0 \leqslant j_{1}, j_{2}, k_{1} \leqslant 2$ are defined by

$$
\left\{\begin{array}{lll}
\square_{x x}^{0}:=\frac{\Delta\left(x x\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} & \square_{x y^{1}}^{0}:=\frac{\Delta\left(x y^{1}\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} & \square_{x y^{2}}^{0}:=\frac{\Delta\left(x y^{2}\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}  \tag{3.21}\\
\square_{y^{1} y^{1}}^{0}:=\frac{\Delta\left(y^{1} y^{1}\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} & \square_{y^{1} y^{2}}^{0}:=\frac{\Delta\left(y^{1} y^{2}\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} & \square_{y^{2} y^{2}}^{0}:=\frac{\Delta\left(y^{2} y^{2}\left|y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} \\
\square_{x x}^{1}:=\frac{\Delta\left(x|x x| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} & \square_{x y^{1}}^{1}:=\frac{\Delta\left(x\left|x y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} & \square_{x y^{2}}^{1}:=\frac{\Delta\left(x\left|x y^{2}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} \\
\square_{y^{1} y^{1}}^{1}:=\frac{\Delta\left(x\left|y^{1} y^{1}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} & \square_{y^{1} y^{2}}^{1}:=\frac{\Delta\left(x\left|y^{1} y^{2}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} & \square_{y^{2} y^{2}}^{1}:=\frac{\Delta\left(x\left|y^{2} y^{2}\right| y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} \\
\square_{x x}^{2}:=\frac{\Delta\left(x\left|y^{1}\right| x x\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} & \square_{x y^{1}}^{2}:=\frac{\Delta\left(x\left|y^{1}\right| x y^{1}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} & \square_{x y^{2}}^{2}:=\frac{\Delta\left(x\left|y^{1}\right| x y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} \\
\square_{y^{1} y^{1}}^{2}:=\frac{\Delta\left(x\left|y^{1}\right| y^{1} y^{1}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} & \square_{y^{1} y^{2}}^{2}:=\frac{\Delta\left(x\left|y^{1}\right| y^{1} y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)} & \square_{y^{2} y^{2}}^{2}:=\frac{\Delta\left(x\left|y^{1}\right| y^{2} y^{2}\right)}{\Delta\left(x\left|y^{1}\right| y^{2}\right)}
\end{array}\right.
$$

Obviously, the square functions are symmetric with respect to the lower indices: $\square_{y_{1} y^{l_{2}}}^{k_{1}}=\square_{y^{l_{2}} y^{l_{1}}}^{k_{1}}$. Here, the upper index designates the column
upon which the second order derivative appears, itself being encoded by the two lower indices. Even if this is hidden in the notation, we shall remember that the square functions are explicit rational expressions in terms of the second order jet of the transformation $(x, y) \mapsto(X(x, y), Y(x, y))$. At this point, we may summarize what we have established so far.

Lemma 3.22. The system of two second order ordinary differential equations $y_{x x}^{1}=F^{1}\left(x, y, y_{x}\right)$ and $y_{x x}^{2}=F^{2}\left(x, y, y_{x}\right)$ is equivalent, under a point transformation, to the flat system $Y_{X X}^{1}=0$ and $Y_{X X}^{2}=0$ if and only if there exist three local $\mathbb{K}$-analytic functions $X(x, y), Y^{1}(x, y)$ and $Y^{2}(x, y)$ such that it may be written under the form

$$
\left\{\begin{align*}
0= & y_{x x}^{1}+\square_{x x}^{1}+y_{x}^{1} \cdot\left(2 \square_{x y^{1}}^{1}-\square_{x x}^{0}\right)+y_{x}^{2} \cdot\left(2 \square_{x y^{2}}^{1}\right)+  \tag{3.23}\\
& +y_{x}^{1} y_{x}^{1} \cdot\left(\square_{y^{1} y^{1}}^{1}-2 \square_{x y^{1}}^{0}\right)+y_{x}^{1} y_{x}^{2} \cdot\left(2 \square_{y^{1} y^{2}}^{1}-2 \square_{x y^{2}}^{0}\right)+y_{x}^{2} y_{x}^{2} \cdot\left(\square_{y^{2} y^{2}}^{1}\right)+ \\
& +y_{x}^{1} y_{x}^{1} y_{x}^{1} \cdot\left(-\square_{y^{1} y^{1}}^{0}\right)+y_{x}^{1} y_{x}^{1} y_{x}^{2} \cdot\left(-2 \square_{y^{1} y^{2}}^{0}\right)+y_{x}^{1} y_{x}^{2} y_{x}^{2} \cdot\left(-\square_{y^{2} y^{2}}^{0}\right), \\
0= & y_{x x}^{2}+\square_{x x}^{2}+y_{x}^{1} \cdot\left(2 \square_{x y^{1}}^{2}\right)+y_{x}^{2} \cdot\left(2 \square_{x y^{2}}^{2}-\square_{x x}^{0}\right)+ \\
& +y_{x}^{1} y_{x}^{1} \cdot\left(\square_{y^{1} y^{1}}^{2}\right)+y_{x}^{1} y_{x}^{2} \cdot\left(2 \square_{y^{1} y^{2}}^{2}-2 \square_{x y^{1}}^{0}\right)+y_{x}^{2} y_{x}^{2} \cdot\left(\square_{y^{2} y^{2}}^{2}-2 \square_{x y^{2}}^{0}\right)+ \\
& +y_{x}^{1} y_{x}^{1} y_{x}^{2} \cdot\left(-\square_{y^{1} y^{1}}^{0}\right)+y_{x}^{1} y_{x}^{2} y_{x}^{2} \cdot\left(-2 \square_{y^{1} y^{2}}^{0}\right)+y_{x}^{2} y_{x}^{2} y_{x}^{2} \cdot\left(-\square_{y^{2} y^{2}}^{0}\right) .
\end{align*}\right.
$$

3.24. Second Lie prolongation of a vector field. At this point, instead of proceeding further with the case $m=2$, it is now time to pass to the general case $m \geqslant 2$. First of all, we would like to remind from [GM2003] the complete explicit expression of the point prolongation to the second order jet space of a general vector field of the form $L=X \frac{\partial}{\partial x}+\sum_{j=1}^{m} Y^{j} \frac{\partial}{\partial y^{j}}$ : it is a vector field of the form

$$
\begin{equation*}
L^{(2)}=X \frac{\partial}{\partial x}+\sum_{j=1}^{m} Y^{j} \frac{\partial}{\partial y^{j}}+\sum_{j=1}^{m} \mathbf{R}_{1}^{j} \frac{\partial}{\partial y_{x}^{j}}+\sum_{j=1}^{m} \mathbf{R}_{2}^{j} \frac{\partial}{\partial y_{x x}^{j}}, \tag{3.25}
\end{equation*}
$$

where the coefficients $\mathbf{R}_{1}^{j}$ and $\mathbf{R}_{2}^{j}$ are polynomials in the jet space variables having as coefficients certain specific linear combinations of first and second
order derivatives of $X$ and of the $Y^{j}$ :
(3.26)

$$
\left\{\begin{aligned}
\mathbf{R}_{1}^{j}= & Y_{x}^{j}+\sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot\left[Y_{y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} X_{x}\right]+\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot\left[-\delta_{l_{1}}^{j} X_{y^{l_{2}}}\right] \\
\mathbf{R}_{2}^{j}= & Y_{x x}^{j}+\sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot\left[2 Y_{x y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} X_{x x}\right]+ \\
& +\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot\left[Y_{y^{l_{1} y^{l_{2}}}}^{j}-\delta_{l_{1}}^{j} X_{x y^{l_{2}}}-\delta_{l_{2}}^{j} X_{x y^{l_{1}}}\right]+ \\
& +\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} \sum_{l_{3}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} y_{x}^{l_{3}} \cdot\left[-\delta_{l_{1}}^{j} X_{y^{l_{2} y^{l_{3}}}}\right]+ \\
& +\sum_{l_{1}=1}^{m} y_{x x}^{l_{1}} \cdot\left[Y_{y^{l_{1}}}^{j}-2 \delta_{l_{1}}^{j} X_{x}\right]+ \\
& +\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x x}^{l_{2}} \cdot\left[-\delta_{l_{1}}^{j} X_{y^{l_{2}}}-2 \delta_{l_{2}}^{j} X_{y^{l_{1}}}\right]
\end{aligned}\right.
$$

However, since the notations in [GM2003] are different and since the general case of $n \geqslant 1$ independent variables and $m \geqslant 1$ dependent variables is considered there, it is certainly easier to reconstiture formulas (3.26) directly by means of the inductive formulas described in [Ol1986], [BK1989]).

Analogously to the observation made in Section 2, we guess that there exists a formal correspondence between the terms of $\mathbf{R}_{2}^{j}$ not involving $y_{x x}^{l}$ and the explicit form of the equation $y_{x x}^{j}=F^{j}\left(x, y, y_{x}\right)$ equivalent to $Y_{X X}^{j}=$ 0 . In the case $m=2$, we claim that this formal correspondence also holds true. Indeed, it suffices to write formula (3.26) for $\mathbf{R}_{2}^{j}$ modulo the $y_{x x}^{l}$, which yields two expressions in total analogy with the two explicit polynomials appearing in the right-hand side of (3.23):
(3.27)

$$
\left\{\begin{aligned}
\mathbf{R}_{2}^{1}\left(\bmod y_{x x}^{l}\right) \equiv & Y_{x x}^{1}+y_{x}^{1} \cdot\left\{2 Y_{x y^{1}}^{1}-X_{x x}\right\}+y_{x}^{2} \cdot\left\{2 Y_{x y^{2}}^{1}\right\}+y_{x}^{1} y_{x}^{1} \cdot\left\{Y_{y^{1} y^{1}}^{1}-2 X_{x y^{1}}\right\}+ \\
& +y_{x}^{1} y_{x}^{2} \cdot\left\{2 Y_{y^{1} y^{2}}^{1}-2 X_{x y^{2}}\right\}+y_{x}^{2} y_{x}^{2} \cdot\left\{Y_{y^{2} y^{2}}^{1}\right\}+ \\
& +y_{x}^{1} y_{x}^{1} y_{x}^{1} \cdot\left\{-X_{y^{1} y^{1}}\right\}+y_{x}^{1} y_{x}^{1} y_{x}^{2} \cdot\left\{-2 X_{y^{1} y^{2}}\right\}+y_{x}^{1} y_{x}^{2} y_{x}^{2} \cdot\left\{-X_{y^{2} y^{2}}\right\} \\
\mathbf{R}_{2}^{2}\left(\bmod y_{x x}^{l}\right) \equiv & Y_{x x}^{2}+y_{x}^{1} \cdot\left\{2 Y_{x y^{1}}^{2}\right\}+y_{x}^{2} \cdot\left\{2 Y_{x y^{2}}^{2}-X_{x x}\right\}+y_{x}^{1} y_{x}^{1} \cdot\left\{Y_{y^{1} y^{1}}^{2}\right\}+ \\
& +y_{x}^{1} y_{x}^{2} \cdot\left\{2 Y_{y^{1} y^{2}}^{2}-2 X_{x y^{1}}\right\}+y_{x}^{2} y_{x}^{2} \cdot\left\{Y_{y^{2} y^{2}}^{2}-2 X_{x y^{2}}\right\}+ \\
& +y_{x}^{1} y_{x}^{1} y_{x}^{2} \cdot\left\{-X_{y^{1} y^{1}}\right\}+y_{x}^{1} y_{x}^{2} y_{x}^{2} \cdot\left\{-2 X_{y^{1} y^{2}}\right\}+y_{x}^{2} y_{x}^{2} y_{x}^{2} \cdot\left\{-X_{y^{2} y^{2}}\right\}
\end{aligned}\right.
$$

Except for inductive inspiration (see the formulation of Lemma 3.32 below), this observation will not be used further. At this stage, it helps at least
to maintain a strong intuitive control of the correctness of the underlying combinatorics.
3.28. System equivalent to the flat system. By induction, we therefore guess that the analogy holds for general $m \geqslant 2$, namely we guess the following combinatorics, which requires some preliminaries.

As in the beginning of $\S 3.1$, let $x \in \mathbb{K}$, let $y=\left(y^{1}, \ldots, y^{m}\right) \in \mathbb{K}^{m}$, let $(x, y) \mapsto(X(x, y), Y(x, y))$ be a local $\mathbb{K}$-analytic transformation defined in a neighborhood of the origin in $\mathbb{K}^{m+1}$ and assume that the system $y_{x x}^{j}=$ $F^{j}\left(x, y, y_{x}\right), j=1, \ldots, m$, is equivalent to the flat system $Y_{X X}^{j}=0, j=$ $1, \ldots, m$. By assumption, the Jacobian matrix of the equivalence equals the identity matrix at the origin. Remind that we identify $x$ with $y^{0}$. For all $k_{1}, l_{1}, l_{2}=0, \ldots, m$, we define a modification

$$
\begin{equation*}
\Delta\left(x|\ldots|^{k_{1}} y^{l_{1}} y^{l_{2}}|\ldots| y^{m}\right) \tag{3.29}
\end{equation*}
$$

of the Jacobian determinant as follows. We replace the $k_{1}$-th column of the determinant (3.2), which consists of first order derivatives ${ }_{y^{k_{1}}}$, by a column which consists of second order derivatives $y_{y^{l_{1}} y^{l_{2}}}$. In (3.29), the notation $\left.\right|^{k_{1}}$ designates the $k_{1}$-th column, the first one being labelled by $k_{1}=0$ and the last one by $k_{1}=m$. With this notation at hand, we may define the square functions

$$
\begin{equation*}
\square_{y^{k_{1}} y^{l_{2}}}^{k_{2}}:=\frac{\Delta\left(x|\ldots|^{k_{1}} y^{l_{1}} y^{l_{2}}|\ldots| y^{m}\right)}{\Delta\left(x|\ldots|^{k_{1}} y^{k_{1}} \mid \ldots y^{m}\right)} \tag{3.30}
\end{equation*}
$$

which are rational expressions in the second order jet of the transformation $(x, y) \mapsto(X(x, y), Y(x, y))$. As before, the denominator is the Jacobian determinant of the change of coordinates.

Since, according to (3.26), the expression of $\mathbf{R}_{2}^{j}\left(\bmod y_{x x}^{l}\right)$ is

$$
\left\{\begin{align*}
\mathbf{R}_{2}^{j}\left(\bmod y_{x x}^{l}\right)=Y_{x x}^{j} & +\sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot\left[2 Y_{x y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} X_{x x}\right]+  \tag{3.31}\\
& +\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot\left[Y_{y^{l_{1}} y^{l_{2}}}^{j}-\delta_{l_{1}}^{j} X_{x y^{l_{2}}}-\delta_{l_{2}}^{j} X_{x y^{l_{1}}}\right]+ \\
& +\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} \sum_{l_{3}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} y_{x}^{l_{3}} \cdot\left[-\delta_{l_{1}}^{j} X_{y^{l_{2} y^{l_{3}}}}\right]
\end{align*}\right.
$$

and since, in the cases $m=1$ and $m=2$, we have already observed strong analogies between (3.31) and the complete explicit expression of the system $y_{x x}^{j}=F^{j}\left(x, y, y_{x}\right)$ equivalent to the flat system $Y_{X X}^{j}=0$, we guess that the following lemma is formally true.

Lemma 3.32. The system $y_{x x}^{j}=F^{j}\left(x, y, y_{x}\right), j=1, \ldots, m$, is equivalent to the flat system $Y_{X X}^{j}=0, j=1, \ldots, m$, if and only if there exist local $\mathbb{K}$-analytic functions $X(x, y)$ and $Y^{j}(x, y), j=1, \ldots, m$, such that it may be written under the specific form

$$
\left\{\begin{align*}
0=y_{x x}^{j}+\square_{x x}^{j} & +\sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot\left[2 \square_{x y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} \square_{x x}^{0}\right]+  \tag{3.33}\\
& +\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot\left[\square_{y^{l_{1}} y^{l_{2}}}^{j}-\delta_{l_{1}}^{j} \square_{x y^{l_{2}}}^{0}-\delta_{l_{2}}^{j} \square_{x l^{l_{1}}}^{0}\right]+ \\
& +y_{x}^{j} \sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot\left[-\square_{y^{l_{1} y^{l_{2}}} 0}^{0}\right] .
\end{align*}\right.
$$

The complete proof of this lemma involves only linear algebra considerations, although with rather massive terms. This makes it rather lengthy. Consequently, we postpone it to the final Section 5 below.
3.34. First auxiliary system. Clearly, if we set

$$
\left\{\begin{align*}
G^{j} & :=-\square_{x x}^{j},  \tag{3.35}\\
H_{l_{1}}^{j} & :=-2 \square_{x y^{l_{1}}}^{j}+\delta_{l_{1}}^{j} \square_{x x}^{0}, \\
L_{l_{1}, l_{2}}^{j} & :=-\square_{y_{1} y^{l_{2}}}^{j}+\delta_{l_{1}}^{j} \square_{x y^{l_{2}}}^{0}+\delta_{l_{2}}^{j} \square_{x y^{l_{1}}}^{0}, \\
M_{l_{1}, l_{2}} & :=\square_{y^{l_{1} l_{2}},}^{0},
\end{align*}\right.
$$

we immediately see that the first condition of Theorem 1.7 holds true. Moreover, we claim that there are $m+1$ more square functions than functions $G^{j}$, $H_{l_{1}}^{j}, L_{l_{1}, l_{2}}^{j}$ and $M_{l_{1}, l_{2}}$. Indeed, taking account of the symmetries, we enumerate:

$$
\left\{\begin{array}{lr}
\#\left\{\square_{x x}^{j}\right\}=m, & \#\left\{\square_{x x}^{0}\right\}=1  \tag{3.36}\\
\#\left\{\square_{x y^{l_{1}}}^{j}\right\}=m^{2}, & \#\left\{\square_{x y^{l_{1}}}^{0}\right\}=m \\
\#\left\{\square_{y^{l_{1} y_{2}}}^{j}\right\}=\frac{m^{2}(m+1)}{2}, & \#\left\{\square_{y^{l_{1} y^{l_{2}}} 0}^{0}\right\}=\frac{m(m+1)}{2}
\end{array}\right.
$$

whereas
(3.37)

$$
\left\{\begin{array}{lr}
\#\left\{G^{j}\right\}=m, & \#\left\{H_{l_{1}}^{j}\right\}=m^{2} \\
\#\left\{L_{l_{1}, l_{2}}^{j}\right\}=\frac{m^{2}(m+1)}{2}, & \#\left\{M_{l_{1}, l_{2}}\right\}=\frac{m(m+1)}{2}
\end{array}\right.
$$

Similarly as in Section 2, for $j, l_{1}, l_{2}=0,1, \ldots, m$, let us introduce functions $\Pi_{l_{1}, l_{2}}^{j}$ of $\left(x, y^{1}, \ldots, y^{m}\right)$, symmetric with respect to the lower indices,
and let us seek necessary and sufficient conditions in order that there exist solutions $(X, Y)$ to the first auxiliary system defined precisely by:

$$
\left\{\begin{array}{lll}
\square_{x x}^{0}=\Pi_{0,0}^{0}, & \square_{x y^{l_{1}}}^{0}=\Pi_{0, l_{1}}^{0}, & \square_{y_{1} y^{l_{2}}}^{0}=\Pi_{l_{1}, l_{2}}^{0},  \tag{3.38}\\
\square_{x x}^{j}=\Pi_{0,0}^{j}, & \square_{x y^{l_{1}}}^{j}=\Pi_{0, l_{1}}^{j}, & \square_{y^{l_{1} y^{l_{2}}}}^{j}=\Pi_{l_{1}, l_{2}}^{j} .
\end{array}\right.
$$

3.39. Compatibility conditions for the first auxiliary system. As in Section 2, the compatibility conditions for this system will simply be obtained by computing the cross differentiations. The following statement generalizes Lemma 2.31 and also provides a proof of it, in the case $m=1$.

Lemma 3.40. For all $j, l_{1}, l_{2}, l_{3}=0,1, \ldots, m$, we have the cross differentiation relations

$$
\begin{equation*}
\left(\square_{y^{l_{1} y^{l_{2}}}}^{j}\right)_{y^{l_{3}}}-\left(\square_{y^{l_{1} y^{l_{3}}}}^{j}\right)_{y^{l_{2}}}=-\sum_{k=0}^{m} \square_{y^{l_{1} l^{l_{2}}}}^{k} \cdot \square_{y^{l_{3} y^{k}}}^{j}+\sum_{k=0}^{m} \square_{y^{l_{1} y^{l_{3}}}}^{k} \cdot \square_{y^{l_{2} y^{k}}}^{j} \tag{3.41}
\end{equation*}
$$

Proof. To begin with, as a preliminary, let us generalize the Plücker identity (2.28). Let $C_{1}, C_{2}, \ldots, C_{m}, D, E$ be $(m+2)$ column vectors in $\mathbb{K}^{m}$ and introduce the following notation for the $m \times(m+2)$ matrix consisting of these vectors:

$$
\begin{equation*}
\left[C_{1}\left|C_{2}\right| \cdots\left|C_{m}\right| D \mid E\right] . \tag{3.42}
\end{equation*}
$$

Extracting columns from this matrix, we shall construct $m \times m$ determinants which are modification of the following "fundamental" determinant

$$
\begin{equation*}
\left\|C_{1}|\cdots| C_{m}\right\| \equiv\left\|C_{1}|\cdots|{ }^{j_{1}} C_{j_{1}}|\cdots|{ }^{j_{2}} C_{j_{2}}|\cdots| C_{m}\right\| . \tag{3.43}
\end{equation*}
$$

Here and in the sequel, we use a double vertical line in the beginning and in the end to denote a determinant. Also, we emphasize two distinct columns, the $j_{1}$-th and the $j_{2}$-th, where $j_{2}>j_{1}$, since we will modify them. For instance in this matrix, let us replace these two columns by the column $D$ and by the column $E$, which yields the determinant

$$
\begin{equation*}
\left\|C_{1}|\cdots|{ }^{j_{1}} D|\cdots|{ }^{j_{2}} E|\cdots| C_{m}\right\| . \tag{3.44}
\end{equation*}
$$

In this notation, one should understand that only the $j_{1}$-th and the $j_{2}$-th columns are distinct from the columns of the fundamental $m \times m$ determinant (3.43). With this notation at hand, we can now formulate and prove a preliminary lemma that will be useful later.

Lemma 3.45. The following quadratic identity between determinants holds
true:
(3.46)

$$
\left\{\begin{array}{l}
\left\|C_{1}|\cdots|{ }^{j_{1}} D|\cdots|{ }^{j_{2}} E|\cdots| C_{n}\right\| \cdot\left\|C_{1}|\cdots|{ }^{j_{1}} C_{j_{1}}|\cdots|{ }^{j_{2}} C_{j_{2}}|\cdots| C_{n}\right\|= \\
\quad=\left\|\left.C_{1}|\cdots|{ }^{j_{1}} D|\cdots|\right|^{j_{2}} C_{j_{2}}|\cdots| C_{n}\right\| \cdot\left\|C_{1}|\cdots|{ }^{j_{1}} C_{j_{1}}|\cdots| \cdots\left|{ }^{j_{2}} E\right| \cdots \mid C_{n}\right\|- \\
-\left\|C_{1}|\cdots|{ }^{j_{1}} E\left|\cdots{ }^{j_{2}} C^{j_{2}}\right| \cdots\left|C_{n}\|\cdot\| C_{1}\right| \cdots\left|{ }^{j_{1}} C_{j_{1}}\right| \cdots| |^{j_{2}} D|\cdots| C_{n}\right\| .
\end{array}\right.
$$

Proof. After some permutations of columns, this identity amounts to

$$
\left\{\begin{array}{l}
\left\|C_{1}|\cdots| C_{m-2}|D| E\right\| \cdot\left\|C_{1}|\cdots| C_{m-2}\left|C_{m-1}\right| C_{m}\right\|=  \tag{3.47}\\
=\left\|C_{1}|\cdots| C_{m-2}|D| C_{m}\right\| \cdot\left\|C_{1}|\cdots| C_{m-2}\left|C_{m-1}\right| E\right\|- \\
-\left\|C_{1}|\cdots| C_{m-2}|E| C_{m}\right\| \cdot\left\|C_{1}|\cdots| C_{m-2}\left|C_{m-1}\right| D\right\|
\end{array}\right.
$$

To establish this identity, we introduce some notation. If $A$ and $B$ are vertical vectors in $\mathbb{K}^{m}$ and if $i_{1}, i_{2}=1, \ldots, m$ with $i_{1}<i_{2}$, we denote

$$
\Delta_{i_{1}, i_{2}}^{2}(A \mid B):=\left|\begin{array}{cc}
A_{i_{1}} & B_{i_{1}}  \tag{3.48}\\
A_{i_{2}} & B_{i_{2}}
\end{array}\right|
$$

If $\left\|A_{1}\left|A_{2}\right| A_{3}|\cdots| A_{m}\right\|$ is a $m \times m$ determinant, and if $i_{1}, i_{2}=1, \ldots, m$ with $i_{1}<i_{2}$, we denote by $M_{i_{1}, i_{2}}^{m-2}\left(A_{3}|\cdots| A_{m}\right)$ the $(m-2) \times(m-2)$ determinant obtained from the matrix $\left[A_{3}|\cdots| A_{m}\right]$ by erasing the $i_{1}$-th line and the $i_{2}$-th line. Without proof, we recall an elementary classical formula

$$
\left\{\begin{array}{l}
\left\|A_{1}\left|A_{2}\right| A_{3}|\cdots| A_{m}\right\|=  \tag{3.49}\\
\quad=\sum_{1 \leqslant i_{1}<i_{2} \leqslant m}(-1)^{i_{1}+i_{2}-1} \Delta_{i_{1}, i_{2}}^{2}\left(A_{1} \mid A_{2}\right) \cdot M_{i_{1}, i_{2}}^{m-2}\left(A_{3}|\cdots| A_{m}\right),
\end{array}\right.
$$

which may be established by developing the determinant $\left\|A_{1}\left|A_{2}\right| A_{3}|\cdots| A_{m}\right\|$ with respect to its first column, and then re-developing all the obtained $(m-1) \times(m-1)$ determinants with respect to their first columns. To establish (3.47), we start with an equivalent version of the identity (2.28):

$$
\left\{\begin{align*}
\Delta_{i_{1}, i_{2}}^{2}(D \mid E) \cdot \Delta_{i_{3}, i_{4}}^{2}\left(C_{1} \mid C_{2}\right)= & \Delta_{i_{1}, i_{2}}^{2}\left(D \mid C_{2}\right) \cdot \Delta_{i_{3}, i_{4}}^{2}\left(C_{1} \mid E\right)-  \tag{3.50}\\
& -\Delta_{i_{1}, i_{2}}^{2}\left(E \mid C_{2}\right) \cdot \Delta_{i_{3}, i_{4}}^{2}\left(C_{1} \mid D\right),
\end{align*}\right.
$$

where $1 \leqslant i_{1}<i_{2} \leqslant m$ and $1 \leqslant i_{3}<i_{4} \leqslant m$. Multiplying by $(-1)^{i_{1}+i_{2}+i_{3}+i_{4}-2}$, multiplying by $M_{i_{1}, i_{2}}^{m-2}\left(C_{3}|\cdots| C_{m}\right)$, and multiplying by $M_{i_{3}, i_{4}}^{m-2}\left(C_{3}|\cdots| C_{m}\right)$ applying the double summation
$\sum_{1 \leqslant i_{1}<i_{2} \leqslant m} \sum_{1 \leqslant i_{3}<i_{1} \leqslant m}$, we get
(3.51)

$$
\left\{\begin{array}{l}
\sum_{1 \leqslant i_{1}<i_{2} \leqslant m} \sum_{1 \leqslant i_{3}<i_{i} \leqslant m}(-1)^{i_{1}+i_{2}+i_{3}+i_{4}-2} \Delta_{i_{1}, i_{2}}^{2}(D \mid E) \cdot \Delta_{i_{3}, i_{4}}^{2}\left(C_{1} \mid C_{2}\right) . \\
=\sum_{1 \leqslant i_{1}<i_{2} \leqslant m} \sum_{1 \leqslant i_{3}<i_{4} \leqslant m}^{m-i_{2}}\left(C_{3}|\cdots| C_{m}\right) \cdot M_{i_{3}, i_{4}}^{m-2}\left(C_{3}|\cdots| C_{m}\right)= \\
\left.\quad-\Delta_{i_{1}, i_{2}}^{2}\left(E \mid C_{2}\right) \cdot \Delta_{i_{3}, i_{4}}^{2}\left(C_{1} \mid D\right)\right] \cdot M_{i_{1}, i_{2}}^{m-2}\left(C_{3}|\cdots| C_{m}\right) \cdot M_{i_{3}, i_{4}}^{m-2}\left(C_{3}|\cdots| C_{m}\right) .
\end{array}\right.
$$

Thanks to the relation (3.49), this last identity coincides exactly with the desired identity (3.47). The proof is complete.

We can now establish Lemma 3.40. As a preliminary observation, by the Leibniz rule for the differentiation of a determinant, we must differentiate every column:

$$
\left\{\begin{align*}
{\left[\Delta\left(y^{l_{1}} y^{k_{1}}|\cdots| y^{l_{m}} y^{k_{m}}\right)\right]_{y^{j}}=} & \Delta\left(y^{j} y^{l_{1}} y^{k_{1}}|\cdots| y^{l_{m}} y^{k_{m}}\right)+\cdots+  \tag{3.52}\\
& +\Delta\left(y^{l_{1}} y^{k_{1}}|\cdots| y^{j} y^{l_{m}} y^{k_{m}}\right) .
\end{align*}\right.
$$

Using also the rule for the differentiation of a quotient, we may endeavour to compute the cross differentiations $\left(\square_{y^{l_{1} y^{l_{2}}}}^{j}\right)_{y^{l_{3}}}-\left(\square_{y_{1} y^{y_{3}}}^{j}\right)_{y^{l_{2}}}$ of the left-hand side of (3.41). This will generalize (2.25). Sometimes in the computation, we shall abbreviate the Jacobian determinant $\Delta\left(y^{0}|\cdots| y^{m}\right)$ using the shorter notation $\Delta$; as before, a product between two elements of $\mathbb{K}$ will
often be denoted by the sign ".", for clarity. Here is the computation:
(3.53)

$$
\begin{aligned}
& \left(\left(\square_{y^{l_{1} l^{l_{2}}}}^{j}\right)_{y^{l_{3}}}-\left(\square_{y^{l_{1} y^{l_{3}}}}^{j}\right)_{y^{l_{2}}}=\right. \\
& =\frac{\partial}{\partial y^{l_{3}}}\left(\frac{\Delta\left(y^{0}|\cdots|{ }^{j} y^{l_{1}} y^{l_{2}}|\cdots| y^{m}\right)}{\Delta\left(y^{0}|\cdots| y^{m}\right)}\right)-\frac{\partial}{\partial y^{l_{2}}}\left(\frac{\Delta\left(y^{0}|\cdots|{ }^{j} y^{l_{1}} y^{l_{3}}|\cdots| y^{m}\right)}{\Delta\left(y^{0}|\cdots| y^{m}\right)}\right) \\
& =\frac{1}{[\Delta]^{2}}\left[\begin{array}{l}
\Delta\left(y^{0} y^{l_{3}}|\cdots|{ }^{j} y^{l_{1}} y^{l_{2}}|\cdots| y^{m}\right) \cdot \Delta+\cdots+ \\
+\Delta\left(y^{0}|\cdots|{ }^{j} y^{l_{3}} y^{l_{1}} y^{l_{2}}|\cdots| y^{m}\right) \cdot \Delta \\
+\Delta\left(y^{0}|\cdots|{ }^{j} y^{l_{1}} y^{l_{2}}|\cdots| y^{l_{3}} y^{m}\right) \cdot \Delta- \\
-\Delta\left(y^{0}|\cdots|{ }^{j} y^{l_{1}} y^{l_{2}}|\cdots| y^{m}\right) \cdot\left[\Delta\left(y^{0} y^{l_{3}}|\cdots| y^{m}\right)+\cdots+\right. \\
\left.+\Delta\left(y^{0}|\cdots| y^{l_{3}} y^{m}\right)\right]
\end{array}\right]- \\
& -\frac{1}{[\Delta]^{2}}\left[\begin{array}{l}
\Delta\left(y^{0} y^{l_{2}}|\cdots|{ }^{j} y^{l_{1}} y^{l_{3}}|\cdots| y^{m}\right) \cdot \Delta+\cdots+ \\
+\Delta\left(y^{0}|\cdots|{ }^{j} y^{l_{2}} y^{l_{1}} y^{l_{3}}|\cdots| y^{m}\right) \cdot \Delta \\
+\Delta\left(y^{0}|\cdots|{ }^{j} y^{l_{1}} y^{l_{3}}|\cdots| y^{l_{2}} y^{m}\right) \cdot \Delta- \\
-\Delta\left(y^{0}|\cdots|{ }^{j} y^{l_{1}} y^{l_{3}}|\cdots| y^{m}\right) \cdot\left[\Delta\left(y^{0} y^{l_{2}}|\cdots| y^{m}\right)+\cdots+\right. \\
\left.\quad+\Delta\left(y^{0}|\cdots| y^{l_{2}} y^{m}\right)\right]
\end{array}\right]
\end{aligned}
$$

Crucially, we observe that all the determinants involving a third order derivative upon one of their columns kill each other and disappear: we have underlined them with (a) appended. However, it still remains plenty of determinants involving a second order derivative upon two different columns. We must transform all of them and express them in terms of determinants involving a second order derivative upon only one column. To this aim, as an application of our preliminary Lemma 3.45, we have the following relations, valid for $j_{1}, j_{2}, l_{1}, l_{2}, l_{3}, l_{4}=0, \ldots, m$ and $j_{1}<j_{2}$ : (3.54)

$$
\left\{\begin{array}{l}
\Delta\left(y^{0}|\cdots|{ }^{j_{1}} y^{l_{1}} y^{l_{2}}|\cdots|{ }^{j_{2}} y^{l_{3}} y^{l_{4}}|\cdots| y^{m}\right) \cdot \Delta\left(y^{0}|\cdots|{ }^{j_{1}} y^{j_{1}}|\cdots|{ }^{j_{2}} y^{j_{2}}|\cdots| y^{m}\right)= \\
=\Delta\left(\left.y^{0}|\cdots|{ }^{j_{1}} y^{l_{1}} y^{l_{2}}|\cdots|\right|_{2} ^{j_{2}} y^{j_{2}}|\cdots| y^{m}\right) \cdot \Delta\left(y^{0}|\cdots|{ }^{j_{1}} y^{j_{1}}|\cdots|{ }^{j_{2}} y^{l_{3}} y^{l_{4}}|\cdots| y^{m}\right) \\
-\Delta\left(y^{0}|\cdots|{ }^{j_{1}} y^{l_{3}} y^{l_{4}}|\cdots|{ }^{j_{2}} y^{j_{2}}|\cdots| y^{m}\right) \cdot \Delta\left(y^{0}|\cdots|{ }^{j_{1}} y^{j_{1}}|\cdots|{ }^{j_{2}} y^{l_{1}} y^{l_{2}}|\cdots| y^{m}\right)
\end{array}\right.
$$

With these formulas, we may transform the lines number $3,4,5$ and $8,9,10$ of (3.53). Also, we observe that the lines 6, 7 and 11, 12 of (3.53) involve determinants having a single second order derivative. Taking account of the $\frac{1}{[\Delta]^{2}}$ factor, we deduce that the lines 6,7 and 11,12 of (3.53) may already be expressed as sums of square functions. Achieving all these transformations,
we may rewrite (3.53) as follows
(3.55)

$$
\begin{aligned}
& \left(\square_{y^{l_{1} l^{l_{2}}}}^{j}\right)_{y^{l_{3}}}-\left(\square_{y^{l_{1} y_{3}}}^{j}\right)_{y^{l_{2}}}= \\
& =\frac{1}{[\Delta]^{2}}\left[\begin{array}{l}
\Delta\left(y^{0} y^{l_{3}}|\cdots|{ }^{j} y^{j}|\cdots| y^{m}\right) \cdot \Delta\left(y^{0}|\cdots|{ }^{j} y^{l_{1}} y^{l_{2}}|\cdots| y^{m}\right)- \\
-\Delta\left(y^{l_{1}} y^{l_{2}}|\cdots|{ }^{j} y^{j}|\cdots| y^{m}\right) \cdot \Delta\left(y^{0}|\cdots|{ }^{j} y^{0} y^{l_{3}}|\cdots| y^{m}\right)+ \\
+\cdots+ \\
+\Delta\left(y^{0}|\cdots|{ }^{j} y^{l_{1}} y^{l_{2}}|\cdots| y^{m}\right) \cdot \Delta\left(y^{0}|\cdots|{ }^{j} y^{j}|\cdots| y^{l_{3}} y^{m}\right)- \\
-\Delta\left(y^{0}|\cdots|{ }^{j} y^{l_{3}} y^{m}|\cdots| y^{m}\right) \cdot \Delta\left(y^{0}|\cdots|{ }^{j} y^{j}|\cdots| y^{l_{1}} y^{l_{2}}\right)
\end{array}\right]- \\
& -\sum_{k=0}^{m} \square_{y^{l_{1}} y^{l_{2}}}^{j} \square_{y^{l_{3}} y^{k}}^{k}- \\
& -\frac{1}{[\Delta]^{2}}\left[\begin{array}{l}
\Delta\left(y^{0} y^{l_{2}}|\cdots|{ }^{j} y^{j}|\cdots| y^{m}\right) \cdot \Delta\left(y^{0}|\cdots|{ }^{j} y^{l_{1}} y^{l_{3}}|\cdots| y^{m}\right)- \\
-\Delta\left(y^{l_{1}} y^{l_{3}}|\cdots|{ }^{j} y^{j}|\cdots| y^{m}\right) \cdot \Delta\left(y^{0}|\cdots|{ }^{j} y^{0} y^{l_{2}}|\cdots| y^{m}\right)+ \\
+\cdots+ \\
+\Delta\left(y^{0}|\cdots|{ }^{j} y^{l_{1}} y^{l_{3}}|\cdots| y^{m}\right) \cdot \Delta\left(y^{0}|\cdots|{ }^{j} y^{j}|\cdots| y^{l_{2}} y^{m}\right)- \\
-\Delta\left(y^{0}|\cdots|{ }^{j} y^{l_{2}} y^{m}|\cdots| y^{m}\right) \cdot \Delta\left(y^{0}|\cdots|{ }^{j} y^{j}|\cdots| y^{l_{1}} y^{l_{3}}\right)
\end{array}\right]+ \\
& +\sum_{k=0}^{m} \square_{y^{l_{1} y^{l_{3}}}}^{j} \square_{y^{l_{2} y^{k}}}^{k} .
\end{aligned}
$$

Notice that two pairs of "cdots" terms $+\cdots+$ appearing in the lines 3, 4 and 8,9 of (3.53) are replaced by a single "cdots" term $+\cdots+$ in the lines 4 and 9 of (3.55). Importantly, we point that in the "middle" of the two "cdots" terms $+\cdots+$ appearing in the lines 4 and 10 of (3.55) just above, there are two terms which do not occur: they simply correspond to the two underlined terms having (appended appearing in the lines 4 and 9 (3.53).

Now, taking account of the factor $\frac{1}{[\Delta]^{2}}$, we can re-express all the terms of (3.55) as sums of square functions:

$$
\left\{\begin{array}{l}
\left(\square_{y^{l_{1} l^{l_{2}}}}^{j}\right)_{y^{l_{3}}}-\left(\square_{y^{l_{1} y_{3} l_{3}}}^{j}\right)_{y^{l_{2}}}=  \tag{3.56}\\
\quad=\sum_{k=0 ; k \neq j}^{m} \square_{y^{l_{3}} y^{k}}^{k} \square_{y^{l_{1} y_{2}}}^{j}-\sum_{k=0 ; k \neq j}^{m} \square_{y^{l_{1} y^{l_{2}}}}^{k} \square_{y^{l_{3} y^{k}}}^{j}- \\
\quad-\sum_{k=0}^{m} \square_{y^{l_{1} y^{l_{2}}}}^{j} \square_{y^{l_{3} y^{k}}}^{k}- \\
\quad-\sum_{k=0 ; k \neq j}^{m} \square_{y^{l_{2} y^{k}}}^{k} \square_{y^{l_{1} y_{3}}}^{j}+\sum_{k=0 ; k \neq j}^{m} \square_{y^{l_{1} y_{3}}}^{k} \square_{y^{l_{2} y^{k}}}^{j}+ \\
\quad+\sum_{k=0}^{m} \square_{y^{l_{1} y_{3}}}^{j} \square_{y^{l_{2} y^{k}}}^{k} .
\end{array}\right.
$$

Finally, we observe that in the two pairs of sums having $k \neq j$ appearing in the lines 2 and 4 just above, we can include the term $k=j$ in each pair, because these two terms are immediately killed inside the corresponding pair. In conclusion, after a final obvious killing of four (among six) complete sums in this modification of (3.56), we obtain the desired formula (3.41), with two sums. This completes the proof of Lemma 3.40 and also at the same occasion, the proof of Lemma 2.31.
3.57. Compatibility conditions for the first auxiliary system. According to the (approximate) identities (3.3), taking account of the explicit definitions (3.30) of the square functions, we have

$$
\left\{\begin{array}{rll}
\square_{x x}^{0} \cong X_{x x}, & \square_{x y^{l_{1}}}^{0} \cong X_{x y^{l_{1}}}, & \square_{y^{l_{1} l_{2}}}^{0} \cong X_{y^{l_{1} y^{l}}}  \tag{3.58}\\
\square_{x x}^{j} \cong Y_{x x}^{j}, & \square_{x y^{l_{1}}}^{j} \cong Y_{x y^{l_{1}}}^{j}, & \square_{y^{l_{1} y^{l_{2}}}}^{j} \cong Y_{y^{l_{1} y^{l_{2}}}}^{j}
\end{array}\right.
$$

Consequently, the first auxiliary system (3.38) looks approximatively like a complete second order system of partial differential equations in the $(m+1)$ independent variables $(x, y)$ and in the $(m+1)$ dependent variables $(X, Y)$. By means of elementary algebraic operations, one may transform this system in a true second order complete system, solved with respect to the top order derivatives, namely of the form

$$
\left\{\begin{array}{cll}
X_{x x}=\Lambda_{0,0}^{0}, & X_{x y^{l_{1}}}=\Lambda_{0, l_{1}}^{0}, & X_{y^{l_{1}} y^{l_{2}}}=\Lambda_{l_{1}, l_{2}}^{0},  \tag{3.59}\\
Y_{x x}^{j}=\Lambda_{0,0}^{j}, & Y_{x y^{l_{1}}}^{j}=\Lambda_{0, l_{1}}^{j}, & Y_{y^{l_{1}} y^{l_{2}}}^{j}=\Lambda_{l_{1}, l_{2}}^{j},
\end{array}\right.
$$

where the $\Lambda_{j_{1}, j_{2}}^{k_{1}}$ are local $\mathbb{K}$-analytic functions of $\left(x, y^{l_{1}}, X, Y^{j}, X_{x}, X_{y^{l_{1}}}, Y_{x}^{j}, Y_{y^{l_{1}}}^{j}\right)$. For such a system, the compatibility conditions [which are necessary and sufficient for the existence of a solution $(X, Y)]$ follow by obvious cross
differentiation. Coming back to the system 3.38, these compatibility conditions amount to the quadratic-like compatibility conditions expressed in Lemma 3.40. In conclusion, we have proved the following intermediate statement.

Proposition 3.60. There exist functions $X, Y^{j}$ solving the first auxiliary system (3.38) of nonlinear second order partial differential equations if and only if the right-hand side functions $\Pi_{l_{1}, l_{2}}^{j}(x, y)$ satisfy the quadratic compatibility conditions

$$
\begin{equation*}
\frac{\partial \Pi_{l_{1}, l_{2}}^{j}}{\partial y^{l_{3}}}-\frac{\partial \Pi_{l_{1}, l_{3}}^{j}}{\partial y^{l_{2}}}=-\sum_{k=0}^{m} \Pi_{l_{1}, l_{2}}^{k} \cdot \Pi_{l_{3}, k}^{j}+\sum_{k=0}^{m} \Pi_{l_{1}, l_{3}}^{k} \cdot \Pi_{l_{2}, k}^{j} \tag{3.61}
\end{equation*}
$$

for $j, l_{1}, l_{2}, l_{3}=0,1, \ldots, m$.
3.62. Principal unknowns. As there are $(m+1)$ more square (or Pi ) functions than the functions $G^{j}, H_{l_{1}}^{j}, L_{l_{1}, l_{2}}^{j}$ and $M_{l_{1}, l_{2}}$ defined by (3.35), we cannot invert directly the linear system (3.35) (which is of maximal rank). Hence we must choose $(m+1)$ specific square functions, calling them principal unknowns, and similarly as in $\S 2.34$, the best choice is to choose $\square_{x x}^{0}$ and $\square_{x x}^{j}$, for $j=1, \ldots, m$. For clarity, it will be useful to adopt the notational equivalences

$$
\begin{equation*}
\Theta^{0} \equiv \Pi_{0,0}^{0} \quad \text { and } \quad \Theta^{j} \equiv \Pi_{j, j}^{j} \tag{3.63}
\end{equation*}
$$

Then we may quasi-inverse the system (3.35), which yields :

$$
\left\{\begin{array}{l}
\Pi_{0,0}^{j}=\square_{x x}^{j}=-G^{j},  \tag{3.64}\\
\Pi_{0, l_{1}}^{j}=\square_{x y^{l_{1}}}^{j}=-\frac{1}{2} H_{l_{1}}^{j}+\frac{1}{2} \delta_{l_{1}}^{j} \Theta^{0}, \\
\Pi_{l_{1}, l_{2}}^{j}=\square_{y^{l_{1} l^{l_{2}}}}^{j}=-L_{l_{1}, l_{2}}^{j}+\frac{1}{2} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}}^{l_{2}}+\frac{1}{2} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}}+\frac{1}{2} \delta_{l_{1}}^{j} \Theta^{l_{2}}+\frac{1}{2} \delta_{l_{2}}^{j} \Theta^{l_{1}} \\
\Pi_{0, l_{1}}^{0}=\square_{x y^{l_{1}}}^{0}=\frac{1}{2} L_{l_{1}, l_{1}}^{l_{1}}+\frac{1}{2} \Theta^{l_{1}}, \\
\Pi_{l_{1}, l_{2}}^{0}=\square_{y^{l_{1} y^{l_{2}}}}^{0}=M_{l_{1}, l_{2}} .
\end{array}\right.
$$

Before replacing these new expressions of the functions $\Pi_{0,0}^{j}, \Pi_{0, l_{1}}^{j}, \Pi_{l_{1}, l_{2}}^{j}$, $\Pi_{l_{1}, l_{2}}^{0}$ and $\Pi_{0, l_{1}}^{l_{1}}$ into the compatibility conditions (3.61), it is necessary to expound first (3.61), taking account of the original splitting of the indices in the two sets $\{0\}$ and $\{1,2, \ldots, m\}$. This yields six families of compatibility
conditions, totally equivalent to the compact identities (3.61):

$$
\left\{\begin{align*}
\left(\Pi_{0,0}^{j}\right)_{y^{l_{1}}}-\left(\Pi_{0, l_{1}}^{j}\right)_{x} & =-\Pi_{0,0}^{0} \Pi_{l_{1}, 0}^{j}-\sum_{k=1}^{m} \Pi_{0,0}^{k} \Pi_{l_{1}, k}^{j}+\Pi_{0, l_{1}}^{0} \Pi_{0,0}^{j}+\sum_{k=1}^{m} \Pi_{0, l_{1}}^{k} \Pi_{0, k}^{j},  \tag{3.65}\\
\left(\Pi_{l_{1}, l_{2}}^{j}\right)_{x}-\left(\Pi_{l_{1}, 0}^{j}\right)_{y^{l_{2}}} & =-\Pi_{l_{1}, l_{2}}^{0} \Pi_{0,0}^{j}-\sum_{k=1}^{m} \Pi_{l_{1}, l_{2}}^{k} \Pi_{0, k}^{j}+\Pi_{l_{1}, 0}^{0} \Pi_{l_{2}, 0}^{j}+\sum_{k=1}^{m} \Pi_{l_{1}, 0}^{k} \Pi_{l_{2}, k}^{j}, \\
\left(\Pi_{l_{1}, l_{2}}^{j}\right)_{y^{l_{3}}}-\left(\Pi_{l_{1}, l_{3}}^{j}\right)_{y^{l_{2}}} & =-\Pi_{l_{1}, l_{2}}^{0} \Pi_{l_{3}, 0}^{j}-\sum_{k=1}^{m} \Pi_{l_{1}, l_{2}}^{k} \Pi_{l_{3}, k}^{j}+\Pi_{l_{1}, l_{3}}^{0} \Pi_{l_{2}, 0}^{j}+\sum_{k=1}^{m} \Pi_{l_{1}, l_{3}}^{k} \Pi_{l_{2}, k}^{j}, \\
\left(\Pi_{0,0}^{0}\right)_{y^{l_{1}}}-\left(\Pi_{0, l_{1}}^{0}\right)_{x} & =-\underline{\Pi_{0,0}^{0} \Pi_{l_{1}, 0}^{0}} \mathbf{a}-\sum_{k=1}^{m} \Pi_{0,0}^{k} \Pi_{l_{1}, k}^{0}+\underline{\Pi_{0, l_{1}}^{0} \Pi_{0,0}^{0}} \times \sum_{k=1}^{m} \Pi_{0, l_{1}}^{k} \Pi_{0, k}^{0}, \\
\left(\Pi_{l_{1}, l_{2}}^{0}\right)_{x}-\left(\Pi_{l_{1}, 0}^{0}\right)_{y^{l_{2}}} & =-\Pi_{l_{1}, l_{2}}^{0} \Pi_{0,0}^{0}-\sum_{k=1}^{m} \Pi_{l_{1}, l_{2}}^{k} \Pi_{0, k}^{0}+\Pi_{l_{1}, 0}^{0} \Pi_{l_{2}, 0}^{0}+\sum_{k=1}^{m} \Pi_{l_{1}, 0}^{k} \Pi_{l_{2}, k}^{0}, \\
\left(\Pi_{l_{1}, l_{2}}^{0}\right)_{y_{3}}-\left(\Pi_{l_{1}, l_{3}}^{0}\right)_{y^{l_{2}}} & =-\Pi_{l_{1}, l_{2}}^{0} \Pi_{l_{3,0}}^{0}-\sum_{k=1}^{m} \Pi_{l_{1}, l_{2}}^{k} \Pi_{l_{3}, k}^{0}+\Pi_{l_{1}, l_{3}}^{0} \Pi_{l_{2}, 0}^{0}+\sum_{k=1}^{m} \Pi_{l_{1}, l_{3}}^{k} \Pi_{l_{2}, k}^{0} .
\end{align*}\right.
$$

3.66. Convention about sums. Up to the end of Section 4, we shall abbreviate any sum $\sum_{k=1}^{m}$ or $\sum_{p=1}^{m}$ as $\sum_{k}$ or $\sum_{p}$. Such sums will appear very frequently. For all other sums, we shall precisely write down the domain of variation of the summation index.
3.67. Continuation. Thus, we have to replace (3.64) in the six identities (3.65). Firstly, let us expose all the intermediate steps in dealing with
the first identity $(3.65)_{1}$. Replacing plainly (3.64) in $(3.65)_{1}$, we get:

$$
\left\{\begin{align*}
&\left(\Pi_{0,0}^{j}\right)_{y^{l_{1}}}-\left(\Pi_{0, l_{1}}^{j}\right)_{x}=G_{y^{l_{1}}}^{j}+\frac{1}{2} H_{l_{1}, x}^{j}-\frac{1}{2} \delta_{l_{1}}^{j} \Theta_{x}^{0}=  \tag{3.68}\\
&= \frac{1}{2} \Theta^{0} H_{l_{1}}^{j}-\frac{1}{2} \delta_{l_{1}}^{j} \Theta^{0} \Theta^{0}- \\
&-\sum_{k}\left(-G^{k}\right)\left(-L_{l_{1}, k}^{j}+\frac{1}{2} \delta_{l_{1}}^{j} L_{k, k}^{k}+\frac{1}{2} \delta_{k}^{j} L_{l_{1}, l_{1}}^{l_{1}}+\frac{1}{2} \delta_{l_{1}}^{j} \Theta^{k}+\frac{1}{2} \delta_{k}^{j} \Theta^{k}\right)+ \\
&+\left(\frac{1}{2} L_{l_{1}, l_{1}}^{l_{1}}+\frac{1}{2} \Theta^{l_{1}}\right)\left(-G^{j}\right)+ \\
&+\sum_{k}\left(-\frac{1}{2} H_{l_{1}}^{k}+\frac{1}{2} \delta_{l_{1}}^{k} \Theta^{0}\right)\left(-\frac{1}{2} H_{k}^{j}+\frac{1}{2} \delta_{k}^{j} \Theta^{0}\right)= \\
&= \frac{1}{2} H_{l_{1}}^{j} \Theta^{0}-\frac{1}{2} \delta_{l_{1}}^{j} \Theta^{0} \Theta^{0}-\sum_{k} G^{k} L_{l_{1}, k}^{j}+\frac{1}{2} \delta_{l_{1}}^{j} \sum_{k} G^{k} L_{k, k}^{k}+\frac{1}{2} G^{j} L_{l_{1}, l_{1}}^{l_{1}} \\
&+\frac{1}{2} \delta_{l_{1}}^{j} \sum_{k} G^{k} \Theta^{k}+\frac{1}{2} G^{j} \Theta^{l_{1}}-\frac{1}{2} G^{j} L_{l_{1}, l_{1}}^{l_{1}}(\text { b } \\
& \underline{\frac{1}{2}} G^{j} \Theta^{l_{1}} \\
&-\frac{1}{4} \sum_{k} H_{l_{1}}^{k} H_{k}^{j}- \\
& \underline{\frac{1}{4} H_{l_{1}}^{j} \Theta^{0}}-\frac{1}{4} H_{l_{1}}^{j} \Theta^{0}+\frac{1}{4} \delta_{l_{1}}^{j} \Theta^{0} \Theta^{0} .
\end{align*}\right.
$$

Eliminating the underlined vanishing terms with the letters $a, b, c$ and $d$ appended, multiplying by -2 and reorganizing the identity so as to put the term $\delta_{l_{1}}^{j} \Theta_{x}^{0}$ solely in the left-hand side, we obtain the relation

$$
\left\{\begin{align*}
\delta_{l_{1}}^{j} \Theta_{x}^{0}=-2 G_{y^{l_{1}}}^{j} & +H_{l_{1}, x}^{j}+2 \sum_{k} G^{k} L_{l_{1}, k}^{j}-\delta_{l_{1}}^{j} \sum_{k} G^{k} L_{k, k}^{k}-  \tag{3.69}\\
& -\frac{1}{2} \sum_{k} H_{l_{1}}^{k} H_{k}^{j}-\delta_{l_{1}}^{j} \sum_{k} G^{k} \Theta^{k}+\frac{1}{2} \delta_{l_{1}}^{j} \Theta^{0} \Theta^{0} .
\end{align*}\right.
$$

3.70. Conventions for simplifications of formal expressions. Before proceeding further, let us explain how we will organize the computations with the formal expressions we shall encounter until the end of Section 4. Our main goal is to devise a methodology of writing formal computations which enables to check every computation visually, without being forced to rebuild any intermediate step. In fact, it would be unsatisfactory to just claim that Theorem 1.7 (3) follows by hidden massive formal computations, so that we have to guide the rigorous and demanding reader until the very extremal branches of our coral tree of formal computations.

As an example, suppose that we have to simplify the equation $0=A_{x}+$ $B_{y}+A-B+2 C-\frac{1}{3} D-\frac{2}{3} A+\frac{1}{6} D+E+B-2 C$. In the beginning, the terms $A_{x}$ and $B_{y}$ are differentiated once and they do not simplify with other terms. To distinguish them, we underline them plainly and we copy the nine
remaining terms afterwards:

$$
\left\{\begin{aligned}
& 0= \underline{A_{x}+B_{y}}+ \\
&+\underline{A}-\underline{B}_{(\mathrm{a})}+\underline{2 C}(\mathrm{~b}) \\
& \underline{\frac{1}{3} D}-\frac{2}{3} A_{1}+\frac{1}{6} D \\
& \square \underline{E} \sqrt{3}+\underline{B}_{(\mathrm{a})}-\underline{2 C_{(b)}} .
\end{aligned}\right.
$$

Here, each remaining term is also underlined, with a number or with a letter appended. For reasons of typographical readability, we never underline the sign, + or - of each term; however, it should be understood that every term always includes its (not underlined) sign. Until the end of Section 4, we shall use the roman alphabetic letters $a, b, c$, etc. inside an octagon $\bigcirc$ to exhibit the vanishing terms. As readily checked by the eyes, we indeed have $-\underline{B}_{(a+}{ }^{+}$ $\underline{B}_{(\mathrm{a})}=0$ and $\underline{2 C}_{(\mathrm{b}-}-\underline{2 C}_{(\mathrm{b})}=0$. Also, until the end of Section 4, we shall use the numbers 1,2 , 3 , etc. inside a square $\square$ to exhibit the remaining terms, collected in a certain order. The numbers have the following signification: after the simplifications, the equation (3.71) may be written

$$
\left\{\begin{align*}
0= & \frac{A_{x}+B_{y}+}{}  \tag{3.72}\\
& +\frac{1}{3} A-\frac{1}{6} D+E .
\end{align*}\right.
$$

Here, the plainly underlined terms $A_{x}+B_{y}$ do not count in the numbering (their number is zero, for instance) and the first term of the second line $\frac{1}{3} A$ correspond to the addition of all terms 1 in (3.69). Analogously, the second term $-\frac{1}{6} D$ correspond to the addition of all terms 2 in (3.69). Again, this guiding facilitates the checking of the correctness of the computation, using simply the eyes. No hidden delicate computational step is "left to the reader" for the convenience of the writer.

This principle will be constantly used until the end of Section 4; it has been systematically used in [M2004] and it could be applied in various other contexts. Again, the advantage is that it enables to check the correctness of all the formal computations just by reading, without having to write anything more. This is also useful for the author.
3.73. Choice of an ordering. Until the end of Section 4, we shall have to deal with terms $G, H, L, M, \Theta$ together with indices and partial derivatives up to order two. In order to organize the formal expressions in a way which provides an easier deciphering, it is convenient to introduce an order between these differential monomials. In a symbolic index-free notation, we choose:

$$
\begin{equation*}
G<H<L<M<\Theta \tag{3.74}
\end{equation*}
$$

It follows for instance that $G<G H<G L<H H L<H L M \Theta$. Also, if a sum appears, we choose: $G M<\sum G M$.

Here, we have only considered terms of order zero, without partial differentiation. The first order partial differentiations are $(\cdot)_{x}$ and $(\cdot)_{y}$, again in symbolic notation, dropping the indices. We choose:

$$
\begin{equation*}
G_{x}<G_{y}<H_{x}<H_{y}<L_{x}<L_{y}<M_{x}<M_{y}<\Theta_{x}<\Theta_{y}<G<\cdots \tag{3.75}
\end{equation*}
$$

For second order derivatives, we choose:

$$
\begin{equation*}
G_{x x}<G_{x y}<G_{y y}<H_{x x}<\cdots<\Theta_{y y}<G_{x}<\cdots . \tag{3.76}
\end{equation*}
$$

As a final general example including indices we have the inequalities

$$
\begin{equation*}
H_{l_{1}, y^{l_{2}}}^{j}<L_{l_{1}, l_{2}, x}^{j}<G^{j} M_{l_{1}, l_{2}}<\sum_{k=1}^{m} G^{k} M_{l_{2}, k}<\sum_{k=1}^{m} H_{k}^{l_{2}} H_{l_{2}, l_{2}}^{k}, \tag{3.77}
\end{equation*}
$$

extracted from (II) of Theorem 1.7 (3).
In the sequel, we shall call

- terms of order 0 monomials like $G, H, L M, G H M$;
- terms of order 1 monomials like $G_{x}, G_{x} M, L_{x} \Theta$;
- terms of order 2 monomials like $G_{x y}, L_{y y}, M_{x x}$ (our terms of order two will always be linear),
according to the top order partial derivatives.
3.78. A mean of checking intuitively the validity of partial differential relations. Before replacing (3.64) in the five remaining identities $(3.65)_{2}$, $(3.65)_{3},(3.65)_{4},(3.65)_{5}$ and $(3.65)_{6}$, let us observe that if we assume that $j \neq l_{1}$ in (3.69), then all the terms involving $\Theta$ vanish, so that we obtain the nontrivial partial differential equations:

$$
\begin{equation*}
0=\underline{-2 G_{y_{1}}^{j}+H_{l_{1}, x}^{j}}+2 \sum_{k} G^{k} L_{l_{1}, k}^{j}-\frac{1}{2} \sum_{k} H_{l_{1}}^{k} H_{k}^{j}, \tag{3.79}
\end{equation*}
$$

for $j \neq l_{1}$. Here, we have underlined the first order terms plainly, in order to distinguish them from the terms of order zero. These equations coincides with (I) of Theorem 1.7 (3), again specialized with $j \neq l_{1}$. Importantly, we notice that the choice of indices $j \neq l_{1}$ is possible only if $m \geqslant 2$. Thus, we have derived a subpart of (I) as a necessary condition for the point equivalence to $Y_{X X}^{j}=0, j=1, \ldots, m \geqslant 2$. These first order equations show at once that there is a strong difference with the case $m=1$.

How can we confirm (at least informally) that the functions $G^{j}, H_{l_{1}}^{j}$ and $L_{l_{1}, l_{2}}^{j}$ given by (3.35) in terms of $X$ and $Y^{j}$ do indeed satisfy these equations for $j \neq l_{1}$ ? Dropping the zero order terms in (3.79) above, we obtain an approximated equation

$$
\begin{equation*}
0 \equiv-2 G_{y_{1}}^{j}+H_{l_{1}, x}^{j} . \tag{3.80}
\end{equation*}
$$

Here, the sign $\equiv$ precisely means: "modulo zero order terms". We claim that this approximated equation is a consequence of the existence of $X, Y^{j}$.

Indeed, according to the approximation (3.58), together with the definition (3.35) of the functions $G^{j}$ and $H_{l_{1}}^{j}$, we have

$$
\left\{\begin{align*}
G^{j} & =-\square_{x x}^{j} \cong-Y_{x x}^{j}  \tag{3.81}\\
H_{l_{1}}^{j} & =-2 \square_{x y^{l_{1}}}^{j} \cong-2 Y_{x y^{l_{1}}}^{j} .
\end{align*}\right.
$$

Differentiation of the first line with respect to $y^{l_{1}}$ and of the second line with respect to $x$ yields:

$$
\begin{equation*}
G_{y^{l_{1}}}^{j} \cong-Y_{x x y^{l_{1}}}^{j} \quad \text { and } \quad H_{l_{1}, x}^{j} \cong-2 Y_{x y^{l_{1} x}}^{j}, \tag{3.82}
\end{equation*}
$$

so that we indeed have $0 \equiv-2 G_{y^{1_{1}}}^{j}+H_{l_{1}, x}^{j}$, approximatively and modulo the derivatives of order 0,1 and 2 of the functions $X, Y^{j}$.

Similar verifications have been effected constantly in our manuscript in order to control the truth of the formal computations that we shall expose until the end of Section 4.
3.83. Continuation. From now on and up to the end of Section 4, the hardest computational core of the proof may - at last - be developed. Further amazing computational obstacles will be encountered.

Replacing plainly (3.64) in $(3.65)_{2}$, we get:

$$
\begin{align*}
& \left(\Pi_{l_{1}, l_{2}}^{j}\right)_{x}-\left(\Pi_{l_{1}, 0}^{j}\right)_{y^{l_{2}}}=-L_{l_{1}, l_{2}, x}^{j}+\frac{1}{2} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}, x}^{l_{2}}+\frac{1}{2} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}, x}^{l_{1}}+\frac{1}{2} \delta_{l_{1}}^{j} \Theta_{x}^{l_{2}}+  \tag{3.84}\\
& +\frac{1}{2} \delta_{l_{2}}^{j} \Theta_{x}^{l_{1}}+\frac{1}{2} H_{l_{1}, y^{l_{2}}}^{j}-\frac{1}{2} \delta_{l_{1}}^{j} \Theta_{y^{l_{2}}}^{0}= \\
& =-\Pi_{l_{1}, l_{2}}^{0} \cdot \Pi_{0,0}^{j}-\sum_{k} \Pi_{l_{1}, l_{2}}^{k} \cdot \Pi_{0, k}^{j}+\Pi_{l_{1}, 0}^{0} \cdot \Pi_{l_{2}, 0}^{j}+\sum_{k} \Pi_{l_{1}, 0}^{k} \cdot \Pi_{l_{2}, k}^{j}= \\
& =M_{l_{1}, l_{2}} G^{j}-\sum_{k}\left(-L_{l_{1}, l_{2}}^{k}+\frac{1}{2} \delta_{l_{1}}^{k} L_{l_{2}, l_{2}}^{l_{2}}+\frac{1}{2} \delta_{l_{2}}^{k} L_{l_{1}, l_{1}}^{l_{1}}+\frac{1}{2} \delta_{l_{1}}^{k} \Theta^{l_{2}}+\frac{1}{2} \delta_{l_{2}}^{k} \Theta^{l_{1}}\right) . \\
& \cdot\left(-\frac{1}{2} H_{k}^{j}+\frac{1}{2} \delta_{k}^{j} \Theta^{0}\right)+ \\
& +\left(\frac{1}{2} L_{l_{1}, l_{1}}^{l_{1}}+\frac{1}{2} \Theta^{l_{1}}\right) \cdot\left(-\frac{1}{2} H_{l_{2}}^{j}+\frac{1}{2} \delta_{l_{2}}^{j} \Theta^{0}\right)+ \\
& +\sum_{k}\left(-\frac{1}{2} H_{l_{1}}^{k}+\frac{1}{2} \delta_{l_{1}}^{k} \Theta^{0}\right) . \\
& \cdot\left(-L_{l_{2}, k}^{j}+\frac{1}{2} \delta_{l_{2}}^{j} L_{k, k}^{k}+\frac{1}{2} \delta_{k}^{j} L_{l_{2}, l_{2}}^{l_{2}}+\frac{1}{2} \delta_{l_{2}}^{j} \Theta^{k}+\frac{1}{2} \delta_{k}^{j} \Theta^{l_{2}}\right) .
\end{align*}
$$

Developing the products and ordering each monomial, we get:

$$
\begin{align*}
& =\underline{G^{j} M_{l_{1}, l_{2}}} \square^{1}-\underline{2}-\underline{\frac{1}{2} \sum_{k} H_{k}^{j} L_{l_{1}, l_{2}}^{k}}+\underline{\frac{1}{4} H_{l_{1}}^{j} L_{l_{2}, l_{2}}^{l_{2}}(\mathrm{a})}+\underline{\frac{1}{4} H_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}}(\mathrm{~b})}+  \tag{3.85}\\
& +\underline{\frac{1}{4} H_{l_{1}}^{j} \Theta^{l_{2}}}+\underline{\text { c }}^{\frac{1}{4} H_{l_{2}}^{j} \Theta^{l_{1}}} \text { (d) }+\underline{\frac{1}{2} L_{l_{1}, l_{2}}^{j} \Theta^{0}} \text { (e) }-\underline{\frac{1}{4} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{0}}{ }^{(f)}- \\
& -\underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{0}}-\underline{\frac{1}{4} \delta_{l_{1}}^{j} \Theta^{0} \Theta^{l_{2}}} \text { (h) }-\underline{\frac{1}{4} \delta_{l_{2}}^{j} \Theta^{0} \Theta^{l_{1}}} \text { (i) } \underline{\frac{1}{4} H_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}}}(\text { b }+
\end{align*}
$$

$$
\begin{aligned}
& -\frac{1}{4} \sum_{k} H_{l_{1}}^{k} L_{k, k}^{k}-\underline{\frac{1}{4} H_{l_{1}}^{j} L_{l_{2}, l_{2}}^{l_{2}}(a)}-\frac{1}{4} \delta_{l_{2}}^{j} \sum_{k} H_{l_{1}}^{k} \Theta^{k}-\frac{1}{4} H_{l_{1}}^{j} \Theta^{l_{2}} \text { (c) } \\
& -\underline{\frac{1}{2} L_{l_{2}, l_{1}}^{j} \Theta^{0}}\left(\text { e) }+\underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{0}}(\mathrm{~g})+\underline{\frac{1}{4} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{0}}\left(\underset{\square}{\frac{1}{4} \delta_{l_{2}}^{j} \Theta^{0} \Theta^{l_{1}}}+\right.\right. \\
& +\underline{\frac{1}{4} \delta_{l_{1}}^{j} \Theta^{0} \Theta^{l_{2}}} \text { (h) }
\end{aligned}
$$

We simplify according to our general principles and we reorganize the equality between the first two lines of (3.84) and (3.85) so as to put all terms $\Theta_{x}$ in the left-hand side of the equality and to put all remaining terms in the right-hand side, respecting the order of $\S 3.73$. We get:

$$
\begin{align*}
& \frac{1}{2} \delta_{l_{1}}^{j} \Theta_{x}^{l_{2}}+\frac{1}{2} \delta_{l_{2}}^{j} \Theta_{x}^{l_{1}}-\frac{1}{2} \delta_{l_{1}}^{j} \Theta_{y^{l_{2}}}^{0}= \\
&=-\frac{1}{2} H_{l_{1}, y^{l_{2}}}^{j}+L_{l_{1}, l_{2}, x}^{j}-\frac{1}{2} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}, x}^{l_{2}}-\frac{1}{2} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}, x}^{l_{1}}+ \\
&+G^{j} M_{l_{1}, l_{2}}-\frac{1}{2} \sum_{k} H_{k}^{j} L_{l_{1}, l_{2}}^{k}+\frac{1}{2} \sum_{k} H_{l_{1}}^{k} L_{l_{2}, k}^{j}-  \tag{3.86}\\
&-\frac{1}{4} \delta_{l_{2}}^{j} \sum_{k} H_{l_{1}}^{k} L_{k, k}^{k}-\frac{1}{4} \delta_{l_{2}}^{j} \sum_{k} H_{l_{1}}^{k} \Theta^{k}+\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{0}+ \\
&+\frac{1}{4} \delta_{l_{2}}^{j} \Theta^{0} \Theta^{l_{1}} .
\end{align*}
$$

Here, we have underlined plainly the four first order terms appearing in the second line.

Next, replacing plainly (3.64) in (3.65) $)_{3}$, we get:
(3.87)

$$
\begin{aligned}
\left(\Pi_{l_{1}, l_{2}}^{j}\right. & )_{y^{l_{3}}}-\left(\Pi_{l_{1}, l_{3}}^{j}\right)_{y^{l_{2}}}= \\
= & -L_{l_{1}, l_{2}, y^{l_{3}}}^{j}+\frac{1}{2} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}, y^{l_{3}}}^{l_{2}}+\frac{1}{2} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}, y^{l_{3}}}^{l_{1}}+\frac{1}{2} \delta_{l_{1}}^{j} \Theta_{y^{l_{3}}}^{l_{2}}+\frac{1}{2} \delta_{l_{2}}^{j} \Theta_{y^{l_{3}}}^{l_{1}}+ \\
& +L_{l_{1}, l_{3}, y^{l_{2}}}^{j}-\frac{1}{2} \delta_{l_{1}}^{j} L_{l_{3}, l_{3}, y^{l_{2}}}^{l_{3}}-\frac{1}{2} \delta_{l_{3}}^{j} L_{l_{1}, l_{1}, y^{l_{2}}}^{l_{1}}-\frac{1}{2} \delta_{l_{1}}^{j} \Theta_{y^{l_{2}}}^{l_{3}}-\frac{1}{2} \delta_{l_{3}}^{j} \Theta_{y^{l_{2}}}^{l_{1}}= \\
= & -\Pi_{l_{1}, l_{2}}^{0} \cdot \Pi_{l_{3}, 0}^{j}-\sum_{k} \Pi_{l_{1}, l_{2}}^{k} \cdot \Pi_{l_{3}, k}^{j}+\Pi_{l_{1}, l_{3}}^{0} \cdot \Pi_{l_{2}, 0}^{j}+\sum_{k} \Pi_{l_{1}, l_{3}}^{k} \cdot \Pi_{l_{2}, k}^{j}= \\
= & -M_{l_{1}, l_{2}} \cdot\left(-\frac{1}{2} H_{l_{3}}^{j}+\frac{1}{2} \delta_{l_{3}}^{j} \Theta^{0}\right)-\sum_{k}\left(-L_{l_{1}, l_{2}}^{k}+\frac{1}{2} \delta_{l_{1}}^{k} L_{l_{2}, l_{2}}^{l_{2}}+\right. \\
& \left.+\frac{1}{2} \delta_{l_{2}}^{k} L_{l_{1}, l_{1}}^{l_{1}}+\frac{1}{2} \delta_{l_{1}}^{k} \Theta^{l_{2}}+\frac{1}{2} \delta_{l_{2}}^{k} \Theta^{l_{1}}\right) \cdot\left(-L_{l_{3}, k}^{j}+\frac{1}{2} \delta_{l_{3}}^{j} L_{k, k}^{k}+\right. \\
& \left.+\frac{1}{2} \delta_{k}^{j} L_{l_{3}, l_{3}}^{l_{3}}+\frac{1}{2} \delta_{l_{3}}^{j} \Theta^{k}+\frac{1}{2} \delta_{k}^{j} \Theta^{l_{3}}\right)+ \\
& +M_{l_{1}, l_{3}} \cdot\left(-\frac{1}{2} H_{l_{2}}^{j}+\frac{1}{2} \delta_{l_{2}}^{j} \Theta^{0}\right)+\sum_{k}^{k}\left(-L_{l_{1}, l_{3}}^{k}+\frac{1}{2} \delta_{l_{1}}^{k} L_{l_{3}, l_{3}}^{l_{3}}+\right. \\
& \left.+\frac{1}{2} \delta_{l_{3}}^{k} L_{l_{1}, l_{1}}^{l_{1}}+\frac{1}{2} \delta_{l_{1}}^{k} \Theta^{l_{3}}+\frac{1}{2} \delta_{l_{3}}^{k} \Theta^{l_{1}}\right) \cdot\left(-L_{l_{2}, k}^{j}+\frac{1}{2} \delta_{l_{2}}^{j} L_{k, k}^{k}+\right. \\
& \left.+\frac{1}{2} \delta_{k}^{j} L_{l_{2} l_{2}}^{l_{2}}+\frac{1}{2} \delta_{l_{2}}^{j} \Theta^{k}+\frac{1}{2} \delta_{k}^{j} \Theta^{l_{2}}\right) .
\end{aligned}
$$

Developing the products and ordering each monomial, we get:

$$
\begin{aligned}
& \text { (3.88) } \\
& =\underline{\frac{1}{2} H_{l_{3}}^{j} M_{l_{1}, l_{2}}}-1 \underline{\frac{1}{2} \delta_{l_{3}}^{j} M_{l_{1}, l_{2}} \Theta^{0}}-\sqrt{16}-\sum_{k} L_{l_{1}, l_{2}}^{k} L_{l_{3}, k}^{j}+\frac{1}{2} \delta_{l_{3}}^{j} \sum_{k} L_{l_{1}, l_{2}}^{k} L_{k, k}^{k}+ \\
& +\underline{\frac{1}{2} L_{l_{1}, l_{2}}^{j} L_{l_{3}, l_{3}}^{l_{3}}}+\underline{\text { (a) }}+\underline{\frac{1}{2} \delta_{l_{3}}^{j} \sum_{k} L_{l_{1}, l_{2}}^{k} \Theta^{k}}+\underline{\frac{1}{2} L_{l_{1}, l_{2}}^{j} \Theta^{l_{3}}} \text { (b) } \underline{\underline{\frac{1}{2}} L_{l_{2}, l_{2}}^{l_{2}} L_{l_{3}, l_{1}}^{j} \text { (c) }}- \\
& -\underline{\frac{1}{4} \delta_{l_{3}}^{j} L_{l_{2}, l_{2}}^{l_{2}} L_{l_{1}, l_{1}}^{l_{1}}} 4_{4}^{\frac{1}{4} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}}^{l_{2}} L_{l_{3}, l_{3}}^{l_{3}}} \text { (d) }-\underline{\frac{1}{4} \delta_{l_{3}}^{j} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{l_{1}}} \text { (e) }-\underline{\frac{1}{4} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{l_{3}}} \oplus+ \\
& +\underline{\frac{1}{2} L_{l_{1}, l_{1}}^{l_{1}} L_{l_{3}, l_{2}}^{j}}\left(\mathrm{~g}-\underline{\frac{1}{4} \delta_{l_{3}}^{j} L_{l_{1}, l_{1}}^{l_{1}} L_{l_{2}, l_{2}}^{l_{2}}}(\mathbb{h}) \underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}} L_{l_{3}, l_{3}}^{l_{3}}}\left(\mathbb{i}-\underline{\frac{1}{4} \delta_{l_{3}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{2}}}-\right.\right. \\
& -\underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{3}}}(\mathbb{K}) \underline{\frac{1}{2} L_{l_{3}, l_{1}}^{j} \Theta^{l_{2}}}-\underline{\frac{1}{4} \delta_{l_{3}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{2}}}-\underline{10}-\underline{\frac{1}{4} \delta_{l_{1}}^{j} L_{l_{3}, l_{3}}^{l_{3}} \Theta^{l_{2}}}- \\
& -\underline{\frac{1}{4} \delta_{l_{3}}^{j} \Theta^{l_{2}} \Theta^{l_{1}}}-\underline{(n)}-\underline{\frac{1}{4} \delta_{l_{1}}^{j} \Theta^{l_{2}} \Theta^{l_{3}}}()^{(0)}+\underline{\frac{1}{2} L_{l_{3}, l_{2}}^{j} \Theta^{l_{1}}}\left(\text { (D) }-\underline{\frac{1}{4} \delta_{l_{3}}^{j} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{l_{1}}}-\right. \\
& -\underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{3}, l_{3}}^{l_{3}} \Theta^{l_{1}}}\left(\underline{\text { q }}-\underline{\frac{1}{4} \delta_{l_{3}}^{j} \Theta^{l_{1}} \Theta^{l_{2}}}-\underline{18}-\underline{\frac{1}{4} \delta_{l_{2}}^{j} \Theta^{l_{1}} \Theta^{l_{3}}}-\right.
\end{aligned}
$$

$$
\begin{aligned}
& -\underline{\frac{1}{2} L_{l_{1}, l_{3}}^{j} L_{l_{2}, l_{2}}^{l_{2}}}(\mathrm{c}) \underline{\frac{1}{2} \delta_{l_{2}}^{j} \sum_{k} L_{l_{1}, l_{3}}^{k} \Theta^{k}}-\underline{14}-\underline{\frac{1}{2} L_{l_{1}, l_{3}}^{j} \Theta^{l_{2}}}(1)-\underline{\frac{1}{2} L_{l_{3}, l_{3}}^{l_{3}} L_{l_{2}, l_{1}}^{j}}(\text { a })+ \\
& +\underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{3}, l_{3}}^{l_{3}} L_{l_{1}, l_{1}}^{l_{1}}} \underline{3}+\underline{\frac{1}{4} \delta_{l_{1}}^{j} L_{l_{3}, l_{3}}^{l_{3}} L_{l_{2}, l_{2}}^{l_{2}}} \text { (d) }+\underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{3}, l_{3}}^{l_{3}} \Theta^{l_{1}}}+\underline{(\mathrm{a})}+\underline{\mathrm{A}}_{\underline{4} \delta_{l_{1}}^{j} L_{l_{3}, l_{3}}^{l_{3}} \Theta^{l_{2}}}^{(\mathrm{m}}- \\
& -\underline{\frac{1}{2} L_{l_{1}, l_{1}}^{l_{1}} L_{l_{2}, l_{3}}^{j}} \underset{(\mathrm{~g}}{1}+\underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}} L_{l_{3}, l_{3}}^{l_{3}}} \underset{i}{ }+\underline{\frac{1}{4} \delta_{l_{3}}^{j} L_{l_{1}, l_{1}}^{l_{1}} L_{l_{2}, l_{2}}^{l_{2}}}(\mathrm{~h}) \underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{3}}}+ \\
& +\underline{\frac{1}{4} \delta_{l_{3}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{2}}}-\underline{(\mathrm{j})}-\underline{\frac{1}{2} L_{l_{2}, l_{1}}^{j} \Theta^{l_{3}}}+\underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{3}}}+\underline{\frac{1}{4}}+\underline{\delta_{l_{1}}^{j} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{l_{3}}}+ \\
& +\underline{\frac{1}{4} \delta_{l_{2}}^{j} \Theta^{l_{3}} \Theta^{l_{1}}}+\underline{\underline{17}}+\underline{\frac{1}{4} \delta_{l_{1}}^{j} \Theta^{l_{3}} \Theta^{l_{2}}}-\underline{\frac{1}{2} L_{l_{2}, l_{3}}^{j} \Theta^{l_{1}}}(\mathrm{p}) \underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{3}, l_{3}}^{l_{3}} \Theta^{l_{1}}}+ \\
& +\underline{\frac{1}{4} \delta_{l_{3}}^{j} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{l_{1}}}+\underline{\frac{1}{4} \delta_{l_{2}}^{j} \Theta^{l_{1}} \Theta^{l_{3}}}+\underline{\mathrm{r})}+\underline{\frac{1}{4} \delta_{l_{3}}^{j} \Theta^{l_{1}} \Theta^{l_{2}}} .
\end{aligned}
$$

We simplify and we reorganize the equality between the second and third lines of (3.88) and (3.85) so as to put all terms $\Theta_{y}$ in the left-hand side of the equality and to put all remaining terms in the right-hand side, respecting
the order of $\S 3.73$. We get:
(3.89)
$\frac{1}{2} \delta_{l_{1}}^{j} \Theta_{y^{l_{3}}}^{l_{2}}-\frac{1}{2} \delta_{l_{1}}^{j} \Theta_{y^{l_{2}}}^{l_{3}}+\frac{1}{2} \delta_{l_{2}}^{j} \Theta_{y^{l_{3}}}^{l_{1}}-\frac{1}{2} \delta_{l_{3}}^{j} \Theta_{y^{l_{2}}}^{l_{1}}=$

$$
\begin{aligned}
= & L_{l_{1}, l_{2}, y^{l_{3}}}^{j}-L_{l_{1}, l_{3}, y^{l_{2}}}^{j}+\frac{1}{2} \delta_{l_{1}}^{j} L_{l_{3}, l_{3}, y^{l_{2}}}^{l_{3}}-\frac{1}{2} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}, y^{l_{3}}}^{l_{2}}+ \\
& +\frac{1}{2} \delta_{l_{3}}^{j} L_{l_{1}, l_{1}, y^{2}}^{l_{1}}-\frac{1}{2} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}, y^{l_{3}}}^{l_{1}}+ \\
& +\frac{1}{2} H_{l_{3}}^{j} M_{l_{1}, l_{2}}-\frac{1}{2} H_{l_{2}}^{j} M_{l_{1}, l_{3}}+\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{3}, l_{3}}^{l_{3}} L_{l_{1}, l_{1}}^{l_{1}}-\frac{1}{4} \delta_{l_{3}}^{j} L_{l_{2}, l_{2}}^{l_{2}} L_{l_{1}, l_{1}}^{l_{1}}+ \\
& +\sum_{k} L_{l_{1}, l_{3}}^{k} L_{l_{2}, k}^{j}-\sum_{k} L_{l_{1}, l_{2}}^{k} L_{l_{3}, k}^{j}+ \\
& +\frac{1}{2} \delta_{l_{3}}^{j} \sum_{k} L_{l_{1}, l_{2}}^{k} L_{k, k}^{k}-\frac{1}{2} \delta_{l_{2}}^{j} \sum_{k} L_{l_{1}, l_{3}}^{k} L_{k, k}^{k}+
\end{aligned}
$$

$$
+\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{3}}-\frac{1}{4} \delta_{l_{3}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{2}}+\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{3}, l_{3}}^{l_{3}} \Theta^{l_{1}}-\frac{1}{4} \delta_{l_{3}}^{j} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{l_{1}}+
$$

$$
+\frac{1}{2} \delta_{l_{3}}^{j} \sum_{k} L_{l_{1}, l_{2}}^{k} \Theta^{k}-\frac{1}{2} \delta_{l_{2}}^{j} \sum_{k} L_{l_{1}, l_{3}}^{k} \Theta^{k}+
$$

$$
+\frac{1}{2} \delta_{l_{2}}^{j} M_{l_{1}, l_{3}} \Theta^{0}-\frac{1}{2} \delta_{l_{3}}^{j} M_{l_{1}, l_{2}} \Theta^{0}+\frac{1}{4} \delta_{l_{2}}^{j} \Theta^{l_{3}} \Theta^{l_{1}}-\frac{1}{4} \delta_{l_{3}}^{j} \Theta^{l_{2}} \Theta^{l_{1}}
$$

Next, replacing plainly (3.64) in (3.65) ${ }_{4}$, we get:

$$
\begin{align*}
& \left(\Pi_{0,0}^{0}\right)_{y^{l_{1}}}-\left(\Pi_{0, l_{1}}^{0}\right)_{x}=  \tag{3.90}\\
& =\Theta_{y^{l_{1}}}^{0}-\frac{1}{2} L_{l_{1}, l_{1}, x}^{l_{1}}-\frac{1}{2} \Theta_{x}^{l_{1}}= \\
& \left.=-\underline{\Pi_{0,0}^{0} \cdot \Pi_{l_{1}, 0}^{0}} \text { (a) }-\sum_{k} \Pi_{0,0}^{k} \cdot \Pi_{l_{1}, k}^{0}+\underline{\Pi_{0, l_{1}}^{0} \cdot \Pi_{0,0}^{0}(\mathrm{a}}\right)+\sum_{k} \Pi_{0, l_{1}}^{k} \cdot \Pi_{0, k}^{0}= \\
& =-\sum_{k}\left(-G^{k}\right) \cdot\left(M_{l_{1}, k}\right)+\sum_{k}\left(-\frac{1}{2} H_{l_{1}}^{k}+\frac{1}{2} \delta_{l_{1}}^{k} \Theta^{0}\right) \text {. } \\
& \cdot\left(\frac{1}{2} L_{k, k}^{k}+\frac{1}{2} \Theta^{k}\right)= \\
& =\sum_{k} G^{k} M_{l_{1}, k}-\frac{1}{4} \sum_{k} H_{l_{1}}^{k} L_{k, k}^{k}-\frac{1}{4} \sum_{k} H_{l_{1}}^{k} \Theta^{k}+\frac{1}{4} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{0}+\frac{1}{4} \Theta^{0} \Theta^{l_{1}} .
\end{align*}
$$

Reorganizing the equality so as to put the terms $\Theta_{x}$ and $\Theta_{y}$ alone in the left-hand side, we get:

$$
\begin{align*}
-\frac{1}{2} \Theta_{x}^{l_{1}}+\Theta_{y l_{1}}^{0}= & \frac{1}{2} L_{l_{1}, l_{1}, x}^{l_{1}}+  \tag{3.91}\\
& +\sum_{k} G^{k} M_{l_{1}, k}-\frac{1}{4} \sum_{k} H_{l_{1}}^{k} L_{k, k}^{k}-\frac{1}{4} \sum_{k} H_{l_{1}}^{k} \Theta^{k}+ \\
& +\frac{1}{4} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{0}+\frac{1}{4} \Theta^{0} \Theta^{l_{1}} .
\end{align*}
$$

Next, replacing plainly (3.64) in $(3.65)_{5}$, we get:

$$
\begin{align*}
&\left(\Pi_{l_{1}, l_{2}}^{0}\right)_{x}-\left(\Pi_{l_{1}, 0}^{0}\right)_{y^{l_{2}}}=  \tag{3.92}\\
&= M_{l_{1}, l_{2}, x}-\frac{1}{2} L_{l_{1}, l_{1}, y^{l_{2}}}^{l_{1}}-\frac{1}{2} \Theta_{y^{l_{2}}}^{l_{1}}= \\
&=-\Pi_{l_{1}, l_{2}} \cdot \Pi_{0,0}^{0}-\sum_{k} \Pi_{l_{1}, l_{2}}^{k} \cdot \Pi_{0, k}^{0}+\Pi_{l_{1}, 0}^{0} \cdot \Pi_{l_{2}, 0}^{0}+\sum_{k} \Pi_{l_{1}, 0}^{k} \cdot \Pi_{l_{2}, k}^{0}= \\
&=-M_{l_{1}, l_{2}} \Theta^{0}-\sum_{k}\left(-L_{l_{1}, l_{2}}^{k}+\frac{1}{2} \delta_{l_{1}}^{k} L_{l_{2}, l_{2}}^{l_{2}}+\frac{1}{2} \delta_{l_{2}}^{k} L_{l_{1}, l_{1}}^{l_{1}}+\right. \\
&\left.+\frac{1}{2} \delta_{l_{1}}^{k} \Theta^{l_{2}}+\frac{1}{2} \delta_{l_{2}}^{k} \Theta^{l_{1}}\right) \cdot\left(\frac{1}{2} L_{k, k}^{k}+\frac{1}{2} \Theta^{k}\right)+ \\
&+\left(\frac{1}{2} L_{l_{1}, l_{1}}^{l_{1}}+\frac{1}{2} \Theta^{l_{1}}\right) \cdot\left(\frac{1}{2} L_{l_{2}, l_{2}}^{l_{2}}+\frac{1}{2} \Theta^{l_{2}}\right)+ \\
&+\sum_{k}\left(-\frac{1}{2} H_{l_{1}}^{k}+\frac{1}{2} \delta_{l_{1}}^{k} \Theta^{0}\right) \cdot M_{l_{2}, k}=
\end{align*}
$$

$$
=-\underline{M_{l_{1}, l_{2}} \Theta^{0}}+\underset{7^{7}}{+\frac{1}{2} \sum_{k} L_{l_{1}, l_{2}}^{k} L_{k, k}^{k}+\frac{1}{2} \sum_{k} L_{l_{1}, l_{2}}^{k} \Theta^{k}-\frac{1}{4} L_{l_{2}, l_{2}}^{l_{2}} L_{l_{1}, l_{1}}^{l_{1}}[2]}-
$$

$$
-\underline{\frac{1}{4} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{l_{1}}}-\underline{(\mathrm{a}}-\underline{\frac{1}{4} L_{l_{1}, l_{1}}^{l_{1}} L_{l_{2}, l_{2}}^{l_{2}}} \text { (b) }-\underline{\frac{1}{4} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{2}}} \text { (c) }-\underbrace{\frac{1}{4} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{2}}}_{4}-
$$

$$
-\underline{\frac{1}{4} \Theta^{l_{1}} \Theta^{l_{2}}} \text { (d) }-\underline{\frac{1}{4} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{l_{1}}} 5_{5}^{-\frac{1}{4} \Theta^{l_{1}} \Theta^{l_{2}}} \sqrt[8]{8}+\underline{\frac{1}{4} L_{l_{1}, l_{1}}^{l_{1}} L_{l_{2}, l_{2}}^{l_{2}}}(\text { b })
$$

$$
+\underline{\frac{1}{4} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{2}}}\left(\mathrm{c}^{2}+\underline{\frac{1}{4} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{l_{1}}}\left(\underline{\text { a })}+\underline{\frac{1}{4} \Theta^{l_{1}} \Theta^{l_{2}}}-\underline{\frac{1}{2} \sum_{k} H_{l_{1}}^{k} M_{l_{2}, k}+}\right.\right.
$$

$$
+{\underline{\frac{1}{2}} M_{l_{2}, l_{1}} \Theta^{0}}_{\square}^{7}
$$

Multiplying by -2 and reorganizing the equality, we get:

$$
\begin{align*}
\Theta_{y^{l_{2}}}^{l_{1}}= & -\frac{L_{l_{1}, l_{1}, y^{l_{2}}}^{l_{1}}+2 M_{l_{1}, l_{2}, x}+}{} \\
& +\sum_{k} H_{l_{1}}^{k} M_{l_{2}, k}+\frac{1}{2} L_{l_{1}, l_{1}}^{l_{1}} L_{l_{2}, l_{2}}^{l_{2}}-\sum_{k} L_{l_{1}, l_{2}}^{k} L_{k, k}^{k}+ \\
& +\frac{1}{2} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{2}}+\frac{1}{2} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{l_{1}}-\sum_{k} L_{l_{1}, l_{2}}^{k} \Theta^{k}+  \tag{3.93}\\
& +M_{l_{1}, l_{2}} \Theta^{0}+\frac{1}{2} \Theta^{l_{1}} \Theta^{l_{2}}
\end{align*}
$$

Next, replacing plainly (3.64) in (3.65) ${ }_{4}$, we get:

$$
\begin{align*}
\left(\Pi_{l_{1}, l_{2}}^{0}\right. & )_{y^{l_{3}}}-\left(\Pi_{l_{1}, l_{3}}^{0}\right)_{y_{2}}=  \tag{3.94}\\
= & M_{l_{1}, l_{2}, y_{3}}^{l_{3}}-M_{l_{1}, l_{3}, y^{l_{2}}}= \\
= & -\Pi_{l_{1}, l_{2}}^{0} \cdot \Pi_{l_{3}, 0}^{0}-\sum_{k} \Pi_{l_{1}, l_{2}}^{k} \cdot \Pi_{l_{3}, k}^{0}+\Pi_{l_{1}, l_{3}}^{0} \cdot \Pi_{l_{2}, 0}^{0}+\sum_{k} \Pi_{l_{1}, l_{3}}^{k} \cdot \Pi_{l_{2}, k}^{0}= \\
= & -M_{l_{1}, l_{2}}\left(\frac{1}{2} L_{l_{3}, l_{3}}^{l_{3}}+\frac{1}{2} \Theta^{l_{3}}\right)-\sum_{k} M_{l_{3}, k}\left(-L_{l_{1}, l_{2}}^{k}+\frac{1}{2} \delta_{l_{1}}^{k} L_{l_{2}, l_{2}}^{l_{2}}+\right. \\
& \left.+\frac{1}{2} \delta_{l_{2}}^{k} L_{l_{1}, l_{1}}^{l_{1}}+\frac{1}{2} \delta_{l_{1}}^{k} \Theta^{l_{2}}+\frac{1}{2} \delta_{l_{2}}^{k} \Theta^{l_{1}}\right)+ \\
& +M_{l_{1}, l_{3}}\left(\frac{1}{2} L_{l_{2}, l_{2}}^{l_{2}}+\frac{1}{2} \Theta^{l_{2}}\right)+\sum_{k} M_{l_{2}, k}\left(-L_{l_{1}, l_{3}}^{k}+\frac{1}{2} \delta_{l_{1}}^{k} L_{l_{3}, l_{3}}^{l_{3}}+\right. \\
& \left.+\frac{1}{2} \delta_{l_{3}}^{k} L_{l_{1}, l_{1}}^{l_{1}}+\frac{1}{2} \delta_{l_{1}}^{k} \Theta^{l_{3}}+\frac{1}{2} \delta_{l_{3}}^{k} \Theta^{l_{1}}\right) .
\end{align*}
$$

Developing the products and ordering each monomial, we get:
(3.95)

$$
\begin{aligned}
& =\underline{-\frac{1}{2} L_{l_{3}, l_{3}}^{l_{3}} M_{l_{1}, l_{2}}} \text { (a) }-\underline{\frac{1}{2} M_{l_{1}, l_{2}} \Theta^{l_{3}}} \text { (b) } \xlongequal[\sum_{k} L_{l_{1}, l_{2}}^{k} M_{l_{3}, k}]{\underline{1}-\frac{\frac{1}{2} L_{l_{2}, l_{2}}^{l_{2}} M_{l_{3}, l_{1}}}{(\text { c }}-} \\
& -\underline{\frac{1}{2} L_{l_{1}, l_{1}}^{l_{1}} M_{l_{3}, l_{2}}}\left(\text { (d) }-\underline{\frac{1}{2}} M_{l_{3, l_{1}}} \Theta^{l_{2}} \text { (e) }-\underline{\frac{1}{2} M_{l_{3, l_{2}}} \Theta^{l_{1}}}(\ddagger) \underline{\frac{1}{2} L_{l_{2}, l_{2}}^{l_{2}} M_{l_{3}, l_{1}}}+\right. \\
& +\underline{\frac{1}{2} M_{l_{1}, l_{3}} \Theta^{l_{2}}} \text { (e) } \underline{\sum_{k} L_{l_{1}, l_{3}}^{k} M_{l_{2}, k}}+\underline{\frac{1}{2} L_{l_{3}, l_{3}}^{l_{3}} M_{l_{2}, l_{1}}} \text { (a) } \underline{\frac{1}{2} L_{l_{1}, l_{1}}^{l_{1}} M_{l_{2}, l_{3}}} \text { (d) }+ \\
& \left.+{\underline{\frac{1}{2}} M_{l_{2}, l_{1}} \Theta^{l_{3}}}_{(\mathrm{b})}^{\underline{\frac{1}{2}} M_{l_{2}, l_{3}} \Theta^{l_{1}}} \underset{( }{f}\right)
\end{aligned}
$$

Simplifying, we obtain the family (IV) in the statement of Theorem 1.7 (3):

$$
\begin{equation*}
0=\underline{M_{l_{1}, l_{2}, y^{l_{3}}}-M_{l_{1}, l_{3}, y^{l_{2}}}}-\sum_{k} L_{l_{1}, l_{2}}^{k} M_{l_{3}, k}+\sum_{k} L_{l_{1}, l_{3}}^{k} M_{l_{2}, k} \tag{3.96}
\end{equation*}
$$

3.97. Solving $\Theta_{x}^{0}, \Theta_{y^{l_{1}}}^{0}, \Theta_{x}^{l_{1}}$ and $\Theta_{y^{l_{2}}}^{l_{1}}$. It is now easy to solve all first order partial derivatives of the functions $\Theta^{0}$ and $\Theta^{l}$. Equation (3.93) already provides the solution for $\Theta_{y^{l_{2}}}^{l_{1}}$. We state the result as an independent proposition.

Proposition 3.98. As a consequence of the six families of equations (3.69), (3.86), (3.89), (3.91), (3.93) and (3.96) the first order derivatives $\Theta_{x}^{0}, \Theta_{y^{l_{1}}}^{0}$, $\Theta_{x}^{l_{1}}$ and $\Theta_{y^{l_{2}}}^{l_{1}}$ of the principal unknowns are given by:

$$
\left\{\begin{align*}
\Theta_{x}^{0}= & -2 G_{y^{l_{1}}}^{l_{1}}+H_{l_{1}, x}^{l_{1}}+  \tag{3.99}\\
& +2 \sum_{k} G^{k} L_{l_{1}, k}^{l_{1}}-\sum_{k} G^{k} L_{k, k}^{k}-\frac{1}{2} \sum_{k} H_{l_{1}}^{k} H_{k}^{l_{1}}- \\
& -\sum_{k} G^{k} \Theta^{k}+\frac{1}{2} \Theta^{0} \Theta^{0} .
\end{align*}\right.
$$

$$
\left\{\begin{align*}
\Theta_{y^{l_{1}}}^{0}= & \frac{2}{3} L_{l_{1}, l_{1}, x}^{l_{1}}-\frac{1}{3} H_{l_{1}, y^{l_{1}}}^{l_{1}}+  \tag{3.100}\\
& +\frac{2}{3} G^{l_{1}} M_{l_{1}, l_{1}}+\frac{4}{3} \sum_{k} G^{k} M_{l_{1}, k}-\frac{1}{3} \sum_{k} H_{k}^{l_{1}} L_{l_{1}, l_{1}}^{k}+ \\
& +\frac{1}{3} \sum_{k} H_{l_{1}}^{k} L_{l_{1}, k}^{l_{1}}-\frac{1}{2} \sum_{k} H_{l_{1}}^{k} L_{k, k}^{k}-\frac{1}{2} \sum_{k} H_{l_{1}}^{k} \Theta^{k}+ \\
& +\frac{1}{2} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{0}+\frac{1}{2} \Theta^{0} \Theta^{l_{1}} .
\end{align*}\right.
$$

$$
\left\{\begin{align*}
\Theta_{x}^{l_{1}}= & -\frac{2}{3} H_{l_{1}, y^{l_{1}}}^{l_{1}}+\frac{1}{3} L_{l_{1}, l_{1}, x}^{l_{1}}+  \tag{3.101}\\
& +\frac{4}{3} G^{l_{1}} M_{l_{1}, l_{1}}+\frac{2}{3} \sum_{k} G^{k} M_{l_{1}, k}-\frac{2}{3} \sum_{k} H_{k}^{l_{1}} L_{l_{1}, l_{1}}^{k}+ \\
& +\frac{2}{3} \sum_{k} H_{l_{1}}^{k} L_{l_{1}, k}^{l_{1}}-\frac{1}{2} \sum_{k} H_{l_{1}}^{k} L_{k, k}^{k}-\frac{1}{2} \sum_{k} H_{l_{1}}^{k} \Theta^{k}+ \\
& +\frac{1}{2} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{0}+\frac{1}{2} \Theta^{0} \Theta^{l_{1}} .
\end{align*}\right.
$$

$$
\left\{\begin{align*}
\Theta_{y^{l_{2}}}^{l_{1}}= & -\frac{L_{l_{1}, l_{1}, y^{l_{2}}}^{l_{1}}+2 M_{l_{1}, l_{2}, x}+}{} \\
& +\sum_{k} H_{l_{1}}^{k} M_{l_{2}, k}+\frac{1}{2} L_{l_{1}, l_{1}}^{l_{1}} L_{l_{2}, l_{2}}^{l_{2}}-\sum_{k} L_{l_{1}, l_{2}}^{k} L_{k, k}^{k}+  \tag{3.102}\\
& +\frac{1}{2} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{2}}+\frac{1}{2} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{l_{1}}-\sum_{k} L_{l_{1}, l_{2}}^{k} \Theta^{k}+ \\
& +M_{l_{1}, l_{2}} \Theta^{0}+\frac{1}{2} \Theta^{l_{1}} \Theta^{l_{2}} .
\end{align*}\right.
$$

We notice that the right-hand side of (3.99) should be independent of $l_{1}$; this phenomenon will be explained in a while.
Proof. For $\Theta_{x}^{0}$ in (3.99), it suffices to put $j:=l_{1}$ in (3.69).
To obtain $\Theta_{y^{l_{1}}}^{0}$, we put $j:=l_{2}$ and $l_{2}:=l_{1}$ in (3.86), which yields:

$$
\begin{align*}
\Theta_{x}^{l_{1}}-\frac{1}{2} \Theta_{y^{l_{1}}}^{0}= & -\frac{1}{2} H_{l_{1}, y^{l_{1}}}^{l_{1}}+ \\
& +G^{l_{1}} M_{l_{1}, l_{1}}-\frac{1}{2} \sum_{k} H_{k}^{l_{1}} L_{l_{1}, l_{1}}^{k}+ \\
& +\frac{1}{2} \sum_{k} H_{l_{1}}^{k} L_{l_{1}, k}^{l_{1}}-\frac{1}{4} \sum_{k} H_{l_{1}}^{k} L_{k, k}^{k}-  \tag{3.103}\\
& -\frac{1}{4} \sum_{k} H_{l_{1}}^{k} \Theta^{k}+\frac{1}{4} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{0}+\frac{1}{4} \Theta^{0} \Theta^{l_{1}} .
\end{align*}
$$

We may easily solve $\Theta_{y^{{ }_{1}^{1}}}^{0}$ and $\Theta_{x}^{l_{1}}$ thanks to this equation (3.103) and thanks to (3.91): indeed, to obtain (3.100), it suffices to compute $\frac{4}{3} \cdot(3.91)+\frac{2}{3} \cdot$ (3.103); to obtain (3.101), it suffices to compute $\frac{2}{3} \cdot(3.91)+\frac{4}{3} \cdot(3.103)$. Finally, (3.102) is a copy of (3.93). This completes the proof.
3.104. Appearance of the crucial four families of first order partial differential relations (I), (II), (III) and (IV) of Theorem 1.7 (3). However, in solving $\Theta_{x}^{0}, \Theta_{y^{l_{1}}}^{0}, \Theta_{x}^{l_{1}}$ and $\Theta_{y^{l_{2}}}^{l_{1}}$ from our six families of equations (3.69), (3.86), (3.89), (3.91), (3.93) and (3.96), only a subpart of these equations has been used. We notice that the two families of equations (3.91) and (3.93) have been used completely and that the family of equations (3.96), which does not involve $\Theta$, coincides precisely with the system (IV) of Theorem 1.7 (3). To insure that $\Theta_{x}^{0}, \Theta_{y^{l_{1}}}^{0}, \Theta_{x}^{l_{1}}$ and $\Theta_{y^{l_{2}}}^{l_{1}}$ as written in Proposition 3.98 are true solutions, it is necessary and sufficient that they satisfy the remaining equations. Thus, we have to replace these solutions (3.99), (3.100), (3.101) and (3.102) in the three remaining families (3.69), (3.86) and (3.89).

Firstly, let us insert inside (3.69) the value of $\Theta_{x}^{0}$ given by the equation (3.99), in which the index $l_{1}$ is replaced in advance by an arbitrary
index $l_{2}$. We get:
(3.105)

$$
\begin{aligned}
0= & -\frac{2 G_{y_{1}}^{j}+2 \delta_{l_{1}}^{j} G_{y^{l_{2}}}^{l_{2}}+H_{l_{1}, x}^{j}-\delta_{l_{1}}^{j} H_{l_{2}, x}^{l_{2}}+}{} \\
& +2 \sum_{k} G^{k} L_{l_{1}, k}^{j}-2 \delta_{l_{1}}^{j} \sum_{k} G^{k} L_{l_{2}, k}^{l_{2}}-\delta_{l_{1}}^{j} \sum_{k} G^{k} L_{k, k}^{k}+ \\
& +\delta_{l_{1} \sum_{k}^{j} G^{k} L_{k, k}^{k}-\frac{1}{2} \sum_{k} H_{l_{1}}^{k} H_{k}^{j}+\frac{1}{2} \delta_{l_{1}}^{j} \sum_{k} H_{l_{2}}^{k} H_{k}^{l_{2}}-} \\
& -\delta_{l_{1}}^{j} \sum_{k} G^{k} \Theta^{k}+\delta_{l_{1}}^{j} \sum_{k} G^{k} \Theta^{k}+\frac{1}{2} \delta_{l_{1}}^{j} \Theta^{0} \Theta^{0}-\text { (b) } \underline{\frac{1}{2} \delta_{l_{1}}^{j} \Theta^{0} \Theta^{0}} \text { (c). }
\end{aligned}
$$

We simplify, which yields the family (I) of partial differential relations of Theorem 1.7 (3):

$$
\begin{align*}
0= & -\frac{2 G_{y^{l_{1}}}^{j}+2 \delta_{l_{1}}^{j} G_{y^{l_{2}}}^{l_{2}}+H_{l_{1}, x}^{j}-\delta_{l_{1}}^{j} H_{l_{2}, x}^{l_{2}}+}{}+ \\
& +2 \sum_{k} G^{k} L_{l_{1}, k}^{j}-2 \delta_{l_{1}}^{j} \sum_{k} G^{k} L_{l_{2}, k}^{l_{2}}+  \tag{3.106}\\
& -\frac{1}{2} \sum_{k} H_{l_{1}}^{k} H_{k}^{j}+\frac{1}{2} \delta_{l_{1}}^{j} \sum_{k} H_{l_{2}}^{k} H_{k}^{l_{2}} .
\end{align*}
$$

Secondly, let us insert inside (3.86) the values of $\Theta_{x}^{l_{1}}, \Theta_{x}^{l_{2}}$ given by (3.101) and the value of $\Theta_{y^{l_{1}}}^{0}$ given by (3.100). We place all the terms in the righthand side of the equality and we place the first order terms in the beginning (first three lines just below). We obtain:
(3.107)

$$
\begin{aligned}
& 0=-\underline{\frac{1}{2} H_{l_{1}, y^{l_{2}}}^{j}} \underline{1}+\underline{L_{l_{1}, l_{2}, x}^{j}}\left[\underline{\underline{2}}-\underline{\frac{1}{2} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}, x}^{l_{2}}} 5 \underline{5}-\frac{1}{2} \delta_{l_{1}}^{j} L_{l_{1}, l_{1}, x}^{l_{1}} \square^{6}+\right. \\
& +\underline{\frac{1}{3} \delta_{l_{1}}^{j} H_{l_{2}, y^{l_{2}}}^{l_{2}}-\frac{\frac{1}{6} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}, x}^{l_{2}}}{5}+\underline{\frac{1}{3} \delta_{l_{2}}^{j} H_{l_{1}, y^{l_{1}}}^{3}} \underline{3}_{\underline{6}}^{l_{1}}-\frac{1}{\delta_{l_{2}}^{j}} L_{l_{1}, l_{1}, x}^{l_{1}}}+ \\
& +\underline{\frac{1}{3} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}, x}^{l_{2}}} 5 \underline{\underline{6} \delta_{l_{1}}^{j} H_{l_{2}, y^{l_{2}}}^{l_{2}}}{ }^{2}+ \\
& \left.+\underline{G^{j} M_{l_{1}, l_{2}}}\right]^{-\frac{1}{2} \sum_{k} H_{k}^{j} L_{l_{1}, l_{2}}^{k}}+\frac{12}{2} \sum_{k} H_{l_{1}}^{k} L_{l_{2}, k}^{j}-\frac{1}{4} \delta_{l_{2}}^{j} \sum_{k} H_{l_{1}}^{k} L_{k, k}^{k}- \\
& -\frac{1}{4} \delta_{l_{2}}^{j} \sum_{k} H_{l_{1}}^{k} \Theta^{k}+\underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{0}}+\underline{\frac{1}{4} \delta_{l_{2}}^{j} \Theta^{0} \Theta^{l_{1}}-} \text { (d) } \\
& -\frac{2}{3} \delta_{l_{1}}^{j} G^{l_{2}} M_{l_{2}, l_{2}}-8 \xrightarrow[10]{8}-\frac{1}{3} \delta_{l_{1}}^{j} \sum_{k} G^{k} M_{l_{2}, k}+\frac{1}{3} \delta_{l_{1}}^{j} \sum_{k} H_{k}^{l_{2}} L_{l_{2}, l_{2}}^{k} \quad-\frac{1}{3} \delta_{l_{1}}^{j} \sum_{k} H_{l_{2}}^{k} L_{l_{2}, k}^{l_{2}} \quad 1{ }_{14}^{15}+
\end{aligned}
$$

$$
\begin{aligned}
& +{\underline{(\mathrm{t}}{ }^{\frac{1}{4} \delta_{l_{1}}^{j} \sum_{k} H_{l_{2}}^{k} L_{k, k}^{k}}+\underline{\frac{1}{4} \delta_{l_{1}}^{j} \sum_{k} H_{l_{2}}^{k} \Theta^{k}-\underline{\frac{1}{4} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{0}}(\mathrm{~g})}-\underline{\frac{1}{4} \delta_{l_{1}}^{j} \Theta^{0} \Theta^{l_{2}}}-{ }^{\text {(h) }}-}_{\text {(e) }}
\end{aligned}
$$

$$
\begin{aligned}
& +\underline{\frac{1}{4} \delta_{l_{2}}^{j} \sum_{k} H_{l_{1}}^{k} L_{k, k}^{k}}+\underline{\frac{1}{4} \delta_{l_{2}}^{j} \sum_{k} H_{l_{1}}^{k} \Theta^{k}-\left(\text { (a) }-\underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{0}} \text { (c) }-\underline{\frac{1}{4} \delta_{l_{2}}^{j} \Theta^{0} \Theta^{0}}+{ }^{\text {(d) }}+\right.} \\
& +\underline{\frac{1}{3} \delta_{l_{1}}^{j} G^{l_{2}} M_{l_{2}, l_{2}}}+\underset{8}{8}+\frac{2}{3} \delta_{l_{1}}^{j} \sum_{k} G^{k} M_{l_{2}, k}-\frac{1}{6} \delta_{l_{1}}^{j} \sum_{k} H_{k}^{l_{2}} L_{l_{2}, l_{2}}^{k}+\sqrt{14}+\frac{1}{6} \delta_{l_{1}}^{j} \sum_{k} H_{l_{2}}^{k} L_{l_{2}, k}^{l_{2}}{ }_{15}^{15}- \\
& \underline{-\frac{1}{4} \delta_{l_{1}}^{j} \sum_{k} H_{l_{2}}^{k} L_{k, k}^{k}} \text { (e) } \underline{-\frac{1}{4} \delta_{l_{1}}^{j} \sum_{k} H_{l_{2}}^{k} \Theta^{k}}+\underbrace{\frac{1}{4} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{0}}_{\text {f) }}(\mathrm{g}) \underline{\frac{1}{4} \delta_{l_{1}}^{j} \Theta^{0} \Theta^{l_{2}}} \text { (h) }
\end{aligned}
$$

Simplifying and ordering, we obtain the family (II) of partial differential relations of Theorem 1.7 (3):

$$
\begin{aligned}
\text { (3.108) } & =\frac{\frac{1}{2} H_{l_{1}, y^{l_{2}}}^{j}+\frac{1}{6} \delta_{l_{1}}^{j} H_{l_{2}, y^{l_{2}}}^{l_{2}}+\frac{1}{3} \delta_{l_{2}}^{j} H_{l_{1}, l_{1}}^{l_{1}}}{0} \\
& +\frac{L_{l_{1}, l_{2}, x}^{j}-\frac{1}{3} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}, x}^{l_{2}}-\frac{2}{3} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}, x}^{l_{1}}}{j} \\
& +G^{j} M_{l_{1}, l_{2}}-\frac{1}{3} \delta_{l_{1}}^{j} G^{l_{2}} M_{l_{2}, l_{2}}-\frac{2}{3} \delta_{l_{2}}^{j} G^{l_{1}} M_{l_{1}, l_{1}}+\frac{1}{3} \delta_{l_{1}}^{j} \sum_{k} G^{k} M_{l_{2}, k}- \\
& -\frac{1}{3} \delta_{l_{2}}^{j} \sum_{k} G^{k} M_{l_{1}, k}-\frac{1}{2} \sum_{k} H_{k}^{j} L_{l_{1}, l_{2}}^{k}+\frac{1}{2} \sum_{k} H_{l_{1}}^{k} L_{l_{2}, k}^{j}+ \\
& +\frac{1}{6} \delta_{l_{1}}^{j} \sum_{k} H_{k}^{l_{2}} L_{l_{2}, l_{2}}^{k}-\frac{1}{6} \delta_{l_{1}}^{j} \sum_{k} H_{l_{2}}^{k} L_{l_{2}, k}^{l_{2}}+ \\
& +\frac{1}{3} \delta_{l_{2}}^{j} \sum_{k} H_{k}^{l_{1}} L_{l_{1}, l_{1}}^{k}-\frac{1}{3} \delta_{l_{2}}^{j} \sum_{k} H_{l_{1}}^{k} L_{l_{1}, k}^{l_{1}} .
\end{aligned}
$$

Thirdly, let us insert inside (3.89) the values of $\Theta_{y^{l_{3}}}^{l_{2}}$, of $\Theta_{y^{l_{2}}}^{l_{3}}$, of $\Theta_{y^{l_{3}}}^{l_{1}}$ and of $\Theta_{y^{l_{2}}}^{l_{1}}$ given by (3.102). We place all the terms in the right-hand side of the equality and we place the first order terms in the beginning (first four lines
just below). We obtain:
(3.109)

$$
+\underline{\frac{1}{2} H_{l_{3}}^{j} M_{l_{1}, l_{2}}}\left[\underline{5}-\underline{\frac{1}{2} H_{l_{2}}^{j} M_{l_{1}, l_{3}}} \underline{6}+\underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{3}, l_{3}}^{l_{3}} L_{l_{1}, l_{1}}^{l_{1}}}\left(\mathrm{f}-\underline{\frac{1}{4} \delta_{l_{3}}^{j} L_{l_{2}, l_{2}}^{l_{2}} L_{l_{1}, l_{1}}^{l_{1}}}(\mathrm{~g})+\right.\right.
$$

$$
+\sum_{\boxed{11}} L_{l_{1}, l_{3}}^{k} L_{l_{2}, k}^{j}-\sum_{12} L_{l_{1}, l_{2}}^{k} L_{l_{3}, k}^{j}+
$$

$$
+\underset{\frac{1}{2}_{\delta_{3}}^{j} \sum_{k} L_{l_{1}, l_{2}}^{k} L_{k, k}^{k}}{(\mathrm{~h})}-\frac{1}{2} \delta_{l_{2}}^{j} \sum_{k} L_{l_{1}, l_{3}}^{k} L_{k, k}^{k}+
$$

$$
+\underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{3}}}-\underline{\frac{1}{\mathrm{j}}}-\underline{\frac{1}{4} \delta_{l_{3}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{2}}}+\underline{\frac{1}{4}} \delta_{l_{2}}^{j} L_{l_{3}, l_{3}}^{l_{3}} \Theta^{l_{1}}\left(\mathbb{1}-\underline{\frac{1}{4}} \delta_{l_{3}}^{j} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{l_{1}}(\mathbb{m})\right.
$$

$$
+\underline{\frac{1}{2} \delta_{l_{3}}^{j} \sum_{k} L_{l_{1}, l_{2}}^{k} \Theta^{k}-\frac{1}{2} \delta_{l_{2}}^{j} \sum_{k} L_{l_{1}, l_{3}}^{k} \Theta^{k}+}
$$

$$
+\underline{\frac{1}{2}} \delta_{l_{2}}^{j} M_{l_{1}, l_{3}} \Theta^{0}-(\mathbb{p}) \underline{\frac{1}{2} \delta_{l_{3}}^{j} M_{l_{1}, l_{2}} \Theta^{0}}+\underline{\frac{1}{4} \delta_{l_{2}}^{j} \Theta^{l_{3}} \Theta^{l_{1}}}\left(\mathbb{\mathrm { r }}-\underline{\frac{1}{4} \delta_{l_{3}}^{j} \Theta^{l_{2}} \Theta^{l_{1}}}-\right.
$$

$$
-\underline{\frac{1}{4} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{l_{3}}}\left(\underline{\mathrm{v}}-\underline{\frac{1}{4} \delta_{l_{1}}^{j} L_{l_{3}, l_{3}}^{l_{3}} \Theta^{l_{2}}}\left(\underline{\omega}+{\underline{\frac{1}{2}} \delta_{l_{1}}^{j} \sum_{k} L_{l_{2}, l_{3}}^{k} \Theta^{k}-}_{\otimes}^{\otimes}\right.\right.
$$

$$
-\underline{\frac{1}{2} \delta_{l_{1}}^{j} M_{l_{2}, l_{3}} \Theta^{0}}(\underline{y}) \underline{\frac{1}{4} \delta_{l_{1}}^{j} \Theta^{l_{2}} \Theta^{l_{3}}}-
$$

$$
-\underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{l_{3}}}-\underline{(\mathrm{j}}-\underline{\frac{1}{4} \delta_{l_{2}}^{j} L_{l_{3}, l_{3}}^{l_{3}} \Theta^{l_{1}}}+(\mathbb{1}) \underline{\frac{1}{2} \delta_{l_{2}}^{j} \sum_{k} L_{l_{1}, l_{3}}^{k} \Theta^{k}-} \text { (0) }
$$

$$
-\underline{\frac{1}{2} \delta_{l_{2}}^{j} M_{l_{1}, l_{3}} \Theta^{0}} \underset{(\mathrm{P}}{-\frac{1}{4} \delta_{l_{2}}^{j} \Theta^{l_{3}} \Theta^{l_{1}}}+
$$

$$
\begin{aligned}
& 0=\underline{\frac{1}{2} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}, y^{l_{3}}}^{l_{2}}} \text { (a) }-\underline{\delta_{l_{1}}^{j} M_{l_{2}, l_{3}, x}} \text { (b) }+\underline{\frac{1}{2} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}, y^{l_{3}}}^{l_{1}} \text { (c) }}-\underline{\delta_{l_{2}}^{j} M_{l_{1}, l_{3}, x}} 4^{-} \\
& -\underline{\frac{1}{2} \delta_{l_{1}}^{j} L_{l_{3}, l_{3}, y^{l_{2}}}^{l_{2}}(\text { d })}+\underline{\delta_{l_{1}}^{j} M_{l_{3}, l_{2}, x}} \text { (b) }-\underline{\frac{1}{3} \delta_{l_{3}}^{j} L_{l_{1}, l_{1}, y^{l_{2}}}^{l_{1}}}(\text { e }) \underline{\delta_{l_{3}}^{j} M_{l_{1}, l_{2}, x}} \underline{3}^{+} \\
& +\underline{L_{l_{1}, l_{2}, y^{l_{3}}}^{j}} \underline{1}-\underline{L_{l_{1}, l_{3}, y^{l_{2}}}^{j}} 2+\underline{\frac{1}{2} \delta_{l_{1}}^{j} L_{l_{3}, l_{3}, y^{l_{2}}}^{l_{2}}} \text { (d) }-\underline{\frac{1}{2} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}, y^{l_{3}}}^{l_{2}}}\left(\begin{array}{l}
\text { a }
\end{array}+\right. \\
& +\underline{\frac{1}{3} \delta_{l_{3}}^{j} L_{l_{1}, l_{1}, y^{l_{2}}}^{l_{1}}}\left(\text { e }-\underline{\frac{1}{2} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}, y^{l_{3}}}^{l_{1}}} \text { (c) }+\right.
\end{aligned}
$$

$$
\begin{aligned}
& \underbrace{+\frac{1}{2} \delta_{l_{1}}^{j} \sum_{k} H_{l_{3}}^{k} M_{l_{2}, k}}_{7}+\underset{(4)}{\frac{1}{4} \delta_{l_{1}}^{j} L_{l_{3}, l_{3}}^{l_{3}} L_{l_{2}, l_{2}}^{l_{2}}}(\mathbb{t}) \xrightarrow{\frac{1}{2} \delta_{l_{1}}^{j} \sum_{k} L_{l_{3}, l_{2}} L_{k, k}^{k}+} \\
& +\underline{\frac{1}{4} \delta_{l_{1}}^{j} L_{l_{3}, l_{3}}^{l_{3}} \Theta^{l_{2}}} \underline{\omega}+\underline{\frac{1}{4} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}}^{l_{2}} \Theta^{l_{3}}}\left(\underline{\mathrm{v}}-\underline{\frac{1}{2} \delta_{l_{1}}^{j} \sum_{k} L_{l_{3}, l_{2}}^{k} \Theta^{k}+}\right. \\
& +\underline{\underline{2} \delta_{l_{1}}^{j} M_{l_{3}, l_{2}} \Theta_{(y)}^{0}}+\underline{\frac{1}{4} \delta_{l_{1}}^{j} \Theta^{l_{3}} \Theta^{l_{2}}}+
\end{aligned}
$$

$$
\begin{aligned}
& +\underline{\frac{1}{2} \delta_{l_{3}}^{j} M_{l_{1}, l_{2}} \Theta_{(9)}^{0}}+\underline{\frac{1}{4} \delta_{l_{3}}^{j} \Theta^{l_{1}} \Theta^{l_{2}}} \text { (5) }
\end{aligned}
$$

Simplifying and ordering, we obtain the family (III) of partial differential relations of Theorem 1.7 (3):

$$
\begin{align*}
0= & \frac{L_{l_{1}, l_{2}, y_{3}}^{j}-L_{l_{1}, l_{3}, y^{l_{2}}}^{j}+\delta_{l_{3}}^{j} M_{l_{1}, l_{2}, x}-\delta_{l_{2}}^{j} M_{l_{1}, l_{3}, x}+}{} \\
& +\frac{1}{2} H_{l_{3}}^{j} M_{l_{1}, l_{2}}-\frac{1}{2} H_{l_{2}}^{j} M_{l_{1}, l_{3}}+ \\
& +\frac{1}{2} \delta_{l_{1}}^{j} \sum_{k} H_{l_{3}}^{k} M_{l_{2}, k}-\frac{1}{2} \delta_{l_{1}}^{j} \sum_{k} H_{l_{2}}^{k} M_{l_{3}, k}+  \tag{3.110}\\
& +\frac{1}{2} \delta_{l_{3}}^{j} \sum_{k} H_{l_{1}}^{k} M_{l_{2}, k}-\frac{1}{2} \delta_{l_{2}}^{j} \sum_{k} H_{l_{1}}^{k} M_{l_{3}, k}+ \\
& +\sum_{k} L_{l_{1}, l_{3}}^{k} L_{l_{2}, k}^{j}-\sum_{k} L_{l_{1}, l_{2}}^{k} L_{l_{3}, k}^{j} .
\end{align*}
$$

3.111. Arguments for the proof of Theorem 1.7 (3): necessity and sufficiency of (I), (II), (III), (IV). Let us summarize the implications that have been established so far, from the beginning of Section 3. Recall that $m \geqslant 2$.

- There exist functions $X, Y^{j}$ of $(x, y)$ transforming the system $y_{x x}^{j}=$ $F^{j}\left(x, y, y_{x}\right), j=1, \ldots, m$, to the free particle system $Y_{X X}^{j}=0, j=$ $1, \ldots, m$.
$\Downarrow$
- There exist functions $\Pi_{l_{1}, l_{2}}^{j}$ of $(x, y), 0 \leqslant j, l_{1}, l_{2} \leqslant m$, satisfying the first auxiliary system (3.38) of partial differential equations.
$\Downarrow$
- There exist (principal unknowns) functions $\Theta^{0}, \Theta^{j}$ satisfying the six families of partial differential equations (3.69), (3.86), (3.89), (3.91), (3.93) and (3.96).
$\Downarrow$
- The functions $G^{j}, H_{l_{1}}^{j}, L_{l_{1}, l_{2}}^{j}$ and $M_{l_{1}, l_{2}}$ satisfy the four families of partial differential equations (I), (II), (III) and (IV) of Theorem 1.7 (3).

The four families of first order partial differential equations (3.99), (3.100), (3.101) and (3.102) satisfied by the principal unknowns will be called the second auxiliary system. It is a complete system.

To achieve the proof of Theorem 1.7 (3), we have to establish the reverse implications. More precisely:

- Some given functions $G^{j}, H_{l_{1}}^{j}, L_{l_{1}, l_{2}}^{j}=L_{l_{2}, l_{1}}^{j}$ and $M_{l_{1}, l_{2}}=M_{l_{2}, l_{1}}$ of $(x, y)$ satisfy the four families of partial differential equations (I), (II), (III) and (IV) of Theorem 1.7 (3), or equivalently, the partial differential equations (3.106), (3.108), (3.110) and (3.96).
$\Downarrow$
- There exist functions $\Theta^{0}, \Theta^{j}$ satisfying the second auxiliary system (3.99), (3.100), (3.101) and (3.102).
$\Downarrow$
- These solution functions $\Theta^{0}, \Theta^{j}$ satisfy the six families of partial differential equations (3.69), (3.86), (3.89), (3.91), (3.93) and (3.96).
$\Downarrow$
- There exist functions $\Pi_{l_{1}, l_{2}}^{j}$ of $(x, y), 0 \leqslant j, l_{1}, l_{2} \leqslant m$, satisfying the first auxiliary system (3.38) of partial differential equations.
$\Downarrow$
- There exist functions $X, Y^{j}$ of $(x, y)$ transforming the system $y_{x x}^{j}=$ $F^{j}\left(x, y, y_{x}\right), j=1, \ldots, m$, to the free particle system $Y_{X X}^{j}=0, j=$ $1, \ldots, m$.

The above last three implications have been already implicitely established in the preceding paragraphs, as may be checked by inspecting Lemma 3.40 and the formal computations after §3.62.

Thus, it remains only to establish the first implication in the above reverse list. Since the second auxiliary system (3.99), (3.100), (3.101) and (3.102) is complete and of first order, a necessary and sufficient condition for the existence of solutions follows by writing out the following four families of cross-differentiations:

$$
\left\{\begin{array}{l}
0=\left(\Theta_{x}^{0}\right)_{y^{l_{1}}}-\left(\Theta_{y^{l_{1}}}^{0}\right)_{x},  \tag{3.112}\\
0=\left(\Theta_{y^{l_{1}}}^{0}\right)_{y^{l_{2}}}-\left(\Theta_{y^{l_{2}}}^{0}\right)_{y^{l_{1}}}, \\
0=\left(\Theta_{x}^{l_{1}}\right)_{y^{l_{2}}}-\left(\Theta_{y^{l_{2}}}^{l_{1}}\right)_{x}, \\
0=\left(\Theta_{y^{l_{2}}}^{l_{1}}\right)_{y^{l_{3}}}-\left(\Theta_{y^{l_{3}}}^{l_{1}}\right)_{y^{l_{2}}} .
\end{array}\right.
$$

In the hardest techical part of this paper (Section 4 below), we verify that these four families of compatibility conditions are a consequence of (I), (II), (III) and (IV). For reasons of space, we shall in fact only study the first family of compatibility conditions, i.e. the first line of (3.112). In the our manuscript, we have treated the remaining three families of compatibility conditions similarly and completely, up to the very end of every branch of the coral tree of computations. However, we would like to mention that typesetting the remaining three cases would add at least fifty pages of Latex to Section 4. Thus, we prefer to expose thoroughly the treatment of the first family of compatibility conditions, explaining implicitely how to guess the treatment of the remaining three.

## §4. COMPATIBILITY CONDITIONS FOR THE SECOND AUXILIARY SYSTEM

So, we have to develope the first line of (3.112): we replace $\Theta_{x}^{0}$ by its expression (3.99), we differentiate it with respect to $y^{l_{1}}$, we replace $\Theta_{y^{l_{1}}}^{0}$ by its expression (3.100), we differentiate it with respect to $x$ and we substract. We get:
(4.1)

$$
\begin{aligned}
& 0=\left(\Theta_{x}^{0}\right)_{y^{l_{1}}}-\left(\Theta_{y^{l_{1}}}^{0}\right)_{x} \\
& =-\underline{\underline{-2 G_{y^{1} y^{l_{1}}}^{l_{1}}+H_{l_{1}, x y^{l_{1}}}^{l_{1}}}+} \\
& +2 \sum_{k} G_{y^{l_{1}}}^{k} L_{l_{1}, k}^{l_{1}}+2 \sum_{k} G^{k} L_{l_{1}, k, y^{l_{1}}}^{l_{1}}-\sum_{k} G_{y^{l_{1}}}^{k} L_{k, k}^{k}-\sum_{k} G^{k} L_{k, k, y^{l_{1}}}^{k}- \\
& -\frac{1}{2} \sum_{k} H_{l_{1}, y^{l_{1}}}^{k} H_{k}^{l_{1}}-\frac{1}{2} \sum_{k} H_{l_{1}}^{k} H_{k, y^{l_{1}}}^{l_{1}}-\sum_{k} G_{y^{l_{1}}}^{k} \Theta^{k}-\sum_{k} G^{k} \underline{\Theta_{l^{l_{1}}}^{k}}+ \\
& +\Theta^{0} \underline{\Theta_{y^{l_{1}}}^{0}}- \\
& -\underline{\underline{\frac{2}{3}} L_{l_{1}, l_{1}, x x}^{l_{1}}}+\frac{1}{3} H_{l_{1, y} y_{1} x}^{l_{1}}- \\
& -\frac{2}{3} G_{x}^{l_{1}} M_{l_{1}, l_{1}}-\frac{2}{3} G^{l_{1}} M_{l_{1}, l_{1}, x}-\frac{4}{3} \sum_{k} G_{x}^{k} M_{l_{1}, k}-\frac{4}{3} \sum_{k} G^{k} M_{l_{1}, k, x}+ \\
& +\frac{1}{3} \sum_{k} H_{k, x}^{l_{1}} L_{l_{1}, l_{1}}^{k}+\frac{1}{3} \sum_{k} H_{k}^{l_{1}} L_{l_{1}, l_{1}, x}^{k}-\frac{1}{3} \sum_{k} H_{l_{1}, x}^{k} L_{l_{1}, k}^{l_{1}}-\frac{1}{3} \sum_{k} H_{l_{1}}^{k} L_{l_{1}, k, x}^{l_{1}}+ \\
& +\frac{1}{2} \sum_{k} H_{l_{1}, x}^{k} L_{k, k}^{k}+\frac{1}{2} \sum_{k} H_{l_{1}}^{k} L_{k, k, x}^{k}+\frac{1}{2} \sum_{k} H_{l_{1}, x}^{k} \Theta^{k}+\frac{1}{2} \sum_{k} H_{l_{1}}^{k} \underline{\Theta_{x}^{k}}- \\
& -\frac{1}{2} L_{l_{1}, l_{1}, x} \Theta^{0}-\frac{1}{2} L_{l_{1}, l_{1}} \underline{\Theta_{x}^{0}}-\frac{1}{2} \underline{\Theta_{x}^{0}} \Theta^{l_{1}}-\frac{1}{2} \Theta^{0} \underline{\Theta_{x}^{l_{1}}} .
\end{aligned}
$$

Here, we underline twice the second order terms. Also, we have underlined once the six terms: $\underline{\Theta_{y^{l_{1}}}^{k}}, \underline{\Theta_{y_{1}}^{0}}, \underline{\Theta_{x}^{k}}, \underline{\Theta_{x}^{0}}, \underline{\Theta_{x}^{0}}$ and $\underline{\Theta_{x}^{l_{1}}}$. They must be
replaced by their values given in (3.99), (3.100), (3.101) and (3.102). In this replacement, some double sums appear. As before, we use the first index $k=1, \ldots, m$ for single summation and then the second index $p=1, \ldots, m$ for double summation. Finally, we put all the second order terms in the first line, not disturbing the order of appearance of the 73 remaining terms. We get:

$$
+\underline{\underline{\frac{2}{3}} L_{l_{1}, l_{1}, x}^{l_{1}} \Theta^{0}}(\mathrm{~g})-\underline{\frac{1}{3} H_{l_{1}, y_{1}}^{l_{1}} \Theta^{0}}(\mathrm{~h})
$$

$$
+\underline{\frac{2}{3} G^{l_{1}} M_{l_{1}, l_{1}} \Theta^{0}}+\underline{(i)} \underline{\frac{4}{3} \sum_{k} G^{k} M_{l_{1}, k} \Theta^{0}}-\underline{\frac{1}{3} \sum_{k} H_{k}^{l_{1}} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{0}}+\underline{\frac{1}{3} \sum_{k} H_{l_{1}}^{k} L_{l_{1}, k}^{l_{1}} \Theta^{0}}-
$$

$$
-\underline{\frac{1}{2} \sum_{k} H_{l_{1}}^{k} L_{k, k}^{k} \Theta^{0}}-\underline{(1)}-\underline{\frac{1}{2} \sum_{k} H_{l_{1}}^{k} \Theta^{k} \Theta^{0}}+\underline{\frac{1}{2} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{0} \Theta^{0}}(\mathbb{n})+\underline{\frac{1}{2} \Theta^{0} \Theta^{0} \Theta^{l_{1}}} \text { (o) }
$$

$$
-\frac{\frac{2}{3} G_{x}^{l_{1}} M_{l_{1}, l_{1}}}{1}-\underline{\frac{2}{3} G^{l_{1}} M_{l_{1}, l_{1}, x}} \underset{17}{-\frac{4}{3} \sum_{k} G_{x}^{k} M_{l_{1}, k}}-2-\frac{4}{3} \sum_{k} G^{k} M_{l_{1}, k, x} \quad 1+
$$

$$
+\frac{\frac{1}{3} \sum_{k} H_{k, x}^{l_{1}} L_{l_{1}, l_{1}}^{k}}{9}+\frac{1}{3} \sum_{k} H_{k}^{l_{1}} L_{l_{1}, l_{1}, x}^{k}-\frac{1}{3} \sum_{k}^{14} H_{l_{1}, x}^{k} L_{l_{1}, k}^{l_{1}}-\sqrt{7} \xrightarrow[k]{-\frac{1}{3} \sum_{k} H_{l_{1}}^{k} L_{l_{1}, k, x}^{l_{1}}}+
$$

$$
+\frac{\frac{1}{2} \sum_{k} H_{l_{1}, x}^{k} L_{k, k}^{k}}{8}+\frac{1}{2} \sum_{k} H_{l_{1}}^{k} L_{k, k, x}^{k}+15 \xrightarrow[k]{\frac{1}{2} \sum_{k} H_{l_{1}, x}^{k} \Theta^{k}-}
$$

$$
\begin{align*}
& 0=\underline{-2 G_{y^{l_{1}} y^{l_{1}}}^{l_{1}}+\frac{4}{3} H_{l_{1}, x y^{l_{1}}}^{l_{1}}-\frac{2}{3} L_{l_{1}, l_{1}, x x}^{l_{1}}}+ \tag{4.2}
\end{align*}
$$

$$
\begin{aligned}
& -\frac{1}{2} \sum_{k} H_{l_{1, y^{l_{1}}}^{k}}^{l} H_{k}^{l_{1}} \quad-\frac{1}{2} \sum_{k} H_{l_{1}}^{k} H_{k, y^{l_{1}}}^{l_{1}} \quad-\sum_{k}^{11} \xrightarrow{10} G_{y^{l_{1}}}^{k} \Theta^{k}+ \\
& +\sum_{1}^{\sum_{k} G^{k} L_{k, k, y^{l_{1}}}^{k} \text { (a) }-2 \sum_{k} G^{k} M_{k, l_{1}, x}-\sum_{k} \sum_{p} G^{k} H_{k}^{p} M_{l_{1}, p} \quad-\quad 19} \\
& -\underline{\underbrace{}_{k}} \begin{array}{r}
\frac{1}{2} \sum_{k} G^{k} L_{k, k}^{k} L_{l_{1}, l_{1}}^{l_{1}} \\
\text { (b) } \\
+\sum_{k} \sum_{p} G^{k} L_{k, l_{1}}^{p} L_{p, p}^{p}-\frac{1}{2} \sum_{k} G^{k} L_{k, k}^{k} \Theta^{l_{1}}- \\
\text { (c) }
\end{array}
\end{aligned}
$$

$$
-\frac{1}{3} \sum_{k} H_{l_{1}}^{k} H_{k, y^{k}}^{k}+\frac{1}{6} \sum_{k} H_{l_{1}}^{k} L_{k, k, x}^{k}+\frac{2}{3} \sum_{k} H_{l_{1}}^{k} G^{k} M_{k, k}+
$$

$$
+\frac{1}{3} \sum_{k} \sum_{p} H_{l_{1}}^{k} G^{p} M_{k, p}-\frac{1}{3} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{p}^{k} L_{k, k}^{p}+\frac{1}{3} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{k}^{p} L_{k, p}^{k}-
$$

$$
-\frac{1}{4} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{k}^{p} L_{p, p}^{p}-\frac{1}{4} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{k}^{p} \Theta^{p}+\frac{1}{4} \sum_{k} H_{l_{1}}^{k} L_{k, k}^{k} \Theta^{0}+
$$

$$
+\underline{\frac{1}{4} \sum_{k} H_{l_{1}}^{k} \Theta^{0} \Theta^{k}-\underline{\frac{1}{2} L_{l_{1}, l_{1}, x}^{l_{1}} \Theta^{0}}(\mathbb{g})+}
$$

$$
\left.+\underline{G_{y^{1}}^{l_{1}} L_{l_{1}, l_{1}}^{l_{1}}}\right]^{-\frac{1}{2} H_{l_{1}, x}^{l_{1}} L_{l_{1}, l_{1}}^{l_{1}}}{ }^{6}-
$$

$$
\xlongequal[\sum_{k} G^{k} L_{l_{1}, l_{1}}^{l_{1}} L_{l_{1}, k}^{l_{1}}]{\underline{22}}+\underset{k}{\frac{1}{2} \sum_{k} G^{k} L_{l_{1}, l_{1}}^{l_{1}} L_{k, k}^{k}}+{ }^{(b)} \xrightarrow[k]{\frac{1}{4} \sum_{k} H_{l_{1}}^{k} H_{k}^{l_{1}} L_{l_{1}, l_{1}}^{l_{1}}}+
$$

$$
+\underline{G_{y^{l_{1}}}^{l_{1}} \Theta^{l_{1}}}(\beta) \underline{\frac{1}{2} H_{l_{1}, x}^{l_{1}} \Theta^{l_{1}}}(\delta)
$$

$$
-\underline{\frac{1}{4} \Theta^{0} \Theta^{0} \Theta^{l_{1}}}{ }^{0}+
$$

$$
+{\underline{\frac{1}{3}} H_{l_{1}, y^{l_{1}}}^{l_{1}} \Theta^{0}}_{(h}-\underline{\frac{1}{6} L_{l_{1}, l_{1}, x}^{l_{1}} \Theta^{0}}-
$$

$$
-\underline{\frac{2}{3} G^{l_{1}} M_{l_{1}, l_{1}} \Theta^{0}}-\underline{(i)}-\underline{\frac{1}{3} \sum_{k} G^{k} M_{l_{1}, k} \Theta^{0}}+\underbrace{\frac{1}{3} \sum_{k} H_{k}^{l_{1}} L_{l_{1}, l_{1}}^{k} \Theta^{0}}_{\text {(e) }}-\underline{(\mathrm{j}}-\underline{\frac{1}{3} \sum_{k} H_{l_{1}}^{k} L_{l_{1}, k}^{l_{1}} \Theta^{0}}+
$$

$$
+\underline{\frac{1}{4} \sum_{k} H_{l_{1}}^{k} L_{k, k}^{k} \Theta^{0}}+\underline{\frac{1}{4} \sum_{k} H_{l_{1}}^{k} \Theta^{0} \Theta^{k}}-\underline{(1)}-\underline{\frac{1}{4} L_{l_{1}, l_{1}}^{l_{1}} \Theta^{0} \Theta^{0}}(\mathbb{n})-\underline{\frac{1}{4} \Theta^{0} \Theta^{0} \Theta^{l_{1}}}(0)
$$

As usual, all the terms underlined with the 15 roman alphabetic letters $\mathrm{a}, \mathrm{b}, \ldots, \mathrm{n}$, o appended vanish evidently. Furthermore, we claim that the eight terms underlined with the 8 Greek alphabetic letters $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta$
and $\theta$ also vanish:

$$
\begin{align*}
0=?= & -\sum_{k} G_{y^{1_{1}}}^{k} \Theta^{k}+G_{y^{l_{1}}}^{l_{1}} \Theta^{l_{1}}+\frac{1}{2} \sum_{k} H_{l_{1}, x}^{k} \Theta^{k}-\frac{1}{2} H_{l_{1}, x}^{l_{1}} \Theta^{l_{1}}+  \tag{4.3}\\
& +\sum_{k} \sum_{p} G^{k} L_{k, l_{1}}^{p} \Theta^{p}-\sum_{k} G^{k} L_{l_{1}, k}^{l_{1}} \Theta^{l_{1}}-\frac{1}{4} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{k}^{p} \Theta^{p}+\frac{1}{4} \sum_{k} H_{l_{1}}^{k} H_{k}^{l_{1}} \Theta^{l_{1}} .
\end{align*}
$$

Indeed, it suffices to observe that this identity coincides with

$$
\begin{equation*}
0=\frac{1}{2} \sum_{k} \Theta^{k}\left(\left.(3.106)\right|_{j:=k ; l_{1}:=l_{1} ; l_{2}:=l_{1}}\right) . \tag{4.4}
\end{equation*}
$$

Simplifying then (4.2), we get the explicit formulation of the first family of compatibility conditions for the second auxiliary system:

$$
\begin{align*}
0=? & =\frac{2 G_{y^{l_{1}} y_{1} l_{1}}^{l_{1}}+\frac{4}{3} H_{l_{1}, x y l_{1}}^{l_{1}}-\frac{2}{3} L_{l_{1}, l_{1}, x x}^{l_{1}}-}{} \\
& -\frac{2}{3} G_{x}^{l_{1}} M_{l_{1}, l_{1}}-\frac{4}{3} \sum_{k} G_{x}^{k} M_{l_{1}, k}+G_{y^{l_{1}}}^{l_{1}} L_{l_{1}, l_{1}}^{l_{1}}+ \\
& +2 \sum_{k} G_{y^{l_{1}}}^{k} L_{l_{1}, k}^{l_{1}}-\sum_{k} G_{y^{l_{1}}}^{k} L_{k, k}^{k}- \\
& -\frac{1}{2} H_{l_{1}, x}^{l_{1}} L_{l_{1}, l_{1}}^{l_{1}}-\frac{1}{3} \sum_{k} H_{l_{1}, x}^{k} L_{l_{1}, k}^{l_{1}}+\frac{1}{2} \sum_{k} H_{l_{1}, x}^{k} L_{k, k}^{k}+  \tag{4.5}\\
& +\frac{1}{3} \sum_{k} H_{k, x}^{l_{1}} L_{l_{1}, l_{1}}^{k}-\frac{1}{2} \sum_{k} H_{l_{1, y} l_{1}}^{k} H_{k}^{l_{1}}-\frac{1}{2} \sum_{k} H_{k, y^{l_{1}}}^{l_{1}} H_{l_{1}-}^{k}- \\
& -\frac{1}{3} \sum_{k} H_{k, y^{k}}^{k} H_{l_{1}}^{k}-
\end{align*}
$$

$$
\begin{aligned}
& -\frac{1}{3} \sum_{k} L_{l_{1}, k, x}^{l_{1}} H_{l_{1}}^{k}+\frac{1}{3} \sum_{k} L_{l_{1}, l_{1}, x}^{k} H_{k}^{l_{1}}+\frac{2}{3} \sum_{k} L_{k, k, x}^{k} H_{l_{1}}^{k}+ \\
& +2 \sum_{k} L_{l_{1}, k, y^{l_{1}}}^{l_{1}} G^{k}- \\
& -\frac{2}{3} M_{l_{1}, l_{1}, x} G^{l_{1}}-\frac{10}{3} \sum_{k} M_{k, l_{1}, x} G^{k}- \\
& -\sum_{k} \sum_{p} G^{k} H_{k}^{p} M_{l_{1}, p}+\frac{2}{3} \sum_{k} G^{k} H_{l_{1}}^{k} M_{k, k}+\frac{1}{3} \sum_{k} \sum_{p} G^{p} H_{l_{1}}^{k} M_{k, p}- \\
& -\sum_{k} G^{k} L_{l_{1}, k}^{l_{1}} L_{l_{1}, l_{1}}^{l_{1}}+\sum_{k} \sum_{p} G^{k} L_{k, l_{1}}^{p} L_{p, p}^{p}- \\
& -\frac{1}{3} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{p}^{k} L_{k, k}^{p}+\frac{1}{3} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{k}^{p} L_{k, p}^{k}- \\
& -\frac{1}{4} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{k}^{p} L_{p, p}^{p}+\frac{1}{4} \sum_{k} H_{l_{1}}^{k} H_{k}^{l_{1}} L_{l_{1}, l_{1}}^{l_{1}} .
\end{aligned}
$$

We can now state the main technical lemma of this section and of this paper.
Lemma 4.6. The second order partial differential relations (4.5) hold true for $l=1, \ldots, m$, and they are a consequence, by differentiations and by linear combinations, of the fundamental first order partial differential equations (3.106), (3.108), (3.110) and (3.96).
4.7. Reconstitution of the appropriate linear combinations. The remaining of Section 4 is entirely devoted to the proof of this statement. From the manual computational point of view, the difficulty of the task is due to the fact that one has to manipulate formal expressions having from 10 to 50 terms. So the real question is: how can we reconstitute the linear combinations and the differentiations which lead to the goal (4.5) from the data (3.106), (3.108), (3.110) and (3.96)?.

The main trick is to first neglect the first order and the zero order terms in the goal (4.5). Using the symbol " $\equiv$ " to denote "modulo first order and the zero order terms", we formulate the following sub-goal:

$$
\begin{equation*}
0 \equiv ? \equiv \underline{\underline{-2} G_{y^{l_{1}} y^{l_{1}}}^{l_{1}}+\frac{4}{3} H_{l_{1}, x y^{l_{1}}}^{l_{1}}-\frac{2}{3} L_{l_{1}, l_{1}, x x}^{l_{1}}}, \tag{4.8}
\end{equation*}
$$

for $l_{1}=1, \ldots, m$. Before estabilishing that these partial differential relations are a consequence of the data (3.106), (3.108), (3.110) and (3.96) (written with a similar sign $\equiv$ ), let us check that they are a consequence of the existence of the change of coordinates $(x, y) \mapsto(X, Y)$ (however, recall that, in establishing the reverse implications of §3.111, we still do not know that such a change of coordinates really exists); this will confirm the coherence and the validity of our computations. Importantly, we have
been able to achieve systematic corrections of our computations by always checking them alongside with the existence of the change of coordinates $(x, y) \mapsto(X, Y)$.

Coming back to the definition (3.35) and to the approximation (3.58), we have:

$$
\begin{align*}
G^{l_{1}} & =-\square_{x x}^{l_{1}} \cong-Y_{x x}^{l_{1}}, \\
H_{l_{1}}^{l_{1}} & =-2 \square_{x y^{l_{1}}}^{l_{1}}+\square_{x x}^{0} \cong-2 Y_{x y^{l_{1}}}^{l_{1}}+X_{x x},  \tag{4.9}\\
L_{l_{1}, l_{1}}^{l_{1}} & =-\square_{y^{l_{1}} y^{l_{1}}}^{l_{1}}+2 \square_{x y^{l_{1}}}^{0} \cong-Y_{y^{1} y^{l_{1}}}^{l_{1}}+2 X_{x y^{l_{1}}} .
\end{align*}
$$

Differentiating the first two lines with respect to $y^{l_{1}}$ and the third line with respect to $x$, and replacing the sign $\cong$ by the sign $\equiv$ (in a non-rigorous way, this corresponds essentially to neglecting the derivatives of order $0,1,2$ and 3 of $X, Y^{j}$ and to neglecting the difference between the Jacobian matrix of the transformation and the identity matrix), we get:

$$
\begin{align*}
& \xlongequal{G_{y^{l_{1}} y_{1}^{l_{1}}}^{l_{1}}} \equiv-Y_{x x y^{\prime} y^{l_{1}}}^{l_{1}}, \\
& \xlongequal{\overline{H_{l_{1}, x y^{l_{1}}}^{l_{1}}}} \equiv-2 Y_{x y^{l_{1} x y^{l_{1}}}}^{l_{1}}+X_{x x x y^{l_{1}}} \text {, }  \tag{4.10}\\
& \xlongequal{L_{l_{1}, l_{1}, x x}^{l_{1}}} \equiv-Y_{y^{1} y^{l_{1}}{ }^{l_{1}}}^{l_{1}}+2 X_{x y}{ }^{l_{1} x x} \text {. }
\end{align*}
$$

Hence the linear combination $-2 \cdot(4.10)_{1}+\frac{4}{3} \cdot(4.10)_{2}-\frac{2}{3}(4.10)_{3}$ yields the desired result:

$$
\begin{align*}
& 0 \equiv ? \equiv \underline{\underline{-2} G_{y^{l_{1}} y^{l_{1}}}^{l_{1}}+\frac{4}{3} H_{l_{1}, x y^{l_{1}}}^{l_{1}}-\frac{2}{3}} L_{l_{1}, l_{1}, x x}^{l_{1}}  \tag{4.11}\\
& \equiv \underline{2 Y_{x x y^{l_{1} y_{1}}}^{l_{1}} \text { (a) }}-\underline{\frac{8}{3} Y_{x y^{l_{1} x y^{l_{1}}}}^{l_{1}} \text { (a) }}+\underline{\frac{4}{3} X_{x x x y^{l_{1}}}} \text { (b) } \underline{\frac{2}{3} Y_{y^{l_{1} y^{l_{1} x x}}{ }^{l_{1}}}^{\text {(a) }}-\underline{\frac{4}{3} X_{x y^{l_{1} x x}}} \text { (b) }} \\
& \equiv 0, \quad \text { indeed! }
\end{align*}
$$

Thanks to this straightforward computation, we guess that the approximate partial differential relations (4.8) are a consequence of the approximate relations (3.106), (3.108), (3.110) and (3.96), namely:
(4.12)

$$
\begin{aligned}
(3.106)^{\bmod }: & 0 \equiv-2 G_{y_{1}}^{j}+2 \delta_{l_{1}}^{j} G_{y^{l_{2}}}^{l_{2}}+H_{l_{1}, x}^{j}-\delta_{l_{1}}^{j} H_{l_{2}, x}^{l_{2}}, \\
(3.108)^{\bmod }: & 0 \equiv- \\
& -\frac{1}{2} H_{l_{1}, y^{l_{2}}}^{j}+\frac{1}{6} \delta_{l_{1}}^{j} H_{l_{2}, y^{l_{2}}}^{l_{2}}+\frac{1}{3} \delta_{l_{2}}^{j} H_{l_{1}, y_{1}}^{l_{1}}+ \\
& +L_{l_{1}, l_{2}, x}^{j}-\frac{1}{3} \delta_{l_{1}}^{j} L_{l_{2}, l_{2}, x}^{l_{2}}-\frac{2}{3} \delta_{l_{2}}^{j} L_{l_{1}, l_{1}, x}^{l_{1}}, \\
(3.110)^{\bmod }: & 0 \equiv L_{l_{1}, l_{2}, y_{3}}^{j}-L_{l_{1}, l_{3}, y^{l_{2}}}^{j}+\delta_{l_{3}}^{j} M_{l_{1}, l_{2}, x}-\delta_{l_{2}}^{j} M_{l_{1}, l_{3}, x}, \\
(3.96)^{\bmod }: & 0 \equiv M_{l_{1}, l_{2}, y_{3}}-M_{l_{1}, l_{3}, y^{l_{2}}} .
\end{aligned}
$$

Here, the sign $\equiv$ means "modulo zero order terms". Before proceding further, recall the correspondence between partial differential relations:

$$
\begin{align*}
(\mathrm{I}) & =(3.106), \\
(\mathrm{II}) & =(3.108), \\
(\mathrm{III}) & =(3.110),  \tag{4.13}\\
(\mathrm{IV}) & =(3.96) .
\end{align*}
$$

However, these couples of equivalent identities are written slightly differently, as may be read by comparison. To fix ideas and to facilitate the eyechecking of our subsequent computations, we shall only use and refer to the exact writing of (3.106), of (3.108), of (3.110) and of (3.96).
4.14. Construction of a guide. So we want to show that the approximate relation (4.8) is a consequence, by differentiations and by linear combinations, of the approximate identities (4.12). The interest of working with approximate identities is that formal computations are lightened substantially. After having discovered which linear combinations and which differentiations are appropriate, i.e. after having constructed a "guide", in $\S 4.22$ below, we shall write down the complete computations, including all zero order terms, following our guide.

We shall use two indices $l_{1}$ and $l_{2}$ with $1 \leqslant l_{1}, l_{2} \leqslant m$ and, crucially, $l_{2} \neq l_{1}$. Again, the assumption $m \geqslant 2$ is used strongly.

Firstly, put $j:=l_{1}$ in $(3.106)^{\mathrm{mod}}$ with $l_{2} \neq l_{1}$ and differentiate with respect to $y^{l_{1}}$ :

$$
\begin{equation*}
0 \equiv-2 G_{y^{l_{1}} y^{l_{1}}}^{l_{1}}+2 G_{y^{l_{2}} y^{l_{1}}}^{l_{2}}+H_{l_{1}, x y^{l_{1}}}^{l_{1}}-H_{l_{2}, x y^{l_{1}}}^{l_{2}} . \tag{4.15}
\end{equation*}
$$

Secondly, put $j:=l_{2}$ in $(3.106)^{\text {mod }}$ with $l_{1} \neq l_{2}$ and differentiate with respect to $y^{l_{2}}$ :

$$
\begin{equation*}
0 \equiv-2 G_{y^{l_{1}} y^{l_{2}}}^{l_{2}}+H_{l_{1}, x y^{l_{2}}}^{l_{2}} . \tag{4.16}
\end{equation*}
$$

Thirdly, put $j:=l_{2}$ in $(3.108)^{\mathrm{mod}}$ with $l_{1} \neq l_{2}$ and differentiate with respect to $x$ :

$$
\begin{equation*}
0 \equiv-\frac{1}{2} H_{l_{1}, l^{l_{2} x}}^{l_{2}}+\frac{1}{3} H_{l_{1}, y^{l_{1} x}}^{l_{1}}+L_{l_{1}, l_{2}, x x}^{l_{2}}-\frac{2}{3} L_{l_{1}, l_{1}, x x}^{l_{1}} . \tag{4.17}
\end{equation*}
$$

Fourthly, put $j:=l_{1}$ in $(3.108)^{\text {mod }}$ with $l_{2} \neq l_{1}$ and differentiate with respect to $x$ :

$$
\begin{equation*}
0 \equiv-\frac{1}{2} H_{l_{1}, y^{l_{2} x}}^{l_{1}}+\frac{1}{6} H_{l_{2}, y^{l_{2} x}}^{l_{2}}+L_{l_{1}, l_{2}, x x}^{l_{1}}-\frac{1}{3} L_{l_{2}, l_{2}, x x}^{l_{2}} . \tag{4.18}
\end{equation*}
$$

Fithly, permute the indices $\left(l_{1}, l_{2}\right) \mapsto\left(l_{2}, l_{1}\right)$ :

$$
\begin{equation*}
0 \equiv-\frac{1}{2} H_{l_{2}, y^{l_{1} x}}^{l_{2}}+\frac{1}{6} H_{l_{1}, y^{l_{1} x}}^{l_{1}}+L_{l_{2}, l_{1}, x x}^{l_{2}}-\frac{1}{3} L_{l_{1}, l_{1}, x x}^{l_{1}} . \tag{4.19}
\end{equation*}
$$

Finally, compute the linear combination $(4.15)+(4.16)+2 \cdot(4.17)-2 \cdot(4.19)$ :

$$
\begin{align*}
& 0 \equiv-\underline{-2 G_{y^{l_{1} y_{1}}}^{l_{1}}} \underline{1}+\underline{2 G_{y^{l_{1}} y^{l_{2}}}^{l_{2}}} \text { (a) }+H_{l_{1}, x y^{l_{1}}}^{l_{1}}-\underline{H_{l_{2}, x y^{l_{1}}}^{l_{1}}(b)}- \\
& -\underline{2 G_{y^{l_{1}} y^{l_{2}}}^{l_{2}}} \text { (a) }+\underline{H_{l_{1}, x y^{l_{2}}}^{l_{2}}} \text { (c) } \\
& \left.-\underline{H_{l_{1}, x y^{l_{2}}}^{l_{2}}} \text { (c) }+{\underline{\frac{2}{3}} H_{l_{1}, l_{1}, x y^{l_{1}}}^{l_{1}}}_{2}^{2}+\underline{2 L_{l_{1}, l_{2}, x x}^{l_{2}}}(\text { ( }) ~-\frac{4}{3} L_{l_{1}, l_{1}, x x}^{l_{1}}\right]^{3}+  \tag{4.20}\\
& +\underline{H_{l_{2}, x y^{l_{1}}}^{l_{2}}(\mathrm{~b})}-\frac{\frac{1}{3} H_{l_{1}, x y^{l_{1}}}^{l_{1}}}{2}-\underline{2 L_{l_{2}, l_{1}, x x}^{l_{2}}(\mathrm{~d})}+\underline{\frac{2}{3} L_{l_{1}, l_{1}, x x}^{l_{1}}}{ }_{3} \text {. }
\end{align*}
$$

We indeed get the desired approximate identity:

$$
\begin{equation*}
0 \equiv-2 G_{y^{l_{1}} y^{l_{1}}}^{l_{1}}+\frac{4}{3} H_{l_{1}, x y^{l_{1}}}^{l_{1}}-\frac{2}{3} L_{l_{1}, l_{1}, x x}^{l_{1}} . \tag{4.21}
\end{equation*}
$$

4.22. Complete computation. Now that the guide is constructed, we can achieve the complete computations.

Firstly, put $j:=l_{1}$ in (3.106) with $l_{2} \neq l_{1}$ and differentiate with respect to $y^{l_{1}}$ :

$$
\begin{align*}
0 \equiv & -2 G_{y^{l_{1} y_{1}}+2}^{l_{1}}+2 G_{y^{l_{2}} y^{l_{1}}}^{l_{2}}+H_{l_{1}, x y^{l_{1}}}^{l_{1}}-H_{l_{2}, x y^{l_{1}}}^{l_{2}}+  \tag{4.23}\\
& +2 \sum_{k} G_{y^{l_{1}}}^{k} L_{l_{1}, k}^{l_{1}}+2 \sum_{k} G^{k} L_{l_{1}, k, y^{l_{1}}}^{l_{1}}-2
\end{align*} \sum_{k} G_{y^{l_{1}}}^{k} L_{l_{2}, k}^{l_{2}}-2 \sum_{k} G^{k} L_{l_{2}, k, y^{l_{1}}-}^{l_{2}}-1+\frac{1}{2} \sum_{k} H_{l_{1}, y^{l_{1}}}^{k} H_{k}^{l_{1}}-\frac{1}{2} \sum_{k} H_{l_{1}}^{k} H_{k, y^{l_{1}}}^{l_{1}}+\frac{1}{2} \sum_{k} H_{l_{2}, y_{1}}^{k} H_{k}^{l_{2}}+\frac{1}{2} \sum_{k} H_{l_{2}}^{k} H_{k, y^{l_{1}}}^{l_{2}} .
$$

Secondly, put $j:=l_{2}$ in (3.106) with $l_{1} \neq l_{2}$ and differentiate with respect to $y^{l_{2}}$ :
$0 \equiv \underline{\underline{-2 G_{y^{l_{1}} y^{l_{2}}}^{l_{2}}+H_{l_{1}, x y^{l_{2}}}^{l_{2}}}+}$
$2 \sum_{k} G_{y^{l_{2}}}^{k} L_{l_{1}, k}^{l_{2}}+2 \sum_{k} G^{k} L_{l_{1}, k, y^{l_{2}}}^{l_{2}}-\frac{1}{2} \sum_{k} H_{l_{1}, y^{l_{2}}}^{k} H_{k}^{l_{2}}-\frac{1}{2} \sum_{k} H_{l_{1}}^{k} H_{k, y^{l_{2}}}^{l_{2}}$.

Thirdly, put $j:=l_{2}$ in (3.108) with $l_{1} \neq l_{2}$ and differentiate with respect to $x$ :

$$
\begin{align*}
0 \equiv & -\frac{1}{\underline{2} H_{l_{1}, y^{2} x}^{l_{2}}+\frac{1}{3} H_{l_{1, y} l_{1} x}^{l_{1}}+L_{l_{1}, l_{2}, x x}^{l_{2}}-\frac{2}{3} L_{l_{1}, l_{1}, x x}^{l_{1}}+}  \tag{4.25}\\
& +G_{x}^{l_{2}} M_{l_{1}, l_{2}}+G_{x}^{l_{2}} M_{l_{1}, l_{2}, x}-\frac{2}{3} G_{x}^{l_{1}} M_{l_{1}, l_{1}}-\frac{2}{3} G^{l_{1}} M_{l_{1}, l_{1}, x}- \\
& -\frac{1}{3} \sum_{k} G_{x}^{k} M_{l_{1}, k}-\frac{1}{3} \sum_{k} G^{k} M_{l_{1}, k, x}-\frac{1}{2} \sum_{k} H_{k, x}^{l_{2}} L_{l_{1}, l_{2}}^{k}-\frac{1}{2} \sum_{k} H_{k}^{l_{2}} L_{l_{1}, l_{2}, x}^{k}+ \\
& +\frac{1}{2} \sum_{k} H_{l_{1}, x}^{k} L_{l_{2}, k}^{l_{2}}+\frac{1}{2} \sum_{k} H_{l_{1}}^{k} L_{l_{2}, k, x}^{l_{2}}+\frac{1}{3} \sum_{k} H_{k, x}^{l_{1}} L_{l_{1}, l_{1}}^{k}+\frac{1}{3} \sum_{k} H_{k}^{l_{1}} L_{l_{1}, l_{1}, x}^{k}- \\
& -\frac{1}{3} \sum_{k} H_{l_{1}, x}^{k} L_{l_{1}, k}^{l_{1}}-\frac{1}{3} \sum_{k} H_{l_{1}}^{k} L_{l_{1}, k, x}^{l_{1}} .
\end{align*}
$$

Fourthly, put $j:=l_{1}$ in (3.108) with $l_{2} \neq l_{1}$, differentiate with respect to $x$ and permute the indices $\left(l_{1}, l_{2}\right) \mapsto\left(l_{2}, l_{1}\right)$ :

$$
\begin{align*}
0 \equiv & -\frac{1}{2} H_{l_{2}, y_{1} x}^{l_{2}}+\frac{1}{6} H_{l_{1}, l_{1} x}^{l_{1}}+L_{l_{2}, l_{1}, x x}^{l_{2}}-\frac{1}{3} L_{l_{1}, l_{1}, x x}^{l_{1}}+  \tag{4.26}\\
& +G_{x}^{l_{2}} M_{l_{2}, l_{1}}+G^{l_{2}} M_{l_{2}, l_{1}, x}-\frac{1}{3} G_{x}^{l_{1}} M_{l_{1}, l_{1}}-\frac{1}{3} G^{l_{1}} M_{l_{1}, l_{1}, x}+ \\
& +\frac{1}{3} \sum_{k} G_{x}^{k} M_{l_{1}, k}+\frac{1}{3} \sum_{k} G^{k} M_{l_{1}, k, x}-\frac{1}{2} \sum_{k} H_{k, x}^{l_{2}} L_{l_{2}, l_{1}}^{k}-\frac{1}{2} \sum_{k} H_{k}^{l_{2}} L_{l_{2}, l_{1}, x}^{k}+ \\
& +\frac{1}{2} \sum_{k} H_{l_{2}, x}^{k} L_{l_{1}, k}^{l_{2}}+\frac{1}{2} \sum_{k} H_{l_{2}}^{k} L_{l_{1}, k, x}^{l_{2}}+\frac{1}{6} \sum_{k} H_{k, x}^{l_{1}} L_{l_{1}, l_{1}}^{k}+\frac{1}{6} \sum_{k} H_{k}^{l_{1}} L_{l_{1}, l_{1}, x}^{k}- \\
& -\frac{1}{6} \sum_{k} H_{l_{1}, x}^{k} L_{l_{1}, k}^{l_{1}}-\frac{1}{6} \sum_{k} H_{l_{1}}^{k} L_{l_{1}, k, x}^{l_{1}} .
\end{align*}
$$

Finally, compute the linear combination $(4.23)+(4.24)+2 \cdot(4.25)-2 \cdot(4.26)$ :

$$
\begin{aligned}
0= & -2 G_{y^{l_{1}} y^{l_{1}}}^{l_{1}}+\frac{4}{3} H_{l_{1}, x y^{l_{1}}}^{l_{1}}-\frac{2}{3} L_{l_{1}, l_{1}, x x}^{l_{1}}- \\
& -\frac{2}{3} G_{x}^{l_{1}} M_{l_{1}, l_{1}}-\frac{4}{3} \sum_{k} G_{x}^{k} M_{l_{1}, k}+ \\
& +2 \sum_{k} G_{y^{l_{1}}}^{k} L_{l_{1}, k}^{l_{1}}-2 \sum_{k} G_{y^{1}}^{k} L_{l_{2}, k}^{l_{2}}+2 \sum_{k} G_{y^{l_{2}}}^{k} L_{l_{1}, k}^{l_{2}}+ \\
& +\sum_{k} H_{l_{1}, x}^{k} L_{l_{2}, k}^{l_{2}}+\frac{1}{3} \sum_{k} H_{k, x}^{l_{1}} L_{l_{1}, l_{1}}^{k}-\frac{1}{3} \sum_{k} H_{l_{1}, x}^{k} L_{l_{1}, k}^{l_{1}}-\sum_{k} H_{l_{2}, x}^{k} L_{l_{1}, k}^{l_{2}}+ \\
& +\frac{1}{2} \sum_{k} H_{l_{2}, y^{l_{1}}}^{k} H_{k}^{l_{2}}+\frac{1}{2} \sum_{k} H_{k, y^{l_{1}}}^{l_{2}} H_{l_{2}}^{k}-\frac{1}{2} \sum_{k} H_{l_{1}, y^{l_{1}}}^{k} H_{k}^{l_{1}}- \\
& -\frac{1}{2} \sum_{k} H_{l_{1}}^{k} H_{k, y^{l_{1}}}^{l_{1}}-\frac{1}{2} \sum_{k} H_{l_{1}, y^{l_{2}}}^{k} H_{k}^{l_{1}}-\frac{1}{2} \sum_{k} H_{l_{1}}^{k} H_{k, y^{l_{2}}}^{l_{2}}+ \\
& +\sum_{k} L_{l_{2}, k, x}^{l_{2}} H_{l_{1}}^{k}+\frac{1}{3} \sum_{k} L_{l_{1}, l_{1}, x}^{k} H_{k}^{l_{1}}-\frac{1}{3} \sum_{k} L_{l_{1}, k, x}^{l_{1}} H_{l_{1}}^{k}-\sum_{k} L_{l_{1}, k, x}^{l_{2}} H_{l_{2}}^{k}+ \\
& +2 \sum_{k} L_{l_{1}, k, y^{l_{1}}}^{l_{1}} G^{k}+2 \sum_{k} L_{l_{1}, k, y^{l_{2}}}^{l_{2}} G^{k}-2 \sum_{k} L_{l_{2}, k, y^{l_{1}}}^{l_{2}} G^{k}- \\
& -\frac{2}{3} M_{l_{1}, l_{1}, x} G^{l_{1}}-\frac{4}{3} \sum_{k} M_{l_{1}, k, x} G^{k} .
\end{aligned}
$$

In this partial differential relation, importantly, the second order terms are exactly the same as in our goal (4.5). Unfortunately, the first order and the zero order terms are not the same.
4.28. Formulation of a new goal. Thus, in order to get rid of the second order expression $2 G_{y^{1_{1}} y^{l_{1}}}^{l_{1}}+\frac{4}{3} H_{l_{1}, x y^{l_{1}}}^{l_{1}}-\frac{2}{3} L_{l_{1}, l_{1}, x x}^{l_{1}}$, we substract: (4.5) - (4.27). In the result, we write the first order terms in a certain way, adapted in advance to our subsequent computations. For this substraction yielding (4.29) just below, we have not underlined the terms in (4.5) and in (4.27). However, they may be underlined with a pencil to check that the result (4.29) is
correct. We get:
(4.29)
$0=$ ? $=$

$$
\begin{aligned}
= & -\sum_{k} G_{y^{l_{1}}}^{k} L_{k, k}^{k}+G_{y^{l_{1}}}^{l_{1}} L_{l_{1}, l_{1}}^{l_{1}}+\frac{1}{2} \sum_{k} H_{l_{1}, x}^{k} L_{k, k}^{k}-\frac{1}{2} H_{l_{1}, x}^{l_{1}} L_{l_{1}, l_{1}}^{l_{1}}+ \\
& +\frac{2 \sum_{k} G_{y^{l_{1}}}^{k} L_{l_{2}, k}^{l_{2}}-\sum_{k} H_{l_{1}, x}^{k} L_{l_{2}, k}^{l_{2}}-}{2 \sum_{k} G_{y^{l_{2}}}^{k} L_{l_{1}, k}^{l_{2}}+\sum_{k} H_{l_{2}, x}^{k} L_{l_{1}, k}^{l_{2}}+} \\
& +\frac{\frac{1}{2} \sum_{k} H_{k, y^{l_{2}}}^{l_{2}} H_{l_{1}}^{k}-\frac{1}{3} \sum_{k} H_{k, y^{k}}^{k} H_{l_{1}}^{k}-\sum_{k} L_{l_{2}, k, x}^{l_{2}} H_{l_{1}}^{k}+\frac{2}{3} \sum_{k} L_{k, k, x}^{k} H_{l_{1}}^{k}-}{} \\
& -\frac{\frac{1}{2} \sum_{k} H_{k, y^{l_{1}}}^{l_{2}} H_{l_{2}}^{k}+\sum_{k} L_{l_{1}, k, x}^{l_{2}} H_{l_{2}}^{k}+}{} \\
& +\frac{\frac{1}{2} \sum_{k} H_{l_{1}, y^{l_{2}}}^{k} H_{k}^{l_{2}}-\frac{1}{2} \sum_{k} H_{l_{2}, y^{l_{1}}}^{k} H_{k}^{l_{2}}+}{} \\
& +\frac{2 \sum_{k} L_{l_{2}, k, y^{l_{1}}}^{l_{2}} G^{k}-2 \sum_{l_{1}, k, y^{l_{2}}} G^{k}-2 \sum_{k}^{l_{2}} M_{k, l_{1}, x} G^{k}-}{} \\
& -\sum_{k} \sum_{p} G^{k} H_{k}^{p} M_{l_{1}, p}+\frac{2}{3} \sum_{k} G^{k} H_{l_{1}}^{k} M_{k, k}+\frac{1}{3} \sum_{k} \sum_{p} G^{p} H_{l_{1}}^{k} M_{k, p}- \\
& -\sum_{k} G^{k} L_{l_{1}, l_{1}}^{l_{1}} L_{l_{1}, k}^{l_{1}}+\sum_{k} \sum_{p} G^{k} L_{k, l_{1}}^{p} L_{p, p}^{p}-\frac{1}{3} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{p}^{k} L_{k, k}^{p}+ \\
& +\frac{1}{3} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{k}^{p} L_{k, p}^{k}-\frac{1}{4} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{k}^{p} L_{p, p}^{p}+\frac{1}{4} \sum_{k} H_{l_{1}}^{k} H_{k}^{l_{1}} L_{l_{1}, l_{1}}^{l_{1}} .
\end{aligned}
$$

We have underlined plainly the first order terms appearing in lines $1,2,3,4$, 5, 6 and 7 .
4.30. Reconstitution of the subgoal (4.29) from (3.106), from (3.108) and from (3.110). Now, it suffices to establish that the first order partial differential relations (4.29) for $1 \leqslant l_{1}, l_{2} \leqslant m$ and $l_{2} \neq l_{1}$ (crucial assumption) are a consequence of (3.106), of (3.108) and of (3.110) by linear combinations. The auxiliary index $l_{2}$, which is absent in the goal (4.5), will disappear at the end. Differentiations will not be applied anymore. Also, the partial differential relations (3.96), which were not used above, will neither be used in the sequel. However, they are strongly used in the treatment of the remaining three compatibility conditions $(3.112)_{2},(3.112)_{3}$ and $(3.112)_{4}$, the detail of which we do not copy in the typesetted paper. Finally, the construction of a guide for the subgoal (4.29) may be guessed similarly as in §4.14 above. We shall provide the final computations directly, without any guide:
they consists of the seven partial differential relations (4.32), (4.33), (4.35), (4.37), (4.39), (4.43) and (4.45) below. At the end, we shall make the addition (4.47) below, producing the desired subgoal (4.29) := (4.32) $+(4.33)$ $+(4.35)+(4.37)+(4.39)+(4.43)+(4.45)$, with the numerotation of terms corresponding to the order of appearance of the terms of (4.29), as usual.

Firstly, put $j:=k, l_{1}:=l_{1}$ and $l_{2}:=l_{1}$ in (3.106):
(4.31)

$$
\begin{aligned}
0= & -2 G_{y^{l_{1}}}^{k}+2 \delta_{l_{1}}^{k} G_{y^{l_{1}}}^{l_{1}}+H_{l_{1}, x}^{k}-\delta_{l_{1}}^{k} H_{l_{1}, x}^{l_{1}}+ \\
& +2 \sum_{p} G^{p} L_{l_{1}, p}^{k}-2 \delta_{l_{1}}^{k} \sum_{p} G^{p} L_{l_{1}, p}^{l_{1}}-\frac{1}{2} \sum_{p} H_{l_{1}}^{p} H_{p}^{k}+\frac{1}{2} \delta_{l_{1}}^{k} \sum_{p} H_{l_{1}}^{p} H_{p}^{l_{1}} .
\end{aligned}
$$

Apply the operator $\frac{1}{2} \sum_{k} L_{k, k}^{k}(\cdot)$ to the preceding equality, namely compute $\frac{1}{2} \sum_{k} L_{k, k}^{k} \cdot(4.31)$. This yields:

$$
\begin{align*}
& 0=-\frac{\sum_{k} G_{y^{l_{1}}}^{k} L_{k, k}^{k}+G_{y^{l_{1}}}^{l_{1}}+\frac{1}{2} \sum_{k} H_{l_{1}, x}^{k} L_{k, k}^{k}-\frac{1}{2} H_{l_{1}, x}^{l_{1}} L_{l_{1}, l_{1}}^{l_{1}}+}{}  \tag{4.32}\\
&+\sum_{k} \sum_{p} G^{p} L_{k, k}^{k} L_{l_{1}, p}^{k}-\sum_{p} G^{p} L_{l_{1}, l_{1}}^{l_{1}} L_{l_{1}, p}^{k}-\frac{1}{4} \sum_{k} \sum_{p} H_{l_{1}}^{p} H_{p}^{k} L_{k, k}^{k}+ \\
&+\frac{1}{4} \sum_{p} H_{l_{1}}^{p} H_{p}^{l_{1}} L_{l_{1}, l_{1}}^{l_{1}} .
\end{align*}
$$

Secondly, apply the operator $-\sum_{k} L_{l_{2}, k}^{l_{2}}(\cdot)$ to (4.32), namely compute $-\sum_{k} L_{l_{2}, k}^{l_{2}} \cdot(4.32)$. This yields:

$$
\begin{align*}
0= & 2 \sum_{k} G_{y_{1}}^{k} L_{l_{2}, k}^{l_{2}}-2 G_{y_{1}^{l_{1}}}^{l_{1}} L_{l_{2}, l_{1}}^{l_{2}}-\sum_{k} H_{l_{1}, x}^{k} L_{l_{2}, k}^{l_{2}}+H_{l_{1}, x}^{l_{1}} L_{l_{2}, l_{1}}^{l_{2}}-  \tag{33}\\
& -2 \sum_{k} \sum_{p} G^{p} L_{l_{2}, k}^{l_{2}} L_{l_{1}, p}^{k}+2 \sum_{p} G^{p} L_{l_{2}, l_{1}}^{l_{2}} L_{l_{1}, p}^{l_{1}}+\frac{1}{2} \sum_{k} \sum_{p} H_{l_{1}}^{p} H_{p}^{k} L_{l_{2}, k}^{l_{2}}- \\
& -\frac{1}{2} \sum_{p} H_{l_{1}}^{p} H_{p}^{l_{1}} L_{l_{2}, l_{1}}^{l_{2}} .
\end{align*}
$$

Thirdly, put $j:=k, l_{1}:=l_{2}$ and $l_{2}:=l_{1}$ with $l_{2} \neq l_{1}$ in (3.106):
(4.34)

$$
\begin{aligned}
0= & -2 G_{y^{l_{2}}}^{k}+2 \delta_{l_{2}}^{k} G_{y^{l_{1}}}^{l_{1}}+H_{l_{2}, x}^{k}-\delta_{l_{2}}^{k} H_{l_{1}, x}^{l_{1}}+ \\
& +2 \sum_{p} G^{p} L_{l_{2}, p}^{k}-2 \delta_{l_{2}}^{k} \sum_{p} G^{p} L_{l_{1}, p}^{l_{1}}-\frac{1}{2} \sum_{p} H_{l_{2}}^{p} H_{p}^{k}+\frac{1}{2} \delta_{l_{2}}^{k} \sum_{p} H_{l_{1}}^{p} H_{p}^{l_{1}} .
\end{aligned}
$$

Next, apply the operator $\sum_{k} L_{l_{1}, k}^{l_{2}}(\cdot)$ to (4.34), namely compute $\sum_{k} L_{l_{1}, k}^{l_{2}}$. (4.34). This yields:

$$
\begin{align*}
0= & -2 \sum_{k} G_{y^{l_{2}}}^{k} L_{l_{1}, k}^{l_{2}}+2 G_{y^{l_{1}}}^{l_{1}} L_{l_{1}, l_{2}}^{l_{2}}+\sum_{k} H_{l_{2}, x}^{k} L_{l_{1}, k}^{l_{2}}-H_{l_{1}, x}^{l_{1}} L_{l_{1}, l_{2}}^{l_{2}}+  \tag{4.35}\\
& +2 \sum_{k} \sum_{p} G^{p} L_{l_{2}, p}^{k} L_{l_{1}, k}^{l_{2}}-2 \sum_{p} G^{p} L_{l_{1}, p}^{l_{1}} L_{l_{1}, l_{2}}^{l_{2}}-\frac{1}{2} \sum_{k} \sum_{p} H_{l_{2}}^{p} H_{p}^{k} L_{l_{1}, k}^{l_{2}}+ \\
& +\frac{1}{2} \sum_{p} H_{l_{1}}^{p} H_{p}^{l_{1}} L_{l_{1}, l_{2}}^{l_{2}} .
\end{align*}
$$

Fourthly, put $j:=l_{2}, l_{1}:=k$ and $l_{2}:=l_{2}$ in (3.108):

$$
\begin{align*}
0= & -\frac{1}{2} H_{k, y^{l_{2}}}^{l_{2}}+\frac{1}{6} \delta_{k}^{l_{2}} H_{l_{2}, y^{l_{2}}}^{l_{2}}+\frac{1}{3} H_{k, y^{k}}^{k}+L_{k, l_{2}, x}^{l_{2}}-  \tag{4.36}\\
& -\frac{1}{3} \delta_{l_{2}}^{k} L_{l_{2}, l_{2}, x}^{l_{2}}-\frac{2}{3} L_{k, k, x}^{k}+ \\
& +G^{l_{2}} M_{k, l_{2}}-\frac{1}{3} \delta_{l_{2}}^{k} G^{l_{2}} M_{l_{2}, l_{2}}-\frac{2}{3} G^{k} M_{k, k}+\frac{1}{3} \delta_{k}^{l_{2}} \sum_{p} G^{p} M_{l_{2}, p}- \\
& -\frac{1}{3} \sum_{p} G^{p} M_{k, p}-\frac{1}{2} \sum_{p} H_{p}^{l_{2}} L_{k, l_{2}}^{p}+\frac{1}{2} \sum_{p} H_{k}^{p} L_{l_{2, p}}^{l_{2}}+\frac{1}{6} \delta_{k}^{l_{2}} \sum_{p} H_{p}^{l_{2}} L_{l_{2}, l_{2}}^{p}- \\
& -\frac{1}{6} \delta_{k}^{l_{2}} \sum_{p} H_{l_{2}}^{p} L_{l_{2}, p}^{l_{2}}+\frac{1}{3} \sum_{p} H_{p}^{k} L_{k, k}^{p}-\frac{1}{3} \sum_{p} H_{k}^{p} L_{k, p}^{k} .
\end{align*}
$$

Next, apply the operator $-\sum_{k} H_{l_{1}}^{k}(\cdot)$ to (4.36), namely compute $-\sum_{k} H_{l_{1}}^{k}$. (4.36). This yields:

$$
\begin{aligned}
0= & \frac{1}{2} \sum_{k} H_{k, y^{l_{2}}}^{l_{2}} H_{l_{1}}^{k}-\frac{1}{6} H_{l_{2}, y^{l_{2}}}^{l_{2}} H_{l_{1}}^{l_{2}}-\frac{1}{3} \sum_{k} H_{k, y^{k}}^{k} H_{l_{1}-}^{k}- \\
& -\sum_{k} L_{k, l_{2}, x}^{l_{2}} H_{l_{1}}^{k}+\frac{1}{3} L_{l_{2}, l_{2}, x}^{l_{2}} H_{l_{1}}^{l_{2}}+\frac{2}{3} \sum_{k} L_{k, k, x}^{k} H_{l_{1}}^{k}- \\
& -\sum_{k} G^{l_{2}} H_{l_{1}}^{k} M_{k, l_{2}}+\frac{1}{3} G^{l_{2}} H_{l_{1}}^{l_{2}} M_{l_{2}, l_{2}}+\frac{2}{3} \sum_{k} G^{k} H_{l_{1}}^{k} M_{k, k}- \\
& -\frac{1}{3} \sum_{p} G^{p} H_{l_{1}}^{l_{2}} M_{l_{2}, p}+\frac{1}{3} \sum_{k} \sum_{p} G^{p} H_{l_{1}}^{k} M_{k, p}+\frac{1}{2} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{p}^{l_{2}} L_{k, l_{2}}^{p}- \\
& -\frac{1}{2} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{k}^{p} L_{l_{2}, p}^{l_{2}}-\frac{1}{6} \sum_{p} H_{l_{1}}^{l_{2}} H_{p}^{l_{2}} L_{l_{2}, l_{2}}^{p}+\frac{1}{6} \sum_{p} H_{l_{1}}^{l_{2}} H_{l_{2}}^{p} L_{l_{2}, p}^{l_{2}}- \\
& -\frac{1}{3} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{p}^{k} L_{k, k}^{p}+\frac{1}{3} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{k}^{p} L_{k, p}^{k} .
\end{aligned}
$$

Fithly, put $j:=l_{2}, l_{1}:=k$ and $l_{2}:=l_{1}$ in (3.108):
(4.38)

$$
\begin{aligned}
0= & -\frac{1}{2} H_{k, y^{l_{1}}}^{l_{2}}+\frac{1}{6} \delta_{k}^{l_{2}} H_{l_{1}, y_{1}}^{l_{1}}+L_{k, l_{1}, x}^{l_{2}}-\frac{1}{3} \delta_{k}^{l_{2}} L_{l_{1}, l_{1}, x}^{l_{1}}+ \\
& +G^{l_{2}} M_{k, l_{1}}-\frac{1}{3} \delta_{k}^{l_{2}} G^{l_{1}} M_{l_{1}, l_{1}}+\frac{1}{3} \delta_{k}^{l_{2}} \sum_{p} G^{p} M_{l_{1}, p}-\frac{1}{2} \sum_{p} H_{p}^{l_{2}} L_{k, l_{1}}^{p}+ \\
& +\frac{1}{2} \sum_{p} H_{k}^{p} L_{l_{1}, p}^{l_{2}}+\frac{1}{6} \delta_{k}^{l_{2}} \sum_{p} H_{p}^{l_{1}} L_{l_{1}, l_{1}}^{p}-\frac{1}{6} \delta_{k}^{l_{2}} \sum_{p} H_{l_{1}}^{p} L_{l_{1}, p}^{l_{1}} .
\end{aligned}
$$

Next, apply the operator $\sum_{k} H_{l_{2}}^{k}(\cdot)$ to (4.38), namely compute $\sum_{k} H_{l_{2}}^{k}$. (4.38). This yields:

$$
\begin{align*}
0= & -\frac{1}{2} \sum_{k} H_{k, y^{l_{1}}}^{l_{2}} H_{l_{2}}^{k}+\frac{1}{6} H_{l_{1}, y_{1}}^{l_{1}} H_{l_{2}}^{l_{2}}+\sum_{k} L_{k, l_{1}, x}^{l_{2}} H_{l_{2}}^{k}-  \tag{4.39}\\
& -\frac{1}{3} L_{l_{1}, l_{1}, x}^{l_{1}} H_{l_{2}}^{l_{2}}+ \\
& +\sum_{k} G^{l_{2}} H_{l_{2}}^{k} M_{k, l_{1}}-\frac{1}{3} G^{l_{1}} H_{l_{2}}^{l_{2}} M_{l_{1}, l_{1}}+\frac{1}{3} \sum_{p} G^{p} H_{l_{2}}^{l_{2}} M_{l_{1}, p}- \\
& -\frac{1}{2} \sum_{k} \sum_{p} H_{l_{2}}^{k} H_{p}^{l_{2}} L_{k, l_{1}}^{p}+\frac{1}{2} \sum_{k} \sum_{p} H_{l_{2}}^{k} H_{k}^{p} L_{l_{1}, p}^{l_{2}}+\frac{1}{6} \sum_{p} H_{l_{2}}^{l_{2}} H_{p}^{l_{1}} L_{l_{1}, l_{1}}^{p}- \\
& -\frac{1}{6} \sum_{p} H_{l_{2}}^{l_{2}} H_{l_{1}}^{p} L_{l_{1}, p}^{l_{1}} .
\end{align*}
$$

Sixthly, we form the expression:

$$
\begin{equation*}
\left.(3.108)\right|_{j:=k ; l_{1}:=l_{1} ; l_{2}:=l_{2}}-\left.(3.108)\right|_{j:=k ; l_{1}:=l_{2} ; l_{2}:=l_{1}} . \tag{4.40}
\end{equation*}
$$

Writing term by term the substractions, we get:

$$
\begin{aligned}
& \text { (4.41) } \\
& 0=-\underline{\frac{1}{2} H_{l_{1}, y^{l_{2}}}^{k}} \underline{1}+\underline{\frac{1}{2} H_{l_{2}, y^{l_{1}}}^{k}} \underline{2}+\underline{\frac{1}{6} \delta_{l_{1}}^{k} H_{l_{2}, y^{l_{2}}}^{l_{2}}} \underline{3}-\frac{\frac{1}{6} \delta_{l_{2}}^{k} H_{l_{1}, y^{l_{1}}}^{l_{1}}}{4}+ \\
& \left.+\underline{\frac{1}{3} \delta_{l_{2}}^{k} H_{l_{1}, y^{l_{1}}}^{l_{1}}} 4_{4}-\frac{1}{3} \delta_{l_{1}}^{k} H_{l_{2}, y^{l_{2}}}^{l_{2}}\right]+\underline{L_{l_{1}, l_{2}, x}^{k}} \text { (a) }-\underline{L_{l_{2}, l_{1}, x}^{k}} \text { (a) } \\
& -\underline{\frac{1}{3} \delta_{l_{1}}^{k} L_{l_{2}, l_{2}, x}^{l_{2}}} \underline{5}+\underline{\frac{1}{3} \delta_{l_{2}}^{k} L_{l_{1}, l_{1}, x}^{l_{1}}} \boxed{6}-\underline{\frac{2}{3} \delta_{l_{2}}^{k} L_{l_{1}, l_{1}, x}^{l_{1}}}{ }_{6}^{6}+\frac{2}{3} \delta_{l_{1}}^{k} L_{l_{2}, l_{2}, x}^{l_{2}}{ }_{5}^{5}+ \\
& +\underline{G^{k} M_{l_{1}, l_{2}}(\mathrm{~b})}-\underline{G^{k} M_{l_{2}, l_{1}}}(\mathrm{~b})-\frac{1}{3} \delta_{l_{1}}^{k} G^{l_{2}} M_{l_{2}, l_{2}}\left[7 \underline{\frac{1}{3} \delta_{l_{2}}^{k} G^{l_{1}} M_{l_{1}, l_{1}}}{ }^{8}\right. \\
& -\underline{\frac{2}{3} \delta_{l_{2}}^{k} G^{l_{1}} M_{l_{1}, l_{1}}}+\underset{8}{8}+\frac{\frac{2}{3} \delta_{l_{1}}^{k} G^{l_{2}} M_{l_{2}, l_{2}}}{7}+\frac{\frac{1}{3} \delta_{l_{1}}^{k} \sum_{p} G^{p} M_{l_{2}, p}}{9}-\frac{1}{3} \delta_{l_{2}}^{k} \sum_{p} G^{p} M_{l_{1}, p}-
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{2} \sum_{p} H_{l_{1}}^{p} L_{l_{2}, p}^{k}-11 \xrightarrow[p]{12}+\frac{1}{2} \sum_{p} H_{l_{2}}^{p} L_{l_{1}, p}^{k}+12 \delta_{l_{1}}^{k} \sum_{p} H_{p}^{l_{2}} L_{l_{2}, l_{2}}^{p}-\frac{1}{6} \delta_{l_{2}}^{k} \sum_{p} H_{p}^{l_{1}} L_{l_{1}, l_{1}}^{p}- \\
& -\frac{1}{6} \delta_{l_{1}}^{k} \sum_{p} H_{l_{2}}^{p} L_{l_{2}, p}^{l_{2}}+\frac{1}{6} \delta_{l_{2}}^{k} \sum_{p} H_{l_{1}}^{p} L_{l_{1}, p}^{l_{1}}+16 \xrightarrow{16}+\frac{1}{3} \delta_{l_{2}}^{k} \sum_{p} H_{p}^{l_{1}} L_{l_{1}, l_{1}}^{p}- \\
& -\frac{1}{3} \delta_{l_{1}}^{k} \sum_{p} H_{p}^{l_{2}} L_{l_{2}, l_{2}}^{p}-\frac{1}{3} \delta_{l_{2}}^{k} \sum_{p} H_{l_{1}}^{p} L_{l_{1}, p}^{l_{1}}+\frac{1}{3} \delta_{l_{1}}^{k} \sum_{p} H_{l_{2}}^{p} L_{l_{2}, p}^{l_{2}}{ }^{16}
\end{aligned}
$$

## Simplifying, we get:

(4.42)

$$
\begin{aligned}
0= & \frac{1}{2} H_{l_{1}, y^{l_{2}}}^{k}+\frac{1}{2} H_{l_{2}, y^{l_{1}}}^{k}-\frac{1}{6} \delta_{l_{1}}^{k} H_{l_{2}, y^{l_{2}}}^{l_{2}}+\frac{1}{6} \delta_{l_{2}}^{k} H_{l_{1}, y^{l_{1}}}^{l_{1}}+ \\
& +\frac{1}{3} \delta_{l_{1}}^{k} L_{l_{2}, l_{2}, x}^{l_{2}}-\frac{1}{3} \delta_{l_{2}}^{k} L_{l_{1}, l_{1}, x}^{l_{1}}+ \\
& +\frac{1}{3} \delta_{l_{1}}^{k} G^{l_{2}} M_{l_{2}, l_{2}}-\frac{1}{3} \delta_{l_{2}}^{k} G^{l_{1}} M_{l_{1}, l_{1}}+\frac{2}{3} \delta_{l_{1}}^{k} \sum_{p} G^{p} M_{l_{2}, p}-\frac{2}{3} \delta_{l_{2}}^{k} \sum_{p} G^{p} M_{l_{1}, p}+ \\
& +\frac{1}{2} \sum_{p} H_{l_{1}}^{p} L_{l_{2}, p}^{k}-\frac{1}{2} \sum_{p} H_{l_{2}}^{p} L_{l_{1}, p}^{k}-\frac{1}{6} \delta_{l_{1}}^{k} \sum_{p} H_{p}^{l_{2}} L_{l_{2}, l_{2}}^{p}+\frac{1}{6} \delta_{l_{2}}^{k} \sum_{p} H_{p}^{l_{1}} L_{l_{1}, l_{1}}^{p}+ \\
& +\frac{1}{6} \delta_{l_{1}}^{k} \sum_{p} H_{l_{2}}^{p} L_{l_{2}, p}^{l_{2}}-\frac{1}{6} \delta_{l_{2}}^{k} \sum_{p} H_{l_{1}}^{p} L_{l_{1}, p}^{l_{1}} .
\end{aligned}
$$

Next, apply the operator $-\sum_{k} H_{k}^{l_{2}}(\cdot)$ to (4.42), namely compute $-\sum_{(4.43)} H_{k}^{l_{2}} \cdot(4.42)$. This yields:

$$
\begin{align*}
0= & \frac{1}{2} \sum_{k} H_{l_{1}, y^{l_{2}}}^{k} H_{k}^{l_{2}}-\frac{1}{2} \sum_{k} H_{l_{2}, y^{l_{1}}}^{k} H_{k}^{l_{2}}+\frac{1}{6} H_{l_{2}, y^{l_{2}}}^{l_{2}} H_{l_{1}}^{l_{2}}-  \tag{43}\\
& -\frac{1}{6} H_{l_{1}, y_{1} l_{1}}^{l_{1}} H_{l_{2}}^{l_{2}}-\frac{1}{3} L_{l_{2}, l_{2}, x}^{l_{2}} H_{l_{1}}^{l_{2}}+\frac{1}{3} L_{l_{1}, l_{1}, x}^{l_{1}} H_{l_{2}}^{l_{2}}- \\
& -\frac{1}{3} G^{l_{2}} H_{l_{1}}^{l_{2}} M_{l_{2}, l_{2}}+\frac{1}{3} G^{l_{1}} H_{l_{2}}^{l_{2}} M_{l_{1}, l_{1}}-\frac{2}{3} \sum_{p} G^{p} H_{l_{1}}^{l_{2}} M_{l_{2}, p}+ \\
& +\frac{2}{3} \sum_{p} G^{p} H_{l_{2}}^{l_{2}} M_{l_{1}, p}-\frac{1}{2} \sum_{k} \sum_{p} H_{k}^{l_{2}} H_{l_{1}}^{p} L_{l_{2}, p}^{k}+\frac{1}{2} \sum_{k} \sum_{p} H_{k}^{l_{2}} H_{l_{2}}^{p} L_{l_{1}, p}^{k}+ \\
& +\frac{1}{6} \sum_{p} H_{l_{1}}^{l_{2}} H_{p}^{l_{2}} L_{l_{2}, l_{2}}^{p}-\frac{1}{6} \sum_{p} H_{l_{2}}^{l_{2}} H_{p}^{l_{1}} L_{l_{1}, l_{1}}^{p}-\frac{1}{6} \sum_{p} H_{l_{1}}^{l_{2}} H_{l_{2}}^{p} L_{l_{2}, p}^{l_{2}}+ \\
& +\frac{1}{6} \sum_{p} H_{l_{2}}^{l_{2}} H_{l_{1}}^{p} L_{l_{1}, p}^{l_{1}} .
\end{align*}
$$

Seventhly, put $j:=l_{2}, l_{1}:=k, l_{2}:=l 2$ and $l_{3}:=l_{1}$ in (3.108):
(4.44)

$$
\begin{aligned}
0= & \frac{L_{k, l_{2}, y^{l_{1}}}^{l_{2}}-L_{k, l_{1}, y^{l_{2}}}^{l_{2}}-M_{k, l_{1}, x}+}{} \\
& +\frac{1}{2} H_{l_{1}}^{l_{2}} M_{k, l_{2}}-\frac{1}{2} H_{l_{2}}^{l_{2}} M_{k, l_{1}}+\frac{1}{2} \delta_{k}^{l_{2}} \sum_{p} H_{l_{1}}^{p} M_{l_{2}, p}-\frac{1}{2} \delta_{k}^{l_{2}} \sum_{p} H_{l_{2}}^{p} M_{l_{1}, p}- \\
& -\frac{1}{2} \sum_{p} H_{k}^{p} M_{l_{1}, p}+\sum_{p} L_{k, l_{1}}^{p} L_{l_{2}, p}^{l_{2}}-\sum_{p} L_{k, l_{2}}^{p} L_{l_{1}, p}^{l_{2}},
\end{aligned}
$$

and then apply the operator $2 \sum_{k} G^{k}(\cdot)$ :

$$
\begin{align*}
0= & \sum_{k} L_{k, l_{2}, y_{1}}^{l_{2}} G^{k}-2 \sum_{k} L_{k, l_{1}, y^{l_{2}}}^{l_{2}} G^{k}-2 \sum_{k} M_{k, l_{1}, x} G^{k}+  \tag{4.45}\\
& +\sum_{k} G^{k} H_{l_{1}}^{l_{2}} M_{l_{1}, l_{2}}-\sum_{k} G^{k} H_{l_{2}}^{l_{2}} M_{k, l_{1}}+\sum_{p} G^{l_{2}} H_{l_{1}}^{p} M_{l_{2}, p}- \\
& -\sum_{p} G^{l_{2}} H_{l_{2}}^{p} M_{l_{1}, p}-\sum_{k} \sum_{p} G^{k} H_{k}^{p} M_{l_{1}, p}+2 \sum_{k} \sum_{p} G^{k} L_{k, l_{1}}^{p} L_{l_{2}, p}^{l_{2}}- \\
& -2 \sum_{k} \sum_{p} G^{k} L_{k, l_{2}}^{p} L_{l_{1}, p}^{l_{2}}
\end{align*}
$$

Finally, achieve the addition

$$
\begin{equation*}
(4.32)+(4.33)+(4.35)+(4.37)+(4.39)+(4.43)+(4.45) . \tag{4.46}
\end{equation*}
$$

We copy these seven formal expression, we underline the vanishing terms and we number the remaining terms so as to respect the order of appearance
of the terms of the subgoal (4.29):

## (4.47)

$$
\begin{aligned}
& +\sum_{k} \sum_{p} G^{p} L_{k, k}^{k} L_{l_{1}, p}^{k}-\sum_{p} G^{p} L_{l_{1}, l_{1}}^{l_{1}} L_{l_{1}, p}^{k} \quad-\frac{1}{4} \sum_{k} \sum_{p} H_{l_{1}}^{p} H_{p}^{k} L_{k, k}^{k} \quad+ \\
& +{ }^{\frac{1}{4}} \sum_{p} H_{l_{1}}^{p} H_{p}^{l_{1}} L_{l_{1}, l_{1}}^{l_{1}}+
\end{aligned}
$$

$$
\begin{aligned}
& +2 \sum_{k} G_{y_{1}}^{k} L_{l_{2}, k}^{l_{2}} \underbrace{-2 G_{y_{1}}^{l_{1}} L_{l_{2}, l_{1}}^{l_{2}}}_{5} \text { (a) }-\xrightarrow{\sum_{k} H_{l_{1}, x}^{k} L_{l_{2}, k}^{l_{2}}}+\underline{6}+\xrightarrow[l_{l_{1}, x}^{l_{1}} L_{l_{2}, l_{1}}^{l_{2}} \text { (b) }]{ }- \\
& \underline{-2 \sum_{k} \sum_{p} G^{p} L_{l_{2}, k}^{l_{2}} L_{l_{1}, p}^{k}}+(\mathrm{g}) \underline{p} \sum_{p}^{p} G_{l_{2}, l_{1}}^{l_{2}} L_{l_{1}, p}^{l_{1}}\left(\mathrm{~h}^{-2}+\frac{1}{2} \sum_{k} \sum_{p} H_{l_{1}}^{p} H_{p}^{k} L_{l_{2}, k}^{l_{2}}-\right. \\
& -\underline{\underbrace{}_{p}}{ }^{\frac{1}{2} \sum_{l_{1}}^{p} H_{p}^{l_{1}} L_{l_{2}, l_{1}}^{l_{2}}-}
\end{aligned}
$$

$$
\begin{aligned}
& +\underbrace{\frac{1}{2} \sum_{p} H_{l_{1}}^{p} H_{p}^{l_{1}} L_{l_{1}, l_{2}}^{l_{2}}+}
\end{aligned}
$$

$$
\begin{aligned}
& +\underbrace{+\frac{1}{2} \sum_{k, y^{2}} H_{l_{1}}^{l_{2}} H_{l_{1}}^{k}}_{\sqrt{9}}-\frac{\frac{1}{6} H_{l_{2}, y^{l_{2}}}^{l_{2}} H_{l_{1}}^{l_{2}}}{(c)}-\frac{\frac{1}{3} \sum_{k} H_{k, y^{k}}^{k} H_{l_{1}}^{k}}{10}
\end{aligned}
$$

$$
\begin{aligned}
& \underline{-\sum_{k} G^{l_{2}} H_{l_{1}}^{k} M_{k, l_{2}}}+\underline{\frac{1}{3} G^{l_{2}} H_{l_{1}}^{l_{2}} M_{l_{2}, l_{2}}}(\curvearrowleft) \underline{\frac{2}{3} \sum_{k} G^{k} H_{l_{1}}^{k} M_{k, k}-} \\
& -\frac{\frac{1}{3} \sum_{p} G^{p} H_{l_{1}}^{l_{2}} M_{l_{2}, p}}{-(0)+\frac{1}{3} \sum_{k} \sum_{p} G^{p} H_{l_{1}}^{k} M_{k, p}}+\frac{1}{2} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{p}^{l_{2}} L_{k, l_{2}}^{p}-
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{3} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{p}^{k} L_{k, k}^{p}+\frac{1}{3} \sum_{k} \sum_{p} H_{l_{1}}^{k} H_{k}^{p} L_{k, p}^{k}-
\end{aligned}
$$

$$
\begin{aligned}
& \underline{-\frac{1}{2} \sum_{k} H_{k, y^{l_{1}}}^{l_{2}} H_{l_{2}}^{k}}+\underset{13}{\frac{1}{6} H_{l_{1}, y^{l_{1}}}^{l_{1}} H_{l_{2}}^{l_{2}}}(\text { e }) \xrightarrow[\sum_{k,}]{L_{k, l_{1}, x}^{l_{2}} H_{l_{2}}^{k}}- \\
& -\underline{\frac{1}{3} L_{l_{1}, l_{1}, x}^{l_{1}} H_{l_{2}}^{l_{2}}}(\ddagger
\end{aligned}
$$

$$
\begin{aligned}
& \underline{-\frac{1}{2} \sum_{k} \sum_{p} H_{l_{2}}^{k} H_{p}^{l_{2}} L_{k, l_{1}}^{p}+\left(\mathbb{v} \sum_{k}^{\frac{1}{2}} \sum_{p} \sum_{l_{2}}^{k} H_{k}^{p} L_{l_{1}, p}^{l_{2}}+\left(\mathbb{1}+\sum_{p} H_{l_{2}}^{l_{2}} H_{p}^{l_{1}} L_{l_{1}, l_{1}}^{p}-\right.\right.} \\
& \underbrace{-\frac{1}{6} \sum_{p} H_{l_{2}}^{l_{2}} H_{l_{1}}^{p} L_{l_{1}, p}^{l_{1}}+}
\end{aligned}
$$

$$
\begin{aligned}
& +\underbrace{\frac{1}{2} \sum_{15} H_{l_{1}, y^{l_{2}}}^{k} H_{k}^{l_{2}}-\frac{1}{2} \sum_{k} H_{l_{2}, y^{l_{1}}}^{k} H_{k}^{l_{2}}+\frac{1}{16} H_{l_{2}, y^{l_{2}}}^{l_{2}} H_{l_{1}}^{l_{2}}(c)}_{k} \\
& -\underline{\frac{1}{6} H_{l_{1}, y_{1}}^{l_{1}} H_{l_{2}}^{l_{2}}}\left(\text { e }-\underline{\frac{1}{3} L_{l_{2}, l_{2}, x}^{l_{2}} H_{l_{1}}^{l_{2}}}\left(\text { (d) }+\underline{\frac{1}{3} L_{l_{1}, l_{1}, x}^{l_{1}} H_{l_{2}}^{l_{2}}}(\underset{\text { f }}{ }-\right.\right. \\
& -\underline{\frac{1}{3} G^{l_{2}} H_{l_{1}}^{l_{2}} M_{l_{2}, l_{2}}}(\mathbb{n}) \underline{\frac{1}{3} G^{l_{1}} H_{l_{2}}^{l_{2}} M_{l_{1}, l_{1}}}(\mathbb{t}) \underline{\left(\frac{2}{3} \sum_{p} G^{p} H_{l_{1}}^{l_{2}} M_{l_{2}, p}\right.}+ \\
& +\underbrace{\frac{2}{3} \sum_{p} G^{p} H_{l_{2}}^{l_{2}} M_{l_{1}, p}-\frac{1}{2} \sum_{k} \sum_{p} H_{k}^{l_{2}} H_{l_{1}}^{p} L_{l_{2}, p}^{k}}_{\text {(U) }} \text { (p) }+\underbrace{\frac{1}{2} \sum_{k} \sum_{p} H_{k}^{l_{2}} H_{l_{2}}^{p} L_{l_{1}, p}^{k}+}_{\text {(v) }}
\end{aligned}
$$

$$
\begin{aligned}
& +\underbrace{\frac{1}{6} \sum_{l_{2}} H_{l_{2}}^{l_{2}} H_{l_{1}}^{p} L_{l_{1}, p}^{l_{1}}+}_{\mathbb{p}} \\
& +2 \sum_{k} L_{k, l_{2}, y^{l_{1}}}^{l_{2}} G^{k} \quad-2 \sum_{k} L_{k, l_{1}, y^{l_{2}}}^{l_{2}} G^{k} \quad-2 \sum_{k}^{18} M_{k, l_{1}, x} G^{k}+ \\
& +\underbrace{\text { (U) }}_{\left(\sum_{k} G^{k} H_{l_{1}}^{l_{2}} M_{l_{1}, l_{2}}-\left(\sum_{k} G^{k} H_{l_{2}}^{l_{2}} M_{k, l_{1}}\right.\right.}+\underline{\sum_{p} G^{l_{2}} H_{l_{1}}^{p} M_{l_{2}, p}}- \\
& -\underbrace{-\sum_{p} G^{l_{2}} H_{l_{2}}^{p} M_{l_{1}, p}}_{p} \text { (5) } \xrightarrow[\sum_{k}]{\sum_{p} \sum_{p} G^{k} H_{k}^{p} M_{l_{1}, p}}+2 \sum_{k} \sum_{p} G^{k} L_{k, l_{1}}^{p} L_{l_{2}, p}^{l_{2}}- \\
& -2 \sum_{k} \sum_{p} G^{k} L_{k, l_{2}}^{p} L_{l_{1}, p}^{l_{2}} \text { (k }
\end{aligned}
$$

In conclusion, there is exact coincidence with the subgoal (4.29). The proof that the first family $(3.112)_{1}$ of compatibility conditions of the second auxiliary system (3.99), (3.100), (3.101) and (3.102) are a consequence of (I), (II), (III) and (IV) of Theorem 1.7 (3) is complete. Granted that the treatment of the other three families of compatibility conditions $(3.112)_{2}$, $(3.112)_{3}$ and $(3.112)_{4}$ is similar (and as well painful), we consider that the proof of the equivalence between (1) and (3) in Theorem 1.7 is complete, now.

## §5. GENERAL FORM OF THE POINT TRANSFORMATION OF THE FREE PARTICLE SYSTEM

This section is devoted to the exposition of a complete proof of Lemma 3.32. To start with, we must develope the fundamental equations (3.10), for $j=1, \ldots, m$. Recalling that the total differentiation
operator is given by $D=\frac{\partial}{\partial x}+\sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot \frac{\partial}{\partial y^{l_{1}}}+\sum_{l_{1}=1}^{m} y_{x x}^{l_{1}} \cdot \frac{\partial}{\partial y_{x}^{l_{1}}}$, we compute first

$$
\left\{\begin{align*}
D D X & =D\left[X_{x}+\sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot X_{y^{l_{1}}}\right]  \tag{5.1}\\
& =X_{x x}+2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot X_{x y^{l_{1}}}+\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot X_{y^{l_{1}} y^{l_{2}}}+\sum_{l_{1}=1}^{m} y_{x x}^{l_{1}} \cdot X_{y^{l_{1}}},
\end{align*}\right.
$$

and
(5.2)

$$
\left\{\begin{aligned}
D D Y^{j} & =D\left[Y_{x}^{j}+\sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot Y_{y^{l_{1}}}^{j}\right] \\
& =Y_{x x}^{j}+2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot Y_{x y^{l_{1}}}^{j}+\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot Y_{y^{l_{1}} y^{l_{2}}}^{j}+\sum_{l_{1}=1}^{m} y_{x x}^{l_{1}} \cdot Y_{y^{l_{1}}}^{j} .
\end{aligned}\right.
$$

Now, we can develope the equation $0=-D Y^{j} \cdot D D X+D X \cdot D D Y^{j}$, which yields

$$
\begin{align*}
& 0=-\left[Y_{x}^{j}+\sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot Y_{y^{l_{1}}}^{j}\right] \cdot\left[X_{x x}+2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot X_{x y^{l_{1}}}+\right. \\
& \left.+\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot X_{y^{l_{1}} y^{l_{2}}}+\sum_{l_{1}=1}^{m} y_{x x}^{l_{1}} \cdot X_{y^{l_{1}}}\right]+  \tag{5.3}\\
& +\left[X_{x}+\sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot X_{y^{l_{1}}}\right] \cdot\left[Y_{x x}^{j}+2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot Y_{x y^{l_{1}}}^{j}+\right. \\
& \left.+\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot Y_{y_{1}^{l_{1}} y^{l_{2}}}^{j}+\sum_{l_{1}=1}^{m} y_{x x}^{l_{1}} \cdot Y_{y^{l_{1}}}^{j}\right]=
\end{align*}
$$

$$
\begin{aligned}
=- & X_{x x} Y_{x}^{j}+Y_{x x}^{j} X_{x}+ \\
& +\sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot\left[-2 X_{x y^{l_{1}}} Y_{x}^{j}+2 Y_{x y^{l_{1}}}^{j} X_{x}-\right. \\
& \left.\quad-X_{x x} Y_{y^{l_{1}}}^{j}+Y_{x x}^{j} X_{y^{l_{1}}}\right]+ \\
+ & \sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot\left[-X_{y^{l_{1}} y^{l_{2}}} Y_{x}^{j}+Y_{y^{l_{1} y^{l_{2}}}}^{j} X_{x}-\right. \\
& \left.-2 X_{x y^{l_{2}}} Y_{y^{l_{1}}}^{j}+2 Y_{x y^{l_{2}}}^{j} X_{y^{l_{1}}}\right]+ \\
& +\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} \sum_{l_{3}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} y_{x}^{l_{3}} \cdot\left[-X_{y^{l_{2} y_{3}}} Y_{y^{l_{1}}}^{j}+Y_{y^{l_{2} y^{l_{3}}}}^{j} X_{y^{l_{1}}}\right]+ \\
& +\sum_{l_{1}=1}^{m} y_{x x}^{l_{1}} \cdot\left[-X_{y^{l_{1}}} Y_{x}^{j}+Y_{y^{l_{1}}}^{j} X_{x}\right]+ \\
& +\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x x}^{l_{1}} y_{x}^{l_{2}} \cdot\left[-X_{y^{l_{1}}} Y_{y^{l_{2}}}^{j}+Y_{y^{l_{1}}}^{j} X_{y^{l_{2}}}\right]
\end{aligned}
$$

The goal is to show that after solving these $m$ equations for $j=1, \ldots, m$ with respect to the $y_{x x}^{l}, l=1, \ldots, m$, one obtains the expression (3.33) of Lemma 3.32, or equivalently, using the $\Delta$ notation instead of the square notation, one obtains

$$
\left\{\begin{align*}
0= & y_{x x}^{j} \cdot \Delta\left(x\left|y^{1}\right| \cdots \mid y^{m}\right)+\Delta\left(x\left|y^{1}\right| \cdots\left|{ }^{j} x x\right| \cdots \mid y^{m}\right)+ \\
& +\sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot\left[2 \Delta\left(x\left|y^{1}\right| \cdots\left|{ }^{j} x y^{l_{1}}\right| \cdots \mid y^{m}\right)-\right. \\
& \left.-\delta_{l_{1}}^{j} \Delta\left(x x\left|y^{1}\right| \cdots \mid y^{m}\right)\right]+  \tag{5.4}\\
& +\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot\left[\Delta\left(x\left|y^{1}\right| \cdots\left|{ }^{j} y^{l_{1}} y^{l_{2}}\right| \cdots \mid y^{m}\right)-\right. \\
& \left.-2 \delta_{l_{1}}^{j} \Delta\left(x y^{l_{2}}\left|y^{1}\right| \cdots \mid y^{m}\right)\right]+ \\
& +\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} \sum_{l_{3}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} y_{x}^{l_{3}}\left[-\delta_{l_{1}}^{j} \Delta\left(y^{l_{2}} y^{l_{3}}\left|y^{1}\right| \cdots \mid y^{m}\right)\right] .
\end{align*}\right.
$$

Unfortunately, the equations (5.3) are not solved with respect to the $y_{x x}^{j}$, because in its last line, we notice that the $y_{x x}^{l_{2}}$ are mixed with the $y_{x}^{l_{1}}$. Consequently, we have to solve a linear system of $m$ equations with the unknowns
$y_{x x}^{j}$ of the form
(5.5)
$\left\{0=A^{j}+\sum_{l_{1}=1}^{m} y_{x x}^{l_{1}} \cdot\left[-X_{y^{l_{1}}} Y_{x}^{j}+Y_{y^{l_{1}}}^{j} X_{x}+\sum_{l_{2}=1}^{m} y_{x}^{l_{2}} \cdot\left[-X_{y^{l_{1}}} Y_{y^{l_{2}}}^{j}+Y_{y^{l_{1}}}^{j} X_{y^{l_{2}}}\right],\right]\right.$,
for $j=1, \ldots, m$, where $A^{j}$ is an abbreviation for the terms appearing in the lines $5,6,7,8,9$ and 10 of (5.3), or even more compactly, changing the index $j$ to the index $k$

$$
\begin{equation*}
\left\{0=A^{k}+\sum_{l_{1}=1}^{m} y_{x x}^{l_{1}} \cdot B_{l_{1}}^{k},\right. \tag{5.6}
\end{equation*}
$$

for $k=1, \ldots, m$, where $B_{l_{1}}^{k}$ is an abbreviation for the terms in the brackets in (5.5).

Thanks to the assumption that the determinant (3.2) is the identity determinant at $(x, y)=(0,0)$, we deduce that the determinant of the $m \times m$ matrix $\left(B_{l_{1}}^{k}\right)_{1 \leqslant k \leqslant m}^{1 \leqslant l_{1} \leqslant m}$ is also the identity determinant at $\left(x, y, y_{x}\right)=(0,0,0)$. It follows that the determinant of the $m \times m$ matrix $\left(B_{l_{1}}^{k}\right)_{1 \leqslant l_{1} \leqslant m}^{1 \leqslant k \leqslant m}$ is nonvanishing in a neighborhood of the origin in the first order jet space. Consequently, we can apply the rule of Cramer to solve the $y_{x x}^{j}$ explicitely interms of the $A^{k}$ and of the $B_{l_{1}}^{k}$ as follows

$$
\left\{y_{x x}^{j}=-\frac{\left|\begin{array}{ccccc}
B_{1}^{1} & \cdots & A^{1} & \cdots & B_{m}^{1}  \tag{5.7}\\
\cdots & \cdots & \cdots & \cdots & \cdots \\
B_{1}^{m} & \cdots & A^{m} & \cdots & B_{m}^{m}
\end{array}\right|}{\left|\begin{array}{ccccc}
B_{1}^{1} & \cdots & B^{1} & \cdots & B_{m}^{1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
B_{1}^{m} & \cdots & B^{m} & \cdots & B_{m}^{m}
\end{array}\right|}\right.
$$

where on the numerator, the only modification of the determinant of the matrix $\left(B_{l_{1}}^{k}\right)_{1 \leqslant l}^{1 \leqslant k \leqslant m}$ is the replacement of its $j$-th column by the column vector $A$. We have to show that after replacing the $A^{k}$ and the $B_{l_{1}}^{k}$ by their complete expressions, one indeed obtains the desired equation (5.4). As in (3.43), we shall introduce a notation for the two $m \times m$ determinants appearing in (5.6): we write this quotient under the form

$$
\begin{equation*}
\left\{y_{x x}^{j}=-\frac{\left\|B_{1}^{k}|\cdots|^{j} A^{k}|\cdots| B_{m}^{k}\right\|}{\left\|B_{1}^{k}|\cdots| B_{j}^{k}|\cdots| B_{m}^{k}\right\|},\right. \tag{5.8}
\end{equation*}
$$

where it is understood that $B_{1}^{k}, \ldots, B_{j}^{k}, \ldots, B_{m}^{k}$ and $A^{k}$ are column vectors whose index $k$ (for their lines) varies from 1 to $m$. This representation of determinants emphasizing only its columns will be appropriate for later manipulations.

Our first task is to compute the determinant in the denominator of (5.8). Recalling that we make the notational identification $y^{0} \equiv x$, it will be convenient to reexpress the $B_{l_{1}}^{k}$ in a slightly compacter form, using the total differentiation operator $D$ :

$$
\left\{\begin{align*}
B_{l_{1}}^{k} & =-X_{y^{l_{1}}} Y_{x}^{k}+Y_{y^{l_{1}}}^{k} X_{x}+\sum_{l_{2}=1}^{m} y_{x}^{l_{2}} \cdot\left[-X_{y^{l_{1}}} Y_{y^{l_{2}}}^{k}+Y_{y^{l_{1}}}^{k} X_{y^{l_{2}}}\right]=  \tag{5.9}\\
& =Y_{y^{1_{1}}}^{k} \cdot D X-X_{y^{l_{1}}} \cdot D Y^{k}
\end{align*}\right.
$$

Lemma 5.10. We have the following expression for the determinant of the matrix $\left(B_{l_{1}}^{k}\right)_{1 \leqslant l_{1} \leqslant m}^{1 \leqslant k \leqslant m}$ :

$$
\left\{\begin{array}{l}
\left\|Y_{y^{1}}^{k} \cdot D X-X_{y^{1}} \cdot D Y^{k}|\cdots| Y_{y^{m}}^{k} \cdot D X-X_{y^{m}} \cdot D Y^{k}\right\|=  \tag{5.11}\\
=[D X]^{m-1} \cdot \Delta\left(x\left|y^{1}\right| \cdots \mid y^{m}\right)
\end{array}\right.
$$

Proof. By multilinearity, we may develope the determinant written in the first line of (5.11). Since it contains two terms in each columns, we should obtain a sum of $2^{m}$ determinants. However, since the obtained determinants vanish as soon as the column $D Y^{k}$ (multiplied by various factors $X_{y^{l}}$ ) appears at least two different places, it remains only $(m+1)$ nonvanishing determinants, those for which the column $D Y^{k}$ appears at most once:
(5.12)

$$
\left\{\begin{array}{l}
\left\|Y_{y^{1}}^{k} \cdot D X-X_{y^{1}} \cdot D Y^{k}|\cdots| Y_{y^{m}}^{k} \cdot D X-X_{y^{m}} \cdot D Y^{k}\right\|= \\
=[D X]^{m} \cdot\left\|Y_{y^{1}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-[D X]^{m-1} X_{y^{1}} \cdot\left\|D Y^{k}\left|Y_{y^{2}}^{k}\right| \cdots \mid Y_{y^{m}}^{k}\right\|-\cdots- \\
\quad-[D X]^{m-1} X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots| Y_{y^{m-1}}^{k} \mid D Y^{k}\right\|
\end{array}\right.
$$

To establish the desired expression appearing in the second line of (5.11), we factor out by $[D X]^{m-1}$ and we develope all the remaining total differentiation operators $D$. Since $y^{0} \equiv x$, we have $y_{x}^{0}=1$, and this enables us to contract $X_{x}+\sum_{l_{1}}^{m} y_{x}^{l_{1}} X_{y^{l_{1}}}$ as $\sum_{l_{1}=0}^{m} y_{x}^{l_{1}} X_{y^{l_{1}}}$. So, we achieve the following
computation (further explanations and comments just afterwards):
(5.13)

$$
\begin{aligned}
& \left\{=[D X]^{m-1} \cdot\left\{\sum_{l_{1}=0}^{m} y_{x}^{l_{1}} X_{y^{l_{1}}} \cdot\left\|Y_{y^{1}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\right.\right. \\
& -\sum_{l_{1}=0}^{m} y_{x}^{l_{1}} X_{y^{1}} \cdot\left\|Y_{y^{l_{1}}}^{k}\left|Y_{y^{2}}^{k}\right| \cdots \mid Y_{y^{m}}^{k}\right\|- \\
& \left.-\cdots-\sum_{l_{1}=0}^{m} y_{x}^{l_{1}} X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots| Y_{y^{m-1}}^{k} \mid Y_{y^{1_{1}}}^{k}\right\|\right\} \\
& =[D X]^{m-1} \cdot\left\{\sum_{l_{1}=0}^{m} y_{x}^{l_{1}} X_{y^{l_{1}}} \cdot\left\|Y_{y^{1}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-X_{y^{1}} \cdot\left\|Y_{x}^{k}\left|Y_{y^{2}}^{k}\right| \cdots \mid Y_{y^{m}}^{k}\right\|-\right. \\
& -y_{x}^{1} X_{y^{1}} \cdot\left\|Y_{y^{1}}^{k}\left|Y_{y^{2}}^{k}\right| \cdots \mid Y_{y^{m}}^{k}\right\|-\cdots- \\
& \left.-X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots| Y_{y^{m-1}}^{k}\left|Y_{x}^{k}\left\|-y_{x}^{m} X_{y^{m}} \cdot\right\| Y_{y^{1}}^{k}\right| \cdots\left|Y_{y^{m-1}}^{k}\right| Y_{y^{m}}^{k}\right\|\right\} \\
& =[D X]^{m-1} \cdot\left\{X_{x} \cdot\left\|Y_{y^{1}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-X_{y^{1}} \cdot\left\|Y_{x}^{k}\left|Y_{y^{2}}^{k}\right| \cdots \mid Y_{y^{m}}^{k}\right\|-\cdots-\right. \\
& \left.-X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots| Y_{y^{m-1}}^{k} \mid Y_{x}^{k}\right\|\right\} \\
& =[D X]^{m-1} \cdot\left\{\Delta\left(x\left|y^{1}\right| \cdots \mid y^{m}\right)\right\} .
\end{aligned}
$$

For the passage to the equality of line 4 , using the fact that a determinant having two identical columns vanishes, we observe that in each of the $m$ sums $\sum_{l_{1}=0}^{m}$ appearing in lines 2 and 3 (including the cdots), there remains only two non-vanishing determinants. For the passage to the equality of line 7, we just sum up all the linear combinations of determinants appearing in lines 4,5 and 6 . Finally, for the passage to the equality of line 9 , we recognize the development of the fundamental Jacobian determinant (3.2) along its first line $\left(X_{x}, X_{y^{1}}, \ldots, X_{y^{m}}\right)$, modulo some permutations of columns in the $m \times m$ minors. The proof is complete.

Our second task, similar but computationnally more heavy, is to compute the determinant in the numerator of (5.8). First of all, we have to re-express the $A^{k}$ defined implicitely between (5.3) and (5.5) using the total differentiation operator to contract them as follows

$$
\left\{\begin{align*}
A^{k}= & D X \cdot Y_{x x}^{k}-D Y^{k} \cdot X_{x x}+2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot\left[D X \cdot Y_{x y^{l_{1}}}^{k}-D Y^{k} \cdot X_{x y^{l_{1}}}\right]+  \tag{5.14}\\
& +\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot\left[D X \cdot Y_{y^{l_{1}} y^{l_{2}}}^{k}-D Y^{k} \cdot X_{y^{l_{1}} y^{l_{2}}}\right]
\end{align*}\right.
$$

Replacing this expression of $A^{k}$ in (5.8), taking account of the expression of the denominator already obtained in the second line of (5.11) and abbreviating $\Delta\left(x\left|y^{1}\right| \cdots \mid y^{m}\right)$ as $\Delta$, we may write (5.8) in length and then develope it by linearity as follows

$$
\begin{align*}
& \left\{\begin{array}{l}
y_{x x}^{j}=\frac{-1}{[D X]^{m-1} \cdot \Delta} .
\end{array}\right. \\
& \| Y_{y^{1}}^{k} \cdot D X-X_{y^{1}} \cdot D Y^{k} \mid \cdots  \tag{5.15}\\
& \left.\cdots\right|^{j} D X \cdot Y_{x x}^{k}-D Y^{k} \cdot Y_{x x}^{k}+ \\
& +2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot\left[D X \cdot Y_{x y^{l_{1}}}^{k}-D Y^{k} \cdot X_{x y^{l_{1}}}\right]+ \\
& +\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot\left[D X \cdot Y_{y^{l_{1}} y^{l_{2}}}^{l_{2}}-D Y^{k} \cdot X_{y^{l_{1}} y^{l_{2}}}\right] \mid \cdots \\
& \left.\cdots \mid Y_{y^{m}}^{k} \cdot D X-X_{y^{m}} \cdot D Y^{k} \|\right] \\
& \| Y_{y^{1}}^{k} \cdot D X-X_{y^{1}} \cdot D Y^{k} \mid \cdots \\
& \cdots{ }^{j} D X \cdot Y_{x x}^{k}-D Y^{k} \cdot X_{x x} \mid \cdots \\
& \cdots \mid Y_{y^{m}}^{k} \cdot D X-X_{y^{m}} \cdot D Y^{k} \|+ \\
& \begin{array}{c}
+2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot \| Y_{y^{1}}^{k} \cdot D X-X_{y^{1}} \cdot D Y^{k} \mid \cdots \\
\left.\cdots\right|^{j} D X \cdot Y_{x y^{l_{1}}}^{k}-D Y^{k} \cdot X_{x y^{1_{1}}} \mid \cdots
\end{array} \\
& \cdots \mid Y_{y^{m}}^{k} \cdot D X-X_{y^{m}} \cdot D Y^{k} \|+ \\
& +\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot \| Y_{y^{1}}^{k} \cdot D X-X_{y^{1}} \cdot D Y^{k} \mid \cdots \\
& \cdots \cdot{ }^{j} D X \cdot Y_{y^{l_{1}} y^{l_{2}}}^{l^{2}}-D Y^{k} \cdot X_{y^{l_{1} l^{l_{2}}}} \mid \cdots \\
& \left.\cdots \mid Y_{y^{m}}^{k} \cdot D X-X_{y^{m}} \cdot D Y^{k} \|\right\rfloor
\end{align*}
$$

As it is delicate to read, let us say that lines 2,3 and 4 just express the $j$-th colum $\left|{ }^{j} A^{k}\right|$ of the determinant $\left\|B_{1} k|\cdots| j A^{k}|\cdots| B_{m}^{k}\right\|$, after replacement of $A^{k}$ by its complete expression (5.14).

In lines $6,7,8$; in lines $9,10,11$; and in lines $12,13,14$, there are three families of $m \times m$ determinants containing a linear combination (soustraction) having exactly two terms in each column. As in the proof of Lemma 5.10, by multilinarity, we have to develope each such determinant. In principle, for each development, we should get $2^{m}$ terms, but since the obtained determinants vanish as soon as the column $D Y^{k}$ (modulo a multiplication by some factor) appears at least twice, it remains only $(m+1)$ nonvanishing determinants, those for which the column $D Y^{k}$ appears at most
once. In addition, for each of the obtained determinant, the factor $[D X]^{m-1}$ appears (sometimes even the factor $[D X]^{m}$ ), so that this factor compensates the factor $[D X]^{m-1}$ in the numerator. In sum, the continuation of the huge computation yields:

$$
\begin{aligned}
& \\
& y_{x x}^{j}=-\frac{1}{\Delta} .
\end{aligned}
$$

$$
\begin{equation*}
\left[D X \cdot\left\|Y_{y^{1}}^{k}|\cdots|^{j} Y_{x x}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\right. \tag{5.16}
\end{equation*}
$$

$$
-X_{y^{1}} \cdot\left\|\left.D Y^{k}|\cdots|\right|^{j} Y_{x x}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\cdots-
$$

$$
-X_{x x} \cdot\left\|Y_{y^{1}}^{k}\left|\cdots{ }^{j} D Y^{k}\right| \cdots \mid Y_{y^{m}}^{k}\right\|-\cdots-
$$

$$
-X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x x}^{k}|\cdots| D Y^{k}\right\|+
$$

$$
+2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} D X \cdot\left\|Y_{y^{1}}^{k}|\cdots|^{j} Y_{x y^{l_{1}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-
$$

$$
-2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} X_{y^{1}} \cdot\left\|D Y^{k}|\cdots|{ }^{j} Y_{x y^{l_{1}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\cdots-
$$

$$
-2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} X_{x y^{l_{1}}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|^{j} D Y^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\cdots-
$$

$$
-2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }_{x y}^{j} Y_{x l_{1}}^{k}|\cdots| D Y^{k}\right\|+
$$

$$
+\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} D X \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{y^{l_{1}} y_{2}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-
$$

$$
\begin{aligned}
& -\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} X_{y^{1}} \cdot\left\|D Y^{k}|\cdots|{ }^{j} Y_{y^{l_{1}} y^{l_{2}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\cdots- \\
& -\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} X_{y^{l_{1}} y^{l_{2}}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} D Y^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\cdots-
\end{aligned}
$$

$$
-\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|^{j} Y_{y^{1_{1}} y^{l_{2}}}^{k}|\cdots| D Y^{k}\right\|
$$

To establish the desired expression (5.4), we must develope all the total differentiation operators $D$ of the terms $D X$ placed as factor and of the terms $D Y^{k}$ placed in various columns of determinants. We notice that in developing $D Y^{k}$, we obtain columns $Y_{y^{l}}^{k}$ (multiplied by the factor $y_{x}^{l}$ ) and for only three (or two) values of $l=0,1, \ldots, m$, this column does not already appear in the corresponding determinant, so that $(m-1)$ determinants vanish and only 3 (or 2 ) remain nonzero. Taking account of these simplifications, we
have the continuation

$$
\begin{equation*}
-y_{x x}^{j} \cdot \Delta=\mathrm{I}+\mathrm{II}+\mathrm{III}, \tag{5.17}
\end{equation*}
$$

where the term I is the development of lines $1,2,3,4$ of (5.16); the term II is the development of lines $5,6,7,8$ of (5.16); and the term III is the development of lines $9,10,11,12$ of (5.16). So we get firstly (further explanations follows):

$$
\begin{align*}
& \sum_{l=0}^{m} y_{x}^{l} X_{y^{l}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x x}^{k}|\cdots| Y_{y^{m}}^{k}\right\| \\
& -X_{y^{1}} \cdot\left\|Y_{x}^{k}|\cdots|{ }^{j} Y_{x x}^{k}|\cdots| Y_{y^{m}}^{k}\right\|- \\
& -y_{x}^{1} X_{y^{1}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x x}^{k}|\cdots| Y_{y^{m}}^{k}\right\| \\
\mathrm{I}:= & -y_{x}^{j} X_{y^{j}} \cdot\left\|Y_{y^{j}}^{k}|\cdots|{ }^{j} Y_{x x}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\cdots-  \tag{5.18}\\
& -X_{x x} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x}^{k}|\cdots| Y_{y^{m}}^{k}\right\|- \\
& -y_{x}^{j} X_{x x} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{y^{j}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\cdots- \\
& -X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x x}^{k}|\cdots| Y_{x}^{k}\right\|- \\
& -y_{x}^{m} X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x x}^{k}|\cdots| Y_{y^{m}}^{k}\right\| \\
& -y_{x}^{j} X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x x}^{k}|\cdots| Y_{y^{j}}^{k}\right\|,
\end{align*}
$$

and secondly (we discuss afterwards the annihilation of the underlined terms):

$$
\begin{aligned}
& 2 \sum_{l_{1}=1}^{m} \sum_{l=0}^{m} y_{x}^{l_{1}} y_{x}^{l} X_{y^{l}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x y^{l_{1}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|
\end{aligned}
$$

and where thirdly (we are nearly the end of the proof):

$$
\begin{align*}
& \sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} \sum_{l=0}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} y_{x}^{l} X_{y^{l}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{y^{l_{1}} y^{l_{2}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|_{-3}-  \tag{5.20}\\
& -\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} X_{y^{1}} \cdot\left\|Y_{x}^{k}|\cdots|{ }^{j} Y_{y^{l_{1}} y^{l_{2}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|- \\
& -\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} y_{x}^{1} X_{y^{1}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{y^{l_{1}} y^{l_{2}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|_{3}- \\
& -\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} y_{x}^{j} X_{y^{1}} \cdot\left\|Y_{y^{j}}^{k}|\cdots|{ }^{j} Y_{y^{l_{1}} y^{l_{2}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\cdots- \\
& \text { III }:=-\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} X_{y^{l_{1} y^{l_{2}}}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x}^{k}|\cdots| Y_{y^{m}}^{k}\right\|- \\
& -\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} y_{x}^{j} X_{y^{l_{1} y^{l_{2}}}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{y^{j}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\cdots- \\
& -\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|^{j} Y_{y^{1} y^{l_{2}}}^{k}|\cdots| Y_{x}^{k}\right\|- \\
& -\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} y_{x}^{m} X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|^{j} Y_{y^{l_{1} y^{l_{2}}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|_{3}^{3}- \\
& -\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} y_{x}^{j} X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }_{y^{j} y^{l_{2}}}^{k}|\cdots| Y_{y^{j}}^{k}\right\| .
\end{align*}
$$

Now, we explain the annihilation of the underlined terms. Consider I: in the first sum $\sum_{l=0}^{m}$, all the terms except only the two corresponding to $l=0$ and to $l=j$ are annihilated by the other terms with 1 appended: indeed, one must take account of the fact that in the expression of $I$, we have two sums represented by some colots, the nature of which was defined without ambiguity in the passage from (5.15) to (5.16).

Similar simplifications occur for II and for III. Consequently, we obtain firstly:

$$
\begin{align*}
& X_{x} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x x}^{k}|\cdots| Y_{y^{m}}^{k}\right\|+ \\
& +y_{x}^{j} X_{y^{j}} \cdot\left\|Y_{y^{2}}^{k}|\cdots|{ }^{j} Y_{x x}^{k}|\cdots| Y_{y^{m}}^{k}\right\|- \\
& -X_{y^{1}} \cdot\left\|Y_{x}^{k}|\cdots|{ }^{j} Y_{x x}^{k}|\cdots| Y_{y^{m}}^{k}\right\|- \\
& -y_{x}^{j} X_{y^{j}} \cdot\left\|Y_{y^{j}}^{k}|\cdots|{ }^{j} Y_{x x}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\cdots-  \tag{5.21}\\
& -X_{x x} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x}^{k}|\cdots| Y_{y^{m}}^{k}\right\|- \\
& -y_{x}^{j} X_{x x} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{y^{j}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\cdots- \\
& -X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x x}^{k}|\cdots| Y_{x}^{k}\right\|- \\
& -y_{x}^{j} X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x x}^{k}|\cdots| Y_{y^{j}}^{k}\right\| ;
\end{align*}
$$

just above, the first two lines consist of the two terms in the sum underlined at the first line of (5.18) which are not annihilated; secondly we obtain:

$$
\begin{align*}
& 2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} X_{x} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x y^{l_{1}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|+ \\
& +2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} y_{x}^{j} X_{y^{j}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|^{j} Y_{x y^{l_{1}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|- \\
& -2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} X_{y^{1}} \cdot\left\|Y_{x}^{k}|\cdots|^{j} Y_{x y^{l_{1}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|- \\
& -2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} y_{x}^{j} X_{y^{1}} \cdot\left\|Y_{y^{j}}^{k}|\cdots|^{j} Y_{x y^{l_{1}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\cdots- \\
& -2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} X_{x y^{l_{1}} \cdot} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-  \tag{5.22}\\
& -2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} y_{x}^{j} X_{x y^{l_{1}}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|^{j} Y_{x}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\cdots- \\
& -2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} X_{y^{m}} \cdot\left\|\left.Y_{y^{1}}^{k}|\cdots|\right|^{j} Y_{x y^{l_{1}}}^{k}|\cdots| Y_{x}^{k}\right\|- \\
& -2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} y_{x}^{j} X_{y^{m}} \cdot\left\|\left.Y_{y^{1}}^{k}|\cdots|\right|^{j} Y_{x y^{l_{1}}}^{k}|\cdots| Y_{y^{j}}^{k}\right\| ;
\end{align*}
$$

similarly, the first two lines above consist of the two terms in the sum underlined at the first line of (5.19) which are not annihilated; and thirdly we
obtain:
(5.23)

$$
\begin{aligned}
& \sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} X_{x} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{y^{l_{1}} y^{l_{2}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|+ \\
& +\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} y_{x}^{j} X_{y^{j}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{y^{l_{1}} y^{l_{2}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|- \\
\text { III }:= & -\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} X_{y^{1}} \cdot\left\|Y_{x}^{k}|\cdots|{ }^{j} Y_{y^{l_{1}} y^{l_{2}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|- \\
& -\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} y_{x}^{j} X_{y^{1}} \cdot\left\|Y_{y^{j}}^{k}|\cdots|{ }^{j} Y_{y^{l_{1}} y^{l_{2}}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\cdots- \\
& -\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} X_{y^{l_{1}} y^{l_{2}}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{x}^{k}|\cdots| Y_{y^{m}}^{k}\right\|- \\
& -\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} y_{x}^{j} X_{y^{l_{1} y^{l}}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{y^{j}}^{k}|\cdots| Y_{y^{m}}^{k}\right\|-\cdots-
\end{aligned}
$$

$$
-\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{y^{l_{1}} y^{l_{2}}}^{k}|\cdots| Y_{x}^{k}\right\|-
$$

$$
-\sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} y_{x}^{j} X_{y^{m}} \cdot\left\|Y_{y^{1}}^{k}|\cdots|{ }^{j} Y_{y^{l_{1}} y^{l_{2}}}^{k}|\cdots| Y_{y^{j}}^{k}\right\| .
$$

Collecting the odd lines of (5.21), we obtain exactly $(m+1)$ terms which correspond to the development of the determinant $\Delta\left(x\left|\cdots{ }^{j} x x\right| \cdots \mid y^{m}\right)$ along its first line, modulo permutations of columns of the associated $m \times m$ minors; collecting the even lines of (5.21), we obtain exactly $(m+1)$ terms which correspond to the development of the determinant $-y_{x}^{j} \cdot \Delta\left(x x\left|y^{1}\right| \cdots \mid y^{m}\right)$ along its first lines, modulo permutations of columns of the associated $m \times m$ minors. Similar observations hold about II and III.

In, we may rewrite the final expressions of these three terms: firstly

$$
\left\{\begin{align*}
& \mathrm{I}= \Delta\left(x|\cdots|^{j} x x|\cdots| y^{m}\right)-y_{x}^{j} \cdot \Delta\left(x x\left|y^{1}\right| \cdots \mid y^{m}\right)  \tag{5.24}\\
& \mathrm{II}=2 \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot \Delta\left(x|\cdots|^{j} x y^{l_{1}}|\cdots| y^{m}\right)- \\
&-2 y_{x}^{j} \sum_{l_{1}=1}^{m} y_{x}^{l_{1}} \cdot \Delta\left(x y^{l_{1}}\left|y^{1}\right| \cdots \mid y^{m}\right) \\
& \mathrm{III}= \sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot \Delta\left(x|\cdots|^{j} y^{l_{1}} y^{l_{2}}|\cdots| y^{m}\right)- \\
& \quad-y_{x}^{j} \sum_{l_{1}=1}^{m} \sum_{l_{2}=1}^{m} y_{x}^{l_{1}} y_{x}^{l_{2}} \cdot \Delta\left(y^{l_{1}} y^{l_{2}}\left|y^{1}\right| \cdots \mid y^{m}\right)
\end{align*}\right.
$$

Coming back to (5.17), we obtain the desired expression (5.4).
The proof of the - technical, though involving only linear algebra Lemma 3.32 is complete.

## REFERENCES

[BK1989] Bluman, G.W.; Kumei, S.: Symmetries and differential equations, Springer-Verlag, New-York, 1989.
[Ca1924] Cartan, É.: Sur les variétés à connexion projective, Bull. Soc. Math. France 52 (1924), 205-241.
[Ch1939] CHERN, S.-S.: Sur la géométrie d'un système d'équations différentielles du second ordre, Bull. Sci. Math. 63 (1939), 206-212.
[CMS1996] Crampin, M.; Martínez, E.; Sarlet, W.: Linear connections for systems of second-order ordinary differential equations, Ann. Inst. H. Poincaré Phys. Théor. 65 (1996), no. 2, 223-249.
[Do2000] Doubrov, B.: Contact invariants of ordinary differential equations, RIMS Kokyuroku 1150 (2000), 105-113.
[DNP2005] Dridi, R.; Neut, S.; Petitot, M.: Élie Cartan's geometrical vision or how to avoid expression swell, arxiv.org/abs/math.DG/0504203.
[Fe1995] FELS, M.: The equivalence problem for systems of second-order ordinary differential equations, Proc. London Math. Soc. 71 (1995), no. 2, 221-240.
[G1989] GARDNER, R.B.: The method of equivalence and its applications, CBMSNSF Regional Conference Series in Applied Mathematics 58 (SIAM, Philadelphia, 1989), 127 pp.
[GM2003] GaUssier, H.; MERKER, J.: Symmetries of partial differential equations, J. Korean Math. Soc. 40 (2003), no. 3, 517-561.
[GG1983] GonZÁLEZ GASCÓN, F.; GonZÁLEZ LóPEZ, A.: Symmetries of differential equations, IV. J. Math. Phys. 24 (1983), 2006-2021.
[GL1988] GonZÁLEZ López, A.: Symmetries of linear systems of second order differential equations, J. Math. Phys. 29 (1988), 1097-1105.
[GKO1992] González López, A.; Kamran, N.; Olver, P.J.: Lie algebras of vector fields in the real plane, Proc. London Math. Soc. 64 (1992), 339-368.
[GTW1989] Grissom, C.; Thompson, G.; Wilkens, G.: Linearization of second order ordinary differential equations via Cartan's equivalence method, J. Diff. Eq. 77 (1989), no. 1, 1-15.
[Gr2000] Grossman, D.A.: Torsion-free path geometries and integrable second order ode systems, Selecta Math. (N.S.) 6 (2000), no. 4, 399-442.
[Ha1937] HAChTroudi, M.: Les espaces d'éléments à connexion projective normale, Actualités Scientifiques et Industrielles, 565, Paris, Hermann, 1937.
[H2001] Hawkins, T.: Emergence of the theory of Lie groups, Springer-Verlag, Berlin, 2001.
[HK1989] HsU, L.; KAMRAN, N.: Classification of second order ordinary differential equations admitting Lie groups of fibre-preserving point symmetries, Proc. London Math. Soc. 58 (1989), no. 3, 387-416.
[IB 1992] Ibragimov, N.H.: Group analysis of ordinary differential equations and the invariance principle in mathematical physics, Russian Math. Surveys 47:4 (1992), 89-156.
[IB 1999] Ibragimov, N.H.: Elementary Lie group analysis and ordinary differential equations, Mathematical methods in practice, John Wiley \& Sons, Chichester, 1999, xviii+347 pp.
[Le1980] LEACH, P.G.L.: $\mathrm{Sl}(3, \mathbb{R})$ and the repulsive oscillator, J. Phys. A 13 (1980), 1991-2000.
[Lie1880] LIE, S.: Theorie der Transformationsgruppen, Math. Ann. 16 (1880), 441528; translated in English and commented in: Ackerman, M.; Hermann, R.: Sophus Lie's 1880 Transformation Group paper, Math. Sci. Press, Brookline, Mass., 1975.
[Lie1883] LIE, S.: Klassifikation und Integration vo gewöhnlichen Differentialgleichungen zwischen $x$, $y$, die eine Gruppe von Transformationen gestaten I-IV. In: Gesammelte Abhandlungen, Vol. 5, B.G. Teubner, Leipzig, 1924, pp. 240310; 362-427, 432-448.
[MS2001] MAhomed, F.M.; Soh, C.W.: Linearization criteria for a system of secondorder differential equations, Internat. J. Non-Linear Mech. 36 (2001), no. 4, 671-677.
[Ma2003] MARDARE, S.: On isometric immersions of a Riemannian space under a weak regularity assumption, C. R. Acad. Sci. Paris, Sér. I 337 (2003), 785790.
[M2004] MERKER, J.: Explicit differential characterization of PDE systems pointwise equivalent to $Y_{X^{j_{1} X^{j_{2}}}}=0,1 \leqslant j_{1}, j_{2} \leqslant n \geqslant 2$, arxiv.org/abs/math.DG/0411637.
[N2003] NEUT, S.: Implantation et nouvelles applications de la méthode d'équivalence d'Élie Cartan, Thèse, Université Lille 1, October 2003.
[NS2003] NUROWSKy, P.; Sparling, G.A.J.: 3-dimensional Cauchy-Riemann structures and $2^{\text {nd }}$ order ordinary differential equations, e-print arXiv:math.DG/0306331.
[Ol1986] OLVER, P.J.: Applications of Lie groups to differential equations. Springer Verlag, New York, 1986. xxvi+497 pp.
[OL1995] OLVER, P.J.: Equivalence, Invariance and Symmetries. Cambridge University Press, Cambridge, 1995, xvi+525 pp.
[Ste1982] Sternberg, S.: Differential geometry. Chelsea, New York, 1982.
[Stk2000] Stormark, O.: Lie's structural approach to PDE systems, Encyclopædia of mathematics and its applications, vol. 80, Cambridge University Press, Cambridge, 2000, xv+572 pp.
[TR1896] Tresse, A.: Détermination des invariants ponctuels de l'équation différentielle du second ordre $y^{\prime \prime}=\omega\left(x, y, y^{\prime}\right)$, Hirzel, Leipzig, 1896.

# Nonalgebraizable real analytic tubes in $\mathbb{C}^{n} \mathbf{x}$ 

Joël Merker (with H. Gaussier)


#### Abstract

We give necessary conditions for certain real analytic tube generic submanifolds in $\mathbb{C}^{n}$ to be locally algebraizable. As an application, we exhibit families of real analytic non locally algebraizable tube generic submanifolds in $\mathbb{C}^{n}$. During the proof, we show that the local CR automorphism group of a minimal, finitely nondegenerate real algebraic generic submanifold is a real algebraic local Lie group. We may state one of the main results as follows. Let $M$ be a real analytic hypersurface tube in $\mathbb{C}^{n}$ passing through the origin, having a defining equation of the form $v=\varphi(y)$, where $(z, w)=(x+i y, u+i v) \in \mathbb{C}^{n-1} \times \mathbb{C}$. Assume that $M$ is Levi nondegenerate at the origin and that the real Lie algebra of local infinitesimal CR automorphisms of $M$ is of minimal possible dimension $n$, i.e. generated by the real parts of the holomorphic vector fields $\partial_{z_{1}}, \ldots, \partial_{z_{n-1}}, \partial_{w}$. Then $M$ is locally algebraizable only if every second derivative $\partial_{y_{k} y_{l}}^{2} \varphi$ is an algebraic function of the collection of first derivatives $\partial_{y_{1}} \varphi, \ldots, \partial_{y_{m}} \varphi$.


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Mathematische Zeitschrift 247 (2004), no. 2, 337-383

## §1. Introduction

A real analytic submanifold $M$ in $\mathbb{C}^{n}$ is called algebraic if it can be represented locally by the vanishing of a collection of Nash algebraic real analytic functions. We say that $M$ is locally algebraizable at one of its points $p$ if there exist some local holomorphic coordinates centered at $p$ in which $M$ is algebraic. For instance, every totally real, real analytic submanifold in $\mathbb{C}^{n}$ of dimension $k \leq n$ is locally biholomorphic to a $k$-dimensional linear real plane, hence locally algebraizable. Also, every complex manifold is locally algebraizable. Although every real analytic submanifold $M$ is clearly locally equivalent to its tangent plane by a real analytic (in general
not holomorphic) equivalence, the question whether $M$ is biholomorphically equivalent to a real algebraic submanifold is subtle. In this article, we study the question whether every real analytic CR submanifold is locally algebraizable. One of the interests of algebraizability lies in the reflection principle, which is better understood in the algebraic category. Indeed, in the fundamental works of Pinchuk [Pi1975], [Pi1978] and of Webster [We 1977], [We 1978] and in the recent works of Sharipov-Sukhov [SS1996], Huang-Ji [HJ1998], Verma [Ve1999], Coupet-Pinchuk-Sukhov [CPS2000], and Shafikov [Sha2000], [Sha2002], the extendability of germs of CR mappings with target in a real algebraic hypersurface is achieved. On the contrary, even if some results previously shown under an algebraization hypothesis were proved recently under general assumptions (see the strong result obtained by Diederich-Pinchuk [DP2003]), most of the results cited above are still open in the case of a real analytic target hypersurface.
1.1. Brief history of the question. By the work of Moser and Webster [MW1983, Thm. 1], it is known that every real analytic two-dimensional surface $S \subset \mathbb{C}^{2}$ at an isolated elliptic (in the sense of Bishop) complex tangency $p \in S$ is biholomorphic to one of the surfaces $S_{\gamma, \delta, s}:=\left\{\left(z_{1}, z_{2}\right) \in\right.$ $\left.\mathbb{C}^{2}: y_{2}=0, x_{2}=z_{1} \bar{z}_{1}+\left(\gamma+\delta\left(x_{2}\right)^{s}\right)\left(z_{1}^{2}+\bar{z}_{1}^{2}\right)\right\}$, where $p$ corresponds to the origin, where $0<\gamma<1 / 2$ is Bishop's invariant and where $\delta= \pm 1$ and $s \in \mathbb{N}$ or $\delta=0$. The quantities $\gamma, \delta, s$ form a complete system of biholomorphic invariants for the surface $S$ near $p$. In particular, every elliptic surface $S \subset \mathbb{C}^{2}$ is locally algebraizable. To the authors' knowledge, it is unknown whether there exist nonalgebraizable hyperbolic surfaces in $\mathbb{C}^{2}$. In fact, very few examples of nonalgebraizable submanifolds are known. In [Eb1996], the author constructed a nonminimal (and non Levi-flat) real analytic hypersurface $M$ through the origin in $\mathbb{C}^{2}$ which is not locally algebraizable (cf. [BER2000, p. 330]). In a recent article [HJY] the authors prove that the strongly pseudoconvex real analytic hypersurface $\operatorname{Im} w=e^{|z|^{2}}-1$ passing through the origin in $\mathbb{C}^{2}$ is not locally algebraizable at any of its points. Using an associated projective structure bundle $\mathscr{Y}$ introduced by Chern, they show that for every rigid algebraic hypersurface in $\mathbb{C}^{n}$, there exists an algebraic dependence relation between seven explicit Cartan-type holomorphic invariant functions on $\mathscr{Y}$. However a computational approach shows that when $M$ is of the specific form $\operatorname{Im} w=e^{|z|^{2}}-1$, no algebraic relation can be satisfied by these seven invariants.
1.2. Presentation of the main results. Our aim is to present a geometrical approach of the problem, valid in arbitrary dimension and in arbitrary codimension, and to exhibit a large class of nonalgebraizable real analytic generic submanifolds. We consider the class $\mathscr{T}_{n}^{d}$ of generic real analytic submanifolds in $\mathbb{C}^{n}$ passing through the origin, of codimension $d \geq 1$
and of CR dimension $m=n-d \geq 1$, whose local CR automorphism group is $n$-dimensional, generated by the real parts of $n$ holomorphic vector fields having holomorphic coefficients $X_{1}, \ldots, X_{n}$ which are linearly independent at the origin and which commute: $\left[X_{i_{1}}, X_{i_{2}}\right]=0$. We shall call $\mathscr{T}_{n}^{d}$ the class of strong tubes of codimension $d$. Indeed, since there exists a straightened system of coordinates $t=\left(t_{1}, \ldots, t_{n}\right)$ over $\mathbb{C}^{n}$ in which $X_{i}=\partial_{t_{i}}$, we observe that every submanifold $M \in \mathscr{T}_{n}^{d}$ is tubifiable at the origin. By this, we mean that there exist holomorphic coordinates $t=(z, w)=(x+i y, u+i v) \in \mathbb{C}^{m} \times \mathbb{C}^{d}$ vanishing at the origin in which $M$ is represented by $d$ equations of the form $v_{j}=\varphi_{j}(y)$. Hence $M$ is a tube, i.e. a product of the submanifold $\left\{v_{j}=\varphi_{j}(y), j=1, \ldots, d\right\} \subset \mathbb{R}_{y, v}^{n}$ by the $n$-dimensional real space $\mathbb{R}_{x, u}^{n}$. Since $M \in \mathscr{T}_{n}^{d}$, the only infinitesimal CR automorphisms of $M$ are the real parts of the vector fields $\partial_{z_{1}}, \ldots, \partial_{z_{m}}, \partial_{w_{1}}, \ldots, \partial_{w_{d}}$, explaining the terminology. Notice that not every tube belongs to the class $\mathscr{T}_{n}^{d}$. For instance in codimension $d=1$, the Heisenberg sphere $v=\sum_{k=1}^{n-1} y_{k}^{2}$ and more generally the Levi nondegenerate quadrics $v=\sum_{k=1}^{n-1} \varepsilon_{k} y_{k}^{2}$, where $\varepsilon_{k}= \pm 1$, have a CR automorphism group of dimension $(n+1)^{2}-1>n$ and so do not belong to $\mathscr{T}_{n}^{1}$. We assume that $M \in \mathscr{T}_{n}^{d}$ is minimal at the origin, namely the local CR orbit of 0 in $M$ contains a neighborhood of 0 in $M$. Furthermore, we assume that $M \in \mathscr{T}_{n}^{d}$ is finitely nondegenerate at 0 , namely that there exists an integer $\ell \geq 1$ such that $\operatorname{Span}\left\{\bar{L}^{\beta} \nabla_{t}\left(r_{j}\right)(0,0): \beta \in \mathbb{N}^{m},|\beta| \leq \ell, j=1, \ldots, d\right\}=\mathbb{C}^{n}$, where $r_{j}(t, \bar{t})=0, j=1, \ldots, d$ are arbitrary real analytic defining functions for $M$ near 0 satisfying $\partial r_{1} \wedge \cdots \wedge \partial r_{d} \neq 0$ on $M$, where $\nabla_{t}\left(r_{j}\right)(t, \bar{t})$ is the holomorphic gradient with respect to $t$ of $r_{j}$ and where $\bar{L}^{\beta}$ denotes $\left(\bar{L}_{1}\right)^{\beta_{1}} \cdots\left(\bar{L}_{m}\right)^{\beta_{m}}$ for an arbitrary basis $\bar{L}_{1}, \ldots, \bar{L}_{m}$ of $(0,1)$-vector fields tangent to $M$ in a neighborhood of 0 . In particular Levi nondegenerate hypersurfaces are finitely nondegenerate. Finally, assuming only that $\varphi_{j}(0)=0, j=1, \ldots, d$, we shall observe in Lemma 3.2 below that a tube $v_{j}=\varphi_{j}(y)$ of codimension $d$ is finitely nondegenerate at the origin if and only if there exist multi-indices $\beta_{*}^{1}, \ldots, \beta_{*}^{m} \in \mathbb{N}^{m}$ with $\left|\beta_{*}^{k}\right| \geq 1$ and integers $1 \leq j_{*}^{1}, \ldots, j_{*}^{m} \leq d$ such that the real mapping

$$
\begin{equation*}
\psi(y):=\left(\frac{\partial^{\left|\beta_{*}^{1}\right|} \varphi_{j ⿱}(y)}{\partial y^{\beta_{*}^{1}}}, \ldots, \frac{\partial^{\left|\beta_{*}^{m}\right|} \varphi_{j_{*}^{m}}(y)}{\partial y_{*}^{\beta_{*}^{m}}}\right)=: y^{\prime} \in \mathbb{R}^{m} \tag{1.1}
\end{equation*}
$$

is of rank $m$ at the origin in $\mathbb{R}^{m}$. Our main theorem provides a necessary condition for the local algebraizability of strong tubes :

Theorem 1.1. Let $M$ be a real analytic generic tube of codimension $d$ in $\mathbb{C}^{n}$ given in coordinates $(z, w)=(x+i y, u+i v) \in \mathbb{C}^{m} \times \mathbb{C}^{d}$ by the equations $v_{j}=\varphi_{j}(y)$, where $\varphi_{j}(0)=0, j=1, \ldots, d$. Assume that $M$ is minimal and finitely nondegenerate at the origin, so the real mapping
$\psi(y)=y^{\prime}$ defined by (1.1) is of rank $m$ at the origin in $\mathbb{R}_{y}^{m}$ and let $y=\psi^{\prime}\left(y^{\prime}\right)$ denote the local inverse in $\psi(y)$. Assume that $M \in \mathscr{T}_{n}^{d}$, namely $M$ is a strong tube of codimension $d$. If $M$ is locally algebraizable at the origin, then all the derivative functions $\partial_{y_{k}^{\prime}} \psi_{l}^{\prime}\left(y^{\prime}\right)$, where $1 \leq k, l \leq m$, are real algebraic functions of $y^{\prime}$. Equivalently, every second derivative $\partial_{y_{k} y_{l}}^{2} \varphi_{j}(y)$ is an algebraic function of the collection of first derivatives $\partial_{y_{1}} \varphi_{j}, \ldots, \partial_{y_{m}} \varphi_{j}$.

By contraposition, every real analytic strong tube $M \in \mathscr{T}_{n}^{d}$ for which one of the derivative functions $\partial_{y_{k}^{\prime}} \psi_{l}^{\prime}$ is not real algebraic is not locally algebraizable. We will argue in $\S 8$ that this is generically the case in the sense of Baire. It is however natural to look for explicit examples of nonalgebraizable real analytic submanifolds in $\mathbb{C}^{n}$. Since the real parts of the vector fields $\partial_{z_{1}}, \ldots, \partial_{z_{m}}, \partial_{w_{1}}, \ldots, \partial_{w_{d}}$ are infinitesimal CR automorphisms of every tube $v=\varphi(y)$, we must provide some sufficient conditions insuring that the dimension of the Lie algebra of such a tube is exactly $n$. We shall establish in §§7-8 below:

Corollary 1.2. The tube hypersurface $M_{\chi_{1}, \ldots, \chi_{n-1}}$ in $\mathbb{C}^{n}$ of equation $v=$ $\sum_{k=1}^{n-1}\left[\varepsilon_{k} y_{k}^{2}+y_{k}^{6}+y_{k}^{9} y_{1} \cdots y_{k-1}+y_{k}^{n+8} \chi_{k}\left(y_{1}, \ldots, y_{n-1}\right)\right]$, where $\chi_{1}, \ldots, \chi_{n-1}$ are arbitrary real analytic functions, belongs to the class $\mathscr{T}_{n}^{1}$ of strong tubes. Two such tubes $M_{\chi_{1}, \ldots, \chi_{n-1}}$ and $M_{\widehat{\chi}_{1}, \ldots, \widehat{\chi}_{n-1}}$ are biholomorphically equivalent if and only if $\chi_{j}=\widehat{\chi}_{j}$ for every $j$. Furthermore, for a generic choice in $\chi_{1}, \ldots, \chi_{n-1}$ in the sense of Baire (to be precised in §8), $M_{\chi_{1}, \ldots, \chi_{n-1}}$ is not locally algebraizable at the origin.

Here we annihilate some Taylor coefficients in $\varphi$ and keep some others to be nonzero to insure that $M_{\chi}$ is a strong tube. Furthermore, the terms $y_{k}^{9} y_{1} \cdots y_{k-1}$ insure that the $M_{\chi}$ are pairwise not biholomorphically equivalent. Using a classical direct algorithm (cf. [Bs1991], [St1991]), or the Lie theory of symmetries of differential equations, combined with Theorem 1.1 we may provide some other explicit strong tubes which are not locally algebraizable (see §§7-8 for the proof):

Corollary 1.3. The following five explicit tubes belong to $\mathscr{T}_{2}^{1}$ and are not locally algebraizable at the origin : $v=\sin \left(y^{2}\right), v=\tan \left(y^{2}\right), v=e^{e^{y}-1}-1$, $v=\sinh \left(y^{2}\right)$ and $v=\tanh \left(y^{2}\right)$.

In these five examples, the algebraic independence in $\partial_{y} \varphi$ and in $\partial_{y y}^{2} \varphi$ is clear; however, checking that each hypersurface is indeed a strong tube requires some formal computations, see §7. One may also check by a direct computation that in a neighborhood of every point $p=\left(z_{p}, w_{p}\right)$ with $z_{p} \neq 0$, the hypersurface $M_{\mathrm{HJY}}$ of global equation $\operatorname{Im} w=e^{|z|^{2}}-1$ is a strong tube (see §7.5). Since it can be represented in a neighborhood of $p$ under the tube form $v^{\prime}=e^{\left|z_{p}\right|^{2}\left(e^{y^{\prime}}-1\right)}-1$ by means of the local change of
coordinates $z^{\prime}=2 i \ln \left(z / z_{p}\right), w^{\prime}=\left(w-w_{p}\right) e^{-\left|z_{p}\right|^{2}}$, applying Theorem 1.1 and inspecting the function $e^{\left|z_{p}\right|^{2}\left(e^{y^{\prime}}-1\right)}-1$, we may check that it is not algebraizable at such points $p$ with $z_{p} \neq 0$ (see §7.5). It follows trivially that the hypersurface $M_{\mathrm{HJY}}$ is also not locally algebraizable at all the points $p$ with $z_{p}=0$, giving the result of [HJY, Theorem 1.1]. Using the same strategy as for Theorem 1.1, we obtain more generally the following criterion:

Theorem 1.4. Let $M_{\varphi}: v=\varphi(z \bar{z})$ be a Levi nondegenerate real analytic hypersurface in $\mathbb{C}^{2}$ passing through the origin whose Lie algebra of local infinitesimal CR automorphisms is generated by $\partial_{w}$ and $i z \partial_{z}$. If $M_{\varphi}$ is locally algebraizable at the origin, then the first derivative $\partial_{r} \varphi$ in $\varphi(r \in \mathbb{R})$ is algebraic. For instance, the following seven explicit examples are not locally algebraizable at the origin: $v=e^{z \bar{z}}-1, v=\sin (z \bar{z}), v=\tan (z \bar{z})$, $v=\sinh (z \bar{z}), v=\tanh (z \bar{z}), v=\sin (\sin (z \bar{z}))$ and $v=e^{e^{z \bar{z}}-1}-1$.

Finally, using the same recipe as for Theorems 1.1 and 1.4, we shall provide a very simple criterion for the local nonalgebraizability of some hypersurfaces having a local Lie CR automorphism group of dimension equal to one exactly. We consider the class $\mathscr{R}_{n}$ of Levi nondegenerate real analytic hypersurfaces passing through the origin in $\mathbb{C}^{n}(n \geq 2)$ such that the Lie algebra of infinitesimal CR automorphisms of $M$ is generated by exactly one holomorphic vector field $X_{1}$ with holomorphic coefficients not all vanishing at the origin. We call $\mathscr{R}_{n}$ the class of strongly rigid hypersurfaces, in order to distinguish them from the so-called rigid ones whose local CR automorphism group may be of dimension larger than 1 . By straightening $X_{1}$, we may assume that $X_{1}=\partial_{w}$ and that $M$ is given by a real analytic equation of the form $v=\varphi(z, \bar{z})=\varphi\left(z_{1}, \ldots, z_{n-1}, \bar{z}_{1}, \ldots, \bar{z}_{n-1}\right)$. By making some elementary changes of coordinates ( $c f . \S 3.3$ ), we can furthermore assume without loss of generality that $\varphi(z, \bar{z})=\sum_{k=1}^{n-1} \varepsilon_{k}\left|z_{k}\right|^{2}+\chi(z, \bar{z})$, where $\varepsilon_{k}= \pm 1$ and $\chi(0, \bar{z}) \equiv \partial_{z_{k}} \chi(0, \bar{z}) \equiv 0$.

Theorem 1.5. Let $M: v=\varphi(z, \bar{z})=\sum_{k=1}^{n-1} \varepsilon_{k}\left|z_{k}\right|^{2}+\chi(z, \bar{z})$ be a strongly rigid hypersurface in $\mathbb{C}^{n}$ with $\chi(0, \bar{z}) \equiv \partial_{z_{k}} \chi(0, \bar{z}) \equiv 0$. If $M$ is locally algebraizable at the origin, then all the first derivatives $\partial_{z_{k}} \varphi$ are algebraic functions of $(z, \bar{z})$.

This criterion enables us to exhibit a whole family of non locally algebraizable hypersurfaces in $\mathbb{C}^{n}$ :

Corollary 1.6. The rigid hypersurfaces $M_{\chi_{1}, \ldots, \chi_{n-1}}$ in $\mathbb{C}^{n}$ of equation $v=$ $\sum_{k=1}^{n-1}\left[\varepsilon_{k}\left|z_{k}\right|^{2}+\left|z_{k}\right|^{10}+\left|z_{k}\right|^{14}+\left|z_{k}\right|^{16}\left(z_{k}+\bar{z}_{k}\right)+\left|z_{k}\right|^{18}\left|z_{1}\right|^{2} \cdots\left|z_{k-1}\right|^{2}+\right.$
$\left.\left|z_{k}\right|^{2 n+16} \chi_{k}(z, \bar{z})\right]$, where the $\chi_{k}$ are arbitrary real analytic functions, belong to the class $\mathscr{R}_{n}$ of strongly rigid hypersurfaces. Two such tubes $M_{\chi_{1}, \ldots, \chi_{n-1}}$ and $M_{\widehat{\chi}_{1}, \ldots, \widehat{\chi}_{n-1}}$ are biholomorphically equivalent if and only if
$\chi_{k}=\widehat{\chi}_{k}$ for $k=1, \ldots, n-1$. Furthermore, for a generic choice of a ( $n-1$ )-tuple of real analytic functions $\left(\chi_{1}, \ldots, \chi_{n-1}\right)$ in the sense of Baire (to be precised in §8), $M_{\chi_{1}, \ldots, \chi_{n-1}}$ is not locally algebraizable at the origin.

Finally, by computing generators of the Lie algebra of local infinitesimal CR automorphisms of some explicit examples, we obtain:

Corollary 1.7. The following seven explicit examples of hypersurfaces in $\mathbb{C}^{2}$ are strongly rigid and are not locally algebraizable at the origin $: v=$ $z \bar{z}+z^{2} \bar{z}^{2} \sin (z+\bar{z}), v=z \bar{z}+z^{2} \bar{z}^{2} \exp (z+\bar{z}), v=z \bar{z}+z^{2} \bar{z}^{2} \cos (z+\bar{z})$, $v=z \bar{z}+z^{2} \bar{z}^{2} \tan (z+\bar{z}), v=z \bar{z}+z^{2} \bar{z}^{2} \sinh (z+\bar{z}), v=z \bar{z}+z^{2} \bar{z}^{2} \cosh (z+\bar{z})$ and $v=z \bar{z}+z^{2} \bar{z}^{2} \tanh (z+\bar{z})$.
1.3. Content of the paper. To prove Theorem 1.1 we consider an algebraic equivalent $M^{\prime}$ of $M$. The main technical part of the proof consists in showing that an arbitrary real algebraic element $M^{\prime}$ of $\mathscr{T}_{n}^{d}$ can be straightened in some local complex algebraic coordinates $t^{\prime} \in \mathbb{C}^{n}$ in order that its infinitesimal CR automorphisms are the real parts of $n$ holomorphic vector fields of the form $X_{i}^{\prime}=c_{i}^{\prime}\left(t_{i}^{\prime}\right) \partial_{t_{i}^{\prime}}, i=1, \ldots, n$, where the variables are separated and the functions $c_{i}^{\prime}\left(t_{i}^{\prime}\right)$ are algebraic. For this, we need to show that the automorphism group of a minimal finitely nondegenerate real algebraic generic submanifold in $\mathbb{C}^{n}$ is a local real algebraic Lie group, a notion defined in $\S 2.3$. A large part of this article $(\S \S 4,5,6)$ is devoted to provide an explicit representation formula for the local biholomorphic self-transformations of a minimal finitely nondegenerate generic submanifold, see especially Theorem 2.1 and Theorem 4.1. Finally, using the specific simplified form of the vector fields $X_{i}^{\prime}$ and assuming that there exists a biholomorphic equivalence $\Phi: M \rightarrow M^{\prime}$ satisfying $\Phi_{*}\left(\partial_{t_{i}}\right)=c_{i}^{\prime}\left(t_{i}^{\prime}\right) \partial_{t_{i}^{\prime}}$, we show by elementary computations that all the first order derivatives of the mapping $\psi^{\prime}\left(y^{\prime}\right)$ must be algebraic. We follow a similar strategy for the proofs of Theorems 1.4 and 1.5. Finally, in §§7-8, we provide the proofs of Corollaries 1.2, 1.3, 1.6 and 1.7.
1.4. Acknowledgment. We acknowledge interesting discussions with Michel Petitot François Boulier at the University of Lille 1.

## §2. Preliminaries

We recall in this section the basic properties of the objects we will deal with.
2.1. Nash algebraic functions and manifolds. In this subsection, let $\mathbb{K}=$ $\mathbb{R}$ or $\mathbb{C}$. Let $\left(x_{1}, \ldots, x_{n}\right)$ denote coordinates over $\mathbb{K}^{n}$. Throughout the article, we shall use the norm $|x|:=\max \left(\left|x_{1}\right|, \ldots,\left|x_{n}\right|\right)$ for $x \in \mathbb{K}^{n}$. Let $\mathscr{K}$ be an open polydisc centered at the origin in $\mathbb{K}^{n}$, namely $\mathscr{K}=$
$\left\{x \in \mathbb{K}^{n}:|x|<\rho\right\}$ for some $\rho>0$. Let $f: \mathscr{K} \rightarrow \mathbb{K}$ be a $\mathbb{K}$ analytic function, defined by a power series converging normally in $\mathscr{K}$. We say that $f$ is (Nash) $\mathbb{K}$-algebraic if there exists a nonzero polynomial $P\left(X_{1}, \ldots, X_{n}, F\right) \in \mathbb{K}\left[X_{1}, \ldots, X_{n}, F\right]$ in $(n+1)$ variables such that the relation $P\left(x_{1}, \ldots, x_{n}, f\left(x_{1}, \ldots, x_{n}\right)\right)=0$ holds for all $\left(x_{1}, \ldots, x_{n}\right) \in \mathscr{K}$. If $\mathbb{K}=\mathbb{R}$, we say that $f$ is real algebraic. If $\mathbb{K}=\mathbb{C}$, we say that $f$ is complex algebraic. The category of $\mathbb{K}$-algebraic functions is stable under elementary algebraic operations, under differentiation and under composition. Furthermore, implicit solutions of $\mathbb{K}$-algebraic equations (for which the real analytic implicit function theorem applies) are again $\mathbb{K}$-algebraic mappings. The theory of $\mathbb{K}$-algebraic manifolds is then defined by the usual axioms of manifolds, for which the authorized changes of chart are $\mathbb{K}$-algebraic mappings only (cf. [Za1995]). In this paper, we shall very often use the stability of algebraicity under differentiation.
2.2. Infinitesimal CR automorphisms. Let $M \subset \mathbb{C}^{n}$ be a generic submanifold of codimension $d \geq 1$ and CR dimension $m=n-d \geq 1$. Let $p \in M$, let $t=\left(t_{1}, \ldots, t_{n}\right)$ be some holomorphic coordinates vanishing at $p$ and for some $\rho>0$, let $\Delta_{n}(\rho):=\left\{t \in \mathbb{C}^{n}:|t|<\rho\right\}$ be an open polydisc centered at $p$. We consider the Lie algebra $\mathfrak{H o l}\left(\Delta_{n}(\rho)\right)$ of holomorphic vector fields of the form $X=\sum_{j=1}^{n} a_{j}(t) \partial / \partial t_{j}$, where the $a_{j}$ are holomorphic functions in $\Delta_{n}(\rho)$. Here, $\mathfrak{H o l}\left(\Delta_{n}(\rho)\right)$ is equipped with the usual Jacobi-Lie bracket operation. We may consider the complex flow $\exp (\sigma X)(q)$ of a vector field $X \in \mathfrak{H o l}\left(\Delta_{n}(\rho)\right)$. It is a holomorphic map of the variables $(\sigma, q)$ which is well defined in some connected open neighborhood of $\{0\} \times \Delta_{n}(\rho)$ in $\mathbb{C} \times \Delta_{n}(\rho)$.

Let $K$ denote the real vector field $K:=X+\bar{X}$, considered as a real vector field over $\mathbb{R}^{2 n} \cong \mathbb{C}^{n}$. Again, the real flow of $K$ is defined in some connected open neighborhood of $\{0\} \times \Delta_{n}(\rho)^{\mathbb{R}}$ in $\mathbb{R} \times \Delta_{n}(\rho)^{\mathbb{R}}$. We remind the following elementary relation between the flow of $K$ and the flow of $X$. For a real time parameter $\sigma:=s \in \mathbb{R}$, the flow $\exp (s X)(q)$ coincides with the real flow of $X+\bar{X}$, namely $\exp (s X)(q)=\exp (s(X+\bar{X}))\left(q^{\mathbb{R}}\right)$, where for $q \in \mathbb{C}^{n}$, we denote $q^{\mathbb{R}}$ the corresponding real point in $\mathbb{R}^{2 n}$. In the sequel, we shall always identify $\Delta_{n}(\rho)$ and its real counterpart $\Delta_{n}(\rho)^{\mathbb{R}}$.

Let now $\mathfrak{H o l}\left(M, \Delta_{n}(\rho)\right)$ denote the real subalgebra of the vector fields $X \in \mathfrak{H o l}\left(\Delta_{n}(\rho)\right)$ such that $X+\bar{X}$ is tangent to $M \cap \Delta_{n}(\rho)$. We also denote by $\mathfrak{A u t}_{C R}\left(M, \Delta_{n}(\rho)\right)$ the Lie algebra of vector fields of the form $X+\bar{X}$, where $X$ belongs to $\mathfrak{H o l}\left(M, \Delta_{n}(\rho)\right)$, so $\mathfrak{A u t}_{C R}\left(M, \Delta_{n}(\rho)\right)=2 \operatorname{Re} \mathfrak{H o l}\left(M, \Delta_{n}(\rho)\right)$. By the above considerations, the local flow $\exp (s X)(q)$ of $X$ with $s \in \mathbb{R}$ real makes a one-parameter family of local biholomorphic transformations of $M$. In the sequel, we shall always identify $\mathfrak{H o l}\left(M, \Delta_{n}(\rho)\right)$ and $\mathfrak{A u t}_{C R}\left(M, \Delta_{n}(\rho)\right)$, namely we shall
identify $X$ and $X+\bar{X}$ and say by some abuse of language that $X$ itself is an infinitesimal CR automorphism.

In the algebraic category, the main drawback of infinitesimal CR automorphism is that they do not have algebraic flow. For instance, the complex dilatation vector field $X=i z \partial_{z}$ has transcendent flow, even if it is an infinitesimal CR automorphism of every algebraic hypersurface in $\mathbb{C}^{2}$ whose equation is of the form $v=\varphi(z \bar{z})$, even if the coefficient of $X$ is algebraic. Thus instead of infinitesimal CR automorphisms which generate oneparameter groups of biholomorphic transformations of $M$, we shall study algebraically dependent one-parameter families of biholomorphic transformations (not necessarily making a one parameter group). To begin with, we need to introduce some precise definitions about local algebraic Lie transformation groups.
2.3. Local Lie group actions in the $\mathbb{K}$-algebraic category. Often in real or in complex analytic geometry, the interest cannot be focalized on global Lie transformation groups, but only on local transformations which are close to the identity. For instance, the transformation group of a small piece of a real analytic CR manifold in $\mathbb{C}^{n}$ which is not contained in a global, large or compact CR manifold is almost never a true, global transformation group. Consequently the usual axioms of Lie transformation groups must be localized. Philosophically speaking, the local point of view is often the most adequate and the richest one, because a given analytico-geometric object often possesses much more local invariant than global invariants, if any. Historically speaking, the local Lie transformation groups were first studied, before the introduction of the now classical notion of global Lie group. Especially, in his first masterpiece work [Lie1880] on the subject, Sophus Lie essentially dealt with local "Lie" groups: he classified all continuous local transformation groups acting on an open subset of $\mathbb{C}^{2}$. This general classification provided afterwards in the years 1880-1890 many applications to the local study of differential equations: local normal forms, local solvability, etc.

In this paragraph we define precisely local actions of local Lie groups and we focus especially on the $\mathbb{K}$-algebraic category.

Let $c \in \mathbb{N}_{*}$, let $g=\left(g_{1}, \ldots, g_{c}\right) \in \mathbb{K}^{c}$ and let two positive numbers satisfy $0<\delta_{2}<\delta_{1}$. We formulate the desired definition by means of the two precise polydiscs $\Delta_{c}\left(\delta_{2}\right) \subset \Delta_{c}\left(\delta_{1}\right) \subset \mathbb{K}^{c}$. A local $\mathbb{K}$-algebraic Lie group of dimension $c$ consists of the following data:
(1) A $\mathbb{K}$-algebraic multiplication mapping $\mu: \Delta_{c}\left(\delta_{2}\right) \times \Delta_{c}\left(\delta_{2}\right) \rightarrow \Delta_{c}\left(\delta_{1}\right)$ which is locally associative $\left(\mu\left(g, \mu\left(g^{\prime}, g^{\prime \prime}\right)\right)=\mu\left(\mu\left(g, g^{\prime}\right), g^{\prime \prime}\right)\right)$, whenever $\mu\left(g^{\prime}, g^{\prime \prime}\right) \in \Delta_{c}\left(\delta_{2}\right), \mu\left(g, g^{\prime}\right) \in \Delta_{c}\left(\delta_{2}\right)$ and which satisfies $\mu(0, g)=\mu(g, 0)=g$, where the origin $0 \in \mathbb{K}^{c}$ corresponds to the identity element in the group structure.
(2) A $\mathbb{K}$-algebraic inversion mapping $\iota: \Delta_{c}\left(\delta_{2}\right) \rightarrow \Delta_{c}\left(\delta_{1}\right)$ satisfying $\mu(g, \iota(g))=\mu(\iota(g), g)=0$ and $\iota(0)=0$ whenever $\iota(g) \in \Delta_{c}\left(\delta_{2}\right)$.

Here, the integer $c \in \mathbb{N}_{*}$ is the dimension of $G$. We shall say that composition and inversion are defined locally in a neighborhood of the identity element. In the $\mathbb{K}$-analytic category, the corresponding definition is similar.

Now, we can define the notion of local $\mathbb{K}$-algebraic Lie group action. Let $n \in \mathbb{N}_{*}$, let $x=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{K}^{n}$ and let two positive numbers satisfy $0<$ $\rho_{2}<\rho_{1}$. Let $G$ be a local $\mathbb{K}$-algebraic Lie group as defined just above. We shall formulate the desired definition by means of the two precise polydiscs $\Delta_{n}\left(\rho_{2}\right) \subset \Delta_{n}\left(\rho_{1}\right)$. This pair of polydiscs represents a local $\mathbb{K}$-algebraic manifold up to changes of $\mathbb{K}$-algebraic coordinates. A local $\mathbb{K}$-algebraic Lie group action on a local $\mathbb{K}$-algebraic manifold consists of a $\mathbb{K}$-algebraic action mapping $x^{\prime}=\Phi(x ; g)$ defined over $\Delta_{n}\left(\rho_{2}\right) \times \Delta_{c}\left(\delta_{2}\right)$ with values in $\Delta_{n}\left(\rho_{1}\right)$ which satisfies:
(1) $\left.\Phi\left(\Phi(x ; g) ; g^{\prime}\right)\right)=\Phi\left(x ; \mu\left(g, g^{\prime}\right)\right)$ whenever $\Phi(x ; g) \in \Delta_{n}\left(\rho_{2}\right)$ and $\mu\left(g, g^{\prime}\right) \in \Delta_{c}\left(\delta_{2}\right)$, where the local group multiplication $\mu\left(g, g^{\prime}\right)$ is $\mathbb{K}$ algebraic as above;
(2) $\Phi(x ; e)=x$ and $\Phi(\Phi(x ; g) ; \iota(g))=x$ whenever $\Phi(x ; g) \in \Delta_{n}\left(\rho_{2}\right)$ and $\iota(g) \in \Delta_{c}\left(\rho_{2}\right)$, where the inverse group mapping $g \mapsto \iota(g)$ is $\mathbb{K}$-algebraic as above.

In this definition, it is allowed to suppose that $x \in \mathbb{C}^{n}$ and $g \in \mathbb{R}^{c}$, which is the case to be considered in the sequel. By differentiation, every local $\mathbb{K}$-algebraic action gives rise to vector fields defined over $\Delta_{n}\left(\rho_{2}\right)$ which are infinitesimal generators of the action. Indeed, let us consider the algebraically dependent one-parameter families of complex algebraic biholomorphic transformations $\Phi\left(x ; 0, \ldots, 0, g_{i}, 0, \ldots, 0\right)=$ : $\Phi_{i}\left(x ; g_{i}\right) \equiv\left(\Phi_{i, 1}\left(x ; g_{i}\right), \ldots, \Phi_{i, n}\left(x ; g_{i}\right)\right) \in \mathbb{K}^{n}$, which we shall also denote by $\Phi_{i, g_{i}}(x)$. In general, such a family does not make a one-parameter group of transformations, but we can nevertheless introduce the vector fields $X_{i}\left(\Phi_{i, g_{i}}(x) ; g_{i}\right):=\partial_{g_{i}} \Phi_{i}\left(x ; g_{i}\right)=\sum_{l=1}^{n} \partial_{g_{i}} \Phi_{i, l}\left(x ; g_{i}\right) \partial / \partial x_{l}$. We notice that the coefficients of these vector fields do in general depend on the group parameter $g_{i} \in G$.

In fact, in the algebraic category, there is no hope to modify the coordinates on the group in order that the infinitesimal generators of the action are independent of the parameter coordinates $g_{j}$. For instance, the trivial one-dimensional action (complex dilatation) defined by $(z, w) \mapsto((1+$ $g) z, w)=: \Phi(z, w ; g)$, where $(z, w) \in \mathbb{C}^{2}$ and $g \in \mathbb{C}$ is clearly an algebraic action. Here, the infinitesimal generator $X(x ; g)=(1+g)^{-1} z \partial_{z}$ depends on the parameter $g$. The only way to avoid the dependence upon $g$ of the coefficient of $X$ is to change coordinates on the group by setting $1+g:=e^{\sigma}, \sigma \in$ $\mathbb{C}$, whence the action is represented by $(z, w) \mapsto\left(e^{\sigma} z, w\right)=: \Phi(z, w ; \sigma)$.

Indeed, from the group property $\Phi\left(\Phi(z, w ; \sigma) ; \sigma^{\prime}\right) \equiv \Phi\left(z, w ; \sigma+\sigma^{\prime}\right)$, it is classical and immediate to deduce that if we define the parameter independent vector field $X^{0}(z, w):=\left.\partial_{\sigma} \Phi(z, w ; \sigma)\right|_{\sigma=0}=z \partial_{z}$, then it holds that $\partial_{\sigma} \Phi(z, w ; \sigma)=e^{\sigma} z \partial_{z}=X^{0}(\Phi(z, w ; \sigma))$. So the infinitesimal generator of the action is independent of the parameter $g$. However, the main trouble here is that the algebraicity of the action is necessarily lost since the flow of $X^{0}$ is not algebraic (the reader may check that each right (or left) invariant vector field on an algebraic local Lie group defines in general a nonalgebraic one-parameter subgroup, e.g. for $\mathrm{SO}(2, \mathbb{R}), \mathrm{SL}(2, \mathbb{C})$ ).

Consequently we may allow the infinitesimal generators of an algebraic local Lie group action $x^{\prime}=\Phi(x ; g)$, defined by $X_{i}\left(x ; g_{i}\right)$ := $\left[\partial_{g_{i}} \Phi_{i}\right]\left(\Phi_{i, g_{i}}^{-1}(x) ; g_{i}\right)$ to depend on the group parameter $g_{i}$, even if the families $\left(\Phi_{i, g_{i}}(x)\right)_{g_{i} \in \mathbb{K}}$ do not constitute one-dimensional subgroups of transformations.
2.4. Algebraicity of complex flow foliations. Suppose now that $M$ is a real algebraic generic submanifold in $\mathbb{C}^{n}$, for instance a hypersurface which is Levi nondegenerate at a "center" point $p \in M$ corresponding to the origin in the coordinates $t=\left(t_{1}, \ldots, t_{n}\right)$. Let $X \in \mathfrak{H o l}(M)$ be an infinitesimal CR automorphism. Even if, for fixed real $s$, the biholomorphic mapping $t \mapsto \exp (s X)(t)$ is complex algebraic, i.e. the $n$ components of this biholomorphism are complex algebraic functions by Webster's theorem [We1977], we know by considering the infinitesimal CR automorphism $X_{1}:=i(z+1) \partial_{z}$ of the strong tube $\operatorname{Im} w=|z+1|^{2}+|z+1|^{6}-2$ in $\mathbb{C}^{2}$ passing through the origin, that the flow of $X$ is not necessarily algebraic with respect to all variables $(s, t)$.

Nevertheless, we shall show that the local CR automorphism group of $M$ is a local algebraic Lie group whose general transformations are of the form $t^{\prime}=H\left(t ; e_{1}, \ldots, e_{c}\right)$, where $t \in \mathbb{C}^{n}$ and $\left(e_{1}, \ldots, e_{c}\right) \in \mathbb{R}^{c}$ and where $H$ is algebraic with respect to all its variables. Thus the "time" dependent vector fields defined by $X_{i}\left(t ; e_{i}\right):=\left[\partial_{e_{i}} H_{i}\right]\left(H_{i, e_{i}}^{-1}(t) ; e_{i}\right)$, where $H_{i, e_{i}}(t):=$ $H_{i}\left(t ; e_{i}\right):=H\left(t ; 0, \ldots, 0, e_{i}, 0, \ldots, 0\right)$, have an algebraic flow, simply given by $\left(t, e_{i}\right) \mapsto H_{i}\left(t ; e_{i}\right)$. It follows that each foliation defined by the complex integral curves of the time dependent complex vector fields $X_{i}, i=1, \ldots, c$, is a complex algebraic foliation, see $\S 3$ below. Now, we can state the main technical theorem of this paper, whose proof is postponed to $\S 4, \S 5$ and $\S 6$.

Theorem 2.1. Let $M \subset \mathbb{C}^{n}$ be a real algebraic connected geometrically smooth generic submanifold of codimension $d \geq 1$ and $C R$ dimension $m=$ $n-d \geq 1$. Let $p \in M$ and assume that $M$ is finitely nondegenerate and minimal at $p$. Then for every sufficiently small nonempty open polydisc $\Delta_{1}$ centered at $p$, the following three properties hold:
(1) The complex Lie algebra $\mathfrak{H o l}\left(M, \Delta_{1}\right)$ is of finite dimension $c \in \mathbb{N}$ which depends only on the local geometry of $M$ in a neighborhood of $p$.
(2) There exists a nonempty open polydisc $\Delta_{2} \subset \Delta_{1}$ also centered at $p$ and $a \mathbb{C}^{n}$-valued mapping $H(t ; e)=H\left(t ; e_{1}, \ldots, e_{c}\right)$ with $H(t ; 0) \equiv t$ which is defined in a neighborhood of the origin in $\mathbb{C}^{n} \times \mathbb{R}^{c}$ and which is algebraic with respect to both its variables $t \in \mathbb{C}^{n}$ and $e \in \mathbb{R}^{c}$ such that for every holomorphic map $h: \Delta_{2} \rightarrow \Delta_{1}$ with $h\left(\Delta_{2} \cap M\right) \subset$ $\Delta_{1} \cap M$ which is sufficiently close to the identity map, there exists a unique $e \in \mathbb{R}^{c}$ such that $h(t)=H(t ; e)$.
(3) The mapping $(t, e) \mapsto H(t ; e)$ constitutes a $\mathbb{K}$-algebraic local Lie transformation group action. More precisely, there exist a local multiplication mapping $\left(e, e^{\prime}\right) \mapsto \mu\left(e, e^{\prime}\right)$ and a local inversion mapping $e \mapsto \iota(e)$ such that $H, \mu$ and $\iota$ satisfy the axioms of local algebraic Lie group action as defined in §2.3.
(4) The c "time dependent" holomorphic vector fields

$$
\begin{equation*}
X_{i}\left(t ; e_{i}\right):=\left[\partial_{e_{i}} H_{i}\right]\left(H_{i, e_{i}}^{-1}(t) ; e_{i}\right), \tag{2.1}
\end{equation*}
$$

where $H_{i, e_{i}}(t):=H_{i}\left(t ; e_{i}\right):=H\left(t ; 0, \ldots, 0, e_{i}, 0, \ldots, 0\right)$, have algebraic coefficients and have an algebraic flow, given by $\left(t, e_{i}\right) \mapsto$ $H_{i}\left(t ; e_{i}\right)$.
In the case where $M$ is real analytic, the same theorem holds true with the word "algebraic" everywhere replaced by the word "analytic".

A special case of Theorem 2.1 was proved in [BER1999b] where, apparently, the authors do not deal with the notion of local Lie groups and consider the isotropy group of the point $p$, namely the group of holomorphic self-maps of $M$ fixing $p$. The consideration of the complete local Lie group of biholomorphic self-maps of a piece of $M$ in a neighborhood of $p$ not only the isotropy group of $p$ ) is crucial for our purpose, since we shall have to deal with strong tubes $M \in \mathscr{T}_{n}^{d}$ for which the isotropy group of $p \in M$ is trivial. Sections $\S 4, \S 5$ and $\S 6$ are devoted to the proof of Theorem 4.1, a precise statement of Theorem 2.1. We mention that our method of proof of Theorem 2.1 gives a non optimal bound for the dimension of $\mathfrak{H o l}\left(M, \Delta_{1}\right)$. To our knowledge, the upper bound $c \leq(n+1)^{2}-1$ is optimal only in codimension $d=1$ and in the Levi nondegenerate case.

## §3. Proof of Theorem 1.1

We take in this section Theorem 2.1 for granted. As explained in $\S 1.3$ above, we shall conduct the proof of Theorem 1.1 in two essential steps ( $\S 3.1$ and 3.2). The strategy for the proof of Theorems 1.4 and 1.5 is similar and we prove them in $\S \S 3.3$ and 3.4. Let $M \in \mathscr{T}_{n}^{d}$ be a strong tube of
codimension $d$ passing through the origin in $\mathbb{C}^{n}$ given by the equations $v_{j}=$ $\varphi_{j}(y), j=1, \ldots, d$. Assume that $M$ is biholomorphically equivalent to a real algebraic generic submanifold $M^{\prime}$.

First step. We show that an arbitrary real algebraic element $M^{\prime} \in \mathscr{T}_{n}^{d}$ can be straightened in some local complex algebraic coordinates $t^{\prime}=\left(t_{1}^{\prime}, \ldots, t_{n}^{\prime}\right) \in$ $\mathbb{C}^{n}$ in order that its infinitesimal CR automorphisms are the $n$ holomorphic vector fields of the specific form $X_{i}^{\prime}=c_{i}^{\prime}\left(t_{i}^{\prime}\right) \partial_{t_{i}^{\prime}}, i=1, \ldots, n$, where the functions $c_{i}^{\prime}\left(t_{i}^{\prime}\right)$ are algebraic.

Second step. Assuming that there exists a biholomorphic equivalence $\Phi$ : $M \rightarrow M^{\prime}$ satisfying $\Phi_{*}\left(\partial_{t_{i}}\right)=c_{i}^{\prime}\left(t^{\prime}\right) \partial_{t_{i}^{\prime}}$, we prove by direct computation that all the first order derivatives of the mapping $\psi^{\prime}\left(y^{\prime}\right)$ must be algebraic.
3.1. Proof of the first step. Let $t^{\prime}=\Phi(t)$ be such an equivalence, with $\Phi(0)=0$ and $M^{\prime}:=\Phi(M)$ real algebraic. Let $X_{i}:=\partial_{t_{i}}, i=1, \ldots, n$, be the $n$ infinitesimal CR automorphisms of $M$ and set $X_{i}^{\prime}:=\Phi_{*}\left(X_{i}\right)$. Of course, we have $\left[X_{i_{1}}^{\prime}, X_{i_{2}}^{\prime}\right]=\Phi_{*}\left(\left[X_{i_{1}}, X_{i_{2}}\right]\right)=0$, so the CR automorphism group of $M^{\prime}$ is also $n$-dimensional and commutative. Let us choose complex algebraic coordinates $t^{\prime}$ in a neighborhood of $0 \in M^{\prime}$ such that $\left.X_{i}^{\prime}\right|_{0}=$ $\partial_{t_{i}^{\prime}} \mid 0$. Let us apply Theorem 2.1 to the real algebraic submanifold $M^{\prime}$, noting all the datas with dashes. There exists an algebraic mapping $H^{\prime}\left(t^{\prime} ; e\right)=$ $H^{\prime}\left(t^{\prime} ; e_{1}, \ldots, e_{n}\right)$ such that every local biholomorphic self-map of $M^{\prime}$ writes uniquely $t^{\prime} \mapsto H^{\prime}\left(t^{\prime} ; e\right)$, for some $e \in \mathbb{R}^{n}$. In particular, for every $i=$ $1, \ldots, n$ and every small $s \in \mathbb{R}$, there exists $e_{s} \in \mathbb{R}^{n}$ depending on $s$ such that $\exp \left(s X_{i}^{\prime}\right)\left(t^{\prime}\right) \equiv H^{\prime}\left(t^{\prime} ; e_{s}\right)$. From the commutativity of the flows of the $X_{i}^{\prime}$, i.e. $\operatorname{from} \exp \left(s_{1} X_{i_{1}}^{\prime}\left(\exp \left(s_{2} X_{i_{2}}^{\prime}\left(t^{\prime}\right)\right)\right)\right) \equiv \exp \left(s_{2} X_{i_{2}}^{\prime}\left(\exp \left(s_{1} X_{i_{1}}^{\prime}\left(t^{\prime}\right)\right)\right)\right)$, we get

$$
\begin{equation*}
H^{\prime}\left(H^{\prime}\left(t^{\prime} ; e_{2}\right) ; e_{1}\right) \equiv H^{\prime}\left(H^{\prime}\left(t^{\prime} ; e_{1}\right) ; e_{2}\right) \tag{3.1}
\end{equation*}
$$

This shows that the biholomorphisms $t^{\prime} \mapsto H_{e}^{\prime}\left(t^{\prime}\right):=H^{\prime}\left(t^{\prime} ; e\right)$ commute pairwise. In particular, if we define

$$
\begin{equation*}
G_{i}^{\prime}\left(t^{\prime} ; e_{i}\right):=H^{\prime}\left(t^{\prime} ; 0, \ldots, 0, e_{i}, 0, \ldots, 0\right), \tag{3.2}
\end{equation*}
$$

we have $G_{i_{1}}^{\prime}\left(G_{i_{2}}^{\prime}\left(t^{\prime} ; e_{2}\right) ; e_{1}\right) \equiv G_{i_{2}}^{\prime}\left(G_{i_{1}}^{\prime}\left(t^{\prime} ; e_{1}\right) ; e_{2}\right)$.
Next, after making a linear change of coordinates in the $e$-space, we can insure that $\left.\partial_{e_{i}} G_{i}^{\prime}\left(0 ; e_{i}\right)\right|_{e_{i}=0}=\left.\partial_{t_{i}^{\prime}}\right|_{0}=\left.X_{i}^{\prime}\right|_{0}$ for $i=1, \ldots, n$. Finally, complexifying the real variable $e_{i}$ in a complex variable $\epsilon_{i}$, we get mappings $G_{i}^{\prime}\left(t_{i}^{\prime} ; \epsilon_{i}\right)$ which are complex algebraic with respect to both variables $t^{\prime} \in \mathbb{C}^{n}$ and $\epsilon_{i} \in \mathbb{C}$ and which commute pairwise. We can now state and prove the following crucial proposition (where we have dropped the dashes) according to which we can straighten commonly the $n$ one-parameter families of biholomorphisms $t^{\prime} \mapsto G_{i}^{\prime}\left(t^{\prime} ; \epsilon_{i}\right)$.

Proposition 3.1. Let $t \mapsto G_{i}\left(t ; \epsilon_{i}\right), i=1, \ldots, n$, be $n$ one complex parameter families of complex algebraic biholomorphic maps from a neighborhood of 0 in $\mathbb{C}^{n}$ onto a neighborhood of 0 in $\mathbb{C}^{n}$ satisfying $G_{i}(t ; 0) \equiv t$, $\left.\partial_{\epsilon_{i}} G_{i}\left(0 ; \epsilon_{i}\right)\right|_{\epsilon_{i}=0}=\left.\partial_{t_{i}}\right|_{0}$ and pairwise commuting: $G_{i_{1}}\left(G_{i_{2}}\left(t ; \epsilon_{2}\right) ; \epsilon_{1}\right) \equiv$ $G_{i_{2}}\left(G_{i_{1}}\left(t ; \epsilon_{1}\right) ; \epsilon_{2}\right)$. Then there exists a complex algebraic biholomorphism of the form $t^{\prime} \mapsto \Phi^{\prime}\left(t^{\prime}\right)=$ : t of $\mathbb{C}^{n}$ fixing the origin with $d \Phi^{\prime}(0)=$ Id such that if we set $G_{i}^{\prime}\left(t^{\prime} ; \epsilon_{i}\right):=\Phi^{\prime-1}\left(G_{i}\left(\Phi^{\prime}\left(t^{\prime}\right) ; \epsilon_{i}\right)\right)$, where $t^{\prime}=\Phi(t)$ denote the inverse of $t=\Phi^{\prime}\left(t^{\prime}\right)$, then we have

$$
\begin{equation*}
G_{i}^{\prime}\left(t^{\prime} ; \epsilon_{i}\right) \equiv\left(t_{1}^{\prime}, \ldots, t_{i-1}^{\prime}, G_{i, i}^{\prime}\left(t_{i}^{\prime} ; \epsilon_{i}\right), t_{i+1}^{\prime}, \ldots, t_{n}^{\prime}\right), \tag{3.3}
\end{equation*}
$$

where the functions $G_{i, i}^{\prime}$ are complex algebraic, depend only on $t_{i}^{\prime}$ (and on $\left.\epsilon_{i}\right)$ and satisfy $G_{i, i}^{\prime}\left(t_{i}^{\prime} ; 0\right) \equiv t_{i}^{\prime}$ and $\left.\partial_{\epsilon_{i}} G_{i, i}^{\prime}\left(0 ; \epsilon_{i}\right)\right|_{\epsilon_{i}=0}=1$.
Proof. First of all, we define the complex algebraic biholomorphism

$$
\begin{equation*}
\Phi_{1}^{\prime}: \quad\left(t_{1}^{\prime}, t_{2}^{\prime}, \ldots, t_{n}^{\prime}\right) \longmapsto G_{1}\left(0, t_{2}^{\prime}, \ldots, t_{n}^{\prime} ; t_{1}^{\prime}\right)=: t . \tag{3.4}
\end{equation*}
$$

We have $d \Phi_{1}^{\prime}(0)=$ Id, because $G_{1}(t ; 0) \equiv t$ and $\left.\partial_{\epsilon_{1}} G_{1}\left(0 ; \epsilon_{1}\right)\right|_{\epsilon_{1}=0}=\left.\partial_{t_{1}}\right|_{0}$. Furthermore, since $\left.\partial_{\epsilon_{1}} G_{1}\left(0 ; \epsilon_{1}\right)\right|_{\epsilon_{1}=0}$ is transversal to $\left\{\left(0, t_{2}, \ldots, t_{n}\right)\right\}$, it also follows that a small neighborhood of the origin in $\mathbb{C}_{t}^{n}$ is algebraically foliated by the $(n-1)$-parameter family of complex curves $\mathscr{C}_{t_{2}^{\prime}, \ldots, t_{n}^{\prime}}^{\prime}:=$ $\left\{G_{1}\left(0, t_{2}^{\prime}, \ldots, t_{n}^{\prime} ; t_{1}^{\prime}\right):\left|t_{1}^{\prime}\right|<\delta\right\}$ where $\delta>0$ is small and $t_{2}^{\prime}, \ldots, t_{n}^{\prime}$ are fixed. The existence of this foliation shows that the relation

$$
\begin{equation*}
t^{*} \sim t \text { iff there exists } \epsilon_{1} \text { such that } t^{*}=G_{1}\left(t ; \epsilon_{1}\right) \tag{3.5}
\end{equation*}
$$

is a local equivalence relation, whose equivalence classes are the leaves $\mathscr{C}_{t_{2}^{\prime}, \ldots, t_{n}^{\prime}}^{\prime}($ see Figure 1).


Consequently, as we clearly have

$$
\begin{equation*}
\left(0, t_{2}^{\prime}, \ldots, t_{n}^{\prime}\right) \sim G_{1}\left(0, t_{2}^{\prime}, \ldots, t_{n}^{\prime} ; t_{1}^{\prime}\right) \sim G_{1}\left(G_{1}\left(0, t_{2}^{\prime}, \ldots, t_{n}^{\prime} ; t_{1}^{\prime}\right) ; \epsilon_{1}\right), \tag{3.6}
\end{equation*}
$$

using the transitivity of the relation $\sim$, it follows that there exists a complex number $\varepsilon_{t^{\prime}, \epsilon_{1}}$ depending on $t^{\prime}$ and on $\epsilon_{1}$ such that

$$
\begin{equation*}
G_{1}\left(G_{1}\left(0, t_{2}^{\prime}, \ldots, t_{n}^{\prime} ; t_{1}^{\prime}\right) ; \epsilon_{1}\right)=G_{1}\left(0, t_{2}^{\prime}, \ldots, t_{n}^{\prime} ; \varepsilon_{t^{\prime}, \epsilon_{1}}\right) . \tag{3.7}
\end{equation*}
$$

By the very definition (3.4) of $\Phi_{1}^{\prime}$, this is equivalent to

$$
\begin{equation*}
\Phi_{1}\left(G_{1}\left(\Phi_{1}^{\prime}\left(t^{\prime}\right) ; \epsilon_{1}\right)\right)=\left(\varepsilon_{t^{\prime}, \epsilon_{1}}, t_{2}^{\prime}, \ldots, t_{n}^{\prime}\right) \tag{3.8}
\end{equation*}
$$

where $t^{\prime}=\Phi_{1}(t)$ denotes the inverse of $t=\Phi_{1}^{\prime}\left(t^{\prime}\right)$. Finally, since the left hand side of (3.8) is clearly a complex algebraic mapping of $\left(t^{\prime} ; \epsilon_{1}\right)$, it follows that there exists a complex algebraic function $G_{1,1}^{\prime}\left(t^{\prime} ; \epsilon_{1}\right)$ such that we can write

$$
\begin{equation*}
\Phi_{1}\left(G_{1}\left(\Phi_{1}^{\prime}\left(t^{\prime}\right) ; \epsilon_{1}\right)\right) \equiv\left(G_{1,1}^{\prime}\left(t^{\prime} ; \epsilon_{1}\right), t_{2}^{\prime}, \ldots, t_{n}^{\prime}\right) \tag{3.9}
\end{equation*}
$$

So we have straightened the first family by means of $\Phi_{1}^{\prime}$.
Next, we drop the dashes and we restart with $G_{1}\left(t ; \epsilon_{1}\right)=$ $\left(G_{1,1}\left(t ; \epsilon_{1}\right), t_{2}, \ldots, t_{n}\right)$. Then, similarly as above, by introducing the complex algebraic biholomorphism

$$
\begin{equation*}
\Phi_{2}^{\prime}:\left(t_{1}^{\prime}, t_{2}^{\prime}, t_{3}^{\prime}, \ldots, t_{n}^{\prime}\right) \longmapsto G_{2}\left(t_{1}^{\prime}, 0, t_{3}^{\prime}, \ldots, t_{n}^{\prime} ; t_{2}^{\prime}\right), \tag{3.10}
\end{equation*}
$$

which satisfies $d \Phi_{2}^{\prime}(0)=\mathrm{Id}$, and by denoting by $t^{\prime}=\Phi_{2}(t)$ the inverse of $t=\Phi_{2}^{\prime}\left(t^{\prime}\right)$, we get again that if we set $G_{2}^{\prime}\left(t^{\prime} ; \epsilon_{2}\right):=\Phi_{2}\left(G_{2}\left(\Phi_{2}^{\prime}\left(t^{\prime}\right) ; \epsilon_{2}\right)\right)$, then

$$
\begin{equation*}
G_{2}^{\prime}\left(t^{\prime} ; \epsilon_{2}\right) \equiv\left(t_{1}^{\prime}, G_{2,2}^{\prime}\left(t^{\prime} ; \epsilon_{2}\right), t_{3}^{\prime}, \ldots, t_{n}^{\prime}\right), \tag{3.11}
\end{equation*}
$$

where the complex algebraic function $G_{2,2}^{\prime}\left(t^{\prime} ; \epsilon_{2}\right)$ satisfies $\left.\partial_{\epsilon_{2}} G_{2,2}^{\prime}\left(0 ; \epsilon_{2}\right)\right|_{\epsilon_{2}=0}=1$ and $G_{2,2}^{\prime}\left(t^{\prime} ; 0\right) \equiv t_{2}^{\prime}$.

We also have to consider the modification of the first family of biholomorphisms $G_{1}^{\prime}\left(t^{\prime} ; \epsilon_{1}\right):=\Phi_{2}\left(G_{1}\left(\Phi_{2}^{\prime}\left(t^{\prime}\right) ; \epsilon_{1}\right)\right)$. Using in an essential way the commutativity, we may compute

$$
\left\{\begin{align*}
\Phi_{2}^{\prime}\left(G_{1}^{\prime}\left(t^{\prime} ; \epsilon_{1}\right)\right) & =G_{1}\left(\Phi_{2}^{\prime}\left(t^{\prime}\right) ; \epsilon_{1}\right)  \tag{3.12}\\
& =G_{1}\left(G_{2}\left(t_{1}^{\prime}, 0, t_{3}^{\prime}, \ldots, t_{n}^{\prime} ; t_{2}^{\prime}\right) ; \epsilon_{1}\right) \\
& =G_{2}\left(G_{1}\left(t_{1}^{\prime}, 0, t_{3}^{\prime}, \ldots, t_{n}^{\prime} ; \epsilon_{1}\right) ; t_{2}^{\prime}\right) \\
& =G_{2}\left(G_{1,1}\left(t_{1}^{\prime}, 0, t_{3}^{\prime}, \ldots, t_{n}^{\prime} ; \epsilon_{1}\right), 0, t_{3}^{\prime}, \ldots, t_{n}^{\prime} ; t_{2}^{\prime}\right) \\
& =\Phi_{2}^{\prime}\left(G_{1,1}^{\prime}\left(t_{1}^{\prime}, 0, t_{3}^{\prime}, \ldots, t_{n}^{\prime} ; \epsilon_{1}\right), t_{2}^{\prime}, t_{3}^{\prime}, \ldots, t_{n}^{\prime}\right)
\end{align*}\right.
$$

It follows that

$$
\begin{equation*}
G_{1}^{\prime}\left(t^{\prime} ; \epsilon_{1}\right) \equiv\left(G_{1,1}\left(t_{1}^{\prime}, 0, t_{3}^{\prime}, \ldots, t_{n}^{\prime} ; \epsilon_{1}\right), t_{2}^{\prime}, t_{3}^{\prime}, \ldots, t_{n}^{\prime}\right) \tag{3.13}
\end{equation*}
$$

whence

$$
\begin{equation*}
G_{1,1}^{\prime}\left(t^{\prime} ; \epsilon_{1}\right):=G_{1,1}\left(t_{1}^{\prime}, 0, t_{3}^{\prime}, \ldots, t_{n}^{\prime} ; \epsilon_{1}\right) \tag{3.14}
\end{equation*}
$$

does not depend on $t_{2}^{\prime}$. Finally, inserting (3.11) and (3.13) in the commutativity relation $G_{1}^{\prime}\left(G_{2}^{\prime}\left(t^{\prime} ; \epsilon_{2}\right) ; \epsilon_{1}\right) \equiv G_{2}^{\prime}\left(G_{1}^{\prime}\left(t^{\prime} ; \epsilon_{1}\right) ; \epsilon_{2}\right)$, we find

$$
\left\{\begin{array}{l}
G_{1,1}^{\prime}\left(t_{1}^{\prime}, G_{2,2}^{\prime}\left(t^{\prime} ; \epsilon_{2}\right), t_{3}^{\prime}, \ldots, t_{n}^{\prime} ; \epsilon_{1}\right) \equiv G_{1,1}^{\prime}\left(t^{\prime} ; \epsilon_{1}\right),  \tag{3.15}\\
G_{2,2}^{\prime}\left(G_{1,1}^{\prime}\left(t^{\prime} ; \epsilon_{1}\right), t_{2}^{\prime}, t_{3}^{\prime}, \ldots, t_{n}^{\prime} ; \epsilon_{2}\right) \equiv G_{2,2}^{\prime}\left(t^{\prime} ; \epsilon_{2}\right) .
\end{array}\right.
$$

The first relation gives nothing, since we already know that $G_{1,1}^{\prime}$ is independent of $t_{2}^{\prime}$. By differentiating the second relation with respect to $\epsilon_{1}$ at $\epsilon_{1}=0$, we find that $G_{2,2}^{\prime}$ is independent of $t_{1}^{\prime}$.

In summary, after the change of coordinates $\Phi_{2}^{\prime} \circ \Phi_{1}^{\prime}\left(t^{\prime}\right)=t$ which is tangent to the identity map at $t^{\prime}=0$, we obtained that

$$
\left\{\begin{array}{l}
G_{1}^{\prime}\left(t^{\prime} ; \epsilon_{1}\right)=\left(G_{1,1}^{\prime}\left(t_{1}^{\prime}, t_{3}^{\prime}, \ldots, t_{n}^{\prime}\right), t_{2}^{\prime}, t_{3}^{\prime}, \ldots, t_{n}^{\prime}\right)  \tag{3.16}\\
G_{2}^{\prime}\left(t^{\prime} ; \epsilon_{2}\right)=\left(t_{1}^{\prime}, G_{2,2}^{\prime}\left(t_{2}^{\prime}, t_{3}^{\prime}, \ldots, t_{n}^{\prime}\right), t_{3}^{\prime}, \ldots, t_{n}^{\prime}\right)
\end{array}\right.
$$

Using these arguments, the proof of Proposition 3.1 clearly follows by induction.

Now, we come back to our CR manifold $M^{\prime}$ having the one-parameter families of algebraic biholomorphisms $G_{i}^{\prime}\left(t^{\prime} ; e_{i}\right)$ given by (3.2) and pairwise commuting. Applying Proposition 3.1, after a change of complex algebraic coordinates of the form $t^{\prime}=\Psi^{\prime \prime}\left(t^{\prime \prime}\right)$, we may assume that the $G_{i}^{\prime \prime}\left(t^{\prime \prime} ; \epsilon_{i}\right)$ are algebraic and can be written in the specific form

$$
\begin{equation*}
G_{i}^{\prime \prime}\left(t^{\prime \prime} ; \epsilon_{i}\right) \equiv\left(t_{1}^{\prime \prime}, \ldots, t_{i-1}^{\prime \prime}, G_{i, i}^{\prime \prime}\left(t_{i}^{\prime \prime} ; \epsilon_{i}\right), t_{i+1}^{\prime \prime}, \ldots, t_{n}^{\prime \prime}\right), \tag{3.17}
\end{equation*}
$$

with $\left.\partial_{\epsilon_{i}} G_{i, i}^{\prime \prime}\left(0 ; \epsilon_{i}\right)\right|_{\epsilon_{i}=0}=1$. Let $t^{\prime \prime}=\Psi^{\prime}\left(t^{\prime}\right)$ denote the inverse of $t^{\prime}=$ $\Psi^{\prime \prime}\left(t^{\prime \prime}\right)$. We thus have $t^{\prime \prime}=\Psi^{\prime}\left(t^{\prime}\right)=\Psi^{\prime}(\Phi(t))$, where we remind that $t^{\prime}=$ $\Phi(t)$ provides the equivalence between the strong tube $M$ and the algebraic CR generic $M^{\prime}$.

Since $\Psi^{\prime}$ is algebraic, the image $M^{\prime \prime}:=\Psi^{\prime}\left(M^{\prime}\right)$ is also algebraic. Let $r_{j}^{\prime}\left(t^{\prime}, \bar{t}^{\prime}\right)=0, j=1, \ldots, d$, be defining equations for $M^{\prime}$. Then $r_{j}^{\prime \prime}\left(t^{\prime \prime}, \bar{t}^{\prime \prime}\right):=$ $r_{j}^{\prime}\left(\Psi^{\prime \prime}\left(t^{\prime \prime}\right), \overline{\Psi^{\prime \prime}\left(t^{\prime \prime}\right)}\right)=0$ are defining equations for $M^{\prime \prime}$. By assumption, for $\epsilon_{i}:=e_{i} \in \mathbb{R}$ real, the family of algebraic biholomorphisms $G_{i}^{\prime}\left(t^{\prime} ; \epsilon_{i}\right)$ maps a small piece of $M^{\prime}$ through the origin into $M^{\prime}$. It follows trivially that $G_{i}^{\prime \prime}\left(t^{\prime \prime} ; \epsilon_{i}\right) \equiv \Psi^{\prime}\left(G_{i}^{\prime}\left(\Psi^{\prime \prime}\left(t^{\prime \prime}\right) ; \epsilon_{i}\right)\right)$ maps a small piece of $M^{\prime \prime}$ through the origin into $M^{\prime \prime}$. Furthermore, since $d \Psi^{\prime \prime}(0)=I d$, it follows that if we denote $X_{i}^{\prime \prime}:=\Psi^{\prime}\left(X_{i}^{\prime}\right)$, then $\left.X_{i}^{\prime \prime}\right|_{0}=\left.\partial_{t_{i}^{\prime \prime}}\right|_{0}$.

Next, thanks to the specific form (3.17), by differentiating $\left.\partial_{\epsilon_{i}} G_{i}^{\prime \prime}\left(t^{\prime \prime} ; \epsilon_{i}\right)\right|_{\epsilon_{i}=0}$, we get $n$ vector fields of the form $Z_{i}^{\prime \prime}=c_{i}^{\prime \prime}\left(t_{i}^{\prime \prime}\right) \partial_{t_{i}^{\prime \prime}}$. By construction, the functions $c_{i}^{\prime \prime}\left(t_{i}^{\prime \prime}\right)$ are algebraic and satisfy $c_{i}^{\prime \prime}(0)=1$. Differentiating with respect to $e_{i}$ the identity $r_{j}^{\prime \prime}\left(G_{i}^{\prime \prime}\left(t^{\prime \prime} ; e_{i}\right), \overline{G_{i}^{\prime \prime}\left(t^{\prime \prime} ; e_{i}\right)}\right)=0$ for $r_{j}^{\prime \prime}\left(t^{\prime \prime}, \bar{t}^{\prime \prime}\right)=0$, i.e. for $t^{\prime \prime} \in M^{\prime \prime}$, we see that $Z_{i}^{\prime \prime}$ is tangent to $M^{\prime \prime}$, i.e. we see that $Z_{i}^{\prime \prime}$ is an infinitesimal CR automorphism of $M^{\prime \prime}$. Consequently, there exist real constants $\lambda_{i, l}$ such that $Z_{i}^{\prime \prime}=\sum_{l=1}^{n} \lambda_{i, l} X_{l}^{\prime \prime}$. Since $\left.Z_{i}^{\prime \prime}\right|_{0}=\left.X_{i}^{\prime \prime}\right|_{0}=\left.\partial_{t_{i}^{\prime \prime}}\right|_{0}$, we have in fact $\lambda_{i, l}=1$ for $i=l$ and $\lambda_{i, l}=0$ for $i \neq l$. So $Z_{i}^{\prime \prime}=X_{i}^{\prime \prime}$ and we have shown that

$$
\begin{equation*}
\left(\Psi^{\prime} \circ \Phi\right)_{*}\left(X_{i}\right)=X_{i}^{\prime \prime}=Z_{i}^{\prime \prime}=c_{i}^{\prime \prime}\left(t_{i}^{\prime \prime}\right) \partial_{t_{i}^{\prime \prime}} . \tag{3.18}
\end{equation*}
$$

We shall call a CR generic manifold $M^{\prime \prime}$ having infinitesimal CR automorphisms of the form $X_{i}^{\prime \prime}=c_{i}^{\prime \prime}\left(t_{i}^{\prime \prime}\right) \partial_{t_{i}^{\prime \prime}}$ with $c_{i}^{\prime \prime}(0) \neq 0$ a pseudotube. Such a pseudotube is not in general a product by $\mathbb{R}^{n}$. In fact, there is no hope to tubify all algebraic peudotubes in algebraic coordinates, as shows the elementary example $\operatorname{Im} w=|z+1|^{2}+|z+1|^{6}-2$ having infinitesimal CR automorphisms $\partial_{w}$ and $i(z+1) \partial_{z}$, since the only change of coordinates for which $\Phi_{*}\left(\partial_{w}\right)=\partial_{w^{\prime}}$ and $\Phi_{*}\left(i(z+1) \partial_{z}\right)=\partial_{z^{\prime}}$ is $z+1=e^{i z^{\prime}}$, $w=w^{\prime}$, which transforms $M$ into $M^{\prime}$ of nonalgebraic defining equation $\operatorname{Im} w^{\prime}=e^{-2 y^{\prime}}+e^{-6 y^{\prime}}$.

The constructions of this paragraph may be represented by the following symbolic picture.


Summary and conclusion of the first step. To conclude, let us denote for simplicity $M^{\prime \prime}$ again by $M^{\prime}$, the coordinates $t^{\prime \prime}$ again by $t^{\prime}$ and $t^{\prime \prime}=\Psi^{\prime} \circ$ $\Phi(t)$ by $t^{\prime}=\Phi(t)$. After the above straightenings, we have shown that the infinitesimal CR automorphisms $X_{i}^{\prime}:=\Phi_{*}\left(X_{i}\right)$ of the algebraic generic manifold $M^{\prime}$ are of the sympathetic form $X_{i}^{\prime}=c_{i}^{\prime}\left(t_{i}^{\prime}\right) \partial_{t_{i}^{\prime}}, i=1, \ldots, n$, with algebraic coefficients $c_{i}^{\prime}\left(t_{i}^{\prime}\right)$ satisfying $c_{i}^{\prime}(0)=1$.
3.2. Proof of the second step. We characterize first finite nondegeneracy for tubes of codimension $d$ in $\mathbb{C}^{n}$.

Lemma 3.2. Let $M$ be a tube of codimension $d$ in $\mathbb{C}^{n}$ equipped with coordinates $(z, w)=(x+i y, u+i v) \in \mathbb{C}^{m} \times \mathbb{C}^{d}$ given by the equations $v_{j}=\varphi_{j}(y), j=1, \ldots, d$, where $\varphi_{j}(0)=0$. Then $M$ is finitely nondegenerate at the origin if and only if there exist m multi-indices $\beta_{*}^{1}, \ldots, \beta_{*}^{m} \in \mathbb{N}^{m}$
with $\left|\beta_{*}^{k}\right| \geq 1$ and integers $j_{*}^{1}, \ldots, j_{*}^{m}$ with $1 \leq j_{*}^{k} \leq d$ such that the real mapping

$$
\begin{equation*}
\psi(y):=\left(\frac{\partial^{\mid \beta_{*}^{1}} \varphi_{j_{*}^{1}}(y)}{\partial y^{\beta_{*}^{1}}}, \ldots, \frac{\partial^{\left|\beta_{*}^{m}\right|} \varphi_{j^{m}}(y)}{\partial y_{*}^{\beta_{*}^{m}}}\right)=: y^{\prime} \in \mathbb{R}^{m} \tag{3.19}
\end{equation*}
$$

is of rank $m$ at the origin in $\mathbb{R}^{m}$.
Proof. We follow the definition of finite nondegeneracy given in §1.2. Let $r_{j}(t, \bar{t}):=v_{j}-\varphi_{j}(y)=0$ be the defining equations of $M$. Let $\bar{L}_{k}:=$ $\partial_{\bar{z}_{k}}+\sum_{j=1}^{d} \varphi_{j, \bar{z}_{k}} \partial_{\bar{w}_{j}}, k=1, \ldots, m$, be a basis of $(1,0)$-vector fields tangent to $M$. We write the first order terms in the Taylor series of $\varphi_{j}(y)$ as $\varphi_{j}(y)=$ $\sum_{l=1}^{n} \lambda_{j, l} y_{l}+\mathrm{O}\left(|y|^{2}\right)$. Then the holomorphic gradient of $r_{j}$ is given by (3.20)

$$
\left\{\begin{aligned}
\nabla_{t}\left(r_{j}\right) & =\left(\partial_{z_{1}} r_{j}, \ldots, \partial_{z_{m}} r_{j}, \partial_{w_{1}} r_{j}, \ldots, \partial_{w_{d}} r_{j}\right) \\
& =i 2^{-1}\left(\partial_{y_{1}} \varphi_{j}, \ldots, \partial_{y_{m}} \varphi_{j}, 0, \ldots, 0,-1,0, \ldots, 0\right) \\
& =i 2^{-1}\left(\lambda_{j, 1}, \cdots, \lambda_{j, m}, 0, \ldots, 0,-1,0, \ldots, 0\right), \text { at the origin. }
\end{aligned}\right.
$$

On the other hand, since for $\beta=\left(\beta_{1}, \ldots, \beta_{m}\right) \in \mathbb{N}^{m}$ with $|\beta| \geq 1$ the order $|\beta|$ derivation $\bar{L}^{\beta}:=\bar{L}_{1}^{\beta_{1}} \cdots \bar{L}_{m}^{\beta_{m}}$ acts on functions of $y$ as the operator $(2 i)^{-|\beta|} \partial_{y}^{\beta}$, we can compute

$$
\left\{\begin{align*}
\bar{L}^{\beta}\left(\nabla_{t}\left(r_{j}\right)\right) & =\left(\bar{L}^{\beta} \partial_{z_{1}} \varphi_{j}, \ldots, \bar{L}^{\beta} \partial_{z_{m}} \varphi_{j}, 0, \ldots, 0, \ldots, 0\right)  \tag{3.21}\\
& =i^{-|\beta|+1} 2^{-|\beta|-1}\left(\partial_{y}^{\beta} \partial_{y_{1}} \varphi_{j}, \ldots, \partial_{y}^{\beta} \partial_{y_{m}} \varphi_{j}, 0, \ldots, 0, \ldots, 0\right)
\end{align*}\right.
$$

By inspecting the expressions (3.20) and (3.21), we see that $\operatorname{Span}\left\{\left(\bar{L}^{\beta}\left(\nabla_{t}\left(r_{j}\right)\right)\right)(0): \beta \in \mathbb{N}^{m}, j=1, \ldots, d\right\}=\mathbb{C}^{n}$ if and only if $\operatorname{Span}\left\{\left(\partial_{y}^{\beta} \partial_{y_{1}} \varphi_{j}(0), \ldots, \partial_{y}^{\beta} \partial_{y_{m}} \varphi_{j}(0)\right): \beta \in \mathbb{N}^{m},|\beta| \geq 1, j=\right.$ $1, \ldots, d\}=\mathbb{R}^{m}$. This last condition is clearly equivalent to the one stated in Lemma 3.2.

We can prove now that the inverse mapping $\psi^{\prime}\left(y^{\prime}\right)$ of the mapping $\psi(y)$ defined by (1.1) (or (3.19)) has algebraic first order derivatives. By Step 1, there exists a biholomorphic transformation $\Phi$ mapping the strong tube $M$ onto the algebraic pseudotube $M^{\prime}$ with the property that $\Phi_{*}\left(\partial_{t_{i}}\right)=c_{i}^{\prime}\left(t_{i}^{\prime}\right) \partial_{t_{i}^{\prime}}$. Writing $\Phi(t)=\left(h_{1}(t), \ldots, h_{n}(t)\right)$, we have $\Phi_{*}\left(\partial_{t_{i}}\right)=$ $\sum_{l=1}^{n} h_{l, t_{i}}(t) \partial_{t_{l}^{\prime}}=c_{i}^{\prime}\left(t_{i}^{\prime}\right) \partial_{t_{i}^{\prime}}$, so $h_{i}(t)$ depends only on $t_{i}$ which yields $\Phi(t)=\left(h_{1}\left(t_{1}\right), \ldots, h_{n}\left(t_{n}\right)\right)$. We shall use the convenient notation $t_{i}^{\prime}=$ $h_{i}\left(t_{i}\right)$ and $t_{i}=h_{i}^{\prime}\left(t_{i}^{\prime}\right)$ for the inverse $h_{i}^{\prime}:=h_{i}^{-1}, i=1, \ldots, n$. If accordingly, $\Phi^{\prime}\left(t^{\prime}\right)=t$ denotes the inverse of $\Phi(t)=t^{\prime}$, we have $\Phi_{*}^{\prime}\left(c_{i}^{\prime}\left(t_{i}^{\prime}\right) \partial_{t_{i}^{\prime}}\right)=$ $c_{i}^{\prime}\left(t_{i}^{\prime}\right) h_{i, t_{i}^{\prime}}^{\prime}\left(t_{i}^{\prime}\right) \partial_{t_{i}}=\partial_{t_{i}}$, which shows that $c_{i}^{\prime}\left(t_{i}^{\prime}\right) h_{i, t_{i}^{\prime}}^{\prime}\left(t_{i}^{\prime}\right) \equiv 1$. Since $c_{i}^{\prime}(0)=$ 1 , we see that $h_{i}^{\prime}\left(t_{i}^{\prime}\right)=\int_{0}^{t_{i}^{\prime}} 1 /\left[c_{i}^{\prime}(\sigma)\right] d \sigma$ is the complex primitive of an algebraic function. This observation will be important.

After a permutation of the coordinates, we may assume that $M^{\prime}$ is given in the coordinates $t^{\prime}=\left(z^{\prime}, w^{\prime}\right) \in \mathbb{C}^{m} \times \mathbb{C}^{d}$ by the real defining equations $\operatorname{Im} w_{j}^{\prime}=\varphi_{j}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \operatorname{Re} w^{\prime}\right), j=1, \ldots, d$, where the functions $\varphi_{j}^{\prime}$ are algebraic and vanish at the origin. Solving in terms of $w^{\prime}$ by means of the algebraic implicit function theorem, we can represent $M^{\prime}$ by the algebraic complex defining equations

$$
\begin{equation*}
w_{j}^{\prime}=\bar{\Theta}_{j}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right), \quad j=1, \ldots, d \tag{3.22}
\end{equation*}
$$

where $\bar{\Theta}^{\prime}$ satisfies the vectorial functional equation $w^{\prime} \equiv$ $\bar{\Theta}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \Theta^{\prime}\left(\bar{z}^{\prime}, z^{\prime}, w^{\prime}\right)\right.$ ) (which we shall not use). According to the splitting $\left(z^{\prime}, w^{\prime}\right)$ of coordinates, it is convenient to modify our previous notation by writing $z_{k}=f_{k}^{\prime}\left(z_{k}^{\prime}\right), k=1, \ldots, m$ and $w_{j}=g_{j}^{\prime}\left(w_{j}^{\prime}\right), j=1, \ldots, d$ instead of $t_{i}=h_{i}^{\prime}\left(t_{i}^{\prime}\right), i=1, \ldots, n$, and also

$$
\left\{\begin{array}{rlrl}
X_{k}^{\prime} & =a_{k}^{\prime}\left(z_{k}^{\prime}\right) \partial_{z_{k}^{\prime}}, & k=1, \ldots, m, &  \tag{3.23}\\
Y_{k}^{\prime}(0)=1 \\
Y_{j}^{\prime}=b_{j}^{\prime}\left(w_{j}^{\prime}\right) \partial_{w_{j}^{\prime}}, & j=1, \ldots, d, & b_{j}^{\prime}(0)=1
\end{array}\right.
$$

instead of $X_{i}^{\prime}=c_{i}^{\prime}\left(t_{i}^{\prime}\right) \partial_{t_{i}^{\prime}}$. The relation $c_{i}^{\prime}\left(t_{i}^{\prime}\right) h_{i, t_{i}^{\prime}}^{\prime}\left(t_{i}^{\prime}\right) \equiv 1$ rewrites down in the form

$$
\left\{\begin{align*}
a_{k}^{\prime}\left(z_{k}^{\prime}\right) f_{k, z_{k}^{\prime}}^{\prime}\left(z_{k}^{\prime}\right) & \equiv 1  \tag{3.24}\\
b_{j}^{\prime}\left(w_{j}^{\prime}\right) g_{j, w_{j}^{\prime}}^{\prime}\left(w_{j}^{\prime}\right) & \equiv 1
\end{align*}\right.
$$

We remind that the derivatives of the $f_{k}^{\prime}$ and of the $g_{j}^{\prime}$ are algebraic. Let now $t^{\prime}=\left(z^{\prime}, w^{\prime}\right) \in M^{\prime}$, thus satisfying (3.22). Then $h^{\prime}\left(t^{\prime}\right)=\left(f^{\prime}\left(z^{\prime}\right), g^{\prime}\left(w^{\prime}\right)\right)$ belongs to $M$, namely we have for $j=1, \ldots, d$ :

$$
\begin{equation*}
\frac{g_{j}^{\prime}\left(w_{j}^{\prime}\right)-\bar{g}_{j}^{\prime}\left(\bar{w}_{j}^{\prime}\right)}{2 i}=\varphi_{j}\left(\frac{f_{1}^{\prime}\left(z_{1}^{\prime}\right)-\bar{f}_{1}^{\prime}\left(\bar{z}_{1}^{\prime}\right)}{2 i}, \ldots, \frac{f_{m}^{\prime}\left(z_{m}^{\prime}\right)-\bar{f}_{m}^{\prime}\left(\bar{z}_{m}^{\prime}\right)}{2 i}\right) \tag{3.25}
\end{equation*}
$$

where $i=\sqrt{-1}$ here. Replacing $w_{j}^{\prime}$ by $\bar{\Theta}_{j}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)$ in the left hand side, we get the following identity between converging power series of the $2 m+d$ complex variables $\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)$ :
(3.26)
$\frac{g_{j}^{\prime}\left(\bar{\Theta}_{j}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)\right)-\bar{g}_{j}^{\prime}\left(\bar{w}_{j}^{\prime}\right)}{2 i} \equiv \varphi_{j}\left(\frac{f_{1}^{\prime}\left(z_{1}^{\prime}\right)-\bar{f}_{1}^{\prime}\left(\bar{z}_{1}^{\prime}\right)}{2 i}, \ldots, \frac{f_{m}^{\prime}\left(z_{m}^{\prime}\right)-\bar{f}_{m}^{\prime}\left(\bar{z}_{m}^{\prime}\right)}{2 i}\right)$.
Let us differentiate this identity with respect to $z_{k}^{\prime}$, for $k=1, \ldots, m$. Taking into account the relations (3.24), we obtain
$\frac{a_{k}^{\prime}\left(z_{k}^{\prime}\right) \bar{\Theta}_{j, z_{k}^{\prime}}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)}{b_{j}^{\prime}\left(\bar{\Theta}_{j}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)\right)} \equiv \frac{\partial \varphi_{j}}{\partial y_{k}}\left(\frac{f_{1}^{\prime}\left(z_{1}^{\prime}\right)-\bar{f}_{1}^{\prime}\left(\bar{z}_{1}^{\prime}\right)}{2 i}, \ldots, \frac{f_{m}^{\prime}\left(z_{m}^{\prime}\right)-\bar{f}_{m}^{\prime}\left(\bar{z}_{m}^{\prime}\right)}{2 i}\right)$.

Clearly, the left hand side is an algebraic function $\mathscr{A}_{j, k}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)$. Then differentiating again with respect to the variables $z_{k}^{\prime}$ the relations (3.27), we see that for every multi-index $\beta \in \mathbb{N}^{m}$ with $|\beta| \geq 1$, and every $j=1, \ldots, d$, there exists an algebraic function $\mathscr{A}_{j, \beta}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)$ such that the following identity holds:

$$
\begin{equation*}
\mathscr{A}_{j, \beta}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right) \equiv \frac{\partial^{\beta_{1}+\cdots+\beta_{m}} \varphi_{j}}{\partial y_{1}^{\beta_{1}} \cdots \partial_{y_{m}}^{\beta_{m}}}\left(\frac{f_{1}^{\prime}\left(z_{1}^{\prime}\right)-\bar{f}_{1}^{\prime}\left(\bar{z}_{1}^{\prime}\right)}{2 i}, \ldots, \frac{f_{m}^{\prime}\left(z_{m}^{\prime}\right)-\bar{f}_{m}^{\prime}\left(\bar{z}_{m}^{\prime}\right)}{2 i}\right) . \tag{3.28}
\end{equation*}
$$

Differentiating (3.28) with respect to $\bar{w}^{\prime}$, we see immediately that $\mathscr{A}_{j, \beta}^{\prime}$ is in fact independent of $\bar{w}^{\prime}$. Furthermore, we see that $\mathscr{A}_{j, \beta}^{\prime}$ is real, namely $\mathscr{A}_{j, \beta}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right) \equiv \overline{\mathscr{A}}_{j, \beta}^{\prime}\left(\bar{z}^{\prime}, z^{\prime}\right)$. Now we extract from (3.28) the $m$ identities written for $\beta:=\beta_{*}^{k}, j:=j_{*}^{k}, k=1, \ldots, m$ and we use the invertibility of the mapping $\psi$ defined in (3.19) (recall that $\psi^{\prime}\left(y^{\prime}\right)=y$ denotes the inverse of $y^{\prime}=\psi(y)$ ), which yields

$$
\begin{equation*}
\frac{f_{k}^{\prime}\left(z_{k}^{\prime}\right)-\bar{f}_{k}^{\prime}\left(\bar{z}_{k}^{\prime}\right)}{2 i} \equiv \psi_{k}^{\prime}\left(\mathscr{A}_{j_{*}^{1}, \beta_{*}^{1}}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right), \ldots, \mathscr{A}_{j_{*}^{m}, \beta_{*}^{m}}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right)\right), \tag{3.29}
\end{equation*}
$$

for $k=1, \ldots, m$. For simplicity, we shall write $\mathscr{A}_{k}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right)$ instead of $\mathscr{A}_{j_{*}^{k}, \beta_{*}^{k}}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right)$. Finally, we differentiate (3.29) with respect to $z_{k}^{\prime}$, which yields, taking into account (3.24):

$$
\left\{\begin{align*}
\frac{1}{2 i a_{k}^{\prime}\left(z_{k}^{\prime}\right)} & \equiv \sum_{l=1}^{m} \frac{\partial \psi_{k}^{\prime}}{\partial y_{l}^{\prime}}\left(\mathscr{A}_{1}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right), \ldots, \mathscr{A}_{m}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right)\right) \frac{\partial \mathscr{A}_{l}^{\prime}}{\partial z_{k}^{\prime}}\left(z^{\prime}, \bar{z}^{\prime}\right),  \tag{3.30}\\
0 & \equiv \sum_{l=1}^{m} \frac{\partial \psi_{k}^{\prime}}{\partial y_{l}^{\prime}}\left(\mathscr{A}_{1}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right), \ldots, \mathscr{A}_{m}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right)\right) \frac{\partial \mathscr{A}_{l}^{\prime}}{\partial z_{\widetilde{k}}^{\prime}}\left(z^{\prime}, \bar{z}^{\prime}\right), \quad \widetilde{k} \neq k .
\end{align*}\right.
$$

It follows from these relations (3.30) viewed in matrix form that the constant matrix $\left(\frac{\partial \mathscr{\mathscr { C } _ { l } ^ { \prime }}}{\partial z_{k}^{\prime}}(0,0)\right)_{1 \leq l, k \leq m}$ is invertible, because the diagonal matrix $\left(\delta_{k}^{\widetilde{k}}\left[2 i a_{k}^{\prime}\left(z_{k}^{\prime}\right)\right]^{-1}\right)_{1 \leq k, \tilde{k} \leq m}$ is evidently invertible at $z_{k}^{\prime}=0$ (recall $a_{k}^{\prime}(0)=1$ ). Consequently, there exist algebraic functions $\mathscr{B}_{k, l}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right)$ so that

$$
\begin{equation*}
\frac{\partial \psi_{k}^{\prime}}{\partial y_{l}^{\prime}}\left(\mathscr{A}_{1}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right), \ldots, \mathscr{A}_{m}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right)\right) \equiv \mathscr{B}_{k, l}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right) \tag{3.31}
\end{equation*}
$$

Next, setting $\widetilde{y}_{k}^{\prime}=\mathscr{A}_{k}^{\prime}\left(i y^{\prime},-i y^{\prime}\right), k=1, \ldots, m$ we see, from the invertibility of the matrix $\left(\frac{\partial \mathscr{\mathscr { L } _ { k } ^ { \prime }}}{\partial z_{l}^{\prime}}(0,0)\right)_{1 \leq k, l \leq m}$ and from the reality of $\mathscr{A}_{k}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right)$, that the Jacobian determinant at the origin of the mapping $y^{\prime} \mapsto \mathscr{A}^{\prime}\left(i y^{\prime},-i y^{\prime}\right)=\widetilde{y}_{k}^{\prime}$ is nonzero. Thus there are real algebraic functions $\mathscr{C}_{k}^{\prime}$ so that we can express
$y^{\prime}$ in terms of $\widetilde{y}^{\prime}$ as $y_{k}^{\prime}=\mathscr{C}_{k}^{\prime}\left(\widetilde{y}^{\prime}\right)$. Finally, we get

$$
\begin{equation*}
\frac{\partial \psi_{k}^{\prime}}{\partial y_{l}^{\prime}}\left(\widetilde{y}_{1}^{\prime}, \ldots, \widetilde{y}_{m}^{\prime}\right)=\mathscr{B}_{k, l}^{\prime}\left(i \mathscr{C}^{\prime}\left(\widetilde{y}^{\prime}\right),-i \mathscr{C}^{\prime}\left(\widetilde{y}^{\prime}\right)\right), \tag{3.32}
\end{equation*}
$$

where the right hand sides are algebraic; this shows that the partial derivatives $\partial_{y^{\prime} l} \psi_{k}^{\prime}$ are algebraic functions of $\tilde{y}^{\prime}$.

To obtain the equivalent formulation of Theorem 1.1, we observe the following.

Lemma 3.3. For every $k, l=1, \ldots, m$, the functions $\partial_{y_{k}^{\prime}} \psi_{l}^{\prime}\left(y^{\prime}\right)$ are algebraic functions of $y^{\prime}$ if and only if for every $k_{1}, k_{2}=1, \ldots, m$, the second derivative $\partial_{y_{k_{1}} y_{k_{2}}}^{2}(y)$ is an algebraic function of $\psi(y)=$ $\left(\partial_{y_{1}} \varphi(y), \ldots, \partial_{y_{m}} \varphi(y)\right)$.

Proof. Differentiating the identities $y_{k} \equiv \psi_{k}^{\prime}(\psi(y)), k=1, \ldots, m$, with respect to $y_{l}$, we get

$$
\begin{equation*}
\delta_{k}^{l} \equiv \sum_{j=1}^{m} \partial_{y_{j}^{\prime}} \psi_{k}^{\prime}(\psi(y)) \partial_{y_{j}} \psi_{j}(y) \equiv \sum_{j=1}^{m} \partial_{y_{j}^{\prime}} \psi_{k}^{\prime}\left(y^{\prime}\right) \partial_{y_{m y l}}^{2} \varphi(y) . \tag{3.33}
\end{equation*}
$$

Applying Cramer's rule, we see that there exist universal rational functions $R_{k, l}$ such that

$$
\left\{\begin{align*}
\partial_{y_{k} y_{l}}^{2} \varphi(y) & \left.\equiv R_{k, l}\left(\left\{\partial_{y_{k_{2}}^{\prime}} \psi_{k_{1}}^{\prime}\left(y^{\prime}\right)\right\}_{1 \leq k_{1}, k_{2} \leq m}\right\}\right)  \tag{3.34}\\
& \left.\equiv R_{k, l}\left(\left\{\partial_{y_{k_{2}}^{\prime}} \psi_{k_{1}}^{\prime}\left(\partial_{y_{1}} \varphi(y), \ldots, \partial_{y_{m}} \varphi(y)\right)\right\}_{1 \leq k_{1}, k_{2} \leq m}\right\}\right)
\end{align*}\right.
$$

This implies the equivalence of Lemma 3.3.
In conclusion, taking Theorem 2.1 for granted, the proof of Theorem 1.1 is now complete.
3.3. Proof of Theorem 1.5. Let $M: v=\varphi(z, \bar{z})$ be a rigid Levi nondegenerate hypersurface in $\mathbb{C}^{n}$ passing through the origin. We may assume that $v=\sum_{k=1}^{n-1} \varepsilon_{k}\left|z_{k}\right|^{2}+\varphi^{3}(z, \bar{z})$, where $\varepsilon_{k}= \pm 1$ and we may write $\varphi^{3}(z, \bar{z})=$ $\sum_{k=1}^{n-1}\left[\bar{z}_{k} \varphi_{k}^{3}(z)+z_{k} \bar{\varphi}_{k}^{3}(\bar{z})\right]+\varphi^{4}(z, \bar{z})$, with $\varphi^{4}(0, \bar{z}) \equiv \varphi_{z_{k}}^{4}(0, \bar{z}) \equiv 0$ and $\varphi_{k}^{3}=\mathrm{O}(2)$. After making the change of coordinates $z_{k}^{\prime}:=z_{k}+\varepsilon_{k} \varphi_{k}^{3}(z)$, $w^{\prime}:=w$, we come to the simple equation $v^{\prime}=\sum_{k=1}^{n-1} \varepsilon_{k}\left|z_{k}^{\prime}\right|^{2}+\chi^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right)$, where $\chi^{\prime}\left(0, \bar{z}^{\prime}\right) \equiv \chi_{z_{k}^{\prime}}\left(0, \bar{z}^{\prime}\right) \equiv 0$, considered in Theorem 1.5.

Assume that $M$ is strongly rigid, locally algebraizable and let $M^{\prime}$ be an algebraic equivalent of $M$. Let $t^{\prime}=h(t)$ be such an equivalence, or in our previous notation $z^{\prime}=f(z, w)$ and $w^{\prime}=g(z, w)$. We note $z=f^{\prime}\left(z^{\prime}, w^{\prime}\right)$ and $w=g^{\prime}\left(z^{\prime}, w^{\prime}\right)$ the inverse equivalence. Since $M$ is strongly rigid, namely $\mathfrak{H o l}(M)$ is generated by the single vector field $X_{1}:=\partial_{w}$, it follows that $\mathfrak{H o l}\left(M^{\prime}\right)$ is also one-dimensional, generated by the single vector
field $X_{1}^{\prime}:=h_{*}\left(X_{1}\right)$. Taking again Theorem 2.1 for granted and proceeding as in the first step of the proof of Proposition 3.1, we may algebraically straighten the complex foliation induced by $X_{1}^{\prime}$ to the "vertical" foliation by $w^{\prime}$-lines. Equivalently, we may assume that $X_{1}^{\prime}=b^{\prime}\left(z^{\prime}, w^{\prime}\right) \partial_{w^{\prime}}$ with $b^{\prime}$ algebraic and $b^{\prime}(0)=1$. The assumption $h_{*}\left(\partial_{w}\right)=b^{\prime}\left(z^{\prime}, w^{\prime}\right) \partial_{w^{\prime}}$ yields that $f^{\prime}\left(z^{\prime}, w^{\prime}\right)$ is independant of $w^{\prime}$ and that $b^{\prime}\left(z^{\prime}, w^{\prime}\right) g_{w^{\prime}}^{\prime}\left(z^{\prime}, w^{\prime}\right) \equiv 1$, so that as in (3.24) above, the derivative $g_{w^{\prime}}^{\prime}$ is algebraic. Let $w^{\prime}=\bar{\Theta}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)$ be the complex defining equation of $M^{\prime}$ in these coordinates. The assumption $h^{\prime}\left(M^{\prime}\right)=M$ yields the following power series identity

$$
\begin{equation*}
g^{\prime}\left(z^{\prime}, \bar{\Theta}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)\right)-\bar{g}^{\prime}\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right) \equiv 2 i \varphi\left(f^{\prime}\left(z^{\prime}\right), \bar{f}^{\prime}\left(\bar{z}^{\prime}\right)\right) \tag{3.35}
\end{equation*}
$$

By differentiating this identity with respect to $z_{k}^{\prime}$, we get
$\partial_{z_{k}^{\prime}} g^{\prime}\left(z^{\prime}, \bar{\Theta}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)\right)+\frac{\partial_{z_{k}^{\prime}} \bar{\Theta}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)}{b^{\prime}\left(\bar{\Theta}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)\right)} \equiv 2 i \sum_{l=1}^{n-1} \partial_{z_{l}} \varphi\left(f^{\prime}\left(z^{\prime}\right), \bar{f}^{\prime}\left(\bar{z}^{\prime}\right)\right) \partial_{z_{k}^{\prime}} f_{l}^{\prime}\left(z^{\prime}\right)$.
We notice that the second term in the left hand side of (3.36) is algebraic. By differentiating in turn (3.36) with respect to $\bar{z}_{k}^{\prime}$ and using the algebraicity of $\partial_{z_{k}^{\prime} w^{\prime}}^{2} g^{\prime}\left(z^{\prime}, \bar{\Theta}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)\right)$, we obtain that there exist algebraic functions $\mathscr{A}_{k_{1}, k_{2}}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right)$ such that

$$
\begin{equation*}
\mathscr{A}_{k_{1}, k_{2}}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right) \equiv \sum_{l_{1}, l_{2}=1}^{n-1} \partial_{z_{l_{1}} \bar{z}_{2}}^{2} \varphi\left(f^{\prime}\left(z^{\prime}\right), \bar{f}^{\prime}\left(\bar{z}^{\prime}\right)\right) \partial_{z_{k_{1}}^{\prime}} f_{l_{1}}^{\prime}\left(z^{\prime}\right) \partial_{\bar{z}_{k_{2}}^{\prime}} \bar{f}_{l_{2}}^{\prime}\left(\bar{z}^{\prime}\right) . \tag{3.37}
\end{equation*}
$$

Without loss of generality, we may assume that $h^{\prime}$ is tangent to the identity map at $t^{\prime}=0$. Then setting $\bar{z}^{\prime}:=0$ in (3.37) and using the fact that $\partial_{z_{1} \bar{z}_{2}}^{2} \varphi(z, 0)=\delta_{l_{1}}^{l_{2}} \varepsilon_{l_{1}}+\partial_{z_{l_{1}} \bar{z}_{l_{2}}}^{2} \chi(z, 0) \equiv \delta_{l_{1}}^{l_{2}} \varepsilon_{l_{1}}$ by the properties of $\chi$ in Theorem 1.5 we get, since $\partial_{\bar{z}_{k_{2}}^{\prime}} \bar{f}_{l_{2}}^{\prime}(0)=\delta_{l_{2}}^{k_{2}}$ :

$$
\begin{equation*}
\mathscr{A}_{k_{1}, k_{2}}^{\prime}\left(z^{\prime}, 0\right) \equiv \varepsilon_{k_{2}} \partial_{z_{k_{1}}^{\prime}} f_{k_{2}}^{\prime}\left(z^{\prime}\right) \tag{3.38}
\end{equation*}
$$

which shows that all the first order derivatives $\partial_{z_{k}^{\prime}} f_{l}^{\prime}\left(z^{\prime}\right)$ are algebraic.
Next, since the canonical transformation to normalizing coordinates is algebraic and preserves the "horizontal" coordinates $z^{\prime}(c f$. [CM1974]), hence does not perturb the complex foliation induced by $X_{1}^{\prime}$, we may also assume that $M^{\prime}$ is given in normal coordinates, namely that the function $\Theta^{\prime}$ satisfies $\Theta^{\prime}\left(0, \bar{z}^{\prime}, \bar{w}^{\prime}\right) \equiv \Theta^{\prime}\left(z^{\prime}, 0, \bar{w}^{\prime}\right) \equiv \bar{w}^{\prime}$. Since the coordinates are normal for both $M$ and $M^{\prime}$, it follows by setting $\bar{z}^{\prime}:=0$ and $\bar{w}^{\prime}:=0$ in (3.35) that $g^{\prime}\left(z^{\prime}, 0\right) \equiv 0$. Consequently, $\partial_{z_{k}^{\prime}} g^{\prime}\left(z^{\prime}, 0\right) \equiv 0$. Finally, by setting $z^{\prime}:=0$ and $\bar{w}^{\prime}:=0$ in (3.36), we see that the first term in the left hand side vanishes and that the second term is algebraic with respect to $\bar{z}^{\prime}$, so we obtain that there
exist algebraic functions $\overline{\mathscr{B}}_{k}^{\prime}\left(\bar{z}^{\prime}\right)$ such that

$$
\begin{equation*}
\overline{\mathscr{B}}_{k}^{\prime}\left(\bar{z}^{\prime}\right) \equiv \sum_{l=1}^{n-1} \partial_{z_{l}} \varphi\left(0, \bar{f}^{\prime}\left(\bar{z}^{\prime}\right)\right) \partial_{z_{k}^{\prime}} f_{l}^{\prime}(0) \equiv \varepsilon_{k} \bar{f}_{k}^{\prime}\left(\bar{z}^{\prime}\right) . \tag{3.39}
\end{equation*}
$$

We have proved that the components $f_{k}^{\prime}\left(z^{\prime}\right)$ are all algebraic.
Finally, coming back to the relation (3.36), we want to prove that the derivatives $\partial_{z_{l}} \varphi\left(f^{\prime}\left(z^{\prime}\right), \bar{f}^{\prime}\left(\bar{z}^{\prime}\right)\right)$ are all algebraic. However, the first term of (3.36) is not algebraic in general. Fortunately, using the fact that $\bar{\Theta}^{\prime}=\bar{w}^{\prime}+\mathrm{O}(2)$, we see that there exists a unique algebraic solution $\bar{w}^{\prime}=\bar{\Lambda}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right)$ of the implicit equation $\bar{\Theta}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)=0$, namely satisfying $\bar{\Theta}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{\Lambda}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right)\right) \equiv 0$. Then by replacing $\bar{w}^{\prime}$ by $\bar{\Lambda}^{\prime}$ in (3.36), we get that there exist algebraic functions $\mathscr{C}_{k}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right)$ such that

$$
\begin{equation*}
\mathscr{C}_{k}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right) \equiv \sum_{l=1}^{n-1} \partial_{z_{l}} \varphi\left(f^{\prime}\left(z^{\prime}\right), \bar{f}^{\prime}\left(\bar{z}^{\prime}\right)\right) \partial_{z_{k}^{\prime}} f_{l}^{\prime}\left(z^{\prime}\right) \tag{3.40}
\end{equation*}
$$

Since $f^{\prime}$ is tangent to the identity map, we can solve by Cramer's rule this linear system for the derivatives $\partial_{z_{l}} \varphi$, which yields that the $\partial_{z_{l}} \varphi\left(f^{\prime}\left(z^{\prime}\right), \bar{f}^{\prime}\left(\bar{z}^{\prime}\right)\right)$ are all algebraic. Since $f^{\prime}\left(z^{\prime}\right)$ is also algebraic, we obtain in sum that the derivatives $\partial_{z_{l}} \varphi(z, \bar{z})$ are all algebraic. In conclusion, taking Theorem 2.1 for granted, the proof of Theorem 1.5 is complete.
3.4. Proof of Theorem 1.4. Let $M: v=\varphi(z \bar{z})$ in $\mathbb{C}^{2}$ with $\mathfrak{H o l}(M)$ generated by $\partial_{w}$ and $i z \partial_{z}$. Without loss of generality, we can assume that $\varphi(r)=r+\mathrm{O}\left(r^{2}\right)$. Let $M^{\prime}$ be an algebraic equivalent of $M$. Let $t=h^{\prime}\left(t^{\prime}\right)$, or $z=f^{\prime}\left(z^{\prime}, w^{\prime}\right), w=g^{\prime}\left(z^{\prime}, w^{\prime}\right)$ be a local holomorphic equivalence satisfying $h^{\prime}\left(M^{\prime}\right)=M$. Let $t^{\prime}=h(t)$ be its inverse. Then $\mathfrak{H o l}\left(M^{\prime}\right)$ is two-dimensional and generated by $h_{*}\left(\partial_{w}\right)$ and $h_{*}\left(i z \partial_{z}\right)$. First of all, using the algebraicity of the CR automorphism group of $M^{\prime}$ and proceeding as in the proof of Proposition 3.1, we can prove that there exist two generators of $\mathfrak{H o l}\left(M^{\prime}\right)$ of the form $X_{1}^{\prime}=b^{\prime}\left(w^{\prime}\right) \partial_{w^{\prime}}$ and $X_{2}^{\prime}=a^{\prime}\left(z^{\prime}\right) \partial_{z^{\prime}}$ where $b^{\prime}$ and $a^{\prime}$ are algebraic and satisfy $b^{\prime}(0)=1$ and $a^{\prime}\left(z^{\prime}\right)=i z^{\prime}+\mathrm{O}\left(z^{\prime 2}\right)$. Furthermore, we may assume that $h^{\prime}$ is tangent to the identity map and that $h_{*}^{\prime}\left(b^{\prime}\left(w^{\prime}\right) \partial_{w^{\prime}}\right)=\partial_{w}$ and $h_{*}^{\prime}\left(a^{\prime}\left(z^{\prime}\right) \partial_{z^{\prime}}\right)=i z \partial_{z}$. As in (3.24), it follows that $b^{\prime}\left(w^{\prime}\right) g_{w^{\prime}}^{\prime}\left(w^{\prime}\right) \equiv 1$ and $a^{\prime}\left(z^{\prime}\right) f_{z^{\prime}}^{\prime}\left(z^{\prime}\right) \equiv i f^{\prime}\left(z^{\prime}\right)$. Let $w^{\prime}=\bar{\Theta}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)$ be the complex algebraic equation of $M^{\prime}$. Then we get the following power series identity:

$$
\begin{equation*}
g^{\prime}\left(\bar{\Theta}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)\right)-\bar{g}^{\prime}\left(\bar{w}^{\prime}\right) \equiv 2 i \varphi\left(f^{\prime}\left(z^{\prime}\right) \bar{f}^{\prime}\left(\bar{z}^{\prime}\right)\right) \tag{3.41}
\end{equation*}
$$

which yields after differentiating with respect to $z^{\prime}$ :

$$
\left\{\begin{aligned}
\left(\frac{3.42)}{\Theta_{z^{\prime}}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right) /\left[b^{\prime}\left(\bar{\Theta}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)\right)\right]}\right. & \equiv 2 i \partial_{z^{\prime}} f\left(z^{\prime}\right) \bar{f}^{\prime}\left(\bar{z}^{\prime}\right) \partial_{r} \varphi\left(f^{\prime}\left(z^{\prime}\right) \bar{f}^{\prime}\left(\bar{z}^{\prime}\right)\right) \\
& \equiv-2 f^{\prime}\left(z^{\prime}\right) \bar{f}^{\prime}\left(\bar{z}^{\prime}\right) \partial_{r} \varphi\left(f^{\prime}\left(z^{\prime}\right) \bar{f}^{\prime}\left(\bar{z}^{\prime}\right)\right) /\left[a^{\prime}\left(z^{\prime}\right)\right]
\end{aligned}\right.
$$

Here, we consider the function $\varphi$ as a function $\varphi(r)$ of the real variable $r \in \mathbb{R}$. Since the left hand side is an algebraic function and $a^{\prime}\left(z^{\prime}\right)$ is also algebraic, there exists an algebraic function $\mathscr{A}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right)$ such that we can write

$$
\begin{equation*}
\mathscr{A}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}\right) \equiv f^{\prime}\left(z^{\prime}\right) \bar{f}^{\prime}\left(\bar{z}^{\prime}\right) \partial_{r} \varphi\left(f^{\prime}\left(z^{\prime}\right) \bar{f}^{\prime}\left(\bar{z}^{\prime}\right)\right) . \tag{3.43}
\end{equation*}
$$

Next, using the property $\varphi(r)=r+\mathrm{O}\left(r^{2}\right)$, differentiating (3.43) with respect to $\bar{z}^{\prime}$ at $\bar{z}^{\prime}=0$, we obtain that $f^{\prime}\left(z^{\prime}\right)$ is algebraic. Coming back to (3.43), this yields that $\partial_{r} \varphi\left(f^{\prime}\left(z^{\prime}\right) \bar{f}^{\prime}\left(\bar{z}^{\prime}\right)\right)$ is algebraic. Since $f^{\prime}\left(z^{\prime}\right)$ is also algebraic, we finally obtain that $\partial_{r} \varphi(r)$ is algebraic. Excepting the examples which will be treated in $\S 7.5$, the proof of Theorem 1.4 is complete.

The next three sections are devoted to the statement of Theorem 4.1, which implies directly Theorem 2.1 (§4), and to its proof (§§5-6).

## §4. Local Lie group structure for the CR automorphism GROUP

4.1. Local representation of a real algebraic generic submanifold. We consider a connected real algebraic (or more generally, real analytic) generic submanifold $M$ in $\mathbb{C}^{n}$ of codimension $d \geq 1$ and CR dimension $m=$ $n-d \geq 1$. Pick a point $p \in M$ and consider some holomorphic coordinates $t=\left(t_{1}, \ldots, t_{n}\right)=\left(z_{1}, \ldots, z_{m}, w_{1}, \ldots, w_{d}\right) \in \mathbb{C}^{m} \times \mathbb{C}^{d}$ vanishing at $p$ in which $T_{0} M=\{\operatorname{Im} w=0\}$. If we denote $w=u+i v$, it follows that there exists (Nash) real algebraic power series $\varphi_{j}(z, \bar{z}, u)$ with $\varphi_{j}(0)=0$ and $d \varphi_{j}(0)=0$ such that the defining equations of $M$ are of the form $v_{j}=\varphi_{j}(z, \bar{z}, u), j=1, \ldots, d$ in a neighborhood of the origin. By means of the algebraic implicit function theorem, we can solve with respect to $\bar{w}$ the equations $w_{j}-\bar{w}_{j}=2 i \varphi_{j}(z, \bar{z},(w+\bar{w}) / 2), j=1, \ldots, d$, which yields $\bar{w}_{j}=\Theta_{j}(\bar{z}, z, w)$ for some power series $\Theta_{j}$ which are complex algebraic with respect to their $2 m+d$ variables. Here, we have $\Theta_{j}=w_{j}+\mathrm{O}(2)$, since $T_{0} M=\{\operatorname{Im} w=0\}$. Without loss of generality, we shall assume that the coordinates are normal, namely the functions $\Theta_{j}(\bar{z}, z, w)$ satisfy $\Theta_{j}(0, z, w) \equiv w_{j}$ and $\Theta_{j}(\bar{z}, 0, w) \equiv w_{j}$. It may be shown that the power series $\Theta_{j}=w_{j}+\mathrm{O}(2)$ satisfy the vectorial functional equation $\Theta(\bar{z}, z, \bar{\Theta}(z, \bar{z}, \bar{w})) \equiv \bar{w}$ in $\mathbb{C}\{z, \bar{z}, \bar{w}\}^{d}$ and conversely that to every such power series mapping satisfying this vectorial functional equation, there corresponds a unique real algebraic generic manifold $M$ (cf. for instance the manuscript [GM2001c] for the details). So we can equivalently
take $\bar{w}_{j}=\Theta_{j}(\bar{z}, z, w)$ or $w_{j}=\bar{\Theta}_{j}(z, \bar{z}, \bar{w})$ as complex defining equations for $M$.

For arbitrary $\rho>0$, we shall often consider the open polydisc $\Delta_{n}(\rho):=$ $\left\{t \in \mathbb{C}^{n}:|t|<\rho\right\}$ where we denote by $|t|:=\max _{1 \leq i \leq n}\left|t_{i}\right|$ the usual polydisc norm. Without loss of generality, we may assume that the power series $\Theta_{j}$ converge normally in the polydic $\Delta_{2 m+d}\left(2 \rho_{1}\right)$, where $\rho_{1}>0$. In fact, we shall successively introduce some other positive constants (radii) $0<\rho_{5}<\rho_{4}<\rho_{3}<\rho_{2}<\rho_{1}$ afterwards. Finally, we define $M$ as:

$$
\begin{equation*}
M=\left\{(z, w) \in \Delta_{n}\left(\rho_{1}\right): \bar{w}_{j}=\Theta_{j}(\bar{z}, z, w), j=1, \ldots, d\right\} . \tag{4.1}
\end{equation*}
$$

Next, let $\rho_{2}$ arbitrary with $0<\rho_{2}<\rho_{1}$. For $h^{\prime}, h \in \mathscr{O}\left(\Delta_{n}\left(\rho_{1}\right), \mathbb{C}^{n}\right)$, we define

$$
\begin{equation*}
\left\|h^{\prime}-h\right\|_{\rho_{2}}:=\sup \left\{\left|h^{\prime}(t)-h(t)\right|: t \in \Delta_{n}\left(\rho_{2}\right)\right\} . \tag{4.2}
\end{equation*}
$$

For $k \in \mathbb{N}$, we shall also consider the $\mathscr{C}^{k}$ norms

$$
\begin{equation*}
\left\|J^{k} h^{\prime}-J^{k} h\right\|_{\rho_{2}}:=\sup \left\{\left|\partial_{t}^{\alpha} h^{\prime}(t)-\partial_{t}^{\alpha} h(t)\right|: t \in \Delta_{n}\left(\rho_{2}\right), \alpha \in \mathbb{N}^{n},|\alpha| \leq k\right\} \tag{4.3}
\end{equation*}
$$

For $k \in \mathbb{N}$ and $t \in \Delta_{n}\left(\rho_{1}\right)$, we denote by $J^{k} h(t)$ the collection of partial derivatives $\left(\partial_{t}^{\alpha} h_{i}(t)\right)_{1 \leq i \leq n,|\alpha| \leq k}$ of length $\leq k$ of the components $h_{1}, \ldots, h_{n}$, so $J^{k} h(t) \in \mathbb{C}^{N_{n, k}}$, where $N_{n, k}:=n \frac{(n+k)!}{n!k!}$. In particular, the expression $J^{k} h(0)=\left(\partial_{t}^{\alpha} h_{i}(0)\right)_{1 \leq i \leq n,|\alpha| \leq k}$ denotes the $k$-jet of $h$ at 0 . So, the space of $k$-jets at the origin of holomorphic mappings $h \in \mathscr{O}\left(\Delta_{n}\left(\rho_{1}\right), \mathbb{C}^{n}\right)$ may be identified with the complex linear space $\mathbb{C}^{N_{n, k}}$. We denote the natural coordinates on $\mathbb{C}^{N_{n, k}}$ by $\left(J_{i}^{\alpha}\right)_{1 \leq i \leq n,|\alpha| \leq k}$. Sometimes, we abbreviate this collection of coordinates by $J^{k} \equiv\left(J_{i}^{\alpha}\right)_{1 \leq i \leq n,|\alpha| \leq k}$. Finally, we denote by $J_{\mathrm{Id}}^{k}$ the $k$-jet at the origin of the identity mapping. We introduce the important set of holomorphic self-mappings of $M$ defined by

$$
\left\{\begin{align*}
\mathscr{H}_{M, k, \varepsilon}^{\rho_{2}, \rho_{1}}:=\left\{h \in \mathscr{O}\left(\Delta_{n}\left(\rho_{1}\right), \mathbb{C}^{n}\right):\right. & \left\|J^{k} h-J_{\mathrm{Id}}^{k}\right\|_{\rho_{2}}<\varepsilon,  \tag{4.4}\\
& \left.h\left(M \cap \Delta_{n}\left(\rho_{2}\right)\right) \subset M \cap \Delta_{n}\left(\rho_{1}\right)\right\} .
\end{align*}\right.
$$

Here, $k \in \mathbb{N}$ and $\varepsilon>0$ is a small positive number that we shall shrink many times in the sequel.


Figure 3: Nest of polydiscs centered at $0 \in M$

We may now state the main theorem of $\S 4, \S 5$ and $\S 6$, namely Theorem 4.1, which provides a complete parametrized description of the set $\mathscr{H}_{M, k, \varepsilon}^{\rho_{2}, \rho_{1}}$ of local biholomorphic self-mappings of $M$, with $k$ equal to an integer $\kappa_{0}$ depending on $M$. During the course of the (rather long) proof, for technical reasons, we shall have to introduce first a third positive radius $\rho_{3}$ with $0<\rho_{3}<\rho_{2}<\rho_{1}$ which is related to the finite nondegeneracy of $M$, and then afterwards a fourth positive radius $\rho_{4}$ with $0<\rho_{4}<\rho_{3}<\rho_{2}<\rho_{1}$, which is related to the minimality of $M$. This is why the radius notation " $\rho_{4}$ " appears after " $\rho_{2}$ " and " $\rho_{1}$ " without mention of " $\rho_{3}$ " ( $c f$. FIGURE 3).

Theorem 4.1. Assume that the real algebraic generic submanifold $M$ defined by (4.1) is minimal and finitely nondegenerate at the origin. As above, fix two radii $\rho_{1}$ and $\rho_{2}$ with $0<\rho_{2}<\rho_{1}$. Then there exists an even integer $\kappa_{0} \in \mathbb{N}_{*}$ which depends only on the local geometry of $M$ near the origin, there exists $\varepsilon>0$, there exists $\rho_{4}>0$ with $\rho_{4}<\rho_{2}$, there exists a complex algebraic $\mathbb{C}^{n}$-valued mapping $H\left(t, J^{\kappa_{0}}\right)$ which is defined for $t \in \mathbb{C}^{n}$ with $|t|<\rho_{4}$ and for $J^{\kappa_{0}} \in \mathbb{C}^{N_{n, \kappa_{0}}}$ (where $\left.N_{n, \kappa_{0}}=n \frac{\left(n+\kappa_{0}\right)!}{n!\kappa_{0}!}\right)$ with $\left|J^{\kappa_{0}}-J_{\mathrm{Id}}^{\kappa_{0}}\right|<\varepsilon$ and which depends only on $M$ and there exists a geometrically smooth real algebraic totally real submanifold $E$ of $\mathbb{C}^{N_{n, \kappa_{0}}}$ passing through the identity jet $J_{\text {Id }}^{\kappa_{0}}$ which depends only on $M$, which is defined by

$$
\begin{equation*}
E=\left\{J^{\kappa_{0}}:\left|J^{\kappa_{0}}-J_{\mathrm{Id}}^{\kappa_{0}}\right|<\varepsilon, C_{l}\left(J^{\kappa_{0}}, \overline{J^{\kappa_{0}}}\right)=0, l=1, \ldots, v\right\}, \tag{4.5}
\end{equation*}
$$

where the $C_{l}\left(J^{\kappa_{0}}, \overline{J^{\kappa_{0}}}\right), l=1, \ldots, v$, are real algebraic functions defined on the polydisc $\left\{\left|J^{\kappa_{0}}-J_{\mathrm{Id}}^{\kappa_{0}}\right|<\varepsilon\right\}$, and which can be constructed algorithmically by means only of the defining equations of $M$, such that the following six statements hold:
(1) Every local biholomorphic self-mapping $h \in \mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$ of $M$ (which is defined on the large polydisc $\left.\Delta_{n}\left(\rho_{1}\right)\right)$ is represented by

$$
\begin{equation*}
h(t)=H\left(t, J^{\kappa_{0}} h(0)\right), \tag{4.6}
\end{equation*}
$$

on the smallest polydisc $\Delta_{n}\left(\rho_{4}\right)$. In particular, each $h \in \mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$ is a complex algebraic biholomorphic mapping. Furthermore, the $\kappa_{0}$-jet of $h$ at the origin belongs to the real algebraic submanifold $E$, namely we have $C_{l}\left(J^{\kappa_{0}} h(0), J^{\kappa_{0}} h(0)\right)=0, l=1, \ldots, v$.
(2) Conversely, shrinking $\varepsilon$ if necessary, given an arbitrary jet $J^{\kappa_{0}}$ in $E$ there exists a smaller positive radius $\rho_{5}<\rho_{4}$ such that the mapping defined by $h(t):=H\left(t, J^{\kappa_{0}}\right)$ for $|t|<\rho_{5}$ sends $M \cap \Delta_{n}\left(\rho_{5}\right) C R$ diffeomorphically onto its image which is contained in $M \cap \Delta_{n}\left(\rho_{4}\right)$. We may therefore say that the set $\mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$ of local biholomorphic selfmappings of $M$ is parametrized by the real algebraic submanifold $E$.
(3) For every choice of two smaller positive radii $\widetilde{\rho}_{1} \leq \rho_{1}$ and $\widetilde{\rho}_{2} \leq \rho_{2}$ with $\widetilde{\rho}_{2}<\widetilde{\rho}_{1}$, there exists a positive radius $\widetilde{\rho}_{4} \leq \rho_{4}$ with $\widetilde{\rho}_{4}<\widetilde{\rho}_{2}$, and a positive $\widetilde{\varepsilon} \leq \varepsilon$ such that the same complex algebraic mapping $H\left(t, J^{\kappa_{0}}\right)$ as in statement (1) above represents all local biholomorphic selfmappings $\widetilde{h} \in \mathscr{H}_{M, \kappa_{0}, \widetilde{\rho_{2}}}^{\widetilde{\rho}_{2}} \widetilde{\rho}_{1}$ of $M$, namely we have $\widetilde{h}(t)=H\left(t, J^{\kappa_{0}} \widetilde{h}(0)\right)$ for all $|t|<\widetilde{\rho}_{4}$ as in (4.6). Furthermore, the corresponding real algebraic totally real submanifold $\widetilde{E}$ coincides with $E$ in the polydisc $\left\{\left|J^{\kappa_{0}}-J_{\mathrm{Id}}^{\kappa_{0}}\right|<\widetilde{\varepsilon}\right\}$ and it is defined by the same real algebraic equations $C_{l}\left(J^{\kappa_{0}}, \overline{J^{\kappa_{0}}}\right)=0, l=1, \ldots, v$, as in equation (4.5). In fact, the algebraic mapping $H\left(t, J^{\kappa_{0}}\right)$ and the real algebraic totally real submanifold $E$ depend only on the local geometry of $M$ in a neighborhood of the origin, namely on the germ of $M$ at 0 .
(4) The set $\mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$, equipped with the law of composition of holomorphic mappings, is a real algebraic local Lie group. More precisely, let the positive integer $c_{0}$ denote the real dimension of $E$, which is independent of $\rho_{1}, \rho_{2}$ and consider a parametrization

$$
\begin{equation*}
\mathbb{R}^{c_{0}} \ni e=\left(e_{1}, \ldots, e_{c_{0}}\right) \mapsto j_{\kappa_{0}}(e) \in E \subset \mathbb{C}^{N_{n, \kappa_{0}}} \tag{4.7}
\end{equation*}
$$

of the real algebraic totally real submanifold $E$. Then there exist a real algebraic associative local multiplication mapping $\left(e, e^{\prime}\right) \mapsto \mu\left(e, e^{\prime}\right)$ and a real algebraic local inversion mapping $e \mapsto \iota(e)$ such that if we define $H(t ; e):=H\left(t, j_{\kappa_{0}}(e)\right)$, then $H\left(H(t ; e) ; e^{\prime}\right) \equiv H\left(t ; \mu\left(e, e^{\prime}\right)\right)$ and $H(t ; e)^{-1} \equiv H(t ; \iota(e))$, with the local Lie transformation group axioms, as defined in §2.3, being satisfied by $H, \mu$ and $\iota$.
(5) For $i=1, \ldots, c_{0}$, consider the one-parameter families of transformations defined by $H\left(t ; 0, \ldots, 0, e_{i}, 0, \ldots, 0\right)=$ : $H_{i}\left(t ; e_{i}\right)=$ : $H_{i, e_{i}}(t)$. Then for each $i=1, \ldots, c_{0}$, the vector field $\left.X_{i}\right|_{\left(t ; e_{i}\right)}:=$
$\left[\partial_{i, e_{i}} H_{e_{i}}\left(t^{\prime}\right)\right]_{t^{\prime}=H_{e_{i}^{-1}}^{-1}(t)}$, is defined for $t \in \Delta_{n}\left(\rho_{5}\right)$ and $\left|e_{i}\right|<\varepsilon$, has algebraic coefficients depending on the "time" parameter $e_{i}$, and has an algebraic flow, since this coincides with the algebraic mapping $\left(t, e_{i}\right) \mapsto H_{i, e_{i}}(t)$.
(6) Let $\rho_{5}$ be as in statement (2). Then the dimension $c_{0}$ of the real Lie algebra $\mathfrak{H o l}\left(M, \Delta_{n}\left(\rho_{5}\right)\right)$ is finite, bounded by the fixed integer $N_{n, \kappa_{0}}:=$ $n \frac{\left(n+\kappa_{0}\right)!}{n!\kappa_{0}!}$. Furthermore, each vector field $X \in \mathfrak{H o l}\left(M, \Delta_{n}\left(\rho_{5}\right)\right)$ has complex algebraic coefficients.

If $M$ is real analytic, the same theorem holds with the word "algebraic" replaced everywhere by the word "analytic".

We shall explain below how the integer $\kappa_{0}$ is related to the minimality and to the finite nondegeneracy of $M$ at the origin. The next $\S 5$ and $\S 6$ are devoted to the proof Theorem 4.1, namely the existence of the mapping $H\left(t, J^{\kappa_{0}}\right)$, the existence of the real algebraic totally real submanifold $E$ and the completion of the proof of properties (1-6).

## §5. Minimality And Finite NONDEGENERACY

### 5.1. Local CR geometry of complexified real analytic generic submani-

 folds. Let $\zeta \in \mathbb{C}^{m}$ and $\xi \in \mathbb{C}^{d}$ denote some independent coordinates corresponging to the complexification of the variables $\bar{z}$ and $\bar{w}$, which we denote symbolically by $\zeta:=(\bar{z})^{c}$ and $\xi:=(\bar{w})^{c}$, where the letter "c" stands for the word "complexified". We also write $\tau:=(\bar{t})^{c}$, so $\tau=(\zeta, \xi) \in \mathbb{C}^{n}$. The extrinsic complexification $\mathscr{M}:=(M)^{c}$ of $M$ is the complex submanifold of codimension $d$ defined by$$
\begin{equation*}
\mathscr{M}:=\left\{(z, w, \zeta, \xi) \in \Delta_{n}\left(\rho_{1}\right) \times \Delta_{n}\left(\rho_{1}\right): \xi=\Theta(\zeta, z, w)\right\} . \tag{5.1}
\end{equation*}
$$

If $M$ is (real, Nash) algebraic, so is $\mathscr{M}$. As remarked, we can choose the equivalent defining equation $w=\bar{\Theta}(z, \zeta, \xi)$ for $\mathscr{M}$. In the remainder of $\S 5$, we shall essentially deal with $\mathscr{M}$ instead of $M$. In fact, $M$ clearly imbeds in $\mathscr{M}$ as the intersection of $\mathscr{M}$ with the antiholomorphic diagonal $\underline{\Lambda}:=$ $\left\{(t, \tau) \in \mathbb{C}^{n} \times \mathbb{C}^{n}: \tau=\bar{t}\right\}$.

Following [Me1998], [Me2001], we shall complexify a conjugate pair of generating families of CR vector fields tangent to $\bar{M}$, namely $L_{1}, \ldots, L_{m}$ of type $(1,0)$ and their conjugates $\bar{L}_{1}, \ldots, \bar{L}_{m}$ which are of type $(0,1)$. Here, we can explicitely choose the generators $L_{k}=\partial / \partial z_{k}+\sum_{j=1}^{d}\left[\partial \bar{\Theta}_{j} / \partial z_{k}(z, \bar{z}, \bar{w})\right] \partial / \partial w_{j}$ for $k=1, \ldots, m$. Then their complexification yields a pair of collections of $m$ vector fields defined
over $\Delta_{n}\left(\rho_{1}\right) \times \Delta_{n}\left(\rho_{1}\right)$ by

$$
\left\{\begin{align*}
\mathscr{L}_{k}:=\frac{\partial}{\partial z_{k}}+\sum_{j=1}^{d} \frac{\partial \bar{\Theta}_{j}}{\partial z_{k}}(z, \zeta, \xi) \frac{\partial}{\partial w_{j}}, & k=1, \ldots, m,  \tag{5.2}\\
\underline{\mathscr{L}_{k}}:=\frac{\partial}{\partial \zeta_{k}}+\sum_{j=1}^{d} \frac{\partial \Theta_{j}}{\partial \zeta_{k}}(\zeta, z, w) \frac{\partial}{\partial \xi_{j}}, & k=1, \ldots, m .
\end{align*}\right.
$$

The reader may check directly that $\mathscr{L}_{k}\left(w_{j}-\bar{\Theta}_{j}(z, \zeta, \xi)\right) \equiv 0$, which shows that the vector fields $\mathscr{L}_{k}$ are tangent to $\mathscr{M}$. Similarly, $\mathscr{L}_{k}\left(\xi_{j}-\right.$ $\left.\Theta_{j}(\zeta, z, w)\right) \equiv 0$, so the vector fields $\mathscr{L}_{k}$ are also tangent to $\mathscr{M}$. Furthemore, we may check the commutation relations $\left[\mathscr{L}_{k}, \mathscr{L}_{k^{\prime}}\right]=0$ and [ $\left.\mathscr{L}_{k}, \mathscr{L}_{k^{\prime}}\right]=0$ for all $k, k^{\prime}=1, \ldots, m$. It follows from the Frobenius theorem that the two $m$-dimensional distributions spanned by each of these two collections of $m$ vector fields has the integral manifold property. This is not surprising since the vector fields $\mathscr{L}_{k}$ are the vector fields tangent to the intersection of $\mathscr{M}$ with the sets $\left\{\tau=\tau_{p}=c t.\right\}$, which are clearly $m$ dimensional complex integral manifolds. Following [Me1998], [Me2001], we denote these manifolds by $\mathscr{S}_{\tau_{p}}:=\left\{\left(t, \tau_{p}\right): w=\bar{\Theta}\left(z, \zeta_{p}, \xi_{p}\right)\right\}$, where $\tau_{p}$ is a constant, and we call them complexified Segre varieties. Similarly, the integral manifolds of the vector fields $\mathscr{L}_{k}$ are the conjugate complexified Segre varieties $\underline{\mathscr{L}}_{t_{p}}:=\left\{\left(t_{p}, \tau\right): \xi=\Theta\left(\zeta, z_{p}, w_{p}\right)\right\}$, where $t_{p}$ is fixed. The union of the manifolds $\mathscr{S}_{\tau_{p}}$ induces a local complex algebraic foliation $\mathscr{F}$ of $\mathscr{M}$ by $m$-dimensional leaves. Similarly, there is a second foliation $\underline{\mathscr{F}}$ whose leaves are the $\mathscr{S}_{t_{p}}$.

The following symbolic picture summarizes our constructions. However, we warn the reader that the codimension $d \geq 1$ of the union of the two foliations $\mathscr{F}$ and $\mathscr{F}$ in $\mathscr{M}$ is not visible in this two-dimensional figure.


Now, we introduce the "multiple" flows of the two collections of conjugate vector fields $\left(\mathscr{L}_{k}\right)_{1 \leq k \leq m}$ and $\left(\mathscr{L}_{k}\right)_{1 \leq k \leq m}$. For an arbitrary point
$p=\left(w_{p}, z_{p}, \zeta_{p}, \xi_{p}\right) \in \mathscr{M}$ and for an arbitrary complex "multitime" parameter $z_{1}=\left(z_{1,1}, \ldots, z_{1, m}\right) \in \mathbb{C}^{m}$, we define
$\left\{\begin{aligned} \mathscr{L}_{z_{1}}\left(z_{p}, w_{p}, \zeta_{p}, \xi_{p}\right) & :=\exp \left(z_{1} \mathscr{L}\right)(p):=\exp \left(z_{1,1} \mathscr{L}_{1}\left(\cdots\left(\exp \left(z_{1, m} \mathscr{L}_{m}(p)\right)\right) \cdots\right)\right):= \\ & :=\left(z_{p}+z_{1}, \bar{\Theta}\left(z_{p}+z_{1}, \zeta_{p}, \xi_{p}\right), \zeta_{p}, \xi_{p}\right) .\end{aligned}\right.$
With this formal definition, there exists a maximal connected open subset $\Omega$ of $\mathscr{M} \times \mathbb{C}^{m}$ containing $\mathscr{M} \times\{0\}$ such that $\mathscr{L}_{z_{1}}(p) \in \mathscr{M}$ for all $\left(z_{1}, p\right) \in \Omega$. Analogously, for $\left(\zeta_{1}, p\right)$ running in a similar open subset $\underline{\Omega}$, we may also define the map

$$
\begin{equation*}
\underline{\mathscr{L}}_{\zeta_{1}}\left(z_{p}, w_{p}, \zeta_{p}, \xi_{p}\right):=\left(z_{p}, w_{p}, \zeta_{p}+\zeta_{1}, \Theta\left(\zeta_{p}+\zeta_{1}, z_{p}, w_{p}\right)\right) . \tag{5.4}
\end{equation*}
$$

We notice that the two maps given by (5.3) and (5.4) are holomorphic in their variables. Since $M$ is real algebraic, they are moreover complex algebraic.
5.2. Segre chains. Let us start from the point $p$ being the origin and let us move alternately in the direction of $\mathscr{S}$ or of $\underline{\mathscr{S}}$, namely we consider the two maps $\Gamma_{1}\left(z_{1}\right):=\mathscr{L}_{z_{1}}(0)$ and $\underline{\Gamma}_{1}\left(z_{1}\right):=\mathscr{L}_{z_{1}}(0)$. Next, we start from these endpoints and we move in the other direction, namely, we consider the two maps

$$
\begin{equation*}
\Gamma_{2}\left(z_{1}, z_{2}\right):=\mathscr{L}_{z_{2}}\left(\mathscr{L}_{z_{1}}(0)\right), \quad \underline{\Gamma}_{2}\left(z_{1}, z_{2}\right):=\mathscr{L}_{z_{2}}\left(\underline{\mathscr{L}}_{z_{1}}(0)\right) \tag{5.5}
\end{equation*}
$$

where $z_{1}, z_{2} \in \mathbb{C}^{m}$. Also, we define $\Gamma_{3}\left(z_{1}, z_{2}, z_{3}\right):=\mathscr{L}_{z_{3}}\left(\mathscr{L}_{z_{2}}\left(\mathscr{L}_{z_{1}}(0)\right)\right)$, etc. By induction, for every positive integer $k$, we obtain two maps $\Gamma_{k}\left(z_{1}, \ldots, z_{k}\right)$ and $\underline{\Gamma}_{k}\left(z_{1}, \ldots, z_{k}\right)$. In the sequel, we shall often use the notation $z_{(k)}:=\left(z_{1}, \ldots, z_{k}\right) \in \mathbb{C}^{m k}$. Since $\Gamma_{k}(0)=\underline{\Gamma}_{k}(0)=0$, for every $k \in \mathbb{N}_{*}$, there exists a sufficiently small open polydisc $\Delta_{m k}\left(\delta_{k}\right)$ centered at the origin in $\mathbb{C}^{m k}$ with $\delta_{k}>0$ such that $\Gamma_{k}\left(z_{(k)}\right)$ and $\underline{\Gamma}_{k}\left(z_{(k)}\right)$ belong to $\mathscr{M}$ for all $z_{(k)} \in \Delta_{m k}\left(\delta_{k}\right)$.

We also exhibit a simple link between the maps $\Gamma_{k}$ and $\underline{\Gamma}_{k}$. Let $\sigma$ be the antiholomorphic involution defined by $\sigma(t, \tau):=(\bar{\tau}, \bar{t})$. Since $w=$ $\bar{\Theta}(z, \zeta, \xi)$ if and only if $\xi=\Theta(\zeta, z, w)$, this involution maps $\mathscr{M}$ onto $\mathscr{M}$ and it also fixes the antidiagonal $\underline{\Lambda}$ pointwise. Using the definitions (5.3) and (5.4), we see readily that $\sigma\left(\mathscr{L}_{z_{1}}(0)\right)=\mathscr{L}_{\bar{z}_{1}}(0)$. It follows generally that $\sigma\left(\Gamma_{k}\left(z_{(k)}\right)\right)=\underline{\Gamma}_{k}\left(\overline{z_{(k)}}\right)$.

Next, we observe that $\Gamma_{k+1}\left(z_{(k)}, 0\right)=\Gamma_{k}\left(z_{(k)}\right)$, since $\mathscr{L}_{0}$ and $\mathscr{L}_{0}$ coincide with the identity map. So the ranks at the origin of the maps $\Gamma_{k}$ increase with $k$.

Definition 5.1. The real analytic generic manifold $M$ is said to be minimal at $p$ if the maps $\Gamma_{k}$ are of (maximal possible) rank equal to $2 m+d=$ $\operatorname{dim}_{\mathbb{C}} \mathscr{M}$ at the origin in $\Delta_{m k}\left(\delta_{k}\right)$ for all $k$ large enough.

The following fundamental properties are established in [Me1998], [Me2001].

Theorem 5.2. The minimality of $M$ at 0 is a biholomorphically invariant property. It depends neither on the choice of a defining equation for $M$ nor on the choice of a system of generating complexified CR vector fields $\left(\mathscr{L}_{k}\right)_{1 \leq k \leq m}$ and $\left(\mathscr{L}_{k}\right)_{1 \leq k \leq m}$. Also, minimality is equivalent to the fact that the Lie algebra generated by the complexified CR vector fields $\left(\mathscr{L}_{k}\right)_{1 \leq k \leq m}$ and $\left(\mathscr{L}_{k}\right)_{1 \leq k \leq m}$ spans $T \mathscr{M}$ in a neighborhood of 0 . Furthermore, there exists an invariant integer $\nu_{0}$, called the Segre type of $M$ at 0 satisfying $\nu_{0} \leq d+1$ which is the smallest integer such that the mappings $\Gamma_{k}$ and $\underline{\Gamma}_{k}$ are of generic rank equal to $2 m+d$ over $\Delta_{m k}\left(\delta_{k}\right)$ for all $k \geq \nu_{0}+1$. Finally, with this integer $\nu_{0}$, the odd integer $\mu_{0}:=2 \nu_{0}+1$, called the Segre type $\mathscr{M}$ at 0 is the smallest integer such that the mappings $\Gamma_{k}$ and $\underline{\Gamma}_{k}$ are of rank equal to $2 m+d$ at the origin in $\Delta_{m k}\left(\delta_{k}\right)$.

Let $\mu_{0}:=2 \nu_{0}+1$ be the Segre type of $\mathscr{M}$ at 0 (notice that this is always odd). In the remainder of this section, we assume that $M$ is minimal at 0 and we exploit the rank condition on $\Gamma_{k}$. More precisely we choose a positive $\eta$ with $0<\eta \leq \delta_{\mu_{0}}$ such that $\Gamma_{\mu_{0}}$ has rank $2 m+d$ at every point of the polydisc $\Delta_{m \mu_{0}}(\eta)$. Without loss of generality, we can also assume that $\Gamma_{\mu_{0}}\left(\Delta_{m \mu_{0}}(\eta)\right)$ contains $\mathscr{M} \cap\left(\Delta_{n}\left(\rho_{4}\right) \times \Delta_{n}\left(\rho_{4}\right)\right)$. Simple examples in the hypersurface case show that $\rho_{4} \ll \rho_{1}$ and in fact, one has necessarily an inequality of the form $\rho_{4} \leq\left(\rho_{1}\right)^{N}$, where $N$ is a certain integer depending on the vector fields $\left(\mathscr{L}_{k}\right)_{1 \leq k \leq m}$ and $\left(\mathscr{L}_{k}\right)_{1 \leq k \leq m}(c f$. [Be1996]).
5.3. Finite nondegeneracy. The last ingredient for Theorem 4.1 consists in developing the equations of $M$ in powers of $\bar{z}$ as follows

$$
\begin{equation*}
\bar{w}_{j}=\sum_{\beta \in \mathbb{N}^{m}}(\bar{z})^{\beta} \Theta_{j, \beta}(t), \quad j=1, \ldots, d, \tag{5.6}
\end{equation*}
$$

where the functions $\Theta_{j, \beta}(t)$ are holomorphic in the polydisc $\Delta_{n}\left(2 \rho_{1}\right)$. So we may introduce the holomorphic maps $\psi_{k}(t):=\left(\Theta_{j, \beta}(t)\right)_{1 \leq j \leq d,|\beta| \leq k}$ with values in $\mathbb{C}^{\frac{d(m+k)!}{m!k!}}$. Obviously, the ranks at the origin of the $\psi_{k}$ increase with $k$.

Definition 5.3. The generic manifold $M$ is said to be finitely nondegenerate at 0 if there exists a positive integer $k$ such that the rank at the origin of the map $\psi_{k}$ is equal to $n$.

It may be checked that this definition depends neither on the system of coordinates nor on the choice of a collection of $d$ defining equations for $M$ and that it coincides with the definition given in $\S 1.2$. If $M$ is finitely nondegenerate at 0 we denote by $\ell_{0}$ the smallest integer $k$ given by definition 5.3 and we say that $M$ is $\ell_{0}$-nondegenerate at the origin.

Finite nondegeneracy is interesting for the following reason. In the sequel, we shall have to consider an infinite collection of equations of the form

$$
\begin{equation*}
\Theta_{j, \beta}(t)+\sum_{\gamma \in \mathbb{N}_{*}^{m}}(\zeta)^{\gamma} \frac{(\beta+\gamma)!}{\beta!\gamma!} \Theta_{j, \beta+\gamma}(t)=\omega_{j, \beta} \tag{5.7}
\end{equation*}
$$

where $\mathbb{N}_{*}^{m}:=\mathbb{N}^{m} \backslash\{0\}$, where $j$ runs from 1 to $d$, where $\beta$ runs in $\mathbb{N}^{m}$ and where the right hand sides $\omega_{j, \beta}$ are independent complex variables. For $\beta=0$, the equations (5.7) write simply $\Theta_{j}(\zeta, t)=\omega_{j, 0}$. By definition, if $M$ is $\ell_{0}$-nondegenerate at 0 , there exists $n$ integers $j_{*}^{1}, \ldots, j_{*}^{n}$ with $1 \leq$ $j_{*}^{i} \leq d$ and $n$ multi-indices $\beta_{*}^{1}, \ldots, \beta_{*}^{n} \in \mathbb{N}^{m}$ with $\left|\beta_{*}^{i}\right| \leq \ell_{0}$ such that the local holomorphic self-mapping $t \mapsto\left(\Theta_{j_{*}^{k}, \beta_{*}^{k}}(t)\right)_{1 \leq k \leq n}$ of $\mathbb{C}^{n}$ is of rank $n$ at the origin. Considering the equations (5.7) for $j=j_{*}^{1}, \ldots, j_{*}^{n}$ and $\beta=\beta_{*}^{1}, \ldots, \beta_{*}^{n}$ and applying the implicit function theorem, we observe that we can solve $t$ in terms of $\left(\zeta, \omega_{j *, \beta_{*}}, \ldots, \omega_{j_{*}^{n}, \beta_{*}^{n}}\right)$ by means of a holomorphic mapping, namely

$$
\begin{equation*}
t=\Psi\left(\tau, \omega_{j_{*}^{1}, \beta_{*}^{1}}, \ldots, \omega_{j_{*}^{n}, \beta_{*}^{n}}\right) . \tag{5.8}
\end{equation*}
$$

Without loss of generality, we may assume that $\Psi$ is holomorphic for $|\zeta|<$ $\widetilde{\rho}_{3}$ and $\left|\omega_{j_{*}^{i}, \beta_{*}^{i}}\right|<\widetilde{\rho}_{3}$, where $0<\widetilde{\rho}_{3}<\rho_{2}<\rho_{1}$.

## §6. Algebraicity of local CR automorphism groups

6.1. Fundamental reflection identity for the mapping. So $M$ is $\ell_{0}-$ nondegenerate at the origin. Recall that $\mu_{0}=2 \nu_{0}+1$ is the Segre type of $\mathscr{M}$ and introduce the new integer $\kappa_{0}:=\ell_{0}\left(\mu_{0}+1\right)$. Notice that $\kappa_{0}$ is even. Let us take an arbitrary local holomorphic self map $h$ of $M$ close to the identity in the set $\mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$, i.e. with $k:=\kappa_{0}$ in the definition (4.4). We denote the map $h$ by $\left(h_{1}, \ldots, h_{n}\right)=\left(f_{1}, \ldots, f_{m}, g_{1}, \ldots, g_{d}\right)$, according to the splitting $t=(z, w)$ of the coordinates. The complexification $h^{c}:=(h, \bar{h})$ induces a local holomorphic self map of the complexification $\mathscr{M}$. More precisely, for all $(t, \tau) \in \mathscr{M}$ with $|t|,|\tau| \leq \rho_{2}$, we have $(h(t), \bar{h}(\tau)) \in \mathscr{M}$ and $|h(t)|,|h(\tau)|<\rho_{1}$, so we can write

$$
\begin{equation*}
\bar{g}_{j}(\tau)=\Theta_{j}(\bar{f}(\tau), h(t)), \tag{6.1}
\end{equation*}
$$

for $j=1, \ldots, d$. Since $h$ is a biholomorphism and $T_{0}^{c} M=\{w=0\}$, it follows that the determinant

$$
\begin{equation*}
\operatorname{det}\left(\mathscr{L}_{k} \bar{f}_{l}(\tau)\right)_{1 \leq k, l \leq n}, \tag{6.2}
\end{equation*}
$$

which is a $\mathbb{K}$-analytic function of $(t, \tau) \in \mathscr{M}$, does not vanish at the origin. Shrinking $\varepsilon$ if necessary, we can assume that for every holomorphic map $h \in \mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$, the determinant (6.2) does not vanish for all $|t|,|\tau|<\rho_{2}$. We
now differentiate (6.2) by applying the vector fields $\mathscr{L}_{1}, \ldots, \mathscr{L}_{m}$, which gives

$$
\begin{equation*}
\underline{\mathscr{L}}_{k} \bar{g}_{j}(\tau)=\sum_{l=1}^{m} \frac{\partial \Theta_{j}}{\partial \zeta_{l}}(\bar{f}(\tau), h(t)) \underline{\mathscr{L}}_{k} \bar{f}_{l}(\tau), \tag{6.3}
\end{equation*}
$$

for $k=1, \ldots, m$ and $j=1, \ldots, d$. For fixed $j$, we consider the $m$ equations (6.3) as an affine system satisfied by the partial derivatives $\partial \Theta_{j} / \partial \zeta_{l}$. By Cramer's rule, there exists universal polynomials $\Omega_{j, k}$ in their variables such that

$$
\begin{equation*}
\frac{\partial \Theta_{j}}{\partial \zeta_{k}}(\bar{f}(\tau), h(t))=\frac{\Omega_{j, k}\left(\left\{\mathscr{L}_{l} \bar{h}(\tau)\right\}_{1 \leq l \leq m}\right)}{\operatorname{det}\left(\underline{\mathscr{L}}_{k} \bar{f}_{l}(\tau)\right)_{1 \leq k, l \leq n}} \tag{6.4}
\end{equation*}
$$

for all $(t, \tau) \in \mathscr{M}$ with $|t|,|\tau|<\rho_{2}$ and for $k=1, \ldots, m, j=1, \ldots, d$.
Applying the derivations $\underline{\mathscr{L}}_{k}$ to (6.4) we see by induction that for every multi-index $\beta \in \mathbb{N}_{*}^{m}$ and for every $j=1, \ldots, d$, there exists a universal polynomial $\Omega_{j, \beta}$ in its variables such that

$$
\begin{equation*}
\frac{1}{\beta!} \frac{\partial^{|\beta|} \Theta_{j}}{\partial \zeta^{\beta}}(\bar{f}(\tau), h(t))=\frac{\Omega_{j, \beta}\left(\left\{\underline{\mathscr{L}}^{\gamma} \bar{h}(\tau)\right\}_{|\gamma| \leq|\beta|}\right)}{\left[\operatorname{det}\left(\underline{\mathscr{L}}_{k} \bar{f}_{l}(\tau)\right)_{1 \leq k, l \leq n}\right]^{2|\beta|-1}}, \tag{6.5}
\end{equation*}
$$

for all $(t, \tau) \in \mathscr{M}$ with $|t|,|\tau|<\rho_{2}$. Here, for $\gamma=\left(\gamma_{1}, \ldots, \gamma_{m}\right) \in \mathbb{N}^{m}$, we denote by $\underline{L}^{\gamma}$ the derivation $\left(\mathscr{L}_{1}\right)^{\gamma_{1}} \ldots\left(\mathscr{L}_{m}\right)^{\gamma_{m}}$. Next, denoting by $\omega_{j, \beta}(t, \tau)$ the right hand side of (6.5) and developing the left hand side in power series using (5.7), we may write

$$
\begin{equation*}
\Theta_{j, \beta}(h(t))+\sum_{\gamma \in \mathbb{N}_{*}^{m}}(\bar{f}(\tau))^{\gamma} \Theta_{j, \beta+\gamma}(h(t))=\omega_{j, \beta}(t, \tau) . \tag{6.6}
\end{equation*}
$$

Recall that $M$ is $\ell_{0}$-nondegenerate at 0 . Using (5.8), we can solve $h(t)$ in terms of the derivatives of $h(\tau)$, namely

$$
\left\{\begin{align*}
h(t) & =\Psi\left(\bar{f}(\tau), \frac{\Omega_{j_{*}^{1}, \beta_{*}^{1}}\left(\left\{\mathscr{L}^{\gamma} \bar{h}(\tau)\right\}_{|\tau| \leq\left|\beta_{*}^{1}\right|}\right)}{\left[\operatorname{det}\left(\mathscr{L}_{k} \bar{f}_{l}(\tau)\right)_{1 \leq k, l \leq n}\right]^{2\left|\beta_{*}\right|-1}}, \ldots\right.  \tag{6.7}\\
& \left.\cdots, \frac{\Omega_{j_{*}^{n}, \beta_{*}^{n}}\left(\left\{\mathscr{L}^{\gamma} \bar{h}(\tau)\right\}_{|\gamma| \leq\left|\beta_{k}^{n}\right|}\right)}{\left[\operatorname{det}\left(\mathscr{L}_{k} \bar{f}_{l}(\tau)\right)_{1 \leq k, l \leq n}\right]^{2\left|\beta_{*}^{n}\right|-1}}\right)= \\
& =\Psi\left(\bar{f}(\tau), \omega_{j_{*}^{1}, \beta_{*}^{1}}(t, \tau), \ldots, \omega_{j_{*}^{n}, \beta_{*}^{n}}(t, \tau)\right) .
\end{align*}\right.
$$

Here, the maximal length of the multi-indices $\beta_{*}^{1}, \ldots, \beta_{*}^{n}$ is equal to $\ell_{0}$. According to (5.8), the representation (6.7) of $h(t)$ holds provided $|\bar{g}(\tau)|<\widetilde{\rho}_{3}$ and $\left|\omega_{j_{*}^{i}, \beta_{*}^{i}}\right|<\widetilde{\rho}_{3}$. Since the coordinates are normal, we have $\Theta_{j}(\bar{z}, 0,0) \equiv$ 0 , or equivalently $\Theta_{j, \beta}(0)=0$ for all $j=1, \ldots, d$ and all $\beta \in \mathbb{N}^{m}$. It follows from (6.6) and from $h(0)=0$ that $\omega_{j, \beta}(0)=0$, for all $j=1, \ldots, d$ and all $\beta \in \mathbb{N}^{m}$. Consequently, there exists a radius $\rho_{3} \sim \widetilde{\rho}_{3}$ with $0<\rho_{3}<\rho_{2}<\rho_{1}$
such that $\left|\omega_{j_{k}^{i}, \beta_{*}^{i}}(t, \tau)\right|<\widetilde{\rho}_{3}, i=1, \ldots, n$ and such that $|\bar{g}(\tau)|<\widetilde{\rho}_{3}$ for all $h \in \mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$ and for all $(t, \tau) \in \mathscr{M}$ with $|t|,|\tau|<\rho_{3}$.

In conclusion, the relation (6.7) holds for all $h \in \mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$ and for all $(t, \tau) \in \mathscr{M}$ with $|t|,|\tau|<\rho_{3}$.

Next, using the explicit expressions of the vector fields $\mathscr{L}_{k}$ given in (5.2), we may develop the higher order derivatives $\underline{L}^{\gamma} \bar{h}(\tau)$ as polynomials in the $|\gamma|$-jet $\left(\partial_{\tau}^{\gamma^{\prime}} \bar{h}(\tau)\right)_{\left|\gamma^{\prime}\right| \leq|\gamma|}$ of $\bar{h}(\tau)$ with coefficients being certain holomorphic functions of $(t, \tau)$ obtained as certain polynomials with respect to the partial derivatives of the functions $\Theta_{j}(\zeta, t)$.

To be more explicit in this desired new representation of (6.7), we remind first our jet notation. For each $i=1, \ldots, n$ and each $\alpha \in \mathbb{N}^{n}$, we introduced a new independent coordinate $J_{i}^{\alpha}$ corresponding to the partial derivative $\partial_{\tau}^{\alpha} \bar{h}_{i}(\tau)$ (or $\partial_{t}^{\alpha} h_{i}(t)$ ). The space of $k$-jets of holomorphic mappings $\bar{h}(\tau)$ is then the complex space $\mathbb{C}^{\frac{(n+k)!}{n!k!}}$ with coordinates $\left(J_{i}^{\alpha}\right)_{1 \leq i \leq n,|\alpha| \leq k}$. It will be convenient to use the abbreviations $J^{k}:=\left(J_{i}^{\alpha}\right)_{1 \leq i \leq n,|\alpha| \leq k}$ and $J^{k} \bar{h}(\tau):=$ $\left(\partial_{\tau}^{\alpha} \bar{h}_{i}(\tau)\right)_{1 \leq i \leq n,|\alpha| \leq k}$.

So pursuing with (6.7), we argue that for every $\gamma \in \mathbb{N}^{m}$, there exists a polynomial in the jet $J^{|\gamma|} \bar{h}(\tau)$ with holomorphic cooeficients depending only on $\Theta$ such that

$$
\begin{equation*}
\underline{\mathscr{L}}^{\gamma} \bar{h}(\tau) \equiv P_{\gamma}\left(t, \tau, J^{|\gamma|} \bar{h}(\tau)\right) . \tag{6.8}
\end{equation*}
$$

Putting all these expressions in (6.7), we obtain an important relation between $h$ and the $\ell_{0}$-jet of $\bar{h}$ which we may now summarize. At first, as $\kappa_{0}=\ell_{0}\left(\mu_{0}+1\right) \geq \ell_{0}$, observe that for every $h \in \mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$, we have $\left\|J^{\ell_{0}} h-J_{\mathrm{Id}}^{\ell_{0}}\right\|_{\rho_{2}} \leq\left\|J^{\kappa_{0}} h-J_{\mathrm{Id}}^{\kappa_{0}}\right\|_{\rho_{2}} \leq \varepsilon$. Shrinking $\varepsilon$ if necessary, we have proved the following lemma.

Lemma 6.1. There exists a complex algebraic $\mathbb{C}^{n}$-valued mapping $\Pi\left(t, \tau, J^{\ell_{0}}\right)$ defined for $|t|,|\tau|<\rho_{3}$ and for $\left|J^{\ell_{0}}-J_{\mathrm{Id}}^{\ell_{0}}\right|<\varepsilon$ which depends only on the defining functions $\xi_{j}-\Theta_{j}(\zeta, t)$ of $\mathscr{M}$, such that for every local holomorphic self-mapping $h \in \mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$ of $M$ (hence satisfying $\left.\left\|J^{\ell_{0}} h-J_{\text {Id }}^{\ell_{0}}\right\|_{\rho_{2}}<\varepsilon\right)$, the relation

$$
\begin{equation*}
h(t)=\Pi\left(t, \tau, J^{\ell_{0}} \bar{h}(\tau)\right) \tag{6.9}
\end{equation*}
$$

holds for all $(t, \tau) \in \mathscr{M}$ with $|t|,|\tau|<\rho_{3}$.
6.2. Reflection identity for arbitrary jets. Let now $\Upsilon_{j}$ and $\Upsilon_{j}$ be the vector fields tangent to $\mathscr{M}$ defined by

$$
\begin{equation*}
\Upsilon_{j}:=\frac{\partial}{\partial w_{j}}+\sum_{l=1}^{d} \Theta_{l, w_{j}}(\zeta, t) \frac{\partial}{\partial \xi_{l}}, \quad \Upsilon_{j}:=\frac{\partial}{\partial \xi_{j}}+\sum_{l=1}^{d} \bar{\Theta}_{l, \xi_{j}}(z, \tau) \frac{\partial}{\partial w_{l}}, \tag{6.10}
\end{equation*}
$$

for $j=1, \ldots, d$. We observe that the collection of $2 m+d$ vector fields $\mathscr{L}_{k}, \mathscr{L}_{k}, \Upsilon_{j}$ span $T \mathscr{M}$. The same holds for the collection $\mathscr{L}_{k}, \mathscr{L}_{k}, \Upsilon_{j}$. We also have the commutation relations $\left[\Upsilon_{j}, \mathscr{L}_{k}\right]=0$ and $\left[\Upsilon_{j}, \mathscr{L}_{k}\right]=0$. We observe that $\Upsilon^{\gamma} h(t)=\partial_{w}^{\gamma} h(t)$ for all $\gamma \in \mathbb{N}^{d}$. Let $\alpha=(\beta, \gamma) \in \mathbb{N}^{m} \times \mathbb{N}^{d}$. By expanding $\mathscr{L}^{\beta} \Upsilon^{\gamma} h(t)$ using the explicit expressions (5.2), we obtain a polynomial $Q_{\beta, \gamma}\left(t, \tau,\left(\partial_{t}^{\alpha^{\prime}} h(t)\right)_{\left|\alpha^{\prime}\right| \leq|\alpha|}\right)$, where $Q_{\beta, \gamma}$ is a polynomial in its last variables with coefficients depending on $\bar{\Theta}$ and its partial derivatives. Conversely, since $\left.\mathscr{L}_{k}\right|_{0}=\partial_{z_{k}}$ at the origin, we can invert these formulas, so there exist polynomials $P_{\alpha}$ in their last variables with coefficients depending only on $\bar{\Theta}$ such that

$$
\begin{equation*}
\partial_{t}^{\alpha} h(t)=P_{\alpha}\left(t, \tau,\left(\mathscr{L}^{\beta^{\prime}} \Upsilon^{\gamma^{\prime}} h(t)\right)_{\left|\beta^{\prime}\right| \leq|\beta|,\left|\gamma^{\prime}\right| \leq|\gamma|}\right) . \tag{6.11}
\end{equation*}
$$

Lemma 6.2. For every $\ell \in \mathbb{N}$, there exists a complex algebraic mapping $\Pi_{\ell}$ with values in $\mathbb{C}^{N_{n, \ell}}$ defined for $|t|,|\tau|<\rho_{3}$ and $\left|J^{\ell_{0}}-J_{\mathrm{Id}}^{\ell_{0}}\right|<\varepsilon$ which is relatively polynomial with respect to the higher order jets $J_{i}^{\alpha}$ with $|\alpha| \geq$ $\ell_{0}+1, i=1, \ldots, n$, such that for every local holomorphic self-mapping $h \in \mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$, the two conjugate relations

$$
\left\{\begin{align*}
J^{\ell} h(t) & =\Pi_{\ell}\left(t, \tau, J^{\ell_{0}+\ell} \bar{h}(\tau)\right),  \tag{6.12}\\
J^{\ell} \bar{h}(\tau) & =\bar{\Pi}_{\ell}\left(\tau, t, J^{\ell_{0}+\ell} h(t)\right)
\end{align*}\right.
$$

hold for all $(t, \tau) \in \mathscr{M}$ with $|t|,|\tau|<\rho_{3}$.
Proof. Applying the derivations $\mathscr{L}^{\beta} \Upsilon^{\gamma}$ to (6.9), and using the chain rule, we obtain

$$
\begin{equation*}
\mathscr{L}^{\beta} \Upsilon^{\gamma} h(t)=\Pi_{\beta, \gamma}\left(t, \tau, J^{\left.\ell_{0}+|\beta|+|\gamma| \bar{h}(\tau)\right), ~}\right. \tag{6.13}
\end{equation*}
$$

where the function $\Pi_{\beta, \gamma}$ (as the function $\Pi$ ) is holomorphic for $|t|,|\tau|<\rho_{3}$ and $\left|J^{\ell_{0}}-J_{\text {Id }}^{\ell_{0}}\right|<\varepsilon$ and relatively polynomial with respect to the jets $J_{i}^{\alpha}$ with $|\alpha| \geq \ell_{0}+1$. Applying (6.11), we obtain the function $\Pi_{\ell}$, which completes the proof.
6.3. Substitutions of reflection identities. Let $\pi_{t}(t, \tau):=t$ and $\pi_{\tau}(t, \tau):=$ $\tau$ denote the two canonical projections. We write $h^{c}(t, \tau):=(h(t), \bar{h}(\tau))$. We make the following slight abuse of notation: instead of rigorously writing $h\left(\pi_{t}(t, \tau)\right.$ ), we write $h(t, \tau)=h(t)$ and $\bar{h}(t, \tau)=\bar{h}(\tau)$.

Let $x \in \mathbb{C}^{\nu}$ and let $\mathscr{Q}(x)=\left(\mathscr{Q}_{1}(x), \ldots, \mathscr{Q}_{2 n}(x)\right) \in \mathbb{C}\{x\}^{2 n}$. As the multiple flow of $\mathscr{L}$ given by (5.3) does not act on the $(z, w)$ variables, we have the trivial but important property $h\left(\mathscr{L}_{z_{1}}(\mathscr{Q}(x))\right)=h(\mathscr{Q}(x))$. More generally, for every multi-index $\alpha \in \mathbb{N}^{n}$, we have $\partial_{t}^{\alpha} h\left(\mathscr{L}_{z_{1}}(\mathscr{Q}(x))\right)=$ $\partial_{t}^{\alpha} h(\mathscr{Q}(x))$. Analogously, we have $\partial_{\tau}^{\alpha} \bar{h}\left(\mathscr{L}_{z_{1}}(\mathscr{Q}(x))\right)=\partial_{\tau}^{\alpha} \bar{h}(\mathscr{Q}(x))$. Since
for $k$ even, we have $\Gamma_{k}\left(z_{(k)}\right)={\underset{\mathscr{L}}{z_{k}}}\left(\Gamma_{k-1}\left(z_{(k-1)}\right)\right)$, the following two properties hold:

$$
\begin{cases}J^{\ell} h\left(\Gamma_{k}\left(z_{(k)}\right)\right)=J^{\ell} h\left(\Gamma_{k-1}\left(z_{(k-1)}\right)\right), & \text { if } k \text { is even }  \tag{6.14}\\ J^{\ell} \bar{h}\left(\Gamma_{k}\left(z_{(k)}\right)\right)=J^{\ell} \bar{h}\left(\Gamma_{k-1}\left(z_{(k-1)}\right)\right), & \text { if } k \text { is odd }\end{cases}
$$

Let now $\kappa_{0}:=\ell_{0}\left(\mu_{0}+1\right)$ be the product of the Levi type with the Segre type of $\mathscr{M}$ plus 1 and consider the open subset of the $\kappa_{0}$-order jet space $\mathbb{C}^{N_{n, \kappa_{0}}}$ defined by the inequality $\left|J^{\kappa_{0}}-J_{\mathrm{Id}}^{\kappa_{0}}\right|<\varepsilon$. Let $z_{(k)} \in \Delta_{m k}$ as in §5.6 above. Since the maps $\Gamma_{k}$ are holomorphic and satisfy $\Gamma_{k}(0)=0$, we may choose $\delta>0$ sufficiently small in order that the following two conditions are satisfied for every $k \leq \mu_{0}$ and for and for every $\left|z_{(k)}\right|<\delta$ :

$$
\begin{equation*}
\left|\Gamma_{k}\left(z_{(k)}\right)\right|<\rho_{3} \quad \text { and } \quad\left|J^{\kappa_{0}} h\left(\Gamma_{k}\left(z_{(k)}\right)\right)-J_{\mathrm{Id}}^{\kappa_{0}}\right|<\varepsilon \tag{6.15}
\end{equation*}
$$

This choice of $\delta$ is convenient to make several susbtitutions by means of formulas (6.12). The formulas (6.16) that we will obtain below strongly differ from the previous formulas (6.12), because they depend on the jet of $h$ at the origin only.

Lemma 6.3. Shrinking $\varepsilon$ if necessary, for every integer $k \leq \mu_{0}+1$ and for every integer $\ell \geq 0$, there exists a complex algebraic mapping $\Pi_{\ell, k}$ with values in $\mathbb{C}^{N_{n, \ell}}$ defined for $|t|,|\tau|<\rho_{3}$ and for $\left|J^{k \ell_{0}}-J_{\text {Id }}^{k \ell_{0}}\right|<\varepsilon$, which is relatively polynomial with respect to the higher order jets $J_{i}^{\alpha}$ with $|\alpha| \geq$ $k \ell_{0}+1, i=1, \ldots, n$, and which depends only on the defining functions $\xi_{j}-$ $\Theta_{j}(\zeta, t)$ of $\mathscr{M}$, such that the following two families of conjugate identities are satisfied

$$
\begin{cases}J^{\ell} h\left(\Gamma_{k}\left(z_{(k)}\right)\right)=\Pi_{\ell, k}\left(\Gamma_{k}\left(z_{(k)}\right), J^{k \ell_{0}+\ell} \bar{h}(0)\right), & \text { if } k \text { is odd; }  \tag{6.16}\\ J^{\ell} \bar{h}\left(\Gamma_{k}\left(z_{(k)}\right)\right)=\overline{\Pi_{\ell, k}}\left(\Gamma_{k}\left(z_{(k)}\right), J^{k \ell_{0}+\ell} \bar{h}(0)\right), & \text { if } k \text { is even }\end{cases}
$$

Proof. For $k=1$, replacing $(t, \tau)$ by $\Gamma_{1}\left(z_{(1)}\right)$ in the first relation (6.12) and using the second property (6.14), we get

$$
\left\{\begin{align*}
J^{\ell} h\left(\Gamma_{1}\left(z_{(1)}\right)\right) & =\Pi_{\ell}\left(\Gamma_{1}\left(z_{(1)}\right), J^{\ell_{0}+\ell} \bar{h}\left(\Gamma_{1}\left(z_{(1)}\right)\right)\right)=  \tag{6.17}\\
& =\Pi_{\ell}\left(\Gamma_{1}\left(z_{(1)}\right), J^{\ell_{0}+\ell} \bar{h}(0)\right),
\end{align*}\right.
$$

so the lemma holds true for $k=1$ if we simply choose $\Pi_{\ell, 1}:=\Pi_{\ell}$. By induction, suppose that the lemma holds true for $k \leq \mu_{0}$. To fix the ideas, let us assume that this $k$ is even (the odd case is completely similar). Then replacing the arguments $(t, \tau)$ in the first relation (6.12) by $\Gamma_{k+1}\left(z_{(k+1)}\right)$, using again the second property (6.14), and using the induction assumption, namely using the conjugate of the second relation (6.16) with $\ell$ replaced by
$\ell_{0}+\ell$, we get

$$
\begin{aligned}
& \left\{\begin{aligned}
J^{\ell} h\left(\Gamma_{k+1}\left(z_{(k+1)}\right)\right) & =\Pi_{\ell}\left(\Gamma_{k+1}\left(z_{(k+1)}\right), J^{\ell_{0}+\ell} \bar{h}\left(\Gamma_{k+1}\left(z_{(k+1)}\right)\right)\right)= \\
& =\Pi_{\ell}\left(\Gamma_{k+1}\left(z_{(k+1)}\right), J^{\ell_{0}+\ell} \bar{h}\left(\Gamma_{k}\left(z_{(k)}\right)\right)\right)= \\
& =\Pi_{\ell}\left(\Gamma_{k+1}\left(z_{(k+1)}\right), \overline{\Pi_{\ell_{0}+\ell, k}}\left(\Gamma_{k}\left(z_{(k)}\right), J^{k \ell_{0}+\ell_{0}+\ell} \bar{h}(0)\right)\right)=: \\
& =: \Pi_{\ell, k+1}\left(\Gamma_{k+1}\left(z_{(k+1)}\right), J^{(k+1) \ell_{0}+\ell} \bar{h}(0)\right),
\end{aligned}\right.
\end{aligned}
$$

which yields the desired formula at level $k+1$. For the above formal composition formulas to be correct, we possibly have to shrink $\varepsilon$. Finally, a direct inspection of relative polynomialness shows that $\Pi_{\ell, k+1}$ is polynomial with respect to the jet variables $J_{i}^{\alpha}$ with $|\alpha| \geq(k+1) \ell_{0}+1, i=1, \ldots, n$. The proof of Lemma 6.21 is complete.
6.4. Algebraic parameterization of CR mappings by their jet at the origin. Finally, as in the paragraph after Theorem 5.2, we choose $\rho_{4}>0$ sufficiently small such that $\Gamma_{\mu_{0}}$ maps the polydisc $\Delta_{m \mu_{0}}(\eta)$ submersively onto an open neighborhood of the origin in $\mathscr{M}$ which contains the open subset $\mathscr{M} \cap\left(\Delta_{n}\left(\rho_{4}\right) \times \Delta_{n}\left(\rho_{4}\right)\right)$. From the relation $\Gamma_{\mu_{0}+1}\left(z_{\left(\mu_{0}\right)}, 0\right) \equiv \Gamma_{\mu_{0}}\left(z_{\left(\mu_{0}\right)}\right)$, it follows trivially that $\Gamma_{\mu_{0}+1}$ also induces a submersion from $\Delta_{m\left(\mu_{0}+1\right)}(\eta)$ onto $\mathscr{M} \cap\left(\Delta_{n}\left(\rho_{4}\right) \times \Delta_{n}\left(\rho_{4}\right)\right)$. It follows that the composition $\pi_{t} \circ \Gamma_{\mu_{0}+1}$ also maps submersively the polydisc $\Delta_{m\left(\mu_{0}+1\right)}(\eta)$ onto an open neighborhood of the origin in $\mathbb{C}^{n}$ which contains $\Delta_{n}\left(\rho_{4}\right)$. Consequently, in the representation obtained in Lemma 6.21 with $\ell=0$ and $k:=\mu_{0}+1=2 \nu_{0}+2$ (which is even), namely in the representation

$$
\begin{equation*}
\bar{h}\left(\Gamma_{\mu_{0}+1}\left(z_{\left(\mu_{0}+1\right)}\right)\right)=\overline{\Pi_{0, \mu_{0}+1}}\left(\Gamma_{\mu_{0}+1}\left(z_{\left(\mu_{0}+1\right)}\right), J^{\left(\mu_{0}+1\right) \ell_{0}} \bar{h}(0)\right), \tag{6.19}
\end{equation*}
$$

we can write an arbitrary $t \in \Delta_{n}\left(\rho_{4}\right)$ in the form $\Gamma_{\mu_{0}+1}\left(z_{\left(\mu_{0}+1\right)}\right)$, and finally, conjugating (6.19), we obtain a complex algebraic mapping $H$ with the property that $h(t)=H\left(t, J^{\left(\mu_{0}+1\right) \ell_{0}} h(0)\right)$. We may now summarize what we have proved so far.

Theorem 6.4. Let $M$ be a real algebraic generic submanifold in $\mathbb{C}^{n}$ passing through the origin, of codimension $d \geq 1$ and of CR dimension $m=n-d \geq$ 1. Assume that $M$ is $\ell_{0}$-nondegenerate at 0 . Assume that $M$ is minimal at 0 , let $\nu_{0}$ be the Segre type of $M$ at 0 and let $\mu_{0}:=2 \nu_{0}+1$ be the Segre type of $\mathscr{M}$ at 0 . Let $\kappa_{0}:=\left(\mu_{0}+1\right) \ell_{0}$. Let $t=(z, w) \in \mathbb{C}^{m} \times \mathbb{C}^{d}$ be holomorphic coordinates vanishing at 0 with $T_{0} M=\{\operatorname{Im} w=0\}$ and let $\rho_{1}>0$ be such that $M$ is represented by the complex analytic defining equations $\xi_{j}=\Theta_{j}(\zeta, t), j=1, \ldots, d$ in $\Delta_{n}\left(\rho_{1}\right)$. Then there exist $\varepsilon>0$, $\rho_{4}>0$ and there exists a complex algebraic $\mathbb{C}^{n}$-valued mapping $H\left(t, J^{\kappa_{0}}\right)$ defined for $|t|<\rho_{4}$ and for $\left|J^{\kappa_{0}}-J_{\mathrm{Id}}^{\kappa_{0}}\right|<\varepsilon$ which satisfies $H\left(t, J_{\mathrm{Id}}^{\kappa_{0}}\right) \equiv t$ and which depends only on the defining functions $\bar{w}_{j}-\Theta_{j}(\bar{z}, t)$ of $\mathscr{M}$, such
that for every local holomorphic self-mapping $h$ of $M$ belonging to $\mathscr{H}_{M, \kappa_{0}, \varepsilon^{\prime}}^{\rho_{2}, \rho_{1}}$, we have the representation formula

$$
\begin{equation*}
h(t)=H\left(t, J^{\kappa_{0}} h(0)\right), \tag{6.20}
\end{equation*}
$$

for all $t \in \mathbb{C}^{n}$ with $|t|<\rho_{4}$. Furthermore the mapping $H$ depends neither on the choice of smaller radii $\widetilde{\rho}_{1} \leq \rho_{1}, \widetilde{\rho}_{2} \leq \rho_{2}, \widetilde{\rho}_{3} \leq \rho_{3}$ and $\widetilde{\rho}_{4} \leq \rho_{4}$ satisfying $0<\widetilde{\rho}_{4}<\widetilde{\rho}_{3}<\widetilde{\rho}_{2}<\widetilde{\rho}_{1}$ nor on the choice of a smaller constant $\widetilde{\varepsilon}<\varepsilon$, so that the first sentence of property (3) in Theorem 4.1 holds true. Finally, if $M$ is real analytic, the same statement holds with the word "algebraic" everywhere replaced by the word "analytic".

It remains now to construct the submanifold $E$ whose existence is stated in Theorem 4.1 and to establish that $\mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$ may be endowed with the structure of a local real algebraic Lie group.
6.5. Local real algebraic Lie group structure. In order to construct this submanifold $E$, we introduce the $\kappa_{0}$-th jet mapping $\mathscr{J}^{\kappa_{0}}: \mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}} \rightarrow$ $\mathbb{C}^{N_{n, \kappa_{0}}}$ defined by $\mathscr{J}^{\kappa_{0}}(h):=\left(\partial_{t}^{\alpha} h(0)\right)_{|\alpha| \leq \kappa_{0}}=J^{\kappa_{0}} h(0)$. The following lemma is crucial.

Lemma 6.5. Shrinking $\varepsilon$ if necessary, the set

$$
\begin{equation*}
E:=\mathscr{J}^{\kappa_{0}}\left(\mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}\right)=\left\{J^{\kappa_{0}} h(0): h \in \mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}\right\} \tag{6.21}
\end{equation*}
$$

is a real algebraic totally real submanifold of the polydisc $\left\{J^{\kappa_{0}} \in \mathbb{C}^{N_{n, \kappa_{0}}}\right.$ : $\left.\left|J^{\kappa_{0}}-J_{\mathrm{Id}}^{\kappa_{0}}\right|<\varepsilon\right\}$.

Proof. Let $h \in \mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$. Substituting the representation formula $h(t)=$ $H\left(t, J^{\kappa_{0}} h(0)\right)$ given by Theorem 6.4 in the defining equations of $M$, we get

$$
\begin{equation*}
r_{j}\left(H\left(t, J^{\kappa_{0}} h(0)\right), \bar{H}\left(\tau, J^{\kappa_{0}} \bar{h}(0)\right)\right)=0 \tag{6.22}
\end{equation*}
$$

for $j=1, \ldots, d$ and $(t, \tau) \in \mathscr{M}$ with $|t|,|\tau|<\rho_{4}$. As $(t, \tau) \in \mathscr{M}$, we replace $\xi$ by $\Theta(\zeta, t)$ and we use the $2 m+d$ coordinates $(t, \zeta)$ on $\mathscr{M}$. So, by expanding the functions (6.22) in power series with respect to $(t, \zeta)$, we can write

$$
\begin{equation*}
r_{j}\left(H\left(t, J^{\kappa_{0}}\right), \bar{H}\left(\zeta, \Theta(\zeta, t), \overline{J^{\kappa_{0}}}\right)\right)=\sum_{\alpha \in \mathbb{N}^{n}, \beta \in \mathbb{N}^{m}} t^{\alpha} \zeta^{\beta} C_{j, \alpha, \beta}\left(J^{\kappa_{0}}, \overline{J^{\kappa_{0}}}\right) . \tag{6.23}
\end{equation*}
$$

Here, we obtain an infinite collection of complex-valued real algebraic functions $C_{j, \alpha, \beta}$ defined in $\left\{\left|J^{\kappa_{0}}-J_{\text {Id }}^{\kappa_{0}}\right|<\varepsilon\right\}$ with the property that a mapping $H\left(t, J^{\kappa_{0}}\right)$ sends $M \cap \Delta_{n}\left(\rho_{4}\right)$ into $M$ if and only if

$$
\begin{equation*}
C_{j, \alpha, \beta}\left(J^{\kappa_{0}}, \overline{J^{\kappa_{0}}}\right)=0, \quad \forall j, \alpha, \beta . \tag{6.24}
\end{equation*}
$$

Consequently, the set $E$ defined by the vanishing of all the equations (6.24) is a real algebraic subset.

It follows from the representation formula (6.20) that the mapping $\mathscr{J}^{\kappa_{0}}$ is injective and from the Cauchy integral formula that $\mathscr{J}^{\kappa_{0}}$ is continuous on its domain of definition $\mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$ endowed with the topology of uniform convergence on compact sets.

On the reverse side, let $J^{\kappa_{0}} \in E$. Then the mapping $h(t):=H\left(t, J^{\kappa_{0}}\right)$ defined for $|t|<\rho_{4}$ maps $M \cap \Delta_{n}\left(\rho_{4}\right)$ into $M$. Applying Theorem 6.4 to this mapping $h(t)$, with $\rho_{1}$ replaced by $\rho_{4}$, we deduce that there exists a radius $\rho_{6}<\rho_{4}$ such that we can represent $h(t)=H\left(t, J^{\kappa_{0}} h(0)\right)$ for $|t|<\rho_{6}$, with the same mapping $H$, as stated in the end of Theorem 6.4. By differentiating this representation with respect to $t$ at $t=0$, we deduce that $J^{\kappa_{0}} h(0)=\left(\left[\partial_{t}^{\alpha} H\left(t, J^{\kappa_{0}} h(0)\right)\right]_{t=0}\right)_{|\alpha| \leq \kappa_{0}}$. Consequently, since $h(t)=H\left(t, J^{\kappa_{0}}\right)$ by definition, we get $J^{\kappa_{0}}=\left(\left[\partial_{t}^{\alpha} H\left(t, J^{\kappa_{0}}\right)\right]_{t=0}\right)_{|\alpha| \leq \kappa_{0}}$. In conclusion, we proved that $\mathscr{J}^{\kappa_{0}}\left(H\left(t, J^{\kappa_{0}}\right)\right)=J^{\kappa_{0}}$ for every $J^{\kappa_{0}} \in E$, so $\mathscr{J}^{\kappa_{0}}$ has a continuous local inverse on $E$, formally defined by $H\left(t, J^{\kappa_{0}}\right)$.

It follows from the above two paragraphs that the mapping $\mathscr{J}^{\kappa_{0}}$ is a local homeomorphism from a neighborhood of the identity in $\mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$ onto its image $E$.

Furthermore, we claim that the real algebraic subset $E$ is in fact geometrically smooth at every point, namely it is a real algebraic submanifold. Indeed, let $J_{1}^{\kappa_{0}}$ be a regular point of $E$ where $E$ is of maximal geometrical dimension $c_{0}$, with $J_{1}^{\kappa_{0}}$ arbitrarily close to the identity jet $J_{\mathrm{Id}}^{\kappa_{0}}$. Let $h_{1} \in \mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}}$ such that $J_{1}^{\kappa_{0}}=\mathscr{J}^{\kappa_{0}}\left(h_{1}\right)$. Let $\mathscr{U}_{1}$ be a small neighborhood of $J_{1}^{\kappa_{0}}$ in $\mathbb{C}^{N_{n, \kappa_{0}}}$ in which $E \cap \mathscr{U}_{1}$ is a regular $c_{0}$-dimensional real algebraic submanifold and consider the complex algebraic mapping defined over $\mathscr{U}_{1}$ by

$$
\begin{equation*}
\mathscr{F}_{1}\left(J^{\kappa_{0}}\right):=\left(\left[\partial_{t}^{\alpha}\left(h_{1}^{-1}\left(H\left(t, J^{\kappa_{0}}\right)\right)\right)\right]_{t=0}\right)_{|\alpha| \leq \kappa_{0}} \in \mathbb{C}^{N_{n, \kappa_{0}}} . \tag{6.25}
\end{equation*}
$$

We have $\mathscr{F}_{1}\left(J_{1}^{\kappa_{0}}\right)=J_{\text {Id }}^{\kappa_{0}}$ and the restriction of $\mathscr{F}_{1}$ to $E \cap \mathscr{U}_{1}$ induces a homeomorphism onto its image, which is a neighborhood of $J_{\mathrm{Id}}^{\kappa_{0}}$ in $E$. We remind that the mapping $J^{\kappa_{0}} \rightarrow\left(\left[\partial_{t}^{\alpha}\left(H\left(t, J^{\kappa_{0}}\right)\right)\right]_{t=0}\right)_{|\alpha| \leq \kappa_{0}}$ restricted to $E \cap \mathscr{U}_{1}$ is the identity and consequently of constant rank equal to $c_{0}$. As $h_{1}$ is invertible, it follows from the chain rule by developing (6.25) that $\left.\mathscr{F}_{1}\right|_{E \cap \mathscr{U}_{1}}$ is also of locally constant rank equal to $c_{0}$. This proves that $E$ is a $c_{0}$-dimensional real algebraic submanifold in $\mathbb{C}^{N_{n, \kappa_{0}}}$ through $J_{\mathrm{Id}}^{\kappa_{0}}$. More generally, this reasoning shows that $E$ is geometrically smooth at every point.

Finally, applying Lemma 6.3 with the odd integer $k=\mu_{0}=2 \nu_{0}+1$ (instead of $k=\mu_{0}+1$ ), we get a new, different representation formula $h(t)=\widetilde{H}\left(t, J^{\ell_{0} \mu_{0}} \bar{h}(0)\right)$ (notice $\bar{h}(0)$ ). Accordingly, we can define a real algebraic submanifold $\widetilde{E}$. It is clear that we can identify $E$ and $\widetilde{E}$, since they both parametrize the local biholomorphic self-mappings of $M$, so they are algebraically equivalent by means of the natural projection from the $\ell_{0}\left(\mu_{0}+\right.$ 1 )-th jet space onto the $\ell_{0} \mu_{0}$-th jet space. Next, we see by differentiating
$h(t)=\widetilde{H}\left(t, J^{\ell_{0} \mu_{0}} \bar{h}(0)\right)$ with respect to $t$ that

$$
\begin{equation*}
J^{\ell_{0} \mu_{0}} h(0)=\left(\left[\partial_{t}^{\alpha} \widetilde{H}\left(t, J^{\ell_{0} \mu_{0}} \bar{h}(0)\right)\right]_{t=0}\right)_{|\alpha| \leq \ell_{0} \mu_{0}} . \tag{6.26}
\end{equation*}
$$

Consequently, if $K$ is the holomorphic map defined by

$$
\begin{equation*}
K\left(J^{\ell_{0} \mu_{0}}\right):=\left(\left[\partial_{t}^{\alpha} \widetilde{H}\left(t, J^{\ell_{0} \mu_{0}}\right]_{t=0}\right)_{|\alpha| \leq \ell_{0} \mu_{0}}\right), \tag{6.27}
\end{equation*}
$$

we get the equality $J^{\ell_{0} \mu_{0}}=K\left(\overline{J^{\ell_{0} \mu_{0}}}\right)$ for every $J^{\ell_{0} \mu_{0}} \in \widetilde{E}$, which proves that $\widetilde{E}$ is totally real. It follows that $E$ is totally real, which completes the proof.

Lemma 6.6. The submanifold $E$ is naturally equipped with a local real algebraic Lie group structure in a neighborhood of $J_{\mathrm{Id}}^{\kappa_{0}}$.

Proof. Indeed, let us parametrize $E$ by a real algebraic mapping

$$
\begin{equation*}
\mathbb{R}^{c_{0}} \ni\left(e_{1}, \ldots, e_{c_{0}}\right) \longmapsto j_{\kappa_{0}}(e) \in \mathbb{C}^{N_{n, \kappa_{0}}} \tag{6.28}
\end{equation*}
$$

where $c_{0}$ is the dimension of $E$. Here, to avoid excessive formal complexity, we shall avoid to mention all the polydiscs of variation of the variables. For $e \in E$, we shall use the notation

$$
\begin{equation*}
H(t ; e):=H\left(t, j_{\kappa_{0}}(e)\right) . \tag{6.29}
\end{equation*}
$$

Let $e \in E$ and $e^{\prime} \in E$, set $J^{\kappa_{0}}:=j_{\kappa_{0}}(e)$ and $J^{\kappa_{0}}:=j_{\kappa_{0}}\left(e^{\prime}\right)$. Then we can define the Lie group multiplication $\mu_{J}$ by

$$
\begin{equation*}
\mu_{J}\left({ }^{\prime} J^{\kappa_{0}}, J^{\kappa_{0}}\right):=\left(\left[\partial_{t}^{\alpha}\left(H\left(H\left(t, J^{\kappa_{0}}\right),{ }^{\prime} J^{\kappa_{0}}\right)\right)\right]_{t=0}\right)_{|\alpha| \leq \kappa_{0}} . \tag{6.30}
\end{equation*}
$$

Accordingly, in terms of the coordinates $\left(e_{1}, \ldots, e_{c_{0}}\right)$ on $E$, the Lie group multiplication $\mu$ is defined by

$$
\begin{equation*}
\mu\left(e, e^{\prime}\right):=\left(j_{\kappa_{0}}\right)^{-1}\left(\mu_{J}\left(j_{\kappa_{0}}\left(e^{\prime}\right), j_{\kappa_{0}}(e)\right)\right) \in \mathbb{R}^{c_{0}} \tag{6.31}
\end{equation*}
$$

It follows from the algebraicity of the mappings $H$ and $j_{\kappa_{0}}$ that the mappings $\mu_{J}$ and $\mu$ are algebraic.

We must check the associativity of $\mu$, namely $\mu\left(\mu\left(e, e^{\prime}\right), e^{\prime \prime}\right)=$ $\mu\left(e, \mu\left(e^{\prime}, e^{\prime \prime}\right)\right)$. So we set $h(t):=H\left(t, j_{\kappa_{0}}(e)\right), h^{\prime}(t):=H\left(t, j_{\kappa_{0}}\left(e^{\prime}\right)\right)$ and $h^{\prime \prime}(t):=H\left(t, j_{\kappa_{0}}\left(e^{\prime \prime}\right)\right)$. By the definition (6.30), we have $\mu_{J}\left(j_{\kappa_{0}}(e), j_{\kappa_{0}}\left(e^{\prime}\right)\right)=J^{\kappa_{0}}\left(h \circ h^{\prime}\right)(0)$. Applying then Theorem 6.4, we get $H\left(t, J^{\kappa_{0}}\left(h \circ h^{\prime}\right)(0)\right) \equiv\left(h \circ h^{\prime}\right)(t)$. Consequently, using again (6.30) and the associativity of the composition of mappings, we may compute (6.32)

$$
\left\{\begin{aligned}
\mu_{J}\left(\mu_{J}\left(j_{\kappa_{0}}(e), j_{\kappa_{0}}\left(e^{\prime}\right)\right), j_{\kappa_{0}}\left(e^{\prime \prime}\right)\right) & =\mu_{J}\left(J^{\kappa_{0}}\left(\left(h \circ h^{\prime}\right)(0), j_{\kappa_{0}}\left(e^{\prime \prime}\right)\right)\right. \\
& =J^{\kappa_{0}}\left(\left(h \circ h^{\prime}\right) \circ h^{\prime \prime}\right)(0) \\
& =J^{\kappa_{0}}\left(h \circ\left(h^{\prime} \circ h^{\prime \prime}\right)\right)(0) \\
& =\mu_{J}\left(j_{\kappa_{0}}(e), J^{\kappa_{0}}\left(h^{\prime} \circ h^{\prime \prime}\right)(0)\right) \\
& =\mu_{J}\left(j_{\kappa_{0}}(e), \mu_{J}\left(j_{\kappa_{0}}(e), j_{\kappa_{0}}\left(e^{\prime}\right)\right)\right),
\end{aligned}\right.
$$

which proves the associativity.
Finally, we may define an algebraic inversion mapping $\iota$ as follows. First of all, for $J^{\kappa_{0}}$ close to $J_{\mathrm{Id}}^{\kappa_{0}}$, the mapping $h(t):=H\left(t, J^{\kappa_{0}}\right)=$ $t+\sum_{\alpha \in \mathbb{N}^{n}} t^{\alpha} H_{\alpha}\left(J^{\kappa_{0}}\right)$ is an invertible algebraic biholomorphic mapping. Here, the coefficients $H_{\alpha}\left(J^{\kappa_{0}}\right)$ are algebraic functions of $J^{\kappa_{0}}$ which vanish at $J_{\mathrm{Id}}^{\kappa_{0}}$ (since $H\left(t, J_{\mathrm{Id}}^{\kappa_{0}}\right) \equiv t$ in Theorem 6.4). From the algebraic implicit function theorem, it follows that the local inverse $h^{-1}(t)$ writes uniquely in the form $h^{-1}(t)=t+\sum_{\alpha \in \mathbb{N}^{n}} t^{\alpha} \widetilde{H}_{\alpha}\left(J^{\kappa_{0}}\right)=: \widetilde{H}\left(t, J^{\kappa_{0}}\right)$, where the $\widetilde{H}_{\alpha}\left(J^{\kappa_{0}}\right)$ are algebraic functions of $J^{\kappa_{0}}$ also satisfying $\widetilde{H}_{\alpha}\left(J_{\text {Id }}^{\kappa_{0}}\right)=0$. Consequently, choosing $e \in E$ such that $J^{\kappa_{0}}=j_{\kappa_{0}}(e)$, we can define

$$
\begin{equation*}
\left.\iota_{J}\left(J^{\kappa_{0}}\right):=\left(\left[\partial_{t}^{\alpha} \widetilde{H}\left(t, J^{\kappa_{0}}\right)\right]_{t=0}\right)\right)_{|\alpha| \leq \kappa_{0}} . \tag{6.33}
\end{equation*}
$$

Accordingly, in terms of the coordinates $\left(e_{1}, \ldots, e_{c_{0}}\right)$ on $E$, the Lie group inverse mapping is defined by

$$
\begin{equation*}
\iota(e):=\left(j^{\kappa_{0}}\right)^{-1}\left(i_{J}\left(j_{\kappa_{0}}(e)\right)\right) . \tag{6.34}
\end{equation*}
$$

Of course, with this definition we have $\iota_{J}\left(J_{\mathrm{Id}}^{\kappa_{0}}\right)=J_{\mathrm{Id}}^{\kappa_{0}}$. Finally, we leave to the reader to verify that $\mu_{J}\left(j_{\kappa_{0}}(e), i_{J}\left(j_{\kappa_{0}}(e)\right)\right)=J_{\text {Id }}^{\kappa_{0}}$. This completes the proof of property (4) of Theorem 4.1.

End of proof of Theorem 4.1. We notice that statement (5) does not need to be proved. Furthermore that the dimensional inequality $c_{0} \leq \frac{\left(n+\kappa_{0}\right)!}{n!\kappa_{0}!}$ in (6) follows from the fact each local biholomorphic mapping in the local Lie group $\mathscr{H}_{M, \kappa_{0}, \varepsilon}^{\rho_{2}, \rho_{1}} \cong E$ writes uniquely as $h(t)=H\left(t, J^{\kappa_{0}} h(0)\right)$, so the complex dimension of the local Lie group $E$ is $\leq \frac{\left(n+\kappa_{0}\right)!}{n!\kappa_{0}!}$, the dimension of the $\kappa_{0}$-th jet space. As $E$ is totally real, the real dimension of $E$ is also $\leq \frac{\left(n+\kappa_{0}\right)!}{n!\kappa_{0}!}$. Finally, it follows that the real local Lie algebra of vector fields $\mathfrak{H o l}\left(M, \Delta_{n}\left(\rho_{5}\right)\right)$ is of dimension $\leq \frac{\left(n+\kappa_{0}\right)!}{n!\kappa_{0}!}$. The proof of Theorem 4.1 is complete.

## §7. DESCRIPTION OF EXPLICIT FAMILIES OF STRONG TUBES IN $\mathbb{C}^{n}$

7.1. Introduction. Theorems $1.1,1.4$ and 1.5 provide sufficient conditions for some real analytic real submanifold in $\mathbb{C}^{n}$ to be not locally algebraizable. For the sake of completeness, we exhibit explicit examples of such nonalgebraizable submanifolds which are effectively strong tubes and effectively nonalgebraizable, proving corollaries 1.2, 1.3, 1.6 and 1.7. Consequently we will deal with the two following families of nonalgebraizable real analytic Levi nondegenerate hypersurfaces in $\mathbb{C}^{n}(n \geq 2)$ : the Levi nondegenerate strong tube hypersurfaces in $\mathbb{C}^{n}$ and the strongly rigid hypersurfaces in $\mathbb{C}^{n}$. For heuristic reasons, we shall sometimes start with the case $n=2$ and treat the general case $n \geq 2$ afterwards. In fact, our goal will be to construct
infinite families of pairwise non biholomorphically equivalent and non locally algebraizable hypersurfaces. Our computations for the construction of families of manifolds with a control on the structure of their automorphism group are all based on the Lie theory of symmetries of differential equations. For the convenience of the reader, we recall briefly the procedure (see [Su2001a,b], [GM2001a,b,c] for more details).
7.2. Hypersurfaces and differential equations. Let $M$ be a real analytic hypersurface in $\mathbb{C}^{n}$. Assume that $M$ is Levi nondegenerate at one of its points $p$. Then there exist some local holomorphic coordinates $(z, w)=$ $(z, u+i v) \in \mathbb{C}^{n-1} \times \mathbb{C}$ vanishing at $p$ such that $M$ is given by the real analytic equation

$$
\begin{equation*}
v=\varphi(z, \bar{z}, u)=\varepsilon_{1}\left|z_{1}\right|^{2}+\cdots+\varepsilon_{n-1}\left|z_{n-1}\right|^{2}+\psi(z, \bar{z}, u) \tag{7.1}
\end{equation*}
$$

where $\varepsilon_{k}= \pm 1, k=1, \ldots, n-1$ and where $\psi=\mathrm{O}(3)$. Passing to the extrinsic complexification $\mathscr{M}$ of $M$, we may consider the variables $\bar{z}$ and $\bar{w}$ as independent complex parameters $\zeta \in \mathbb{C}^{n-1}$ and $\xi \in \mathbb{C}$. Then the associated complex defining equation is of the form
(7.2) $w=\bar{\Theta}(z, \zeta, \xi)=\xi+2 i\left(\varepsilon_{1} z_{1} \zeta_{1}+\cdots+\varepsilon_{n-1} z_{n-1} \zeta_{n-1}+\bar{\Xi}(z, \zeta, \xi)\right)$,
where $\bar{\Xi}=\mathrm{O}(3)$. By [Me1998] (cf. §5.1 above), for $\tau_{p}=\left(\zeta_{p}, \xi_{p}\right)$ fixed, the family of complexified Segre varieties $\mathscr{S}_{\tau_{p}}:=\left\{\left(t, \tau_{p}\right): w=\bar{\Theta}\left(z, \tau_{p}\right)\right\}$ is invariantly and biholomorphically attached to $M$.

Following [Se1931] and [Su2001a,b], we may consider this family as a family of graphs of the solutions of a second order completely integrable system of partial differential equations as follows. By differentiating the left and the right hand sides of (7.2) with respect to $z_{k}$, we get

$$
\begin{equation*}
\partial_{z_{k}} w=\partial_{z_{k}} \bar{\Theta}(z, \tau)=2 i\left(\varepsilon_{k} \zeta_{k}+\partial_{z_{k}} \bar{\Xi}(z, \tau)\right), \tag{7.3}
\end{equation*}
$$

for $k=1, \ldots, n-1$. Here, we consider $w$ as a function of $z$. Using the analytic implicit function theorem to solve $\tau$ in the $1+(n-1)=n$ equations (7.2) and (7.3), we may express $\tau$ in terms of $w$, of $z$ and of the first order derivative $w_{z_{l}}$, which yields

$$
\begin{equation*}
\tau=\Pi\left(z, w,\left(\partial_{z_{l}} w\right)_{1 \leq l \leq n-1}\right), \tag{7.4}
\end{equation*}
$$

where $\Pi$ is holomorphic in its variables. If we take the second derivative $w_{z_{k_{1}} z_{k_{2}}}$ of $w$ and replace the value of $\tau$, we get the desired system of partial differential equations:

$$
\begin{align*}
\partial_{z_{k_{1}} z_{k_{2}}}^{2} w & =\partial_{z_{k_{1}} z_{k_{2}}}^{2} \bar{\Theta}(z, \tau)=\partial_{z_{k_{1}} z_{k_{2}}}^{2} \bar{\Theta}\left(z, \Pi\left(z, w,\left(\partial_{z_{l}} w\right)_{1 \leq l \leq n-1}\right)\right)=:  \tag{7.5}\\
& =: F_{k_{1}, k_{2}}\left(z, w,\left(\partial_{z_{l}} w\right)_{1 \leq l \leq n-1}\right) .
\end{align*}
$$

Here, $k_{1}, k_{2}=1, \ldots, n-1$ and the $F_{k_{1}, k_{2}} \equiv F_{k_{2}, k_{1}}$ are holomorphic in their variables. We denote by $\mathscr{E}_{M}$ this system of partial differential equations (here, to construct $\mathscr{E}_{M}$, we have used the Levi nondegeneracy of $M$ but we note that if $M$ were finitely nondegenerate the same conclusion would be true, by considering some derivatives of $w$ of larger order). Since the solutions of $\mathscr{E}_{M}$ are precisely the complexified Segre varieties $\mathscr{S}_{\tau}$, the system $\mathscr{E}_{M}$ is completely integrable.

To study the local geometry of $M$, we may consider on one hand the real Lie algebra of infinitesimal CR automorphisms of $M$ ( $c f$. §2.2), namely $\mathfrak{A} \mathfrak{u t}_{C R}(M)=2 \operatorname{Re} \mathfrak{H o l}(M)$. On the other hand, following the general ideas of Lie (cf. the modern restitution by Olver in [Ol1986, Ch 2]), we may consider the Lie algebra of infinitesimal generators of the local symmetry group of the system of partial differential equations $\mathscr{E}_{M}$, which we shall denote by $\mathfrak{S y m}\left(\mathscr{E}_{M}\right)$. By definition, $\mathfrak{S y m}\left(\mathscr{E}_{M}\right)$ consists of holomorphic vector fields in the $(z, w)$-space whose local flow transforms the graph of every solution of $\mathscr{E}_{M}$ (namely a complexified Segre variety) into the graph of another solution of $\mathscr{E}_{M}$ (namely into another complexified Segre variety). The link between $\mathfrak{A u t}_{C R}(M)$ and $\mathfrak{S y m}\left(\mathscr{E}_{M}\right)$ is as follows: by [Ca1932, p. 30-32], one can prove that $\mathfrak{A u t}_{C R}(M)$ is a maximally real subspace of $\mathfrak{S y m}\left(\mathscr{E}_{M}\right)$ (see also [Su2001a,b], [GM2001a,b,c]).

The computation of explicit generators of $\mathfrak{S y m}\left(\mathscr{E}_{M}\right)$ may be performed using the Lie theory of symmetries of differential equations. By inspecting some examples, it appears that dealing with $\mathfrak{S y m}\left(\mathscr{E}_{M}\right)$ generally shortens the complexity of the computation of $\mathfrak{A u t} t_{C R}(M)$ by at least one half.

The Lie procedure to compute $\mathfrak{S y m}\left(\mathscr{E}_{M}\right)$ is as follows. Let $J_{n-1,1}^{2}(\mathbb{C})$ denote the space of second order jets of a function $w\left(z_{1}, \ldots, z_{n-1}\right)$ of $(n-1)$ complex variables, equipped with independent coordinates $\left(z, w, W_{l}^{1}, W_{k_{1}, k_{2}}^{2}\right)$ corresponding to $\left(z, w, w_{z_{l}}, w_{z_{k_{1}} z_{k_{2}}}\right)$, where $l=1, \ldots, n-1$, where $k_{1}, k_{2}=1, \ldots, n-1$, and where we of course identify $W_{k_{1}, k_{2}}^{2}$ with $W_{k_{2}, k_{1}}^{2}$. To the system $\mathscr{E}_{M}$, we associate the complex submanifold of $J_{n-1,1}^{2}(\mathbb{C})$ defined by replacing the derivatives of $w$ by the independent jet variables in the system $\mathscr{E}_{M}$, which yields ( $c f$. (7.5)):

$$
\begin{equation*}
W_{k_{1}, k_{2}}^{2}=F_{k_{1}, k_{2}}\left(z, w,\left(W_{l}^{1}\right)_{1 \leq l \leq n-1}\right), \tag{7.6}
\end{equation*}
$$

for $k_{1}, k_{2}=1, \ldots, n-1$. Let $\Delta_{M}$ denote this submanifold. By Lie's theory, every vector field $X=\sum_{k=1}^{n-1} Q^{k}(z, w) \partial_{z_{k}}+R(z, w) \partial_{w}$ defined in a neighborhood of the origin in $\mathbb{C}^{n}$ can be uniquely lifted to a vector field $X^{(2)}$ in $J_{n-1,1}^{2}(\mathbb{C})$, which is called the second prolongation of $X$ (by definition, the lift $X^{(2)}$ shows how the flow of $X$ transforms second order jets of graphs of
functions $w(z)$ ). The coefficients $R_{l}^{1}$ and $R_{k_{1}, k_{2}}^{2}$ of the second prolongation

$$
\begin{equation*}
X^{(2)}=\sum_{k=1}^{n-1} Q^{k} \frac{\partial}{\partial z_{k}}+R \frac{\partial}{\partial w}+\sum_{l=1}^{n-1} R_{l}^{1} \frac{\partial}{\partial W_{l}^{1}}+\sum_{k_{1}, k_{2}=1}^{n-1} R_{k_{1}, k_{2}}^{2} \frac{\partial}{\partial W_{k_{1}, k_{2}}^{2}} \tag{7.7}
\end{equation*}
$$

are completely determined by the following universal formulas (cf. [O11986], [Su2001a, b], [GM2001a]):
(7.8)

$$
\left\{\begin{aligned}
R_{l}^{1} & =\partial_{z_{l}} R+\sum_{m_{1}}\left[\delta_{l}^{m_{1}} \partial_{w} R-\partial_{z_{l}} Q^{m_{1}}\right] W_{m_{1}}^{1}+\sum_{m_{1}, m_{2}}\left[-\delta_{l}^{m_{1}} \partial_{w} Q^{m_{2}}\right] W_{m_{1}}^{1} W_{m_{2}}^{1} \\
R_{k_{1}, k_{2}}^{2} & =\partial_{z_{k_{1}} z_{k_{2}}}^{2} R+\sum_{m_{1}}\left[\delta_{k_{1}}^{m_{1}} \partial_{z_{k_{2}} w}^{2} R+\delta_{k_{2}}^{m_{1}} \partial_{z_{k_{1}} w}^{2} R-\partial_{z_{k_{1}} z_{k_{2}}}^{2} Q^{m_{1}}\right] W_{m_{1}}^{1}+ \\
& +\sum_{m_{1}, m_{2}}\left[\delta_{k_{1}, k_{2}}^{m_{1}, m_{2}} \partial_{w^{2}}^{2} R-\delta_{k_{1}}^{m_{1}} \partial_{z_{k_{2}} w}^{2} Q^{m_{2}}-\delta_{k_{2}}^{m_{1}} \partial_{z_{k_{1}} w}^{2} Q^{m_{2}}\right] W_{m_{1}}^{1} W_{m_{2}}^{1}+ \\
& +\sum_{m_{1}, m_{2}, m_{3}}\left[-\delta_{k_{1}, k_{2}}^{m_{1}, m_{2}} \partial_{w^{2}}^{2} Q^{m_{3}}\right] W_{m_{1}}^{1} W_{m_{2}}^{1} W_{m_{3}}^{1}+ \\
& +\sum_{m_{1}, m_{2}}\left[\delta_{k_{1}, k_{2}}^{m_{1}, m_{2}} \partial_{w} R-\delta_{k_{1}}^{m_{1}} \partial_{z_{k_{2}}} Q^{m_{2}}-\delta_{k_{2}}^{m_{1}} \partial_{z_{k_{1}}} Q^{m_{2}}\right] W_{m_{1}, m_{2}}^{2}+ \\
& +\sum_{m_{1}, m_{2}, m_{3}}\left[-\delta_{k_{1}, k_{2}}^{m_{1}, m_{2}} \partial_{w} Q^{m_{3}}-\delta_{k_{1}, k_{2}}^{m_{2}, m_{3}} \partial_{w} Q^{m_{1}}-\delta_{k_{1}, k_{2}}^{m_{3}, m_{1}} \partial_{w} Q^{m_{2}}\right] W_{m_{1}}^{1} W_{m_{2}, m_{3}}^{2} .
\end{aligned}\right.
$$

In these formulas, by $\delta_{l}^{m}$ we denote the Kronecker symbol equal to 1 if $l=$ $m$ and to 0 otherwise. The multiple Kronecker symbol $\delta_{l_{1}, l_{2}}^{m_{1}, m_{2}}$ is defined to be the product $\delta_{l_{1}}^{m_{1}} \cdot \delta_{l_{2}}^{m_{2}}$. Finally, in the sums $\sum_{m_{1}}, \sum_{m_{1}, m_{2}}$ and $\sum_{m_{1}, m_{2}, m_{3}}$, the integers $m_{1}, m_{2}, m_{3}$ run from 1 to $n-1$. We would like to mention that in [GM2001a], we also provide some explicit expression of the $k$-th prolongation $X^{(k)}$ for $k \geq 3$.

Then the Lie criterion states that a holomorphic vector field $X$ belongs to $\mathfrak{S y m}\left(\mathscr{E}_{M}\right)$ if and only if its second prolongation $X^{(2)}$ is tangent to $\Delta_{M}$ ([Ol1986, Ch 2]). This gives the following equations:

$$
\begin{equation*}
R_{k_{1}, k_{2}}^{2}-\sum_{k=1}^{n-1} Q^{k} \partial_{z_{k}} F_{k_{1}, k_{2}}-R \partial_{w} F_{k_{1}, k_{2}}-\sum_{l=1}^{n-1} R_{l}^{1} \partial_{W_{l}^{1}} F_{k_{1}, k_{2}} \equiv 0 \tag{7.9}
\end{equation*}
$$

where $1 \leq k_{1}, k_{2} \leq n-1$ and where each occurence of $W_{l_{1}, l_{2}}^{2}$ is replaced by its value $F_{l_{1}, l_{2}}$ on $\Delta_{M}$. By developping (7.9) in power series with respect to the variables $W_{l}^{1}$, we get an expression of the form

$$
\begin{equation*}
\sum_{l_{1}, \ldots, l_{n-1} \geq 0} W_{l_{1}}^{1} \cdots W_{l_{n-1}}^{1} \Phi_{l_{1}, \ldots, l_{n-1}} \equiv 0 \tag{7.10}
\end{equation*}
$$

where each term $\Phi_{l_{1}, \ldots, l_{n-1}}$ is a certain linear partial differential expression involving the derivatives of $Q^{1}, \ldots, Q^{n-1}, R$ up to order two with coefficients being holomorphic functions of $(z, w)$. The determination of a system of generators $X_{1}, \ldots, X_{c}$ of $\mathfrak{S y m}\left(\mathscr{E}_{M}\right)$ is obtained by solving the infinite collection of these linear partial differential equations $\Phi_{l_{1}, \ldots, l_{n-1}}=0$ (cf. [Ol1986], [Su2001a,b], [GM2001a,b,c]). We shall apply this general procedure to provide different families of nonalgebraizable real analytic hypersurfaces in $\mathbb{C}^{n}$.

### 7.3. Hypersurfaces in $\mathbb{C}^{2}$ with control of their $\mathbf{C R}$ automorphism group.

 The goal of this paragraph is to construct some classes of strong tubes, namely tubes having the smallest possible CR automorphism group. We start with the case $n=2$ and study afterwards the case $n \geq 3$ in the next subparagraph. Let $M_{\chi}$ be the strong tube hypersurface in $\mathbb{C}^{2}$ defined by the equation$$
\begin{equation*}
M_{\chi}: \quad v=\varphi(y):=y^{2}+y^{6}+y^{9}+y^{10} \chi(y) . \tag{7.11}
\end{equation*}
$$

where $\chi$ is a real analytic function defined in a neighborhood of the origin in $\mathbb{R}$.

Lemma 7.1. The hypersurfaces $M_{\chi}$ are pairwise not biholomorphically equivalent strong tubes.

Proof. To check that $M_{\chi}$ is a strong tube, it suffices to show that every hypersurface of the form $v=y^{2}+y^{6}+\mathrm{O}\left(y^{9}\right)$ is a strong tube (the term $y^{9}$ will be used afterwards). Writing $v=(w-\bar{w}) / 2 i$ and $y=(z-\bar{z}) / 2 i$, considering $w$ as a function of $z$ and $\bar{w}, \bar{z}$ as constants, the differentiation of $w$ with respect to $z$ in (7.11) yields:

$$
\begin{equation*}
\partial_{z} w=2 y+6 y^{5}+\mathrm{O}\left(y^{8}\right) . \tag{7.12}
\end{equation*}
$$

The implicit function theorem yields:

$$
\begin{equation*}
y=(1 / 2) \partial_{z} w-\left(3 / 2^{5}\right)\left(\partial_{z} w\right)^{5}+\mathrm{O}\left(\left(\partial_{z} w\right)^{8}\right) . \tag{7.13}
\end{equation*}
$$

One further differentiation of equation (7.12) with respect to $z$ gives:

$$
\begin{equation*}
\partial_{z z}^{2} w=-i-(15 i) y^{4}+\mathrm{O}\left(y^{7}\right) . \tag{7.14}
\end{equation*}
$$

Replacing $y$ in this equation by its value obtained in (7.13), we obtain the following second order ordinary equation $\mathscr{E}_{M}$ satisfied by $\partial_{z} w$ and $\partial^{2}{ }_{z} z w$ :

$$
\begin{equation*}
\partial_{z z}^{2} w=-i-\left(15 i / 2^{4}\right)\left(\partial_{z} w\right)^{4}+\mathrm{O}\left(\left(\partial_{z} w\right)^{7}\right) . \tag{7.15}
\end{equation*}
$$

In the four dimensional jet space $J_{1,1}^{2}(\mathbb{C})$ equipped with the coordinates $\left(z, w, W^{1}, W^{2}\right)$ the equation of the corresponding complex hypersurface $\Delta_{M}$ is of course:

$$
\begin{equation*}
W^{2}=-i-\left(15 i / 2^{4}\right)\left(W^{1}\right)^{4}+\mathrm{O}\left(\left(W^{1}\right)^{7}\right) \tag{7.16}
\end{equation*}
$$

Then the Lie criterion states that a holomorphic vector field $X=Q \partial_{z}+$ $R \partial_{w}$ belongs to $\mathfrak{S y m}\left(\mathscr{E}_{M}\right)$ if and only if its second prolongation $X^{(2)}=$ $Q \partial_{z}+R \partial_{w}+R^{1} \partial_{W^{1}}+R^{2} \partial_{W^{2}}$ is tangent to $\Delta_{M}$, where the coefficients $R^{1}$ and $R^{2}$ are given by the formulas (7.8) specified for $n=2$, namely:
(7.17)

$$
\left\{\begin{array}{l}
R^{1}=\partial_{z} R+\left[\partial_{w} R-\partial_{z} Q\right] W^{1}-\partial_{w} Q\left(W^{1}\right)^{2} \\
R^{2}=\partial_{z z}^{2} R+\left[2 \partial_{z w}^{2} R-\partial_{z z}^{2} Q\right] W^{1}+\left[\partial_{w w}^{2} R-2 \partial_{z w}^{2} Q\right]\left(W^{1}\right)^{2}-\partial_{w w}^{2} Q\left(W^{1}\right)^{3}+ \\
\quad \quad+\left[\partial_{w} R-2 \partial_{z} Q\right] W^{2}-3 \partial_{w} Q W^{1} W^{2} .
\end{array}\right.
$$

The tangency condition yields the following equation which is satisfied on $\Delta_{M}$, i.e. after replacing $W^{2}$ by its value given by (7.16):

$$
\begin{equation*}
R^{2}+\left(15 i / 2^{2}\right) R^{1}\left(W^{1}\right)^{3}+\mathrm{O}\left(\left(W^{1}\right)^{6}\right)=0 \tag{7.18}
\end{equation*}
$$

By expanding equation (7.18) in powers of $W^{1}$ up to order five, we obtain the following system of six linear partial differential equations which must be satisfied by the derivatives of $Q$ and $R$ up to order two:

$$
\begin{cases}\left(e_{0}\right): & \partial_{z z}^{2} R-i\left(\partial_{w} R-2 \partial_{z} Q\right) \equiv 0  \tag{7.19}\\ \left(e_{1}\right): & 2 \partial_{z w}^{2} R-\partial_{z z}^{2} Q \equiv 0 \\ \left(e_{2}\right): & \partial_{w w}^{2} R-2 \partial_{z w}^{2} Q \equiv 0 \\ \left(e_{3}\right): & -\partial_{w w}^{2} Q+\frac{15 i}{2^{2}} \partial_{z} R \equiv 0 \\ \left(e_{4}\right): & -\frac{15 i}{2^{4}}\left(\partial_{w} R-2 \partial_{z} Q\right)+\frac{15 i}{2^{2}}\left(\partial_{w} R-\partial_{z} Q\right) \equiv 0 \\ \left(e_{5}\right): & -\frac{15 i}{2^{4}}\left(-3 \partial_{w} Q\right)-\frac{15 i}{2^{2}}\left(\partial_{w} Q\right) \equiv 0\end{cases}
$$

It follows from the equation $\left(e_{5}\right)$ that $\partial_{w} Q \equiv 0$ which implies $\partial_{w w}^{2} Q \equiv 0$. Then by equation $\left(e_{3}\right)$ we obtain $\partial_{z} R \equiv 0$, implying $\partial_{z z}^{2} R \equiv 0$. From equation $\left(e_{0}\right)$ we get $\partial_{w} R \equiv 2 \partial_{z} Q$ and, from equation $\left(e_{4}\right)$, we get $\partial_{w} R \equiv \partial_{z} Q$. Consequently $\partial_{z} R \equiv \partial_{w} R \equiv \partial_{z} Q \equiv \partial_{w} Q \equiv 0$. Since the two vector fields $\partial_{z}$ and $\partial_{w}$ evidently belong to $\mathfrak{S y m}\left(\mathscr{E}_{M}\right)$, it follows that $\operatorname{dim}_{\mathbb{C}} \mathfrak{S y m}\left(\mathscr{E}_{M}\right)=$
 generated by $\partial_{w}+\partial_{\bar{w}}$ and $\partial_{z}+\partial_{\bar{z}}$.

Next, let $\chi(y)$ and $\chi^{\prime}\left(y^{\prime}\right)$ be two real analytic functions, and assume that $M_{\chi}$ and $M_{\chi^{\prime}}^{\prime}$ are biholomorphically equivalent. Let $t^{\prime}=h(t)$ be such an equivalence. Reasoning as in $\S 4$ and taking into account that both are strong tubes, we see that $h_{*}\left(\partial_{z}\right)$ and $h_{*}\left(\partial_{w}\right)$ must be linear combinations of $\partial_{z^{\prime}}$ and $\partial_{w^{\prime}}$ with real coefficients. It follows that $h$ must be linear, of the form $z^{\prime}=$ $a z+b w, w^{\prime}=c z+d w$, where $a, b, c$ and $d$ are real. Since $T_{0} M_{\chi}=\{v=0\}$
and $T_{0} M_{\chi^{\prime}}^{\prime}=\left\{v^{\prime}=0\right\}$, we have $c=0$. Next, in the equation

$$
\left\{\begin{array}{l}
d\left(y^{2}+y^{6}+y^{9}+y^{10} \chi(y)\right) \equiv\left[a y+b\left(y^{2}+y^{6}+y^{9}+y^{10} \chi(y)\right)\right]^{2}+  \tag{7.20}\\
+\left[a y+b\left(y^{2}+y^{6}+y^{9}+y^{10} \chi(y)\right)\right]^{6}+\left[a y+b\left(y^{2}+y^{6}+y^{9}+y^{10} \chi(y)\right)\right]^{9}+ \\
+\left[a y+b\left(y^{2}+y^{6}+y^{9}+y^{10} \chi(y)\right)\right]^{10} \chi^{\prime}\left(a y+b\left(y^{2}+y^{6}+y^{9}+y^{10} \chi(y)\right)\right)
\end{array}\right.
$$

we firstly see that $b=0$, and then from

$$
\begin{equation*}
d\left(y^{2}+y^{6}+y^{9}+y^{10} \chi(y)\right) \equiv a^{2} y^{2}+a^{6} y^{6}+a^{9} y^{9}+a^{10} y^{10} \chi^{\prime}(a y) \tag{7.21}
\end{equation*}
$$

we see that $a=d=1$. In other words, $h=\mathrm{Id}$, whence $y^{\prime}=y$ and $\chi^{\prime}\left(y^{\prime}\right) \equiv \chi(y)$. This proves Lemma 7.1.

In the remainder of $\S 7$, we shall exhibit other classes of hypersurfaces with a control on their CR automorphism group. Since the computations are generally similar, we shall summarize them.

### 7.4. Some classes of strong tube hypersurfaces in $\mathbb{C}^{n}$. Generalizing

 Lemma 7.1, we may state:Lemma 7.2. The real analytic hypersurfaces $M_{\chi_{1}, \ldots, \chi_{n-1}} \subset \mathbb{C}^{n}$ of equation

$$
\begin{equation*}
v=\sum_{k=1}^{n-1}\left[\varepsilon_{k} y_{k}^{2}+y_{k}^{6}+y_{k}^{9} y_{1} \cdots y_{k-1}+y_{k}^{n+8} \chi_{k}\left(y_{1}, \ldots, y_{n-1}\right)\right] \tag{7.22}
\end{equation*}
$$

where $\varepsilon_{k}= \pm 1$, are pairwise not biholomorphically equivalent strong tubes.
Proof. The associated system of partial differential equations is of the form
(7.23)
$\left\{\begin{array}{l}\partial_{z_{k} z_{k}}^{2} w=-i \varepsilon_{k}-\left(15 i / 2^{4}\right)\left(\partial_{z_{k}} w\right)^{4}+\mathrm{O}\left(\left(\partial_{z_{1}} w\right)^{7}\right)+\cdots+\mathrm{O}\left(\left(\partial_{z_{n-1}} w\right)^{7}\right), \\ \partial_{z_{k_{1}} z_{k_{2}}}^{2} w=0, \quad \text { for } k_{1} \neq k_{2} .\end{array}\right.$
Using the formulas (7.8) and inspecting the coefficients of the monomials in the $W_{l}^{1}$ up to order five in the $(n-1)$ equations extracted from the set of Lie equations

$$
\left\{\begin{array}{l}
R_{k, k}^{2}+\left(15 i / 2^{2}\right)\left(W_{k}^{2}\right) R_{k}^{1}+\mathrm{O}\left(\left(W_{1}^{1}\right)^{6}\right)+\cdots+\mathrm{O}\left(\left(W_{n-1}^{1}\right)^{6}\right)=0  \tag{7.24}\\
R_{k_{1}, k_{2}}^{2}=0, \\
\text { for } k_{1} \neq k_{2}
\end{array}\right.
$$

we get $\partial_{z_{l}} R \equiv \partial_{w} R \equiv \partial_{z_{l}} Q^{k} \equiv \partial_{w} Q^{k} \equiv 0$ for $l, k=1, \ldots, n-1$. Thus, $M_{\chi_{1}, \ldots, \chi_{n-1}}$ is a strong tube.

Next, reasoning as in the end of the proof of Lemma 7.1, we see first that an equivalence between $M_{\chi_{1}, \ldots, \chi_{n-1}}$ and $M_{\chi_{1}^{\prime}, \ldots, \chi_{n-1}^{\prime}}^{\prime}$ must be of the form $z_{k}^{\prime}=\sum_{l=1}^{n-1} \lambda_{k}^{l} z_{l}, w^{\prime}=\mu w$, where $\lambda_{k}^{l}, 1 \leq l, k \leq n-1$ and $\mu$ are real. Inspecting the terms of degree $9,10, \ldots, n+7$, we get $\lambda_{k}^{l}=0$ if $k \neq l$, i.e.
$y_{k}^{\prime}=\lambda_{k}^{k} y_{k}$ and $w^{\prime}=\mu w$. Finally, $\lambda_{k}^{k}=1$ and $\mu=1$, which completes the proof.
7.5. Families of strongly rigid hypersurfaces. Alongside the same recipe, we can study some classes of hypersurfaces of the form $v=\varphi(z \bar{z})$.

Lemma 7.3. The Lie algebra $\mathfrak{H o l}\left(M_{\chi}\right)$ of the rigid real analytic hypersurfaces $M_{\chi}$ in $\mathbb{C}^{2}$ of equation $v=\varphi(z \bar{z})=z \bar{z}+z^{5} \bar{z}^{5}+z^{7} \bar{z}^{7}+z^{8} \bar{z}^{8} \chi(z \bar{z})$ is two-dimensional and generated by $\partial_{w}$ and $i z \partial_{z}$. Furthermore, $M_{\chi}$ is biholomorphically equivalent to $M_{\chi^{\prime}}^{\prime}$ if and only if $\chi=\chi^{\prime}$.
Proof. The associated differential equation is of the form

$$
\begin{equation*}
\partial_{z z}^{2} w=\left[5 z^{3} / 4\right]\left(\partial_{z} w\right)^{5}-\left[21 z^{5} / 32\right]\left(\partial_{z} w\right)^{7}+\mathrm{O}\left(\left(\partial_{z} w\right)^{9}\right) \tag{7.25}
\end{equation*}
$$

Extracting from the associated Lie equations (7.10) the coefficients of the monomials $\left(W^{1}\right)^{4},\left(W^{1}\right)^{5},\left(W^{1}\right)^{6}$ and $\left(W^{1}\right)^{7}$, we obtain four equations which are solved by $z \partial_{z} Q-Q \equiv 0, \partial_{w} Q \equiv 0, \partial_{z} R \equiv 0$ and $\partial_{w} R \equiv 0$. Next, if $M_{\chi}$ and $M_{\chi^{\prime}}^{\prime}$ are biholomorphically equivalent, reasoning as in $\S 4$, taking into account that $h_{*}\left(i z \partial_{z}\right)$ and $h_{*}\left(\partial_{w}\right)$ are linear combinations of $i z^{\prime} \partial_{z^{\prime}}$ and $\partial_{w^{\prime}}$ with real coefficients, we see first that $z^{\prime}=\lambda z e^{\gamma w / 2 i}$ and $w^{\prime}=\mu w$ for some three real constants $\gamma, \lambda \neq 0$ and $\mu \neq 0$. Replacing $z^{\prime}$ and $w^{\prime}$ in the equation of $M_{\chi^{\prime}}^{\prime}$, we get $\gamma=0, \mu=1$ and $\lambda \pm 1$. In other words, $z^{\prime}= \pm z$ and $w^{\prime}=w$, which entails $\chi^{\prime}\left(z^{\prime} \bar{z}^{\prime}\right) \equiv \chi(z \bar{z})$, as claimed.

Perturbing this family we may exhibit other strongly rigid hypersurfaces :
Lemma 7.4. The Lie algebra $\mathfrak{H o l}\left(M_{\chi}\right)$ of the real analytic hypersurfaces $M_{\chi}$ in $\mathbb{C}^{2}$ of equation $v=\varphi(z, \bar{z})=z \bar{z}+z^{5} \bar{z}^{5}+z^{7} \bar{z}^{7}+z^{8} \bar{z}^{8}(z+\bar{z})+$ $z^{10} \bar{z}^{10} \chi(z, \bar{z})$ is one-dimensional and generated by $\partial_{w}$. Furthermore, $M_{\chi}$ is biholomorphically equivalent to $M_{\chi^{\prime}}^{\prime}$ if and only if $\chi=\chi^{\prime}$.
Proof. We already know that $z \partial_{z} Q-Q \equiv \partial_{w} Q \equiv \partial_{z} R \equiv \partial_{w} R \equiv 0$. Extracting from the associated Lie equations (7.10) the coefficient of the monomials $\left(W^{1}\right)^{8}$, we also get $Q \equiv 0$. Next, let $M_{\chi}$ and $M_{\chi^{\prime}}^{\prime}$ be biholomorphically equivalent. Let $t^{\prime}=h(t)$ be such an equivalence. Us$\operatorname{ing} h_{*}\left(\partial_{w}\right)=\mu \partial_{w^{\prime}}$, where $\mu \in \mathbb{R}$ is nonzero, we get $z^{\prime}=f(z)$ and $w^{\prime}=\mu w+g(z)$. Next from the equation
(7.26)
$\left\{\begin{array}{l}\mu\left(z \bar{z}+z^{5} \bar{z}^{5}+z^{7} \bar{z}^{7}+z^{8} \bar{z}^{8}(z+\bar{z})+\mathrm{O}\left(z^{9} \bar{z}^{9}\right)\right)+[g(z)-\bar{g}(\bar{z})] / 2 i \equiv \\ \equiv f(z) \bar{f}(\bar{z})+f(z)^{5} \bar{f}(\bar{z})^{5}+f(z)^{7} \bar{f}(\bar{z})^{7}+f(z)^{8} \bar{f}(\bar{z})^{8}(f(z)+\bar{f}(\bar{z}))+\mathrm{O}\left(z^{9} \bar{z}^{9}\right),\end{array}\right.$
we get firstly $f(z)=\sqrt{|\mu|} e^{i \theta} z$ by differentiating with respect to $\bar{z}$ at $\bar{z}=0$ and secondly $\mu=e^{i \theta}=1$, which completes the proof.

We provide a second family of strongly rigid hypersurfaces in $\mathbb{C}^{2}$ with a one-dimensional Lie algebra :

Lemma 7.5. The Lie algebra $\mathfrak{H o l}\left(M_{\chi}\right)$ of the real analytic hypersurfaces $M_{\chi} \subset \mathbb{C}^{2}$ of equation $v=z \bar{z}+z^{5} \bar{z}^{5}(z+\bar{z})+z^{10} \bar{z}^{10} \chi(z, \bar{z})$ is onedimensional and generated by $\partial_{w}$. Furthermore $M_{\chi}$ is biholomorphically equivalent to $M_{\chi^{\prime}}^{\prime}$ if and only if $\chi=\chi^{\prime}$.

Proof. The derivatives $\partial_{z} w$ and $\partial_{z^{2}}^{2} w$ of $w$ with respect to $z$ are given by :

$$
\left\{\begin{align*}
\partial_{z} w & \left.=2 i \bar{z}+12 i z^{5} \bar{z}^{5}+10 i z^{4} \bar{z}^{6}+\mathrm{O}\left(\bar{z}^{10}\right)\right)  \tag{7.27}\\
\partial_{z z}^{2} w & =60 i z^{4} \bar{z}^{5}+40 i z^{3} \bar{z}^{6}+\mathrm{O}\left(\bar{z}^{10}\right)
\end{align*}\right.
$$

Replacing $\bar{z}$ in the second equation by its expression given by the first equation we obtain the following second order differential equation, interpreted in the jet space:
(7.28)
$W_{2}=\left[15 z^{4} / 8\right]\left(W^{1}\right)^{5}-\left[5 i z^{3} / 8\right]\left(W^{1}\right)^{6}-\left[225 z^{9} / 64\right]\left(W^{1}\right)^{9}+\mathrm{O}\left(\left(W^{1}\right)^{10}\right)$.
Solving the partial differential equations involving $Q, \partial_{z} Q, \partial_{w} Q, \partial_{z} R$ and $\partial_{w} R$ given in the coefficients of $\left(W^{1}\right)^{4},\left(W^{1}\right)^{5},\left(W^{1}\right)^{6},\left(W^{1}\right)^{7}$ and $\left(W^{1}\right)^{9}$ we obtain $Q \equiv \partial_{z} Q \equiv \partial_{w} Q \equiv \partial_{z} R \equiv \partial_{w} R \equiv 0$ which is the desired information. Finally, proceeding exactly as in the end of the proof of Lemma 7.4, we see that the $M_{\chi}$ are pairwise biholomorphically not equivalent.

The dimension of $\mathfrak{H o l}(M)$ for the five examples of Corollary 1.3, for the seven examples of Theorem 1.4, for the seven examples of Corollary 1.7 and for the hypersurface $v=e^{z \bar{z}}-1$ at a point $p$ with $z_{p} \neq 0$ was computed with the package diffalg of Maple Release 6. Since at a point $p$ with $z_{p} \neq 0$ the hypersurface $v=e^{z \bar{z}}-1$ is biholomorphically equivalent to the hypersurface $M_{a}$ of equation $v=\varphi^{a}(y):=e^{a\left(e^{y}-1\right)}-1$ with $a=\left|z_{p}\right|^{2}$, this defines a strong tube. Applying Theorem 1.1 and Lemma 3.3, we see that $M_{a}$ is not locally algebraizable at the origin, because $\varphi_{y y}^{a}(y)=a e^{y} e^{a\left(e^{y}-1\right)}+a^{2} e^{2 y} e^{a\left(e^{y}-1\right)}$ and $\varphi_{y}^{a}(y)=a e^{y} e^{a\left(e^{y}-1\right)}$ are algebraically independent. Finally all the examples of Corollary 1.3, Theorem 1.4 and Corollary 1.6 are not locally algebraic since they satisfy the required transcendence conditions.

## §8. Analyticity versus algebraicity

Intuitively there seems to be much more analytic mappings, manifolds and varieties than algebraic ones. Our goal is to elaborate a precise statement about this. By complexification, every local real analytic object yields a local complex analytic object, so we shall only work in the holomorphic category. Let $\Delta_{n}$ be the complex polydisc of radius one in $\mathbb{C}^{n}$ and $\bar{\Delta}_{n}$ its closure. Let $k \in \mathbb{N}$. We consider the space $\mathscr{O}^{k}\left(\bar{\Delta}_{n}\right):=\mathscr{O}\left(\Delta_{n}\right) \cap \mathscr{C}^{k}\left(\bar{\Delta}_{n}\right)$ of holomorphic functions extending up to the boundary as a function of class $\mathscr{C}^{k}$ . This is a Banach space for the $\mathscr{C}^{k}$ norm $\|\varphi\|_{k}:=\sum_{l=0}^{k} \sup _{z \in \bar{\Delta}_{n}}\left|\varphi_{z^{l}}(z)\right|$.

The last statements of Corollaries 1.2 and 1.6 are a direct consequence of the following lemma.
Lemma 8.1. The set of holomorphic functions $\varphi \in \mathscr{O}^{k}\left(\bar{\Delta}_{n}\right)$ such that there exists a polynomial $P$ such that

$$
\begin{equation*}
P\left(z, j^{k} \varphi(z)\right) \equiv 0 \tag{8.1}
\end{equation*}
$$

is of first category, namely it can be represented as the countable union of nowhere dense closed subsets. Conversely, the set of functions $\varphi \in \mathscr{O}^{k}\left(\bar{\Delta}_{n}\right)$ such that there is no algebraic dependence relation like (8.1) is generic in the sense of Baire, namely it can be represented as the countable intersection of everywhere dense open subsets.

Proof. Let $N \in \mathbb{N}$. Consider the set $F_{N}$ of functions $\varphi$ such that there exists a polynomial of degree $N$ satisfying (8.1). It suffices to show that $F_{N}$ is closed and that its complement is everywhere dense. Suppose that a sequence $\left(\varphi^{(m)}\right)_{m \in \mathbb{N}}$ converges to $\varphi \in \mathscr{O}^{k}\left(\bar{\Delta}_{n}\right)$. Let the zero-set of a degree $N$ polynomial $P_{N}^{(m)}\left(z, J_{k}\right)$ contain the graph of the $k$-jet of $\varphi^{(m)}$. The coefficients of $P_{N}^{(m)}$ belong to a certain complex projective space $P_{A}(\mathbb{C})$, where the integer $A=A(n, k)$ is independent of $m$. By compactness of $P_{A}(\mathbb{C})$, passing to a subsequence if necessary, the $P_{N}^{(m)}$ converge to a nonzero polynomial $P_{N}$. By continuity, $P_{N}\left(z, j^{k} \varphi(z)\right)=0$ for all $z \in \mathscr{O}\left(\bar{\Delta}_{n}\right)$. We claim that the complement of the union of the $F_{N}$ is dense in $\mathscr{O}^{k}\left(\bar{\Delta}_{n}\right)$. Indeed, let $\varphi(z)$ be such that there exists a degree $N$ polynomial $P$ satisfying (8.1). Fix $z_{0} \in \Delta_{n}$ having rational real and imaginary parts. Then the complex numbers $z_{0}, \partial_{z}^{\alpha} \varphi\left(z_{0}\right),|\alpha| \leq k$, are algebraically dependent. By a Cantorian argument, there exists complex numbers $\chi_{0}^{\alpha}$ arbitrarily close to $\partial_{z}^{\alpha} \varphi\left(z_{0}\right)$ such that $z_{0}, \chi_{0}^{\alpha}$ are algebraically independent. Let $\chi(z)$ be a polynomial with $\partial_{z}^{\alpha}\left(z_{0}\right)=\chi_{0}^{\alpha}-\partial_{t}^{\alpha} \varphi\left(z_{0}\right)$. We can choose $\chi$ to be arbitrarily close to zero in the $\mathscr{C}^{k}\left(\bar{\Delta}_{n}\right)$ norm. Then the function $\varphi(z)+\chi(z)$ is not Nash algebraic.

## References

[Ar1974] ARNold, V.I.: Équations différentielles ordinaires. Champs de vecteurs, groupes à un paramètre, difféomorphismes, flots, systèmes linéaires, stabilité des positions d'équilibre, théorie des oscillations, équations différentielles sur les variétés. Traduit du russe par Djilali Embarek, Éditions Mir, Moscou, 1974. 267pp.
[BER1999] BAOUENDI, M.S.; EbENFELT, P.; RothSchild, L.P.: Rational dependence of smooth and analytic CR mappings on their jets. Math. Ann. 315 (1999), 205-249.
[BER2000] BaOUENDI, M.S.; Ebenfelt, P.; RothSchild, L.P.: Local geometric properties of real submanifolds in complex space, Bull. Amer. Math. Soc. 37 (2000), no.3, 309-336.
[Be1996] Bellaïche, A.: SubRiemannian Geometry, Progress in Mathematics 144, Birkhäuser Verlag, Basel/Switzerland, 1996, 1-78.
[Bs1991] Beloshapka, V.K.: On holomorphic transformations of a quadric, Mat. Sb. 182 (1991), no.2, 203-219; English transl. in Math. USSR Sb. 72 (1992), no.1, 189-205.
[Ca1932] CARTAN, É.: Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes, I, Ann. Math. Pura Appl. 11 (1932), 17-90.
[CM1974] CHERN, S.S.; MOSER, J.K.: Real hypersurfaces in complex manifolds, Acta Math. 133 (1974), no.2, 219-271.
[CPS2000] Coupet, B.; Pinchuk, S.; Sukhov, A.: On partial analyticity of CR mappings, Math. Z. 235 (2000), 541-557.
[DP2003] Diederich, D.; Pinchuk, S.: Regularity of continuous CR-maps in arbitrary dimension, to appear in Michigan Math. J. (2003).
[Eb1996] EbENFELT, P.: On the unique continuation problem for $C R$ mappings into nonminimal hypersurfaces, J. Geom. Anal. 6 (1996), no.3, 385-405.
[GM2001a] Gaussier, H.; MERKER, J.: Estimates on the dimension of the symmetry group of a system of $k$-order partial differential equations, Université de Provence, Prépublication LATP, 12, 2001, 27 pp.
[GM2001b] Gaussier, H.; MERKER, J.: A new example of uniformly Levi degenerate hypersurface in $\mathbb{C}^{3}$, Ark. Mat. (to appear).
[GM2001c] Gaussier, H.; MERKER, J.: Symmetries of differential equations and infinitesimal CR automorphisms of real analytic CR submanifolds of $\mathbb{C}^{n}$, manuscript, 34 pp.
[HJ1998] HUANG, X.; JI, S.: Global holomorphic extension of a local map and a Riemann mapping theorem for algebraic domains, Math. Res. Lett. 5 (1998), no.1-2, 247-260.
[HJY2001] HUANG, X.; JI, S.; YAU, S.T.: An example of a real analytic strongly pseudoconvex hypersurface which is not holomorphically equivalent to any algebraic hypersurface, Ark. Mat. 39 (2001), no.1, 75-93.
[Lie1880] LiE, S.: Theorie der Transformationsgruppen, Math. Ann. 16 (1880), 441528.
[Me1998] MERKER, J.: Vector field construction of Segre sets, Preprint 1998, augmented in 2000. Downloadable at arXiv.org/abs/math.CV/9901010.
[Me2001] MERKER, J.: On the partial algebraicity of holomorphic mappings between two real algebraic sets, Bull. Soc. Math. France 129 (2001), no.3, 547-591.
[MW1983] Moser, J.K.; Webster S.M.: Normal forms for real surfaces in $\mathbb{C}^{2}$ near complex tangents and hyperbolic surface transformations, Acta Math. 150 (1983), no.3-4, 255-296.
[Ol1986] OLVER, P.J.: Applications of Lie groups to differential equations. Springer Verlag, Heidelberg, 1986.
[Pi1975] Pinchuk, S.: On the analytic continuation of holomorphic mappings (Russian), Mat. Sb. (N.S.) 98(140) (1975) no.3(11), 375-392, 416-435, 495-496.
[Pi1978] Pinchuk, S.: Holomorphic mappings of real-analytic hypersurfaces (Russian), Mat. Sb. (N.S.) 105(147) (1978), no. 4, 574-593, 640.
[Se1931] SEGRE, B.: Intorno al problema di Poincaré della rappresentazione pseudoconforme, Rend. Acc. Lincei, VI, Ser. 13 (1931), 676-683.
[Sha2000] SHAFIKOV, R.: Analytic continuation of germs of holomorphic mappings between real hypesurfaces in $\mathbb{C}^{n}$, Michigan Math. J. 47 (2000), no.1, 133-149.
[Sha2002] Shafikov, R.: Analytic continuation of holomorphic correspondences and equivalence of domains in $\mathbb{C}^{n}$, to appear in Inventiones Math.
[SS1996] Sharipov, R.; Sukhov, A.: On CR mappings between algebraic CauchyRiemann manifolds and separate algebraicity for holomorphic functions, Trans. Amer. Math. Soc. 348 (1996), no.2, 767-780.
[St1991] Stanton, N.: Infinitesimal CR automorphisms of rigid hypersurfaces in $\mathbb{C}^{2}$, J. Geom. Anal. 1 (1991), no.3, 231-267.
[Sto2000] Stormark, O.: Lie's structural approach to PDE systems, Encyclopædia of mathematics and its applications, vol. 80, Cambridge University Press, Cambridge, 2000, xv+572 pp.s
[Su2001a] Sukhov, A.: Segre varieties and Lie symmetries, Math. Z. 238 (2001), no.3, 483-492.
[Su2001b] SuKhov, A.: On transformations of analytic CR structures, Pub. Irma, Lille 2001, Vol. 56, no. II.
[Ve1999] VERMA, K: Boundary regularity of correspondences in $\mathbb{C}^{2}$, Math. Z. 231 (1999), no.2, 253-299.
[We1977] WEBSTER, S.M.: On the mapping problem for algebraic real hypersurfaces, Invent. Math. 43 (1977), no.1, 53-68.
[We1978] WEbSTER, S.M.: On the reflection principle in several complex variables, Proc. Amer. Math. Soc. 71 (1978), no.1, 26-28.
[Za1995] Zaitsev, D.: On the automorphism groups of algebraic bounded domains, Math. Ann. 302 (1995), no.1, 105-129.

# Symmetries of partial differential equations 

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#### Abstract

We establish a link between the study of completely integrable systems of partial differential equations and the study of generic submanifolds in $\mathbb{C}^{n}$. Using the recent developments of Cauchy-Riemann geometry we provide the set of symmetries of such a system with a Lie group structure. Finally we determine the precise upper bound of the dimension of this Lie group for some specific systems of partial differential equations.


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## 1. Introduction

To study the geometry of a real analytic Levi nondegenerate hypersurface $M$ in $\mathbb{C}^{2}$, one of the principal ideas of H. Poincaré, of B. Segre and of É. Cartan in the fundamental memoirs [20], [Se1931], [Se1932], [3] was to associate to $M$ a system $\left(\mathscr{E}_{M}\right)$ of (partial) differential equations, in order to solve the so-called equivalence problem. Establishing a natural correspondence between the local holomorphic automorphisms of $M$ and the Lie symmetries of $\left(\mathscr{E}_{M}\right)$ they could use the classification results on differential equations achieved by S. Lie in [5] and pursued by A. Tresse in [Tr1896].

Starting with such a correspondence, we shall establish a general link between the study of a real analytic generic submanifold of codimension $m$ in $\mathbb{C}^{n+m}$ and the study of completely integrable systems of analytic partial differential equations. We shall observe that the recent theories in CauchyRiemann (CR) geometry may be transposed to the setting of partial differential equations, providing some new information on their Lie symmetries.

Indeed consider for $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ a $\mathbb{K}$-analytic system $(\mathscr{E})$ of the following general form:

$$
\begin{equation*}
u_{x^{\alpha}}^{j}(x)=F_{\alpha}^{j}\left(x, u(x),\left(u_{x^{\beta(q)}}^{j(q)}(x)\right)_{1 \leq q \leq p}\right) . \tag{E}
\end{equation*}
$$

Here $x=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{K}^{n}, u=\left(u^{1}, \ldots, u^{m}\right) \in \mathbb{K}^{m}$, the integers $j(1), \ldots, j(p)$ satisfy $1 \leq j(q) \leq m$ for $q=1, \ldots, p$, and $\alpha$ and the multiindices $\beta(1), \ldots, \beta(p) \in \mathbb{N}^{n}$ satisfy $|\alpha|,|\beta(q)| \geq 1$. We also require $(j, \alpha) \neq(j(1), \beta(1)), \ldots,(j(p), \beta(p))$. For $j=1, \ldots, m$ and $\alpha \in \mathbb{N}^{n}$, we denote by $u_{x^{\alpha}}^{j}$ the partial derivative $\partial^{|\alpha|} u^{j} / \partial x^{\alpha}$. We assume that the system $(\mathscr{E})$ is completely integrable, namely that the Pfaffian system naturally associated in the jet space is involutive in the sense of Frobenius. We note that in that case $(\mathscr{E})$ is locally solvable, meaning that through every point $\left(x^{*}, u^{*}, u_{\beta}^{*}, u_{\alpha}^{*}\right)$ in the jet space, satisfying $u_{\alpha}^{*}=F_{\alpha}\left(x^{*}, u^{*}, u_{\beta}^{*}\right)$ (written in a condensed form), there exists a local $\mathbb{K}$-analytic solution $u=u(x)$ of ( $\mathscr{E}$ ) satisfying $u\left(x^{*}\right)=u^{*}$ and $u_{x^{\beta}}\left(x^{*}\right)=u_{\beta}^{*}$. Consequently the Lie theory ([Ol1986]) may be applied to such systems. We shall associate with ( $\mathscr{E})$ the submanifold of solutions $\mathscr{M}$ in $\mathbb{K}^{n+2 m+p}$ given by $\mathbb{K}$-analytic equations of the form

$$
\begin{equation*}
u^{j}=\Omega_{j}(x, \nu, \chi), \quad j=1, \ldots, m \tag{7.28}
\end{equation*}
$$

where $\nu \in \mathbb{K}^{m}$ and where $\chi \in \mathbb{K}^{p}$. Moreover the integer $m+p$ is the number of initial conditions for the general solution $u(x):=\Omega(x, \nu, \chi)$ of $(\mathscr{E})$, whose existence and uniqueness follow from complete integrability. Precisely, the parameters $\nu, \chi$ correspond to the data $u(0),\left(u_{x^{\beta(q)}}^{j(q)}(0)\right)_{1 \leq q \leq p}$. In the special case where the system $(\mathscr{E})$ is constructed from a generic submanifold $M$ as in [Se1931], [24] (see also Subsection 2.2 below), the corresponding submanifold of solutions is exactly the extrinsic complexification of $M$.

A pointwise $\mathbb{K}$-analytic transformation $\left(x^{\prime}, u^{\prime}\right)=\Phi(x, u)$ defined in a neighbourhood of the origin and sufficiently close to the iedntity mapping is called a Lie symmetry of $(\mathscr{E})$ if it transforms the graph of every solution to the graph of an other local solution. A vector field $X=$ $\sum_{l=1}^{n} Q^{l}(x, u) \partial / \partial x_{l}+\sum_{j=1}^{m} R^{j}(x, u) \partial / \partial u^{j}$ is called an infinitesimal symmetry of $(\mathscr{E})$ if for every $s$ close to zero in $\mathbb{K}$ the local diffeomorphism $(x, u) \mapsto \exp (s X)(x, u)$ associated to the flow of $X$ is a Lie symmetry of $\mathscr{E}$. According to [Ol1986] (Chapter 2) the infinitesimal symmetries of $(\mathscr{E})$ form a Lie algebra of vector fields defined in a neighbourhood of the origin in $\mathbb{K}^{n} \times \mathbb{K}^{m}$, denoted by $\mathfrak{S v m}(\mathscr{E})$. Inspired by recent developments in CR geometry we shall provide in Section 2 nondegeneracy conditions on $\mathscr{M}$ insuring firstly that $\mathfrak{S y m}(\mathscr{E})$ may be identified with the Lie algebra $\mathfrak{S y m}(\mathscr{M})$ of vector fields of the form
(7.28)
$\sum_{l=1}^{n} Q^{l}(x, u) \frac{\partial}{\partial x_{l}}+\sum_{j=1}^{m} R^{j}(x, u) \frac{\partial}{\partial u^{j}}+\sum_{j=1}^{m} \Pi^{j}(\nu, \chi) \frac{\partial}{\partial \nu^{j}}+\sum_{q=1}^{p} \Lambda^{q}(\nu, \chi) \frac{\partial}{\partial \chi_{q}}$,
which are tangent to $\mathscr{M}$, and secondly that $\mathfrak{S y m}(\mathscr{M}) \cong \mathfrak{S y m}(\mathscr{E})$ is finite dimensional. The strength of this identification is to provide some (non optimal) bound on the dimension of $\mathfrak{S y m}(\mathscr{E})$ for arbitrary systems of partial differential equations with an arbitrary number of variables, see Theorem 6.4.

In the second part of the paper (Sections 3, 4 and 5), using the classical Lie theory (cf. [5], [Ol1986], [Ol1995] and [BK1989]), we provide an optimal upper bound on the dimension of $\mathfrak{S y m}(\mathscr{E})$ for a completely integrable $\mathbb{K}$ analytic system $(\mathscr{E})$ of the following form:
( $\mathscr{E}) u_{x^{\alpha}}^{j}=F_{\alpha}^{j}\left(x, u(x),\left(u_{x^{\beta}}(x)\right)_{1 \leq|\beta| \leq \kappa-1}\right), \quad \alpha \in \mathbb{N}^{n},|\alpha|=\kappa, j=1, \ldots, m$.
This system is a special case of the system studied in Section 2. For instance the homogeneous system $\left(\mathscr{E}_{0}\right): u_{x_{k_{1}} \cdots x_{k_{\kappa}}}^{j}(x)=0$ is completely integrable. The solutions of $\left(\mathscr{E}_{0}\right)$ are the polynomials of the form $u^{j}(x)=\sum_{\beta \in \mathbb{N}^{n},|\beta| \leq \kappa-1} \lambda_{\beta}^{j} x^{\beta}, j=1, \ldots, m$, where $\lambda_{\beta}^{j} \in \mathbb{K}$ and a Lie symmetry of $\left(\mathscr{E}_{0}\right)$ is a transformation stabilizing the graphs of polynomials of degree $\leq \kappa-1$. We prove the following Theorem:

Theorem 6.4. Let ( $\mathscr{E}$ ) be the $\mathbb{K}$-analytic system of partial differential equations of order $\kappa \geq 2$, with $n$ independent variables and $m$ dependent variables, defined just above. Assume that $(\mathscr{E})$ is completely integrable. Then the Lie algebra $\mathfrak{S y m}(\mathscr{E})$ of its infinitesimal symmetries satisfies the following estimates:

$$
\left\{\begin{array}{rll}
\operatorname{dim}_{\mathbb{K}}(\mathfrak{S y m}(\mathscr{E})) & \leq(n+m+2)(n+m), & \text { if } \kappa=2,  \tag{7.28}\\
\operatorname{dim}_{\mathbb{K}}(\mathfrak{S y m}(\mathscr{E})) & \leq n^{2}+2 n+m^{2}+m C_{n+\kappa-1}^{\kappa-1}, & \text { if } \kappa \geq 3,
\end{array}\right.
$$

where we denote $C_{n+\kappa-1}^{\kappa-1}:=\frac{(n+\kappa-1)!}{n!(\kappa-1)!}$. Moreover the inequalities (7.28) become equalities for the homogeneous system $\left(\mathscr{E}_{0}\right)$.
We remark that there is no combinatorial formula interpolating these two estimates. Theorem 6.4 is a generalization of the following results. For $n=m=1$, S. Lie proved that the dimension of the Lie algebra $\mathfrak{S y m}(\mathscr{E})$ is less than or equal to 8 if $\kappa=2$ and is less than or equal to $\kappa+4$ if $\kappa \geq 3$, these bounds being reached for the homogeneous system (cf. [5]). For $n=1, m \geq 1$ and $\kappa=2$, F. González-Gascón and A. GonzálezLópez proved in [11] that the dimension of $\mathfrak{S y m}(\mathscr{E})$ is less than or equal to $(m+3)(m+1)$. For $n=1, m \geq 1$ and $\kappa=2$, using the equivalence method due to É. Cartan, M. Fels [Fe1995] proved that the dimension of $\mathfrak{S y m}(\mathscr{E})$ is less than or equal to $m^{2}+4 m+3$, with equality if and only if the system $(\mathscr{E})$ is equivalent to the system $u_{x^{2}}^{j}=0, j=1, \ldots, m$. He also generalized this result to the case $n=1, m \geq 1, \kappa=3$. For $n \geq 1, m \geq 1$ and $\kappa=2$, A. Sukhov proved in [24] that the dimension of $\mathfrak{S y m}(\mathscr{E})$ is less than or equal to $(n+m+2)(n+m)$ (the first inequality in Theorem 6.4), with equality for the homogeneous system $u_{x_{k_{1}} x_{k_{2}}}^{j}=0$.

Consequently, for the case $\kappa=2$, we will only give the general form of the Lie symmetries of the homogeneous system ( $\mathscr{E}_{0}$ ) (see Subsection 5.2). We will prove Theorem 6.4 for the case $\kappa \geq 3$. The formulas obtained in Sections 3, 4 and 5 were checked with the help of MAPLE release 6.
Acknowledgment. This article was written while the first author had a six months delegation position at the CNRS. He thanks this institution for providing him this research opportunity. The authors are indebted to Gérard Henry, the computer ingénieur (LATP, UMR 6632 CNRS), for his technical support.

## 2. Submanifold of Solutions

2.1. Preliminary. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Let $n \geq 1$ and let $x=\left(x_{1}, \ldots, x_{n}\right) \in$ $\mathbb{N}$. We denote by $\mathbb{K}\{x\}$ the local ring of $\mathbb{K}$-analytic functions $\varphi=\varphi(x)$ defined in some neighbourhood of the origin in $\mathbb{K}^{n}$. If $\varphi \in \mathbb{K}\{x\}$ we denote by $\bar{\varphi}$ the function in $\mathbb{K}\{x\}$ satisfying $\overline{\varphi(x)} \equiv \bar{\varphi}(\bar{x})$. Recall that a $\mathbb{K}$-analytic function $\varphi$ defined in a domain $U \subset \mathbb{K}^{n}$ is called $\mathbb{K}$-algebraic (in the sense of Nash) if there exists a nonzero polynomial $P=P\left(X_{1}, \ldots, X_{n}, \Phi\right) \in$ $\mathbb{K}\left[X_{1}, \ldots, X_{n}, \Phi\right]$ such that $P(x, \varphi(x)) \equiv 0$ on $U$. All the considerations in this paper will be local: functions, submanifolds and mappings will always be defined in a small connected neighbourhood of some point (most often the origin) in $\mathbb{K}^{n}$.
2.2. System of partial differential equations associated to a generic submanifold of $\mathbb{C}^{n+m}$. Let $M$ be a real algebraic or analytic local submanifold of codimension $m$ in $\mathbb{C}^{n+m}$, passing through the origin. We assume that $M$ is generic, namely $T_{0} M+i T_{0} M=T_{0} \mathbb{C}^{n+m}$. Classically (cf. [BER1999]) there exists a choice of complex linear coordinates $t=(z, w) \in \mathbb{C}^{n} \times \mathbb{C}^{m}$ centered at the origin such that $T_{0} M=\{\operatorname{Im} w=0\}$ and such that there exist $m$ complex algebraic or analytic defining equations representing $M$ as the set of $(z, w)$ in a neighbourhood of the origin in $\mathbb{C}^{n+m}$ which satisfy

$$
\begin{equation*}
w_{1}=\Theta_{1}(z, \bar{z}, \bar{w}), \ldots \ldots, w_{m}=\Theta_{m}(z, \bar{z}, \bar{w}) . \tag{7.28}
\end{equation*}
$$

Furthermore, the mapping $\Theta=\left(\Theta_{1}, \ldots, \Theta_{m}\right)$ satisfies the functional equation

$$
\begin{equation*}
w \equiv \Theta(z, \bar{z}, \bar{\Theta}(\bar{z}, z, w)) \tag{7.28}
\end{equation*}
$$

which reflects the reality of the generic submanifold $M$. It follows in particular from (7.28) that the local holomorphic mapping $\mathbb{C}^{m} \ni \bar{w} \mapsto$ $\left(\Theta_{j}(0,0, \bar{w})\right)_{1 \leq j \leq m} \in \mathbb{C}^{m}$ is of rank $m$ at $\bar{w}=0$.

Generalizing an idea due to B. Segre in [Se1931] and [Se1932], exploited by É. Cartan in [3] and more recently by A. Sukhov in [24], [25], [Su2002], we shall associate to $M$ a system of partial differential equations. For this, we need some general nondegeneracy condition, which generalizes Levi
nondegeneracy. Let $\ell_{0} \in \mathbb{N}$ with $\ell_{0} \geq 1$. We shall assume that $M$ is $\ell_{0}-$ finitely nondegenerate at the origin, cf. [BER1999], [Me2003], [8]. This means that there exist multiindices $\beta(1), \ldots, \beta(n) \in \mathbb{N}^{n}$ with $|\beta(k)| \geq 1$ for $k=1, \ldots, n$ and $\max _{1 \leq k \leq n}|\beta(k)|=\ell_{0}$, and integers $j(1), \ldots, j(n)$ with $1 \leq j(k) \leq m$ for $k=1, \ldots, n$ such that the local holomorphic mapping
(7.28)
$\mathbb{C}^{n+m} \ni(\bar{z}, \bar{w}) \longmapsto\left(\left(\Theta_{j}(0, \bar{z}, \bar{w})\right)_{1 \leq j \leq m},\left(\Theta_{j(k), z^{\beta(k)}}(0, \bar{z}, \bar{w})\right)_{1 \leq k \leq n}\right) \in \mathbb{C}^{m+n}$
is of rank equal to $n+m$ at $(\bar{z}, \bar{w})=(0,0)$. Here, we denote the partial derivative $\partial^{|\beta|} \Theta_{j}(0, \bar{z}, \bar{w}) / \partial z^{\beta}$ simply by $\Theta_{j, z^{\beta}}(0, \bar{z}, \bar{w})$. Then $M$ is Levi nondegenerate at the origin if and only if $\ell_{0}=1$. By complexifying the variables $\bar{z}$ and $\bar{w}$, we get new independent variables $\zeta \in \mathbb{C}^{n}$ and $\xi \in \mathbb{C}^{m}$ together with a complex algebraic or analytic $m$-codimensional submanifold $\mathscr{M}$ in $\mathbb{C}^{2(n+m)}$ of equations

$$
\begin{equation*}
w_{j}=\Theta_{j}(z, \zeta, \xi), \quad j=1, \ldots, m \tag{7.28}
\end{equation*}
$$

called the extrinsic complexification of $M$. In the defining equations (7.28) of $\mathscr{M}$, following [Se1931] and [24], we may consider the "dependent variables" $w_{1}, \ldots, w_{m}$ as algebraic or analytic functions of the "independent variables" $z=\left(z_{1}, \ldots, z_{n}\right)$, with additional dependence on the extra "parameters" $(\zeta, \xi) \in \mathbb{C}^{n+m}$. Then by applying the differential operator $\partial^{|\alpha|} / \partial z^{\alpha}$ to (7.28), we obtain $w_{j, z^{\alpha}}(z)=\Theta_{j, z^{\alpha}}(z, \zeta, \xi)$. Writing these equations for $(j, \alpha)=(j(k), \beta(k))$ with $k=1, \ldots, n$, we obtain a system of $m+n$ equations

$$
\left\{\begin{align*}
w_{j}(z) & =\Theta_{j}(z, \zeta, \xi), \quad j=1, \ldots, m,  \tag{7.28}\\
w_{j(k), z^{\beta(k)}}(z) & =\Theta_{j(k), z^{\beta(k)}}(z, \zeta, \xi), \quad k=1, \ldots, n .
\end{align*}\right.
$$

In this system (7.28), by the assumption of $\ell_{0}$-finite nondegeneracy (7.28), the algebraic or analytic implicit function theorem allows to solve the parameters $(\zeta, \xi)$ in terms of the variables $\left(z_{k}, w_{j}(z), w_{j(k), z^{\beta(k)}}(z)\right)$, providing a local algebraic or analytic $\mathbb{C}^{n+m}$-valued mapping $R$ such that $(\zeta, \xi)=$ $R\left(z_{k}, w_{j}(z), w_{j(k), z^{\beta(k)}}(z)\right)$. Finally, for every pair $(j, \alpha)$ different from $(1,0), \ldots,(m, 0),(j(1), \beta(1)), \ldots,(j(n), \beta(n))$, we may replace $(\zeta, \xi)$ by $R$ in the differentiated expression $w_{j, z^{\alpha}}(z)=\Theta_{j, z^{\alpha}}(z, \zeta, \xi)$. This yields

$$
\begin{align*}
w_{j, z^{\alpha}}(z) & =\Theta_{j, z^{\alpha}}\left(z, R\left(z_{k}, w_{j}(z), w_{j(k), z^{\beta(k)}}(z)\right)\right)  \tag{7.28}\\
& =: F_{j, \alpha}\left(z_{k}, w_{j}(z), w_{j(k), z^{\beta(k)}}(z)\right) .
\end{align*}
$$

This is the system of partial differential equations associated with $\mathscr{M}$. As argued by B. Segre in [Se1931], the geometric study of generic submanifolds of $\mathbb{C}^{n}$ may gain much information from the study of their associated systems of partial differential equations (cf. [24], [25]). The next paragraphs
are devoted to provide a general one-to-one correspondence between completely integrable systems of analytic partial differential equations and their associated "submanifolds of solutions" (to be defined precisely below) like $\mathscr{M}$ above. Afterwards, we shall observe that conversely, the study of systems of analytic partial differential equations also gains much information from the direct study of their associated submanifolds of solutions.

### 2.3. Completely integrable systems of partial differential equations.

Let now $n, m, p \in \mathbb{N}$ with $n, m, p \geq 1$, let $\kappa \in \mathbb{N}$ with $\kappa \geq 2$ and let $u=\left(u^{1}, \ldots, u^{m}\right) \in \mathbb{K}^{m}$. Consider a collection of $p$ multiindices $\beta(1), \ldots, \beta(p) \in \mathbb{N}^{n}$ with $|\beta(q)| \geq 1$ for $q=1, \ldots, p$ and $\max _{1 \leq q \leq p}|\beta(q)|=\kappa-1$. Consider also $p$ integers $j(1), \ldots, j(p)$ with $1 \leq j(q) \leq m$ for $q=1, \ldots, p$. Inspired by (7.28), we consider a general system of partial differential equations of $n$ independent variables $\left(x_{1}, \ldots, x_{n}\right)$ and $m$ dependent variables $\left(u^{1}, \ldots, u^{m}\right)$ which is of the following form:

$$
\begin{equation*}
u_{x^{\alpha}}^{j}(x)=F_{\alpha}^{j}\left(x, u(x),\left(u_{x^{\beta(q)}}^{j(q)}(x)\right)_{1 \leq q \leq p}\right), \tag{E}
\end{equation*}
$$

where $(j, \alpha) \neq(j(1), \beta(1)), \ldots,(j(p), \beta(p))$ and $j=1, \ldots, m,|\alpha| \leq \kappa$. Here, we assume that $u=0$ is a local solution of the system $(\mathscr{E})$ and that the functions $F_{\alpha}^{j}$ are $\mathbb{K}$-algebraic or $\mathbb{K}$-analytic in a neighbourhood of the origin in $\mathbb{K}^{n+m+p}$. Among such systems are included ordinary differential equations of any order $\kappa \geq 2$, systems of second order partial differential equation as studied in [24], etc.

Throughout this article, we shall assume the system ( $\mathscr{E}$ ) completely integrable. By analyzing the application of the Frobenius theorem in jet spaces, one can show (we will not develop this) that the general solution of the system $(\mathscr{E})$ is given by $u(x):=\Omega(x, \nu, \chi)$, where the parameters $\nu \in \mathbb{K}^{n}$ and $\chi \in \mathbb{K}^{n}$ essentially correspond to the "initial conditions" $u(0)$ and $\left(u_{x^{\beta(q)}}^{j(q)}(0)\right)_{1 \leq q \leq p}$, and $\Omega$ is a $\mathbb{K}$-analytic $\mathbb{K}^{n}$-valued mapping. In the case of a generic submanifold as in Subsection 2.2 above, we recover the mapping $\Theta$. In the sequel, we shall use the following terminology: the coordinates $(x, u)$ will be called the variables and the coordinates $(\nu, \chi)$ will be called the parameters or the initial conditions. In Subsection 2.5 below, we shall introduce a certain duality where the rôles between variables and parameters are exchanged.
2.4. Associated submanifold of solutions. The existence of the function $\Omega$ and the analogy with Subsection 2.2 leads us to introduce the submanifold of solutions associated to the completely integrable system $(\mathscr{E})$, which by definition is the $m$-codimensional $\mathbb{K}$-analytic submanifold of $\mathbb{K}^{n+2 m+p}$,
equipped with the coordinates $(x, u, \nu, \chi)$, defined by the Cartesian equations

$$
\begin{equation*}
u_{j}=\Omega_{j}(x, \nu, \chi), \quad j=1, \ldots, m \tag{7.28}
\end{equation*}
$$

Let us denote this submanifold by $\mathscr{M}$. We stress that in general such a submanifold cannot coincide with the complexification of a generic submanifold of $\mathbb{C}^{m+n}$, for instance because $\mathbb{K}$ may be equal to $\mathbb{R}$ or, if $\mathbb{K}=\mathbb{C}$, because the integer $p$ is not necessarily equal to $n$. Also, even if $\mathbb{K}=\mathbb{C}$ and $n=p$, the mapping $\Omega$ does not satisfy a functional equation like (7.28). In fact, it may be easily established that the submanifold of solutions of a completely integrable system of partial differential equations like ( $\mathscr{E}$ ) coincides with the complexification of a generic submanifold if and only if $\mathbb{K}=\mathbb{C}$, $p=n$ and the mapping $\Omega$ satisfies a functional equation like (7.28).

Let now $\mathscr{M}$ be a submanifold of $\mathbb{K}^{n+2 n+p}$ of the form (7.28), but not necessarily constructed as the submanifold of solutions of a system $(\mathscr{E})$. We shall always assume that $\Omega_{j}(0, \nu, \chi) \equiv \nu^{j}$. We say that $\mathscr{M}$ is solvable with respect to the parameters if there exist multiindices $\beta(1), \ldots, \beta(p) \in \mathbb{N}^{n}$ with $|\beta(q)| \geq 1$ for $q=1, \ldots, p$ and integers $j(1), \ldots, j(p)$ with $1 \leq$ $j(q) \leq m$ for $q=1, \ldots, p$ such that the local $\mathbb{K}$-analytic mapping (7.28)
$\mathbb{K}^{m+p} \ni(\nu, \chi) \longmapsto\left(\left(\Omega_{j}(0, \nu, \chi)_{1 \leq j \leq m},\left(\Omega_{j(q), x^{\beta(q)}}(0, \nu, \chi)\right)_{1 \leq q \leq p}\right) \in \mathbb{K}^{m+p}\right.$
is of rank equal to $m+p$ at $(\zeta, \chi)=(0,0)$ (notice that since $\Omega_{j}(0, \nu, \chi) \equiv \nu^{j}$, then the $m$ first components of the mapping (7.28) are already of rank $m$ ). We remark that the submanifold of solutions of a system $(\mathscr{E})$ is automatically solvable with respect to the variables, the multiindices $\beta(q)$ and the integers $j(q)$ being the same as in the arguments of the right hand side terms $F_{\alpha}^{j}$ in $(\mathscr{E})$.
2.5. Dual system of defining equations. Since $\Omega_{j}(0, \nu, \chi) \equiv \nu^{j}$, we may solve the equations (7.28) with respect to $\nu$ by means of the analytic implicit function theorem, getting an equivalent system of equations for $\mathscr{M}$ :

$$
\begin{equation*}
\nu^{j}=\Omega_{j}^{*}(\chi, x, u), \quad j=1, \ldots, m . \tag{7.28}
\end{equation*}
$$

We call this the dual system of defining equations for $\mathscr{M}$. By construction, we have the functional equation

$$
\begin{equation*}
u \equiv \Omega\left(x, \Omega^{*}(\chi, x, u), \chi\right) \tag{7.28}
\end{equation*}
$$

implying the identity $\Omega_{j}^{*}(0, x, u) \equiv u^{j}$. We say that $\mathscr{M}$ is solvable with respect to the variables if there exist multiindices $\delta(1), \ldots, \delta(n) \in \mathbb{N}^{p}$ with $|\delta(l)| \geq 1$ for $l=1, \ldots, n$ and integers $j(1), \ldots, j(n)$ with $1 \leq j(l) \leq m$
for $l=1, \ldots, m$ such that the local $\mathbb{K}$-analytic mapping (7.28)
$\mathbb{K}^{n+m} \ni(x, u) \longmapsto\left(\left(\Omega_{j}^{*}(0, x, u)\right)_{1 \leq j \leq m},\left(\Omega_{j(l), \chi^{\delta(l)}}^{*}(0, x, u)\right)_{1 \leq l \leq n}\right) \in \mathbb{K}^{m+n}$
is of rank equal to $n+m$ at $(x, u)=(0,0)$ (notice that since $\Omega_{j}^{*}(0, x, u) \equiv u^{j}$, the $m$ fisrt components of the mapping (7.28) are already of rank $m$ ).

In the case where $\mathscr{M}$ is the complexification of a generic submanifold then the solvability with respect to the parameters is equivalent to the solvability with respect to the variables since $\Omega^{*} \equiv \bar{\Omega}$. However we notice that a submanifold $\mathscr{M}$ of solutions of a system $(\mathscr{E})$ is not automatically solvable with respect to the variables, as shows the following trivial example.

Example 1. Let $n=2, m=1$ and let $(\mathscr{E})$ denote the system $u_{x_{2}}=0$, $u_{x_{1} x_{1}}=0$, whose general solutions are $u(x)=\nu+x_{1} \chi=: \Omega\left(x_{1}, x_{2}, \nu, \chi\right)$. Notice that the variable $x_{2}$ is absent from the dual equation $\nu=u-x_{1} \chi_{1}=$ : $\Omega^{*}\left(\chi, x_{1}, x_{2}, u\right)$. It follows that $\mathscr{M}$ is not solvable with respect to the variables.
2.6. Symmetries of $(\mathscr{E})$, their lift to the jet space and their lift to the parameter space. We denote by $\mathscr{J}_{n, m}^{\kappa}$ the space of jets of order $\kappa$ of $\mathbb{K}$ analytic mappings $u=u(x)$ from $\mathbb{K}^{n}$ to $\mathbb{K}^{m}$. Let

$$
\begin{equation*}
\left(x_{l}, u^{j}, U_{l_{1}}^{i_{1}}, U_{l_{1}, l_{2}}^{i_{1}}, \ldots, U_{l_{1}, \ldots, l_{k}}^{i_{1}}\right) \in \mathbb{K}^{n+m C_{\kappa+n}^{\kappa}} \tag{7.28}
\end{equation*}
$$

denote the natural coordinates on $\mathscr{J}_{n, m}^{\kappa}$. Here, the superscripts $j, i_{1}$ and the subscripts $l, l_{1}, l_{2}, \ldots, l_{\kappa}$ satisfy $j, i_{1}=1, \ldots, m$ and $l, l_{1}, l_{2}, \ldots, l_{\kappa}=$ $1, \ldots, n$. The independent coordinate $U_{l_{1}, \ldots, l_{\lambda}}^{i_{1}}$ corresponds to the partial derivative $u_{x_{l_{1}} \ldots x_{l_{\lambda}}}^{i_{1}}$. Finally, by symmetry of partial differentiation, we identity every coordinate $U_{l_{1}, \ldots, l_{\lambda}}^{i_{1}}$ with the coordinates $U_{\sigma\left(l_{1}\right), \ldots, \sigma\left(l_{\lambda}\right)}^{i_{1}}$, where $\sigma$ is an arbitrary permutation of the set $\{1, \ldots, \lambda\}$. With these identifications, the $\kappa$-th order jet space $\mathscr{J}_{n, m}^{\kappa}$ is of dimension $n+m C_{\kappa+n}^{\kappa}$, where $C_{p}^{q}:=\frac{p!}{q!(p-q)!}$ denotes the binomial coefficient. Also, we shall sometimes use an equivalent notation for coordinates on $\mathscr{J}_{n, m}^{\kappa}$ :

$$
\begin{equation*}
\left(x_{l}, u^{j}, U_{\beta}^{i}\right) \in \mathbb{K}^{n+m C_{\kappa+n}^{n}} \tag{7.28}
\end{equation*}
$$

where $\beta \in \mathbb{N}^{n}$ satisfies $|\beta| \leq \kappa$ and where the independent coordinate $U_{\beta}^{i}$ corresponds to the partial derivative $u_{x^{\beta}}^{i}$.
associated to the system $(\mathscr{E})$ is the so-called skeleton $\Delta_{\mathscr{E}}$, which is the $\mathbb{K}$-analytic submanifold of dimension $n+m+p$ in $\mathscr{J}_{n, m}^{\kappa}$ simply defined by replacing the partial derivatives of the dependent variables $u^{j}$ by the independent jet variables in $(\mathscr{E})$ :

$$
\begin{equation*}
U_{\alpha}^{j}=F_{\alpha}^{j}\left(x, u,\left(U_{\beta(q)}^{j(q)}\right)_{1 \leq q \leq p}\right), \tag{7.28}
\end{equation*}
$$

for $(j, \alpha) \neq(j(1), \beta(1)), \ldots,(j(p), \beta(p))$ and $j=1, \ldots, m,|\alpha| \leq \kappa$. Clearly, the natural coordinates on the submanifold $\Delta_{\mathscr{E}}$ of $\mathscr{J}_{n, m}^{\kappa}$ are the $n+m+p$ coordinates

$$
\begin{equation*}
\left(x, u,\left(U_{\beta(q)}^{j(q)}\right)_{1 \leq q \leq p}\right) . \tag{7.28}
\end{equation*}
$$

Let $h=h(x, u)$ be a local $\mathbb{K}$-analytic diffeomorphism of $\mathbb{K}^{n+m}$ close to the identity mapping and let $\pi_{\kappa}: \mathscr{J}_{n, m}^{\kappa} \rightarrow \mathbb{K}^{n+m}$ be the canonical projection. According to [O11986] (Chapter 2) there exists a unique lift $h^{(\kappa)}$ of $h$ to $\mathscr{J}_{n, m}^{\kappa}$ such that $\pi_{\kappa} \circ h^{(\kappa)}=h \circ \pi_{\kappa}$. The components of $h^{(\kappa)}$ may be computed by means of universal combinatorial formulas and they are rational functions of the jet variables (7.28), their coefficients being partial derivatives of the components of $h$, see for instance $\S 3.3 .5$ of [BK1989]. By definition, $h$ is a local symmetry of $(\mathscr{E})$ if $h$ transforms the graph of every local solution of $(\mathscr{E})$ into the graph of another local solution of $(\mathscr{E})$. This definition seems to be rather uneasy to handle, because of the abstract quantification of "every local solution", but we have the following concrete characterization for $h$ to be a local symmetry of $(\mathscr{E}), c f$. Chapter 2 in [Ol1986].

## Lemma 8.1. The following conditions are equivalent:

(1) The local transformation $h$ is a local symmetry of $(\mathscr{E})$.
(2) Its $\kappa$-th prolongation $h^{(\kappa)}$ is a local self-transformation of the skeleton $\Delta_{\mathscr{E}}$ of $(\mathscr{E})$.

These considerations have an infinitesimal version. Indeed, let $X=$ $\sum_{l=1}^{n} Q^{l}(x, u) \partial / \partial x_{l}+\sum_{j=1}^{m} R^{j}(x, u) \partial / \partial u^{j}$ be a local vector field with $\mathbb{K}$-analytic coefficients which is defined in a neighbourhood of the origin in $\mathbb{K}^{n+m}$. Let $s \in \mathbb{K}$ and consider the flow of $L$ as the one-parameter family $h_{s}(x, u):=\exp (s X)(x, u)$ of local transformations. We recall that $X$ is an infinitesimal symmetry of $(\mathscr{E})$ if for every small $s \in \mathbb{K}$, the mapping $h_{s}(x, u):=\exp (s X)(x, u)$ is a local symmetry of $(\mathscr{E})$. By differentiating with respect to $s$ the $\kappa$-th prolongation $\left(h_{s}\right)^{(\kappa)}$ of $h_{s}$ at $s=0$, we obtain a unique vector field $X^{(\kappa)}$ on the $\kappa$-th jet space, called the $\kappa$-th prolongation of $X$ and which satisfies $\left(\pi_{k}\right)_{*}\left(X^{(\kappa)}\right)=X$. In Subsections 3.1 and 3.2 below, we shall analyze the combinatorial formulas for the coefficients of $X^{(\kappa)}$, since they will be needed to prove Theorem 6.4.

Let $X_{\mathscr{E}}$ be the projection to the restricted jet space $\mathbb{K}^{m+n+p}$, equipped with the coordinates (7.28), of the restriction of $X^{(\kappa)}$ to $\Delta_{\mathscr{E}}$, namely

$$
\begin{equation*}
X_{\mathscr{E}}:=\left(\pi_{\kappa, p}\right)_{*}\left(\left.X^{(\kappa)}\right|_{\Delta_{\mathscr{E}}}\right) . \tag{7.28}
\end{equation*}
$$

The following Lemma, called the Lie criterion, is the concrete characterization for $X$ to be an infinitesimal symmetry of $(\mathscr{E})$ and is a direct corollary of Lemma 8.1, cf. Chapter 2 in [Ol1986]. This criterion will be central in the next Sections 3, 4 and 5.

Lemma 8.1. The following conditions are equivalent:
(1) The vector field $X$ is an infinitesimal symmetry of ( $\mathscr{E})$.
(2) Its $\kappa$-th prolongation $X^{(\kappa)}$ is tangent to the skeleton $\Delta_{\mathscr{E}}$.

We denote by $\mathfrak{S y m}(\mathscr{E})$ the set of infinitesimal symmetries of $(\mathscr{E})$. Since it may be easily checked that $(c X+d Y)^{(\kappa)}=c X^{(\kappa)}+d Y^{(\kappa)}$ and that $\left[X^{(\kappa)}, Y^{(\kappa)}\right]=([X, Y])^{(\kappa)}$, see Theorem 2.39 in [Ol1986], it follows from Lemma 8.1 (2) that $\mathfrak{S y m}(\mathscr{E})$ is a Lie algebra of locally defined vector fields. Our main question in this section is the following: under which natural conditions is $\mathfrak{S y m}(\mathscr{E})$ finite-dimensional ?

Example 2. We observe that the Lie algebra $\mathfrak{S y m}(\mathscr{E})$ of the system ( $\mathscr{E}$ ) presented in Example 1 is infinite-dimensional, since it includes all vector fields of the form $X=Q^{2}\left(x_{1}, x_{2}, u\right) \partial / \partial x_{2}$, as may be verified. As we will argue in Proposition 3.1 below, this phenomenon is typical, the main reason lying in the first order relation $u_{x_{2}}=0$.

By analyzing the construction of the submanifold of solutions $\mathscr{M}$ associated to the system $(\mathscr{E})$, we may establish the following correspondence (we shall not develop its proof).

Proposition 3.1. To every infinitesimal symmetry $X=$ $\sum_{l=1}^{n} Q^{l}(x, u) \partial / \partial x_{l}+\sum_{j=1}^{m} R^{j}(x, u) \partial / \partial u^{j}$ of $(\mathscr{E})$, there corresponds $a$ unique vector field of the form

$$
\begin{equation*}
\mathscr{X}=\sum_{j=1}^{m} \Pi^{j}(\nu, \chi) \frac{\partial}{\partial \nu^{j}}+\sum_{q=1}^{p} \Lambda^{q}(\nu, \chi) \frac{\partial}{\partial \chi_{q}}, \tag{7.28}
\end{equation*}
$$

whose coefficients depend only on the parameters $(\nu, \chi)$, such that $X+\mathscr{X}$ is tangent to the submanifold of solutions $\mathscr{M}$.

This leads us to define the Lie algebra $\mathfrak{S y m}(\mathscr{M})$ of vector fields of the form
$\sum_{l=1}^{n} Q^{l}(x, u) \frac{\partial}{\partial x_{l}}+\sum_{j=1}^{m} R^{j}(x, u) \frac{\partial}{\partial u^{j}}+\sum_{j=1}^{m} \Pi^{j}(\nu, \chi) \frac{\partial}{\partial \nu^{j}}+\sum_{q=1}^{p} \Lambda^{q}(\nu, \chi) \frac{\partial}{\partial \chi_{q}}$
which are tangent to $\mathscr{M}$. We shall say that the submanifold $\mathscr{M}$ is degenerate if there exists a nonzero vector field of the form $X=\sum_{l=1}^{n} Q^{l}(x, u) \partial / \partial x_{l}+$ $\sum_{j=1}^{m} R^{j}(x, u) \partial / \partial u^{j}$ which is tangent to $\mathscr{M}$, which means that the corresponding $\mathscr{X}$ part is zero. In this case, we claim that $\mathfrak{S y m}(\mathscr{M})$ is infinite dimensional. Indeed there exists then a nonzero vector field $T=\sum_{l=1}^{n} Q^{l}(x, u) \partial / \partial x_{l}+\sum_{j=1}^{m} R^{j}(x, u) \partial / \partial u^{j}$ tangent to $\mathscr{M}$. Consequently, for every $\mathbb{K}$-analytic function $A(x, u)$, the vector field $A(x, u) T$ belongs to $\mathfrak{S y m}(\mathscr{M})$, hence $\mathfrak{S y m}(\mathscr{M})$ is infinite dimensional.

By developing the dual defining functions of $\mathscr{M}$ with respect to the powers of $\chi$, we may write

$$
\begin{equation*}
\nu^{j}=\Omega_{j}^{*}(\chi, x, u)=\sum_{\gamma \in \mathbb{N}^{p}} \chi^{\gamma} \Omega_{j, \gamma}^{*}(x, u), \tag{7.28}
\end{equation*}
$$

where the functions $\Omega_{j, \gamma}^{*}(x, u)$ are $\mathbb{K}$-analytic in a neighbourhood of the origin, we may formulate a criterion for $\mathscr{M}$ to be non degenerate with respect to the variables (whose proof is skipped).

Proposition 3.1. The submanifold $\mathscr{M}$ is not degenerate with respect to the variables if and only if there exists an integer $k$ such that the generic rank of the local $\mathbb{K}$-analytic mapping

$$
\begin{equation*}
(x, u) \longmapsto\left(\Omega_{j, \gamma}^{*}(x, u)\right)_{1 \leq j \leq m, \gamma \in \mathbb{N}^{p},|\gamma| \leq k} \tag{7.28}
\end{equation*}
$$

is equal to $n+m$.
Seeking for conditions which insure that $\mathfrak{S y m}(\mathscr{M})$ is finite-dimensional, it is therefore natural to assume that the generic rank of the mapping (7.28) is equal to $n+m$. Furthermore, to simplify the presentation, we shall assume that the rank at $(x, u)=(0,0)$ (not only the generic rank) of the mapping (7.28) is equal to $n+m$ for $k$ large enough. This is a "Zariskigeneric" assumption. Coming back to (7.28), we observe that this means exactly that $\mathscr{M}$ is solvable with respect to the variables. Then we denote by $\ell_{0}^{*}$ the smallest integer $k$ such that the rank at $(x, u)=(0,0)$ of the mapping (7.28) is equal to $n+m$ and we say that $\mathscr{M}$ is $\ell_{0}^{*}$-solvable with respect to the variables. Also, we denote by $\ell_{0}$ the integer $\max _{1 \leq q \leq p}|\beta(q)|$ and we say that $\mathscr{M}$ is $\ell_{0}$-solvable with respect to the parameters.
2.7. Fundamental isomorphism between $\mathfrak{S y m}(\mathscr{E})$ and $\mathfrak{S y m}(\mathscr{M})$. In the remainder of this Section 2, we shall assume that $\mathscr{M}$ is $\ell_{0}$-solvable with respect to the parameters and $\ell_{0}^{*}$-solvable with respect to the variabes. In this case, viewing the variables $\left(\nu^{1}, \ldots, \nu^{m}\right)$ in the dual equations $\nu^{j}=$ $\Omega_{j}^{*}(\chi, x, u)$ of $\mathscr{M}$ as a mapping of $\chi$ with (dual) "parameters" $(x, u)$ and proceeding as in Subsection 2.2, we may construct a dual system of completely integrable partial differential equations of the form

$$
\begin{equation*}
\nu_{\chi^{\gamma}}^{j}(\chi)=G_{\gamma}^{j}\left(\chi, \nu(\chi),\left(\nu_{\chi^{\delta(l)}}^{j(l)}(\chi)\right)_{1 \leq l \leq n}\right), \tag{*}
\end{equation*}
$$

where $(j, \gamma) \neq(j(1), \delta(1)), \ldots,(j(n), \delta(n))$. This system has its own infinitesimal symmetry Lie algebra $\mathfrak{S y m}\left(\mathscr{E}^{*}\right)$.

Theorem 6.4. If $\mathscr{M}$ is both solvable with respect to the parameters and solvable with respect to the variables, we have the following two isomorphisms:

$$
\begin{equation*}
\mathfrak{S y m}(\mathscr{E}) \cong \mathfrak{S y m}(\mathscr{M}) \cong \mathfrak{S y m}\left(\mathscr{E}^{*}\right) \tag{7.28}
\end{equation*}
$$

namely $X \longleftrightarrow X+\mathscr{X} \longleftrightarrow \mathscr{X}$.
In Subsection 2.10 below, we shall introduce a second geometric condition which is in general necessary for $\mathfrak{S y m}(\mathscr{M})$ to be finite-dimensional.
2.8. Local (pseudo)group $\operatorname{Sym}(\mathscr{M})$ of point transformations of $\mathscr{M}$. We shall study the geometry of a local $\mathbb{K}$-analytic submanifold $\mathscr{M}$ of $\mathbb{K}^{n+2 m+p}$ whose equations and dual equations are of the form

$$
\begin{cases}u^{j}=\Omega_{j}(x, \nu, \chi), & j=1, \ldots, m,  \tag{7.28}\\ \nu^{j}=\Omega_{j}^{*}(\chi, x, u), & j=1, \ldots, m .\end{cases}
$$

Let $t:=(x, u) \in \mathbb{K}^{n+m}$ and $\tau:=(\nu, \chi) \in \mathbb{K}^{n+m}$. We are interested in describing the set of local $\mathbb{K}$-analytic transformations of the space $\mathbb{K}^{n+2 m+p}$ which are of the specific form

$$
\begin{equation*}
(t, \tau) \longmapsto(h(t), \phi(\tau)), \tag{7.28}
\end{equation*}
$$

and which stabilize $\mathscr{M}$, in a neighborhood of the origin. We denote the local Lie pseudogroup of such transformations (possibly infinite-dimensional) by $\operatorname{Sym}(\mathscr{M})$. Importantly, each transformation of $\operatorname{Sym}(\mathscr{M})$ stabilize both the sets $\{t=c t$.$\} and the sets \{\tau=c t$. $\}$. Of course, the Lie algebra of $\operatorname{Sym}(\mathscr{M})$ coincides with $\mathfrak{S y m}(\mathscr{M})$ defined above.
2.9. Fundamental pair of foliations on $\mathscr{M}$. Let $p_{0} \in \mathbb{K}^{n+2 m+p}$ be a fixed point of coordinates $\left(t_{p_{0}}, \tau_{p_{0}}\right)$. Firstly, we observe that the intersection $\mathscr{M} \cap$ $\left\{\tau=\tau_{p_{0}}\right\}$ consists of the $n$-dimensional $\mathbb{K}$-analytic submanifold of equation $u=\Omega\left(x, \tau_{p_{0}}\right)$. As $\tau_{p_{0}}$ varies, we obtain a local $\mathbb{K}$-analytic foliation of $\mathscr{M}$ by $n$-dimensional submanifolds. Let us denote this first foliation by $\mathscr{F}_{p}$ and call it the foliation of $\mathscr{M}$ with respect to parameters. Secondly, and dually, we observe that the intersection $\mathscr{M} \cap\left\{t=t_{p_{0}}\right\}$ consists of the $p$-dimensional $\mathbb{K}$ analytic submanifold of equation $\nu=\Omega^{*}\left(\chi, t_{p_{0}}\right)$. As $t_{p_{0}}$ varies, we obtain a local $\mathbb{K}$-analytic foliation of $\mathscr{M}$ by $p$-dimensional submanifolds. Let us denote this second foliation by $\mathscr{F}_{v}$ and call it the foliation of $\mathscr{M}$ with respect to the variables. We call $\left(\mathscr{F}_{p}, \mathscr{F}_{v}\right)$ the fundamental pair of foliations on $\mathscr{M}$.
2.10. Covering property of the fundamental pair of foliations. We wish to formulate a geometric condition which says that starting from the origin in $\mathscr{M}$ and following alternately the leaves of $\mathscr{F}_{p}$ and the leaves of $\mathscr{F}_{v}$, we cover a neighborhood of the origin in $\mathscr{M}$. Let us introduce two collections
$\left(\mathscr{L}_{k}\right)_{1 \leq k \leq n}$ and $\left(\mathscr{L}_{q}^{*}\right)_{1 \leq q \leq p}$ of vector fields whose integral manifolds coincide with the leaves of $\mathscr{F}_{p}$ and $\mathscr{F}_{v}$ :

$$
\left\{\begin{align*}
\mathscr{L}_{k} & :=\frac{\partial}{\partial x_{k}}+\sum_{j=1}^{m} \frac{\partial \Omega_{j}}{\partial x_{k}}(x, \nu, \chi) \frac{\partial}{\partial u^{j}}, & k=1, \ldots, n,  \tag{7.28}\\
\mathscr{L}_{q}^{*} & :=\frac{\partial}{\partial \chi_{q}}+\sum_{j=1}^{m} \frac{\partial \Omega_{j}^{*}}{\partial \chi_{q}}(\chi, x, u) \frac{\partial}{\partial \nu^{j}}, & k=1, \ldots, n .
\end{align*}\right.
$$

Let $p_{0}$ be a fixed point in $\mathscr{M}$ of coordinates $\left(x_{p_{0}}, u_{p_{0}}, \nu_{p_{0}}, \chi_{p_{0}}\right) \in \mathbb{K}^{n+2 m+p}$, let $x_{1}:=\left(x_{1,1}, \ldots, x_{1, n}\right) \in \mathbb{K}^{n}$ be a "multitime" parameter and define the multiple flow map

$$
\left\{\begin{align*}
\mathscr{L}_{x_{1}}\left(x_{p_{0}}, u_{p_{0}}, \nu_{p_{0}}, \chi_{p_{0}}\right) & :=\exp \left(x_{1} \mathscr{L}\right)\left(p_{0}\right):=\exp \left(x_{1, n} \mathscr{L}_{n}\left(\cdots\left(\exp \left(x_{1,1} \mathscr{L}_{1}\left(p_{0}\right)\right)\right) \cdots\right)\right):=  \tag{7.28}\\
& :=\left(x_{p_{0}}+x_{1}, \Omega\left(x_{p_{0}}+x_{1}, \nu_{p_{0}}, \chi_{p_{0}}\right), \nu_{p_{0}}, \chi_{p_{0}}\right) .
\end{align*}\right.
$$

Similarly, for $\chi=\left(\chi_{1,1}, \ldots, \chi_{1, p}\right) \in \mathbb{K}^{p}$, define the multiple flow map

$$
\begin{equation*}
\mathscr{L}_{\chi_{1}}^{*}\left(x_{p_{0}}, u_{p_{0}}, \nu_{p_{0}}, \chi_{p_{0}}\right):=\left(x_{p_{0}}, u_{p_{0}}, \Omega^{*}\left(\chi_{p_{0}}+\chi_{1}, x_{p_{0}}, u_{p_{0}}\right), \chi_{p_{0}}+\chi_{1}\right) \tag{7.28}
\end{equation*}
$$

We may define now the mappings which correspond to start from the origin and to move alternately along the two foliations $\mathscr{F}_{p}$ and $\mathscr{F}_{v}$. If the first movement consists in moving along the foliation $\mathscr{F}_{v}$, we define

$$
\left\{\begin{align*}
\Gamma_{1}\left(x_{1}\right) & :=\mathscr{L}_{x_{1}}(0),  \tag{7.28}\\
\Gamma_{1}\left(x_{1}, \chi_{1}\right) & :=\mathscr{L}_{\chi_{1}}^{*}\left(\mathscr{L}_{x_{1}}(0)\right), \\
\Gamma_{3}\left(x_{1}, \chi_{1}, x_{2}\right) & :=\mathscr{L}_{x_{2}}\left(\mathscr{L}_{\chi_{1}}^{*}\left(\mathscr{L}_{x_{1}}(0)\right)\right), \\
\Gamma_{4}\left(x_{1}, \chi_{1}, x_{2}, \chi_{2}\right) & :=\mathscr{L}_{\chi_{2}}^{*}\left(\mathscr{L}_{x_{2}}\left(\mathscr{L}_{\chi_{1}}^{*}\left(\mathscr{L}_{x_{1}}(0)\right)\right)\right) .
\end{align*}\right.
$$

Generally, we may define the maps $\Gamma_{k}\left([x \chi]_{k}\right)$, where $[x \chi]_{k}=$ $\left(x_{1}, \chi_{1}, x_{2}, \chi_{2}, \ldots\right)$ with exactly $k$ terms and where each $x_{l}$ belongs to $\mathbb{K}^{n}$ and each $\chi_{l}$ belongs to $\mathbb{K}^{p}$. On the other hand, if the first movement consists in moving along the foliation $\mathscr{F}_{p}$, we start with $\Gamma_{1}^{*}\left(\chi_{1}\right):=\mathscr{L}_{\chi_{1}}^{*}(0)$, $\Gamma_{2}^{*}\left(\chi_{1}, x_{1}\right):=\mathscr{L}_{x_{1}}\left(\mathscr{L}_{\chi_{1}}^{*}(0)\right)$, etc., and generally we may define the maps $\Gamma_{k}^{*}\left([\chi x]_{k}\right)$, where $[\chi x]_{k}=\left(\chi_{1}, x_{1}, \chi_{2}, x_{2}, \ldots\right)$, with exactly $k$ terms. The range of both maps $\Gamma_{k}$ and $\Gamma_{k}^{*}$ is contained in $\mathscr{M}$. We call $\Gamma_{k}$ the $k$-th chain and $\Gamma_{k}^{*}$ the $k$-th dual chain.

Definition 5.3. The pair of foliations $\left(\mathscr{F}_{p}, \mathscr{F}_{v}\right)$ is called covering at the origin if there exists an integer $k$ such that the generic rank of $\Gamma_{k}$ is (maximal possible) equal to $\operatorname{dim}_{\mathbb{K}} \mathscr{M}$. Since the dual $(k+1)$-th chain $\Gamma_{k+1}^{*}$ for $\chi_{1}=0$ identifies with the $k$-th chain $\Gamma_{k}$, it follows that the same property holds for the dual chains.

In terms of Sussmann's approach [27], this means that the local orbit of the two systems of vector fields $\left(\mathscr{L}_{k}\right)_{1 \leq k \leq n}$ and $\left(\mathscr{L}_{q}^{*}\right)_{1 \leq q \leq p}$ is of maximal dimension. Reasoning as in [27] (using the so-called backward trick in Control Theory, see also [Me2003]), it may be shown that there exists the smallest even integer $2 \mu_{0}$ such that the ranks of the two maps $\Gamma_{2 \mu_{0}}$ and $\Gamma_{2 \mu_{0}}^{*}$ at the origin (not only their generic rank) in $\mathbb{K}^{n \mu_{0}+p \mu_{0}}$ are both equal to $\operatorname{dim}_{\mathbb{K}} \mathscr{M}$. This means that $\Gamma_{2 \mu_{0}}$ and $\Gamma_{2 \mu_{0}}^{*}$ are submersive onto a neighborhood of the origin in $\mathscr{M}$. We call $\mu_{0}$ the type of the pair of foliations $\left(\mathscr{F}_{p}, \mathscr{F}_{v}\right)$. It may also be established that $\mu_{0} \leq m+2$.

Example 2.46. We give an example of a submanifold which is both 1 solvable with respect to the parameters and with respect to the variables but whose pair of foliations is not covering: with $n=1, m=2$ and $p=1$, this is given by the two equations $u^{1}=\nu^{1}, u^{2}=\nu^{2}+x \chi_{1}$. Then $\mathfrak{S y m}(\mathscr{M})$ is infinite-dimensional since it contains the vector fields $a\left(u^{1}\right) \partial / \partial u^{1}+a\left(\nu^{1}\right) \partial / \partial \nu^{1}$, where $a$ is an arbitrary $\mathbb{K}$-analytic function. For this reason, we shall assume in the sequel that the pair of foliations $\left(\mathscr{F}_{p}, \mathscr{F}_{v}\right)$ is covering at the origin.
2.11. Estimate on the dimension of the local symmetry group of the submanifold of solutions. We may now formulate the main theorem of this section, which shows that, under suitable nondegeneracy conditions, $\operatorname{Sym}(\mathscr{M})$ is a finite dimensional local Lie group of local transformations. If $t \in \mathbb{K}^{n+m}$, we denote by $|t|:=\max _{1 \leq k \leq n+m}\left|t_{k}\right|$. If $(h, \phi) \in \operatorname{Sym}(\mathscr{M})$ we denote by $J_{t}^{k} h(0)$ the $k$-th order jet of $h$ at the origin and by $J_{\tau}^{k} \phi(0)$ the $k$-th order jet of $\phi$ at the origin. Also, we shall assume that $\mathscr{M}$ is either $\mathbb{K}$-algebraic or $\mathbb{K}$-analytic. Of course, the $\mathbb{K}$-algebraicity of the submanifold of solutions does not follow from the $\mathbb{K}$-algebraicity of the right hand sides $F_{\alpha}^{j}$ of the system of partial differential equations $(\mathscr{E})$.
Theorem 6.4. Assume that the $\mathbb{K}$-algebraic or $\mathbb{K}$-analytic submanifold of solutions $\mathscr{M}$ of the completely integrable system of partial differential equations $(\mathscr{E})$ is both $\ell_{0}$-sovable with respect to the parameters and $\ell_{0}^{*}$-solvable with respect to the variables. Assume that the fundamental pair of foliations $\left(\mathscr{F}_{p}, \mathscr{F}_{v}\right)$ is covering at the origin and let $\mu_{0}$ be its type at the origin. Then there exists $\varepsilon_{0}>0$ such that for every $\varepsilon$ with $0<\varepsilon<\varepsilon_{0}$, the following four properties hold:
(a) The (pseudo)group $\operatorname{Sym}(\mathscr{M})$ of local $\mathbb{K}$-analytic diffeomorphisms defined for $\left\{(t, \tau) \in \mathbb{K}^{n+2 m+p}:|t|<\varepsilon,|\tau|<\varepsilon\right\}$ which are of the form $(t, \tau) \mapsto(h(t), \phi(\tau))$ and which stabilize $\mathscr{M}$ is a local Lie pseudogroup of transformations of finite dimension $d \in \mathbb{N}$.
(b) Let $\kappa_{0}:=\mu_{0}\left(\ell_{0}+\ell_{0}^{*}\right)$. Then there exist two $\mathbb{K}$-algebraic or $\mathbb{K}$-analytic mappings $H_{\kappa_{0}}$ and $\Phi_{\kappa_{0}}$ which depend only on $\mathscr{M}$ and which may be constructed algorithmically by means of the defining equations of $\mathscr{M}$
such that every element $(h, \phi) \in \operatorname{Sym}(\mathscr{M})$, sufficiently close to the identity mapping, may be represented by

$$
\left\{\begin{align*}
h(t) & =H_{\kappa_{0}}\left(t, J_{t}^{\kappa_{0}} h(0)\right),  \tag{7.28}\\
\phi(\tau) & =\Phi_{\kappa_{0}}\left(\tau, J_{\tau}^{\kappa_{0}} \phi(0)\right) .
\end{align*}\right.
$$

Consequently, every element of $\operatorname{Sym}(\mathscr{M})$ is uniquely determined by its $\kappa_{0}-$ th jet at the origin and the dimension $d$ of the Lie algebra $\mathfrak{S v m}(\mathscr{M})$ is bounded by the number of components of the vector $\left(J_{t}^{\kappa_{0}} h(0), J_{\tau}^{\kappa_{0}} \phi(0)\right)$, namely we have
$\operatorname{dim}_{\mathbb{K}} \mathfrak{S y m}(\mathscr{E})=\operatorname{dim}_{\mathbb{K}} \mathfrak{S y m}(\mathscr{M}) \leq(n+m) C_{n+m+\kappa_{0}}^{\kappa_{0}}+(m+p) C_{m+p+\kappa_{0}}^{\kappa_{0}}$.
(c) There exists $\varepsilon^{\prime}$ with $0<\varepsilon^{\prime}<\varepsilon$ and a $\mathbb{K}$-algebraic or $\mathbb{K}$-analytic mapping $\left(H_{\mathscr{M}}, \Phi_{\mathscr{M}}\right)$ which may be constructed algorithmically by means of the defining equations of $\mathscr{M}$, defined in a neighbourhood of the origin in $\mathbb{K}^{n+2 m+p} \times \mathbb{K}^{d}$ with values in $\mathbb{K}^{n+2 m+p}$ and which satifies $\left(H_{\mathscr{M}}(t, 0), \Phi_{\mathscr{M}}(\tau, 0)\right) \equiv(t, \tau)$, such that every element $(h, \phi) \in$ $\operatorname{Sym}(\mathscr{M})$ defined on the set $\left\{(t, \tau) \in \mathbb{K}^{n+2 m+p}:|t|<\varepsilon^{\prime},|\tau|<\varepsilon^{\prime}\right\}$, sufficiently close to the identity mapping and stabilizing $\mathscr{M}$ may be represented as $(h(t), \phi(\tau)) \equiv\left(H_{\mathscr{M}}\left(t, s_{h, \phi}\right), \Phi_{\mathscr{M}}\left(\tau, s_{h, \phi}\right)\right)$ for a unique element $s_{h, \phi} \in \mathbb{K}^{d}$ depending on the mapping $(h, \phi)$.
(d) The mapping $(t, \tau, s) \longmapsto\left(H_{\mathscr{M}}(t, s), \Phi_{\mathscr{M}}(\tau, s)\right)$ defines a local $\mathbb{K}$ algebraic or $\mathbb{K}$-analytic Lie group of local $\mathbb{K}$-algebraic or $\mathbb{K}$-analytic transformations stabilizing $\mathscr{M}$.
2.12. Applications. The proof of Theorem 6.4, which possesses strong similarities with the proof of Theorem 4.1 in [8], will not be presented. It seems that Theorem 6.4, together with the argumentation on the necessity of assumptions that $\mathscr{M}$ be solvable with respect to the variables and that its fundamental pair of foliations be covering, is a new result about the finite-dimensionality of a completely integrable system of partial differential equations having an arbitrary number of independent and dependent variables. The main interest lies in the fact that we obtain the algorithmically constructible representation formula (7.28) together with the local Lie group structure mapping $\left(H_{\mathscr{M}}, \Phi_{\mathscr{M}}\right)$. In particular, we get as a corollary that every transformation $(h(t), \phi(\tau))$ given by a formal power series (not necessarily convergent) is as smooth as the applications $\left(H_{\kappa_{0}}, \Phi_{\kappa_{0}}\right)$ are, namely every formal element of $\operatorname{Sym}(\mathscr{M})$ is necessarily $\mathbb{K}$-algebraic or $\mathbb{K}$-analytic. As a counterpart of its generality, Theorem 3 does not provide optimal bounds, as shows the following illustration.

Example 2.46. Let $n=m=1$, let $\kappa \geq 3$ and let ( $\mathscr{E}$ ) denote the ordinary differential equation $u_{x^{\kappa}}(x)=F\left(x, u(x), u_{x}(x), \ldots, u_{x^{\kappa-1}}(x)\right)$. Then the submanifold of solutions $\mathscr{M}$ is of the form $u=\nu+x \chi_{1}+\cdots+$
$x^{\kappa-1} \chi_{\kappa-1}+\mathrm{O}\left(|x|^{\kappa}\right)+\mathrm{O}\left(|\chi|^{2}\right)$. It may be checked that $\ell_{0}=\kappa-1, \ell_{0}^{*}=1$ and $\mu_{0}=3$, hence $\kappa_{0}=3 \kappa$. Then the dimension estimate in (7.28) is: $\operatorname{dim}_{\mathbb{K}} \mathfrak{S y m}(\mathscr{E}) \leq 2 C_{2+3 \kappa}^{3 \kappa}+\kappa C_{4 \kappa}^{3 \kappa}$. This bound is much larger than the optimal bound $\operatorname{dim}_{\mathbb{K}} \mathfrak{S y m}(\mathscr{E}) \leq \kappa+4$ due to S . Lie (cf. [5]; see also the case $n=m=1$ of Theorem 6.4).

Untill now we focused on providing the set of Lie symmetries of a general system of partial differential equations with a local Lie group structure. As a byproduct we obtained the (non optimal) dimensional upper bound (7.28) of Theorem 6.4. In the next Sections 3, 4 and 5, using the classical Lie algorithm based on the Lie criterion (see Lemma 8.1), we provide an optimal bound for some specific systems of partial differential equations, answering an open problem raised in [Ol1995] page 206.

## 3. LIE THEORY FOR PARTIAL DIFFERENTIAL EQUATIONS

3.1. Prolongation of vector fields to the jet spaces. Consider the following $\mathbb{K}$-analytic system $(\mathscr{E})$ of non linear partial differential equations:

$$
\begin{equation*}
u_{x_{k_{1} \cdots x_{k_{\kappa}}}^{j}}^{j}(x)=F_{k_{1}, \ldots, k_{\kappa}}^{j}\left(x, u(x), u_{x_{l_{1}}}^{i_{1}}(x), \ldots, u_{x_{l_{1}} \cdots x_{l_{\kappa-1}}}^{i_{1}}(x)\right), \tag{7.28}
\end{equation*}
$$

where $1 \leq k_{1} \leq \cdots \leq k_{\kappa} \leq n, 1 \leq j \leq m$, and $F_{k_{1}, \ldots, k_{\kappa}}^{j}$ are analytic functions of $n+m C_{n+\kappa-1}^{\kappa-1}$ variables, defined in a neighbourhood of the origin. We assume that $(\mathscr{E})$ is completely integrable. The Lie theory consists in studying the infinitesimal symmetries $X=\sum_{l=1}^{n} Q^{l}(x, u) \partial / \partial x_{l}+$ $\sum_{j=1}^{m} R^{j}(x, u) \partial / \partial u^{j}$ of $(\mathscr{E})$. Consider the skeleton of $(\mathscr{E})$, namely the complex subvariety $\Delta_{\mathscr{E}}$ of codimension $m C_{\kappa+n-1}^{\kappa}$ in the jet space $\mathscr{J}_{n, m}^{\kappa}$, defined by

$$
\begin{equation*}
U_{k_{1}, \ldots, k_{\kappa}}^{j}=F_{k_{1}, \ldots, k_{\kappa}}^{j}\left(x, u, U_{l_{1}}^{i_{1}}, \ldots, U_{l_{1}, \ldots, l_{\kappa-1}}^{i_{1}}\right) \tag{7.28}
\end{equation*}
$$

where $j, i_{1}=1, \ldots, m$ and $k_{1}, \ldots, k_{\kappa}, l_{1}, \ldots, l_{\kappa-1}=1, \ldots, n$. For $k=$ $1, \ldots, n$ let $D_{k}$ be the $k$-th operator of total differentiation, characterized by the property that for every integer $\lambda \geq 2$ and for every analytic function $P=P\left(x, u, U_{l_{1}}^{i_{1}}, \ldots, U_{l_{1}, \ldots, l_{\lambda-1}}^{i_{1}}\right)$ defined in the jet space $\mathscr{J}_{n, m}^{\lambda-1}$, the operator $D_{k}$ is the unique formal infinite differential operator satisfying the relation

$$
\left\{\begin{array}{l}
{\left[D_{k} P\right]\left(x, u(x), u_{x_{1}}^{i_{1}}(x), \ldots, u_{x_{l_{1}} \cdots x_{l_{\lambda-1}}}^{i_{1}}(x)\right) \equiv}  \tag{7.28}\\
\frac{\partial}{\partial x_{k}}\left[P\left(x, u(x), u_{x_{l_{1}}}^{i_{1}}(x), \ldots, u_{x_{l_{1}} \cdots x_{l_{\lambda-1}}}^{i_{1}}(x)\right)\right] .
\end{array}\right.
$$

Note that this identity involves only the troncature of $D_{k}$ to order $\lambda$, denoted by $D_{k}^{\lambda}$, and defined by

$$
\left\{\begin{align*}
D_{k}^{\lambda}:=\frac{\partial}{\partial x_{k}} & +\sum_{i_{1}=1}^{m} U_{k}^{i_{1}} \frac{\partial}{\partial u^{i_{1}}}+\sum_{i_{1}=1}^{m} \sum_{l_{1}=1}^{n} U_{k, l_{1}}^{i_{1}} \frac{\partial}{\partial U_{l_{1}}^{i_{1}}}+\cdots+  \tag{7.28}\\
& +\sum_{i_{1}=1}^{m} \sum_{l_{1}, \ldots, l_{\lambda-1}=1}^{n} U_{k, l_{1}, \ldots, l_{\lambda-1}}^{i_{1}} \frac{\partial}{\partial U_{l_{1}, \ldots, l_{\lambda-1}}^{i_{1}}} .
\end{align*}\right.
$$

According to Theorem 2.36 of [Ol1986], the prolongation of order $\kappa$ of a vector field $X=\sum_{l=1}^{n} Q^{l}(x, u) \partial / \partial x_{l}+\sum_{j=1}^{m} R^{j}(x, u) \partial / \partial u^{j}$, denoted by $X^{(\kappa)}$, is the unique vector field on the space $\mathscr{J}_{n, m}^{\kappa}$ of the form

$$
\left\{\begin{align*}
X^{(\kappa)}=X+\sum_{j=1}^{m} & \sum_{k_{1}=1}^{n} \mathbf{R}_{k_{1}}^{j} \frac{\partial}{\partial U_{k_{1}}^{j}}+\sum_{j=1}^{m} \sum_{k_{1}, k_{2}=1}^{n} \mathbf{R}_{k_{1}, k_{2}}^{j} \frac{\partial}{\partial U_{k_{1}, k_{2}}^{j}}+\cdots+  \tag{7.28}\\
& +\sum_{j=1}^{m} \sum_{k_{1}, \ldots, k_{\kappa}=1}^{n} \mathbf{R}_{k_{1}, \ldots, k_{\kappa}}^{j} \frac{\partial}{\partial U_{k_{1}, k_{2}, \ldots, k_{\kappa}}^{j}}
\end{align*}\right.
$$

corresponding to the infinitesimal action of the flow of $X$ on the jets of order $\kappa$ of the graphs of maps $u=u(x)$, and whose coefficients are computed recursively by the formulas

For a better comprehension of the general computation, let us start by computing $R^{\kappa}$ in the case $n=m=1$.
3.2. Computation of $R^{\kappa}$ when $n=m=1$. A direct application of the preceding formulas leads to the following classical expressions:
(7.28)

$$
\left\{\begin{array}{l}
\mathbf{R}^{1}=R_{x}+\left[R_{u}-Q_{x}\right] U^{1}+\left[-Q_{u}\right]\left(U_{1}\right)^{2} \\
\mathbf{R}^{2}=R_{x^{2}}+\left[2 R_{x u}-Q_{x^{2}}\right] U^{1}+\left[R_{u^{2}}-2 Q_{x u}\right]\left(U^{1}\right)^{2}+\left[-Q_{u^{2}}\right]\left(U^{1}\right)^{3}+ \\
\quad+\left[R_{u}-2 Q_{x}\right] U^{2}+\left[-3 Q_{u}\right] U^{1} U^{2}
\end{array}\right.
$$

Observe that these expressions are polynomial in the jet variables, their coefficients being differential expressions involving a partial derivative of $R$ (with a positive integer coefficient) and a partial derivative of $Q$ (with a negative integer coefficient). We have also:

$$
\left\{\begin{align*}
\mathbf{R}^{3}=R_{x^{3}} & +\left[3 R_{x^{2} u}-Q_{x^{3}}\right] U^{1}+\left[3 R_{x u^{2}}-3 Q_{x^{2} u}\right]\left(U^{1}\right)^{2}+  \tag{7.28}\\
& +\left[R_{u^{3}}-3 Q_{x u^{2}}\right]\left(U^{1}\right)^{3}+\left[-Q_{u^{3}}\right]\left(U^{1}\right)^{4}+\left[3 R_{x u}-3 Q_{x^{2}}\right] U^{2}+ \\
& +\left[3 R_{u^{2}}-9 Q_{x u}\right] U^{1} U^{2}+\left[-6 Q_{u^{2}}\right]\left(U^{1}\right)^{2} U^{2}+\left[-3 Q_{u}\right]\left(U^{2}\right)^{2}+ \\
& +\left[R_{u}-3 Q_{x}\right] U^{3}+\left[-4 Q_{u}\right] U^{1} U^{3} . \\
\mathbf{R}^{4}=R_{x^{4}} & +\left[4 R_{x^{3} u}-Q_{x^{4}}\right] U^{1}+\left[6 R_{x^{2} u^{2}}-4 Q_{x^{3} u}\right]\left(U^{1}\right)^{2}+ \\
& +\left[4 R_{x u^{3}}-6 Q_{x^{2} u^{2}}\right]\left(U^{1}\right)^{3}+\left[R_{u^{4}}-4 Q_{x u^{3}}\right]\left(U^{1}\right)^{4}+\left[-Q_{u^{4}}\right]\left(U^{1}\right)^{5}+ \\
& +\left[6 R_{x^{2} u}-4 Q_{x^{3}}\right] U^{2}+\left[12 R_{x u^{2}}-18 Q_{x^{2} u}\right] U^{1} U^{2}+ \\
& +\left[6 R_{u^{3}}-24 Q_{x u^{2}}\right]\left(U^{1}\right)^{2} U^{2}+\left[-10 Q_{u^{3}}\right]\left(U^{1}\right)^{3} U^{2}+ \\
& +\left[3 R_{u^{2}}-12 Q_{x u}\right]\left(U^{2}\right)^{2}+\left[-15 Q_{u^{2}}\right] U^{1}\left(U^{2}\right)^{2}+\left[4 R_{x u}-6 Q_{x^{2}}\right] U^{3}+ \\
& +\left[4 R_{u^{2}}-16 Q_{x u}\right] U^{1} U^{3}+\left[-10 Q_{u^{2}}\right]\left(U^{1}\right)^{2} U^{3}+\left[-10 Q_{u}\right] U^{2} U^{3}+ \\
& +\left[R_{u}-4 Q_{x}\right] U^{4}+\left[-5 Q_{u}\right] U^{1} U^{4} .
\end{align*}\right.
$$

Remark that all the brackets involved in equations (7.28) are of the form $\left[\lambda R_{x^{a} u^{b+1}}-\mu Q_{x^{a+1} u^{b}}\right]$, where $\lambda, \mu \in \mathbb{N}$ and $a, b \in \mathbb{N}$.

In what follows we will not need the complete form of $R^{\kappa}$ but only the following partial form:

Lemma 8.1. For $\kappa \geq 4$ :

$$
\left\{\begin{align*}
\mathbf{R}^{\kappa}= & R_{x^{\kappa}}+\left[C_{\kappa}^{1} R_{x^{\kappa-1} u}-Q_{x^{\kappa}}\right] U^{1}+\left[C_{\kappa}^{2} R_{x^{\kappa-2} u}-C_{\kappa}^{1} Q_{x^{\kappa-1}}\right] U^{2}+  \tag{7.28}\\
& +\left[C_{\kappa}^{2} R_{x^{2} u}-C_{\kappa}^{3} Q_{x^{3}}\right] U^{\kappa-2}+\left[C_{\kappa}^{1} R_{x u}-C_{\kappa}^{2} Q_{x^{2}}\right] U^{\kappa-1}+ \\
& +\left[C_{\kappa}^{1} R_{u^{2}}-\kappa^{2} Q_{x u}\right] U^{1} U^{\kappa-1}+\left[-C_{\kappa+1}^{2} Q_{u}\right] U^{2} U^{\kappa-1}+ \\
& +\left[R_{u}-C_{\kappa}^{1} Q_{x}\right] U^{\kappa}+\left[-C_{\kappa+1}^{1} Q_{u}\right] U^{1} U^{\kappa}+ \\
& + \text { Remainder, }
\end{align*}\right.
$$

where the term Remainder denotes the remaining terms in the expansion of $R^{\kappa}$.

We note that the formula (7.28) is valid for $\kappa=3$, comparing with (7.28), with the convention that the terms $U^{\kappa-2}$ and $U^{\kappa-1}$ vanish (they coincide with $U^{1}$ and $U^{2}$ ), and replacing the coefficient $-C_{\kappa+1}^{2} Q_{u}=-C_{4}^{2} Q_{u}=-6 Q_{u}$ of the monomial $U^{2} U^{\kappa-1}$ by $-3 Q_{u}$, as it appears in (7.28). The proof goes by a straightforward computation, applying the recursive definition of this partial formula.
3.3. Computation of $R^{\kappa}$ in the general case. Following the exact same scheme as in the case $n=1$ we give the general partial formula for $R^{\kappa}$. We start with the first three families of coefficients $\mathbf{R}_{k_{1}}^{j}, \mathbf{R}_{k_{1}, k_{2}}^{j}$ and $\mathbf{R}_{k_{1}, k_{2}, k_{3}}^{j}$. Let $\delta_{p}^{q}$ be the Kronecker symbol, equal to 1 if $p=q$ and to 0 if $p \neq q$. More generally, the generalized Kronecker symbols are defined by $\delta_{p_{1}, \ldots, p_{k}}^{q_{1}, \ldots, q_{k}}:=\delta_{p_{1}}^{q_{1}} \delta_{p_{2}}^{q_{2}} \cdots \delta_{p_{k}}^{q_{k}}$.

By convention, the indices $j, i_{1}, i_{2}, \ldots, i_{\lambda}$ run in the set $\{1, \ldots, m\}$, the indices $k, k_{1}, k_{2}, \ldots, k_{\lambda}$ and $l, l_{1}, l_{2}, \ldots, l_{\lambda}$ running in $\{1, \ldots, n\}$. Hence we will write $\sum_{i_{1}=1}^{m} \sum_{i_{2}=1}^{m} \cdots \sum_{i_{\lambda}=1}^{m}$ as $\sum_{i_{1}, \ldots, i_{\lambda}}$ and $\sum_{l_{1}=1}^{n} \sum_{l_{2}=1}^{n} \cdots \sum_{l_{\lambda}=1}^{n}$ as $\sum_{l_{1}, \ldots, l_{\lambda}}$. The letters $i_{1}, i_{2}, \ldots, i_{\lambda}$ and $l_{1}, l_{2}, \ldots, l_{\lambda}$ will always be used for the summations in the development of $\mathbf{R}_{k_{1}, k_{2}, \ldots, k_{\lambda}}^{j}$. We will always use the indices $j$ and $k_{1}, k_{2}, \ldots, k_{\lambda}$ to write the coefficient $\mathbf{R}_{k_{1}, k_{2}, \ldots, k_{\lambda}}^{j}$.

We have:

$$
\left\{\begin{align*}
\mathbf{R}_{k_{1}}^{j}=R_{x_{k_{1}}}^{j}+\sum_{i_{1}} & \sum_{l_{1}}\left[\delta_{k_{1}}^{l_{1}} R_{u^{i_{1}}}^{j}-\delta_{i_{1}}^{j} Q_{x_{k_{1}}}^{l_{1}}\right] U_{l_{1}}^{i_{1}}+  \tag{7.28}\\
& +\sum_{i_{1}, i_{2}} \sum_{l_{1}, l_{2}}\left[-\delta_{i_{2}}^{j} \delta_{k_{1}}^{l_{1}} Q_{u^{i_{1}}}^{l_{2}}\right] U_{l_{1}}^{i_{1}} U_{l_{2}}^{i_{2}} .
\end{align*}\right.
$$

For $\mathbf{R}_{k_{1}, k_{2}}^{j}$ we have:
(7.28)

$$
\left\{\begin{array}{l}
\mathbf{R}_{k_{1}, k_{2}}^{j}=R_{x_{k_{1}} x_{k_{2}}}^{j}+\sum_{i_{1}} \sum_{l_{1}}\left[\delta_{k_{2}}^{l_{1}} R_{x_{k_{1} u^{i_{1}}}^{j}}^{j}+\delta_{k_{1}}^{l_{1}} R_{x_{k_{2}} i_{1}}^{j}-\delta_{i_{1}}^{j} Q_{x_{k_{1} x_{k_{2}}}}^{l_{1}}\right] U_{l_{1}}^{i_{1}}+ \\
\\
+\sum_{i_{1}, i_{2}} \sum_{l_{1}, l_{2}}\left[\delta_{k_{1}, k_{2}}^{l_{1}, l_{2}} R_{u^{i_{1}} u^{i_{2}}}^{j}-\delta_{i_{2}}^{j}\left(\delta_{k_{1}}^{l_{1}} Q_{x_{k_{2}} u^{i_{1}}}^{l_{2}}+\delta_{k_{2}}^{l_{1}} Q_{x_{k_{1}} u_{1}}^{l_{2}}\right)\right] U_{l_{1}}^{i_{1}} U_{l_{2}}^{i_{2}}+ \\
\\
+\sum_{i_{1}, i_{2}, i_{3}} \sum_{l_{1}, l_{2}, l_{3}}\left[-\delta_{i_{3}}^{j} \delta_{k_{1}, k_{2}}^{l_{1}, l_{2}} Q_{u^{i_{1}} u^{i_{2}}}^{l_{3}}\right] U_{l_{1}}^{i_{1}} U_{l_{2}}^{i_{2}} U_{l_{3}}^{i_{3}}+ \\
\\
+\sum_{i_{1}} \sum_{l_{1}, l_{2}}\left[\delta_{k_{1}, k_{2}}^{l_{1}, l_{2}} R_{u^{i_{1}}}^{j}-\delta_{i_{1}}^{j} \delta_{k_{2}}^{l_{1}} Q_{x_{k_{1}}}^{l_{2}}-\delta_{i_{1}}^{j} \delta_{k_{1}}^{l_{1}} Q_{x_{k_{2}}}^{l_{2}}\right] U_{l_{1}, l_{2}}^{i_{1}}+ \\
\\
+\sum_{i_{1}, i_{2}} \sum_{l_{1}, l_{2}, l_{3}}\left[-\delta_{i_{2}}^{j} \delta_{k_{1}, k_{2}}^{l_{1}, l_{2}} Q_{u_{1}^{i_{1}}}^{l_{3}}-\delta_{i_{2}}^{j} \delta_{k_{1}, k_{2}}^{l_{3}, l_{1}} Q_{u^{i_{1}}}^{l_{2}}-\delta_{i_{1}}^{j} \delta_{k_{1}, k_{2}}^{l_{2}, l_{3}} Q_{u^{i_{2}}}^{l_{1}}\right] U_{l_{1}}^{i_{1}} U_{l_{2}, l_{3}}^{i_{2}} .
\end{array}\right.
$$

Since we also treat systems of order $\kappa \geq 3$, it is necessary to compute $\mathbf{R}_{k_{1}, k_{2}, k_{3}}^{j}$. We write this as follows:

$$
\begin{equation*}
\mathbf{R}_{k_{1}, k_{2}, k_{3}}^{j}=\mathrm{I}+\mathrm{II}+\mathrm{III}, \tag{7.28}
\end{equation*}
$$

where the first term I involves only polynomials in $U_{l_{1}}^{i_{1}}$ :

$$
\begin{align*}
& \left(\mathrm{I}=R_{x_{k_{1}} x_{k_{2}} x_{k_{3}}}^{j}+\sum_{i_{1}} \sum_{l_{1}}\left[\delta_{k_{1}}^{l_{1}} R_{x_{k_{2}} x_{k_{3}} u^{i_{1}}}^{j}+\delta_{k_{2}}^{l_{1}} R_{x_{k_{1}} x_{k_{3}} u^{i_{1}}}^{j}+\delta_{k_{3}}^{l_{1}} R_{x_{k_{1}} x_{k_{2}} u^{i_{1}}}^{j}-\right.\right.  \tag{7.28}\\
& \left.-\delta_{i_{1}}^{j} Q_{x_{k_{1}} x_{k_{2}} x_{k_{3}}}^{l_{1}}\right] U_{l_{1}}^{i_{1}}+\sum_{i_{1}, i_{2}} \sum_{l_{1}, l_{2}}\left[\delta_{k_{1}, k_{2}}^{l_{1}, l_{2}} R_{x_{k_{3}} u^{i_{1}} u^{i_{2}}}^{j}+\delta_{k_{3}, k_{1}}^{l_{1}, l_{2}} R_{x_{k_{2}} u^{i_{1}} u^{i_{2}}}^{j}+\right. \\
& +\delta_{k_{2}, k_{3}}^{l_{1}, l_{2}} R_{x_{k_{1}} u^{i_{1}} u^{i_{2}}}^{j}-\delta_{i_{2}}^{j} \delta_{k_{1}}^{l_{1}} Q_{x_{k_{2}} x_{k_{3}} u^{i_{1}}}^{l_{2}}-\delta_{i_{2}}^{j} \delta_{k_{2}}^{l_{1}} Q_{x_{k_{1}} x_{k_{3}} u^{i_{1}}}^{l_{2}}- \\
& \left.-\delta_{i_{2}}^{j} \delta_{k_{3}}^{l_{1}} Q_{x_{k_{1}} x_{k_{2}} u^{i_{1}}}^{l_{2}}\right] U_{l_{1}}^{i_{1}} U_{l_{2}}^{i_{2}}+\sum_{i_{1}, i_{2}, i_{3}} \sum_{l_{1}, l_{2}, l_{3}}\left[\delta_{k_{1},,_{2}, k_{3}}^{l_{1}, l_{2}, l_{3}} R_{u^{i_{1}} u^{i_{2}} u^{i_{3}}}^{j}-\right. \\
& -\delta_{i_{3}}^{j} \delta_{k_{1}, k_{2}}^{l_{1}, l_{2}} Q_{x_{k_{3}} u^{i_{1}} u^{i_{2}}}^{l_{3}}-\delta_{i_{3}}^{j} \delta_{k_{2}, k_{3}}^{l_{1}, l_{2}} Q_{x_{k_{1}} u_{1}^{i_{1}} u^{i_{2}}}^{l_{3}}- \\
& \left.-\delta_{i_{3}}^{j} \delta_{k_{1}, k_{3}}^{l_{1}, l_{2}} Q_{x_{k_{2}} u^{i_{1}} u^{i_{2}}}^{l_{3}}\right] U_{l_{1}}^{i_{1}} U_{l_{2}}^{i_{2}} U_{l_{3}}^{i_{3}}+ \\
& +\sum_{i_{1}, i_{2}, i_{3}, i_{4}} \sum_{l_{1}, l_{2}, l_{3}, l_{4}}\left[-\delta_{i_{4}}^{j} \delta_{k_{1}, k_{2}, k_{3}}^{l_{1}, l_{2}, l_{3}} Q_{u^{i_{1}} u^{i_{2}} u^{i_{3}}}^{l_{3}}\right] U_{l_{1}}^{i_{1}} U_{l_{2}}^{i_{2}} U_{l_{3}}^{i_{3}} U_{l_{4}}^{i_{4}},
\end{align*}
$$

the second term II involves at least once the monomial $U_{l_{1}, l_{2}}^{i_{1}}$ :

$$
\begin{align*}
& \left(\mathrm{II}=\sum_{i_{1}} \sum_{l_{1}, l_{2}}\left[\delta_{k_{1}, k_{2}}^{l_{1}, l_{2}} R_{x_{k_{3}} u^{i_{1}}}^{j}+\delta_{k_{3}, k_{1}}^{l_{1}, l_{2}} R_{x_{k_{2}} u^{i_{1}}}^{j}+\delta_{k_{2}, k_{3}}^{l_{1}, l_{2}} R_{x_{k_{1}} u^{i_{1}}}^{j}-\right.\right.  \tag{7.28}\\
& \left.-\delta_{i_{1}}^{j}\left(\delta_{k_{1}}^{l_{1}} Q_{x_{k_{2}} x_{k_{3}}}^{l_{2}}+\delta_{k_{2}}^{l_{1}} Q_{x_{k_{1}} x_{k_{3}}}^{l_{2}}+\delta_{k_{3}}^{l_{1}} Q_{x_{k_{1}} x_{k_{2}}}^{l_{2}}\right)\right] U_{l_{1}, l_{2}}^{i_{1}}+ \\
& +\sum_{i_{1}, i_{2}} \sum_{l_{1}, l_{2}, l_{3}}\left[\delta_{k_{1}, k_{2}, k_{3}}^{l_{1}, l_{2}, l_{3}} R_{u^{i_{1}} u^{i_{2}}}^{j}+\delta_{k_{1}, k_{2}, k_{3}}^{l_{3}, l_{1}, l_{2}} R_{u^{i_{1}} u^{i_{2}}}^{j}+\delta_{k_{1}, k_{2}, k_{3}}^{l_{2}, l_{3}, l_{1}} R_{u^{i_{1}} u^{i_{2}}}^{j}-\right. \\
& -\delta_{i_{1}}^{j}\left(\delta_{k_{1}, k_{2}}^{l_{2}, l_{3}} Q_{x_{k_{3}} u^{i_{2}}}^{l_{1}}+\delta_{k_{3}, k_{1}}^{l_{2}, l_{3}} Q_{x_{k_{2}} u^{i_{2}}}^{l_{1}}+\delta_{k_{2}, k_{3}}^{l_{2}, l_{3}} Q_{x_{k_{1}} u^{i_{2}}}^{l_{1}}\right)- \\
& -\delta_{i_{2}}^{j}\left(\delta_{k_{1}, k_{2}}^{l_{1}, l_{2}} Q_{x_{k_{3}} u^{i_{1}}}^{l_{3}}+\delta_{k_{3}, k_{1}}^{l_{1}, l_{2}} Q_{x_{k_{2}} u^{i_{1}}}^{l_{3}}+\delta_{k_{2}, k_{3}}^{l_{1}, l_{2}} Q_{x_{k_{1}} u^{i_{1}}}^{l_{3}}+\right. \\
& \left.\left.+\delta_{k_{1}, k_{2}}^{l_{3}, l_{1}} Q_{x_{k_{3}} u^{i_{1}}}^{l_{2}}+\delta_{k_{3}, k_{1}}^{l_{3}, l_{1}} Q_{x_{k_{2}} u^{i_{1}}}^{l_{2}}+\delta_{k_{2}, k_{3}}^{l_{3}, l_{1}} Q_{x_{k_{1}} u_{1}^{i_{1}}}^{l_{2}}\right)\right] U_{l_{1}}^{i_{1}} U_{l_{2}, l_{3}}^{i_{2}}+ \\
& +\sum_{i_{1}, i_{2}, i_{3}} \sum_{l_{1}, l_{2}, l_{3}, l_{4}}\left[-\delta_{i_{3}}^{j}\left(\delta_{k_{1}, k_{2}, k_{3}}^{l_{1}, l_{2}, l_{3}} Q_{u^{i_{1}} u^{i_{2}}}^{l_{4}}+\delta_{k_{1}, k_{2}, k_{3}}^{l_{1} l_{4}, l_{2}} Q_{u^{i_{1}} u^{i_{2}}}^{l_{3}}+\right.\right. \\
& \left.\delta_{k_{1}, k_{2}, k_{3}}^{l_{3}, l_{1}, l_{2}} Q_{u^{i_{1}} u^{i_{2}}}^{l_{4}}\right)-\delta_{i_{1}}^{j}\left(\delta_{k_{1}, k_{2}, k_{3}}^{l_{3}, l_{2}, l_{4}} Q_{u^{i_{2}} u^{i_{3}}}^{l_{1}}+\delta_{k_{1}, k_{2}, k_{3}}^{l_{4}, l_{3}, l_{2}} Q_{u^{i_{2}} u^{i_{3}}}^{l_{1}}+\right. \\
& \left.\left.+\delta_{k_{1}, k_{2}, k_{3}}^{l_{2}, l_{3}, l_{4}} Q_{u^{i_{1}} u^{i_{2}}}^{l_{1}}\right)\right] U_{l_{1}}^{i_{1}} U_{l_{2}}^{i_{2}} U_{l_{3}, l_{4}}^{i_{3}}+\sum_{i_{1}, i_{2}} \sum_{l_{1}, l_{2}, l_{3}, l_{4}}\left[-\delta_{i_{2}}^{j}\left(\delta_{k_{1}, k_{2}, k_{3}}^{l_{1}, l_{2}, l_{3}} Q_{u_{1}}^{l_{4}}+\right.\right. \\
& \left.\left.+\delta_{k_{1}, k_{2}, k_{3}}^{l_{3}, l_{1}, l_{2}} Q_{u_{1}^{i_{1}}}^{l_{4}}+\delta_{k_{1}, k_{2}, k_{3}}^{l_{2}, l_{3}, l_{1}} Q_{u^{i_{1}}}^{l_{4}}\right)\right] U_{l_{1}, l_{2}}^{i_{1}} U_{l_{3}, l_{4}}^{i_{2}}
\end{align*}
$$

and the third term III involves at least once the monomial $U_{l_{1}, l_{2}, l_{3}}^{i_{1}}$ (note that there is no term involving simultaneously $U_{l_{1}, l_{2}}^{i_{1}}$ and $U_{l_{1}, l_{2}, l_{3}}^{i_{1}}$ ):

$$
\left\{\begin{align*}
\text { III }= & \sum_{i_{1}} \sum_{l_{1}, l_{2}, l_{3}}\left[\delta_{k_{1}, k_{2}, k_{3}}^{l_{1}, l_{2}, l_{3}} R_{u^{i_{1}}}^{j}-\delta_{i_{1}}^{j}\left(\delta_{k_{2}, k_{3}}^{l_{1}, l_{2}} Q_{x_{k_{1}}}^{l_{3}}+\delta_{k_{3}, k_{1}}^{l_{1}, l_{2}} Q_{x_{k_{2}}}^{l_{3}}+\right.\right.  \tag{7.28}\\
& \left.\left.+\delta_{k_{1}, k_{2}}^{l_{1}, l_{2}} Q_{x_{k_{3}}}^{l_{3}}\right)\right] U_{l_{1}, l_{2}, l_{3}}^{i_{1}}+\sum_{i_{1}, i_{2}} \sum_{l_{1}, l_{2}, l_{3}, l_{4}}\left[-\delta_{i_{1}}^{j} \delta_{k_{1}, k_{2}, k_{3}}^{l_{2}, l_{3}, l_{4}} Q_{u^{i_{2}}}^{l_{1}}-\right. \\
& \left.-\delta_{i_{2}}^{j}\left(\delta_{k_{1}, k_{2}, k_{3}}^{l_{1}, l_{2}, l_{3}} Q_{u^{i_{1}}}^{l_{4}}+\delta_{k_{1}, k_{2}, k_{3}}^{l_{4}, l_{1}, l_{2}} Q_{u^{i_{1}}}^{l_{3}}+\delta_{k_{1}, k_{2}, k_{3}}^{l_{3}, l_{4}, l_{1}} Q_{u^{i_{1}}}^{l_{2}}\right)\right] U_{l_{1}}^{i_{1}} U_{l_{2}, l_{3}, l_{4}}^{i_{2}} .
\end{align*}\right.
$$

Before giving the partial expression of $R^{\kappa}$ we introduce some notations. For $p \in \mathbb{N}$ with $p \geq 1$, let $\mathfrak{S}_{p}$ be the group of permutations of $\{1,2, \ldots, p\}$. For $q \in \mathbb{N}$ with $1 \leq q \leq p-1$, let $\mathfrak{S}_{p}^{q}$ be the set of permutations $\sigma \in \mathfrak{S}_{p}$ such that $\sigma(1)<\sigma(2)<\cdots<\sigma(q)$ and $\sigma(q+1)<\sigma(q+2)<\cdots<\sigma(p)$. Its cardinal is $C_{p}^{q}$. Let $\mathfrak{C}_{p}$ be the group of cyclic permutations of $\{1,2, \ldots, p\}$. Reasoning recursively from the formula of $\mathbf{R}_{k_{1}, k_{2}, k_{3}}^{j}$ given by (7.28), we may generalize Lemma 8.1:

Lemma 8.1. For every $\kappa \geq 4$ and for every $j=1, \ldots, m, k_{1}, \ldots, k_{\kappa}=$ $1, \ldots, n$, we have:

$$
\begin{equation*}
\mathbf{R}_{k_{1}, k_{2}, \ldots, k_{\kappa}}^{j}=I_{1}+\cdots+I_{9}+\text { Remainder } \tag{7.28}
\end{equation*}
$$

where $I_{1}=R_{x_{k_{1}} x_{k_{2}} \ldots x_{k_{\kappa}}}^{j}$,

$$
\begin{aligned}
& I_{2}=\sum_{i_{1}} \sum_{l_{1}}\left[\sum_{\sigma \in \mathfrak{S}_{\kappa}^{1}} \delta_{k_{\sigma(1)}}^{l_{1}} R_{x_{k_{\sigma(2)}} \cdots x_{k_{\sigma(\kappa)}}}^{j} u^{i_{1}}-\delta_{i_{1}}^{j} Q_{x_{k_{1}} \ldots x_{k_{\kappa}}}^{l_{1}}\right] U_{l_{1}}^{i_{1}}, \\
& I_{3}=\sum_{i_{1}} \sum_{l_{1}, l_{2}}\left[\sum_{\sigma \in \mathfrak{S}_{k}^{2}} \delta_{k_{\sigma(1)}, k_{\sigma(2)}}^{l_{1}, l_{2}} R_{x_{k_{\sigma(3)}}^{j} \cdots x_{k_{\sigma(k)}} u^{i_{1}}}^{j}-\right. \\
& \left.-\delta_{i_{1}}^{j}\left(\sum_{\sigma \in \mathfrak{S}_{\kappa}^{1}} \delta_{k_{\sigma(1)}}^{l_{1}} Q_{x_{k_{\sigma(2)}} \cdots x_{k_{\sigma(\kappa)}}}^{l_{2}}\right)\right] U_{l_{1}, l_{2}}^{i_{1}}, \\
& I_{4}=\sum_{i_{1}} \sum_{l_{1}, \ldots, l_{\kappa-2}}\left[\sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-2}} \delta_{k_{\sigma(1), \ldots, k_{\sigma(\kappa-2)}}^{l_{1}, \ldots \ldots, l_{k-2}} R_{x_{k_{\sigma(\kappa-1)}}}^{j} x_{k_{\sigma(\kappa)}} u^{i_{1}}}-\right. \\
& \left.-\delta_{i_{1}}^{j}\left(\sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-3}} \delta_{k_{\sigma(1), \ldots, k_{\sigma(\kappa-3)}}^{l_{1}, \ldots \ldots, l_{\kappa-3}} Q_{x_{k_{\sigma(\kappa-2)}}}^{l_{k-2}} x_{k_{\sigma(\kappa-1)}} x_{k_{\sigma(\kappa)}}}^{\kappa_{2}}\right)\right] U_{l_{1}, \ldots, l_{\kappa-2}}^{i_{1}},
\end{aligned}
$$

$$
\begin{aligned}
& I_{5}=\sum_{i_{1}} \sum_{l_{1}, \ldots, l_{\kappa-1}}\left[\sum_{\sigma \in \mathfrak{S}_{k}^{\kappa-1}} \delta_{k_{\sigma(1)}, \ldots, k_{\sigma(\kappa-1)}}^{l_{1}, \ldots \ldots, l_{\kappa-1}} R_{x_{k_{\sigma(\kappa)}} i_{1}}^{j}-\right. \\
& \left.-\delta_{i_{1}}^{j}\left(\sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-2}} \delta_{k_{\sigma(1)}, \ldots, k_{\sigma(\kappa-2)} x_{k_{\sigma(\kappa-1)}}^{l_{1}, \ldots \ldots, l_{\kappa-2}} x_{k_{\sigma(\kappa)}}^{l_{\kappa-1}}}^{x_{1}}\right)\right] U_{l_{1}, \ldots, l_{\kappa-1}}^{i_{1}}, \\
& I_{6}=\sum_{i_{1}, i_{2}} \sum_{l_{1}, \ldots, l_{\kappa}}\left[\sum_{\tau \in \mathfrak{C}_{\kappa}} \delta_{k_{1}, \ldots \ldots, k_{\kappa}}^{l_{\tau(1)}, \ldots, l_{\tau(\kappa)}} R_{u^{i_{1}} u^{i_{2}}}^{j}-\delta_{i_{1}}^{j}\left(\sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-1}} \delta_{k_{\sigma(1), \ldots, k_{\sigma(\kappa-1)}}^{l_{1}, \ldots \ldots, l_{\kappa}} Q_{x_{k_{\sigma(\kappa)}}}^{l_{1}} u^{i_{2}}}^{l_{1}}\right)-\right. \\
& \left.-\delta_{i_{2}}^{j}\left(\sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-1}}\left(\delta_{k_{\sigma(1), \ldots, k_{\sigma(\kappa-1)}}^{l_{1}, \ldots, l_{\kappa-1}} Q_{x_{k_{\sigma(\kappa)}}}^{l_{\kappa}} u^{i_{1}}}+\cdots+\delta_{k_{\sigma(1), \ldots, k_{\sigma(\kappa-1)}}^{l_{3}, \ldots \ldots, l_{1}}}^{l_{x_{k_{\sigma(\kappa)}}}^{u^{i_{2}}}}\right)\right)\right] \times \\
& \times U_{l_{1}}^{i_{1}} U_{l_{2}, \ldots, l_{\kappa}}^{i_{2}}, \\
& I_{7}=\sum_{i_{1}, i_{2}} \sum_{l_{3}, \ldots, l_{\kappa+1}}\left[-\delta_{i_{1}}^{j}\left(\delta_{k_{1}, \ldots, k_{\kappa}}^{l_{2}, \ldots, l_{\kappa+1}} Q_{u^{i_{2}}}^{l_{1}}+\cdots+\delta_{k_{1}, \ldots, k_{\kappa}}^{l_{\kappa+1}, \ldots, l_{2}} Q_{u^{i_{2}}}^{l_{2}}\right)-\right. \\
& \left.-\delta_{i_{2}}^{j}\left(\sum_{\tau \in \mathfrak{S}_{\kappa}^{2}} \delta_{k_{1}, \ldots \ldots, k_{\kappa}}^{l_{\tau(1)}, \ldots, l_{\tau(\kappa)}} Q_{u^{i_{1}}}^{l_{\kappa+1}}\right)\right] U_{l_{1}, l_{2}}^{i_{1}} U_{l_{3}, \ldots, l_{\kappa+1}}^{i_{2}}, \\
& I_{8}=\sum_{i_{1}} \sum_{l_{1}, \ldots, l_{\kappa}}\left[\delta_{k_{1}, \ldots, k_{\kappa}}^{l_{1}, \ldots, l_{\kappa}} R_{u^{i_{1}}}^{j}-\delta_{i_{1}}^{j}\left(\sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-1}} \delta_{\left.\left.k_{\sigma(1), \ldots, k_{\sigma(\kappa-1)}}^{l_{1}, \ldots \ldots, l_{\kappa-1}} Q_{x_{k_{\sigma(\kappa)}}}^{l_{\kappa}}\right)\right] U_{l_{1}, \ldots, l_{k}}^{i_{1}}, ~}^{\text {, }}\right.\right. \\
& I_{9}=\sum_{i_{1}, i_{2}} \sum_{l_{1}, \ldots, l_{\kappa+1}}\left[-\delta_{i_{1}}^{j} \delta_{k_{1}, \ldots, k_{\kappa}}^{l_{2}, \ldots, l_{\kappa+1}} Q_{u^{i_{2}}}^{l_{1}}-\delta_{i_{2}}^{j}\left(\delta_{k_{1}, \ldots, k_{\kappa}}^{l_{1}, \ldots, l_{\kappa}} Q_{u_{1}}^{l_{\kappa+1}}+\cdots+\delta_{k_{1}, \ldots, k_{\kappa}}^{l_{3}, \ldots, l_{1}} Q_{u^{i_{1}}}^{l_{2}}\right)\right] \times \\
& \times U_{l_{1}}^{i_{1}} U_{l_{2}, \ldots, l_{\kappa+1}}^{i_{2}}
\end{aligned}
$$

and where the term Remainder denotes the remaining terms in the expansion of $R_{k_{1}, k_{2}, \ldots, k_{\kappa}}^{j}$.

In $I_{6}$ the summation on the upper indices $\left(l_{1}, \ldots, l_{\kappa}\right)$ gets on all the circular permutations of $\{1,2, \ldots, \kappa\}$ except the identity. In $I_{7}$ the summation gets on all the circular permutations of $\{2,3, \ldots, \kappa+1\}$. In $I_{9}$ the summation gets on all the circular permutations of $\{1,2, \ldots, \kappa+1\}$ except the one transforming $\left(l_{1}, l_{2}, \ldots, l_{\kappa+1}\right)$ into $\left(l_{2}, l_{3}, \ldots, l_{1}\right)$. For $\kappa=3$, comparing with (7.28), we see that the formula remains valid, with the same conventions as in the case $n=1$.
3.4. Lie criterion and defining equations of $\mathfrak{S y m}(\mathscr{E})$. We recall the Lie criterion, presented in Subsection 2.6 (see Theorem 2.71 of [Ol1986]):

A vector field $X$ is an infinitesimal symmetry of the completely integrable system $(\mathscr{E})$ if and only if its prolongation $X^{(\kappa)}$ of order $\kappa$ is tangent to the skeleton $\Delta_{\mathscr{E}}$ in the jet space $\mathscr{J}_{n, m}^{\kappa}$.

The set of infinitesimal symmetries of $(\mathscr{E})$ forms a Lie algebra, since we have the relation $\left[X, X^{\prime}\right]^{(\kappa)}=\left[X^{(\kappa)}, X^{\prime(\kappa)}\right](c f .[\mathrm{Ol1986}])$. We will denote by $\mathfrak{S y m}(\mathscr{E})$ this Lie algebra. The aim of the forecoming Section is to obtain precise bounds on the dimension of the Lie algebra $\mathfrak{S y m}(\mathscr{E})$ of infinitesimal symmetries of $(\mathscr{E})$. For simplicity we start with the case $n=m=1$.
4. OPtimal UPPER BOUND ON DIM $\mathbb{K}_{\mathbb{K}} \mathfrak{S y m}(\mathscr{E})$ WHEN $n=m=1$.
4.1. Defining equations for $\mathfrak{S y m}(\mathscr{E})$. Applying the Lie criterion, the tangency condition of $X^{(\kappa)}$ to $\Delta_{\mathscr{E}}$ is equivalent to the identity:

$$
\begin{equation*}
\mathbf{R}^{\kappa}-\left[Q \frac{\partial F}{\partial x}+R \frac{\partial F}{\partial u}+\mathbf{R}^{1} \frac{\partial F}{\partial U^{1}}+\mathbf{R}^{2} \frac{\partial F}{\partial U^{2}}+\cdots+\mathbf{R}^{\kappa-1} \frac{\partial F}{\partial U^{\kappa-1}}\right] \equiv 0 \tag{7.28}
\end{equation*}
$$

on the subvariety $\Delta_{\mathscr{E}}$, that is to a formal identity in $\mathbb{K}\left\{x, u, U^{1}, \ldots, U^{\kappa-1}\right\}$, in which we replace the variable $U^{\kappa}$ by $F\left(x, u, U^{1}, \ldots, U^{\kappa-1}\right)$ in the two monomials $U^{\kappa}$ and $U^{1} U^{\kappa}$ of $\mathbf{R}^{\kappa}$, cf. Lemma 8.1. Expanding $F$ and its partial derivatives in power series of the variables $\left(U^{1}, \ldots, U^{\kappa-1}\right)$ with analytic coefficients in $(x, u)$, we may rewrite (7.28) as follows:

$$
\left\{\begin{align*}
& \sum_{\mu_{1}, \ldots, \mu_{\kappa-1} \geq 0}\left[\Phi_{\mu_{1}, \ldots, \mu_{\kappa-1}}\left(x, u,\left(Q_{x^{k} u^{l}}\right)_{k+l \leq \kappa},\left(R_{x^{k} u^{l}}\right)_{k+l \leq \kappa}\right)\right] \times  \tag{7.28}\\
& \times\left(U^{1}\right)^{\mu_{1}} \ldots\left(U^{\kappa-1}\right)^{\mu_{\kappa-1}} \equiv 0
\end{align*}\right.
$$

where the expressions

$$
\begin{equation*}
\Phi_{\mu_{1}, \ldots, \mu_{\kappa-1}}\left(x, u,\left(Q_{x^{k} u^{l}}\right)_{k+l \leq \kappa},\left(R_{x^{k} u^{l}}\right)_{k+l \leq \kappa}\right) \tag{7.28}
\end{equation*}
$$

are linear with respect to the partial derivatives $\left(\left(Q_{x^{k} u^{l}}\right)_{k+l \leq \kappa},\left(R_{x^{k} u^{l}}\right)_{k+l \leq \kappa}\right)$, with analytic coefficients in $(x, u)$. By construction these coefficients essentially depend on the expansion of $F$. The tangency condition (7.28) is equivalent to the following infinite linear system of partial differential equations, called defining equations of $\mathfrak{S y m}(\mathscr{E})$ :

$$
\begin{equation*}
\Phi_{\mu_{1}, \ldots, \mu_{\kappa-1}}\left(x, u,\left(Q_{x^{k} u^{l}}(x, u)\right)_{k+l \leq \kappa},\left(R_{x^{k} u^{l}}(x, u)\right)_{k+l \leq \kappa}\right)=0, \tag{7.28}
\end{equation*}
$$

satisfied by $(Q(x, u), R(x, u))$. The Lie method consists in studying the solutions of this linear system of partial differential equations.
4.2. Homogeneous system. As mentioned in the introduction, we focus our attention on the case $\kappa \geq 3$. Denote by $\left(\mathscr{E}_{0}\right)$ the homogeneous equation $u_{x^{\kappa}}=0$ of order $\kappa$. The general solution $u=\sum_{l=0}^{\kappa-1} \lambda_{l} x^{l}$ consists of polynomials of degree $\leq \kappa-1$ and the defining equation (7.28) reduces to $\mathbf{R}^{\kappa}=0$. Using the expression (7.28), expanding (7.28), (7.28) and considering only the coefficients of the five monomials ct., $U^{\kappa-2}, U^{\kappa-1}, U^{1} U^{\kappa-1}$ and
$U^{2} U^{\kappa-1}$, we obtain the five following partial differential equations, which are sufficient to determine $\mathfrak{S y m}\left(\mathscr{E}_{0}\right)$ :

$$
\left\{\begin{align*}
R_{x^{\kappa}} & =0  \tag{7.28}\\
R_{x^{2} u}-\frac{(\kappa-2)}{3} Q_{x^{3}} & =0 \\
R_{x u}-\frac{(\kappa-1)}{2} Q_{x^{2}} & =0 \\
R_{u^{2}}-\kappa Q_{x u} & =0 \\
Q_{u} & =0
\end{align*}\right.
$$

The general solution of this system is evidently:

$$
\left\{\begin{array}{l}
Q=A+B x+C x^{2}  \tag{7.28}\\
R=(\kappa-1) C x u+D u+E^{0}+E^{1} x+\cdots+E^{\kappa-1} x^{\kappa-1}
\end{array}\right.
$$

where the $(\kappa+4)$ constants $A, B, C, D, E^{0}, E^{1}, \ldots, E^{\kappa-1}$ are arbitrary. Computing explicitely the flows of the $(\kappa+4)$ generators $\partial / \partial x, x \partial / \partial x$, $x^{2} \partial / \partial x+(\kappa-1) x u \partial / \partial u, u \partial / \partial u, \partial / \partial u, x \partial / \partial u, \ldots, x^{\kappa-1} \partial / \partial u$, we check easily that they stabilize the graphs of polynomials of degree $\leq \kappa-1$. Moreover they span a Lie algebra of dimension $(\kappa+4)$ and the general form of a Lie symmetry is:

$$
\begin{equation*}
(x, u) \longmapsto\left(\frac{\alpha_{0}+\alpha_{1} x}{1+\varepsilon x}, \frac{\beta u+\gamma_{0}+\gamma_{1} x+\cdots+\gamma_{\kappa-1} x^{\kappa-1}}{(1+\varepsilon x)^{\kappa-1}}\right) . \tag{7.28}
\end{equation*}
$$

4.3. Nonhomogeneous system. Consider for $\kappa \geq 3$ the equation (7.28) after replacing the variable $U^{\kappa}$ by $F$. Let $\Phi\left(U^{\lambda}\right)$ denote an arbitrary term of the form $\phi(x, u) U^{\lambda}$, where $\phi(x, u)$ is an analytic function. We consider the five following terms $\Phi($ ct. $), \Phi\left(U^{\kappa-2}\right), \Phi\left(U^{\kappa-1}\right), \Phi\left(U^{1} U^{\kappa-1}\right)$ and $\Phi\left(U^{2} U^{\kappa-1}\right)$. Since some multiplications of monomials appear in the expression (7.28), we must be aware of the fact that $\Phi\left(U^{1} U^{\kappa-1}\right) \equiv$ $\Phi\left(U^{1}\right) \Phi\left(U^{\kappa-1}\right)$ and $\Phi\left(U^{2} U^{\kappa-1}\right) \equiv \Phi\left(U^{2}\right) \Phi\left(U^{\kappa-1}\right)$. Consequently in the expansion of (7.28) we must take into account the seven types of monomials $\Phi($ ct. $), \Phi\left(U^{1}\right), \Phi\left(U^{2}\right), \Phi\left(U^{\kappa-2}\right), \Phi\left(U^{\kappa-1}\right), \Phi\left(U^{1} U^{\kappa-1}\right)$ and $\Phi\left(U^{2} U^{\kappa-1}\right)$. The $(\kappa+1)$ derivatives $\partial F / \partial x, \partial F / \partial u, \partial F / \partial U^{1}, \ldots, \partial F / \partial U^{\kappa-1}$ appearing in the brackets of (7.28), and the term $F$ appearing in the expression of $\mathbf{R}^{\kappa}$ after replacing $U^{\kappa}$ by $F$ (cf. the last two monomials $U^{\kappa}$ and $U^{1} U^{\kappa}$ in (7.28)) may all contain the seven monomials ct., $U^{1}, U^{2}, U^{\kappa-2}, U^{\kappa-1}$, $U^{1} U^{\kappa-1}$ and $U^{2} U^{\kappa-1}$. For $F$ and its $(\kappa+1)$ first derivatives we use the generic simplified notation
$\Phi($ ct. $)+\Phi\left(U^{1}\right)+\Phi\left(U^{2}\right)+\Phi\left(U^{\kappa-2}\right)+\Phi\left(U^{\kappa-1}\right)+\Phi\left(U^{1} U^{\kappa-1}\right)+\Phi\left(U^{2} U^{\kappa-1}\right)$,
to name the seven monomials appearing a priori. Hence, expanding (7.28), picking up the only terms which may contain the five monomials we are interested in, and using the formula of Lemma 8.1 for $\mathbf{R}^{\lambda}(1 \leq \lambda \leq \kappa)$, we obtain the following expression:

$$
\left\{\begin{align*}
& R_{x^{\kappa}}+\left[C_{\kappa}^{2} R_{x^{2} u}-C_{\kappa}^{3} Q_{x^{3}}\right] U^{\kappa-2}+\left[C_{\kappa}^{1} R_{x u}-C_{\kappa}^{2} Q_{x^{2}}\right] U^{\kappa-1}+  \tag{7.28}\\
&+\left[C_{\kappa}^{1} R_{u^{2}}-\kappa^{2} Q_{x u}\right] U^{1} U^{\kappa-1}+\left[-C_{\kappa+1}^{2} Q_{u}\right] U^{2} U^{\kappa-1}+ \\
&+\left\{R_{u}-C_{\kappa}^{1} Q_{x}+\left[-C_{\kappa+1}^{1} Q_{u}\right] U^{1}\right\} \times \\
& \times\left\{\Phi(\text { ct. })+\Phi\left(U^{1}\right)+\Phi\left(U^{2}\right)+\Phi\left(U^{\kappa-2}\right)+\Phi\left(U^{\kappa-1}\right)+\Phi\left(U^{1} U^{\kappa-1}\right)+\Phi\left(U^{2} U^{\kappa-1}\right)\right\}- \\
&-\left\{Q+R+R_{x}+\left[R_{u}-Q_{x}\right] U^{1}+R_{x^{2}}+\left[2 R_{x u}-Q_{x^{2}}\right] U^{1}+\right. \\
&+\left[R_{u}-2 Q_{x}\right] U^{2}+\cdots+R_{x^{\kappa-3}}+\left[C_{\kappa-3}^{1} R_{x^{\kappa-4} u}-Q_{x^{\kappa-3}}\right] U^{1}+ \\
&+\left[C_{\kappa-3}^{2} R_{x^{\kappa-5} u}-C_{\kappa-3}^{1} Q_{x^{\kappa-4}}\right] U^{2}+R_{x^{\kappa-2}}+ \\
&+\left[C_{\kappa-2}^{1} R_{x^{\kappa-3} u}-Q_{x^{\kappa-2}}\right] U^{1}+\left[C_{\kappa-2}^{2} R_{x^{\kappa-4} u}-C_{\kappa-2}^{1} Q_{x^{\kappa-3}}\right] U^{2}+ \\
&+\left[R_{u}-C_{\kappa-2}^{1} Q_{x}\right] U^{\kappa-2}+\left[-C_{\kappa-1}^{1} Q_{u}\right] U^{1} U^{\kappa-2}+ \\
&+R_{x^{\kappa-1}}+\left[C_{\kappa-1}^{1} R_{x^{\kappa-2} u}-Q_{x^{\kappa-1}}\right] U^{1}+ \\
&+\left[C_{\kappa-1}^{2} R_{x^{\kappa-3} u}-C_{\kappa-1}^{1} Q_{x^{\kappa-2}}\right] U^{2}+\left[C_{\kappa-1}^{1} R_{x u}-C_{\kappa-1}^{2} Q_{x^{2}}\right] U^{\kappa-2}+ \\
&+\left[C_{\kappa-1}^{1} R_{u^{2}}-(\kappa-1)^{2} Q_{x u}\right] U^{1} U^{\kappa-2}+\left[R_{u}-C_{\kappa-1}^{1} Q_{x}\right] U^{\kappa-1}+ \\
&\left.+\left[-C_{\kappa}^{1} Q_{u}\right] U^{1} U^{\kappa-1}\right\} \times \\
& \times\left\{\Phi(\text { ct. })+\Phi\left(U^{1}\right)+\Phi\left(U^{2}\right)+\Phi\left(U^{\kappa-2}\right)+\Phi\left(U^{\kappa-1}\right)+\Phi\left(U^{1} U^{\kappa-1}\right)+\Phi\left(U^{2} U^{\kappa-1}\right)\right\} \\
&+\operatorname{Remainder} \equiv 0 .
\end{align*}\right.
$$

Here the term Remainder consists of the monomials, in the jet variables, different from the five ones we are concerned with. The first four lines before the sign "-" develop $\mathbf{R}^{\kappa}$ and the third line consists of the factor $F$ replaced by (7.28). In the last line (note that this is multiplied by the nine preceding lines) we replaced the $(\kappa+1)$ first partial derivatives of $F$ appearing in (7.28) by the term (7.28) which we factorized.

By expanding the product appearing in this expression (7.28), and equaling to zero the coefficients of the five monomials ct., $U^{\kappa-2}, U^{\kappa-1}, U^{1} U^{\kappa-1}$
and $U^{2} U^{\kappa-1}$, we obtain the five following partial differential equations (7.28)

$$
\left\{\begin{aligned}
& R_{x^{\kappa}}=\Pi\left(x, u, Q, Q_{x}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}\right) \\
& C_{\kappa}^{2} R_{x^{2} u}-C_{\kappa}^{3} Q_{x^{3}}=\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}, R_{x u}\right) \\
& C_{\kappa}^{1} R_{x u}-C_{\kappa}^{2} Q_{x^{2}}=\Pi\left(x, u, Q, Q_{x}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}\right) \\
& C_{\kappa}^{1} R_{u^{2}}-\kappa^{2} Q_{x u}=\Pi\left(x, u, Q, Q_{x}, \ldots, Q_{x^{\kappa-1}}, Q_{u}, R, R_{x}, \ldots R_{x^{\kappa-1}}\right. \\
&\left.R_{u}, R_{x u}, \ldots, R_{x^{\kappa-2} u}\right) \\
&-C_{\kappa+2}^{2} Q_{u}=\Pi\left(x, u, Q, Q_{x}, \ldots, Q_{x^{\kappa-2}}, R, R_{x}, \ldots R_{x^{\kappa-1}}\right. \\
&\left.R_{u}, R_{x u}, \ldots, R_{x^{\kappa-3} u}\right)
\end{aligned}\right.
$$

Here by convention $\Pi$ denotes any linear quantity in $Q, R$ and some of their derivatives, of the form

$$
\left\{\begin{array}{l}
\Pi\left(x, u, Q_{x^{a_{1}} u^{b_{1}}}, \ldots, Q_{x^{a_{p}} u^{b_{p}}}, R_{x^{c_{1}} u^{d_{1}}}, \ldots, R_{x^{c_{q}} u^{d_{q}}}\right)=  \tag{7.28}\\
\quad=\sum_{j=1}^{p} \phi_{i}(x, u) Q_{x^{a_{i}} u^{b_{i}}}(x, u)+\sum_{j}^{q} \psi_{j}(x, u) R_{x^{c_{j}} u^{d_{j}}}(x, u),
\end{array}\right.
$$

where $\phi_{i}$ and $\psi_{j}$ are analytic in $(x, u)$. For instance, the differentiation of $\Pi\left(x, u, Q, R, R_{u}\right)$ with respect to $x$ gives the expression $\Pi\left(x, u, Q, Q_{x}, R, R_{x}, R_{x u}\right)$. Let us introduce the following collection of $(\kappa+4)$ partial derivatives of $(Q, R)$ defined by $J:=\left(Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}\right)$. The aim is now to make linear substitutions on the system (7.28) to obtain the system (7.28) where the five second members depend only on the collection $J$. The desired estimate $\operatorname{dim}_{\mathbb{K}} \mathfrak{S y m}(\mathscr{E}) \leq \kappa+4$ will follow from (7.28).

Let us differentiate the third equation of (7.28) with respect to $x$. Dividing by $C_{\kappa}^{1}$ we obtain:
(7.28)

$$
R_{x^{2} u}-\frac{(\kappa-1)}{2} Q_{x^{3}}=\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa}}, R_{u}, R_{x u}\right)
$$

Solving $R_{x^{2} u}$ and $Q_{x^{3}}$ by the second equality in (7.28) and by (4.3) we find

$$
\left\{\begin{align*}
Q_{x^{3}} & =\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa}}, R_{u}, R_{x u}\right),  \tag{7.28}\\
R_{x^{2} u} & =\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa}}, R_{u}, R_{x u}\right) .
\end{align*}\right.
$$

Replacing $R_{x^{k}}$ by its value given by the first equality in (7.28) we obtain for $Q_{x^{3}}$ :

$$
\begin{equation*}
Q_{x^{3}}=\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{k-1}}, R_{u}, R_{x u}\right) \tag{7.28}
\end{equation*}
$$

If we write the third equality in (7.28) as

$$
\begin{equation*}
R_{x u}=\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}\right), \tag{7.28}
\end{equation*}
$$

we may replace $R_{x u}$ in (7.28). This gives the desired dependence of $Q_{x^{3}}$ on the collection $J$ :

$$
\begin{equation*}
Q_{x^{3}}=\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}\right) \tag{7.28}
\end{equation*}
$$

We may now differentiate the equalities (7.28) and (7.28) with respect to $x$ up to the order $l$. At each differentiation we replace $Q_{x^{3}}, R_{x u}$ and $R_{x^{\kappa}}$ by their values in (7.28), in (7.28) and in the first equality in (7.28) respectively. We obtain for $l \in \mathbb{N}$ :

$$
\left\{\begin{align*}
Q_{x^{l+3}} & =\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}\right)  \tag{7.28}\\
R_{x^{l+1} u} & =\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}\right)
\end{align*}\right.
$$

Replacing these values in the fifth equality of (7.28), we obtain

$$
\begin{equation*}
Q_{u}=\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}\right) \tag{7.28}
\end{equation*}
$$

By replacing the fourth equality of (7.28) we obtain finally

$$
\begin{equation*}
R_{u^{2}}=\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}\right) \tag{7.28}
\end{equation*}
$$

To summarize, using the first equality of (7.28), using (7.28), (7.28), (7.28) and (7.28), we obtained the desired system:

$$
\left\{\begin{align*}
R_{x^{\kappa}} & =\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}\right)  \tag{7.28}\\
Q_{u} & =\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}\right) \\
R_{u^{2}} & =\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}\right) \\
R_{x u} & =\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}\right) \\
Q_{x^{3}} & =\Pi\left(x, u, Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}\right) .
\end{align*}\right.
$$

We recall that the terms $\Pi$ are linear expressions of the form (7.28). Let us differentiate every equation of system (7.28) with respect to $x$ at an arbitrary order and let us replace in the right hand side the terms $R_{x^{\kappa}}, R_{x u}$ and $Q_{x^{3}}$ that may appear at each step by their value in (7.28), and then differentiate with respect to $u$ at an arbitrary order. We deduce that all the partial derivatives of the five functions $R_{x^{\kappa}}, Q_{u}, R_{u^{2}}, R_{x u}$ and $Q_{x^{3}}$ are also linear functions of the $(\kappa+4)$ partial derivatives $\left(Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{k-1}}, R_{u}\right)$. Thus the analytic functions $Q$ and $R$ are determined uniquely by the value at the origin of the $(\kappa+4)$ partial derivatives $\left(Q, Q_{x}, Q_{x^{2}}, R, R_{x}, \ldots, R_{x^{\kappa-1}}, R_{u}\right)$. This ends the proof of the inequality $\operatorname{dim}_{\mathbb{K}} \mathfrak{S y m}(\mathscr{E}) \leq \kappa+4$.

## 5. Optimal upper bound on dim $\mathbb{K} \mathfrak{S y m}(\mathscr{E})$ IN THE GENERAL DIMENSIONAL CASE

5.1. Defining equations for $\mathfrak{S y m}(\mathscr{E})$. In the general dimensional case, the tangency condition of the prolongation $X^{\kappa}$ of $X$ to the skeleton gives the
following equations for $j=1, \ldots, m$ and $k_{1}, \ldots, k_{\kappa}=1, \ldots, n$ :

$$
\left\{\begin{array}{l}
\mathbf{R}_{k_{1}, \ldots, k_{\kappa}}^{j}-\left[\sum_{l=1}^{n} Q^{l} \frac{\partial F_{k_{1}, \ldots, k_{\kappa}}^{j}}{\partial x_{l}}+\sum_{i=1}^{m} R^{i} \frac{\partial F_{k_{1}, \ldots, k_{\kappa}}^{j}}{\partial u^{i}}+\right.  \tag{7.28}\\
\left.\quad+\sum_{i_{1}} \sum_{l_{1}} \mathbf{R}_{l_{1}}^{i_{1}} \frac{\partial F_{k_{1}, \ldots, k_{\kappa}}^{j}}{\partial U_{l_{1}}^{i_{1}}}+\cdots+\sum_{i_{1}} \sum_{l_{1}, \ldots, l_{\kappa-1}} \mathbf{R}_{l_{1}, \ldots, l_{\kappa-1}}^{i_{1}} \frac{\partial F_{k_{1}, \ldots, k_{\kappa}}^{j}}{\partial U_{l_{1}, \ldots, l_{\kappa-1}}^{i_{1}}}\right] \equiv 0,
\end{array}\right.
$$

on $\Delta_{\mathscr{E}}$, by replacing the variables $U_{l_{1}, \ldots, l_{\kappa}}^{i_{1}}$ by $F_{l_{1}, \ldots, l_{\kappa}}^{i_{1}}$ wherever they appear. Let us expand $F_{k_{1}, \ldots, k_{\kappa}}^{j}$ and their partial derivatives and use the fact that $\mathbf{R}_{k_{1}, \ldots, k_{\lambda}}^{j}$ are polynomials expressions of the jets variables $\left(U_{l_{1}}^{i_{1}}, \ldots, U_{l_{1}, \ldots, l_{\lambda}}^{i_{1}}\right)$, with coefficients being linear expressions of the partial derivatives of order $\leq \lambda+1$ of $Q^{l}$ and $R^{j}$. We obtain for $j=1, \ldots, m$ and $k_{1}, \ldots, k_{\kappa}=1, \ldots, n$ some identities of the form
(7.28)

$$
\left\{\begin{aligned}
& \sum_{i_{1}, \ldots, l_{1}, \ldots} \Phi_{k_{1}, \ldots, k_{\kappa} ; l_{1}, \ldots \ldots . .}^{j ; i_{1}, \ldots \ldots .}\left(x, u,\left(Q_{x^{\alpha} u^{\beta}}^{l}\right)_{1 \leq l \leq n,|\alpha|+|\beta| \leq \kappa+1},\left(R_{x^{\alpha} u^{\beta}}^{j}\right)_{1 \leq j \leq m,|\alpha|+|\beta| \leq \kappa+1}^{j}\right) \times \\
& \times U_{l_{1}}^{i_{1}} \ldots U_{l_{\mu_{1}}}^{i_{\mu_{1}}} \times U_{l_{\mu_{1}}+1, l_{\mu_{1}}+2}^{i_{\mu_{1}}+1} \cdots U_{l_{\mu_{1}+2 \mu_{2}-1}}^{i_{\mu_{1}+\mu_{2}-1}} U_{l_{\mu_{1}+2 \mu_{2}}}^{i_{\mu_{1}+\mu_{2}}} \times \cdots \cdots \equiv 0,
\end{aligned}\right.
$$

satisfied if and only if the functions $Q^{l}$ and $R^{j}$ are solutions of the following system of partial differential equations
$\Phi_{k_{1}, \ldots, k_{\kappa} ; l_{1}, \ldots \ldots . .}^{j, i_{1}, \ldots \ldots} .\left(x, u,\left(Q_{x^{\alpha} u^{\beta}}^{l}\right)_{1 \leq l \leq n,|\alpha|+|\beta| \leq \kappa+1},\left(R_{x^{\alpha} u^{\beta}}^{j}\right)_{1 \leq j \leq m,|\alpha|+|\beta| \leq \kappa+1}\right)=0$.
5.2. Homogeneous system. We start by giving the general form of the symmetries of the homogeneous system in the case $\kappa=2$. Then we prove the equality $\operatorname{dim}_{\mathbb{K}}\left(\mathfrak{S y m}\left(\mathscr{E}_{0}\right)\right)=n^{2}+2 n+m^{2}+m C_{n+\kappa-1}^{\kappa-1}$ in the case $\kappa \geq 3$.
In the case $\kappa=2$ we obtain:

$$
\left\{\begin{align*}
& Q^{l}(x, u)=A^{l}+\sum_{k_{1}=1}^{n} B_{k_{1}}^{l} x_{k_{1}}+\sum_{i_{1}=1}^{m} C_{i_{1}}^{l} u^{i_{1}}+  \tag{7.28}\\
&+\sum_{k_{1}=1}^{n} D_{k_{1}} x_{l} x_{k_{1}}+\sum_{i_{1}=1}^{m} E_{i_{1}} x_{l} u^{i_{1}} \\
& R^{j}(x, u)=F^{j}+\sum_{k_{1}=1}^{n} G_{k_{1}}^{j} x_{k_{1}}+\sum_{i_{1}=1}^{m} H_{i_{1}}^{j} u^{i_{1}}+ \\
&+\sum_{k_{1}=1}^{n} D_{k_{1}} x_{k_{1}} u^{j}+\sum_{i_{1}=1}^{m} E_{i_{1}} u^{i_{1}} u^{j}
\end{align*}\right.
$$

Here the $(n+m)(n+m+2)$ constants $A^{l}, B_{k_{1}}^{l}, C_{i_{1}}^{l}, D_{k_{1}}, E_{i_{1}}, F^{j}, G_{k_{1}}^{j}, H_{i_{1}}^{j} \in$ $\mathbb{K}$ are arbitrary. Moreover one can check that the vector space spanned by the $(n+m)(n+m+2)$ vector fields

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial x_{k_{1}}}, x_{k_{1}} \frac{\partial}{\partial x_{k_{2}}}, u^{i_{1}} \frac{\partial}{\partial x_{k_{1}}},  \tag{7.28}\\
x_{k_{1}}\left(x_{1} \frac{\partial}{\partial x_{1}}+\cdots+x_{n} \frac{\partial}{\partial x_{n}}+u^{1} \frac{\partial}{\partial u^{1}}+\cdots+u^{m} \frac{\partial}{\partial u^{m}}\right), \\
u^{i_{1}}\left(x_{1} \frac{\partial}{\partial x_{1}}+\cdots+x_{n} \frac{\partial}{\partial x_{n}}+u^{1} \frac{\partial}{\partial u^{1}}+\cdots+u^{m} \frac{\partial}{\partial u^{m}}\right), \\
\frac{\partial}{\partial u^{i_{1}}}, x_{k_{1}} \frac{\partial}{\partial u^{i_{1}}}, u^{i_{1}} \frac{\partial}{\partial u^{i_{2}}}
\end{array}\right.
$$

is stable under the Lie bracket action and that the flow of each of these generators is a Lie symmetry of the system $\left(\mathscr{E}_{0}\right)$. This proves that $\mathfrak{S y m}\left(\mathscr{E}_{0}\right)$ is indeed a Lie algebra with dimension $(n+m)(n+m+2)$. Finally the corresponding transformations close to the identity mapping are projective, represented by the formula:

$$
\left\{\begin{align*}
(x, u) \longmapsto & \left(\left(\frac{\alpha_{l, 0}+\sum_{k=1}^{n} \alpha_{l, k} x_{k}+\sum_{i=1}^{m} \alpha_{l, n+i} u^{i}}{1+\sum_{k=1}^{n} \gamma_{k} x_{k}+\sum_{i=1}^{m} \gamma_{n+i} u^{i}}\right)_{1 \leq l \leq n}\right.  \tag{7.28}\\
& \left.\left(\frac{\beta_{j, 0}+\sum_{k=1}^{n} \beta_{j, k} x_{k}+\sum_{i=1}^{m} \beta_{j, n+i} u^{i}}{1+\sum_{k=1}^{n} \gamma_{k} x_{k}+\sum_{i=1}^{m} \gamma_{n+i} u^{i}}\right)_{1 \leq j \leq m}\right)
\end{align*}\right.
$$

It is clear that these transformations preserve all the solutions of $\left(\mathscr{E}_{0}\right)$ : $u_{x_{k_{1}} x_{k_{2}}}^{j}=0$, the graphs of affine maps from $\mathbb{K}^{n}$ to $\mathbb{K}^{m}$.

In the case $\kappa \geq 3$ we consider the homogeneous system $\left(\mathscr{E}_{0}\right)$ in which the second members $F_{k_{1}, \ldots, k_{\kappa}}^{j}$ vanish identically. Its solutions are the graphs of polynomial maps of degree $\leq(\kappa-1)$ from $\mathbb{K}^{n}$ to $\mathbb{K}^{m}$. The defining equations of its Lie algebra of infinitesimal symmetries are $\mathbf{R}_{k_{1}, \ldots, k_{\kappa}}^{j}=0$, after having replaced the variables $U_{l_{1}, \ldots, l_{\kappa}}^{i_{1}}$ by $0=F_{l_{1}, \ldots, l_{\kappa}}^{i_{1}}$ in $I_{8}$ and $I_{9}$ in (7.28). We will keep in this system the only equations coming from the vanishing of the coefficients of the five families of monomials ct., $U_{l_{1}, \ldots, l_{\kappa-2}}^{i_{1}}, U_{l_{1}, \ldots, l_{\kappa-1}}^{i_{1}}$, $U_{l_{1}}^{i_{1}} U_{l_{2}, \ldots, l_{\kappa}}^{i_{2}}$ and $U_{l_{1}, l_{2}}^{i_{1}} U_{l_{3}, \ldots, l_{\kappa+1}}^{i_{2}}$ (this is inspired from the computations in Subsection 4.2). The coefficients of these five monomials families already appear in the expression (7.28). Moreover we fix $l_{1}=l_{2}=\cdots=l_{\kappa+1}=l$ and $i_{1}=i_{2}$, except for the coefficient of the monomial $U_{l}^{i_{1}} U_{l, \ldots, l}^{i_{2}}$, where we fix first $i_{1}=i_{2}$ and then $i_{1} \neq i_{2}$. This provides the six partial differential
linear equations:
(7.28)

$$
0=\kappa \delta_{k_{1}, \ldots, k_{k}}^{l, \ldots \ldots, l} R_{u^{i_{1}} u^{i_{1}}}^{j}-\kappa \delta_{i_{1}}^{j}\left(\sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-1}} \delta_{k_{\sigma(1)}, \ldots, k_{\sigma(\kappa-1)}}^{l, \ldots \ldots \ldots, l} Q_{x_{k_{\sigma(\kappa)}}}^{l} u^{i_{1}}\right)
$$

$$
0=2 \kappa \delta_{k_{1}, \ldots, k_{\kappa}}^{l, \ldots \ldots, l} R_{u^{i_{1}} u^{i_{2}}}^{j}-\kappa \delta_{i_{1}}^{j}\left(\sum_{\sigma \in \mathfrak{S}_{k}^{\kappa-1}} \delta_{k_{\sigma(1)}^{l}, \ldots, k_{\sigma(\kappa-1)}}^{l, \ldots \ldots \ldots,} Q_{x_{k_{\sigma(\kappa)}} u^{i_{2}}}^{l}\right)-
$$

$$
-\kappa \delta_{i_{2}}^{j}\left(\sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-1}} \delta_{k_{\sigma(1), \ldots, k_{\sigma(\kappa-1)}}^{l, \ldots \ldots \ldots, l} Q_{x_{k_{\sigma(\kappa)}}}^{l} u^{i_{1}}}\right), \quad i_{1} \neq i_{2}
$$

$$
0=-C_{\kappa+1}^{2} \delta_{i_{1}}^{j} \delta_{k_{1}, \ldots, k_{\kappa}}^{l, \ldots \ldots, l} Q_{u^{i_{1}}}^{l}
$$

To solve the system (7.28) we fix the indices $k_{1}=\cdots=k_{\kappa}=l$ and $j=i_{1}$ in the sixth equation, implying $Q_{u^{i_{1}}}^{l}=0$. Hence the terms following $\delta_{i_{1}}^{j}$ and $\delta_{i_{2}}^{j}$ in the fourth and in the fifth equations vanish identically. Let us choose the indices $k_{1}=\cdots=k_{\kappa}$ in the fourth and the fifth equations (this last equation is satisfied only for $i_{1} \neq i_{2}$ ). We obtain first three simple equations, without any restriction on the indices:

$$
\left\{\begin{array}{l}
0=R_{x_{k_{1} x_{k_{2}} \cdots x_{k_{k}}}}^{j}  \tag{7.28}\\
0=Q_{u^{i_{1}}}^{l} \\
0=R_{u^{i_{1} u^{i} i_{2}}}^{j}
\end{array}\right.
$$

Finally we specify the indices in the third equation of (7.28) as follows: $l=k_{\kappa}=\cdots=k_{3}=k_{2}=k_{1}$; then $l=k_{\kappa}=\cdots=k_{3}=k_{2} \neq k_{1} ;$ finally $l=k_{\kappa}=\cdots=k_{3}, k_{3} \neq k_{2}, k_{3} \neq k_{1}$. This gives the three following

$$
\begin{aligned}
& -\delta_{i_{1}}^{j}\left(\sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-3}} \delta_{k_{\sigma(1)}, \ldots, \ldots, k_{\sigma(\kappa-3)}}^{l, \ldots \ldots, l} Q_{x_{k_{\sigma(\kappa-2)}} x_{k_{\sigma(\kappa-1)}} x_{k_{\sigma(k)}}}^{l}\right), \\
& 0=\sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-1}} \delta_{k_{\sigma(1)}, \ldots, k_{\sigma(\kappa-1)}}^{l, \ldots \ldots \ldots, l} R_{x_{k_{\sigma(\kappa)}}}^{j} u^{i_{1}}-
\end{aligned}
$$

equations:

$$
\left\{\begin{array}{l}
0=C_{\kappa}^{1} R_{x_{k_{1}} u^{i_{1}}}^{j}-C_{\kappa}^{2} \delta_{i_{1}}^{j} Q_{x_{k_{1}} x_{k_{1}}}^{k_{1}},  \tag{7.28}\\
0=R_{x_{k_{1}} u^{i_{1}}}^{j}-C_{\kappa-1}^{1} \delta_{i_{1}}^{j} Q_{x_{k_{1} x_{k_{2}}}}^{k_{2}}, \quad k_{2} \neq k_{1}, \\
0=-\delta_{i_{1}}^{j} Q_{x_{k_{1} x_{k_{2}}}}^{k_{3}}, \quad k_{3} \neq k_{1}, \quad k_{3} \neq k_{2} .
\end{array}\right.
$$

We specify the indices in the second equation of (7.28) as follows: $l=$ $k_{\kappa}=\cdots=k_{3}=k_{2}=k_{1}$; then $l=k_{\kappa}=\cdots=k_{3}=k_{2} \neq k_{1}$; then $l=k_{\kappa}=\cdots=k_{3}, k_{3} \neq k_{2}, k_{3} \neq k_{1}$; finally $l=k_{\kappa}=\cdots=k_{4}, l \neq k_{1}$, $l \neq k_{2}, l \neq k_{3}$. This gives the four following equalities:

$$
\left\{\begin{array}{l}
0=C_{\kappa}^{2} R_{x_{k_{1}} x_{k_{1}} u_{1}}^{j}-C_{\kappa}^{3} \delta_{i_{1}}^{j} Q_{x_{k_{1}} x_{k_{1}} x_{k_{1}}}^{k_{1}},  \tag{7.28}\\
0=C_{\kappa-1}^{1} R_{x_{k_{1}} x_{k_{2}} u^{i} i_{1}}^{j}-C_{\kappa-1}^{2} \delta_{i_{1}}^{j} Q_{x_{k_{1}} x_{k_{2}} x_{k_{2}}}^{k_{2}}, \quad k_{2} \neq k_{1}, \\
0=R_{x_{k_{1} x_{k_{2}} u_{1}}^{i_{1}}-C_{\kappa-2}}^{j} \delta_{i_{1}}^{j} Q_{x_{k_{1} x_{k_{2}} x_{k_{3}}}^{k_{3}}, \quad k_{3} \neq k_{1}, \quad k_{3} \neq k_{2},}^{0=-\delta_{i_{1}}^{j} Q_{x_{k_{1} x_{k_{2}} x_{k_{3}}}, \quad l \neq k_{1}, \quad l \neq k_{2}, \quad l \neq k_{3} .}^{l} .} \begin{array}{l}
\end{array}, \quad l \\
0
\end{array}\right.
$$

Let us differentiate now the equations (7.28) with respect to the variables $x_{l}$ as follows: we differentiate $(7.28)_{1}$ with respect to $x_{k_{1}}$; then we differentiate $(7.28)_{2}$ with respect to $x_{k_{2}}$; finally we differentiate $(7.28)_{3}$ with respect to $x_{k_{3}}$. This gives the three following equations:

$$
\left\{\begin{array}{l}
0=C_{\kappa}^{1} R_{x_{k_{1}} x_{k_{1}} u^{i_{1}}}^{j}-C_{\kappa}^{2} \delta_{i_{1}}^{j} Q_{x_{k_{1}} x_{k_{1} x_{k_{1}}}}^{k_{1}},  \tag{7.28}\\
0=R_{x_{k_{1}} x_{k_{2}} u_{1}}^{j}-C_{\kappa-1}^{1} \delta_{i_{1}}^{j} Q_{x_{k_{1} x_{k_{2}} x_{k_{2}}}, \quad k_{2} \neq k_{1},}^{0=-\delta_{i_{1}}^{j} Q_{x_{k_{1}} x_{k_{2}} x_{k_{3}}}^{k_{3}}, \quad k_{3} \neq k_{1}, \quad k_{3} \neq k_{2} .} .
\end{array}\right.
$$

The seven equations given by the systems (7.28) and (7.28) may be considered as three systems of two equations (of two variables) with a nonzero determinant, to which we add the last equation (7.28) ${ }_{4}$. We get immediately:

It follows from these relations and from the relations $Q_{u^{i_{1}}}^{l}=R_{u^{i_{1}} u^{i_{2}}}^{j}=0$ obtained in (7.28) that all the third order partial derivatives of $Q^{l}$ vanish identically, this being also satisfied by the third order partial derivatives of
$R^{j}$ containing at least one partial derivative with respect to $u^{i_{1}}$ :

$$
\left\{\begin{array}{l}
0=Q_{x_{k_{1}} x_{k_{2}} x_{k_{3}}}^{l}=Q_{x_{k_{1}} x_{k_{2}} u^{i_{1}}}^{l}=Q_{x_{k_{1} u^{i} 1}^{l} u^{i_{2}}}^{l}=Q_{u^{i_{1}} u^{i_{2}} u^{i_{3}}}^{j},  \tag{7.28}\\
0=R_{x_{k_{1} x_{k_{2}}} u^{i_{1}}}^{j}=R_{x_{k_{1} u^{1} u^{i_{2}}}^{j}=R_{u^{i_{1} u^{i_{2}} u^{i_{3}}} .}^{j} .} .
\end{array}\right.
$$

It follows from the equations (7.28) and (7.28) that all the functions $Q^{l}$ are polynomials of degree $\leq 2$ with respect to the variables $x_{k_{1}}$ and all the functions $R^{j}$ are a sum of a polynomial of degree $\leq(\kappa-1)$ in the variables $x_{k_{1}}$ and of monomials of the form $u^{i_{1}}$ and $x_{k_{1}} u^{i_{1}}$. Let us develop now the relations (7.28) separately for $j=i_{1}$ and $j \neq i_{1}$. We obtain the five equations:

$$
\left\{\begin{array}{l}
0=C_{\kappa}^{1} R_{x_{k_{1}} u^{i_{1}}}^{i_{1}}-C_{\kappa}^{2} Q_{x_{k_{1}} x_{k_{1}}}^{k_{1}},  \tag{7.28}\\
0=C_{\kappa}^{1} R_{x_{k_{1}} u_{1}}^{j}, \quad j \neq i_{1}, \\
0=R_{x_{k_{1}} u^{i_{1}}}^{i_{1}}-C_{\kappa-1}^{1} Q_{x_{k_{1}} x_{k_{2}}}^{k_{2}}, \quad k_{2} \neq k_{1}, \\
0=R_{x_{k_{1} u_{1}} u_{1}}^{j}, \quad j \neq i_{1}, \\
0=-Q_{x_{k_{1} x_{k_{2}}}}^{k_{3}}, \quad k_{3} \neq k_{1}, \quad k_{3} \neq k_{2} .
\end{array}\right.
$$

According to the equations (7.28), (7.28), (7.28), we have the following form of the general solution:
(7.28)

$$
\left\{\begin{aligned}
Q^{l}(x, u)= & A^{l}+\sum_{k_{1}=1}^{n} B_{k_{1}}^{l} x_{k_{1}}+\sum_{k_{1}=1}^{n} C_{k_{1}} x_{k_{1}} x_{l}, \\
R^{j}(x, u)= & \sum_{k_{1}=1}^{n}(\kappa-1) C_{k_{1}} x_{k_{1}} u^{j}+\sum_{i_{1}=1}^{m} D_{i_{1}}^{j} u^{i_{1}}+E^{j, 0}+\sum_{k_{1}=1}^{n} E_{k_{1}}^{j, 1} x_{k_{1}}+ \\
& +\cdots+\sum_{1 \leq k_{1} \leq \cdots \leq k_{\kappa-1} \leq n} E_{k_{1}, \ldots, k_{k-1}}^{j, \kappa-1} x_{k_{1}} \cdots x_{k_{\kappa-1}} .
\end{aligned}\right.
$$

Here the $n+n^{2}+n+m^{2}+m C_{n+\kappa-1}^{\kappa-1}$ constants $A^{l}, B_{k_{1}}^{l}, C_{k_{1}}, D_{i_{1}}^{j}, E^{j, 0}, E_{k_{1}}^{j, 1}$, $\ldots, E_{k_{1}, \ldots, k_{\kappa-1}}^{j, \kappa-1} \in \mathbb{K}$ are arbitrary. Moreover one can check that the vector space spanned by the vector fields
(7.28)

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial x_{k_{1}}}, x_{k_{1}} \frac{\partial}{\partial x_{k_{2}}}, \\
x_{k_{1}}\left(x_{1} \frac{\partial}{\partial x_{1}}+\cdots+x_{n} \frac{\partial}{\partial x_{n}}+(\kappa-1)\left(u^{1} \frac{\partial}{\partial u^{1}}+\cdots+u^{m} \frac{\partial}{\partial u^{m}}\right)\right), \\
u^{i_{1}} \frac{\partial}{\partial u^{i_{2}}}, \frac{\partial}{\partial u^{i_{1}}}, x_{k_{1}} \frac{\partial}{\partial u^{i_{1}}}, \ldots \cdots, x_{k_{1}} \cdots x_{k_{k_{k-1}}} \frac{\partial}{\partial u^{i_{1}}},
\end{array}\right.
$$

is stable under the Lie bracket action and that the flow of each of these generators is indeed a Lie symmetry of the system $\left(\mathscr{E}_{0}\right)$. Finally the Lie
symmetries of $\left(\mathscr{E}_{0}\right)$ have the following form:

$$
\begin{align*}
& (x, u) \longmapsto\left(\left(\frac{\alpha_{l, 0}+\sum_{k=1}^{n} \alpha_{l, k} x_{k}}{1+\sum_{k=1}^{n} \varepsilon_{k} x_{k}}\right)_{1 \leq l \leq n},\right.  \tag{7.28}\\
& \left.\left(\frac{\sum_{i_{1}=1}^{m} \beta_{i_{1}}^{j} u_{i_{1}}+\gamma^{0, j}+\sum_{k_{1}=1}^{n} \gamma_{k_{1}}^{1, j} x_{k_{1}}+\cdots+\sum_{k_{1} \leq \cdots \leq k_{k-1}} \gamma_{k_{1}, \ldots, k_{k-1}}^{\kappa-1, j} x_{k_{1}} \cdots x_{k_{k-1}}}{\left[1+\sum_{k=1}^{n} \varepsilon_{k} x_{k}\right]^{\kappa-1}}\right)_{1 \leq j \leq m}\right) .
\end{align*}
$$

We note again that these transformations preserve the solutions of $\left(\mathscr{E}_{0}\right)$ : $u_{x_{k_{1}} \cdots x_{k_{\kappa}}}^{j}=0$, namely the graphs of polynomial maps of degree $\leq(\kappa-1)$ from $\mathbb{K}^{n}$ to $\mathbb{K}^{m}$.
5.3. Nonhomogeneous system. Let $\kappa \geq 3$. Let us expand the defining equations (7.28) as done in (7.28). We will write only the coefficients of the five monomial families ct., $U_{l_{1}, \ldots, l_{\kappa-2}}^{i_{1}}, U_{l_{1}, \ldots, l_{\kappa-1}}^{i_{1}}, U_{l_{1}}^{i_{1}} U_{l_{2}, \ldots, l_{\kappa}}^{i_{2}}$ and $U_{l_{1}, l_{2}}^{i_{1}} U_{l_{3}, \ldots, l_{\kappa+1}}^{i_{2}}$. Moreover, we fix always $l_{1}=l_{2}=\cdots=l_{\kappa}=l_{\kappa+1}=l$ and $i_{1}=i_{2}$, except for the fourth family of monomials where we distinguish the two cases $i_{1}=i_{2}$ and $i_{1} \neq i_{2}$. Thus we obtain six linear equations of partial derivatives, the members on the left side (coming from the expression of $\mathbf{R}_{k_{1}, \ldots, k_{\kappa}}^{j_{1}}$ given by Lemma 8.1) coincide with the members on the right hand side of (7.28). Furthermore, the members on the right hand side are exactly the same as those obtained in (7.28), with more indices! We use the letters $l^{\prime}, k_{1}^{\prime}, \ldots, k_{\kappa}^{\prime}=1, \ldots, n$ and $j^{\prime}, i_{1}^{\prime}=1, \ldots, m$ for the indices of the arguments of the expressions $\Pi$, obtaining the six following equations, which generalize the equations (7.28):
$[1]: \quad R_{x_{k_{1} x_{k_{2}} \cdots x_{k_{k}}}}^{j}=\Pi\left(x, u, Q^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}}^{l^{\prime}}, R^{j^{\prime}}, R_{x_{k_{1}^{\prime}}}^{j^{\prime}}, \ldots, R_{\left.x_{k_{1}^{\prime} \cdots x_{k_{k-1}^{\prime}}}^{j^{\prime}}, R_{u_{1}^{i_{1}^{\prime}}}^{j^{\prime}}\right), ~}^{\text {, }}\right.$
$[1]: \quad R_{x_{k_{1} x_{k_{2}} \cdots x_{k_{\kappa}}}^{j}}^{j}=\Pi\left(x, u, Q^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}}^{l^{\prime}}, R^{j^{\prime}}, R_{x_{k_{1}^{\prime}}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime}} \cdots x_{k_{k-1}^{\prime}}}^{j^{\prime}}, R_{u^{\prime}}^{j^{\prime}}\right)$.
[2] : $\quad \sum_{\sigma \in \mathfrak{G}_{\kappa}^{\kappa-2}} \delta_{k_{\sigma(1)}, \ldots, k_{\sigma(\kappa-2)}}^{l, \ldots \ldots \ldots, l} R_{x_{k_{\sigma(\kappa-1)}}^{j} x_{k_{\sigma(\kappa)}} i^{i_{1}}}^{j}$
$-\delta_{i_{1}}^{j}\left(\sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-3}} \delta_{k_{\sigma(1)}, \ldots, \ldots, k_{\sigma(\kappa-3)}}^{l, \ldots \ldots, l} Q_{x_{k_{\sigma(\kappa-2)}}^{l} x_{k_{\sigma(\kappa-1)}} x_{k_{\sigma(k)}}}^{l}\right)=$
$=\Pi\left(x, u, Q^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}}^{l^{\prime}}, Q_{x_{k_{1}^{\prime}} x_{k_{2}^{\prime}}^{\prime}}^{l^{\prime}}, R^{j^{\prime}}, R_{x_{k_{1}^{\prime}}^{\prime}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime}} \cdots x_{k_{k-1}^{\prime}}^{\prime}}^{j^{\prime}}, R_{u^{i_{1}^{\prime}}}^{j^{\prime}}, R_{x_{k_{1}^{\prime}} i^{i_{1}^{\prime}}}^{j^{\prime}}\right)$.

$$
\begin{aligned}
& {[3]: \quad \sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-1}} \delta_{k_{\sigma(1)}, \ldots, \ldots \ldots, k_{\sigma(\kappa-1)}}^{l, \ldots \ldots, l} R_{x_{k_{\sigma(\kappa)}}}^{j} u^{i_{1}}-} \\
& -\delta_{i_{1}}^{j}\left(\sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-2}} \delta_{k_{\sigma(1)}, \ldots, k_{\sigma(\kappa-2)}}^{l, \ldots \ldots \ldots l} Q_{x_{k_{\sigma(\kappa-1)}}^{l}}^{x_{k_{\sigma(\kappa)}}}\right)= \\
& =\Pi\left(x, u, Q^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}^{\prime}}^{l^{\prime}}, R^{j^{\prime}}, R_{x_{k_{1}^{\prime}}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime}} \cdots x_{k_{k-1}^{\prime}}^{\prime}}^{j^{\prime}}, R_{u^{\prime}}^{j^{\prime}}\right) . \\
& \text { [4]: } \quad \kappa \delta_{k_{1}, \ldots, k_{\kappa}}^{l, \ldots \ldots, l} R_{u^{i_{1}} u^{i_{1}}}^{j}-\kappa \delta_{i_{1}}^{j}\left(\sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-1}} \delta_{k_{\sigma(1), \ldots, k_{\sigma(\kappa-1)}}^{l, \ldots \ldots \ldots, l} Q_{x_{k_{\sigma(\kappa)}}}^{l} u^{i_{1}}}^{l}\right)= \\
& =\Pi\left(x, u, Q^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}}^{l^{\prime}}, \ldots, Q_{x_{k_{1}^{\prime}}^{\prime} \cdots x_{k_{k-1}^{\prime}}}^{l^{\prime}}, Q_{u^{i_{1}^{\prime}}}^{l^{\prime}}, R^{j^{\prime}}, R_{x_{k_{1}^{\prime}}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime} \cdots x_{k_{k-1}^{\prime}}}^{j^{\prime}}},\right. \\
& \left., R_{u_{1}^{i_{1}^{\prime}}}^{j^{\prime}}, R_{x_{k_{1}^{\prime}}^{u^{i_{1}^{\prime}}}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime} \cdots x_{k_{\kappa-2}^{\prime}}}^{j^{\prime}}}^{u^{i_{1}^{\prime}}}\right) . \\
& \text { [5] : } \quad 2 \kappa \delta_{k_{1}, \ldots, k_{\kappa}}^{l, \ldots \ldots, l} R_{u^{i_{1}} u^{i_{2}}}^{j}-\kappa \delta_{i_{1}}^{j}\left(\sum_{\sigma \in \mathfrak{S}_{\kappa}^{\kappa-1}} \delta_{k_{\sigma(1), \ldots, k_{\sigma(\kappa-1)}}^{l, \ldots \ldots \ldots, l} Q_{x_{k_{\sigma(\kappa)}}}^{l} u^{i_{2}}}^{l}\right)-
\end{aligned}
$$

$$
\begin{aligned}
& =\Pi\left(x, u, Q^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}}^{l^{\prime}}, \ldots, Q_{x_{k_{1}^{\prime}} \cdots x_{k_{k-1}^{\prime}}^{\prime}}^{l^{\prime}}, Q_{u^{i_{1}^{\prime}}}^{l^{\prime}}, R^{j^{\prime}}, R_{x_{k_{1}^{\prime}}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime}}^{\prime} \cdots x_{k_{k-1}^{\prime}}}^{j^{\prime}},\right. \\
& \left., R_{u^{i_{1}^{\prime}}}^{j^{\prime}}, R_{x_{k_{1}^{\prime}}}^{j^{\prime}}{u^{\prime}}^{j_{1}}, \ldots, R_{x_{k_{1}^{\prime} \cdots x_{k_{k-2}^{\prime}}^{\prime}}^{j^{\prime}}}^{u^{i_{1}^{\prime}}}\right) . \\
& \text { [6] : } \quad-C_{\kappa+1}^{2} \delta_{i_{1}}^{j} \delta_{k_{1}, \ldots, k_{\kappa}}^{l, \ldots \ldots, l} Q_{u^{i_{1}}}^{l}= \\
& =\Pi\left(x, u, Q^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}}^{l^{\prime}}, \ldots, Q_{x_{k_{1}^{\prime}} \cdots x_{k_{k-2}^{\prime}}^{\prime}}^{l^{\prime}}, R^{j^{\prime}}, R_{x_{k_{1}^{\prime}}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime}}^{\prime} \cdots x_{k_{k-1}^{\prime}}^{\prime}}^{j^{\prime}},\right. \\
& \left., R_{u^{i_{1}^{\prime}}}^{j^{\prime}}, R_{x_{k_{1}^{\prime}}^{i^{\prime}}}^{i^{\prime}}, \ldots, R_{x_{k_{1}^{\prime}}^{j^{\prime}}}^{j_{k_{k-3}^{\prime}}} u^{u_{1}^{i_{1}^{\prime}}}\right) .
\end{aligned}
$$

Then we get the following Lemma:

Lemma 8.1. Let $J$ denote the collection of $n+n^{2}+n+m C_{n+\kappa-1}^{\kappa-1}+m^{2}$ partial derivatives

$$
\begin{equation*}
J:=\left(Q^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}}^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}^{\prime} x_{k_{1}^{\prime}}}^{k_{1}^{\prime}}, R^{j^{\prime}}, R_{x_{k_{1}^{\prime}}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime}}^{\prime \cdots x_{k_{\kappa-1}^{\prime}}^{\prime}}}^{j^{\prime}}, R_{u^{\prime}}^{j_{1}^{\prime}}\right) . \tag{7.28}
\end{equation*}
$$

After linear combinations on the system (7.28) we obtain the following equations:

$$
\left\{\begin{array}{l}
\Pi(x, u, J)=R_{x_{k_{1}} \cdots x_{k_{k}}}^{j},  \tag{7.28}\\
\Pi(x, u, J)=Q_{u_{1}}^{l}, \\
\Pi(x, u, J)=R_{u_{1} i_{1} u^{i}}^{j}, \\
\Pi(x, u, J)=Q_{x_{k_{1} x_{k_{2}} x_{k_{3}}}}^{l}, \\
\Pi(x, u, J)=R_{x_{k_{1}} u_{1}}^{j}, \\
\Pi(x, u, J)=Q_{x_{k_{1} x_{k_{2}}}}^{k_{1}}, \quad k_{1} \neq k_{2}, \\
\Pi(x, u, J)=Q_{x_{k_{1}} x_{k_{2}}}^{l}, \quad l \neq k_{1}, \quad l \neq k_{2}
\end{array}\right.
$$

Moreover all the partial derivatives (with respect to $x_{l}$ and $u^{i}$ ) up to order three of the coefficients $Q^{l}$ and $R^{j}$ of the vector field $X \in \mathfrak{S y m}(\mathscr{E})$ are of the form $\Pi(x, u, J)$. Hence every function $Q^{l}$ and $R^{j}$ is uniquely determined by the values at the origin of the $n+n^{2}+n+m C_{n+\kappa-1}^{\kappa-1}+m^{2}$ partial derivatives (7.28). This implies that $\operatorname{dim}_{\mathbb{K}} \mathfrak{S y m}(\mathscr{E}) \leq n^{2}+2 n+m^{2}+$ $m C_{n+\kappa-1}^{\kappa-1}$.

Proof. Since the second part of Lemma 8.1 is immediate let us establish only the identities (7.28). We first specify the indices in the equation (7.28) ${ }_{[3]}$ as follows: $l=k_{\kappa}=\cdots=k_{3}=k_{2}=k_{1}$; then $l=k_{\kappa}=\cdots=k_{3}=k_{2} \neq$ $k_{1}$; and finally $l=k_{\kappa}=\cdots=k_{3}, k_{3} \neq k_{2}, k_{3} \neq k_{1}$. This gives three equations whose members on the right hand side are the same as those in the equation (7.28) and whose members on the left hand side are the same as those in the equation $(7.28)_{[3]}$ :
(7.28)

We remark that these three equations (after specialization of $j=i_{1}$ or of $j \neq i_{1}$ and after some easy linear combinations) provide directly the fifth, sixth and seventh equations of (7.28). In particular we may replace
the values of the partial derivatives $R_{x_{j_{1}^{\prime}}^{i^{\prime} i_{1}^{\prime}}}^{j^{\prime}}$ and $Q_{x_{k_{1}^{\prime}}^{\prime} x_{k_{2}^{\prime}}}^{l^{\prime}}$ with $k_{1}^{\prime} \neq k_{2}^{\prime}$ or $l^{\prime} \neq k_{1}^{\prime}, l^{\prime} \neq k_{2}^{\prime}$ appearing in the expressions $\Pi$ of the second member of (7.28) ${ }_{[1]}$ by their values just obtained from the fifth, the sixth and the seventh equations of (7.28). This gives the first equation of (7.28).

Then we specify the indices in (7.28) ${ }_{[2]}$ as follows: $l=k_{\kappa}=\cdots=k_{3}=$ $k_{2}=k_{1}$; then $l=k_{\kappa}=\cdots=k_{3}=k_{2} \neq k_{1}$; then $l=k_{\kappa}=\cdots=k_{3}$, $k_{3} \neq k_{2}, k_{3} \neq k_{1}$; and finally $l=k_{\kappa}=\cdots=k_{4}, l \neq k_{1}, l \neq k_{2}, l \neq k_{3}$. This gives four equations, whose members on the right hand side are the same as those in (7.28) and the members on the left hand side are the same as those in $(7.28)_{[2]}$ :

$$
\begin{align*}
& \left(\Pi\left(x, u, Q^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}^{\prime}}^{l^{\prime}} Q_{x_{k_{1}^{\prime}} x_{k_{2}^{\prime}}}^{l^{\prime}}, R^{j^{\prime}}, R_{x_{k_{1}^{\prime}}^{\prime}}^{j^{\prime}} \ldots, R_{x_{k_{1}^{\prime}} \cdots x_{k_{k-1}^{\prime}}^{\prime}}^{j^{\prime}}, R_{u_{1}^{i_{1}^{\prime}}}^{j^{\prime}}, R_{x_{k_{1}^{\prime}} u^{\prime} i_{1}^{\prime}}^{j^{\prime}}\right)=\right.  \tag{7.28}\\
& =C_{\kappa}^{2} R_{x_{k_{1}} x_{k_{1}} u^{i_{1}}}^{j}-C_{\kappa}^{3} \delta_{i_{1}}^{j} Q_{x_{k_{1}} x_{k_{1}} x_{k_{1}}}^{k_{1}}, \\
& \Pi\left(x, u, Q^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}}^{l^{\prime}}, Q_{x_{k_{1}^{\prime}} x_{k_{2}^{\prime}}^{\prime}}^{l^{\prime}}, R^{j^{\prime}}, R_{x_{k_{1}^{\prime}}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime}} \cdots x_{k_{k-1}^{\prime}}}^{j^{\prime}}, R_{u_{1}^{i_{1}^{\prime}}}^{j^{\prime}}, R_{x_{k_{1}^{\prime}} i^{i_{1}^{\prime}}}^{j^{\prime}}\right)= \\
& =C_{\kappa-1}^{1} R_{x_{k_{1}} x_{k_{2}} u^{i_{1}}}^{j}-C_{\kappa-1}^{2} \delta_{i_{1}}^{j} Q_{x_{k_{1}} x_{k_{2}} x_{k_{2}}}^{k_{2}}, \quad k_{2} \neq k_{1}, \\
& \Pi\left(x, u, Q^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}}^{l^{\prime}}, Q_{x_{k_{1}^{\prime}} x_{k_{2}^{\prime}}^{\prime}}^{l^{\prime}}, R^{j^{\prime}}, R_{x_{k_{1}^{\prime}}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime} \cdots x_{k_{k-1}^{\prime}}^{\prime}}^{j^{\prime}}, R_{u^{i_{1}}}^{j^{\prime}}, R_{x_{k_{1}^{\prime}}^{\prime}}^{j^{\prime}}}^{i_{1}^{\prime}}\right)= \\
& =R_{x_{k_{1}} x_{k_{2}} u^{i_{1}}}^{j}-C_{\kappa-2}^{1} \delta_{i_{1}}^{j} Q_{x_{k_{1}} x_{k_{2}} x_{k_{3}}}^{k_{3}}, \quad k_{3} \neq k_{1}, \quad k_{3} \neq k_{2}, \\
& \Pi\left(x, u, Q^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}}^{l^{\prime}}, Q_{x_{k_{1}^{\prime}} x_{k_{2}^{\prime}}^{\prime}}^{l^{\prime}}, R^{j^{\prime}}, R_{x_{k_{1}^{\prime}}}^{j^{\prime}}, \ldots, R_{\left.x_{k_{1}^{\prime} \cdots x_{k_{k-1}^{\prime}}}^{j^{\prime}}, R_{u_{1}^{i}}^{j^{\prime}}, R_{x_{k_{1}^{\prime}}^{\prime i_{1}^{\prime}}}^{j^{\prime}}\right)=}=\right. \\
& =-\delta_{i_{1}}^{j} Q_{x_{k_{1}} x_{k_{2}} x_{k_{3}}}^{l}, \quad l \neq k_{1}, \quad l \neq k_{2}, \quad l \neq k_{3} .
\end{align*}
$$

Using the fifth, the sixth and the seventh equations of (7.28) just obtained, we may replace the partial derivatives $R_{x_{j_{1}^{\prime}}^{\prime} i_{1}^{\prime}}^{j^{\prime}}$ and $Q_{x_{k_{1}^{\prime}}^{\prime} x_{k_{2}^{\prime}}^{\prime}}^{l^{\prime}}$ with $k_{1}^{\prime} \neq k_{2}^{\prime}$ or $l^{\prime} \neq k_{1}^{\prime}, l^{\prime} \neq k_{2}^{\prime}$ appearing in the expressions $\Pi$ of (7.28), providing four new equations in which the arguments of $\Pi$ are the desired ones: $(x, u, J)$, where $J$ is defined in (7.28):

Let us differentiate now the equations (7.28) with respect to the variables $x_{l}$ as follows: first we differentiate $(7.28)_{1}$ with respect to $x_{k_{1}}$; then we differentiate $(7.28)_{2}$ with respect to $x_{k_{2}}$; finally we differentiate $(7.28)_{3}$ with respect to $x_{k_{3}}$. The arguments in the expressions $\Pi$ in the equation (7.28) contain now the terms $R_{x_{k_{1}^{\prime}} \cdots x_{k_{k}^{\prime}}^{\prime}}^{j^{\prime}}$; we replace them by their value given in the first equation of (7.28) already obtained. The arguments also contain the terms $R_{x_{j_{1}^{\prime}} i_{1}^{\prime}}^{j_{1}^{\prime}}$ and $Q_{x_{k_{1}^{\prime}} x_{k_{2}^{\prime}}^{\prime}}^{l^{\prime}}$ with $k_{1}^{\prime} \neq k_{2}^{\prime}$ or $l^{\prime} \neq k_{1}^{\prime}, l^{\prime} \neq k_{2}^{\prime}$. We replace them by their value given by the fifth, the sixth and the seventh equations of (7.28). We obtain three new equations in which the arguments of the expressions $\Pi$ are the desired ones: $(x, u, J)$, where $J$ is defined in (7.28):

The seven equations (7.28) and (7.28) may be considered as three systems of two linear equations of two variables with a nonzero determinant, the seventh equation being the last equation in (7.28). We immediately obtain: (7.28)

$$
\left\{\begin{array}{l}
\Pi(x, u, J)=R_{x_{k_{1} x_{k_{1}} u} u_{1}}^{j}=\delta_{i_{1}}^{j} Q_{x_{k_{1}} x_{k_{1}} x_{k_{1}}}^{k_{1}}, \\
\Pi(x, u, J)=R_{x_{k_{1} x_{k_{2}}} u^{i_{1}}}^{j}=\delta_{i_{1}}^{j} Q_{x_{k_{1}} x_{k_{2}} x_{k_{2}}}^{k_{2}}, \quad k_{2} \neq k_{1}, \\
\Pi(x, u, J)=R_{x_{k_{1}} x_{k_{2}} u_{1}}^{j}=\delta_{i_{1}}^{j} Q_{x_{k_{1}} x_{k_{2}} x_{k_{3}}}^{k_{3}}, \quad k_{3} \neq k_{1}, \quad k_{3} \neq k_{2}, \\
\Pi(x, u, J)=\delta_{i_{1}}^{j} Q_{x_{k_{1} x_{k_{2}} x_{k_{3}}}^{l}, \quad k_{3} \neq k_{1}, \quad k_{3} \neq k_{2},} .
\end{array}\right.
$$

giving the fourth equation in (7.28).
It remains now to obtain the second and the third equations in (7.28). Let us write firstly equation $(7.28)_{[6]}$ with the choice of the indices $j=i_{1}$, $l=k_{1}=\cdots=k_{\kappa}$. This gives the equation:

$$
\left\{\begin{array}{c}
Q_{u^{i_{1}}}^{l}=\Pi\left(x, u, Q^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}}^{l^{\prime}}, \ldots, Q_{x_{k_{1}^{\prime}} \cdots x_{k_{k-2}^{\prime}}^{\prime}}^{l^{\prime}}, R^{j^{\prime}}, R_{x_{k_{1}^{\prime}}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime}} \cdots x_{k_{k-1}^{\prime}}^{\prime}}^{j^{\prime}},\right.  \tag{7.28}\\
\left., R_{u_{1}^{i_{1}^{\prime}}}^{j^{\prime}}, R_{x_{k_{1}^{\prime}}^{\prime} u_{1}^{i_{1}}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime} \cdots x_{k_{k-3}^{\prime}}^{\prime}}^{j^{\prime}}}^{i_{1}^{\prime}}\right) .
\end{array}\right.
$$

We observe first that the differentiation with respect to the variables $x_{l}$ of one of the expressions $\Pi(x, u, J)$ remains an expression $\Pi(x, u, J)$. Indeed we see from (7.28) that there appears, in the partial derivative $J_{x_{l}}$, derivatives $Q_{x_{k_{1}^{\prime}} x_{k_{2}^{\prime}}}^{l^{\prime}}$ with $k_{1}^{\prime} \neq k_{2}^{\prime}$ or $l^{\prime} \neq k_{1}^{\prime}, l^{\prime} \neq k_{2}^{\prime}$. We may replace them by their value obtained in the sixth and the seventh equations of (7.28). It
also appears some derivatives $Q_{x_{k_{1}^{\prime}}^{\prime} x_{k_{2}^{\prime}} x_{k_{3}^{\prime}}}$ (we replace them by their value obtained in the fourth equation of (7.28)), some derivatives $R_{x_{k_{1}^{\prime}} \cdots x_{k_{k}^{\prime}}}^{j^{\prime}}$ (we replace them by their value obtained in the first equation of (7.28)) and some derivatives $R_{x_{k_{1}^{\prime}} u_{1}^{i_{1}^{\prime}}}^{j^{\prime}}$ (we replace them by their value obtained in the fifth equation of (7.28)). Consequently we may write:

$$
\begin{equation*}
[\Pi(x, u, J)]_{x_{l}}=\Pi(x, u, J) . \tag{7.28}
\end{equation*}
$$

It follows that any derivative with respect to $x_{l}$ (to any order) of the fourth and the fifth equations of (7.28) provides expressions of the form $\Pi(x, u, J)$. In other words for any integer $\lambda \geq 3$ and any integer $\mu \geq 1$ we have

$$
\left\{\begin{array}{l}
\Pi(x, u, J)=Q_{x_{k_{1}} x_{k_{2}} x_{k_{3}} \cdots x_{k_{\lambda}}}^{l},  \tag{7.28}\\
\Pi(x, u, J)=R_{x_{k_{1}} \cdots x_{k_{\mu}} u^{i_{1}}}^{j}
\end{array}\right.
$$

We may replace then these values in the equation (7.28), replacing also the derivatives $Q_{x_{k_{1}^{\prime}} x_{k_{2}^{\prime}}^{\prime}}^{l^{\prime}}$ with $k_{1}^{\prime} \neq k_{2}^{\prime}$ or $l^{\prime} \neq k_{1}^{\prime}, l^{\prime} \neq k_{2}^{\prime}$ by their values obtained in the sixth and the seventh equations of (7.28). This gives the second equation of (7.28).

We also remark that by a differentiation with respect to the variables $x_{l}$, the second equation $Q_{u^{i_{1}}}^{l}=\Pi(x, u, J)$ just obtained implies, using (7.28):

$$
\begin{equation*}
\Pi(x, u, J)=Q_{x_{k_{1}} u^{i_{1}}}^{l} . \tag{7.28}
\end{equation*}
$$

It remains finally to write (7.28) ${ }_{[4]}$ first with the choice of indices $l=k_{1}=$ $\cdots=k_{\kappa}, j=i_{1}$ then with the choice of indices $l=k_{1}=\cdots=k_{\kappa}, j \neq i_{1}$. We also write (7.28) ${ }_{[5]}$ first with the choice of indices $l=k_{1}=\cdots=k_{\kappa}$, $j=i_{2}$ then with the choice of indices $l=k_{1}=\cdots=k_{\kappa}, j \neq i_{1}, j \neq i_{2}$. We obtain four new equations:

$$
\begin{align*}
& \int R_{u^{i_{1}} u_{1}^{i_{1}}}^{i_{1}}-\kappa Q_{x_{k_{1}} u_{1}^{i_{1}}}^{k_{1}}=\Pi\left(x, u, Q^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}}^{l^{\prime}}, \ldots, Q_{x_{k_{1}^{\prime} \cdots x_{k_{k-1}^{\prime}}}^{l^{\prime}}}, Q_{u^{i_{1}^{\prime}}}^{l^{\prime}}, R^{j^{\prime}}, R_{x_{k_{1}^{\prime}}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime} \cdots x_{k_{k-1}^{\prime}}^{\prime}}^{j^{\prime}},},\right.  \tag{7.28}\\
& \left., R_{u^{i_{1}^{\prime}}}^{j^{\prime}}, R_{x_{k_{1}^{\prime}}^{i^{\prime}} u_{1}^{\prime}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime} \cdots x_{k_{\kappa-2}^{\prime}}}^{j^{\prime}}}^{u^{i_{1}^{\prime}}}\right), \\
& R_{u^{i_{1}} u^{i_{1}}}^{j}=\Pi\left(x, u, Q^{l^{\prime}}, Q_{x_{k_{1}^{\prime}}}^{l^{\prime}}, \ldots, Q_{x_{k_{1}^{\prime}}^{\prime \cdots x_{k_{k-1}^{\prime}}}}^{l^{\prime}}, Q_{u^{i_{1}^{\prime}}}^{l^{\prime}}, R^{j^{\prime}}, R_{x_{k_{1}^{\prime}}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime}}^{\prime} \cdots x_{k_{k-1}^{\prime}}^{\prime}}^{j^{\prime}},\right. \\
& \left.\begin{array}{l}
, R_{u^{i_{1}^{\prime}}}^{j^{\prime}}, R_{x_{k_{1}^{\prime}}^{\prime} u_{1}^{i_{1}}}^{j^{\prime}}, \ldots, R_{x_{k_{1}^{\prime}} \cdots x_{k_{\kappa-2}^{\prime}}}^{j^{\prime}} u^{i_{1}^{\prime}}
\end{array}\right), \quad j \neq i_{1}, \\
& \left., R_{u_{1}^{i_{1}^{\prime}}}^{j^{\prime}}, R_{x_{k_{1}^{\prime}}^{i^{\prime}}}^{i^{\prime}}, \ldots, R_{x_{k_{1}^{\prime} \cdots x_{k_{k-2}^{\prime}}}^{j^{\prime}}}^{u^{i_{1}^{\prime}}}\right), \quad i_{1} \neq i_{2},
\end{align*}
$$

$$
\begin{aligned}
& \left., R_{u^{i_{1}^{\prime}}}^{j^{\prime}}, R_{x_{k_{1}^{\prime}}^{j_{1}^{\prime}}}^{i_{1}^{\prime}}, \ldots, R_{x_{k_{1}^{\prime} \cdots x_{k_{k-2}^{\prime}}}^{j^{\prime}}}^{i^{\prime}}\right), \quad i_{1} \neq i_{2}, \quad j \neq i_{2}, \quad j \neq i_{2} .
\end{aligned}
$$

Using the equations of (7.28) we already obtained (namely all except the second equation), using (7.28) and (7.28), we may simplify these four equations:

$$
\begin{cases}\Pi(x, u, J)=R_{u^{i_{1} i_{1}}}^{i_{1}}, &  \tag{7.28}\\ \Pi(x, u, J)=R_{u^{i_{1}} i_{1}}^{j}, & j \neq i_{1} \\ \Pi(x, u, J)=R_{u^{i_{1}} i^{i_{2}}}^{i_{1}}, & i_{1} \neq i_{2}, \\ \Pi(x, u, J)=R_{u^{i_{1} i^{i_{2}}},}^{j}, & i_{1} \neq i_{2}, \quad j \neq i_{1}, \quad j \neq i_{2}\end{cases}
$$

This gives the second equation of (7.28), completing the proof of Lemma 8.1 and consequently the proof of Theorem 6.4.

## References

[1] Baouendi, M.S.; Ebenfelt, P.; Rothschild, L.P.: Real submanifolds in complex space and their mappings. Princeton Mathematical Series, 47, Princeton University Press Princeton, NJ, 1999, xii+404 pp.
[2] Bluman, G.W.; Kumei, S.: Symmetries and differential equations, Springer Verlag, Berlin, 1989.
[3] Cartan, É.: Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes, I, Annali di Mat. 11 (1932), 17-90.
[4] Chern, S.S.; Moser, J.K.: Real hypersurfaces in complex manifolds, Acta Math. 133 (1974), no.2, 219-271.
[5] F. Engel; Lie, S.: Theorie der Transformationsgruppen, I, II, II, Teubner, Leipzig, 1889, 1891, 1893.
[6] FELS, M.: The equivalence problem for systems of second-order ordinary differential equations, Proc. London Math. Soc. 71 (1995), 221-240.
[7] GAUSSIER, H.; MERKER, J.: A new example of uniformly Levi degenerate hypersurface in $\mathbb{C}^{3}$, Ark. Mat., to appear.
[8] Gaussier, H.; Merker, J.: Nonalgebraizable real analytic tubes in $\mathbb{C}^{n}$, Math. Z., to appear.
[9] Gaussier, H.; MERKER, J.: Sur l'algébrisabilité locale de sous-variétés analytiques réelles génériques de $\mathbb{C}^{n}, \mathrm{C} . \mathrm{R}$. Acad. Sci. Paris Sér. I Math., to appear.
[10] GaUssier, H.; MERKER, J.: Géométrie des sous-variétés analytiques réelles de $\mathbb{C}^{n}$ et symétries de Lie des équations aux dérivées partielles, Bull. Soc. Math. Tunisie, to appear.
[11] GonZÁlez-GAScón, F.; GonZÁlez-López, A.: Symmetries of differential equations, IV. J. Math. Phys. 24 (1983), 2006-2021.
[12] GonZÁlez-López, A.: Symmetries of linear systems of second order differential equations, J. Math. Phys. 29 (1988), 1097-1105.
[13] Ibragimov, N.H.: Group analysis of ordinary differential equations and the invariance principle in mathematical physics, Russian Math. Surveys 47:4 (1992), 89-156.
[14] Lie, S.: Theorie der Transformationsgruppen, Math. Ann. 16 (1880), 441-528.
[15] Merker, J.: Vector field construction of Segre sets, Preprint 1998, augmented in 2000. Downloadable at arXiv.org/abs/math. CV/9901010.
[16] MERKER, J.: On the partial algebraicity of holomorphic mappings between two real algebraic sets, Bull. Soc. Math. France 129 (2001), no.3, 547-591.
[17] MERKER, J.: On the local geometry of generic submanifolds of $\mathbb{C}^{n}$ and the analytic reflection principle, Viniti, to appear.
[18] Olver, P.J.: Applications of Lie groups to differential equations. Springer Verlag, Heidelberg, 1986.
[19] Olver, P.J.: Equivalence, Invariance and Symmetries. Cambridge, Cambridge University Press, 1995, xvi+525 pp.
[20] Poincaré, H.: Les fonctions analytiques de deux variables et la représentation conforme, Rend. Circ. Mat. Palermo, II, Ser. 23, 185-220.
[21] SEGRE, B.: Intorno al problema di Poincaré della rappresentazione pseudoconforme, Rend. Acc. Lincei, VI, Ser. 13 (1931), 676-683.
[22] SEgre, B.: Questioni geometriche legate colla teoria delle funzioni di due variabili complesse, Rendiconti del Seminario di Matematici di Roma, II, Ser. 7 (1932), no. 2, 59-107.
[23] Stormark, O.: Lie's structural approach to PDE systems. Encyclopædia of mathematics and its applications, vol. 80, Cambridge University Press, Cambridge, 2000, $\mathrm{xv}+572 \mathrm{pp}$.
[24] Sukhov, A.: Segre varieties and Lie symmetries, Math. Z. 238 (2001), no.3, 483492.
[25] Sukhov, A.: On transformations of analytic CR structures, Pub. Irma, Lille 2001, Vol. 56, no. II.
[26] Sukhov, A.: CR maps and point Lie transformations, Michigan Math. J. 50 (2002), 369-379.
[27] Sussmann, H.J.: Orbits of families of vector fields and integrability of distributions, Trans. Amer. Math. Soc. 180 (1973), 171-188.
[28] Tresse, A.: Détermination des invariants ponctuels de l'équation différentielle du second ordre $y^{\prime \prime}=\omega\left(x, y, y^{\prime}\right)$, Hirzel, Leipzig, 1896.

## Nonrigid spherical

# real analytic hypersurfaces in $\mathbb{C}^{2}$ 

Joël Merker


#### Abstract

A Levi nondegenerate real analytic hypersurface $M$ of $\mathbb{C}^{2}$ represented in local coordinates $(z, w) \in \mathbb{C}^{2}$ by a complex defining equation of the form $w=\Theta(z, \bar{z}, \bar{w})$ which satisfies an appropriate reality condition, is spherical if and only if its complex graphing function $\Theta$ satisfies an explicitly written sixth-order polynomial complex partial differential equation. In the rigid case (known before), this system simplifies considerably, but in the general nonrigid case, its combinatorial complexity shows well why the two fundamental curvature tensors constructed by Élie Cartan in 1932 in his classification of hypersurfaces have, since then, never been reached in parametric representation.


> arxiv.org/abs/0910.1694/

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## §1. Introduction

A real analytic hypersurface $M$ in $\mathbb{C}^{2}$ is called spherical at one of its points $p$ if there exists a nonempty open neighborhood $U_{p}$ of $p$ in $\mathbb{C}^{2}$ such that $M \cap U_{p}$ is biholomorphic to a piece of the unit sphere $S^{3}=$ $\left\{(z, w):|z|^{2}+|w|^{2}=1\right\}$. When $M$ is connected, sphericality at one point is known to propagate all over $M$, for it is equivalent to the vanishing of two certain real analytic curvature tensors that were constructed by Élie Cartan in [3]. However, the intrinsic computational complexity, in the Cauchy-Riemann (CR for short) context, of Élie Cartan's algorithm to derive an absolute parallelism on some suitable eight-dimensional principal bundle $\mathscr{P} \rightarrow M$ prevents from controlling explicitly all the appearing differential forms. As a matter of fact, the effective computation, in terms of a defining equation for $M$, of the two fundamental differential invariants the vanishing of which characterizes sphericality, appears nowhere in the literature (see e.g. [23,5,11] and the references therein as well), except notably when one makes the assumption that, in some suitable local holomorphic
coordinates $(z, w)=(x+i y, u+i v)$ vanishing at the point $p$, the defining equation is of the so-called rigid form $u=\varphi(x, y)$ with the variable $v$ missing, or even of the so-called (simpler) tube form $u=\varphi(x)$, with the two variables $y$ and $v$ missing, see [11] which showed recently a renewed interest, in CR geometry, for explicit characterizations of sphericality. But in general, a real analytic hypersurface $M \subset \mathbb{C}^{2}$ is represented at $p$ by a real equation $u=\varphi(x, y, v)$ whose graphing function $\varphi$ depends entirely arbitrarily upon $v$ also, and then apparently, the characterization of sphericality is still unknown.

On the other hand, in the studies [12, 13, 14, Me2005a, Me2005b] devoted to the CR reflection principle, it was emphasized that all the adequate invariants of CR mappings between CR manifolds: Pair of Segre foliations, Segre chains, Complexified CR orbits, Jets of complexified Segre varietes, Rigidity of formal CR mappings, Nondegeneracy conditions, CR-reflection function ${ }^{5}$, can be viewed correctly only when $M$ is represented by a socalled complex defining equation of the form:

$$
w=\Theta(z, \bar{z}, \bar{w})
$$

where the function $\Theta \in \mathbb{C}\{z, \bar{z}, \bar{w}\}$, vanishing at the origin, is the unique function obtained by solving with respect to $w$ the equation: $\frac{w+\bar{w}}{2}=$ $\varphi\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2 i}, \frac{w-\bar{w}}{2 i}\right)$; then the fact that $\varphi$ was real is reflected, in terms of this new function $\Theta(z, \bar{z}, \bar{w})$, by the constraint that, together with its complex conjugate $\bar{\Theta}(z, \bar{z}, \bar{w})$, it satisfies the functional equation ${ }^{6}$ :

$$
w \equiv \Theta(z, \bar{z}, \bar{\Theta}(\bar{z}, z, w))
$$

Accordingly, the author suspected since a few years - $c f$. the Open Question 2.35 in [19] - that sphericality of $M$ at $p$ should and could be expressed adequately in terms of $\Theta$. The classical assumption that $M$ be Levi nondegenerate at the point $p$ (see e.g. [11]) - which is the origin of our present system of coordinates $(z, w)$ - may then be expressed here (cf. [Me2005a, Me2005b]) by requiring that $\Theta_{\bar{z}} \Theta_{z \bar{w}}-\Theta_{\bar{w}} \Theta_{z \bar{z}}$ does not vanish at the origin. In particular, this guarantees that the following explicit rational expression whose numerator is a polynomial in the fourth-order jet $J_{z, \bar{z}, \bar{w}}^{4} \Theta$, is well defined and analytic in some sufficiently small neighborhood of the

[^4]origin:
\[

$$
\begin{aligned}
& -2 \Theta_{z z \overline{z w}}\left(\begin{array}{ll}
\left.\Theta_{\bar{z}} \Theta_{\bar{w}} \left\lvert\, \begin{array}{cc}
\Theta_{\bar{z}} & \Theta_{\bar{w}} \\
\Theta_{z \bar{z}} & \Theta_{z \bar{w}}
\end{array}\right.\right)+\Theta_{z z \overline{w w}}\left(\Theta_{\bar{z}} \Theta_{\bar{z}} \left\lvert\, \begin{array}{cc}
\Theta_{\bar{z}} & \Theta_{\bar{w}} \\
\Theta_{z \bar{z}} & \Theta_{z \bar{w}}
\end{array}\right.\right)+\Theta^{-1}
\end{array}\right)+ \\
& +\Theta_{z z \bar{z}}\left(\left.\Theta_{\bar{z}} \Theta_{\bar{z}}\left|\begin{array}{cc}
\Theta_{\bar{w}} & \Theta_{\overline{w w}} \\
\Theta_{z \bar{w}} & \Theta_{z \overline{w w}}
\end{array}\right|-2 \Theta_{\bar{z}} \Theta_{\bar{w}}\left|\begin{array}{cc}
\Theta_{\bar{w}} & \Theta_{\overline{z w}} \\
\Theta_{z \bar{w}} & \Theta_{z \overline{z w}}
\end{array}\right|+\Theta_{\bar{w}} \Theta_{\bar{w}} \right\rvert\, \begin{array}{cc}
\Theta_{\bar{w}} & \Theta_{\bar{z} \bar{z}} \\
\Theta_{z \bar{w}} & \Theta_{z \overline{z z}}
\end{array}\right)+ \\
& \left.+\Theta_{z z \bar{w}}\left(-\Theta_{\bar{z}} \Theta_{\bar{z}}\left|\begin{array}{cc}
\Theta_{\bar{z}} & \Theta_{\overline{w w}} \\
\Theta_{z \bar{z}} & \Theta_{z \overline{w w}}
\end{array}\right|+2 \Theta_{\bar{z}} \Theta_{\bar{w}}\left|\begin{array}{cc}
\Theta_{\bar{z}} & \Theta_{\overline{z w}} \\
\Theta_{z \bar{z}} & \Theta_{z \overline{z w}}
\end{array}\right|-\Theta_{\bar{w}} \Theta_{\bar{w}}\left|\begin{array}{|cc}
\Theta_{\bar{z}} & \Theta_{\overline{z z}} \\
\Theta_{z \bar{z}} & \Theta_{z \bar{z} \bar{z}}
\end{array}\right|\right)\right\} .
\end{aligned}
$$
\]

We hope, then, that the following precise statement will fill a gap in our understanding of the vanishing of CR curvature tensors.
Main (and unique) theorem. An arbitrary, not necessarily rigid, real analytic hypersurface $M \subset \mathbb{C}^{2}$ which is Levi nondegenerate at one of its points $p$ and has a complex defining equation of the form:

$$
w=\Theta(z, \bar{z}, \bar{w})
$$

in some system of local holomorphic coordinates $(z, w) \in \mathbb{C}^{2}$ centered at $p$, is spherical at $p$ if and only if its graphing complex function $\Theta$ satisfies the following explicit sixth-order algebraic partial differential equation:

$$
0 \equiv\left(\frac{-\Theta_{\bar{w}}}{\Theta_{\bar{z}} \Theta_{z \bar{w}}-\Theta_{\bar{w}} \Theta_{z \bar{z}}} \frac{\partial}{\partial \bar{z}}+\frac{\Theta_{\bar{z}}}{\Theta_{\bar{z}} \Theta_{z \bar{w}}-\Theta_{\bar{w}} \Theta_{z \bar{z}}} \frac{\partial}{\partial \bar{w}}\right)^{2}\left[\mathrm{AJ}^{4}(\Theta)\right]
$$

identically in $\mathbb{C}\{z, \bar{z}, \bar{w}\}$.
Here, it is understood that the first-order derivation in parentheses is applied twice to the fourth-order rational differential expression $A J^{4}(\Theta)$. The factor $\frac{1}{\left[\Theta_{\bar{z}} \Theta_{z \bar{w}}-\Theta_{\bar{w}} \Theta_{z \bar{z}}\right]^{7}}$ then appears, and after clearing out this denominator, one obtains a universal polynomial differential expression $A J^{6}(\Theta)$ depending upon the sixth-order jet $J_{z, \bar{z}, \bar{w}}^{6} \Theta$ and having integer coefficients. A partial expansion is provided in Section 5, and the already formidable incompressible length of this expansion perhaps explains the reason why no reference in the literature provides the explicit expressions, in terms of some defining function for $M$, of Élie Cartan's two fundamental differential invariants ${ }^{7}$ which can (in principle) be used to classify real analytic hypersurfaces of $\mathbb{C}^{2}$ up to biholomorphisms, and to at least characterize sphericality.

Suppose in particular for instance that $M$ is rigid, given by a complex equation of the form $w=-\bar{w}+\Xi(z, \bar{z})$, that is to say with $\Theta(z, \bar{z}, \bar{w})$ of the form $-\bar{w}+\Xi(z, \bar{z})$, so that the reality condition simply reads here: $\Xi(z, \bar{z}) \equiv \bar{\Xi}(\bar{z}, z)$. Then as a corollary-exercise, sphericality is explicitly

[^5]characterized by a much simpler partial differential equation that we can write down in expanded form:
\[

$$
\begin{aligned}
0 \equiv & \frac{\Xi_{z^{2} \bar{z}^{4}}}{\left(\Xi_{z \bar{z}}\right)^{4}}-6 \frac{\Xi_{z^{2} \bar{z}^{3}} \Xi_{z \bar{z}^{2}}}{\left(\Xi_{z \bar{z}}\right)^{5}}-4 \frac{\Xi_{z^{2} \bar{z}^{2}} \Xi_{z \bar{z}^{3}}}{\left(\Xi_{z \bar{z}}\right)^{5}}-\frac{\Xi_{z^{2} \bar{z}} \Xi_{z \bar{z}^{4}}}{\left(\Xi_{z \bar{z}}\right)^{5}}+ \\
& +15 \frac{\Xi_{z^{2} \bar{z}^{2}}\left(\Xi_{z \bar{z}^{2}}\right)^{2}}{\left(\Xi_{z \bar{z}}\right)^{6}}+10 \frac{\Xi_{z \bar{z}^{3}} \Xi_{z^{2} \bar{z}} \Xi_{z \bar{z}^{2}}}{\left(\Xi_{z \bar{z}}\right)^{6}}-15 \frac{\Xi_{z^{2} \bar{z}}\left(\Xi_{z \bar{z}^{2}}\right)^{3}}{\left(\Xi_{z \bar{z}}\right)^{7}},
\end{aligned}
$$
\]

and this equation should of course hold identically in $\mathbb{C}\{z, \bar{z}\}$.
Now, here is a summarized description of our arguments of proof. Beniamino Segre ([23]) in 1931 and in fact much earlier Sophus Lie himself in the 1880's (see e.g. Chapter 10 of Volume I of the Theorie der Transformationsgruppen [8]) showed how to elementarily associate a unique secondorder ordinary differential equation:

$$
w_{z z}(z)=\Phi\left(z, w(z), w_{z}(z)\right)
$$

to the Levi nondegenerate equation $w=\Theta(z, \bar{z}, \bar{w})$ by eliminating the two variables $\bar{z}$ and $\bar{w}$, viewed as parameters, from the two equations $w=\Theta$ and $w_{z}=\Theta_{z}$. We check in great details the semi-known result that $M$ is spherical at the origin if and only if its associated differential equation is equivalent, under some appropriate local holomorphic point transformation $(z, w) \longmapsto\left(z^{\prime}, w^{\prime}\right)=\left(z^{\prime}(z, w), w^{\prime}(z, w)\right)$ fixing the origin, to the simplest possible equation $w_{z^{\prime} z^{\prime}}^{\prime}\left(z^{\prime}\right)=0$ having null right-hand side, whose obvious solutions are just the affine complex lines. But since the doctoral dissertation of Arthur Tresse (defended in 1895 under the direction of Lie in Leipzig), it is known that, attached to any such differential equation are two explicit differential invariants:

$$
\begin{aligned}
\mathrm{I}^{1}:= & \Phi_{w_{z} w_{z} w_{z} w_{z}} \quad \text { and: } \\
\mathrm{I}^{2}:= & \mathrm{DD}\left(\Phi_{w_{z} w_{z}}\right)-\Phi_{w_{z}} \mathrm{D}\left(\Phi_{w_{z} w_{z}}\right)-4 \mathrm{D}\left(\Phi_{w w_{z}}\right)+ \\
& +6 \Phi_{w w}-3 \Phi_{w} \Phi_{w_{z} w_{z}}+4 \Phi_{w_{z}} \Phi_{w w_{z}},
\end{aligned}
$$

where $\mathrm{D}:=\partial_{z}+w_{z} \partial_{w}+\Phi\left(z, w, w_{z}\right) \partial_{w_{z}}$,
depending both upon the fourth-order jet of $\Phi$, which, together with all their covariant differentiations, enable one (in principle ${ }^{8}$ ) to completely determine when two arbitrarily given differential equations are equivalent one to another ${ }^{9}$. A very well-known application is: the vanishing of both $I^{1}$ and

[^6]$\mathrm{I}^{2}$ characterizes equivalence to $w_{z^{\prime} z^{\prime}}^{\prime}\left(z^{\prime}\right)=0$. So in order to characterize sphericality, one only has to reexpress the vanishing of $I^{1}$ and of $I^{2}$ in terms of the complex defining function $\Theta(z, \bar{z}, \bar{w})$. For this, we apply the techniques of computational differential algebra developed in [19] which enable us here to explicitly execute the two-ways transfer between algebraic expressions in the jet of $\Phi$ and algebraic expressions in the jet of $\Theta$. It then turns out that the two equations which one obtains by transferring to $\Theta$ the vanishing of $\mathrm{I}^{1}$ and of $\mathrm{I}^{2}$ are conjugate one to another, so that a single equation suffices, and it is precisely the one enunciated in the theorem. In fact, this coincidence is caused by the famous projective duality, explained e.g. by Lie and Scheffers in Chapter 10 of [12] and restituted in modern language in $[1,5]$. It is indeed well known that to any second-order ordinary differential equation $(\mathscr{E}): y_{x x}(x)=F\left(x, y(x), y_{x}(x)\right)$ is canonically associated a certain dual second-order ordinary differential equation, call it ( $\mathscr{E}^{*}$ ): $b_{a a}(a)=F^{*}\left(a, b_{a}(a), b_{a a}(a)\right)$, which has the crucial property that:
\[

$$
\begin{array}{lll} 
& I_{(\mathscr{E})}^{1} & \text { is a nonzero multiple of } \\
\text { and symmetrically also: } & \mathrm{I}_{\left(\mathscr{E}^{*}\right)}^{2} \\
\text { is a nonzero multiple of } & \mathrm{I}_{\left(\mathscr{E}^{*}\right)}^{1} .
\end{array}
$$
\]

The doctoral dissertation [10] of Koppisch (Leipzig 1905) cited only passim by Élie Cartan in [Ca1924] contains the analytical details of this correspondence, which was well reconstituted recently in [5] within the context of projective Cartan connections. But the differential equation which is dual to the one $w_{z z}(z)=\Phi\left(z, w(z), w_{z}(z)\right)$ associated to $w=\Theta(z, \bar{z}, \bar{w})$ is easily seen to be just its complex conjugate $(\overline{\mathscr{E}}): \bar{w}_{\bar{z}( }(\bar{z})=\bar{\Phi}\left(\bar{z}, \bar{w}(\bar{z}), \bar{w}_{\bar{z}}\right)$, and then as a consequence, $I_{(\bar{E})}^{1}=\overline{I_{(\mathscr{E})}^{1}}$ is the conjugate of $I_{(\mathscr{E})}^{1}$, and similarly also $\mathrm{I}_{(\overline{\mathscr{E}})}^{2}=\overline{\mathrm{I}_{(\mathscr{E})}^{2}}$ is the conjugate of $\mathrm{I}_{(\mathscr{E})}^{2}$. So it is no mystery that, as said, the sphericality of $M$ at the origin:

$$
0 \equiv \mathbf{I}_{(\mathscr{E})}^{1} \quad \text { and } \quad 0 \equiv \mathrm{I}_{(\mathscr{E})}^{2}=\text { nonzero } \cdot \mathrm{I}_{(\overline{\mathscr{E}})}^{1}=\text { nonzero } \cdot \overline{\mathbf{I}_{(\mathscr{E})}^{1}},
$$

can in a simpler way be characterized by the vanishing of the two mutually conjugate (complex) equations:

$$
0 \equiv \mathrm{I}_{(\mathscr{E})}^{1} \quad \text { and } \quad 0 \equiv \overline{\bar{I}_{(\mathscr{E})}^{1}},
$$

which of course amount to just one (complex) equation.
To conclude this introduction, we would like to mention firstly that none of our computations - especially those of Sections 4 and 5 - was performed with the help of any computer, and secondly that the effective characterization of sphericality in higher complex dimension $n \geqslant 3$ will appear soon [21].
of the curvature provide a full list of differential invariants for positive definite quadratic infinitesimal metrics.

## §2. SEGRE VARIETIES AND DIFFERENTIAL EQUATIONS

Real analytic hypersurfaces in $\mathbb{C}^{2}$. Let us consider an arbitrary real analytic hypersurface $M$ in $\mathbb{C}^{2}$ and let us localize it around one of its points, say $p \in M$. Then there exist complex affine coordinates:

$$
(z, w)=(x+i y, u+i v)
$$

vanishing at $p$ in which $T_{p} M=\{u=0\}$, so that $M$ is represented in a neighborhood of $p$ by a graphed defining equation of the form:

$$
u=\varphi(x, y, v)
$$

where the real-valued function:

$$
\varphi=\varphi(x, y, v)=\sum_{\substack{k, l, m \in \mathbb{N} \\ k+l+m \geqslant 2}} \varphi_{k, l, m} x^{k} y^{l} v^{m} \in \mathbb{R}\{x, y, u\}
$$

which possesses entirely arbitrary real coefficients $\varphi_{k, l, m}$, vanishes at the origin: $\varphi(0)=0$, together with all its first order derivatives: $0=\partial_{x} \varphi(0)=$ $\partial_{y} \varphi(0)=\partial_{v} \varphi(0)$. All studies in the analytic reflection principle ${ }^{10}$ show without doubt that the adequate geometric concepts: Pair of Segre foliations, Segre chains, Complexified CR orbits, Jets of complexified Segre varietes, Rigidity of formal CR mappings, Nondegeneracy conditions, CR-reflection function, can be viewed correctly only when $M$ is represented by a so-called complex defining equation. Such an equation may be constructed by simply rewriting the initial real equation of $M$ as:

$$
\frac{w+\bar{w}}{2}=\varphi\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2 i}, \frac{w-\bar{w}}{2 i}\right),
$$

and then by solving ${ }^{11}$ the so written equation with respect to $w$, which yields an equation of the shape ${ }^{12}$ :

$$
w=\Theta(z, \bar{z}, \bar{w})=\sum_{\substack{\alpha, \beta, \gamma \in \mathbb{N} \\ \alpha+\beta+\gamma \geqslant 1}} \Theta_{\alpha, \beta, \gamma} z^{\alpha} \bar{z}^{\beta} \bar{w}^{\gamma} \in \mathbb{C}\{\bar{z}, z, w\},
$$

whose right-hand side converges of course near the origin $(0,0,0) \in \mathbb{C} \times$ $\mathbb{C} \times \mathbb{C}$ and has complex coefficients $\Theta_{\alpha, \beta, \gamma} \in \mathbb{C}$. The paradox that any such complex equation provides in fact two real defining equations for the real hypersurface $M$ which is one-codimensional, and also in addition the fact that one could as well have chosen to solve the above equation with respect to $\bar{w}$, instead of $w$, these two apparent "contradictions" are corrected

[^7]by means of a fundamental, elementary statement that transfers to $\Theta$ (in a natural way) the condition of reality:
$$
\overline{\varphi(x, y, u)}=\sum_{k+l+m \geqslant 1} \overline{\varphi_{k, l, m}} \bar{x}^{k} \bar{y}^{l} \bar{v}^{m}=\sum_{k+l+m \geqslant 1} \varphi_{k, l, m} x^{k} y^{l} v^{m}=\varphi(x, y, v)
$$
enjoyed by the initial definining function $\varphi$.
Theorem. ([18], p. 19 ${ }^{13}$ ) The complex analytic function $\Theta=\Theta(z, \bar{z}, \bar{w})$ with $\Theta=-\bar{w}+\mathrm{O}(2)$ together with its complex conjugate ${ }^{14}$ :
$$
\bar{\Theta}=\bar{\Theta}(\bar{z}, z, w)=\sum_{\alpha, \beta, \gamma \in \mathbb{N}} \bar{\Theta}_{\alpha, \beta, \gamma} \bar{z}^{\alpha} z^{\beta} w^{\gamma} \in \mathbb{C}\{\bar{z}, z, w\}
$$
satisfy the two (equivalent by conjugation) functional equations:
\[

$$
\begin{align*}
\bar{w} & \equiv \bar{\Theta}(\bar{z}, z, \Theta(z, \bar{z}, \bar{w})) \\
w & \equiv \Theta(z, \bar{z}, \bar{\Theta}(\bar{z}, z, w)) \tag{7.28}
\end{align*}
$$
\]

Conversely, given a local holomorphic function $\Theta(z, \bar{z}, \bar{w}) \in \mathbb{C}\{z, \bar{z}, \bar{w}\}$, $\Theta=-\bar{w}+\mathrm{O}(2)$ which, in conjunction with its conjugate $\bar{\Theta}(\bar{z}, z, w)$, satisfies this pair of equivalent identities, then the two zero-sets:

$$
\{0=-w+\Theta(z, \bar{z}, \bar{w})\} \quad \text { and } \quad\{0=-\bar{w}+\bar{\Theta}(\bar{z}, z, w)\}
$$

coincide and define a local one-codimensional real analytic hypersurface $M$ passing through the origin in $\mathbb{C}^{2}$.

As before, let $M$ be an arbitrary real analytic hypersurface passing through the origin in $\mathbb{C}^{2}$ equipped with coordinates $(z, w)$, and assume that $T_{0} M=\{u=0\}$. Without loss of generality, we can and we shall assume that the coordinates are chosen in such a way that a certain standard convenient normalization condition holds.
Theorem. ([Me2005a], p. 12) There exists a local complex analytic change of holomorphic coordinates $h:(z, w) \longmapsto\left(z^{\prime}, w^{\prime}\right)=h(z, w)$ fixing the origin and tangent to the identity at the origin of the specific form:

$$
z^{\prime}=z, \quad w^{\prime}=g(z, w)
$$

such that the image $M^{\prime}:=h(M)$ has a new complex defining equation $w^{\prime}=\Theta^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)$ satisfying:

$$
\Theta^{\prime}\left(0, \bar{z}^{\prime}, \bar{w}^{\prime}\right) \equiv \Theta^{\prime}\left(z^{\prime}, 0, \bar{w}^{\prime}\right) \equiv-\bar{w}^{\prime}
$$

[^8]or equivalently, which has a power series expansion of the form:
$$
\Theta^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)=-\bar{w}^{\prime}+\sum_{\alpha \geqslant 1, \beta \geqslant 1} \Theta_{\alpha, \beta, 0}^{\prime} z^{\prime \alpha} \bar{z}^{\prime \beta}+\sum_{\gamma \geqslant 1} \bar{w}^{\prime \gamma} \sum_{\alpha \geqslant 1, \beta \geqslant 1} \Theta_{\alpha, \beta, \gamma}^{\prime} z^{\prime \alpha} \bar{z}^{\prime \beta} .
$$

Levi nondegenerate hypersurfaces. Leaving aside the real defining equation of $M$, let us now rename the complex defining equation of $M$ in such normalized coordinates simply as before: $w=\Theta(z, \bar{z}, \bar{w})$, dropping all the prime signs. Quite concretely, the real analytic hypersurface $M$ is said to be Levi nondegenerate at the origin if the coefficient $\Theta_{1,1,0}$ of $z \bar{z}$, which may be checked to always be real because of the reality condition (7.28), is nonzero. In fact, it is well known that Levi nondegeneracy is a biholomorphically invariant property, see for instance [18], p. 158, but in more conceptual terms, the following general characterization, which may be taken as a definition here, holds true. One then readily checks that it is equivalent to $\Theta_{1,1,0} \neq 0$ in normalized coordinates.
Lemma. ([Me2005a, Me2005b, 19]) The real analytic hypersurface $M \subset$ $\mathbb{C}^{2}$ with $0 \in M$ represented in coordinates $(z, w)$ by a complex defining equation of the form $w=\Theta(z, \bar{z}, \bar{w})$ is Levi nondegenerate at the origin if and only if the map:

$$
(\bar{z}, \bar{w}) \longmapsto\left(\Theta(0, \bar{z}, \bar{w}), \Theta_{z}(0, \bar{z}, \bar{w})\right)
$$

has nonvanishing $2 \times 2$ Jacobian determinant at $(\bar{z}, \bar{w})=(0,0)$.
After a possible real dilation of the $z$-coordinate, we can therefore assume that $\Theta_{1,1,0}=1$, and then we are provided with the following normalization:

$$
\begin{equation*}
w=-\bar{w}+z \bar{z}+z \bar{z} \mathrm{O}(|z|+|\bar{w}|) \tag{7.28}
\end{equation*}
$$

that will be useful shortly. Another, even more convincing argument for consigning to oblivion the real defining equation $u=\varphi(x, y, v)$ dates back to Beniamino Segre [23], who observed that to any real analytic $M$ are associated two deeply linked objects.

1) The nowadays so-called Segre varieties ${ }^{15} S_{\bar{q}}$ associated to any point $q \in \mathbb{C}^{2}$ near the origin of coordinates $\left(z_{q}, w_{q}\right)$ that are the complex curves defined by the equation:

$$
S_{\bar{q}}:=\left\{0=-w+\Theta\left(z, \bar{z}_{q}, \bar{w}_{q}\right)\right\},
$$

quite appropriately in terms of the fundamental complex defining function $\Theta$; this equation is holomorphic just because its antiholomorphic terms are set fixed.

[^9]2) When $M$ is Levi nondegenerate at the origin, a second-order complex ordinary differential equation ${ }^{16}$ of the form:
$$
w_{z z}(z)=\Phi\left(z, w(z), w_{z}(z)\right),
$$
whose solutions are exactly the Segre varieties of $M$, parametrized by the two initial conditions $w(0)$ and $w_{z}(0)$ which correspond bijectively to the antiholomorphic variables $\bar{z}_{q}$ and $\bar{w}_{q}$.

In fact, the recipe for deriving the second-order differential equation associated to a local Levi-nondegenerate $M \subset \mathbb{C}^{2}$ with $0 \in M$ represented by a normalized ${ }^{17}$ equation of the form (7.28) is very simple. Considering that $w=w(z)$ is given in the equation:

$$
w(z)=\Theta(z, \bar{z}, \bar{w})
$$

as a function of $z$ with two supplementary (antiholomorphic) parameters $\bar{z}$ and $\bar{w}$ that one would like to eliminate, we solve with respect to $\bar{z}$ and $\bar{w}$, just by means of the implicit function theorem ${ }^{18}$, the pair of equations:

$$
\left[\begin{array}{rl}
w(z) & =\Theta(z, \bar{z}, \bar{w})=-\bar{w}+z \bar{z}+z \bar{z} \mathrm{O}(|z|+|\bar{w}|) \\
w_{z}(z) & =\Theta_{z}(z, \bar{z}, \bar{w})=\bar{z}+\bar{z} \mathrm{O}(|z|+|\bar{w}|)
\end{array}\right.
$$

the second one being obtained by differentiating the first one with respect to $z$, and this yields a representation:

$$
\bar{z}=\zeta\left(z, w(z), w_{z}(z)\right) \quad \text { and } \quad \bar{w}=\xi\left(z, w(z), w_{z}(z)\right)
$$

for certain two uniquely defined local complex analytic functions $\zeta\left(z, w, w_{z}\right)$ and $\xi\left(z, w, w_{z}\right)$ of three complex variables. By means of these functions, we may then replace $\bar{z}$ and $\bar{w}$ in the second derivative:

$$
\begin{aligned}
w_{z z}(z) & =\Theta_{z z}(z, \bar{z}, \bar{w}) \\
& =\Theta_{z z}\left(z, \zeta\left(z, w(z), w_{z}(z)\right), \xi\left(z, w(z), w_{z}(z)\right)\right) \\
& =: \Phi\left(z, w(z), w_{z}(z)\right)
\end{aligned}
$$

and this defines without ambiguity the associated differential equation. More about differential equations will be said in $\S 3$ below.

[^10]Of course, any spherical real analytic $M \subset \mathbb{C}^{2}$ must be Levi nondegenerate at every point, for the unit 3 -sphere $S^{3} \subset \mathbb{C}^{2}$ is. It is well known that $S^{3}$ minus one of its points, for instance: $S^{3} \backslash\left\{p_{\infty}\right\}$ with $p_{\infty}:=(0,-1)$, is biholomorphic, through the so-called Cayley transform:
$(z, w) \longmapsto\left(\frac{i z}{1+w}, \frac{1-w}{2+2 w}\right)=:\left(z^{\prime}, w^{\prime}\right) \quad$ having inverse: $\quad\left(z^{\prime}, w^{\prime}\right) \longmapsto\left(\frac{-2 i z^{\prime}}{1+2 w^{\prime}}, \frac{1-2 w^{\prime}}{1+2 w^{\prime}}\right)$
to the so-called Heisenberg sphere of equation:

$$
w^{\prime}=-\bar{w}^{\prime}+z^{\prime} \bar{z}^{\prime},
$$

in the target coordinate-space $\left(z^{\prime}, w^{\prime}\right)$, and this model will be more convenient to deal with for our purposes.
Proposition. A Levi nondegenerate local real analytic hypersurface $M$ in $\mathbb{C}^{2}$ is locally biholomorphic to a piece of the Heisenberg sphere (hence spherical) if and only if its associated second-order ordinary complex differential equation is locally equivalent to the Newtonian free particle equation: $w_{z^{\prime} z^{\prime}}^{\prime}\left(z^{\prime}\right)=0$, with identically vanishing right-hand side.

Proof. Indeed, any local equivalence of $M$ to the Heisenberg sphere transforms its differential equation to the one associated with the Heisenberg sphere, and then trivially: $w_{z^{\prime}}^{\prime}\left(z^{\prime}\right)=\bar{z}^{\prime}$, whence $w_{z^{\prime} z^{\prime}}^{\prime}\left(z^{\prime}\right)=0$.

Conversely, if the Segre varieties of $M$ are mapped to the solutions of $w_{z^{\prime} z^{\prime}}^{\prime}\left(z^{\prime}\right)=0$, namely to the complex affine lines of $\mathbb{C}^{2}$, the complex defining equation of the transformed $M^{\prime}$ must necessarily be affine:

$$
\begin{equation*}
w^{\prime}=\bar{\lambda}^{\prime}\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right)+z^{\prime} \bar{\mu}^{\prime}\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right)=: \Theta^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right), \tag{7.28}
\end{equation*}
$$

with certain coefficients that are holomorphic with respect to $\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right)$. Then $\bar{\lambda}^{\prime}(0)=0$ since the origin is fixed, and if $\bar{\mu}^{\prime}(0)$ is nonzero, one performs the linear transformation $z^{\prime} \mapsto z^{\prime}, w^{\prime} \mapsto w^{\prime}-\bar{\mu}^{\prime}(0) z^{\prime}$, which stabilizes both $w_{z^{\prime} z^{\prime}}^{\prime}\left(z^{\prime}\right)=0$ and the form of (7.28), to insure then that $\bar{\mu}^{\prime}(0)=0$.

Next, the second reality condition (7.28) now reads:

$$
w^{\prime} \equiv \bar{\lambda}^{\prime}\left(\bar{z}^{\prime}, \bar{\Theta}^{\prime}\left(\bar{z}^{\prime}, z^{\prime}, w^{\prime}\right)\right)+z^{\prime} \bar{\mu}^{\prime}\left(\bar{z}^{\prime}, \bar{\Theta}^{\prime}\left(\bar{z}^{\prime}, z^{\prime}, w^{\prime}\right)\right),
$$

and by differentiating it with respect to $\bar{z}^{\prime}$, we get, without writing the arguments for brevity:

$$
\begin{aligned}
0 & \equiv \bar{\lambda}_{\bar{z}^{\prime}}^{\prime}+\bar{\Theta}_{\bar{z}^{\prime}}^{\prime} \bar{\lambda}_{\bar{w}^{\prime}}^{\prime}+z^{\prime} \bar{\mu}_{\bar{z}^{\prime}}^{\prime}+z^{\prime} \bar{\Theta}_{\bar{z}^{\prime}}^{\prime} \bar{\mu}_{\bar{w}^{\prime}}^{\prime} \\
& \equiv \bar{\lambda}_{\bar{z}^{\prime}}^{\prime}+\mu^{\prime} \bar{\lambda}_{\bar{w}^{\prime}}^{\prime}+z^{\prime} \bar{\mu}_{\bar{z}^{\prime}}^{\prime}+z^{\prime} \mu^{\prime} \bar{\mu}_{\bar{w}^{\prime}}^{\prime},
\end{aligned}
$$

where we replace $\bar{\Theta}_{\bar{z}^{\prime}}^{\prime}$ in the second line by its value $\mu^{\prime}\left(z^{\prime}, w^{\prime}\right)$. But with all the arguments, this identity reads in full length as the following identity
holding in $\mathbb{C}\left\{\bar{z}^{\prime}, z^{\prime}, w^{\prime}\right\}$ :

$$
\begin{aligned}
& -\bar{\lambda}_{\bar{z}^{\prime}}^{\prime}\left(\bar{z}^{\prime}, \bar{\Theta}^{\prime}\left(\bar{z}^{\prime}, z^{\prime}, w^{\prime}\right)\right)-z^{\prime} \bar{\mu}_{\bar{z}^{\prime}}^{\prime}\left(\bar{z}^{\prime}, \bar{\Theta}^{\prime}\left(\bar{z}^{\prime}, z^{\prime}, w^{\prime}\right)\right) \equiv \\
& \quad \equiv \mu^{\prime}\left(z^{\prime}, w^{\prime}\right) \bar{\lambda}_{\bar{w}^{\prime}}^{\prime}\left(\bar{z}^{\prime}, \bar{\Theta}^{\prime}\left(\bar{z}^{\prime}, z^{\prime}, w^{\prime}\right)\right)+z^{\prime} \mu^{\prime}\left(z^{\prime}, w^{\prime}\right) \bar{\mu}_{\bar{w}^{\prime}}^{\prime}\left(\bar{z}^{\prime}, \bar{\Theta}^{\prime}\left(\bar{z}^{\prime}, z^{\prime}, w^{\prime}\right)\right)
\end{aligned}
$$

For convenience, it is better to take $\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)$ as arguments of this identity instead of $\left(\bar{z}^{\prime}, z^{\prime}, w^{\prime}\right)$, so we simply replace $w^{\prime}$ in it by:

$$
\Theta^{\prime}\left(z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right)
$$

we apply the first reality condition (7.28) and we get what we wanted to pursue the reasonings:

$$
\begin{aligned}
\stackrel{(7.28)}{-\bar{\lambda}_{\bar{z}^{\prime}}^{\prime}}\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right)-z^{\prime} \bar{\mu}_{\bar{z}^{\prime}}^{\prime}\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right) \equiv \mu^{\prime}\left(z^{\prime},\right. & \left.\bar{\lambda}^{\prime}\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right)+z^{\prime} \bar{\mu}^{\prime}\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right)\right) \\
\cdot & {\left[\bar{\lambda}_{\bar{w}^{\prime}}^{\prime}\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right)+z^{\prime} \bar{\mu}_{\bar{w}^{\prime}}^{\prime}\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right)\right] }
\end{aligned}
$$

i.e. an identity holding now in $\mathbb{C}\left\{z^{\prime}, \bar{z}^{\prime}, \bar{w}^{\prime}\right\}$. The left-hand side being affine with respect to $z^{\prime}$, the same must be true of each one of the two factors of the right-hand side. In particular, the second order derivative of the first factor with respect to $z^{\prime}$ must vanish identically:

$$
\begin{aligned}
0 & \equiv \partial_{z^{\prime}} \partial_{z^{\prime}}\left\{\mu^{\prime}\left(z^{\prime}, \bar{\lambda}^{\prime}+z^{\prime} \overline{\mu^{\prime}}\right)\right\} \\
& \equiv \mu_{z^{\prime} z^{\prime}}^{\prime}+2 \bar{\mu}^{\prime} \mu_{z^{\prime} w^{\prime}}^{\prime}+\bar{\mu}^{\prime} \bar{\mu}^{\prime} \mu_{w^{\prime} w^{\prime}}^{\prime}
\end{aligned}
$$

Because $M^{\prime}$ is Levi nondegenerate at the origin, the lemma on p. 192 together with the affine form (7.28) of the defining equation entails that the map:

$$
\begin{equation*}
\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right) \longmapsto\left(\bar{\lambda}^{\prime}\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right), \bar{\mu}^{\prime}\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right)\right) \tag{7.28}
\end{equation*}
$$

has nonvanishing Jacobian determinant at $\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right)=(0,0)$. Consequently, in the above identity (rewritten with some of the arguments):
$0 \equiv \mu_{z^{\prime} z^{\prime}}^{\prime}\left(z^{\prime}, \bar{\lambda}^{\prime}+z^{\prime} \bar{\mu}^{\prime}\right)+2 \bar{\mu}^{\prime} \mu_{z^{\prime} w^{\prime}}^{\prime}\left(z^{\prime}, \bar{\lambda}^{\prime}+z^{\prime} \bar{\mu}^{\prime}\right)+\bar{\mu}^{\prime} \bar{\mu}^{\prime} \mu_{w^{\prime} w^{\prime}}^{\prime}\left(z^{\prime}, \bar{\lambda}^{\prime}+z^{\prime} \bar{\mu}^{\prime}\right)$, we can consider $z^{\prime}, \bar{\lambda}^{\prime}$ and $\bar{\mu}^{\prime}$ as being just three independent variables. Setting $\bar{\mu}^{\prime}=0$, we get $0 \equiv \mu_{z^{\prime} z^{\prime}}^{\prime}\left(z^{\prime}, \bar{\lambda}^{\prime}\right)$, that is to say: $\mu_{z^{\prime} z^{\prime}}\left(z^{\prime}, w^{\prime}\right) \equiv 0$ and then after division of $\bar{\mu}^{\prime}$, we are left with only two terms:

$$
0 \equiv 2 \mu_{z^{\prime} w^{\prime}}^{\prime}\left(z^{\prime}, \bar{\lambda}^{\prime}+z^{\prime} \bar{\mu}^{\prime}\right)+\bar{\mu}^{\prime} \mu_{w^{\prime} w^{\prime}}^{\prime}\left(z^{\prime}, \bar{\lambda}^{\prime}+z^{\prime} \bar{\mu}^{\prime}\right)
$$

Then again $0 \equiv 2 \mu_{z^{\prime} w^{\prime}}\left(z^{\prime}, w^{\prime}\right)$ and finally also $0 \equiv \mu_{w^{\prime} w^{\prime}}\left(z^{\prime}, w^{\prime}\right)$. This means that the function:

$$
\mu^{\prime}\left(z^{\prime}, w^{\prime}\right)=c_{1}^{\prime} z^{\prime}+c_{2}^{\prime} w^{\prime}
$$

with some two constants $c_{1}^{\prime}, c_{2}^{\prime} \in \mathbb{C}$, is linear.

Now, we claim that $c_{2}^{\prime}=0$ in fact. Indeed, setting $\bar{z}^{\prime}=0$ in (7.28), we get:
$-\bar{\lambda}_{\bar{z}^{\prime}}^{\prime}\left(0, \bar{w}^{\prime}\right)-z^{\prime} \bar{c}_{1}^{\prime} \equiv\left\{c_{1}^{\prime} z^{\prime}+c_{2}^{\prime}\left(\bar{\lambda}^{\prime}\left(0, \bar{w}^{\prime}\right)+z^{\prime} \bar{c}_{2}^{\prime} \bar{w}^{\prime}\right)\right\} \cdot\left[\bar{\lambda}_{\bar{w}^{\prime}}^{\prime}\left(0, \bar{w}^{\prime}\right)+z^{\prime} \bar{c}_{2}^{\prime}\right]$.
The coefficient $c_{2}^{\prime} \bar{c}_{2}^{\prime} \bar{c}_{2}^{\prime}$ of $\left(z^{\prime}\right)^{2} \bar{w}^{\prime}$ in the right-hand side must vanish, so $c_{2}^{\prime}=$ 0 . Since the rank at the origin of the map (7.28) equals 2 , necessarily $\mu^{\prime} \not \equiv 0$, so $c_{1}^{\prime} \neq 0$, and then $c_{1}^{\prime}=1$ after a suitable dilation of the $z^{\prime}$-axis. Next, rewriting the identity (7.28):

$$
-\bar{\lambda}_{\bar{z}^{\prime}}^{\prime}\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right)-z^{\prime} \equiv z^{\prime}\left[\bar{\lambda}_{\bar{w}^{\prime}}^{\prime}\left(\bar{z}^{\prime}, \bar{w}^{\prime}\right)\right],
$$

we finally get $\bar{\lambda}_{\bar{z}^{\prime}}^{\prime} \equiv 0$ and $\bar{\lambda}_{\bar{w}^{\prime}}^{\prime} \equiv-1$, which means in conclusion that:

$$
\lambda^{\prime}\left(z^{\prime}, w^{\prime}\right) \equiv-w^{\prime} \quad \text { and } \quad \mu^{\prime}\left(z^{\prime}, w^{\prime}\right) \equiv z^{\prime}
$$

so that the equation of $M^{\prime}$ is the one: $w^{\prime}=-\bar{w}^{\prime}+z^{\prime} \bar{z}^{\prime}$ of the Heisenberg sphere in the target coordinates $\left(z^{\prime}, w^{\prime}\right)$.

Thanks to this proposition, in order to characterize the sphericality of a local real analytic hypersurface $M \subset \mathbb{C}^{2}$ explicitly in terms of its complex defining function $\Theta$, our strategy ${ }^{19}$ will be to:
characterize the local equivalence to $w_{z^{\prime} z^{\prime}}^{\prime}\left(z^{\prime}\right)=0$ of the associated differential equation:

$$
\begin{equation*}
w_{z z}(z)=\Theta_{z z}\left(z, \zeta\left(z, w(z), w_{z}(z)\right), \xi\left(z, w(z), w_{z}(z)\right)\right) \tag{7.28}
\end{equation*}
$$

explicitly in terms of the three functions $\Theta_{z z}, \zeta$ and $\xi$;
eliminate any occurence of the two auxiliary functions $\zeta$ and $\xi$ so as to re-express the obtained result only in terms of the sixth-order jet $J_{z, \bar{z}, \bar{w}}^{6} \Theta$.

## §3. GEOMETRY OF ASSOCIATED SUBMANIFOLDS OF SOLUTIONS

The characterization we will obtain holds in fact inside a broader context than just CR geometry, in terms of what we called in [19] the submanifold of solutions associated to any second-order ordinary differential equation, no matter whether it comes or not from a Levi nondegenerate $M \subset \mathbb{C}^{2}$. In fact, the elementary foundations towards a general theory embracing all systems of completely integrable partial differential equations was laid down [19], especially by producing explicit prolongation formulas for infinitesimal Lie symmetries, with many interesting problems that are still wide open as soon as the number of (independent or dependent) variables increases: construction of Cartan connections; production of differential invariants; full classification according to the Lie symmetry group.

[^11]Fortunately for our present purposes here, the geometry, the classification, and the Lie transformation group features of second order ordinary differential equations are essentially completely understood since the groundbreaking works of Lie [Lie1883], followed by a prized thesis by Tresse [Tr1896] and later by a celebrated memoir of Élie Cartan, see also [GTW1989] and the references therein.

Accordingly, letting $x \in \mathbb{K}$ and $y \in \mathbb{K}$ be two real or complex variables (with hence $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ throughout), consider any second-order ordinary differential equation:

$$
y_{x x}(x)=F\left(x, y(x), y_{x}(x)\right)
$$

having local $\mathbb{K}$-analytic right-hand side $F$, and denote it by $(\mathscr{E})$ for short. In the space of first-order jets of arbitrary graphing functions $y=y(x)$ that we equip with three independent coordinates denoted $\left(x, y, y_{x}\right)$, let us introduce the vector field:

$$
\mathrm{D}:=\frac{\partial}{\partial x}+y_{x} \frac{\partial}{\partial y}+F\left(x, y, y_{x}\right) \frac{\partial}{\partial y_{x}},
$$

whose integral curves inside the three-dimensional space $\left(x, y, y_{x}\right)$ correspond, classically, to solving the equation $y_{x x}(x)=F\left(x, y(x), y_{x}(x)\right)$ by transforming it into a system of two first-order differential equations with the two unknown functions $y(x)$ and $y_{x}(x)$.
Theorem. ([Lie1883, Tr1896, Ca1924, GTW1989, 17]) A second-order ordinary differential equation $y_{x x}=F\left(x, y, y_{x}\right)$ denoted $(\mathscr{E})$ with $\mathbb{K}$-analytic right-hand side possesses two fundamental differential invariants, namely:

$$
\begin{aligned}
\mathrm{I}_{(\mathscr{E})}^{1}:= & F_{y_{x} y_{x} y_{x} y_{x}} \quad \text { and: } \\
\mathrm{I}_{(\mathscr{E})}^{( }:= & \mathrm{DD}\left(F_{y_{x} y_{x}}\right)-F_{y_{x}} \mathrm{D}\left(F_{y_{x} y_{x}}\right)-4 \mathrm{D}\left(F_{y y_{x}}\right)+ \\
& +6 F_{y y}-3 F_{y} F_{y_{x} y_{x}}+4 F_{y_{x}} F_{y y_{x}},
\end{aligned}
$$

while all other differential invariants are deduced from $\mathrm{I}_{(\mathscr{E})}^{1}$ and $\mathrm{I}_{(\mathscr{E})}^{2}$ by covariant (in the sense of Tresse) or coframe (in the sense of Cartan) diffentiations. Moreover, local equivalence to $y_{x^{\prime} x^{\prime}}^{\prime}\left(x^{\prime}\right)=0$ holds under some invertible local $\mathbb{K}$-analytic point transformation:

$$
(x, y) \longmapsto\left(x^{\prime}, y^{\prime}\right)=\left(x^{\prime}(x, y), y^{\prime}(x, y)\right)
$$

if and only if both invariants vanish:

$$
0=I_{(\mathscr{E})}^{1}=I_{(\mathscr{E})}^{2} .
$$

In order to characterize sphericality of an $M \subset \mathbb{C}^{2}$, it is then natural and advisable to study what the vanishing of the above two differential invariants gives when applied to the second order ordinary differential equation (7.28) enjoyed by the defining function $\Theta$. This goal will be pursued in $\S 4$ below.

For the time being, with the aim of extending such a kind of characterization to a broader scope, following §2 of [19], let us now recall how one may in a natural way construct a sumanifold of solutions $\mathscr{M}_{\mathscr{E}}$ associated to the differential equation $(\mathscr{E})$ which, when $(\mathscr{E})$ comes from a Levi nondegenerate local real analytic hypersurface $M \subset \mathbb{C}^{2}$, regives without any modification its complex defining equation $w=\Theta(z, \bar{z}, \bar{w})$.

To begin with, in the first-order jet space $\left(x, y, y_{x}\right)$ that we simply draw as a common three-dimensional space:

we duplicate the two dependent coordinates $\left(y, y_{x}\right)$ by introducing a new subspace of coordinates $(a, b) \in \mathbb{K} \times \mathbb{K}$, and we draw a vertical plane containing the two new axes that are just parallel copies (for the moment, just look at the left-hand side). Then the leaves of the local foliation associated to the integral curves of the vector field D are uniquely determined by their intersection with this plane, because thanks to the presence of $\frac{\partial}{\partial x}$ in D , all these curves are approximately directed by the $x$-axis in a neighborhood of the origin: no tangent vector can be vertical. But we claim that all such intersection points of coordinates $(0, b, a) \in \mathbb{K} \times \mathbb{K} \times \mathbb{K}$ correspond bijectively to the two initial conditions $y(0) \equiv b$ and $y_{x}(0)=a$ for solving uniquely the differential equation. In fact, the flow of D at time $x$ starting from all such points $(0, b, a)$ of the duplicated vertical plane:

$$
\exp (x \mathrm{D})(0, b, a)=:(x, Q(x, a, b), S(x, a, b))
$$

(see again the diagram) expresses itself in terms of two certain local $\mathbb{K}$ analytic functions $Q$ and $S$ that satisfy, by the very definition of the flow of our vector field $\partial_{x}+y_{x} \partial_{y}+F \partial_{y_{x}}$, the following two differential equations:

$$
\frac{d}{d x} Q(x, a, b)=S(x, a, b) \quad \text { and: } \quad \frac{d}{d x} S(x, a, b)=F(x, Q(x, a, b), S(x, a, b))
$$

together with the (obious) initial condition for $x=0$ :

$$
(0, b, a)=\exp (0 \mathrm{D})(0, b, a)=(0, Q(0, a, b), S(0, a, b)) .
$$

We notice passim that $S \equiv Q_{x}$ (no two symbols were in fact needed), and most importantly, we emphasize that in this way, we have viewed in a somewhat geometric-minded way of thinking that the general solution:

$$
y=y(x)=Q\left(x, y_{x}(0), y(0)\right)=Q(x, a, b)
$$

to the original differential equation arises naturally as the first (amongst two) graphing function for the integral curves of $D$ in the first order jet space, these curves being parametrized by $(a, b)$.
Definition. The sumanifold of solutions ${ }^{20} \mathscr{M}_{(\mathscr{E})}$ associated with the secondorder ordinary differential equation $(\mathscr{E}): y_{x x}(x)=F\left(x, y(x), y_{x}(x)\right)$ is the local $\mathbb{K}$-analytic submanifold of the four-dimensional Euclidean space $\mathbb{K}_{x} \times \mathbb{K}_{y} \times \mathbb{K}_{a} \times \mathbb{K}_{b}$ represented as the zero-set:

$$
0=-y+Q(x, a, b)
$$

where $Q(x, a, b)$ is the general local $\mathbb{K}$-analytic solution of $(\mathscr{E})$, satisfying therefore:

$$
Q_{x x}(x, a, b) \equiv F\left(x, Q(x, a, b), Q_{x}(x, a, b)\right)
$$

and $Q(0, a, b)=b, Q_{x}(0, a, b)=a$.
Conversely, let us assume we are given a submanifold $\mathscr{M}$ of $\mathbb{K}_{x} \times \mathbb{K}_{y} \times$ $\mathbb{K}_{a} \times \mathbb{K}_{b}$ of the specific equation $y=Q(x, a, b)$, for a certain local $\mathbb{K}$ analytic function $Q$ of the three variables $(x, a, b)$. Call $(x, y)$ the variables, $(a, b)$ the parameters, and call $\mathscr{M}$ solvable with respect to the parameters (at the origin) if the map:

$$
(a, b) \longmapsto\left(Q(0, a, b), Q_{x}(0, a, b)\right)
$$

has rank two at the central point $(a, b)=(0,0)$. Of course, the submanifold of solutions associated to any second-order ordinary differential equation is solvable with respect to parameters, for in this case $Q(0, a, b) \equiv b$ and $Q_{x}(0, a, b) \equiv a$.

Similarly as what we did for deriving 2) on p. 193, if an arbitrarily given submanifold $\mathscr{M}$ of $\mathbb{K}_{x} \times \mathbb{K}_{y} \times \mathbb{K}_{a} \times \mathbb{K}_{b}$ is assumed to be solvable with respect to parameters, then viewing $y$ in $y=Q(x, a, b)$ as a parametrized function of $x$, the implicit function theorem enables one to solve $(a, b)$ in the two equations:

$$
\left[\begin{array}{rl}
y(x) & =Q(x, a, b) \\
y_{x}(x) & =Q_{x}(x, a, b),
\end{array}\right.
$$

[^12]to yield both a representation for $a$ and and a representation for $b$ of the form:
\[

\left[$$
\begin{array}{l}
a=A\left(x, y(x), y_{x}(x)\right)  \tag{7.28}\\
b=B\left(x, y(x), y_{x}(x)\right),
\end{array}
$$\right.
\]

for certain two local $\mathbb{K}$-analytic functions $A$ and $B$ of three independent variables $\left(x, y, y_{x}\right)$, that one may insert afterwards in the second order derivative:

$$
\begin{aligned}
y_{x x}(x) & =Q_{x x}(x, a, b) \\
& =Q_{x x}\left(x, A\left(x, y(x), y_{x}(x)\right), B\left(x, y(x), y_{x}(x)\right)\right) \\
& =: F\left(x, y(x), y_{x}(x)\right),
\end{aligned}
$$

which yields the differential equation $\left(\mathscr{E}_{\mathscr{M}}\right)$ associated to the submanifold $\mathscr{M}$ solvable with respect to the parameters. In summary:
Proposition. ([19]) There is a one-to-one correspondence:

$$
\left(\mathscr{E}_{\mathscr{M}}\right)=(\mathscr{E}) \longleftrightarrow \mathscr{M}=\mathscr{M}_{(\mathscr{E})}
$$

between second-order ordinary differential equations ( $\mathscr{E}$ ) of the general form:

$$
y_{x x}(x)=F\left(x, y(x), y_{x}(x)\right)
$$

and submanifolds (of solutions) $\mathscr{M}$ of equation:

$$
y=Q(x, a, b)
$$

that are solvable with respect to the parameters, and this correspondence satisfies:

$$
\left(\mathscr{E}_{\mathscr{M}_{(\mathscr{E})}}\right)=(\mathscr{E}) \quad \text { and } \quad \mathscr{M}_{(\mathscr{E} \mathscr{M})}=\mathscr{M}
$$

We now claim that solvability with respect to the parameters is an invariant condition, independently of the choice of coordinates. Indeed, let $y=Q(x, a, b)$ be any submanifold of solutions, call it $\mathscr{M}$, and let:

$$
(x, y, a, b) \longmapsto\left(x^{\prime}(x, y), y^{\prime}(x, y), a, b\right)
$$

be an arbitrary local $\mathbb{K}$-analytic diffeomorphism fixing the origin which leaves untouched the parameters. The vector of coordinates $\left(1, Q_{x}(x, a, b), 0,0\right)$ based at the point $(x, Q(x, a, b), a, b)$ of $\mathscr{M}$ is sent, through such a diffeomorphism, to a vector whose $x^{\prime}$-coordinate equals: $\frac{d}{d x}\left[x^{\prime}(x, Q)\right]=x_{x}^{\prime}+Q_{x} x_{y}^{\prime}$. Therefore the implicit function theorem insures that, provided the expression:

$$
x_{x}^{\prime}(x, y)+Q_{x}(x, a, b) x_{y}^{\prime}(x, y) \neq 0
$$

does not vanish, the image $\mathscr{M}^{\prime}$ of $\mathscr{M}$ through such a diffeomorphism can still be represented, locally in a neighborhood of the origin, as a graph of a similar form:

$$
y^{\prime}=Q^{\prime}\left(x^{\prime}, a, b\right),
$$

for a certain local $\mathbb{K}$-analytic new function $Q^{\prime}=Q^{\prime}\left(x^{\prime}, a, b\right)$. Since $\mathscr{M}: y=$ $Q(x, a, b)$ is sent to $\mathscr{M}^{\prime}: y^{\prime}=Q^{\prime}\left(x^{\prime}, a, b\right)$, it follows that $x^{\prime}(x, y), y^{\prime}(x, y)$, $Q(x, a, b)$ and $Q^{\prime}\left(x^{\prime}, a, b\right)$ are all linked by the following fundamental identity:

$$
\begin{equation*}
y^{\prime}(x, Q(x, a, b)) \equiv Q^{\prime}\left(x^{\prime}(x, Q(x, a, b)), a, b\right) \tag{7.28}
\end{equation*}
$$

which holds in $\mathbb{C}\{x, a, b\}$.
Claim. If $\mathscr{M}$ is solvable with respect to the parameters (at the origin), then $\mathscr{M}^{\prime}$ is also solvable with respect to the parameters (at the origin too), and conversely.

Proof. The assumption that $\mathscr{M}$ is solvable with respect to the parameters is equivalent to the fact that its first order $x$-jet map:

$$
\left.(x, a, b) \longmapsto\left(x, Q(x, a, b), Q_{x}(x, a, b)\right)\right)
$$

is (locally) of rank three. One should therefore look at the same first order jet map attached to $\mathscr{M}^{\prime}$, represented in the right part of the following diagram:

and ask how these two $x$ - and $x^{\prime}$-jet maps can be related to each other, namely search for a map:

$$
\mathbf{X ? :} \quad\left(x, Q, Q_{x}\right) \longmapsto\left(x, Q^{\prime}, Q_{x}^{\prime}\right)
$$

which would close up the diagram and make it commutative.
The answer for the second component of the sought map is simply:

$$
\mathbf{X}_{2}: \quad\left(x, Q, Q_{x}\right) \longmapsto y^{\prime}(x, Q)
$$

since (9) indeed shows that composing the right vertical arrow with the upper horizontal one gives the same result, concerning a second component, as composing the bottom horizontal arrow with the left vertical one.

The answer for the third component of the sought map then proceeds by differentiating with respect to $x$ the fundamental identity (9), which yields, without writing the arguments:

$$
y_{x}^{\prime}+Q_{x} y_{y}^{\prime} \equiv\left[x_{x}^{\prime}+Q_{x} x_{y}^{\prime}\right] Q_{x^{\prime}}^{\prime}
$$

and since $x_{x}^{\prime}+Q_{x} x_{y}^{\prime} \neq 0$ by assumption, it suffices to set:

$$
\mathrm{X}_{3}: \quad\left(x, Q, Q_{x}\right) \longmapsto \frac{y_{x}^{\prime}(x, Q)+Q_{x} y_{y}^{\prime}(x, Q)}{x_{x}^{\prime}(x, Q)+Q_{x} x_{y}^{\prime}(x, Q)}
$$

in order to complete the commutativity of the diagram, namely to get:

$$
Q_{x^{\prime}}^{\prime}(f(x, Q(x, a, b)), a, b) \equiv \frac{y_{x}^{\prime}(x, Q(x, a, b))+Q_{x}(x, a, b) y_{y}^{\prime}(x, Q(x, a, b))}{x_{x}^{\prime}(x, Q(x, a, b))+Q_{x}(x, a, b) x_{y}^{\prime}(x, Q(x, a, b))},
$$

as was required. But now considering instead the inverse diffeomorphisme changes nothing to the reasonings, hence we have at the same time a rightinverse:

of our commutative diagram, so that the $x$-jet map and the $x^{\prime}$-jet map have coinciding ranks at pairs of points which correspond one to another.

We are now in a position to generalize the characterization of sphericality derived earlier on p. 223.
Proposition. A second-order ordinary differential equation $y_{x x}(x)=$ $F\left(x, y(x), y_{x}(x)\right)$ with $\mathbb{K}$-analytic right-hand side is equivalent, under some invertible local $\mathbb{K}$-analytic point transformation $(x, y) \mapsto\left(x^{\prime}, y^{\prime}\right)$, to the free particle Newtonian equation $y_{x^{\prime} x^{\prime}}^{\prime}\left(x^{\prime}\right)=0$ if and only if its associated submanifold of solutions $y=Q(x, a, b)$ is equivalent, under some local $\mathbb{K}$ analytic map in which variables are separated from parameters:

$$
(x, y, a, b) \longmapsto\left(x^{\prime}(x, y), y^{\prime}(x, y), a^{\prime}(a, b), b^{\prime}(a, b)\right)
$$

to the affine submanifold of solutions of equation $y^{\prime}=b^{\prime}+x^{\prime} a^{\prime}$.
Before proceeding to the proof, let us observe that when one looks at a real analytic hypersurface $M \subset \mathbb{C}^{2}$, the corresponding transformation in the parameter space is constrained to be the conjugate transformation of the local biholomorphism:

$$
(z, w, \bar{z}, \bar{w}) \longmapsto\left(z^{\prime}(z, w), w^{\prime}(z, w), \bar{z}^{\prime}(\bar{z}, \bar{w}), \bar{w}^{\prime}(\bar{z}, \bar{w})\right),
$$

while one has more freedom for general differential equations, in the sense that transformations of variables and transformations of parameters are entirely decoupled.

Proof. One direction is clear: if $y=Q(x, a, b)$ is equivalent to:

$$
\begin{equation*}
y^{\prime}=b^{\prime}+x^{\prime} a^{\prime}=b^{\prime}(a, b)+x^{\prime} a^{\prime}(a, b), \tag{7.28}
\end{equation*}
$$

then its associated differential equation $y_{x x}(x)=F\left(x, y(x), y_{x}(x)\right)$ is equivalent, through the same diffeomorphism $(x, y) \mapsto\left(x^{\prime}, y^{\prime}\right)$ of the variables, to the differential equation associated with (7.28), which trivially is: $y_{x^{\prime} x^{\prime}}^{\prime}\left(x^{\prime}\right)=0$.

Conversely, if $y_{x x}(x)=F\left(x, y(x), y_{x}(x)\right)$ is equivalent, through a diffeomorphism $(x, y) \mapsto\left(x^{\prime}, y^{\prime}\right)$, to $y_{x^{\prime} x^{\prime}}^{\prime}\left(x^{\prime}\right)=0$, then its submanifold of solutions $y=Q(x, a, b)$ is transformed to $y^{\prime}=Q^{\prime}\left(x^{\prime}, a, b\right)$ and since $y_{x^{\prime} x^{\prime}}^{\prime}\left(x^{\prime}\right)=0$, the function $Q^{\prime}$ is necessarily of the form:

$$
y^{\prime}=b^{\prime}(a, b)+x^{\prime} a^{\prime}(a, b) .
$$

Because the condition of solvability with respect to the parameters is invariant, the rank of $(a, b) \mapsto\left(a^{\prime}(a, b), b^{\prime}(a, b)\right)$ is again equal to 2 , which concludes the proof.

Coming now back to the wanted characterization of sphericality, our more general goal now amounts to characterize, directly in terms of its fundamental solution function $Q(x, a, b)$, the local equivalence to $y_{x^{\prime} x^{\prime}}^{\prime}\left(x^{\prime}\right)=0$ of a second-order ordinary differential equation $y_{x x}(x)=F\left(x, y(x), y_{x}(x)\right)$. Afterwards at the end, it will suffice to replace $Q(x, a, b)$ simply by $\Theta(z, \bar{z}, \bar{w})$ in the obtained equations.

But before going further, let us explain how a certain generalized projective duality will simplify our task, as already said in the Introduction. Thus, let $(\mathscr{E}): y_{x x}(x)=F\left(x, y(x), y_{x}(x)\right)$ be a differential equation as above having general solution $y=Q(x, a, b)=-b+x a+\mathrm{O}\left(x^{2}\right)$, with initial conditions $b=-y(0)$ and $a=y_{x}(0)$. The implicit function theorem enables us to solve $b$ in the equation $y=Q(x, a, b)$ of the associated submanifold of solutions $\mathscr{M}_{(\mathscr{E})}$ in terms of the other quantities, which yields an equation of the shape:

$$
b=Q^{*}(a, x, y)=-y+a x+\mathrm{O}\left(x^{2}\right),
$$

for some new local $\mathbb{K}$-analytic function $Q^{*}=Q^{*}(a, x, y)$. Then similarly as previously, we may eliminate $x$ and $y$ from the two equations:

$$
\begin{aligned}
& b(a)=Q^{*}(a, x, y)=-y+a x+\mathrm{O}\left(x^{2}\right) \\
& b_{a}(a)=Q_{a}^{*}(a, x, y)=x+\mathrm{O}\left(x^{2}\right) \text {, }
\end{aligned}
$$

that is to say: $x=X\left(a, b(a), b_{a}(a)\right)$ and $y=Y\left(a, b(a), b_{a}(a)\right)$, and we then insert these two solutions in:

$$
\begin{aligned}
b_{a a}(a) & =Q_{a a}^{*}(a, x, y) \\
& =Q_{a a}^{*}\left(a, X\left(a, b(a), b_{a}(a)\right), Y\left(a, b(a), b_{a}(a)\right)\right) \\
& =: F^{*}\left(a, b(a), b_{a}(a)\right) .
\end{aligned}
$$

We shall call the so obtained second-order ordinary differential equation the dual of $y_{x x}(x)=F\left(x, y(x), y_{x}(x)\right)$.

In the case of a hypersurface $M \subset \mathbb{C}^{2}$, solving $\bar{w}$ in the equation $w=$ $\Theta(z, \bar{z}, \bar{w})$ gives nothing else but the conjugate equation $\bar{w}=\bar{\Theta}(\bar{z}, z, w)$,
just by virtue of the reality identities (7.28). It also follows rather trivially that the dual differential equation:

$$
\begin{aligned}
\bar{w}_{\overline{z z}}(\bar{z}) & =\bar{\Theta}_{\overline{z z}}\left(\bar{z}, \bar{\zeta}\left(\bar{z}, \bar{w}(\bar{z}), \bar{w}_{\bar{z}}(\bar{z})\right), \bar{\xi}\left(\bar{z}, \bar{w}(\bar{z}), \bar{w}_{\bar{z}}(\bar{z})\right)\right) \\
& =\bar{\Phi}\left(\bar{z}, \bar{w}(\bar{z}), \bar{w}_{\bar{z}}(\bar{z})\right)
\end{aligned}
$$

is also just the conjugate differential equation.
To the differential equation $y_{x x}=F$ and to its dual $b_{a a}=F^{*}$ are associated two submanifolds of solutions:

$$
\mathscr{M}=\mathscr{M}_{(\mathscr{E})}:=\{(x, y, a, b) \in \mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{K}: y=Q(x, a, b)\}
$$

together with:

$$
\mathscr{M}^{*}=\mathscr{M}_{\left(\mathscr{E}^{*}\right)}:=\left\{(a, b, x, y) \in \mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{K}: \quad b=Q^{*}(a, x, y)\right\}
$$

and as one obviously guesses, the duality, when viewed within submanifolds of solutions, just amounts to permute variables and parameters:

$$
\mathscr{M} \ni(x, y, a, b) \longleftrightarrow(a, b, x, y) \in \mathscr{M}^{*}
$$

In the CR case, if we denote by $\widetilde{z}$ and $\widetilde{w}$ two independent complex variables which correspond to the complexifications of $\bar{z}$ and $\bar{w}$ (respectively of course), the duality takes place between the so-called extrinsic complexification ([13, 14, Me2005a, Me2005b, 18, 19]):

$$
\mathscr{M}=M^{c}:=\{(z, w, \widetilde{z}, \widetilde{w}) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}: w=\Theta(z, \widetilde{z}, \widetilde{w})\}
$$

of $M$ in one hand, and in the other hand, its own transformation ${ }^{21}$ :

$$
\mathscr{M}^{*}=*^{c}\left(M^{c}\right):=\{(\widetilde{z}, \widetilde{w}, z, w) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}: \widetilde{w}=\bar{\Theta}(\widetilde{z}, z, w)\}
$$

under the involution:

$$
*^{c}(z, w, \widetilde{z}, \widetilde{w}):=(\widetilde{z}, \widetilde{w}, z, w)
$$

which clearly is the complexification of the natural antiholomorphic involution:

$$
*(z, w, \bar{z}, \bar{w}):=(\bar{z}, \bar{w}, z, w)
$$

that fixes $M$ pointwise, as it fixes any other real analytic subset of $\mathbb{C}^{2}$. Here, one has $\mathscr{M}^{*}=*(\mathscr{M})$ - which is $\neq \mathscr{M}$ in general - and of course also $\left(\mathscr{M}^{*}\right)^{*}=\mathscr{M}$.

So in terms of the coordinates $(x, a, b)$ on $\mathscr{M}$ and of the coordinates $(a, x, y)$ on $\mathscr{M}^{*}$, the duality is the map:

$$
(x, a, b) \longmapsto(a, x, Q(x, a, b))
$$

with inverse:

$$
(a, x, y) \longmapsto\left(x, a, Q^{*}(a, x, y)\right) .
$$

[^13]But we may also express the duality from the first jet $\left(x, y, y_{x}\right)$-space to the first jet $\left(a, b, b_{a}\right)$-space by simply composing the following three maps, the central one being the duality $\mathscr{M} \rightarrow \mathscr{M}^{*}$ :

$$
\left(\begin{array}{c}
(a, x, y) \\
\downarrow \\
\left(a, Q^{*}(a, x, y), Q_{a}^{*}(a, x, y)\right)
\end{array}\right) \circ((x, a, b) \rightarrow(a, x, Q(x, a, b))) \circ\left(\begin{array}{c}
\left(x, A\left(x, y, y_{x}\right), B\left(x, y, y_{x}\right)\right) \\
\uparrow \\
\left(x, y, y_{x}\right)
\end{array}\right),
$$

which in sum gives us the map:

$$
\left(x, y, y_{x}\right) \longmapsto\left(\begin{array}{cc}
A\left(x, y, y_{x}\right), & Q^{*}\left(A\left(x, y, y_{x}\right), x, Q\left(x, A\left(x, y, y_{x}\right), B\left(x, y, y_{x}\right)\right)\right), \\
& Q_{a}^{*}\left(A\left(x, y, y_{x}\right), x, Q\left(x, A\left(x, y, y_{x}\right), B\left(x, y, y_{x}\right)\right)\right)
\end{array}\right) .
$$

With the approximations, one checks that:

$$
\left(x, y, y_{x}\right) \longmapsto\left(y_{x}+\cdots,-y+x y_{x}+\cdots, x+\cdots\right),
$$

where the remainder terms " $+\cdots$ " are all $\mathrm{O}\left(x^{2}\right)$. For the differential equation $y_{x x}(x)=0$ of affine lines, these remainders disappear completely and we recover the classical projective duality written in inhomogeneous coordinates ([5], pp. 156-157). Furthermore, one shows (see e.g. [5]) that the above duality map within first order jet spaces is a contact transformation, namely through it, the pullback of the standard contact form $d b-b_{a} d a$ in the target space is a nonzero multiple of the standard contact form $d y-y_{x} d x$ in the source space.

But what matters more for us is the following. The two fundamental differential invariants of $b_{a a}(a)=F^{*}\left(a, b(a), b_{a}(a)\right)$ are functions exactly similar to the ones written on p. 197, namely:

$$
\begin{aligned}
\mathrm{I}_{\left(\mathscr{E}^{*}\right)}^{1}:= & F_{b_{a} b_{a} b_{a} b_{a}}^{*} \\
\mathrm{I}_{\left(\mathscr{E}^{*}\right)}^{2}:= & \mathrm{D}^{*} \mathrm{D}^{*}\left(F_{b_{a} b_{a}}^{*}\right)-F_{b_{a}}^{*} \mathrm{D}^{*}\left(F_{b_{a} b_{a}}^{*}\right)-4 \mathrm{D}^{*}\left(F_{b b_{a}}^{*}\right)+ \\
\quad & +6 F_{b b}^{*}-3 F_{b}^{*} F_{b_{a} b_{a}}^{*}+4 F_{b_{a}}^{*} F_{b b_{a}}^{*},
\end{aligned}
$$

where $\mathrm{D}^{*}:=\partial_{a}+b_{a} \partial_{b}+F^{*}\left(a, b, b_{a}\right) \partial_{b_{a}}$. Then according to Koppisch ([10]), through the duality map, $l_{(\mathscr{E})}^{1}$ is transformed to a nonzero multiple of $\mathrm{I}_{\left(\mathscr{E}^{*}\right)}^{2}$, and simultaneously also, $\mathrm{I}_{\left(\mathscr{E}^{\mathscr{E}}\right)}^{2}$ is transformed to a nonzero multiple ${ }^{22}$ of $\left.\right|_{\left(\mathscr{E}^{*}\right)} ^{1}$, so that:

$$
\begin{array}{lll}
0=I_{(\mathscr{E})}^{1} & \Longleftrightarrow & I_{\left(\mathscr{E}^{*}\right)}^{2}=0 \\
0=I_{(\mathscr{E})}^{2} & \Longleftrightarrow & I_{\left(\mathscr{E}^{*}\right)}^{1}=0 .
\end{array}
$$

Consequently, the differential equation $(\mathscr{E}): y_{x x}(x)=F\left(x, y(x), y_{x}(x)\right)$ is equivalent to $y_{x^{\prime} x^{\prime}}^{\prime}\left(x^{\prime}\right)=0$ if and only if:

$$
F_{y_{x} y_{x} y_{x} y_{x}}=0 \quad \text { and } \quad F_{b_{a} b_{a} b_{a} b_{a}}^{*}=0 .
$$

[^14]This observation has essentially no practical interest, because the computation of $F^{*}$ in terms of $F$ relies upon the composition of three maps ... except notably in the CR case, since the duality in this case is complex conjugation: $\Phi^{*}=\bar{\Phi}$. In summary, we have established the following.
Proposition. An arbitrary, not necessarily rigid, real analytic hypersurface $M \subset \mathbb{C}^{2}$ which is Levi nondegenerate at one of its points $p$ and has a complex definining equation of the form:

$$
w=\Theta(z, \bar{z}, \bar{w})
$$

in some system of local holomorphic coordinates $(z, w) \in \mathbb{C}^{2}$ centered at $p$, is spherical at $p$ if and only if the right-hand side $\Phi$ of its uniquely associated second-order ordinary complex differential equation:

$$
w_{z z}(z)=\Phi\left(z, w(z), w_{z}(w)\right)
$$

satisfies the single fourth-order partial differential equation:

$$
0 \equiv \Phi_{w_{z} w_{z} w_{z} w_{z}}\left(z, w, w_{z}\right)
$$

It now only remains to re-express this fourth-order partial differential equation in terms of the complex graphing function $\Theta(z, \bar{z}, \bar{w})$ for $M$. We will achieve this more generally for $F_{y_{x} y_{x} y_{x} y_{x}}$.

## §4. EfFECTIVE DIFFERENTIAL CHARACTERIZATION OF SPHERICALITY IN $\mathbb{C}^{2}$

Reminding the reasonings and notations introduced in a neighborhood of equation (7.28), the transformation:

$$
\left(x, y, y_{x}\right) \longmapsto(x, a, b)
$$

and its inverse are given by the two triples of functions:

$$
\left[\begin{array} { r l } 
{ x } & { = x } \\
{ a } & { = A ( x , y , y _ { x } ) } \\
{ b } & { = B ( x , y , y _ { x } ) }
\end{array} \quad \text { and } \quad \left[\begin{array}{r}
x \\
=x \\
y
\end{array} \quad Q(x, a, b) .\right.\right.
$$

Equivalently, one has the two pairs of identically satisfied equations:

$$
\begin{aligned}
a & \equiv A\left(x, Q(x, a, b), Q_{x}(x, a, b)\right) \quad \text { and } \quad \begin{aligned}
y & \equiv Q\left(x, A\left(x, y, y_{x}\right), B\left(x, y, y_{x}\right)\right) \\
b & \equiv B\left(x, Q(x, a, b), Q_{x}(x, a, b)\right)
\end{aligned} \quad \begin{array}{l} 
\\
y_{x}
\end{array} Q_{x}\left(x, A\left(x, y, y_{x}\right), B\left(x, y, y_{x}\right)\right) .
\end{aligned}
$$

Differentiating the second column of equations with respect to $x$, to $y$ and to $y_{x}$ yields:

$$
\left.\left.\begin{array}{llll}
0 & =Q_{x}+Q_{a} A_{x}+Q_{b} B_{x} & 0 & =Q_{x x}+Q_{x a} A_{x}+Q_{x b} B_{x} \\
1 & = & Q_{a} A_{y}+Q_{b} B_{y} & 0
\end{array}\right)=Q_{x a} A_{y}+Q_{x b} B_{y}\right) .
$$

Then thanks to a straightforward application of the rule of Cramer for $2 \times 2$ linear systems, we derive six useful formulas.
Lemma. ([19], p. 9) All the six first order derivatives $A_{x}, A_{y}, A_{y_{x}}, B_{x}$, $B_{y}, B_{y_{x}}$ of the two functions $A$ and $B$ with respect to their three arguments $\left(x, y, y_{x}\right)$ may be expressed as follows in terms of the second jet $J^{2}(Q)$ of the defining function $Q$ :

$$
\begin{array}{rlrl}
A_{x} & =\frac{Q_{b} Q_{x x}-Q_{x} Q_{x b}}{Q_{a} Q_{x b}-Q_{b} Q_{x a}}, & B_{x}=\frac{Q_{x} Q_{x a}-Q_{a} Q_{x x}}{Q_{a} Q_{x b}-Q_{b} Q_{x a}}, \\
A_{y} & =\frac{Q_{x b}}{Q_{a} Q_{x b}-Q_{b} Q_{x a}}, & B_{y}=\frac{-Q_{x a}}{Q_{a} Q_{x b}-Q_{b} Q_{x a}}, \\
A_{y_{x}}=\frac{-Q_{b}}{Q_{a} Q_{x b}-Q_{b} Q_{x a}}, & B_{y_{x}}=\frac{Q_{a}}{Q_{a} Q_{x b}-Q_{b} Q_{x a}} .
\end{array}
$$

For future abbreviation, we shall denote the single appearing denominator - which evidently is the common determinant of all the three $2 \times 2$ linear systems involved above - simply by a square symbol:

$$
\Delta:=Q_{a} Q_{x b}-Q_{b} Q_{x a}
$$

The two-ways transfer between functions $G$ defined in the $\left(x, y, y_{x}\right)$-space and functions $T$ defined in the ( $x, a, b$ )-space, namely the one-to-one correspondence:

$$
G\left(x, y, y_{x}\right) \longleftrightarrow T(x, a, b)
$$

may be read very concretely as the following two equivalent identities:

$$
\begin{aligned}
G\left(x, y, y_{x}\right) & \equiv T\left(x, A\left(x, y, y_{x}\right), B\left(x, y, y_{x}\right)\right) \\
G\left(x, Q(x, a, b), Q_{x}(x, a, b)\right) & \equiv T(x, a, b)
\end{aligned}
$$

holding in $\mathbb{K}\left\{x, y, y_{x}\right\}$ and in $\mathbb{K}\{x, a, b\}$ respectively. By differentiating the first identity, the chain rule shows how the three first-order derivation operators (basic vector fields) $\partial_{x}, \partial_{y}$ and $\partial_{y_{x}}$ living in the $\left(x, y, y_{x}\right)$-space are transformed into the $(x, a, b)$-space:

$$
\left.\begin{array}{ccc}
\frac{\partial}{\partial x} & =\frac{\partial}{\partial x}+\left(\frac{Q_{b} Q_{x x}-Q_{x} Q_{x b}}{\Delta}\right) \frac{\partial}{\partial a}+\left(\frac{Q_{x} Q_{x a}-Q_{a} Q_{x x}}{\Delta}\right) \frac{\partial}{\partial b} \\
\frac{\partial}{\partial y} & = & \left(\frac{Q_{x b}}{\Delta}\right) \frac{\partial}{\partial a}+ \\
\frac{\partial}{\partial y_{x}} & = & \left(\frac{-Q_{x a}}{\Delta}\right) \frac{\partial}{\partial b} \\
\Delta
\end{array}\right) \frac{\partial}{\partial a}+\quad\left(\frac{Q_{a}}{\Delta}\right) \frac{\partial}{\partial b} .
$$

Lemma. The total differentiation operator $\mathrm{D}=\partial_{x}+y_{x} \partial_{y}+F \partial_{y_{x}}$ associated to $y_{x x}=F\left(x, y, y_{x}\right)$ simply transfers to the basic derivation operator along the $x$-direction:

$$
\mathrm{D} \longleftrightarrow \partial_{x}
$$

Proof. Reading the three formulas just preceding, by adding the first one to the second one multiplied by $y_{x}=Q_{x}$ together with the third one multiplied by $F=Q_{x x}$, one visibly sees that the coefficients of both $\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial b}$ do vanish in the obtained sum, as announced.

Keeping in mind - so as to avoid any confusion - that the same letter $x$ is used to denote simultaneously the independent variable of the differential equation $y_{x x}=F\left(x, y, y_{x}\right)$ and the non-parameter variable of the associated submanifold of solutions $y=Q(x, a, b)$, we may now write this two-ways transfer $\mathrm{D} \longleftrightarrow \partial_{x}$ exactly as we did in the above three equations, namely simply as an equality between two derivations living in the $\left(x, y, y_{x}\right)$-space and in the $(x, a, b)$-space:

$$
\mathrm{D}=\partial_{x}
$$

Lemma. With $G=G\left(x, y, y_{x}\right)$ being any local $\mathbb{K}$-analytic function in the $\left(x, y, y_{x}\right)$-space, the three second-order derivatives $G_{y_{x} y_{x}}, G_{y y_{x}}$ and $G_{y y}$ express as follows in terms of the second-order jet $J_{x, a, b}^{2}(T)$ of the defining function $T$ :

$$
\begin{aligned}
G_{y_{x} y_{x}}= & \frac{Q_{b} Q_{b}}{\Delta^{2}} T_{a a}-\frac{2 Q_{a} Q_{b}}{\Delta^{2}} T_{a b}+\frac{Q_{a} Q_{a}}{\Delta^{2}} T_{b b}+ \\
& +\frac{T_{a}}{\Delta^{3}}\left(Q_{a} Q_{a}\left|\begin{array}{cc}
Q_{b} & Q_{b b} \\
Q_{x b} & Q_{x b b}
\end{array}\right|-2 Q_{a} Q_{b}\left|\begin{array}{cc}
Q_{b} & Q_{a b} \\
Q_{x b} & Q_{x a b}
\end{array}\right|+Q_{b} Q_{b}\left|\begin{array}{cc}
Q_{b} & Q_{a a} \\
Q_{x b} & Q_{x a a}
\end{array}\right|\right)+ \\
& +\frac{T_{b}}{\Delta^{3}}\left(-Q_{a} Q_{a}\left|\begin{array}{cc}
Q_{a} & Q_{b b} \\
Q_{x a} & Q_{x b b}
\end{array}\right|+2 Q_{a} Q_{b}\left|\begin{array}{cc}
Q_{a} & Q_{a b} \\
Q_{x a} & Q_{x a b}
\end{array}\right|-Q_{b} Q_{b}\left|\begin{array}{cc}
Q_{a} & Q_{a a} \\
Q_{x a} & Q_{x a a}
\end{array}\right|\right) \\
G_{y y_{x}}= & -\frac{Q_{b} Q_{x b}}{\Delta^{2}} T_{a a}+\frac{Q_{a} Q_{x b}+Q_{b} Q_{x a}}{\Delta^{2}} T_{a b}-\frac{Q_{a} Q_{x a}}{\Delta^{2}} T_{b b}+ \\
& +\frac{T_{a}}{\Delta^{3}}\left(-Q_{a} Q_{x a}\left|\begin{array}{cc}
Q_{b} & Q_{b b} \\
Q_{x b} & Q_{x b b}
\end{array}\right|+\left(Q_{a} Q_{x b}+Q_{b} Q_{x a}\right)\left|\begin{array}{cc}
Q_{b} & Q_{a b} \\
Q_{x b} & Q_{x a b}
\end{array}\right|-Q_{b} Q_{x b}\left|\begin{array}{cc}
Q_{b} & Q_{a a} \\
Q_{x b} & Q_{x a a}
\end{array}\right|\right)+ \\
& +\frac{T_{b}}{\Delta^{3}}\left(Q_{a} Q_{x a}\left|\begin{array}{cc}
Q_{a} & Q_{b b} \\
Q_{x a} & Q_{x b b}
\end{array}\right|-\left(Q_{a} Q_{x b}+Q_{b} Q_{x a}\right)\left|\begin{array}{cc}
Q_{a} & Q_{a b} \\
Q_{x a} & Q_{x a b}
\end{array}\right|+Q_{b} Q_{x b}\left|\begin{array}{cc}
Q_{a} & Q_{a a} \\
Q_{x a} & Q_{x a a}
\end{array}\right|\right) \\
G_{y y}= & \frac{Q_{x b} Q_{x b}}{\Delta^{2}} T_{a a}-\frac{2 Q_{x a} Q_{x b}}{\Delta^{2}} T_{a b}+\frac{Q_{x a} Q_{x a}}{\Delta^{2}} T_{b b}+ \\
& +\frac{T_{a}}{\Delta^{3}}\left(Q_{x a} Q_{x a}\left|\begin{array}{cc}
Q_{b} & Q_{b b} \\
Q_{x b} & Q_{x b b}
\end{array}\right|-2 Q_{x a} Q_{x b}\left|\begin{array}{cc}
Q_{b} & Q_{a b} \\
Q_{x b} & Q_{x a b}
\end{array}\right|+Q_{x b} Q_{x b}\left|\begin{array}{cc}
Q_{b} & Q_{a a} \\
Q_{x b} & Q_{x a a}
\end{array}\right|\right)+ \\
& +\frac{T_{b}}{\Delta^{3}}\left(-Q_{x a} Q_{x a}\left|\begin{array}{ll}
Q_{a} & Q_{b b} \\
Q_{x a} & Q_{x b b}
\end{array}\right|+2 Q_{x a} Q_{x b}\left|\begin{array}{ll}
Q_{a} & Q_{a b} \\
Q_{x a} & Q_{x a b}
\end{array}\right|-Q_{x b} Q_{x b}\left|\begin{array}{cc}
Q_{a} & Q_{a a} \\
Q_{x a} & Q_{x a a}
\end{array}\right|\right) .
\end{aligned}
$$

Proof. We apply the operator $\frac{\partial}{\partial y_{x}}$, wiewed in the $(x, a, b)$-space, to the first order derivative $G_{y_{x}}$, namely we consider:

$$
\partial_{y_{x}}\left(G_{y_{x}}\right)=\frac{\partial}{\partial y_{x}}\left[-\frac{Q_{b}}{\Delta} T_{a}+\frac{Q_{a}}{\Delta} T_{b}\right]
$$

and we then expand carefully the result by collecting somewhat in advance the obtained terms with respect to the derivatives of $T$ :

$$
\begin{aligned}
G_{y_{x} y_{x}}= & \left(-\frac{Q_{b}}{\Delta} \frac{\partial}{\partial a}+\frac{Q_{a}}{\Delta} \frac{\partial}{\partial b}\right)\left[-\frac{Q_{b}}{\Delta} T_{a}+\frac{Q_{a}}{\Delta} T_{b}\right] \\
= & \left(\frac{Q_{b}}{\Delta} \frac{Q_{a b}}{\Delta}-\frac{Q_{b}}{\Delta} \frac{Q_{b} \Delta_{a}}{\Delta^{2}}\right) T_{a}+\frac{Q_{b}}{\Delta} \frac{Q_{b}}{\Delta} T_{a a}+ \\
& +\left(-\frac{Q_{b}}{\Delta} \frac{Q_{a a}}{\Delta}+\frac{Q_{b}}{\Delta} \frac{Q_{a} \Delta_{a}}{\Delta^{2}}\right) T_{b}-\frac{Q_{b}}{\Delta} \frac{Q_{a}}{\Delta} T_{a b}+ \\
& +\left(-\frac{Q_{a}}{\Delta} \frac{Q_{b b}}{\Delta}+\frac{Q_{a}}{\Delta} \frac{Q_{b} \Delta_{b}}{\Delta^{2}}\right) T_{a}-\frac{Q_{a}}{\Delta} \frac{Q_{b}}{\Delta} T_{a b}+ \\
& +\left(\frac{Q_{a}}{\Delta} \frac{Q_{a b}}{\Delta}-\frac{Q_{a}}{\Delta} \frac{Q_{a} \Delta_{b}}{\Delta^{2}}\right) T_{b}+\frac{Q_{a}}{\Delta} \frac{Q_{a}}{\Delta} T_{b b} .
\end{aligned}
$$

The terms involving $T_{a a}, T_{a b}, T_{b b}$ are exactly the ones exhibited by the lemma for the expression of $G_{y_{x} y_{x}}$. In the four large parentheses which are coefficients of $T_{a}, T_{b}, T_{a}, T_{b}$, we replace the occurences of $\Delta_{a}, \Delta_{a}, \Delta_{b}, \Delta_{b}$ simply by:

$$
\begin{aligned}
\Delta_{a} & =Q_{x b} Q_{a a}+Q_{a} Q_{x a b}-Q_{x a} Q_{a b}-Q_{b} Q_{x a a} \\
\Delta_{b} & =Q_{x b} Q_{a b}+Q_{a} Q_{x b b}-Q_{x a} Q_{b b}-Q_{b} Q_{x a b},
\end{aligned}
$$

and the total sum of terms coefficiented by $T_{a}$ in our expression now becomes:

$$
\begin{gathered}
\frac{T_{a}}{\Delta^{3}}\left(Q_{b} Q_{a b}\left[Q_{a} Q_{x b}-Q_{b} Q_{x a}\right]-Q_{b} Q_{b}\left[Q_{x b} Q_{a a}+Q_{a} Q_{x a b}-Q_{x a} Q_{a b}-Q_{b} Q_{x a a}\right]-\right. \\
\left.\quad-Q_{a} Q_{b b}\left[Q_{a} Q_{x b}-Q_{b} Q_{x a}\right]+Q_{a} Q_{b}\left[Q_{x b} Q_{a b}+Q_{a} Q_{x b b}-Q_{x a} Q_{b b}-Q_{b} Q_{x a b}\right]\right)= \\
=\frac{T_{a}}{\Delta^{3}}\left(Q_{a} Q_{b} Q_{x b} Q_{a b}-\underline{Q_{b} Q_{b} Q_{x a} Q_{a b}(\mathbb{1}}-Q_{b} Q_{b} Q_{x b} Q_{a a}-Q_{a} Q_{b} Q_{b} Q_{x a b}+\right. \\
+\underline{Q_{b} Q_{b} Q_{x a} Q_{a b}+Q_{b} Q_{b} Q_{b} Q_{x a a}-} \\
\quad-Q_{a} Q_{a} Q_{x b} Q_{b b}+\underline{Q_{a} Q_{b} Q_{x a} Q_{b b}}+Q_{a} Q_{b} Q_{x b} Q_{a b}+Q_{a} Q_{a} Q_{b} Q_{x b b}- \\
-\underline{\left.Q_{a} Q_{b} Q_{x a} Q_{b b}-Q_{a} Q_{b} Q_{b} Q_{x a b}\right)=} \\
=\frac{T_{a}}{\Delta^{3}}\left(Q_{a} Q_{a}\left[Q_{b} Q_{x b b}-Q_{x b} Q_{b b}\right]-2 Q_{a} Q_{b}\left[Q_{b} Q_{x a b}-Q_{x b} Q_{a b}\right]+\right. \\
\left.+Q_{b} Q_{b}\left[Q_{b} Q_{x a a}-Q_{x b} Q_{a a}\right]\right),
\end{gathered}
$$

so that we now have effectively reconstituted the three $2 \times 2$ determinants appearing in the second line of the expression claimed by the lemma for the transfer of $G_{y_{x} y_{x}}$ to the $(x, a, b)$-space. The treatment of the coefficient of $\frac{T_{b}}{\Delta^{3}}$ makes only a few differences, hence will be skipped here (but not in the manuscript). Finally, the two remaining expressions for $G_{y y_{x}}$ and for $G_{y y}$ are obtained by performing entirely analogous algebrico-differential computations.
End of the proof of the Main Theorem. Applying the above formula for $G_{y_{x} y_{x}}$ with $x:=z$, with $a:=\bar{z}$, with $b:=\bar{w}$, with $\Delta:=\Theta_{\bar{z}} \Theta_{z \bar{w}}-\Theta_{\bar{w}} \Theta_{z \bar{z}}$,
with $G:=\Phi$ and with $T:=\Theta_{z z}$, we exactly get the expression $\mathrm{AJ}^{4}(\Theta)$ of the Introduction, and then its further derivative $\partial_{y_{x}} \partial_{y_{x}}\left[G_{y_{x} y_{x}}\right]=G_{y_{x} y_{x} y_{x} y_{x}}$ is exactly:

$$
\begin{aligned}
0 & \left.\equiv\left(\frac{-\Theta_{\bar{w}}}{\Theta_{\bar{z}} \Theta_{z \bar{w}}-\Theta_{\bar{w}} \Theta_{z \bar{z}}} \frac{\partial}{\partial \bar{z}}+\frac{\Theta_{\bar{z}}}{\Theta_{\bar{z}} \Theta_{z \bar{w}}-\Theta_{\bar{w}} \Theta_{z \bar{z}}} \frac{\partial}{\partial \bar{w}}\right)^{2}[A\lrcorner^{4}(\Theta)\right] \\
& =: \frac{A J^{6}(\Theta)}{\left[\Theta_{\bar{z}} \Theta_{z \bar{w}}-\Theta_{\bar{w}} \Theta_{z \bar{z}}\right]^{7}} .
\end{aligned}
$$

As we have said, the vanishing of the second invariant of $w_{z z}(z)=$ $\Phi\left(z, w(z), w_{z}(z)\right)$ amounts to the complex conjugation of the above equation, which is then obviously redundant. Thus, the proof of the Main Theorem is now complete, but we will nevertheless discuss in a specific final section what $\mathrm{AJ}^{6}(\Theta)$ would look like in purely expanded form.

## §5. Some complete expansions: <br> EXAMPLES OF EXPRESSION SWELLINGS

Coming back to the non-CR context with the submanifold of solutions $\mathscr{M}_{(\mathscr{E})}=\{y=Q(x, a, b)\}$, let us therefore figure out how to expand the expression differentiated twice:

$$
\begin{aligned}
G_{y_{x} y_{x} y_{x} y_{x}}= & \left(-\frac{Q_{b}}{\Delta} \frac{\partial}{\partial a}+\frac{Q_{a}}{\Delta} \frac{\partial}{\partial b}\right)^{2}\left\{\frac{Q_{b} Q_{b}}{\Delta^{2}} T_{a a}-\frac{2 Q_{a} Q_{b}}{\Delta^{2}} T_{a b}+\frac{Q_{a} Q_{a}}{\Delta^{2}} T_{b b}+\right. \\
& +\frac{T_{a}}{\Delta^{3}}\left(Q_{a} Q_{a}\left|\begin{array}{cc}
Q_{b} & Q_{b b} \\
Q_{x b} & Q_{x b b}
\end{array}\right|-2 Q_{a} Q_{b}\left|\begin{array}{cc}
Q_{b} & Q_{a b} \\
Q_{x b} & Q_{x a b}
\end{array}\right|+Q_{b} Q_{b}\left|\begin{array}{cc}
Q_{b} & Q_{a a} \\
Q_{x b} & Q_{x a a}
\end{array}\right|\right)+ \\
& \left.+\frac{T_{b}}{\Delta^{3}}\left(-Q_{a} Q_{a}\left|\begin{array}{cc}
Q_{a} & Q_{b b} \\
Q_{x a} & Q_{x b b}
\end{array}\right|+2 Q_{a} Q_{b}\left|\begin{array}{cc}
Q_{a} & Q_{a b} \\
Q_{x a} & Q_{x a b}
\end{array}\right|-Q_{b} Q_{b}\left|\begin{array}{cc}
Q_{a} & Q_{a a} \\
Q_{x a} & Q_{x a a}
\end{array}\right|\right)\right\},
\end{aligned}
$$

which would make the Main Theorem a bit more precise and explicit.
First of all, we notice that, in the formulas for $G_{y_{x} y_{x}}$, for $G_{y y_{x}}$, for $G_{y y}$, all the appearing $2 \times 2$ determinants happen to be modifications of the basic Jacobian-like $\Delta$-determinant:

$$
\Delta(a \mid b):=\Delta=\left|\begin{array}{cc}
Q_{a} & Q_{b} \\
Q_{x a} & Q_{x b}
\end{array}\right|
$$

and we will denote them accordingly by employing the following (formally and intuitively clear) notations:

$$
\begin{aligned}
\Delta(b \mid b b):=\left|\begin{array}{cc}
Q_{b} & Q_{b b} \\
Q_{x b} & Q_{x b b}
\end{array}\right| & \Delta(b \mid a b):=\left|\begin{array}{cc}
Q_{b} & Q_{a b} \\
Q_{x b} & Q_{x a b}
\end{array}\right| \quad \Delta(b \mid a a):=\left|\begin{array}{cc}
Q_{b} & Q_{a a} \\
Q_{x b} & Q_{x a}
\end{array}\right| \\
\Delta(a \mid b b):=\left|\begin{array}{cc}
Q_{a} & Q_{b b} \\
Q_{x a} & Q_{x b b}
\end{array}\right| & \Delta(a \mid a b):=\left|\begin{array}{cc}
Q_{a} & Q_{a b} \\
Q_{x a} & Q_{x a b}
\end{array}\right|
\end{aligned} \quad \Delta(a \mid a a):=\left|\begin{array}{cc}
Q_{a} & Q_{a a} \\
Q_{x a} & Q_{x a a}
\end{array}\right|,
$$

the bottom line always coinciding with the differentiation with respect to $x$ of the top line. These abbreviations will be very appropriate for the next
explicit computation, so let us rewrite the formula for $G_{y_{x} y_{x}}$ using this newly introduced formalism:

$$
\begin{aligned}
G_{y_{x} y_{x}}=\frac{1}{\Delta(a \mid b)^{3}}\{ & T_{a a}\left[Q_{b} Q_{b} \Delta(a \mid b)\right]+T_{a b}\left[-2 Q_{a} Q_{b} \Delta(a \mid b)\right]+T_{b b}\left[Q_{a} Q_{a} \Delta(a \mid b)\right]+ \\
& +T_{a}\left[Q_{a} Q_{a} \Delta(b \mid b b)-2 Q_{a} Q_{b} \Delta(b \mid a b)+Q_{b} Q_{b} \Delta(b \mid a a)\right]+ \\
& \left.+T_{b}\left[-Q_{a} Q_{a} \Delta(a \mid b b)+2 Q_{a} Q_{b} \Delta(a \mid a b)-Q_{b} Q_{b} \Delta(a \mid a a)\right]\right\} .
\end{aligned}
$$

Then the twelve partial derivatives with respect to $a$ and with respect to $b$ of all the six determinants $\Delta(* \mid *)$ appearing in the the second line are easy to write down:

$$
\begin{array}{rlrl}
\frac{\partial}{\partial b}[\Delta(b \mid b b)] & =\Delta(b b \mid b b)+\Delta(b \mid b b b) & \frac{\partial}{\partial a}[\Delta(b \mid b b)]=\Delta(a b \mid b b)+\Delta(b \mid a b b) \\
\frac{\partial}{\partial b}[\Delta(b \mid a b)] & =\Delta(b b \mid a b)+\Delta(b \mid a b b) & & \frac{\partial}{\partial a}[\Delta(b \mid a b)]=\Delta(a b \mid a b)+\Delta(b \mid a a b) \\
\frac{\partial}{\partial b}[\Delta(b \mid a a)] & =\Delta(b b \mid a a)+\Delta(b \mid a a b) & & \frac{\partial}{\partial a}[\Delta(b \mid a a)]=\Delta(a b \mid a a)+\Delta(b \mid a a a) \\
\frac{\partial}{\partial b}[\Delta(a \mid b b)] & =\Delta(a b \mid b b)+\Delta(a \mid b b b) & & \frac{\partial}{\partial a}[\Delta(a \mid b b)]=\Delta(a a \mid b b)+\Delta(a \mid a b b) \\
\frac{\partial}{\partial b}[\Delta(a \mid a b)] & =\Delta(a b \mid a b)+\Delta(a \mid a b b) & & \frac{\partial}{\partial a}[\Delta(a \mid a b)]=\Delta(a a \mid a b)+\Delta(a \mid a a b) \\
\frac{\partial}{\partial b}[\Delta(a \mid a a)] & =\Delta(a b \mid a a)+\Delta(a \mid a a b) & & \frac{\partial}{\partial a}[\Delta(a \mid a a)]=\underline{\Delta(a a \mid a a)+\Delta(a \mid a a a),} 0
\end{array}
$$

and the underlined terms vanish for the trivial reason that any $2 \times 2$ determinant, two columns of which coincide, vanishes. Consequently, we may now endeavour the computation of the third order derivative:

$$
G_{y_{x} y_{x} y_{x}}=\left(-\frac{Q_{b}}{\Delta} \frac{\partial}{\partial a}+\frac{Q_{a}}{\Delta} \frac{\partial}{\partial b}\right)\left[G_{y_{x} y_{x}}\right]
$$

When applying the two derivations in parentheses to:

$$
G_{y_{x} y_{x}}=\frac{1}{\Delta^{3}}\{\text { expression }\}
$$

we start out by differentiating $\frac{1}{\Delta^{3}}$ multiplied by expression, and then we differentiate expression. Before any contraction, the full expansion of:

$$
\Delta^{5} G_{y_{x} y_{x} y_{x}}=
$$

(we indeed clear out the denominator $\Delta^{5}$ ) is then:

$$
\begin{aligned}
& =T_{a a}\left[3 Q_{b} Q_{b} Q_{b} \Delta(a \mid b) \Delta(a a \mid b)+3 Q_{b} Q_{b} Q_{b} \Delta(a \mid b) \Delta(a \mid a b)-3 Q_{a} Q_{b} Q_{b} \Delta(a \mid b) \Delta(a b \mid b)-3 Q_{a} Q_{b} Q_{b} \Delta(a \mid b) \Delta(a \mid b b)\right]+ \\
& +T_{a b}\left[-6 Q_{a} Q_{b} Q_{b} \Delta(a \mid b) \Delta(a a \mid b)-6 Q_{a} Q_{b} Q_{b} \Delta(a \mid b) \Delta(a \mid a b)+6 Q_{a} Q_{a} Q_{b} \Delta(a \mid b) \Delta(a b \mid a)+6 Q_{a} Q_{a} Q_{b} \Delta(a \mid b) \Delta(a \mid b b)\right]+ \\
& +T_{b b}\left[3 Q_{a} Q_{a} Q_{b} \Delta(a \mid b) \Delta(a a \mid b)+3 Q_{a} Q_{a} Q_{b} \Delta(a \mid b) \Delta(a \mid a b)-3 Q_{a} Q_{a} Q_{a} \Delta(a \mid b) \Delta(a b \mid b)-3 Q_{a} Q_{a} Q_{a} \Delta(a \mid b) \Delta(a \mid b b)\right]+ \\
& +T_{a}\left[3 Q_{a} Q_{a} Q_{b} \Delta(b \mid b b) \Delta(a a \mid b)+3 Q_{a} Q_{a} Q_{b} \Delta(b \mid b b) \Delta(a \mid a b)-3 Q_{a} Q_{a} Q_{a} \Delta(b \mid b b) \Delta(a b \mid b)-3 Q_{a} Q_{a} Q_{a} \Delta(b \mid b b) \Delta(a \mid b b)-\right. \\
& \quad-6 Q_{a} Q_{b} Q_{b} \Delta(b \mid a b) \Delta(a a \mid b)-6 Q_{a} Q_{b} Q_{b} \Delta(b \mid a b) \Delta(a \mid a b)+6 Q_{a} Q_{a} Q_{b} \Delta(b \mid a b) \Delta(a b \mid b)+6 Q_{a} Q_{a} Q_{b} \Delta(b \mid a b) \Delta(a \mid b b)+ \\
& \left.\quad+3 Q_{b} Q_{b} Q_{b} \Delta(b \mid a a) \Delta(a a \mid b)+3 Q_{b} Q_{b} Q_{b} \Delta(b \mid a a) \Delta(a \mid a b)-3 Q_{a} Q_{b} Q_{b} \Delta(b \mid a a) \Delta(a b \mid b)-3 Q_{a} Q_{b} Q_{b} \Delta(b \mid a a) \Delta(a \mid b b)\right]+
\end{aligned}
$$

$$
\begin{aligned}
& +T_{b}\left[3 Q_{a} Q_{a} Q_{b} \Delta(a \mid b b) \Delta(a a \mid b)+3 Q_{a} Q_{a} Q_{b} \Delta(a \mid b b) \Delta(a \mid a b)-3 Q_{a} Q_{a} Q_{a} \Delta(a \mid b b) \Delta(a b \mid b)-3 Q_{a} Q_{a} Q_{a} \Delta(a \mid b b) \Delta(a \mid b b)-\right. \\
& -6 Q_{a} Q_{b} Q_{b} \Delta(a \mid a b) \Delta(a a \mid b)-6 Q_{a} Q_{b} Q_{b} \Delta(a \mid a b) \Delta(a \mid a b)+6 Q_{a} Q_{a} Q_{b} \Delta(a \mid a b) \Delta(a b \mid b)+6 Q_{a} Q_{a} Q_{b} \Delta(a \mid a b) \Delta(a \mid b b)+ \\
& \left.+3 Q_{b} Q_{b} Q_{b} \Delta(a \mid a a) \Delta(a a \mid b)+3 Q_{b} Q_{b} Q_{b} \Delta(a \mid a a) \Delta(a \mid a b)-3 Q_{a} Q_{b} Q_{b} \Delta(a \mid a a) \Delta(a b \mid b)-3 Q_{a} Q_{b} Q_{b} \Delta(a \mid a a) \Delta(a \mid b b)\right]+ \\
& +\Delta(a \mid b) T_{a a a}\left[-Q_{b} Q_{b} Q_{b} \Delta(a \mid b)\right]+T_{a a b}\left[3 Q_{a} Q_{b} Q_{b} \Delta(a \mid b)\right]+T_{a b b}\left[-3 Q_{a} Q_{a} Q_{b} \Delta(a \mid b)\right]+T_{b b b}\left[Q_{a} Q_{a} Q_{a} \Delta(a \mid b)\right]+ \\
& +\Delta(a \mid b) T_{a a}\left[-2 Q_{b} Q_{b} Q_{a b} \Delta(a \mid b)-Q_{b} Q_{b} Q_{b} \Delta(a a \mid b)-Q_{b} Q_{b} Q_{b} \Delta(a \mid a b)+\right. \\
& \left.+2 Q_{a} Q_{b} Q_{b} \Delta(a \mid b)+Q_{a} Q_{b} Q_{b} Q_{b} \Delta(a b \mid b)+Q_{a} Q_{b} Q_{b} \Delta(a \mid b b)\right]+ \\
& +\Delta(a \mid b) T_{a b}\left[2 Q_{b} Q_{b} Q_{a a} \Delta(a \mid b)+2 Q_{a} Q_{b} Q_{a b} \Delta(a \mid b)+2 Q_{a} Q_{b} Q_{b} \Delta(a a \mid b)+2 Q_{a} Q_{b} Q_{b} \Delta(a \mid a b)-\right. \\
& \left.-2 Q_{a} Q_{b} Q_{a b} \Delta(a \mid b)-2 Q_{a} Q_{a} Q_{b b} \Delta(a \mid b)-2 Q_{a} Q_{a} Q_{b} \Delta(a b \mid b)-2 Q_{a} Q_{a} Q_{b} \Delta(a \mid b b)\right]+ \\
& +\Delta(a \mid b) T_{b b}\left[-2 Q_{a} Q_{b} Q_{a a} \Delta(a \mid b)-Q_{a} Q_{a} Q_{b} \Delta(a a \mid b)-Q_{a} Q_{a} Q_{b} \Delta(a \mid a b)+\right. \\
& \left.+2 Q_{a} Q_{a} Q_{a b} \Delta(a \mid b)+Q_{a} Q_{a} Q_{a} \Delta(a b \mid b)+Q_{a} Q_{a} Q_{a} \Delta(a \mid b b)\right]+ \\
& +\Delta(a \mid b) T_{a a}\left[-Q_{a} Q_{a} Q_{b} \Delta(b \mid b b)+2 Q_{a} Q_{b} Q_{b} \Delta(b \mid a b)-Q_{b} Q_{b} Q_{b} \Delta(b \mid a a)\right]+ \\
& +\Delta(a \mid b) T_{a b}\left[Q_{a} Q_{a} Q_{a} \Delta(b \mid b b)-2 Q_{a} Q_{a} Q_{b} \Delta(b \mid a b)+Q_{a} Q_{b} Q_{b} \Delta(b \mid a a)\right]+ \\
& +\Delta(a \mid b) T_{b a}\left[Q_{a} Q_{a} Q_{b} \Delta(a \mid b b)-2 Q_{a} Q_{b} Q_{b} \Delta(a \mid a b)+Q_{b} Q_{b} Q_{b} \Delta(a \mid a a)\right]+ \\
& +\Delta(a \mid b) T_{b b}\left[-Q_{a} Q_{a} Q_{a} \Delta(a \mid b b)+2 Q_{a} Q_{a} Q_{b} \Delta(a \mid a b)-Q_{a} Q_{b} Q_{b} \Delta(a \mid a a)\right]+ \\
& +\Delta(a \mid b) T_{a}\left[-2 Q_{a} Q_{b} Q_{a a} \Delta(b \mid b b)-Q_{a} Q_{a} Q_{b} \Delta(a b \mid b b)-Q_{a} Q_{a} Q_{b} \Delta(b \mid a b b)+\right. \\
& +2 Q_{b} Q_{b} Q_{a a} \Delta(b \mid a b)+2 Q_{a} Q_{b} Q_{a b} \Delta(b \mid a b)+\underline{2 Q}_{a} Q_{b} Q_{b} \Delta(a b \mid a b)_{0}+2 Q_{a} Q_{b} Q_{b} \Delta(b \mid a a b)- \\
& -2 Q_{b} Q_{b} Q_{a b} \Delta(b \mid a a)-Q_{b} Q_{b} Q_{b} \Delta(a b \mid a a)-Q_{b} Q_{b} Q_{b} \Delta(b \mid a a a)+ \\
& +2 Q_{a} Q_{a} Q_{a b} \Delta(b \mid b b)+\underline{Q a}_{a} Q_{a} Q_{a} \Delta(b b \mid b b)_{0}+Q_{a} Q_{a} Q_{a} \Delta(b \mid b b b)- \\
& -2 Q_{a} Q_{b} Q_{a b} \Delta(b \mid a b)-2 Q_{a} Q_{a} Q_{b b} \Delta(b \mid a b)-2 Q_{a} Q_{a} Q_{b} \Delta(b b \mid a b)-2 Q_{a} Q_{a} Q_{b} \Delta(b \mid a b b)+ \\
& \left.+2 Q_{a} Q_{b} Q_{b b} \Delta(b \mid a a)+Q_{a} Q_{b} Q_{b} \Delta(b b \mid a a)+Q_{a} Q_{b} Q_{b} \Delta(b \mid a a b)\right]+ \\
& +\Delta(a \mid b) T_{b}\left[2 Q_{a} Q_{b} Q_{a a} \Delta(a \mid b b)+Q_{a} Q_{a} Q_{b} \Delta(a a \mid b b)+Q_{a} Q_{a} Q_{b} \Delta(a \mid a b b)-\right. \\
& -2 Q_{b} Q_{b} Q_{a a} \Delta(a \mid a b)-2 Q_{a} Q_{b} Q_{a b} \Delta(a \mid a b)-\underline{2 Q a}_{a} Q_{b} Q_{b} \Delta(a a \mid a b)_{0}-2 Q_{a} Q_{b} Q_{b} \Delta(a \mid a a b)+ \\
& +2 Q_{b} Q_{b} Q_{a b} \Delta(a \mid a a)+Q_{b} Q_{b} Q_{b} \Delta(a a \mid a a)+Q_{b} Q_{b} Q_{b} \Delta(a \mid a a a)- \\
& -2 Q_{a} Q_{a} Q_{a b} \Delta(a \mid b b)-\underline{Q}_{a} Q_{a} Q_{a} \Delta(a b \mid b b)_{0}-Q_{a} Q_{a} Q_{a} \Delta(a \mid b b b)+ \\
& +2 Q_{a} Q_{b} Q_{a b} \Delta(a \mid a b)+2 Q_{a} Q_{a} Q_{b b} \Delta(a \mid a b)+2 Q_{a} Q_{a} Q_{b} \Delta(a b \mid a b)+2 Q_{a} Q_{a} Q_{b} \Delta(a \mid a b b)- \\
& \left.-2 Q_{a} Q_{b} Q_{b b} \Delta(a \mid a a)-Q_{a} Q_{b} Q_{b} \Delta(a b \mid a a)-Q_{a} Q_{b} Q_{b} \Delta(a \mid a a b)\right] .
\end{aligned}
$$

The simplification (collecting all terms) gives:

$$
\begin{gathered}
G_{y_{x} y_{x} y_{x}}=\frac{1}{[\Delta(a \mid b)]^{5}}\left\{\begin{array}{c}
T_{a a a}\left[-Q_{b}^{3} \Delta(a \mid b)^{2}\right]+T_{a a b}\left[3 Q_{a} Q_{b}^{2} \Delta(a \mid b)^{2}\right]+ \\
+T_{a b b}\left[-3 Q_{a}^{2} Q_{b} \Delta(a \mid b)^{2}\right]+T_{b b b}\left[Q_{a}^{3} \Delta(a \mid b)^{2}\right]+
\end{array}\right. \\
+T_{a a}\left[-2 Q_{b}^{2} Q_{a b} \Delta(a \mid b)^{2}+2 Q_{a} Q_{b} Q_{b b} \Delta(a \mid b)^{2}+3 Q_{b}^{3} \Delta(a \mid b) \Delta(a a \mid b)+2 Q_{b}^{3} \Delta(a \mid b) \Delta(a \mid a b)-\right. \\
\left.-4 Q_{a} Q_{b}^{2} \Delta(a \mid b) \Delta(a b \mid b)-2 Q_{a} Q_{b}^{2} \Delta(a \mid b) \Delta(a \mid b b)-Q_{a}^{2} Q_{b} \Delta(a \mid b) \Delta(b \mid b b)\right]+ \\
+T_{a b}\left[-2 Q_{a}^{2} Q_{b b} \Delta(a \mid b)^{2}+2 Q_{b} Q_{b} Q_{a a} \Delta(a \mid b)^{2}+Q_{a}^{3} \Delta(a \mid b) \Delta(b \mid b b)+6 Q_{a}^{2} Q_{b} \Delta(a \mid b) \Delta(a b \mid b)+\right. \\
\left.+Q_{b}^{3} \Delta(a \mid b) \Delta(a \mid a a)-6 Q_{a} Q_{b}^{2} \Delta(a \mid b) \Delta(a \mid a b)+5 Q_{a}^{2} Q_{b} \Delta(a \mid b) \Delta(a \mid b b)-5 Q_{a} Q_{b}^{2} \Delta(a \mid b) \Delta(a a \mid b)\right]+ \\
+T_{b b}\left[-2 Q_{a} Q_{b} Q_{a a} \Delta(a \mid b)^{2}+2 Q_{a}^{2} Q_{a b} \Delta(a \mid b)^{2}-3 Q_{a}^{3} \Delta(a \mid b) \Delta(a \mid b b)-2 Q_{a}^{3} \Delta(a \mid b) \Delta(a b \mid b)+\right. \\
\left.+4 Q_{a}^{2} Q_{b} \Delta(a \mid b) \Delta(a \mid a b)+2 Q_{a}^{2} Q_{b} \Delta(a \mid b) \Delta(a a \mid b)-Q_{a} Q_{b}^{2} \Delta(a \mid b) \Delta(a \mid a a)\right]+
\end{gathered}
$$

$$
\begin{aligned}
+T_{a}[ & 3 Q_{a}^{2} Q_{b} \Delta(a a \mid b) \Delta(b \mid b b)+3 Q_{a}^{2} Q_{b} \Delta(a \mid a b) \Delta(b \mid b b)-3 Q_{a}^{3} \Delta(a b \mid b) \Delta(b \mid b b)-3 Q_{a}^{3} \Delta(a \mid b b) \Delta(b \mid b b)- \\
& -6 Q_{a} Q_{b}^{2} \Delta(a a \mid b) \Delta(b \mid a b)-6 Q_{a} Q_{b}^{2} \Delta(a \mid a b) \Delta(b \mid a b)+6 Q_{a}^{2} Q_{b} \Delta(a b \mid b) \Delta(b \mid a b)+6 Q_{a}^{2} Q_{b} \Delta(a \mid b b) \Delta(b \mid a b)- \\
& -3 Q_{b}^{3} \Delta(a a \mid b) \Delta(b \mid a a)-3 Q_{b}^{3} \Delta(a \mid a b) \Delta(b \mid a a)+3 Q_{a} Q_{b}^{2} \Delta(a b \mid b) \Delta(b \mid a a)+3 Q_{a} Q_{b}^{2} \Delta(a \mid b b) \Delta(b \mid a a)- \\
& -2 Q_{a} Q_{b} Q_{a a} \Delta(a \mid b) \Delta(b \mid b b)+2 Q_{b}^{2} Q_{a a} \Delta(a \mid b) \Delta(b \mid a b)+2 Q_{a} Q_{b} Q_{a b} \Delta(a \mid b) \Delta(b \mid a b)-2 Q_{b}^{2} Q_{a b} \Delta(a \mid b) \Delta(b \mid a a)- \\
& -Q_{a}^{2} Q_{b} \Delta(a \mid b) \Delta(a b \mid b b)-Q_{a}^{2} Q_{b} \Delta(a \mid b) \Delta(b \mid a b b)+2 Q_{a} Q_{b}^{2} \Delta(a \mid b) \Delta(b \mid a a b)-Q_{b}^{3} \Delta(a \mid b) \Delta(a b \mid a a)-Q_{b}^{3} \Delta(a \mid b) \Delta(b \mid a a a)+ \\
& +2 Q_{a}^{2} Q_{a b} \Delta(a \mid b) \Delta(b \mid b b)-2 Q_{a} Q_{b} Q_{a b} \Delta(a \mid b) \Delta(b \mid a b)-2 Q_{a}^{2} Q_{b b} \Delta(a \mid b) \Delta(b \mid a b)+2 Q_{a} Q_{b} Q_{b b} \Delta(a \mid b) \Delta(b \mid a a)+ \\
& \left.+Q_{a}^{3} \Delta(a \mid b) \Delta(b \mid b b b)-2 Q_{a}^{2} Q_{b} \Delta(a \mid b) \Delta(b b \mid a b)-2 Q_{a}^{2} Q_{b} \Delta(a \mid b) \Delta(b \mid a b b)+Q_{a} Q_{b}^{2} \Delta(a \mid b) \Delta(b b \mid a a)+Q_{a} Q_{b}^{2} \Delta(a \mid b) \Delta(b \mid a a b)\right]+ \\
+T_{b}[ & 3 Q_{a}^{2} Q_{b} \Delta(a \mid b b) \Delta(a a \mid b)+3 Q_{a}^{2} Q_{b} \Delta(a \mid b b) \Delta(a \mid a b)-3 Q_{a}^{3} \Delta(a \mid b b) \Delta(a b \mid b)-3 Q_{a}^{3} \Delta(a \mid b b) \Delta(a \mid b b)- \\
& -6 Q_{a} Q_{b}^{2} \Delta(a \mid a b) \Delta(a a \mid b)-6 Q_{a} Q_{b}^{2} \Delta(a \mid a b) \Delta(a \mid a b)+6 Q_{a}^{2} Q_{b} \Delta(a \mid a b) \Delta(a b \mid b)+6 Q_{a} Q_{a} Q_{b} \Delta(a \mid a b) \Delta(a \mid b b)+ \\
& +3 Q_{b}^{2} \Delta(a \mid a a) \Delta(a a \mid b)+3 Q_{b}^{2} \Delta(a \mid a a) \Delta(a \mid a b)-3 Q_{a} Q_{b}^{2} \Delta(a \mid a a) \Delta(a b \mid b)-3 Q_{a} Q_{b}^{2} \Delta(a \mid a a) \Delta(a \mid b b)+ \\
& +2 Q_{a} Q_{b} Q_{a a} \Delta(a \mid b) \Delta(a \mid b b)-2 Q_{b}^{2} Q_{a a} \Delta(a \mid b) \Delta(a \mid a b)-2 Q_{a} Q_{b} Q_{a b} \Delta(a \mid b) \Delta(a \mid a b)+2 Q_{b}^{2} Q_{a b} \Delta(a \mid b) \Delta(a \mid a a)+ \\
& +Q_{a}^{2} Q_{b} \Delta(a \mid b) \Delta(a a \mid b b)+Q_{a}^{2} Q_{b} \Delta(a \mid b) \Delta(a \mid a b b)-2 Q_{a} Q_{b}^{2} \Delta(a \mid b) \Delta(a \mid a a b)+Q_{b}^{3} \Delta(a \mid b) \Delta(a a \mid a a)+Q_{b}^{3} \Delta(a \mid b) \Delta(a \mid a a a)- \\
& -2 Q_{a}^{2} Q_{a b} \Delta(a \mid b) \Delta(a \mid b b)+2 Q_{a} Q_{b} Q_{a b} \Delta(a \mid b) \Delta(a \mid a b)+2 Q_{a}^{2} Q_{b b} \Delta(a \mid b) \Delta(a \mid a b)-2 Q_{a} Q_{b} Q_{b b} \Delta(a \mid b) \Delta(a \mid a a)- \\
& \left.-Q_{a}^{3} \Delta(a \mid b) \Delta(a \mid b b b)+2 Q_{a}^{2} Q_{b} \Delta(a \mid b) \Delta(a b \mid a b)+2 Q_{a}^{2} Q_{b} \Delta(a \mid b) \Delta(a \mid a b b)-Q_{a} Q_{b}^{2} \Delta(a \mid b) \Delta(a b \mid a a)-Q_{a} Q_{b}^{2} \Delta(a \mid b) \Delta(a \mid a a b)\right] .
\end{aligned}
$$

The full expansion of $G_{y_{x} y_{x} y_{x} y_{x}}$ will not be presented here.

## References

[1] Bryant, R.L.: Élie Cartan and geometric duality, Journées Élie Cartan 1998 \& 1999, Inst. É. Cartan 16 (2000), 5-20.
[2] Cartan, É.: Sur les variétés à connexion projective, Bull. Soc. Math. France 52 (1924), 205241.
[3] Cartan, É.: Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes, II, Ann. Scuola Norm. Sup. Pisa 1 (1932), 333-354.
[4] Christoffel, E.B.: Über die Transformation der homogenen Differentialausdrücke zweiten Grades, J. reine angew. Math. 70 (1869), 46-70.
[5] Crampin, M.; Saunders, D.J.: Cartan's concept of duality for second-order ordinary differential equations, J. Geom. Phys. 54 (2005), 146-172.
[6] Engel, F.; Lie, S.: Theorie der transformationsgruppen. Erster Abschnitt. Unter Mitwirkung von Dr. Friedrich Engel, bearbeitet von Sophus Lie, B.G. Teubner, Leipzig, 1888. Reprinted by Chelsea Publishing Co. (New York, N.Y., 1970).
[7] Grissom, C.; Thompson, G.; Wilkens, G.: Linearization of second-order ordinary differential equations via Cartan's equivalence method, J. Diff. Eq. 77 (1989), no. 1, 1-15.
[8] Hsu, L.; Kamran, N.: Classification of second order ordinary differential equations admitting Lie groups of fibre-preserving point symmetries, Proc. London Math. Soc. 58 (1989), no. 3, 387-416.
[9] Isaev, A.V.: Zero CR-curvature equations for rigid and tube hypersurfaces, Complex Variables and Elliptic Equations, 54 (2009), no. 3-4, 317-344.
[10] Koppisch, M.A.: Zur Invariantentheorie der gewöhlichen Differentialgleichungen zweiter Ordnung, Inaugural Dissertation, Leipzig, B.-G. Teubner, 1905.
[11] Lie, S.: Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen $x$, $y$, die eine Gruppe von Transformationen gestaten I-IV. In: Gesammelte Abhandlungen, Vol. 5, B.G. Teubner, Leipzig, 1924, pp. 240-310; 362-427, 432-448.
[12] Lie, S.; Scheffers, G.: Vorlesungen über continuirlichen gruppen, mit geometrischen und anderen anwendungen, B.-G. Teubner, Leipzig, 1893. Reprinted by Chelsea Publishing Co. (New York, N.Y., 1971).
[13] Merker, J.: Convergence of formal biholomorphisms between minimal holomorphically nondegenerate real analytic hypersurfaces, Int. J. Math. Math. Sci. 26 (2001), no. 5, 281-302.
[14] Merker, J.: On the partial algebraicity of holomorphic mappings between real algebraic sets, Bull. Soc. Math. France 129 (2001), no. 3, 547-591.
[15] Merker, J.: On envelopes of holomorphy of domains covered by Levi-flat hats and the reflection principle, Ann. Inst. Fourier (Grenoble) 52 (2002), no. 5, 1443-1523.
[16] Merker, J.: On the local geometry of generic submanifolds of $\mathbb{C}^{n}$ and the analytic reflection principle, Journal of Mathematical Sciences (N. Y.) 125 (2005), no. 6, 751-824.
[17] Merker, J.: Étude de la régularité analytique de l'application de réflexion CR formelle, Annales Fac. Sci. Toulouse, XIV (2005), no. 2, 215-330.
[18] Merker, J.: Explicit differential characterization of the Newtonian free particle system in $m \geqslant$ 2 dependent variables, Acta Mathematicæ Applicandæ, 92 (2006), no. 2, 125-207.
[19] Merker, J.; Porten, E.: Holomorphic extension of CR functions, envelopes of holomorphy and removable singularities, International Mathematics Research Surveys, Volume 2006, Article ID 28295, 287 pages.
[20] Merker, J.: Lie symmetries of partial differential equations and CR geometry, Journal of Mathematical Sciences (N.Y.), to appear (2009), arxiv.org/abs/math/0703130
[21] Merker, J.: Sophus Lie, Friedrich Engel et le problème de Riemann-Helmholtz, Hermann Éditeurs, Paris, 2010, à paraître, 307 pp ., arxiv.org/abs/0910.0801
[22] Merker, J.: Vanishing Hachtroudi curvature and spherical real analytic hypersurfaces, arxiv.org, to appear.
[23] Nurowski, P.; Sparling, G.A.J.: 3-dimensional Cauchy-Riemann structures and $2^{\text {nd }}$ order ordinary differential equations, Class. Quant. Gravity, 20 (2003), 4995-5016.
[24] Segre, B.: Intorno al problema di Poincaré della rappresentazione pseudoconforme, Rend. Acc. Lincei, VI, Ser. 13 (1931), 676-683.
[25] Tresse, A.: Détermination des invariants ponctuels de l'équation différentielle du second ordre $y^{\prime \prime}=\omega\left(x, y, y^{\prime}\right)$, Hirzel, Leipzig, 1896.

# Vanishing Hachtroudi curvature and local equivalence to the Heisenberg pseudosphere 

Joël Merker


#### Abstract

To any completely integrable second-order system of real or complex partial differential equations: $$
y_{x^{k_{1} x^{k_{2}}}}=F_{k_{1}, k_{2}}\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right)
$$ with $1 \leqslant k_{1}, k_{2} \leqslant n$ and with $F_{k_{1}, k_{2}}=F_{k_{2}, k_{1}}$ in $n \geqslant 2$ independent variables $\left(x^{1}, \ldots, x^{n}\right)$ and in one dependent variable $y$, Mohsen Hachtroudi associated in 1937 a normal projective (Cartan) connection, and he computed its curvature. By means of a natural transfer of jet polynomials to the associated submanifold of solutions, what the vanishing of the Hachtroudi curvature gives can be precisely translated in order to characterize when both families of Segre varieties and of conjugate Segre varieties associated to a Levi nondegenerate real analytic hypersurface $M$ in $\mathbb{C}^{n}(n \geqslant 3)$ can be straightened to be affine complex (conjugate) lines. In continuation to a previous paper devoted to the quite distinct $\mathbb{C}^{2}$-case, this then characterizes in an effective way those hypersurfaces of $\mathbb{C}^{n+1}$ in higher complex dimension $n+1 \geqslant 3$ that are locally biholomorphic to a piece of the $(2 n+1)$ dimensional Heisenberg quadric, without any special assumption on their defining equations.


arxiv.org/abs/0910.2861/
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## §1. Introduction

The explicit characterization of pseudosphericality of an arbitrary real analytic local hypersurface sitting in the complex Euclidean space has been (re)studied recently by Isaev in [11], who employed the famous Chern(Moser) tensorial approach [CM1974, Ch1975] to the concerned equivalence problem. But in the growing literature devoted to Lie-group symmetries of Cauchy-Riemann manifolds, only a very few articles underline that, already in his 1937 Ph.D. thesis [Ha1937] under the direction of his Élie Cartan - who was around the same period also the master of Chern - , the

Iranian mathematician Mohsen Hachtroudi (cited briefly only in [Ch1975]) constructed directly an explicit normal projective Cartan connection canonically associated to any completely integrable system of real or complex partial differential equations:

$$
\begin{equation*}
y_{x^{k_{1} x^{k_{2}}}}(x)=F_{k_{1}, k_{2}}\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right) \tag{7.28}
\end{equation*}
$$

in $n \geqslant 2$ independent variables $x^{1}, \ldots, x^{n}$ and in one dependent variable $y$, by endeavouring in a successful way to generalize the celebrated paper [Ca1924]. Chern's clever observation in 1974 that Hachtroudi's 37 years-old approach was intrinsically related to the nascent higherdimensional CR geometry was followed, in his two papers in question, by his technical contribution of redoing (only) parts of Hachtroudi's effective computations, following the alternative (heavier, though essentially equivalent) strategy of constructing a posteriori the projective connection, after having reinterpreted at the beginning the problem in terms of the wide and powerful Cartan Method of Equivalence. Thus, one should be aware, historically speaking, that in the original reference [Ha1937], much more complete geometric and computational aspects were published long before, though they were expressed in a purely analytic and somewhat elliptic language which, unfortunately for us at present times, does not transmit in words and with figures all the underlying geometric meanings which were clear then to Élie Cartan.

Because Hachtroudi was able to write down explicitly his curvature tensors, he deduced the second-order system (7.28) - below - of partial differential equations that the functions $F_{k_{1}, k_{2}}$ should satisfy in order that the system (7.28) be equivalent, through a point transformation $(x, y) \mapsto$ $\left(x^{\prime}, y^{\prime}\right)=\left(x^{\prime}(x, y), y^{\prime}(x, y)\right)$ to the simplest system: $y_{x^{\prime k} k_{1} x^{\prime k}}^{\prime}\left(x^{\prime}\right)=0$, with all right-hand sides being zero. In the present article, a companion and a follower of a preceding one [22] devoted to the quite different $\mathbb{C}^{2}$-case, we will apply, to the higher-dimensional characterization of pseudosphericality, this effective necessary and sufficient condition (7.28) due to Hachtroudi which, however and inexplicably, is totally inextant in the two contributions of Chern. We hope in this way to complete the explicit characterization of pseudosphericality for rigid or even tube hypersurfaces that was obtained recently by Isaev in [11], because apparently, the general (nonrigid) case was still open in the specialized field.

We now start the exposition. Let $M$ be a local real analytic in $\mathbb{C}^{n+1}$. Though the basic definitions, lemmas and propositions of the theory are valid in any complex dimension $n+1 \geqslant 2$, there is a strong computational difference between the two characterizations of sphericality for $n=1$ (compare [21]) and of pseudosphericality for $n \geqslant 2$ (presently), so that, in order
to fix the ideas, it will be assumed throughout the paper - and recalled when necessary - that the CR dimension $n$ is always $\geqslant 2$.

Locally in a neighborhood of one of its points $p$, the hypersurface $M$ may be represented, in any system of local holomorphic coordinates:

$$
t=(w, z) \in \mathbb{C}^{n} \times \mathbb{C}
$$

vanishing at $p$ for which the $w$-axis is not complex-tangent to $M$ at $p$, by a so-called complex defining equation - Section 2 provides further informations - of the form:

$$
\begin{equation*}
w=\Theta(z, \bar{z}, \bar{w})=\Theta(z, \bar{t}) \tag{7.28}
\end{equation*}
$$

or equivalently in a more expanded form which exhibits all the indices:

$$
w=\Theta\left(z_{1}, \ldots, z_{n}, \bar{z}_{1}, \ldots, \bar{z}_{n}, \bar{w}\right)=\Theta\left(z_{1}, \ldots, z_{n}, \bar{t}_{1}, \ldots, \bar{t}_{n}, \bar{t}_{n+1}\right)
$$

Then $M$ localized at $p$ is called pseudospherical (at $p$ ) if it is biholomorphic to a piece of one Heisenberg pseudosphere:

$$
\begin{equation*}
\operatorname{Im} w^{\prime}=\left|z_{1}^{\prime}\right|^{2}+\cdots+\left|z_{q}^{\prime}\right|^{2}-\left|z_{q+1}^{\prime}\right|^{2}-\cdots-\left|z_{n}^{\prime}\right|^{2} \tag{7.28}
\end{equation*}
$$

for some $q$ with $0 \leqslant q \leqslant n$, the number of positive eigenvalues of the nondegenerate Levi form. Next, let us introduce the following Jacobian-like determinant:

$$
\Delta:=\left|\begin{array}{cccc}
\Theta_{\bar{z}_{1}} & \cdots & \Theta_{\bar{z}_{n}} & \Theta_{\bar{w}} \\
\Theta_{z_{1} \bar{z}_{1}} & \cdots & \Theta_{z_{1} \bar{z}_{n}} & \Theta_{z_{1} \bar{w}} \\
\cdot & \cdots & \cdot & \cdot \\
\Theta_{z_{n} \bar{z}_{1}} & \cdots & \Theta_{z_{n} \bar{z}_{n}} & \Theta_{z_{n} \bar{w}}
\end{array}\right|=\left|\begin{array}{cccc}
\Theta_{\bar{t}_{1}} & \cdots & \Theta_{\bar{t}_{n}} & \Theta_{\bar{t}_{n+1}} \\
\Theta_{z_{1} \bar{t}_{1}} & \cdots & \Theta_{z_{1} \bar{t}_{n}} & \Theta_{z_{1} \bar{t}_{n+1}} \\
\cdot & \cdots & \cdot & \cdots \\
\Theta_{z_{n} \bar{t}_{1}} & \cdots & \Theta_{z_{n} \bar{t}_{n}} & \Theta_{z_{n} \bar{t}_{n+1}}
\end{array}\right| .
$$

For any index $\mu \in\{1, \ldots, n, n+1\}$ and for any index $\ell \in\{1, \ldots, n\}$, let also $\Delta_{\left[0_{1+\ell}\right]}^{\mu}$ denote the same determinant, but with its $\mu$-th column replaced by the transpose of the line $(0 \cdots 1 \cdots 0)$ with 1 at the $(1+\ell)$-th place, and 0 elsewhere, its other columns being untouched. One easily convinces oneself (but see also Section 2) that $M$ is Levi-nondegenerate at $p$ - which is the origin of our system of coordinates - if and only if $\Delta$ does not vanish at the origin, whence $\Delta$ is nowhere zero in some sufficiently small neighborhood of the origin. Similarly, for any indices $\mu, \nu, \tau \in\{1, \ldots, n, n+1\}$, denote by $\Delta_{\left[t^{\mu} \bar{t}^{\nu}\right]}^{\tau}$ the same determinant as $\Delta$, but with only its $\tau$-th column replaced by the transpose of the line:

$$
\left(\begin{array}{llll}
\Theta_{\bar{t}^{\mu} \bar{t}^{\nu}} & \Theta_{z_{1}} \bar{t}^{\mu} \bar{t}^{\nu} & \cdots & \Theta_{z_{n}} \bar{t}^{\mu} \bar{t}^{\nu}
\end{array}\right)
$$

other columns being again untouched. All these determinants $\Delta, \Delta_{\left[0_{1+\ell}\right]}^{\mu}$, $\Delta_{\left[\bar{t}^{\mu} \bar{t}^{\nu}\right]}^{\tau}$ are visibly universal differential expressions depending upon the second-order jet $J_{z, \bar{z}, \bar{w}}^{2} \Theta$ and upon the third-order jet $J_{z, \bar{z}, \bar{w}}^{3} \Theta$.
Main Theorem. An arbitrary, not necessarily rigid, real analytic hypersurface $M \subset \mathbb{C}^{n+1}$ with $\underline{n \geqslant 2}$ which is Levi nondegenerate at one of its points
$p$ and has a complex defining equation of the form (7.28) in some system of local holomorphic coordinates $t=(z, w) \in \mathbb{C}^{n} \times \mathbb{C}$ vanishing at $p$, is pseudospherical at $p$ if and only if its complex graphing function $\Theta$ satisfies the following explicit nonlinear fourth-order system of partial differential equations:

$$
\begin{aligned}
& 0 \equiv \sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1}\left[\Delta_{\left[0_{1+\ell_{1}}\right]}^{\mu} \cdot \Delta_{\left[0_{1+\ell_{2}}\right]}^{\nu}\left\{\Delta \cdot \frac{\partial^{4} \Theta}{\partial z_{k_{1}} \partial z_{k_{2}} \partial \bar{t}_{\mu} \partial \bar{t}_{\nu}}-\sum_{\tau=1}^{n+1} \Delta_{\left[t^{\mu} \bar{\tau}^{\nu}\right]}^{\tau} \cdot \frac{\partial^{3} \Theta}{\partial z_{k_{1}} \partial z_{k_{2}} \partial \partial^{\tau}}\right\}-\right. \\
& -\frac{\delta_{k_{1}, \ell_{1}}}{n+2} \sum_{\ell^{\prime}=1}^{n} \Delta_{\left[0_{1+\ell^{\prime}}\right]}^{\mu} \cdot \Delta_{\left[0_{1+\ell_{2}}\right]}^{\nu}\left\{\Delta \cdot \frac{\partial^{4} \Theta}{\partial z_{\ell^{\prime}} \partial z_{k_{2}} \partial \bar{t}_{\mu} \partial \bar{t}_{\nu}}-\sum_{\tau=1}^{n+1} \Delta_{\left[\bar{t}^{\mu} \bar{t}^{\nu}\right]}^{\tau} \cdot \frac{\partial^{3} \Theta}{\partial z_{\ell^{\prime}} \partial z_{k_{2}} \partial \bar{t}^{\tau}}\right\}- \\
& -\frac{\delta_{k_{1}, \ell_{2}}}{n+2} \sum_{\ell^{\prime}=1}^{n} \Delta_{\left[0_{1+\ell_{1}}\right]}^{\mu} \cdot \Delta_{\left[0_{1+\ell^{\prime}}\right]}^{\nu}\left\{\Delta \cdot \frac{\partial^{4} \Theta}{\partial z_{\ell^{\prime}} \partial z_{k_{2}} \partial \bar{t}_{\mu} \partial \bar{t}_{\nu}}-\sum_{\tau=1}^{n+1} \Delta_{\left[t^{\mu} \bar{t}^{\nu}\right]}^{\tau} \cdot \frac{\partial^{3} \Theta}{\partial z_{\ell^{\prime}} \partial z_{k_{2}} \partial \bar{t} \tau}\right\}- \\
& -\frac{\delta_{k_{2}, \ell_{1}}}{n+2} \sum_{\ell^{\prime}=1}^{n} \Delta_{\left[0_{1+\ell^{\prime}}\right]}^{\mu} \cdot \Delta_{\left[0_{1+\ell_{2}}\right]}^{\nu}\left\{\Delta \cdot \frac{\partial^{4} \Theta}{\partial z_{k_{1}} \partial z_{\ell^{\prime}} \partial \bar{t}_{\mu} \partial \bar{t}_{\nu}}-\sum_{\tau=1}^{n+1} \Delta_{\left[\bar{t}^{\mu} \bar{t}^{\nu}\right]}^{\tau} \cdot \frac{\partial^{3} \Theta}{\partial z_{k_{1}} \partial z_{\ell^{\prime}} \partial \bar{t}^{\tau}}\right\}- \\
& -\frac{\delta_{k_{2}, \ell_{2}}}{n+2} \sum_{\ell^{\prime}=1}^{n} \Delta_{\left[0_{1+\ell_{1}}\right]}^{\mu} \cdot \Delta_{\left[0_{1+\ell^{\prime}}\right]}^{\nu}\left\{\Delta \cdot \frac{\partial^{4} \Theta}{\partial z_{k_{1}} \partial z_{\ell^{\prime}} \partial \bar{t}_{\mu} \partial \bar{t}_{\nu}}-\sum_{\tau=1}^{n+1} \Delta_{\left[t^{\mu} \bar{t}^{\nu}\right]}^{\tau} \cdot \frac{\partial^{3} \Theta}{\partial z_{k_{1}} \partial z_{\ell^{\prime}} \partial \bar{t}^{\tau}}\right\}+ \\
& +\frac{1}{(n+1)(n+2)} \cdot\left[\delta_{k_{1}, \ell_{1}} \delta_{k_{2}, \ell_{2}}+\delta_{k_{2}, \ell_{1}} \delta_{k_{1}, \ell_{2}}\right] . \\
& \left.\cdot \sum_{\ell^{\prime}=1}^{n} \sum_{\ell^{\prime \prime}=1}^{n} \Delta_{\left[0_{1+\ell^{\prime}}\right]}^{\mu} \cdot \Delta_{\left[0_{1+\ell^{\prime \prime}}\right.}^{\nu}\left\{\Delta \cdot \frac{\partial^{4} \Theta}{\partial z_{\ell^{\prime}} \partial z_{\ell^{\prime \prime}} \partial \bar{t} \bar{t}_{\mu} \partial \bar{t}_{\nu}}-\sum_{\tau=1}^{n+1} \Delta_{\left.\tau \bar{t}^{\mu} \bar{t}^{\nu}\right]}^{\tau} \cdot \frac{\partial^{3} \Theta}{\partial z_{\ell^{\prime}} \partial z_{\ell^{\prime \prime}} \partial \bar{t}^{\tau}}\right\},\right\}
\end{aligned}
$$

for all pairs of indices $\left(k_{1}, k_{2}\right)$ with $1 \leqslant k_{1}, k_{2} \leqslant n$, and for all pairs of indices $\left(\ell_{1}, \ell_{2}\right)$ with $1 \leqslant \ell_{1}, \ell_{2} \leqslant n$.

The written system is effective: no implicit formal expression is involved and pseudosphericality is characterized directly and only in terms of $\Theta$.

Now, here is a summarized description of our arguments of proof. A bit similarly as for the $\mathbb{C}^{2}$-case - but with major differences afterwards which was already studied in [21], we may associate to any such Levi nondegenerate real analytic local hypersurface $M \subset \mathbb{C}^{n+1}$ of equation $w=\Theta(z, \bar{z}, \bar{w})$ a uniquely defined system of second-order partial differential equations:

$$
\begin{equation*}
w_{z_{k_{1}} z_{k_{2}}}(z)=\Phi_{k_{1}, k_{2}}\left(z, w(z), w_{z}(w)\right) \quad\left(1 \leqslant k_{1}, k_{2} \leqslant n\right) \tag{7.28}
\end{equation*}
$$

with $\Phi_{k_{1}, k_{2}}=\Phi_{k_{2}, k_{1}}$, simply by eliminating the two variables $\bar{z}$ and $\bar{w}$, viewed as parameters, from the set of $n+1$ equations ${ }^{23}$ :
$w(z)=\Theta(z, \bar{z}, \bar{w}), \quad w_{z_{1}}(z)=\frac{\partial \Theta}{\partial z_{1}}(z, \bar{z}, \bar{w}), \ldots \ldots, w_{z_{n}}(z)=\frac{\partial \Theta}{\partial z_{n}}(z, \bar{z}, \bar{w})$,

- the assumption that the Jacobian determinant $\Delta$ is nonvanishing at the origin being precisely the one which guarantees, technically speaking, that the classical (holomorphic) implicit function theorem applies - and
${ }^{23}$ This process appears for instance in the references [8, Ha1937, Ch1975, Su2001, Su2002, 1, 19].
then by replacing the so obtained values for $\bar{z}$ and $\bar{w}$ in all second order derivatives $\frac{\partial^{2} \Theta}{\partial z_{k_{1}} z_{k_{2}}}(z, \bar{z}, \bar{w})$, see (7.28) below. Trivially, this system is completely integrable, for we just derived it from its general solution $w(z):=\Theta(z, \bar{z}, \bar{w})$, where $(\bar{z}, \bar{w})$ are understood as parameters.

As we said, Hachtroudi showed that the curvature of the projective normal (Cartan) connection he associated with the system (7.28) vanishes if and only if the right-hand side functions $F_{k_{1}, k_{2}}$ satisfy the following explicit differential system, which is linear in terms of their second-order derivatives (all of which, notably, appear only with respect to the $y_{x^{\ell}} \ell$ :
(7.28)

$$
\begin{aligned}
0 \equiv & \frac{\partial^{2} F_{k_{1}, k_{2}}}{\partial y_{x_{1} y_{1}} y_{x^{\ell_{2}}}}- \\
& -\frac{1}{n+2} \sum_{\ell^{\prime}=1}^{n}\left(\delta_{k_{1}, \ell_{1}} \frac{\partial^{2} F_{\ell^{\prime}, k_{2}}}{\partial y_{x^{\ell}} \partial y_{x^{\ell_{2}}}}+\delta_{k_{1}, \ell_{2}} \frac{\partial^{2} F_{\ell^{\prime}, k_{2}}}{\partial y_{x_{1} \ell_{1}} \partial y_{x^{\ell^{\prime}}}}+\delta_{k_{2}, \ell_{1}} \frac{\partial^{2} F_{k_{1}, \ell^{\prime}}}{\partial y_{x^{\ell^{\prime}}} \partial y_{x^{\ell} \ell_{2}}}+\delta_{k_{2}, \ell_{2}} \frac{\partial^{2} F_{k_{1}, \ell^{\prime}}}{\partial y_{x^{\ell_{1}}} \partial y_{x^{\ell^{\prime}}}}\right)+ \\
& +\frac{1}{(n+1)(n+2)}\left[\delta_{k_{1}, \ell_{1}} \delta_{k_{2}, \ell_{2}}+\delta_{k_{2}, \ell_{1}} \delta_{k_{1}, \ell_{2}}\right] \sum_{\ell^{\prime}=1}^{n} \sum_{\ell^{\prime \prime}=1}^{n} \frac{\partial^{2} F_{\ell^{\prime}, \ell^{\prime \prime}}}{\partial y_{x^{\prime}} \partial y_{x^{\ell^{\prime \prime}}} \quad\left(1 \leqslant k_{1}, k_{2} \leqslant n\right)} . \quad\left(1 \leqslant \ell_{1}, \ell_{2} \leqslant n\right)
\end{aligned}
$$

Hachtroudi also showed that this latter condition, better known nowadays amongst the Several Complex Variables community as vanishing of Chern(-Moser) curvature to which it indeed amounts, characterizes the local equivalence, through a point transformation $(x, y) \mapsto\left(x^{\prime}, y^{\prime}\right)=$ $\left(x^{\prime}(x, y), y^{\prime}(x, y)\right)$, to the simplest system: $y_{x^{\prime k} k_{1} x^{\prime} k_{2}}^{\prime}\left(x^{\prime}\right)=0$. We then remind the semi-known fact that $M$ is pseudospherical if and only if its associated second-order system (7.28) is equivalent, through a local biholo$\operatorname{morphism}(z, w) \mapsto\left(z^{\prime}, w^{\prime}\right)=\left(z^{\prime}(z, w), w^{\prime}(z, w)\right)$ fixing the origin, to the simplest system $w_{z_{k_{1}}^{\prime} z_{k_{2}}^{\prime}}^{\prime}\left(z^{\prime}\right)=0$. So we may apply to the functions $\Phi_{k_{1}, k_{2}}$ Hachtroudi's vanishing curvature equations (7.28), but still, the $\Phi_{k_{1}, k_{2}}$ are not expressed in terms of $\Theta$, for they were constructed by employing some unpleasant implicit functions when solving above for $\bar{z}$ and $\bar{w}$. Fortunately, here similarly as in [21], we may apply the techniques of computational differential algebra sketched in [19] in order to explicitly express any algebraic expressions in the second-order jet of the $\Phi_{k_{1}, k_{2}}$ in terms of the fourth-order jet of $\Theta$, and the appropriate general equation which we shall need:

$$
\frac{\partial^{2} \Phi_{k_{1}, k_{2}}}{\partial w_{z_{1}} \partial w_{z_{2}}}=\frac{1}{\Delta^{3}} \sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \Delta_{\left[0_{\left.1+\ell_{1}\right]}\right]}^{\mu} \cdot \Delta_{\left[0_{1+\ell_{2}}\right]}^{\nu}\left\{\Delta \cdot \frac{\partial^{4} \Theta}{\partial z_{k_{1}} \partial z_{k_{2}} \partial \bar{t}^{\mu} \partial \bar{t}^{\nu}}-\sum_{\tau=1}^{n+1} \Delta_{\left[t^{\mu} \bar{t}^{\nu}\right]}^{\tau} \cdot \frac{\partial^{3} \Theta}{\partial z_{k_{1}} \partial z_{k_{2}} \partial \bar{t}^{\tau}}\right\}
$$

will be obtained in Section 4 below, after rather lengthy but elementary calculations, parts of which are inspired from [17]. It is now essentially clear how one obtains the (boxed) long fourth-order differential equations stated in the theorem, but in any case, some complete details will be provided at the very end of the paper.

To conclude this extensive introduction which was designed for readers wanting to quickly embrace the contents, we would like to draw the attention on the work [22], whose manual calculations where finalized in manuscript form already in $2003^{24}$, and which will soon confirm the above theorem by following another route, viz. by calculating explicitly the socalled Chern(-Moser) tensor differential forms, which might interest some contemporary CR geometers better than the (essentially equivalent) original Cartan-Hachtroudi(-Tanaka) approach.

## §2. SEGRE VARIETIES AND DIFFERENTIAL EQUATIONS

Real analytic hypersurfaces in $\mathbb{C}^{n+1}$. Let us therefore consider an arbitrary real analytic hypersurface $M$ in $\mathbb{C}^{n+1}$ with $n \geqslant 2$, and let us localize it around one of its points, say $p \in M$. Then there exist complex affine coordinates:
$(z, w)=\left(z_{1}, \ldots, z_{n}, w\right)=\left(x_{1}+\sqrt{-1} y_{1}, \ldots, x_{n}+\sqrt{-1} y_{n}, u+i v\right)=(x+\sqrt{-1} y, u+\sqrt{-1} v)$
vanishing at $p$ in which $T_{p} M=\{u=0\}$, so that $M$ is represented in a neighborhood of $p$ by a graphed defining equation of the form:

$$
u=\varphi(x, y, v)=\varphi\left(x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}, v\right)
$$

where the real-valued function:

$$
\varphi=\varphi(x, y, v)=\sum_{\substack{k \in \mathbb{N} n \\ k \in \mathbb{N},+\mathbb{N}, m \in \mathbb{N} \\ k| |+| |+m \geqslant 2}} \varphi_{k, l, m} x^{k} y^{l} v^{m} \in \mathbb{R}\{x, y, u\},
$$

which possesses entirely arbitrary real coefficients $\varphi_{k, l, m}$, vanishes at the origin: $\varphi(0)=0$, together with all its first order derivatives: $0=\partial_{x^{k}} \varphi(0)=$ $\partial_{y^{\prime}} \varphi(0)=\partial_{v} \varphi(0)$. By simply rewriting this initial real equation of $M$ as:

$$
\frac{w+\bar{w}}{2}=\varphi\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2 \sqrt{-1}}, \frac{w-\bar{w}}{2 \sqrt{-1}}\right),
$$

and then by solving the so written equation with respect to $w$, one obtains an equation of the shape:

$$
w=\Theta(z, \bar{z}, \bar{w})=\sum_{\substack{k \in \mathbb{N}^{n} n, l \in \mathbb{N}^{n}, m \in \mathbb{N} \\|k|+| |+m \geqslant 1}} \Theta_{k, l, m} z^{k} \bar{z}^{l} \bar{w}^{m} \in \mathbb{C}\{\bar{z}, z, w\},
$$

whose right-hand side converges of course near the origin $(0,0,0) \in \mathbb{C}^{n} \times$ $\mathbb{C}^{n} \times \mathbb{C}$ and whose coefficients $\Theta_{k, l, m} \in \mathbb{C}$ are complex. Since $d \varphi(0)=0$, one has $\Theta=-\bar{w}+$ order 2 terms.

The paradox that any such complex equation provides in fact two real defining equations for the real hypersurface $M$ which is one-codimensional,

[^15]and also in addition the fact that one could as well have chosen to solve the above equation with respect to $\bar{w}$, instead of $w$, these two apparent "contradictions" are corrected by means of a fundamental, elementary statement that transfers to $\Theta$ (in a natural way) the condition of reality:
$\overline{\varphi(x, y, u)}=\sum_{|k|+|l|+m \geqslant 1} \overline{\varphi_{k, l, m}} \bar{x}^{k} \bar{y}^{l} \bar{v}^{m}=\sum_{|k|+|l|+m \geqslant 1} \varphi_{k, l, m} x^{k} y^{l} v^{m}=\varphi(x, y, v)$
enjoyed by the initial definining function $\varphi$. In the sequel, we shall work exclusively with $\Theta$; the reader is referred to [21] for justifications and motivations.

Theorem. ([18], p. 19) The complex analytic function $\Theta=\Theta(z, \bar{z}, \bar{w})$ with $\Theta=-\bar{w}+\mathrm{O}(2)$ together with its complex conjugate:

$$
\bar{\Theta}=\bar{\Theta}(\bar{z}, z, w)=\sum_{k \in \mathbb{N}^{n}, l \in \mathbb{N}^{n}, m \in \mathbb{N}} \bar{\Theta}_{k, l, m} \bar{z}^{k} z^{l} w^{m} \in \mathbb{C}\{\bar{z}, z, w\}
$$

satisfy the two (equivalent by conjugation) functional equations:

$$
\begin{align*}
& \bar{w} \equiv \bar{\Theta}(\bar{z}, z, \Theta(z, \bar{z}, \bar{w})), \\
& w \equiv \Theta(z, \bar{z}, \bar{\Theta}(\bar{z}, z, w)) . \tag{7.28}
\end{align*}
$$

Conversely, given a local holomorphic function $\Theta(z, \bar{z}, \bar{w}) \in \mathbb{C}\{z, \bar{z}, \bar{w}\}$, $\Theta=-\bar{w}+\mathrm{O}(2)$ which, in conjunction with its conjugate $\bar{\Theta}(\bar{z}, z, w)$, satisfies this pair of equivalent identities, then the two zero-sets:

$$
\{0=-w+\Theta(z, \bar{z}, \bar{w})\} \quad \text { and } \quad\{0=-\bar{w}+\bar{\Theta}(\bar{z}, z, w)\}
$$

coincide and define a local one-codimensional real analytic hypersurface $M$ passing through the origin in $\mathbb{C}^{n+1}$.

Levi nondegeneracy. Within the hierarchy of nondegeneracy conditions for real hypersurfaces initiated by Diederich and Webster ([DW1980], see also [Me2005a, Me2005b] for generalizations and a unification), Levi nondegeneracy is the most studied. The classical definition may be found in [Bo1991] and in the survey of Chirka [Ch1991], but the following basic equivalent characterization can also be understood as a definition in the present paper. One may show ([Me2005a, Me2005b, 18]) that it is biholomorphically invariant.
Lemma. ([18], p. 28) The real analytic hypersurface $M \subset \mathbb{C}^{n+1}$ with $0 \in M$ represented in coordinates $\left(z_{1}, \ldots, z_{n}, w\right)$ by a complex defining equation of the form $w=\Theta(z, \bar{z}, \bar{w})$ is Levi nondegenerate at the origin if and only if the map:

$$
\left(\bar{z}_{1}, \ldots, \bar{z}_{n}, \bar{w}\right) \longmapsto\left(\Theta(0, \bar{z}, \bar{w}), \frac{\partial \Theta}{\partial z_{1}}(0, \bar{z}, \bar{w}), \ldots, \frac{\partial \Theta}{\partial z_{n}}(0, \bar{z}, \bar{w})\right)
$$

has nonvanishing $(n+1) \times(n+1)$ Jacobian determinant at $(\bar{z}, \bar{w})=(0,0)$.

It follows then that this Jacobian determinant, not restricted to the origin:

$$
\Delta=\Delta(z, \bar{z}, \bar{w}):=\left|\begin{array}{cccc}
\Theta_{\bar{z}_{1}} & \cdots & \Theta_{\bar{z}_{n}} & \Theta_{\bar{w}}  \tag{7.28}\\
\Theta_{z_{1} \overline{1}_{1}} & \cdots & \Theta_{z_{1} \bar{z}_{n}} & \Theta_{z_{1} \bar{w}} \\
\cdot & \cdots & \cdot & \cdot \\
\Theta_{z_{n} \bar{z}_{1}} & \cdots & \Theta_{z_{n} \bar{z}_{n}} & \Theta_{z_{n} \bar{w}}
\end{array}\right|
$$

does not vanish in some small neighborhood of the origin in $\mathbb{C}^{n} \times \mathbb{C}^{n} \times \mathbb{C}$. Levi nondegeneracy at the central point, i.e. $\Delta \neq 0$ locally, will be assumed throughout the present paper.

Associated system of partial differential equations. At least since the publication in 1888 by Lie and Engel in Leipzig of the Theorie der Transformationsguppen, it is known in a very general context - see Chapter 10 of [8] and also [23, Ha1937, Ch1975, Fa1980, 19, 1, 21] - that, to the whole family of Segre varieties:

$$
S_{\bar{z}, \bar{w}}:=\left\{(z, w) \in \mathbb{C}^{n} \times \mathbb{C}: w=\Theta(z, \bar{z}, \bar{w})\right\}
$$

parametrized by the $n+1$ antiholomorphic variables $\left(\bar{z}_{1}, \ldots, \bar{z}_{n}, \bar{w}\right)$, one may canonically associate a completely integrable second-order system of partial differential equations whose general solution is precisely the function $\Theta(z, \bar{z}, \bar{w})$. Indeed, considering $w$ as a function $w=w(z)$ of $\left(z_{1}, \ldots, z_{n}\right)$ in the defining equation of $M$, one differentiates it once with respect to each variable $z_{1}, \ldots, z_{n}$ so that one gets the $n+1$ equations:
$w(z)=\Theta(z, \bar{z}, \bar{w}), \quad w_{z_{1}}(z)=\frac{\partial \Theta}{\partial z_{1}}(z, \bar{z}, \bar{w}), \ldots \ldots, w_{z_{n}}(z)=\frac{\partial \Theta}{\partial z_{n}}(z, \bar{z}, \bar{w})$.
Then by means of the implicit function theorem - which applies precisely thanks to the nonvanishing of $\Delta$-, one may clearly solve for the $n+1$ antiholomorphic "parameters" $(\bar{z}, \bar{w})$, and this procedure provides a representation:
$\bar{z}_{1}=\zeta_{1}\left(z, w(z), w_{z}(z)\right), \ldots, \bar{z}_{n}=\zeta_{n}\left(z, w(z), w_{z}(z)\right), \bar{w}=\xi\left(z, w(z), w_{z}(z)\right)$
with certain $n+1$ uniquely defined local complex analytic functions $\zeta_{i}\left(z, w, w_{z}\right)$ and $\xi\left(z, w, w_{z}\right)$ of $2 n+1$ complex variables. Utilizing these functions, one is then pushed to replace $\bar{z}$ and $\bar{w}$ in all possible second-order derivative:

$$
\begin{align*}
w_{z_{k_{1}} z_{k_{2}}}(z) & =\frac{\partial^{2} \Theta}{\partial z_{k_{1}} \partial z_{k_{2}}}(z, \bar{z}, \bar{w}) \\
& =\frac{\partial^{2} \Theta}{\partial k_{k_{1}} \partial z_{k_{2}}}\left(z, \zeta\left(z, w(z), w_{z}(z)\right), \quad \xi\left(z, w(z), w_{z}(z)\right)\right)  \tag{7.28}\\
& =: \Phi_{k_{1}, k_{2}}\left(z, w(z), w_{z}(z)\right) \quad\left(k_{1}, k_{2}=1 \cdots n\right),
\end{align*}
$$

and this defines without ambiguity the associated system of partial differential equations. It is of second order. It is complete: all second-order derivatives are functions of derivatives of lower order $\leqslant 1$. In a sense to be precised right now, it is also completely integrable because by construction, its general solution is $\Theta(z, \bar{z}, \bar{w})$.

Geometric characterization of pseudosphericality. It is well known that the unit sphere:

$$
S^{2 n+1}=\left\{\left(z_{1}, \ldots, z_{n}, w\right) \in \mathbb{C}^{n} \times \mathbb{C}:\left|z_{1}\right|^{2}+\cdots+\left|z_{n}\right|^{2}+|w|^{2}=1\right\}
$$

in $\mathbb{C}^{n}$ minus one of its points, for instance: $S^{2 n+1} \backslash\left\{p_{\infty}\right\}$ with $p_{\infty}:=$ $(0, \ldots, 0,-1)$, is biholomorphic, through the so-called Cayley transform:

$$
\left(z_{1}, \ldots, z_{n}, w\right) \longmapsto\left(\frac{i z_{1}}{1+w}, \ldots, \frac{i z_{n}}{1+w}, \frac{1-w}{2+2 w}\right)=:\left(z_{1}^{\prime}, \ldots, z_{n}^{\prime}, w^{\prime}\right)
$$

having inverse:

$$
\left(z_{1}^{\prime}, \ldots, z_{n}^{\prime}, w^{\prime}\right) \longmapsto\left(\frac{-2 i z_{1}^{\prime}}{1+2 w^{\prime}}, \ldots, \frac{-2 i z_{n}^{\prime}}{1+2 w^{\prime}}, \frac{1-2 w^{\prime}}{1+2 w^{\prime}}\right)=\left(z_{1}, \ldots, z_{n}, w\right)
$$

to the so-called standard Heisenberg sphere of equation:

$$
w^{\prime}=-\bar{w}^{\prime}+z_{1}^{\prime} \bar{z}_{1}^{\prime}+\cdots+z_{n}^{\prime} \bar{z}_{n}^{\prime}
$$

which sits in the target space $\left(z^{\prime}, w^{\prime}\right)$. Hence in the particular case when the Levi form of $M$ has only positive eigenvalues, namely when $q=n$ in (7.28), it follows clearly that $M$ is spherical in the sense given in the Introduction if and only if there exists a nonempty open neighborhood $U_{0}$ of 0 in $\mathbb{C}^{n+1}$ such that $M \cap U_{0}$ is biholomorphic to a piece of the unit sphere. In general, there are $n-q$ negative eigenvalues in the Levi form, and this justifies adding a "pseudo".
Proposition. A Levi nondegenerate local real analytic hypersurface $M$ in $\mathbb{C}^{n+1}$ is locally biholomorphic to a piece of the Heisenberg pseudosphere (hence pseudospherical) if and only if its associated second-order ordinary complex differential equation is locally equivalent to the second-order system:

$$
w_{z_{k_{1}}^{\prime} z_{k_{2}}^{\prime}}^{\prime}\left(z^{\prime}\right)=0 \quad\left(1 \leqslant k_{1}, k_{2} \leqslant n\right),
$$

with identically vanishing right-hand side.
Proof. The $n=1$ case, treated in great details by a previous reference [21], generalizes here with rather evident adaptations, hence will be skipped. As $n \geqslant 2$ throughout the present paper, one may also argue by slicing $\mathbb{C}^{n+1}$ by all possible copies of $\mathbb{C}^{2}$ which pass through the origin and which contain the $w$-axis, so as to be able to apply the alreaday detailed $n=1$ case.

Geometrically, the local equivalence of $M$ to the Heisenberg pseudosphere means that, through some suitable local biholomorphism $(z, w) \mapsto$
$\left(z^{\prime}, w^{\prime}\right)$ fixing the origin, both its Segre varieties and its conjugate Segre varieties ([Me2005a, Me2005b, 18]):
$S_{\bar{z}, \bar{w}}:=\{(z, w): w=\Theta(z, \bar{z}, \bar{w})\} \quad$ and $\quad \bar{S}_{z, w}:=\{(\bar{z}, \bar{w}): \bar{w}=\bar{\Theta}(\bar{z}, z, w)\}$
are mapped to the Segre and conjugate Segre varieties of the Heisenberg pseudosphere:

$$
S_{\bar{z}^{\prime}, \bar{w}^{\prime}}^{\prime}=\left\{w^{\prime}=-\bar{w}^{\prime}+z^{\prime} \bar{z}^{\prime}\right\} \quad \text { and } \quad\left\{\bar{w}^{\prime}=-w^{\prime}+\bar{z}^{\prime} z^{\prime}\right\}
$$

which, visibly, are plain complex affine lines.

## §3. GEOMETRY OF ASSOCIATED SUBMANIFOLDS OF SOLUTIONS

Completely integrable systems of partial differential equations. The characterization of pseudosphericality we are dealing with holds in a context more general than just CR geometry ${ }^{25}$. Accordingly, let $\mathbb{K}$ denote either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers, let $x=\left(x^{1}, \ldots, x^{n}\right) \in \mathbb{K}^{n}$ with again $n \geqslant 2-$ since the case $n=1$ was already studied in [21] —, let $y \in \mathbb{K}$, and consider a system of the form (7.28). We will assume that it is completely integrable in the sense that the natural commutativity of partial derivatives enjoyed trivially by the left-hand sides:

$$
\begin{gathered}
\partial^{2} y_{x^{k_{1}} x^{k_{2}}} / \partial y_{x^{k_{3}}}=\partial^{2} y_{x^{k_{1}} x^{k_{3}}} / \partial y_{x^{k_{2}}} \\
\left(1 \leqslant k_{1}, k_{2}, k_{3} \leqslant n\right)
\end{gathered}
$$

imposes immediately to the right-hand side functions $F_{k_{1}, k_{2}}$ that they satisfy the so-called compatibility conditions:

$$
\mathrm{D}_{k_{3}}\left(F_{k_{1}, k_{2}}\right)=\mathrm{D}_{k_{2}}\left(F_{k_{1}, k_{3}}\right),
$$

where we have introduced the following $n$ total differentiation operators:

$$
\begin{gathered}
\mathrm{D}_{k}:=\frac{\partial}{\partial x^{k}}+y_{x^{k}} \frac{\partial}{\partial y}+\sum_{\ell=1}^{n} F_{k, l} \frac{\partial}{\partial y_{x^{\ell}} \ell} \\
(1 \leqslant k \leqslant n)
\end{gathered}
$$

living on the first-order jet space $\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right)$. One verifies that these compatibility conditions amount to the fact that the $n$-dimensional tangential distribution spanned by $\mathrm{D}_{1}, \ldots, \mathrm{D}_{n}$ in the $(2 n+1)$-dimensional first-order jet space satisfies the classical Frobenius integrability condition $\left[\mathrm{D}_{k^{\prime}}, \mathrm{D}_{k^{\prime \prime}}\right]=0$, and then the Clebsch-Frobenius theorem tells us that this distribution comes from a local foliation by $n$-dimensional manifolds graphed over the $x$-space that are naturally parametrized by $n+1$ auxiliary constants (transversal directions) - call them $a^{1}, \ldots, a^{n}, b \in \mathbb{K}$-, namely

[^16]the leaves of this local foliation may be explicitly represented as sets of the shape:
\[

$$
\begin{aligned}
& \left\{\left(x^{1}, \ldots, x^{n}, Q\left(x^{1}, \ldots, x^{n}, a^{1}, \ldots, a^{n}, b\right)\right.\right. \\
& \left.\left.\quad S^{1}\left(x^{1}, \ldots, x^{n}, a^{1}, \ldots, a^{n}, b\right), \ldots, S^{n}\left(x^{1}, \ldots, x^{n}, a^{1}, \ldots, a^{n}, b\right)\right)\right\}
\end{aligned}
$$
\]

where $x^{1}, \ldots, x^{n}$ vary freely and where $Q, S^{1}, \ldots, S^{n}$ are certain graphing functions. In fact, the functions $S^{k}$ are the first-order derivatives:

$$
S^{1}=Q_{x^{1}}, \ldots \ldots, S^{n}=Q_{x^{n}}
$$

of the function $Q$, because by definition the integral curves of every vector field $\mathrm{D}_{k}$ must be contained in such leaves, so that one has:

$$
\frac{\partial Q}{\partial x^{k}}=\left.y_{x^{k}}\right|_{\text {any leaf }}=S^{k}
$$

and furthermore also:

$$
\frac{\partial S^{l}}{\partial x^{k}}=\left.F^{k, l}\right|_{\text {any leaf }},
$$

whence we see that the fundamental graphing function $Q=Q(x, a, b)$ happens to be the general solution to the initially given system of partial differential equations:

$$
\begin{gathered}
\left.Q_{x^{k_{1} x^{k_{2}}}}(x, a, b) \equiv F_{k_{1}, k_{2}}\left(x, Q(x, a, b), Q_{x^{1}}(x, a, b), \ldots, Q_{x^{n}}(x, a, b)\right)\right) \\
\left(1 \leqslant k_{1}, k_{2} \leqslant n\right) .
\end{gathered}
$$

In the CR case, the fundamental function which is the general solution to the associated system of partial diffential equations (7.28) is obviously the complex defining function $\Theta(z, \bar{z}, \bar{w})$, where the $n+1$ quantities $(\bar{z}, \bar{w})$, viewed as independent variables, play the role of the constants $(a, b)$.

As in the $n=1$ case, the constants $\left(a^{1}, \ldots, a^{n}, b\right)$ are best interpreted as a set of $n+1$ initial conditions $\left(y_{x^{1}}(0), \ldots, y_{x^{n}}(0),-y(0)\right)$ or integration constants, so that we can assume without loss of generality that the firstorder terms in the fundamental function $Q$ are ${ }^{26}$ :

$$
Q(x, a, b)=-b+x^{1} a^{1}+\cdots+x^{n} a^{n}+\mathrm{O}\left(|x|^{2}\right)
$$

It is then clear that the map:

$$
\begin{align*}
\left(a^{1}, \ldots, a^{n}, b\right) & \longmapsto  \tag{7.28}\\
& =\left(Q(0, a, b), Q_{x^{1}}(0, a, b), \ldots, Q_{x^{n}}(0, a, b)\right) \\
& \left(-b, a^{1}, \ldots, a^{n}\right)
\end{align*}
$$

is of rank $n+1$ at the origin, and this property remains also true whatever one chooses as a fundamental function $Q(x, a, b)$, that is to say, without necessarily assuming it to be normalized as above, which amounts to saying

[^17]that ${ }^{27}$, in the parameter $(a, b)$-space, everything holds invariantly up to any local $\mathbb{K}$-analytic transformation $(a, b) \mapsto\left(a^{\prime}, b^{\prime}\right)$ which does not involve the variables $(x, y)$.

The way how one recovers the system of partial differential equations is very similar to what we did in the CR case (7.28). Suppose indeed a bit more generally that we are given any local $\mathbb{K}$-analytic function $Q=Q(x, a, b)$ having the property that its first-order $x$-jet map (7.28) is of rank $n+1$ at $(a, b)=(0,0)$. Then in the $n+1$ equations:

$$
y(x)=Q(x, a, b), \quad y_{x^{1}}=Q_{x^{1}}(x, a, b), \ldots \ldots, Q_{x^{n}}(x, a, b),
$$

we can solve, by means of the implicit function theorem, for the $n+1$ constants ( $a^{1}, \ldots, a^{n}, b$ ), and this yields a representation:

$$
\begin{aligned}
a^{k} & =A^{k}\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right) \\
b & =B\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right)
\end{aligned}
$$

for certain functions $A^{1}, \ldots, A^{n}, B$ of $(2 n+1)$ variables. Then by replacing these obtained values for the $a^{k}$ and for $b$ in all the possible second-order derivatives:

$$
\begin{aligned}
y_{x^{k_{1} x^{k_{2}}}} & =Q_{x^{k_{1}} x^{k_{2}}}(x, a, b) \\
& =Q_{x^{k_{1}} x^{k_{2}}}\left(x, A\left(x, y, y_{x}\right), B\left(x, y, y_{x}\right)\right) \\
& =F_{k_{1}, k_{2}}\left(x, y, y_{x}\right)
\end{aligned}
$$

it is rigorously clear that one may only recover the functions $F_{k_{1}, k_{2}}$ we started with.

## §4. EfFECTIVE DIFFERENTIAL CHARACTERIZATION OF PSEUDOSPHERICALITY IN $\mathbb{C}^{n+1}$

The $2 n+1$ coordinates of the transformation considered at the moment:

$$
\begin{equation*}
\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right) \longmapsto\left(x^{1}, \ldots, x^{n}, a^{1}, \ldots, a^{n}, b\right) \tag{7.28}
\end{equation*}
$$

and those of its inverse are given by the collection of functions:

$$
\left[\begin{array} { r l } 
{ x ^ { j } } & { = x ^ { j } } \\
{ a ^ { k } } & { = A ^ { k } ( x ^ { 1 } , \ldots , x ^ { n } , y , y _ { x ^ { 1 } } , \ldots , y _ { x ^ { n } } ) } \\
{ b } & { = B ( x ^ { 1 } , \ldots , x ^ { n } , y , y _ { x ^ { 1 } } , \ldots , y _ { x ^ { n } } ) }
\end{array} \quad \text { and } \quad \left[\begin{array}{rl}
x^{j} & =x^{j} \\
y & =Q\left(x^{1}, \ldots, x^{n}, a^{1}, \ldots, a^{n}, b\right) \\
y_{x^{k}} & =Q_{x^{k}}\left(x^{1}, \ldots, x^{n}, a^{1}, \ldots, a^{n}, b\right) .
\end{array}\right.\right.
$$

For uniformity and harmony, we shall admit by convention the equivalences of notation:

$$
b \equiv a^{n+1} \quad \text { and } \quad B \equiv A^{n+1}
$$

[^18]Then by differentiating with respect to $y_{x^{\ell}}$ each one of the following $n+1$ identically satisfied equations:

$$
\begin{aligned}
& y \equiv Q\left(x^{1}, \ldots, x^{n}, A^{1}\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right), \ldots\right. \\
& \left.A^{n}\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right), A^{n+1}\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right)\right) \\
& y_{x^{k}} \equiv Q_{x^{k}}\left(x^{1}, \ldots, x^{n}, A^{1}\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right), \ldots\right. \\
& \\
& \left.A^{n}\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right), A^{n+1}\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right)\right)
\end{aligned}
$$

we get the following $n+n^{2}$ equations:

$$
\begin{aligned}
0 & \equiv Q_{a^{1}} \frac{\partial A^{1}}{\partial y_{x^{\ell}}}+\cdots+Q_{a^{n}} \frac{\partial A^{n}}{\partial y_{x^{\ell} \ell}}+Q_{a^{n+1}} \frac{\partial A^{n+1}}{\partial y_{x^{\ell}}} \\
\delta_{k, \ell} & =Q_{x^{k} a^{1}} \frac{\partial A^{1}}{\partial y_{x^{\ell} \ell}}+\cdots+Q_{x^{k} a^{n}} \frac{\partial A^{n}}{\partial y_{x^{\ell} \ell}}+Q_{x^{k} a^{n+1}} \frac{\partial A^{n+1}}{\partial y_{x^{\ell} \ell}}
\end{aligned}
$$

$$
(k, \ell=1 \cdots n)
$$

Fixing any $\ell \in\{1, \ldots, n\}$, thanks to the assumption (Levi nondegeneracy) that the Jacobian determinant:

$$
\square=\square\left(a^{1}|\cdots| a^{n} \mid a^{n+1}\right):=\left|\begin{array}{cccc}
Q_{a^{1}} & \cdots & Q_{a^{n}} & Q_{a^{n+1}} \\
Q_{x^{1} a^{1}} & \cdots & Q_{x^{1} a^{n}} & Q_{x^{1} a^{n+1}} \\
\vdots & \ddots & \vdots & \vdots \\
Q_{x^{n} a^{1}} & \cdots & Q_{x^{n} a^{n}} & Q_{x^{n} a^{n+1}}
\end{array}\right|
$$

does not vanish, we may solve - just by means of Cramer's rule - for the $n+1$ unknowns $\frac{\partial A^{\mu}}{\partial y_{x} \ell}$, the above system of $n+1$ equations, and this gives us:

$$
\begin{equation*}
\frac{\partial A^{\mu}}{\partial y_{x^{\ell}}}=\frac{\square_{\left[0_{1+\ell}^{\mu}\right]}}{\square}:=\frac{\square\left(a_{1}|\cdots| a^{\mu-1}\left|0_{1+\ell}\right| a^{\mu+1}|\cdots| a^{n+1}\right)}{\square\left(a^{1}|\cdots| a^{\mu-1}\left|a^{\mu}\right| a^{\mu+1}|\cdots| a^{n+1}\right)} \tag{7.28}
\end{equation*}
$$

where $0_{1+\ell}$ is a specific notation to denote the column consisting of $n+1$ zeros piled up, except at the $(1+\ell)$-th level from its top, where instead of 0 , one reads 1 , and where, as our notation with vertical bars helps to guess:

$$
\begin{gathered}
\square_{\left[0_{1+\ell}\right]}^{\mu}=\square\left(\left.a_{1}|\cdots| a^{\mu-1}\right|^{\mu} 0_{1+\ell}\left|a^{\mu+1}\right| \cdots \mid a^{n+1}\right):= \\
:=\left|\begin{array}{ccccccc}
Q_{a^{1}} & \cdots & Q_{a^{\mu-1}} & 0 & Q_{a^{\mu+1}} & \cdots & Q_{a^{n+1}} \\
Q_{x^{1} a^{1}} & \cdots & Q_{x^{1} a^{\mu-1}} & 0 & Q_{x^{1} a^{\mu+1}} & \cdots & Q_{x^{1} a^{n+1}} \\
\cdot & \cdots & \cdot & . & . & \cdots & \cdot \cdot \\
Q_{x^{k} a^{1}} & \cdots & Q_{x^{k} a^{\mu-1}} & 1 & Q_{x^{k} a^{\mu+1}} & \cdots & Q_{x^{k} a^{n+1}} \\
\cdot & \cdots & . & . & . \cdot & \cdots & \cdot \\
Q_{x^{n} a^{1}} & \cdots & Q_{x^{n} a^{\mu-1}} & 0 & Q_{x^{n} a^{\mu+1}} & \cdots & Q_{x^{n} a^{n+1}}
\end{array}\right| .
\end{gathered}
$$

To avoid any ambiguity, we shall sometimes put the integer $\mu$ in the upper index position of the vertical bar to indicate precisely which column is concerned. As is clear, this notation allows one to view and to remember what
are the involved partial derivatives of the fundamental function $Q$ that appear inside each column. In summary, $\square_{\left[0_{1+\ell}\right]}^{\mu}$ comes from $\square$ by changing just its $\mu$-th column, as Cramer's rule classically says.

Next, the two-ways transfer between local functions $G$ defined in the $\left(x, y, y_{x}\right)$-space and local functions $T$ defined in the $(x, a, b)$-space, namely the one-to-one correspondence:

$$
G\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right) \longleftrightarrow T\left(x^{1}, \ldots, x^{n}, a^{1}, \ldots, a^{n}, b\right)
$$

through the diffeomorphism (7.28), may be viewed concretely, in the direction we are interested in, as the following identity:

$$
\begin{aligned}
& \quad G\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right) \equiv \\
& \equiv T\left(x^{1}, \ldots, x^{n},\right. \\
& \\
& \quad A^{1}\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right), \ldots, A^{n}\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right), \\
& \\
& \left.\quad A^{n+1}\left(x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right)\right)
\end{aligned}
$$

holding of course in $\mathbb{C}\left\{x^{1}, \ldots, x^{n}, y, y_{x^{1}}, \ldots, y_{x^{n}}\right\}$. We therefore readily deduce how the derivation $\frac{\partial}{\partial y_{x^{\ell}}}$ is transferred to the $(x, a, b)$-space:

$$
\frac{\partial G}{\partial y_{x^{\ell}}}=\frac{\partial A^{1}}{\partial y_{x^{\ell}} \ell} \cdot \frac{\partial T}{\partial a^{1}}+\cdots+\frac{\partial A^{n}}{\partial y_{x^{\ell}}} \cdot \frac{\partial T}{\partial a^{n}}+\frac{\partial A^{n+1}}{\partial y_{x^{\ell}}} \cdot \frac{\partial T}{\partial a^{n+1}} .
$$

By applying twice any two such derivations $\partial / \partial y_{x^{\ell_{1}}}$ and $\partial / \partial y_{x^{\ell_{2}}}$ to an arbitrary function $G$, we may see, after a few computations, what such a composed differentiation corresponds to, in terms of the function $T$ defined in the ( $x, a, b$ )-space:

$$
\begin{aligned}
\frac{\partial^{2} G}{\partial y_{x^{\ell_{1}}} \partial y_{x^{\ell_{2}}}} & =\left(\sum_{\mu=1}^{n+1} \frac{\partial A^{\mu}}{\partial y_{x^{\ell_{1}}}} \frac{\partial}{\partial a^{\mu}}\right)\left[\sum_{\nu=1}^{n+1} \frac{\partial A^{\nu}}{\partial y_{x}^{\ell_{2}}} \frac{\partial T}{\partial a^{\nu}}\right] \\
& =\sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \frac{\partial A^{\mu}}{\partial y_{x} \ell_{1}} \frac{\partial A^{\nu}}{\partial y_{x^{\ell_{2}}}} \frac{\partial^{2} T}{\partial a^{\mu} \partial a^{\nu}}+\sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \frac{\partial A^{\mu}}{\partial y_{x} \ell_{1}} \frac{\partial}{\partial a^{\mu}}\left[\frac{\partial A^{\nu}}{\partial y_{x^{\ell}} \ell_{2}}\right] \frac{\partial T}{\partial a^{\nu}} .
\end{aligned}
$$

Here, by a helpful formal convention, the three Greek letters $\mu, \nu$ and $\tau$ will be used as summation indices in the total set $\{1, \ldots, n, n+1\}$, while the four Latin letters $i, j, k, \ell$ will always run in the restricted set $\{1, \ldots, n\}$. Replacing then the partial derivatives of the $A^{\mu}$ by their values (7.28) obtained previously, we thus get:

$$
\begin{aligned}
& \frac{\partial^{2} G}{\partial y_{x^{\ell_{1}}} \partial y_{x^{\ell_{2}}}}=\sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \frac{\square_{\left[0_{\left.1+\ell_{1}\right]}^{\mu}\right.}^{\mu} \frac{\square_{\left[0_{1+\ell_{2}}\right]}^{\nu}}{\square} \frac{\partial^{2} T}{\partial a^{\mu} \partial a^{\nu}}+. . \square^{\circ}}{\square} \\
& +\sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1}\left\{\frac{\square_{\left[0_{1+\ell_{1}}\right]}^{\mu}}{\square} \cdot \frac{\square \cdot \frac{\partial}{\partial a^{\mu}}\left(\square_{\left[0_{\left.1+\ell_{2}\right]}\right]}^{\nu}\right)-\square_{\left[0_{\left.1+\ell_{2}\right]}\right]}^{\nu} \cdot \frac{\partial}{\partial a^{\mu}}(\square)}{\square \cdot \square}\right\} \frac{\partial T}{\partial a^{\nu}}
\end{aligned}
$$

Here, the coefficients of the $\frac{\partial^{2} T}{\partial a^{\mu} \partial a^{\nu}}$ will not be touched anymore, but the coefficients of the $\frac{\partial T}{\partial a^{\nu}}$ must be subjected to further transformations towards formal harmony, especially the numerator involving a subtraction.

First of all, let us rewrite in length the concerned partial derivative of the appearing modified Jacobian determinant ${ }^{28}$ :

$$
\begin{aligned}
\frac{\partial}{\partial a^{\mu}}\left(\square_{\left[0_{1+\ell_{2}}\right]}^{\nu}\right)= & \frac{\partial}{\partial a^{\mu}}\left[\square\left(a^{1}|\cdots| a^{\nu-1}\left|0_{\left[1+\ell_{2}\right]}\right| a^{\nu+1}|\cdots| a^{n+1}\right)\right] \\
= & \square\left(a^{1} a^{\mu}|\cdots| a^{\nu-1}\left|0_{\left[1+\ell_{2}\right]}\right| a^{\nu+1}|\cdots| a^{n+1}\right)+\cdots+ \\
& +\square\left(a^{1}|\cdots| a^{\nu-1} a^{\mu}\left|0_{\left[1+\ell_{2}\right]}\right| a^{\nu+1}|\cdots| a^{n+1}\right) \\
& +0+ \\
& +\square\left(a^{1}|\cdots| a^{\nu-1}\left|0_{\left[1+\ell_{2}\right]}\right| a^{\nu+1} a^{\mu}|\cdots| a^{n+1}\right)+\cdots+ \\
& +\square\left(a^{1}|\cdots| a^{\nu-1}\left|0_{\left[1+\ell_{2}\right]}\right| a^{\nu+1}|\cdots| a^{n+1} a^{\mu}\right),
\end{aligned}
$$

and also at the same time the partial derivative of the plain Jacobiant determinant:

$$
\begin{aligned}
\frac{\partial}{\partial a^{\mu}}(\square)= & \frac{\partial}{\partial a^{\mu}}\left[\square\left(a^{1}|\cdots| a^{\nu-1}\left|a^{\nu}\right| a^{\nu+1}|\cdots| a^{n+1}\right)\right] \\
= & \square\left(a^{1} a^{\mu}|\cdots| a^{\nu-1}\left|a^{\nu}\right| a^{\nu+1}|\cdots| a^{n+1}\right)+\cdots+ \\
& +\square\left(a^{1}|\cdots| a^{\nu-1} a^{\mu}\left|a^{\nu}\right| a^{\nu+1}|\cdots| a^{n+1}\right)+ \\
& +\square\left(a^{1}|\cdots| a^{\nu-1}\left|a^{\nu} a^{\mu}\right| a^{\nu+1}|\cdots| a^{n+1}\right)+ \\
& +\square\left(a^{1}|\cdots| a^{\nu-1}\left|a^{\nu}\right| a^{\nu+1} a^{\mu}|\cdots| a^{n+1}\right)+\cdots+ \\
& +\square\left(a^{1}|\cdots| a^{\nu-1}\left|a^{\nu}\right| a^{\nu+1}|\cdots| a^{n+1} a^{\mu}\right) .
\end{aligned}
$$

Consequently, the numerator with a subtraction that we want to simplify may be rewritten in length as follows:

$$
\begin{equation*}
\left.\square \cdot \frac{\partial}{\partial a^{\mu}}\left(\square_{\left[0_{1+\ell_{2}}\right]}^{\nu}\right)-\square_{\left[0_{1+\ell_{2}}\right]}^{\nu}\right] \frac{\partial}{\partial a^{\mu}}(\square)= \tag{7.28}
\end{equation*}
$$

$$
\begin{aligned}
& =\underline{\square\left(a^{1}|\cdots| a^{\nu-1}\left|a^{\nu}\right| a^{\nu+1}|\cdots| a^{n+1}\right) \cdot \square\left(a^{1} a^{\mu}|\cdots| a^{\nu-1}\left|0_{\left[1+\ell_{2}\right]}\right| a^{\nu+1}|\cdots| a^{n+1}\right)_{(2)}}+\cdots+ \\
& +\underline{\square\left(a^{1}|\cdots| a^{\nu-1}\left|a^{\nu}\right| a^{\nu+1}|\cdots| a^{n+1}\right) \cdot \square\left(a^{1}|\cdots| a^{\nu-1} a^{\mu}\left|0_{\left[1+\ell_{2}\right]}\right| a^{\nu+1}|\cdots| a^{n+1}\right)_{(b)}}+ \\
& +0+ \\
& +\underline{\square\left(a^{1}|\cdots| a^{\nu-1}\left|a^{\nu}\right| a^{\nu+1}|\cdots| a^{n+1}\right) \cdot \square\left(a^{1}|\cdots| a^{\nu-1}\left|0_{\left[1+\ell_{2}\right]}\right| a^{\nu+1} a^{\mu}|\cdots| a^{n+1}\right)_{©}^{c}}+\cdots+ \\
& +\underline{\square\left(a^{1}|\cdots| a^{\nu-1}\left|a^{\nu}\right| a^{\nu+1}|\cdots| a^{n+1}\right) \cdot \square\left(a^{1}|\cdots| a^{\nu-1}\left|0_{\left[1+\ell_{2}\right]}\right| a^{\nu+1}|\cdots| a^{n+1} a^{\mu}\right)}{\underset{C}{C}} \text { - }
\end{aligned}
$$

[^19]\[

$$
\begin{aligned}
& -\underline{\square\left(a^{1}|\cdots| a^{\nu-1}\left|0_{\left[1+\ell_{2}\right]}\right| a^{\nu+1}|\cdots| a^{n+1}\right) \cdot \square\left(a^{1} a^{\mu}|\cdots| a^{\nu-1}\left|a^{\nu}\right| a^{\nu+1}|\cdots| a^{n+1}\right)_{(a)}^{(a)}} \text { - } \\
& -\underline{\square}{\left(a^{1}|\cdots| a^{\nu-1}\left|0_{\left[1+\ell_{2}\right]}\right| a^{\nu+1}|\cdots| a^{n+1}\right) \cdot \square\left(a^{1}|\cdots| a^{\nu-1} a^{\mu}\left|a^{\nu}\right| a^{\nu+1}|\cdots| a^{n+1}\right)}_{\text {(b) }}^{\text {- }} \\
& -\square\left(a^{1}|\cdots| a^{\nu-1}\left|0_{\left[1+\ell_{2}\right]}\right| a^{\nu+1}|\cdots| a^{n+1}\right) \cdot \square\left(a^{1}|\cdots| a^{\nu-1}\left|a^{\nu} a^{\mu}\right| a^{\nu+1}|\cdots| a^{n+1}\right) \text { OK }^{-} \\
& -\square\left(a^{1}|\cdots| a^{\nu-1}\left|0_{\left[1+\ell_{2}\right]}\right| a^{\nu+1}|\cdots| a^{n+1}\right) \cdot \square\left(a^{1}|\cdots| a^{\nu-1}\left|a^{\nu}\right| a^{\nu+1} a^{\mu}|\cdots| a^{n+1}\right)(©-\cdots- \\
& -\underline{\square\left(a^{1}|\cdots| a^{\nu-1}\left|0_{\left[1+\ell_{2}\right]}\right| a^{\nu+1}|\cdots| a^{n+1}\right) \cdot \square\left(a^{1}|\cdots| a^{\nu-1}\left|a^{\nu}\right| a^{\nu+1}|\cdots| a^{n+1} a^{\mu}\right)} \text { (d) }
\end{aligned}
$$
\]

The ante-penultimate underlined term "ок" will be kept untouched. To the pairs of (subtracted) $\square$-binomials that are underlined with a, b, c, d appended (including of course all terms present in the four "..."), we need an elementary instance of the Plücker identities.

To state it generally, let $m \geqslant 2$, let $C_{1}, C_{2}, \ldots, C_{m}, D, E$ be $(m+2)$ column vectors in $\mathbb{K}^{m}$ and introduce the following notation for the $m \times$ $(m+2)$ matrix consisting of these vectors:

$$
\left[C_{1}\left|C_{2}\right| \cdots\left|C_{m}\right| D \mid E\right] .
$$

Extracting columns from this matrix, we shall construct $m \times m$ determinants that are modification of the following "ground" determinant:

$$
\left\|C_{1}|\cdots| C_{m}\right\| \equiv\left\|C_{1}|\cdots|{ }^{j_{1}} C_{j_{1}}|\cdots|{ }^{j_{2}} C_{j_{2}}|\cdots| C_{m}\right\| .
$$

We use a double vertical line in the beginning and in the end to denote a determinant. Also, we emphasize two distinct columns, the $j_{1}$-th and the $j_{2}$-th, where $j_{2}>j_{1}$, since we will modify them. For instance in this matrix, let us replace these two columns by the column $D$ and by the column $E$, which yields the determinant:

$$
\left\|C_{1}|\cdots|{ }^{j_{1}} D|\cdots|{ }^{j_{2}} E|\cdots| C_{m}\right\| .
$$

In this notation, one should understand that only the $j_{1}$-th and the $j_{2}$-th columns are distinct from the columns of the fundamental $m \times m$ "ground" determinant.

Lemma. ([17], p. 155) The following quadratic identity between determinants holds true:

$$
\begin{aligned}
& \left\|C_{1}|\cdots|{ }^{j_{1}} D|\cdots|{ }^{j_{2}} E|\cdots| C_{n}\right\| \cdot\left\|C_{1}|\cdots|{ }^{j_{1}} C_{j_{1}}|\cdots|{ }^{j_{2}} C_{j_{2}}|\cdots| C_{n}\right\|= \\
& =\left\|\left.C_{1}|\cdots|\right|^{j_{1}} D|\cdots|{ }^{j_{2}} C_{j_{2}}|\cdots| C_{n}\right\| \cdot\left\|C_{1}\left|\cdots \cdot{ }^{j_{1}} C_{j_{1}}\right| \cdots\left|{ }^{j_{2}} E\right| \cdots \mid C_{n}\right\|- \\
& -\left\|C_{1}|\cdots|{ }^{j_{1}} E|\cdots|{ }^{j_{2}} C^{j_{2}}|\cdots| C_{n}\right\| \cdot\left\|C_{1}|\cdots|{ }^{j_{1}} C_{j_{1}}|\cdots|{ }^{j_{2}} D|\cdots| C_{n}\right\| .
\end{aligned}
$$

Admitting this elementary statement without redoing its proof and applying it to all the above underlined pairs of (subtracted) monomials, after checking that all final signs are "-", we obtain the following neat expression
for (7.28):

$$
\begin{aligned}
& \square \cdot \frac{\partial}{\partial a^{\mu}}\left(\square_{\left[0_{1+\ell_{2}}\right]}^{\nu}\right)-\square_{\left[0_{\left.1+\ell_{2}\right]}^{\nu} \cdot \frac{\partial}{\partial a^{\mu}}(\square)=\right.}^{=} \quad-\square\left(0_{\left[1+\ell_{2}\right]}|\cdots|^{\nu} a^{\nu}|\cdots| a^{n+1}\right) \cdot \square\left(a^{1}|\cdots|^{\nu} a^{1} a^{\mu}|\cdots| a^{n+1}\right)-\cdots- \\
& \quad-\square\left(a^{1}|\cdots|^{\nu} 0_{\left[1+\ell_{2}\right]}|\cdots| a^{n+1}\right) \cdot \square\left(a^{1}|\cdots|^{\nu} a^{\nu} a^{\mu}|\cdots| a^{n+1}\right)-\cdots- \\
& \quad-\square\left(a^{1}|\cdots|^{\nu} a^{\nu}|\cdots| 0_{\left[1+\ell_{2}\right]}\right) \cdot \square\left(a^{1}|\cdots|^{\nu} a^{n+1} a^{\mu}|\cdots| a^{n+1}\right),
\end{aligned}
$$

or equivalently, in contracted form:

$$
\square \cdot \frac{\partial}{\partial a^{\mu}}\left(\square_{\left[0_{1+\ell_{2}}\right]}^{\nu}\right)-\square_{\left[0_{1+\ell_{2}}\right]}^{\nu} \cdot \frac{\partial}{\partial a^{\mu}}(\square)=-\sum_{\tau=1}^{n+1} \square_{\left[0_{1+\ell_{2}}\right]}^{\tau} \cdot \square_{\left[a^{\tau} a^{\mu}\right]}^{\nu}
$$

Thanks to this sidework, coming back to the expression for $\frac{\partial^{2} G}{\partial y_{x} \ell_{1} \partial y_{x^{\ell}} \ell_{2}}$ we left pending above, we obtain:

$$
\begin{aligned}
\frac{\partial^{2} G}{\partial y_{x^{\ell_{1}}} \partial y_{x^{\ell_{2}}}}= & \frac{1}{\square^{2}} \sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1}\left\{\square_{\left[0_{\left.1+\ell_{1}\right]}^{\mu}\right]}^{\mu} \cdot \square_{\left[0_{1+\ell_{2}}\right]}^{\nu}\right\} \frac{\partial^{2} T}{\partial a^{\mu} \partial a^{\nu}}- \\
& -\frac{1}{\square^{3}} \sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \sum_{\tau=1}^{n+1}\left\{\square_{\left[0_{\left.1+\ell_{1}\right]}^{\mu}\right]}^{\mu} \cdot \square_{\left[0_{\left.1+\ell_{2}\right]}^{\tau}\right]}^{\tau} \square_{\left[a^{\mu} a^{\tau}\right]}^{\nu}\right\} \frac{\partial T}{\partial a^{\nu}} .
\end{aligned}
$$

To really finalize this expression, we factor everything by $\frac{1}{\square^{3}}$ and we exchange the two summation indices $\nu$ and $\tau$ in the second line:

$$
\frac{\partial^{2} G}{\partial y_{x^{\ell_{1}}} \partial y_{x^{\ell_{2}}}}=\frac{1}{\square^{3}} \sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \square_{\left[0_{1+\ell_{1}}\right]}^{\mu} \cdot \square_{\left[0_{1+\ell_{2}}\right]}^{\nu}\left\{\square \cdot \frac{\partial^{2} T}{\partial a^{\mu} \partial a^{\nu}}-\sum_{\tau=1}^{n+1} \square_{\left[a^{\mu} a^{\nu}\right]}^{\tau} \cdot \frac{\partial T}{\partial a^{\tau}}\right\} .
$$

End of proof of the Main Theorem. As already explained in the Introduction, one applies to the system (7.28) Hachtroudi's characterization (7.28) of equivalence to the system $w_{z_{k_{1}} z_{k_{2}}^{\prime}}^{\prime}\left(z^{\prime}\right)=0$ with $x:=z$, with $y:=w$, with $a:=\bar{z}$, with $b:=\bar{w}$, with $(a, b):=\bar{t}$, with $Q:=\Theta$, with $\square:=\Delta$, with $G:=\Phi_{k_{1}, k_{2}}$ and with $T:=\frac{\partial^{2} \Theta}{\partial z_{k_{1}} \partial z_{k_{2}}}$. The denominator $\frac{1}{\Delta^{3}}$ can be cleared out, and we simply get the explicit fourth-order partial differential equation satisfied by $\Theta$. This completes the proof of our Main Theorem and the paper may end up now.

## References

[1] Bièche, C.: Le problème d'équivalence locale pour un système scalaire complet d'équations aux dérivées partielles d'ordre deux à $n$ variables indépendantes, Annales de la Faculté des Sciences de Toulouse, XVI (2007), no. 1, 1-36.
[2] Boggess, A.: CR manifolds and the tangential Cauchy-Riemann complex. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1991, xviii+364 pp.
[3] Cartan, É.: Sur les variétés à connexion projective, Bull. Soc. Math. France 52 (1924), 205241.
[4] Chern, S.-S.: On the projective structure of a real hypersurface in $\mathbb{C}^{n+1}$, Math. Scand. 36 (1975), 74-82.
[5] Chern, S.S.; Moser, J.K.: Real hypersurfaces in complex manifolds, Acta Math. 133 (1974), no. 2, 219-271.
[6] Chirka, E.M.: An introduction to the geometry of CR manifolds (Russian), Uspekhi Mat. Nauk 46 (1991), no. 1(277), 81-164, 240; translation in Russian Math. Surveys 46 (1991), no. 1, 95197
[7] Diederich, K.; Webster, S.M.: A reflection principle for degenerate real hypersurfaces, Duke Math. J. 47 (1980), no. 4, 835-843.
[8] Engel, F.; Lie, S.: Theorie der transformationsgruppen. Erster Abschnitt. Unter Mitwirkung von Dr. Friedrich Engel, bearbeitet von Sophus Lie, B.G. Teubner, Leipzig, 1888. Reprinted by Chelsea Publishing Co. (New York, N.Y., 1970).
[9] Faran, J.: Segre families and real hypersurfaces, Invent. Math. 60 (1980), no. 2, 135-172.
[10] Hachtroudi, M.: Les espaces d'éléments à connexion projective normale, Actualités Scientifiques et Industrielles, vol. 565, Paris, Hermann, 1937.
[11] Isaev, A.V.: Zero CR-curvature equations for rigid and tube hypersurfaces, Complex Variables and Elliptic Equations, 54 (2009), no. 3-4, 317-344.
[12] Merker, J.: Convergence of formal biholomorphisms between minimal holomorphically nondegenerate real analytic hypersurfaces, Int. J. Math. Math. Sci. 26 (2001), no. 5, 281-302.
[13] Merker, J.: On the partial algebraicity of holomorphic mappings between real algebraic sets, Bull. Soc. Math. France 129 (2001), no. 3, 547-591.
[14] Merker, J.: On envelopes of holomorphy of domains covered by Levi-flat hats and the reflection principle, Ann. Inst. Fourier (Grenoble) 52 (2002), no. 5, 1443-1523.
[15] Merker, J.: On the local geometry of generic submanifolds of $\mathbb{C}^{n}$ and the analytic reflection principle, Journal of Mathematical Sciences (N. Y.) 125 (2005), no. 6, 751-824.
[16] Merker, J.: Étude de la régularité analytique de l'application de réflexion CR formelle, Annales Fac. Sci. Toulouse, XIV (2005), no. 2, 215-330.
[17] Merker, J.: Explicit differential characterization of the Newtonian free particle system in $m \geqslant$ 2 dependent variables, Acta Mathematicæ Applicandæ, 92 (2006), no. 2, 125-207.
[18] Merker, J.; Porten, E.: Holomorphic extension of CR functions, envelopes of holomorphy and removable singularities, International Mathematics Research Surveys, Volume 2006, Article ID 28295, 287 pages.
[19] Merker, J.: Lie symmetries of partial differential equations and CR geometry, Journal of Mathematical Sciences (N.Y.), to appear (2009), arxiv.org/abs/math/0703130
[20] Merker, J.: Sophus Lie, Friedrich Engel et le problème de Riemann-Helmholtz, Hermann Éditeurs, Paris, 2010, à paraître, 307 pp., arxiv.org/abs/0910.0801
[21] Merker, J.: Nonrigid spherical real analytic hypersurfaces in $\mathbb{C}^{2}$, arxiv.org/abs/0910.1694
[22] Merker, J.: Explicit Chern-Moser tensors, arxiv.org, to appear.
[23] Segre, B.: Intorno al problema di Poincaré della rappresentazione pseudoconforme, Rend. Acc. Lincei, VI, Ser. 13 (1931), 676-683.
[24] Sukhov, A.: Segre varieties and Lie symmetries, Math. Z. 238 (2001), no. 3, 483-492.
[25] Sukhov, A.: CR maps and point Lie transformations, Michigan Math. J. 50 (2002), no. 2, 369-379.

## Lie symmetries

## of partial differential equations

Joël MerkerTable of contents

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## Journal of Mathematical Sciences (N. Y.), to appear

This memoir is divided in three parts ${ }^{29}$. Part I endeavours a general, new theory (inspired by modern CR geometry) of Lie symmetries of completely integrable PDE systems, viewed from their associated submanifold of solutions. Part II builds general combinatorial formulas for the prolongations of vector fields to jet spaces. Part III characterizes explicitly flatness of some systems of second order. The results presented here are original and did not appear in print elsewhere; most formulas of Parts II and III were checked by means of Maple Release 7.

## §1. COMPLETELY INTEGRABLE SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

1.1. General systems. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Let $n \in \mathbb{N}$ with $n \geqslant 1$ and let $x=\left(x^{1}, \ldots, x^{n}\right) \in \mathbb{K}^{n}$. Also, let $m \in \mathbb{N}$ with $m \geqslant 1$ and let $y=$ $\left(y^{1}, \ldots, y^{m}\right) \in \mathbb{K}^{m}$. For $\alpha \in \mathbb{N}^{n}$, we denote by a subscript $y_{x^{\alpha}}$ the partial derivative $\partial^{|\alpha|} y / \partial x^{\alpha}$ of a local map $\mathbb{K}^{n} \ni x \mapsto y(x) \in \mathbb{K}^{m}$.

Let $\kappa \in \mathbb{N}$ with $\kappa \geqslant 1$, let $p \in \mathbb{N}$ with $p \geqslant 1$, choose a collection of $p$ multiindices $\beta(1), \ldots, \beta(p) \in \mathbb{N}^{n}$ with $|\beta(q)| \geqslant 1$ for $q=1, \ldots, p$ and $\max _{1 \leqslant q \leqslant p}|\beta(q)|=\kappa$, and choose integers $j(1), \ldots, j(p)$ with $1 \leqslant j(q) \leqslant$

[^20]$m$ for $q=1, \ldots, p$. In the present Part I, we study the Lie symmetries of a general system of analytic partial differential equations of the form:
\[

$$
\begin{equation*}
y_{x^{\alpha}}^{j}(x)=F_{\alpha}^{j}\left(x, y(x),\left(y_{x^{\beta(q)}}^{j(q)}(x)\right)_{1 \leqslant q \leqslant p}\right), \tag{E}
\end{equation*}
$$

\]

where $j$ with $1 \leqslant j \leqslant m$ and $\alpha \in \mathbb{N}^{n}$ satisfy

$$
\begin{equation*}
(j, \alpha) \neq(j, 0) \quad \text { and } \quad(j, \alpha) \neq(j(q), \beta(q)) . \tag{1.2}
\end{equation*}
$$

In particular, all $(\kappa+1)$-th partial derivatives of the unknown $y=y(x)$ depend on a certain precise set of derivatives of order $\leqslant \kappa$ : the system is complete. In addition, all the other partial derivatives of order $\leqslant \kappa$ do also depend on the same precise set of derivatives.

Here, we assume that $u=0$ is a local solution of the system $(\mathscr{E})$ and that the functions $F_{\alpha}^{j}$ are $\mathbb{K}$-algebraic (in the sense of Nash) or $\mathbb{K}$-analytic, in a neighborhood of the origin in $\mathbb{K}^{n+m+p}$. Even if our concern will be local throughout, we will not introduce any special notation to speak of open subsets and simply refer to various $\mathbb{K}^{\mu}$. We will study five concrete instances, the first three ones being classical.

Example 1.3. With $n=m=\kappa=1$, a second order ordinary differential equation
$\left(\mathscr{E}_{1}\right)$

$$
y_{x x}=F\left(x, y, y_{x}\right),
$$

and more generally $y_{x^{\kappa+1}}=F\left(x, y, y_{x}, \ldots, y_{x^{\kappa}}\right)$, where $x, y \in \mathbb{K}$, see [Lie1883, EL1890, Tr1896, Ca1924, Se1931, Ca1932a, Ol1986, Ar1988, BK1989, GTW1989, HK1989, Ib1992, Ol1995, N2003].

Example 1.4. With $n \geqslant 2, m=1$ and $\kappa=1$, a complete system of second order equations
$\left(\mathscr{E}_{2}\right) \quad y_{x^{i_{1}} x^{i_{2}}}=F_{i_{1}, i_{2}}\left(x^{i}, y, y_{x^{k}}\right), \quad 1 \leqslant i_{1}, i_{2} \leqslant n$,
see [Ha1937, Ch1975, Su2001] and Part III below.
Example 1.5. Dually, with $n=1, m \geqslant 2$ and $\kappa=1$, an ordinary system of second order

$$
\begin{equation*}
y_{x x}^{j}=F^{j}\left(x, y^{j_{1}}, y_{x}^{j_{1}}\right), \quad j=1, \ldots, m, \tag{3}
\end{equation*}
$$

see [Fe1995, Me2004] and the references therein.
Example 1.6. With $n=1, m=2$ and $\kappa=1$, a system of the form

$$
\left\{\begin{align*}
y_{x}^{2} & =F\left(x, y^{1}, y^{2}, y_{x}^{1}\right)  \tag{4}\\
y_{x x}^{1} & =G\left(x, y^{1}, y^{2}, y_{x}^{1}\right) .
\end{align*}\right.
$$

Differentiating the first equation with respect to $x$ and substituting, we get the missing equation:

$$
\begin{align*}
y_{x x}^{2} & =F_{x}+y_{x}^{1} F_{y^{1}}+y_{x}^{2} F_{y^{2}}+y_{x x}^{1} F_{y_{x}^{1}} \\
& =F_{x}+y_{x}^{1} F_{y^{1}}+y_{x}^{2} F_{y^{2}}+G F_{y_{x}^{1}}  \tag{1.7}\\
& =: H\left(x, y^{1}, y^{2}, y_{x}^{1}\right) .
\end{align*}
$$

Example 1.8. With $n=2, m=1$ and $\kappa=2$, a system of the form

$$
\left\{\begin{align*}
y_{x^{2}} & =F\left(x^{1}, x^{2}, y, y_{x^{1}}, y_{x^{1} x^{1}}\right)  \tag{5}\\
y_{x^{1} x^{1} x^{1}} & =G\left(x^{1}, x^{2}, y, y_{x^{1}}, y_{x^{1} x^{1}}\right) .
\end{align*}\right.
$$

Here, five equations are missing. Differentiating the first equation with respect to $x^{1}$ and substituting:

$$
\begin{align*}
y_{x^{1} x^{2}} & =F_{x^{1}}+y_{x^{1}} F_{y}+y_{x^{1} x^{1}} F_{y_{x^{1}}}+y_{x^{1} x^{1} x^{1}} F_{y_{x^{1} x^{1}}} \\
& =F_{x^{1}}+y_{x^{1}} F_{y}+y_{x^{1} x^{1}} F_{y_{x^{1}}}+G F_{y_{x^{1} x^{1}}}  \tag{1.9}\\
& =: H\left(x^{1}, x^{2}, y, y_{x^{1}}, y_{x^{1} x^{1}}\right),
\end{align*}
$$

and then similarly for $y_{x^{2} x^{2}}, y_{x^{1} x^{1} x^{2}}, y_{x^{1} x^{2} x^{2}}, y_{x^{2} x^{2} x^{2}}$.
1.10. Finitely nondegenerate generic submanifolds of $\mathbb{C}^{n+m}$. Examples $1.3,1.4,1.6$ and 1.8 (but not 1.5) are intrinsically linked to real submanifolds of complex submanifolds.

Let $M$ be a real algebraic or analytic local generic $\mathrm{CR}^{30}$ submanifold of $\mathbb{C}^{n+m}$ of codimension $m \geqslant 1$ and of CR dimension $n \geqslant 1$, and let $p \in M$. Classically, there exists local holomorphic coordinates $t=(z, w) \in$ $\mathbb{C}^{n} \times \mathbb{C}^{m}$ centered at $p$ in which $M$ is represented by

$$
\begin{equation*}
w^{j}=\bar{\Theta}^{j}(z, \bar{z}, \bar{w}), \quad j=1, \ldots, m, \tag{1.11}
\end{equation*}
$$

for some local $\mathbb{C}$-analytic map $\Theta=\left(\Theta^{1}, \ldots, \Theta^{m}\right)$ satisfying the identity

$$
\begin{equation*}
w \equiv \bar{\Theta}(z, \bar{z}, \Theta(\bar{z}, z, w)) \tag{1.12}
\end{equation*}
$$

reflecting the fact that $M$ is real.
Definition 1.13. ([BER1999, Me2005a, Me2005b, MP2005]) $M$ is finitely nondegenerate if there exists an integer $\kappa \geqslant 1$ such that the local holomorphic map

$$
\begin{equation*}
(\bar{z}, \bar{w}) \longmapsto\left(\bar{\Theta}_{z^{\beta}}^{j}(0, \bar{z}, \bar{w})\right)_{|\beta| \leqslant \kappa}^{1 \leqslant j \leqslant m} \tag{1.14}
\end{equation*}
$$

is of rank $n+m$ at $(\bar{z}, \bar{w})=(0,0)$.

[^21]From (1.12), the map $\bar{w} \mapsto \bar{\Theta}(0,0, \bar{w})$ is already of rank $m$ at $\bar{w}=0$. One then verifies ([BER1999, Me2005a, Me2005b, MP2005]) that there exist multiindices $\beta(1), \ldots, \beta(n) \in \mathbb{N}^{n}$ with $|\beta(k)| \geqslant 1$ for $k=1, \ldots, n$ and $\max _{1 \leqslant k \leqslant n}|\beta(k)|=\kappa$ together with integers $j(1), \ldots, j(n)$ with $1 \leqslant$ $j(k) \leqslant m$ such that the local holomorphic map
$\mathbb{C}^{n+m} \ni(\bar{z}, \bar{w}) \longmapsto\left(\left(\bar{\Theta}^{j}(0, \bar{z}, \bar{w})\right)^{1 \leqslant j \leqslant m},\left(\bar{\Theta}_{z^{\beta(k)}}^{j(k)}(0, \bar{z}, \bar{w})\right)_{1 \leqslant k \leqslant n}\right) \in \mathbb{C}^{m+n}$
is of rank $n+m$ at $(\bar{z}, \bar{w})=(0,0)$.
1.16. Associated system of partial differential equations. Generalizing an idea which goes back to B. Segre in [Se1931, Se1932] ( $n=m=1$ ), applied by É. Cartan in [Ca1932a] and studied more recently in [Su2001, GM2003a], we may associate to $M$ a system of partial differential equations of the form ( $\mathscr{E}$ ) as follows. Complexifying the variables $\bar{z}$ and $\bar{w}$, we introduce new independent variables $\zeta \in \mathbb{C}^{n}$ and $\xi \in \mathbb{C}^{m}$ together with the complex algebraic or analytic $m$-codimensional submanifold $\mathscr{M}$ of $\mathbb{C}^{2(n+m)}$ defined by

$$
\begin{equation*}
w^{j}=\bar{\Theta}^{j}(z, \zeta, \xi), \quad j=1, \ldots, m \tag{1.17}
\end{equation*}
$$

We then consider the "dependent variables" $w^{j}$ as algebraic or analytic functions of the "independent variables" $z^{k}$, with additional dependence on the extra "parameters" $(\zeta, \xi)$. Then by applying the differentiation $\partial^{|\alpha|} / \partial z^{\alpha}$ to (1.17), we get $w_{z^{\alpha}}^{j}(z)=\bar{\Theta}_{z^{\alpha}}^{j}(z, \zeta, \xi)$. Assuming finite nondegeneracy and writing these equations for $(j, \alpha)=(j(k), \beta(k))$, we obtain a system of $m+n$ equations:

$$
\left\{\begin{align*}
w^{j}(z) & =\bar{\Theta}^{j}(z, \zeta, \xi), \quad j=1, \ldots, m,  \tag{1.18}\\
w_{z^{\beta(k)}}^{j(k)}(z) & =\bar{\Theta}_{z^{\beta(k)}}^{j(k)}(z, \zeta, \xi), \quad k=1, \ldots, n .
\end{align*}\right.
$$

By means of the implicit function theorem we can solve:

$$
\begin{equation*}
(\zeta, \xi)=R\left(z^{k}, w^{j}(z), w_{z^{\beta}(k)}^{j(k)}(z)\right) . \tag{1.19}
\end{equation*}
$$

Finally, for every pair $(j, \alpha)$ different from $(j, 0)$ and from $(j(k), \beta(k))$, we may replace $(\zeta, \xi)$ by $R$ in the differentiated expression $w_{z^{\alpha}}^{j}(z)=$ $\bar{\Theta}_{z^{\alpha}}^{j}(z, \zeta, \xi)$, which yields

$$
\begin{align*}
w_{z^{\alpha}}^{j}(z) & =\bar{\Theta}_{z^{\alpha}}^{j}\left(z, R\left(z^{k}, w^{j}(z), w_{z^{\beta(k)}}^{j(k)}(z)\right)\right)  \tag{1.20}\\
& =: F_{\alpha}^{j}\left(z^{k}, w^{j}(z), w_{z^{\beta(k)}}^{j(k)}(z)\right) .
\end{align*}
$$

This is the system of partial differential equations associated to $M$.

Example 1.21. (Continued) With $n=m=1$, i.e. $M \subset \mathbb{C}^{2}$ and $\kappa=1$, i.e. $M$ is Levi nondegenerate of equation

$$
\begin{equation*}
w=\bar{w}+i z \bar{z}+\mathrm{O}_{3}, \tag{1.22}
\end{equation*}
$$

where $z, \bar{z}$ are assigned weight 1 and $w, \bar{w}$ weight 2, B. Segre [Se 1931] obtained $w_{z z}=F\left(z, w, w_{z}\right)$. J. Faran [Fa1980] found some examples of such equations that cannot come from a $M \subset \mathbb{C}^{2}$. But the following was left unsolved.

Open problem 1.23. Characterize equations $y_{x x}=F\left(x, y, y_{x}\right)$ associated to a real analytic, Levi nondegenerate (i.e. $\kappa=1$ ) hypersurface $M \subset \mathbb{C}^{2}$. Can on read the reality condition (1.12) on $F$ ? In case of success, generalize to arbitrary $M \subset \mathbb{C}^{n+m}$.

Example 1.24. (Continued) Similarly, the system ( $\mathscr{E}_{2}$ ) comes from a Levi nondegenerate hypersurface $M \subset \mathbb{C}^{n+1}$ ([Ha1937, CM1974, Ch1975, Su2001]. Exercise: why $\left(\mathscr{E}_{3}\right)$ cannot come from any $M \subset \mathbb{C}^{\nu}$ ?

Example 1.25. (Continued) With $n=1, m=2$ and $\kappa=1$, the system ( $\mathscr{E}_{4}$ ) comes from a $M \subset \mathbb{C}^{3}$ which is Levi nondegenerate and satisfies

$$
\begin{equation*}
T^{c} M+\left[T^{c} M, T^{c} M\right]+\left[T^{c} M,\left[T^{c} M, T^{c} M\right]\right]=T M \tag{1.26}
\end{equation*}
$$

at the origin, namely which has equations of the following form, after some elementary transformations ([Be1997, BES2005]):

$$
\begin{align*}
& w^{1}=\bar{w}^{1}+i z \bar{z}+\mathrm{O}_{4}, \\
& w^{2}=\bar{w}^{2}+i z \bar{z}(z+\bar{z})+\mathrm{O}_{4}, \tag{1.27}
\end{align*}
$$

where $z, \bar{z}$ are assigned weight 1 and $w^{1}, w^{2}, \bar{w}^{1}, \bar{w}^{2}$ weight 2 .
Example 1.28. (Continued) With $n=2, m=1$ and $\kappa=2$, the system ( $\mathscr{E}_{5}$ ) comes from a hypersurface $M \subset \mathbb{C}^{3}$ of equation ([Eb1998, GM2003b, FK2005a, FK2005b, Eb2006, GM2006]):

$$
\begin{equation*}
w=\bar{w}+i \frac{2 z^{1} \bar{z}^{1}+z^{1} z^{1} \bar{z}^{2}+\bar{z}^{1} \bar{z}^{1} z^{2}}{1-z^{2} \bar{z}^{2}}+\mathrm{O}_{4}, \tag{1.29}
\end{equation*}
$$

where $z_{1}, \bar{z}_{1}, z_{2}, \bar{z}_{2}$ are assigned weight 1 and $w, \bar{w}$ weight 2 , with the assumption that the Levi form has rank exactly one at every point, and with the assumption that $M$ is 2-nondegenerate at 0 .
1.30. Jet spaces, contact forms and Frobenius integrability. Throughout the present Part I, we assume that the system $(\mathscr{E})$ is completely integrable, namely that the Pfaffian system naturally associated to $(\mathscr{E})$ in the appropriate jet space is involutive in the sense of Frobenius. This holds automatically in case ( $\mathscr{E}$ ) comes from a generic submanifold $M \subset \mathbb{C}^{n+m}$. In general, we will construct a submanifold of solutions associated to $(\mathscr{E})$. So, we must explain complete integrability.

We denote by $\mathscr{J}_{n, m}^{\kappa}$ the space of $\kappa$-th jets of maps $\mathbb{K}^{n} \ni x \mapsto y(x) \in \mathbb{K}^{m}$. Let

$$
\begin{equation*}
\left(x^{i}, y^{j}, y_{i_{1}}^{j}, y_{i_{1}, i_{2}}^{j}, \ldots \ldots, y_{i_{1}, i_{2}, \ldots, i_{\kappa}}^{j}\right) \in \mathbb{K}^{n+m+m n+m n^{2}+\cdots+m n^{\kappa}} \tag{1.31}
\end{equation*}
$$

denote the natural coordinates on $\mathscr{J}_{n, m}^{\kappa} \simeq \mathbb{K}^{n+m\left(1+n+\cdots+n^{\kappa}\right)}$. For instance, $\left(x, y, y_{1}\right) \in \mathscr{J}_{1,1}^{1}$. We shall sometimes write them shortly:

$$
\begin{equation*}
\left(x^{i}, y^{j}, y_{\beta}^{j}\right) \in \mathbb{K}^{n+m+m\left(n+\cdots+n^{\kappa}\right)}, \tag{1.32}
\end{equation*}
$$

where $\beta \in \mathbb{N}^{n}$ varies and satisfies $|\beta| \leqslant \kappa$. Sometimes also, we consider these jet coordinates only up to their symmetries $y_{i_{1}, i_{2}, \ldots, i_{\lambda}}^{j}=$ $y_{i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(\lambda)}}^{j}$, where $\sigma$ is a permutation of $\{1,2, \ldots, \lambda\}$, so that $\mathscr{J}_{n, m}^{\kappa} \simeq$ $\mathbb{K}^{n+m C_{n+\kappa}^{\kappa}}$, with $C_{n+\kappa}^{\kappa}:=\frac{(n+\kappa)!}{\kappa!n!}$.

Having these notations at hand, we may develope the canonical system of contact forms on $\mathscr{J}_{n, m}^{\kappa}$ ([Ol1995], [Stk2000]):

$$
\left\{\begin{array}{c}
\theta^{j}:=d y^{j}-\sum_{k=1}^{n} y_{k}^{j} d x^{k}  \tag{1.33}\\
\theta_{i_{1}}^{j}:=d y_{i_{1}}^{j}-\sum_{k=1}^{n} y_{i_{1}, k}^{j} d x^{k} \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
\theta_{i_{1}, \ldots, i_{k-1}}^{j}:=d y_{i_{1}, \ldots, i_{k-1}}^{j}-\sum_{k=1}^{n} y_{i_{1}, \ldots, i_{\kappa-1}, k}^{j} d x^{k} .
\end{array}\right.
$$

For instance, with $n=m=1$ and $\kappa=2$, we have $\theta^{1}=d y-y_{1} d x$ and $\theta_{1}^{1}=$ $d y_{1}-y_{2} d x$. These (linearly independent) one-forms generate a subspace $\mathscr{C} \mathscr{T}_{n, m}^{\kappa}$ of the cotangent $T^{*} \mathscr{J}_{n, m}^{\kappa}$ whose dimension equals $m C_{n+\kappa-1}^{\kappa-1}$. For the duality between forms and vectors, the orthogonal $\left(\mathscr{C} \mathscr{T}_{n, m}^{\kappa}\right)^{\perp}$ in $T \mathscr{J}_{n, m}^{\kappa}$ is spanned by the $n+m C_{n+\kappa-1}^{\kappa}$ vector fields:

$$
\left\{\begin{align*}
D_{i} & :=\frac{\partial}{\partial x^{i}}+\sum_{j_{1}=1}^{m} y_{i}^{j_{1}} \frac{\partial}{\partial y^{j_{1}}}+\cdots+\sum_{j_{1}=1}^{m} \sum_{k_{1}, \ldots, k_{\kappa-1}=1}^{n} y_{i, k_{1}, \ldots, k_{\kappa-1}}^{j_{1}} \frac{\partial}{\partial y_{k_{1}, \ldots, k_{\kappa-1}}^{j_{1}}},  \tag{1.34}\\
T_{i_{1}, \ldots, i_{\kappa}}^{j_{1}} & :=\frac{\partial}{\partial y_{i_{1}, \ldots, i_{\kappa}}^{j_{1}}},
\end{align*}\right.
$$

the first $n$ ones being the total differentiation operators, considered in Part II. For $n=m=1, \kappa=2$, we get $\frac{\partial}{\partial x}+y_{1} \frac{\partial}{\partial y}+y_{2} \frac{\partial}{\partial y_{1}}$ and $\frac{\partial}{\partial y_{2}}$.

Classically ([Ol1986, BK1989, Ol1995]), one associates to ( $\mathscr{E}$ ) its skeleton $\Delta_{\mathscr{E}}$, namely the $(n+m+p)$-dimensional submanifold of $\mathscr{J}_{n, m}^{\kappa+1}$ simply
defined by the graphed equations:

$$
\begin{equation*}
y_{\alpha}^{j}=F_{\alpha}^{j}\left(x, y,\left(y_{\beta(q)}^{j(q)}\right)_{1 \leqslant q \leqslant p}\right), \tag{1.35}
\end{equation*}
$$

for $(j, \alpha) \neq(j, 0)$ and $\neq(j(q), \beta(q))$ with $|\alpha| \leqslant \kappa+1$. Clearly, the natural coordinates on $\Delta_{\mathscr{E}}$ are:

$$
\begin{equation*}
\left(x, y,\left(y_{\beta(q)}^{j(q)}\right)_{1 \leqslant q \leqslant p}\right) \equiv\left(x, y,\left(y_{l_{1}(q), \ldots, l_{\lambda_{q}(q)}}^{j(q)}\right)_{1 \leqslant q \leqslant p}\right) \in \mathbb{K}^{n} \times \mathbb{K}^{m} \times \mathbb{K}^{p} \tag{1.36}
\end{equation*}
$$

where $\lambda_{q}:=|\beta(q)|$ and $\left(l_{1}(q), \ldots, l_{\lambda_{q}}(q)\right):=\beta(q)$.
Next, in view of the form (1.34) of the generators of $\left(\mathscr{C} \mathscr{T}_{n, m}^{\kappa+1}\right)^{\perp}$ and in view of the equations of $\Delta_{\mathscr{E}}$, the intersection

$$
\begin{equation*}
\left(\mathscr{C} \mathscr{T}_{n, m}^{\kappa+1}\right)^{\perp} \cap T \Delta_{\mathscr{E}} \tag{1.37}
\end{equation*}
$$

is a vector subbundle of $T \Delta_{\mathscr{E}}$ that is generated by $n$ linearly independent vector fields obtained by restricting the $D_{i}$ to $\Delta_{\mathscr{E}}$, which yields:

$$
\left\{\begin{align*}
\mathrm{D}_{i}=\frac{\partial}{\partial x^{i}}+ & \sum_{j=1}^{m} \mathbf{A}_{i}^{j}\left(x^{i_{1}}, y^{j_{1}}, y_{\beta\left(q_{1}\right)}^{j\left(q_{1}\right)}\right) \frac{\partial}{\partial y^{j}}+  \tag{1.38}\\
& +\sum_{q=1}^{p} \mathbf{B}_{i}^{q}\left(x^{i_{1}}, y^{j_{1}}, y_{\beta\left(q_{1}\right)}^{j\left(q_{1}\right)}\right) \frac{\partial}{\partial y_{\beta(q)}^{j(q)}},
\end{align*}\right.
$$

$i=1, \ldots, n$, where the coefficients $\mathbf{A}_{i}^{j}$ and $\mathbf{B}_{i}^{q}$ are given by:
$\mathbf{A}_{i}^{j}:=\left\{\begin{array}{l}y_{i}^{j} \text { if the variable } y_{i}^{j} \text { appears among the } p \text { variables } y_{\beta\left(q_{1}\right)}^{j\left(q_{1}\right)} ; \\ F_{i}^{j} \text { otherwise } ;\end{array}\right.$
$\mathbf{B}_{i}^{q}:=\left\{\begin{array}{l}y_{i, l_{1}(q), \ldots, l_{\lambda_{q}}(q)}^{j(q)} \text { if } y_{l_{1}(q), \ldots, l_{\lambda_{q}(q)}^{j(q)}}^{j(q)} \\ F_{i, l_{1}(q), \ldots, l_{\lambda_{q}(q)}(q)}^{j(q)} \text { otherwise. }\end{array}\right.$ appears among the $p$ variables $y_{\beta\left(q_{1}\right)}^{j\left(q_{1}\right)} ;$
Example 1.40. For ( $\mathscr{E}_{1}$ ), we get $\mathrm{D}=\frac{\partial}{\partial x}+y_{1} \frac{\partial}{\partial x}+F\left(x, y, y_{1}\right) \frac{\partial}{\partial y_{2}}$; exercise: treat $\left(\mathscr{E}_{2}\right)$ and $\left(\mathscr{E}_{3}\right)$. For $\left(\mathscr{E}_{4}\right)$, we get $\mathrm{D}=\frac{\partial}{\partial x}+y_{1}^{1} \frac{\partial}{\partial y^{1}}+F \frac{\partial}{\partial y^{2}}+G \frac{\partial}{\partial y_{1}^{1}}$. For $\left(\mathscr{E}_{5}\right)$, whose skeleton is written $y_{2}=F, y_{1,1,1}=G, y_{1,2}=H, y_{1,1,2}=K$, with $F, G, H, K$ being functions of $\left(x^{1}, x^{2}, y, y_{1}, y_{1,1}\right)$, we get

$$
\begin{align*}
& \mathrm{D}_{1}=\frac{\partial}{\partial x^{1}}+y_{1} \frac{\partial}{\partial y}+y_{1,1} \frac{\partial}{\partial y_{1}}+G \frac{\partial}{\partial y_{1,1}}, \\
& \mathrm{D}_{2}=\frac{\partial}{\partial x^{2}}+F \frac{\partial}{\partial y}+H \frac{\partial}{\partial y_{1}}+K \frac{\partial}{\partial y_{1,1}} . \tag{1.41}
\end{align*}
$$

Definition 1.42. The system ( $\mathscr{E}$ ) is completely integrable if the $n$ vector fields (1.38) satisfy the Frobenius integrability condition, namely every Lie
bracket $\left[\mathrm{D}_{i_{1}}, \mathrm{D}_{i_{2}}\right], 1 \leqslant i_{1}, i_{2} \leqslant n$, is a linear combination of the vector fields $\mathrm{D}_{1}, \ldots, \mathrm{D}_{n}$.

Because of their specific form (1.38), we must then have in fact $\left[\mathrm{D}_{i_{1}}, \mathrm{D}_{i_{2}}\right]=0$. For $n=1$, the condition is of course void.

## §2. SUBMANIFOLD OF SOLUTIONS

2.1. Fundamental foliation of the skeleton. As the vector fields $D_{i}$ commute, they equip the skeleton $\Delta_{\mathscr{E}} \simeq \mathbb{K}^{n+m+p}$ with a foliation $\mathrm{F}_{\Delta_{\mathscr{E}}}$ by $n$ dimensional integral manifolds which are (approximately) directed along the $x$-axis. We draw a diagram (see only the left side).


The (abstract, not numerical) integration of $(\mathscr{E})$ is thus straightforwardly completed: the set of solutions coincides with the set of leaves of $\mathrm{F}_{\Delta_{\mathscr{E}}}$. This is the true geometric content, viewed in the appropriate jet space, of the assumption of complete integrability.
2.2. General solution and submanifold of solutions. To construct the submanifold of solutions $\mathscr{M}_{(\mathscr{E})}$ associated to ( $\left.\mathscr{E}\right)$ (sketched in the right hand side), we execute some elementary analytico-geometric constructions.

At first, we duplicate the coordinates $\left(y_{\beta(q)}^{j(q)}, y^{j}\right) \in \mathbb{K}^{p} \times \mathbb{K}^{m}$ by introducing a new subspace of coordinates $(a, b) \in \mathbb{K}^{p} \times \mathbb{K}^{m}$; thus, on the left diagram, we draw a vertical plane together with $a$ - and $b$-axes. The leaves of the foliation $\mathrm{F}_{\Delta_{\mathscr{E}}}$ are uniquely determined by their intersections with this plane, consisting of points of coordinates $(0, a, b) \in \mathbb{K}^{n} \times \mathbb{K}^{p} \times \mathbb{K}^{m}$.

Such points $(0, a, b)$ correspond to the initial conditions $\left(y_{x^{\beta(q)}}^{j(q)}(0), y(0)\right)$ for the general solution of $(\mathscr{E})$. In fact, the (concatenated, multiple) flow of $\left\{\mathrm{D}_{1}, \ldots, \mathrm{D}_{n}\right\}$ is given by
$\exp \left(x^{n} \mathbf{D}_{n}\left(\cdots\left(\exp \left(x^{1} \mathbf{D}_{1}(0, a, b)\right)\right) \cdots\right)\right)=(x, \Pi(x, a, b), \Omega(x, a, b)) \in \mathbb{K}^{n} \times \mathbb{K}^{m} \times \mathbb{K}^{p}$,
for some two local analytic maps $\Pi=\left(\Pi^{1}, \ldots, \Pi^{m}\right)$ and $\Omega=\left(\Omega^{1}, \ldots, \Omega^{p}\right)$ and the next lemma is straightforward.

Lemma 2.4. The general solution of $(\mathscr{E})$ is

$$
\begin{equation*}
y(x):=\Pi(x, a, b) \tag{2.5}
\end{equation*}
$$

where $(a, b)$ varies in $\mathbb{K}^{p} \times \mathbb{K}^{m}$. Furthermore, for $q=1, \ldots, p$ :

$$
\begin{equation*}
\Omega^{q}(x, a, b) \equiv \Pi_{x^{\beta(q)}}^{j(q)}(x, a, b) \tag{2.6}
\end{equation*}
$$

This leads to introducing a fundamental geometric object.
Definition 2.7. The submanifold of solutions $\mathscr{V}_{\mathscr{S}}(\mathscr{E})$ associated to $(\mathscr{E})$ is the analytic submanifold of $\mathbb{K}_{x}^{n} \times \mathbb{K}_{y}^{m} \times \mathbb{K}_{a}^{p} \times \mathbb{K}_{b}^{m}$ defined by the Cartesian equations:

$$
\begin{equation*}
0=-y^{j}+\Pi^{j}(x, a, b), \quad j=1, \ldots, m . \tag{2.8}
\end{equation*}
$$

There is a strong interplay between the study of $(\mathscr{E})$ and the geometry of $\mathscr{V}_{\mathscr{S}}(\mathscr{E})$. By construction, the diffeomorphism:
$\left\{\begin{array}{l}\mathrm{A}: \mathbb{K}^{n+p+m}\left[\text { coordinates }\left(x^{i}, a^{q}, b^{j}\right)\right] \longrightarrow \mathbb{K}^{n+m+p}\left[\text { coordinates }\left(x^{i}, y^{j}, y_{\beta(q)}^{j(q)}\right)\right] \\ \mathrm{A}\left(x^{i}, a^{q}, b^{j}\right):=\left(x^{i}, \Pi^{j}(x, a, b), \Pi_{x^{\beta(q)}}^{j(q)}(x, a, b),\right),\end{array}\right.$
sends the foliation $\mathrm{F}_{\mathrm{v}}$ by the variables $x$ whose leaves are $\{a=\operatorname{cst}$., $b=$ cst. $\}$ (see the diagram), to the previous foliation $\mathrm{F}_{\Delta_{\mathscr{E}}}$.
2.10. PDE system associated to a submanifold. Inversely, let $\mathscr{M}$ be a submanifold of $\mathbb{K}_{x}^{n} \times \mathbb{K}_{y}^{m} \times \mathbb{K}_{a}^{p} \times \mathbb{K}_{b}^{m}$ of the form

$$
\begin{equation*}
y^{j}=\Pi^{j}(x, a, b), \quad j=1, \ldots, m . \tag{2.11}
\end{equation*}
$$

A necessary condition for it to be the complexification of a generic $M \subset$ $\mathbb{C}^{n+m}$ is that $p=n$ (answer to an exercise above).

Definition 2.12. $\mathscr{M}$ is solvable with respect to the parameters if $b \mapsto \Pi(0,0, b)$ of rank $m$ at $b=0$ and if there exist $\kappa \geqslant 1$, multiindices $\beta(1), \ldots, \beta(p) \in \mathbb{N}^{n}$ with $|\beta(q)| \geqslant 1$ for $q=1, \ldots, p$ and $\max _{1 \leqslant q \leqslant p}|\beta(q)|=\kappa$, together with integers $j(1), \ldots, j(p)$ with $1 \leqslant j(q) \leqslant m$ such that the local $\mathbb{K}$-analytic map

$$
\begin{equation*}
\mathbb{K}^{m+p} \ni(a, b) \longmapsto\left(\left(\Pi^{j}(0, a, b)\right)^{1 \leqslant j \leqslant m},\left(\Pi_{x^{\beta(q)}}^{j(q)}(0, a, b)\right)_{1 \leqslant q \leqslant p}\right) \in \mathbb{K}^{m+p} \tag{2.13}
\end{equation*}
$$

is of rank equal to $m+p$ at $(a, b)=(0,0)$

When $\mathscr{M}$ is the submanifold of solutions of a system $(\mathscr{E})$, it is automatically solvable with respect to the variables, the pairs $(j(q), \beta(q))$ being the same as in the arguments of the right hand sides $F_{\alpha}^{j}$ in $(\mathscr{E})$. Proceeding as in $\S 1.16$, we may associate to $\mathscr{M}$ a system of the form $(\mathscr{E})$. Since we need introduce some new notation, let us repeat the argument.

Considering $y=y(x)=\Pi(x, a, b)$ as a function of $x$ with extra parameters $(a, b)$ and applying $\partial^{|\alpha|} / \partial x^{\alpha}$, we get $y_{x^{\alpha}}^{j}(x)=\Pi_{x^{\alpha}}^{j}(x, a, b)$. Writing only the relevant $(m+p)$ equations:

$$
\left\{\begin{array}{l}
y^{j}(x)=\Pi^{j}(x, a, b),  \tag{2.14}\\
y_{x^{\beta(q)}}^{j(q)}=\Pi_{x^{\beta(q)}}^{j(q)}(x, a, b),
\end{array}\right.
$$

the assumption of solvability with respect to parameters enables to get

$$
\left\{\begin{array}{l}
a^{q}=A^{q}\left(x^{i}, y^{j}, y_{\beta\left(q_{1}\right)}^{j\left(q_{1}\right)}\right),  \tag{2.15}\\
b^{j}=B^{j}\left(x^{i}, y^{j_{1}}, y_{\beta(q)}^{j(q)}\right) .
\end{array}\right.
$$

For every $(j, \alpha) \neq(j, 0)$ and $\neq(j(q), \beta(q))$, we then replace $(a, b)$ in $y_{x^{\alpha}}^{j}=$ $\Pi_{x^{\alpha}}^{j}$ :

$$
\begin{align*}
y_{x^{\alpha}}^{j}(x) & =\Pi_{x^{\alpha}}^{j}\left(x, A\left(x^{i}, y^{j_{1}}(x), y_{\beta(q)}^{j(q)}(x)\right), B\left(x^{i}, y^{j_{1}}(x), y_{\beta(q)}^{j(q)}(x)\right)\right)  \tag{2.16}\\
& =: F_{\alpha}^{j}\left(x^{i}, y^{j_{1}}(x), y_{x^{\beta(q)}}^{j(q)}(x)\right) .
\end{align*}
$$

Proposition 2.17. There is a one-to-one correspondence

$$
\begin{equation*}
\left(\mathscr{E}_{\mathscr{M}}\right)=(\mathscr{E}) \longleftrightarrow \mathscr{M}=\mathscr{M}_{(\mathscr{E})}, \tag{2.18}
\end{equation*}
$$

between completely integrable systems of partial differential equations of the general form ( $\mathscr{E}$ ) and submanifolds (of solutions) $\mathscr{M}$ of the form (2.11) which are solvable with respect to the parameters. Of course

$$
\begin{equation*}
\left(\mathscr{E}_{\mathscr{M}_{(\mathscr{E})}}\right)=(\mathscr{E}) \quad \text { and } \quad \mathscr{M}_{\left(\mathscr{E}_{\mathscr{M}}\right)}=\mathscr{M} \tag{2.19}
\end{equation*}
$$

2.20. Transfer of total differentiations. We notice that the auxiliary functions $A^{q}$ and $B^{j}$ enable to express the inverse of A :

$$
\begin{equation*}
\mathrm{A}^{-1}:\left(x^{i_{1}}, y^{j_{1}}, y_{\beta\left(q_{1}\right)}^{j\left(q_{1}\right)}\right) \longmapsto\left(x^{i}, A^{q}\left(x^{i_{1}}, y^{j_{1}}, y_{\beta\left(q_{1}\right)}^{j\left(q_{1}\right)}\right), B^{j}\left(x^{i_{1}}, y^{j_{1}}, y_{\beta\left(q_{1}\right)}^{j\left(q_{1}\right)}\right)\right) . \tag{2.21}
\end{equation*}
$$

More importantly, the total differentiation operator considerably simplifies when viewed on $\mathscr{M}$. This observation is useful for translating differential invariants of $(\mathscr{E})$ as differential invariants of $\mathscr{M}$.
Lemma 2.22. Through A, for $i=1, \ldots, n$, the pull-back of the total differentiation operator $D_{i}$ is simply $\frac{\partial}{\partial x^{i}}$, or equivalently:

$$
\begin{equation*}
\mathrm{A}_{*}\left(\frac{\partial}{\partial x^{i}}\right)=\mathrm{D}_{i} . \tag{2.23}
\end{equation*}
$$

Proof. Let $\ell=\ell\left(x^{i}, y^{j}, y_{\beta(q)}^{j(q)}\right)$ be any function defined on $\Delta_{\mathscr{E}}$. Composing with A yields the function $\Lambda:=\ell \circ \mathrm{A}$, i.e.

$$
\begin{equation*}
\Lambda(x, a, b) \equiv \ell\left(x^{i}, \Pi^{j}(x, a, b), \Pi_{x^{\beta(q)}}^{j(q)}(x, a, b)\right) \tag{2.24}
\end{equation*}
$$

Differentiating with respect to $x^{i}$, we get, dropping the arguments:

$$
\begin{equation*}
\frac{\partial \Lambda}{\partial x^{i}}=\frac{\partial \ell}{\partial x^{i}}+\sum_{j=1}^{m} \Pi_{x^{i}}^{j} \frac{\partial \ell}{\partial y^{j}}+\sum_{q=1}^{p} \Pi_{x^{i} x^{\beta(q)}}^{j(q)} \frac{\partial \ell}{\partial y_{x^{\beta(q)}}^{j(q)}} . \tag{2.25}
\end{equation*}
$$

Replacing the appearing $\Pi_{x^{\alpha}}^{j}$ for which $(j, \alpha) \neq(j, 0)$ and $\neq(j(q), \beta(q))$ by $F_{\alpha}^{j}$, we recover $\mathrm{D}_{i}$ as defined by (1.38), whence $\frac{\partial \Lambda}{\partial x^{i}}=\mathrm{D}_{i} \ell$.
2.26. Transfer of algebrico-differential expressions. The diffeomorphism A may be used to translate algebrico-differential expressions from $\mathscr{M}$ to ( $\mathscr{E}$ ) and vice-versa:

$$
\begin{equation*}
\mathrm{I}_{\mathscr{M}}\left(J_{x, a, b}^{\lambda+\kappa+1} \Pi\right) \longleftrightarrow \mathrm{I}_{(\mathscr{E})}\left(J_{x, y, y_{1}}^{\lambda} F\right) . \tag{2.27}
\end{equation*}
$$

Here, $\lambda \in \mathbb{N}$, the letter $J$ is used to denote jets, and $I=I_{\mathscr{M}}$ or $=I_{(\mathscr{E})}$ is a polynomial or more generally, a quotient of polynomials with respect to its jet arguments. Notice the shift by $\kappa+1$ of the jet orders.

Example 2.28. Suppose $n=m=1$ and $\kappa=1$. Then $F=\Pi_{x x}$. As an exercise, let us compute $F_{x}, F_{y}, F_{y_{1}}$ in terms of $J_{x, a, b}^{3} \Pi$. We start with the identity

$$
\begin{equation*}
F\left(x, y, y_{1}\right) \equiv \Pi_{x x}\left(x, A\left(x, y, y_{1}\right), B\left(x, y, y_{1}\right)\right) \tag{2.29}
\end{equation*}
$$

that we differentiate with respect to $x$, to $y$ and to $y_{1}$ :

$$
\begin{array}{rlrl}
F_{x} & =\Pi_{x x x}+\Pi_{x x a} A_{x}+\Pi_{x x b} B_{x}, \\
F_{y} & = & \Pi_{x x a} A_{y}+\Pi_{x x b} B_{y},  \tag{2.30}\\
F_{y_{1}} & = & \Pi_{x x a} A_{y_{1}}+\Pi_{x x b} B_{y_{1}} .
\end{array}
$$

Thus, we need to compute $A_{x}, A_{y}, A_{y_{1}}, B_{x}, B_{y}, B_{y_{1}}$. This is easy: it suffices to differentiate the two identities that define $A$ and $B$ as implicit functions, namely:

$$
\begin{align*}
y & \equiv \Pi\left(x, A\left(x, y, y_{1}\right), B\left(x, y, y_{1}\right)\right) \quad \text { and } \\
y_{1} & \equiv \Pi_{x}\left(x, A\left(x, y, y_{1}\right), B\left(x, y, y_{1}\right)\right) \tag{2.31}
\end{align*}
$$

with respect to $x$, to $y$ and to $y_{1}$, which gives six new identities:

$$
\begin{array}{lllrl}
0 & =\Pi_{x}+\Pi_{a} A_{x}+\Pi_{b} B_{x}, & & 0 & =\Pi_{x x}+\Pi_{x a} A_{x}+\Pi_{x b} B_{x} \\
1 & = & \Pi_{a} A_{y}+\Pi_{b} B_{y}, & 0 & =  \tag{2.32}\\
0 & = & \Pi_{a} A_{y_{1}}+\Pi_{b} B_{y_{1}}, & 1 & = \\
\Pi_{x a} A_{y}+\Pi_{x b} B_{y} \\
\Pi_{x b} B_{y_{1}}
\end{array}
$$

and to solve each of the three linear systems of two equations located in a line, noticing that their common determinant $\Pi_{b} \Pi_{x a}-\Pi_{a} \Pi_{x b}$ does not
vanish at the origin, since $\Pi=b+x a+\mathrm{O}_{3}$. By elementary Cramer formulas, we get:

$$
\left\{\begin{align*}
A_{x}=\frac{-\Pi_{b} \Pi_{x x}+\Pi_{x} \Pi_{x b}}{\Pi_{b} \Pi_{x a}-\Pi_{a} \Pi_{x b}}, & B_{x}=\frac{-\Pi_{x} \Pi_{x a}+\Pi_{a} \Pi_{x x}}{\Pi_{b} \Pi_{x a}-\Pi_{a} \Pi_{x b}},  \tag{2.33}\\
A_{y}=\frac{-\Pi_{x b}}{\Pi_{b} \Pi_{x a}-\Pi_{a} \Pi_{x b}}, & B_{y}=\frac{\Pi_{x a}}{\Pi_{b} \Pi_{x a}-\Pi_{a} \Pi_{x b}}, \\
A_{y_{1}}=\frac{\Pi_{b}}{\Pi_{b} \Pi_{x a}-\Pi_{a} \Pi_{x b}}, & B_{y_{1}}=\frac{-\Pi_{a}}{\Pi_{b} \Pi_{x a}-\Pi_{a} \Pi_{x b}} .
\end{align*}\right.
$$

Replacing in (2.30), no simplification occurs and we get what we wanted:

$$
\left\{\begin{align*}
F_{x} & =\Pi_{x x x}+\frac{\Pi_{x x a}\left[-\Pi_{b} \Pi_{x x}+\Pi_{x} \Pi_{x b}\right]+\Pi_{x x b}\left[-\Pi_{x} \Pi_{x a}+\Pi_{a} \Pi_{x x}\right]}{\Pi_{b} \Pi_{x a}-\Pi_{a} \Pi_{x b}}  \tag{2.34}\\
F_{y} & =\frac{-\Pi_{x x a} \Pi_{x b}+\Pi_{x x b} \Pi_{x a}}{\Pi_{b} \Pi_{x a}-\Pi_{a} \Pi_{x b}} \\
F_{y_{1}} & =\frac{\Pi_{x x a} \Pi_{b}-\Pi_{x x b} \Pi_{a}}{\Pi_{b} \Pi_{x a}-\Pi_{a} \Pi_{x b}}
\end{align*}\right.
$$

One sees D $F=F_{x}+\Pi_{x} F_{y}+\Pi_{x x} F_{y_{1}}=\Pi_{x x x}$ simply, as predicted by Lemma 2.22.

Second order derivatives $F_{x x}, F_{x y}, F_{x y_{1}}, F_{y y}, F_{y y_{1}}, F_{y_{1} y_{1}}$ have still reasonable complexity, when expressed in terms of $J_{x, a, b}^{4} \Pi$. Beyond, the computations explode.

Open question 2.35. A second order ordinary differential equation $y_{x x}=$ $F\left(x, y, y_{x}\right)$ has two fundamental differential invariants, namely ([Tr1896, Ca1924, GTW1989, Ol1995]):
$\mathrm{I}_{\left(\mathscr{E}_{1}\right)}^{1}:=\frac{\partial^{4} F}{\partial y_{1}^{4}} \quad$ and
$\mathrm{I}_{\left(\mathscr{\varepsilon}_{1}\right)}^{2}:=\mathrm{DD}\left(F_{y_{1} y_{1}}\right)-F_{y_{1}} \mathrm{D}\left(F_{y_{1} y_{1}}\right)-4 \mathrm{D}\left(F_{y y_{1}}\right)+6 F_{y y}-3 F_{y} F_{y_{1} y_{1}}+4 F_{y_{1}} F_{y y_{1}}$.
Compute $I_{\mathscr{M}_{1}}^{1}$ and $I_{\mathscr{M}_{1}}^{2}$.
Although the notion of diffeomorphism is clear and apparently obvious from the intuitive, geometric and conceptual viewpoints, in concrete applications and in explicit computations, it almost never straightforward to transfer algebrico-differential objects.

Open problem 2.37. For general $(\mathscr{E})$ and $\mathscr{M}$, build closed combinatorial formulas executing the double translation (2.27).
2.38. Plan for the sequel. We will endeavour a general theory showing that the study of systems $(\mathscr{E})$ and the study of submanifolds of solutions $\mathscr{M}$ gives
complementary views on the same object. In fact, Lie symmetries, equivalence problems, Cartan connections, normal forms and classification lists may be endeavoured on both sides, yielding essentially equivalent results, though the translation is seldom straightforward. In Section 3, 4 and 5, we review some features from the side ( $\mathscr{E}$ ), before studying some aspects from the side of $\mathscr{M}$. A more systematic and complete approach shall appear as a monography.

## §3. Classification problems

3.1. Transformations of PDE systems. Through a local $\mathbb{K}$-analytic change of variables close to the identity $(x, y) \mapsto \varphi(x, y)=:\left(x^{\prime}, y^{\prime}\right)$, the system $(\mathscr{E})$ transforms to a similar system, with primes:

$$
y_{x^{\prime \alpha}}^{\prime j}\left(x^{\prime}\right)=F_{\alpha}^{\prime j}\left(x^{\prime}, y^{\prime}\left(x^{\prime}\right),\left(y_{x^{\prime \beta(q)}}^{\prime j(q)}\left(x^{\prime}\right)\right)_{1 \leqslant q \leqslant p}\right) .
$$

Example 3.2. Coming back temporarily to the notations of $\S 1.12$ (II), with $n=m=\kappa=1$, assume that $y_{x x}=f\left(x, y, y_{x}\right)$ transforms to $Y_{X X}=F\left(X, Y, Y_{X}\right)$ through a local diffeomorphism $(x, y) \mapsto(X, Y)=$ $(X(x, y), Y(x, y))$. How $F$ is related to $f$ ? By symmetry, it suffices to compute $f$ in terms of $F, X, Y$. The prolongation to $\mathscr{J}_{1,1}^{2}$ of the diffeomorphism has components ([BK1989, Me2004]):

$$
\begin{equation*}
Y_{X}=\frac{Y_{x}+y_{x} Y_{y}}{X_{x}+y_{x} X_{y}} \tag{3.3}
\end{equation*}
$$

and

$$
\begin{align*}
& Y_{X X}=\frac{1}{\left[X_{x}+y_{x} X_{y}\right]^{3}}\left(y_{x x} \cdot\left|\begin{array}{cc}
X_{x} & X_{y} \\
Y_{x} & Y_{y}
\end{array}\right|+\left|\begin{array}{cc}
X_{x} & X_{x x} \\
Y_{x} & Y_{x x}
\end{array}\right|+\right.  \tag{3.4}\\
&+y_{x} \cdot\left\{2\left|\begin{array}{cc}
X_{x} & X_{x y} \\
Y_{x} & Y_{x y}
\end{array}\right|-\left|\begin{array}{cc}
X_{x x} & X_{y} \\
Y_{x x} & Y_{y}
\end{array}\right|\right\}+ \\
&+y_{x} y_{x} \cdot\left\{\left|\begin{array}{cc}
X_{x} & X_{y y} \\
Y_{x} & Y_{y y}
\end{array}\right|-2\left|\begin{array}{cc}
X_{x y} & X_{y} \\
Y_{x y} & Y_{y}
\end{array}\right|\right\}+ \\
&\left.+y_{x} y_{x} y_{x} \cdot\left\{-\left|\begin{array}{cc}
X_{y y} & X_{y} \\
Y_{y y} & Y_{y}
\end{array}\right|\right\}\right) .
\end{align*}
$$

It then suffices to replace $Y_{X X}$ above by $F\left(X, Y, Y_{X}\right)$ and to solve $y_{x x}$ :

$$
\begin{align*}
& y_{x x}=\frac{1}{\left|\begin{array}{cc}
X_{x} & X_{y} \\
Y_{x} & Y_{y}
\end{array}\right|}( \left(\left[X_{x}+y_{x} X_{y}\right]^{3} F\left(X, Y, \frac{Y_{x}+y_{x} Y_{y}}{X_{x}+y_{x} X_{y}}\right)-\left|\begin{array}{cc}
X_{x} & X_{x x} \\
Y_{x} & Y_{x x}
\end{array}\right|+\right.  \tag{3.5}\\
&+y_{x} \cdot\left\{-2\left|\begin{array}{cc}
X_{x} & X_{x y} \\
Y_{x} & Y_{x y}
\end{array}\right|+\left|\begin{array}{cc}
X_{x x} & X_{y} \\
Y_{x x} & Y_{y}
\end{array}\right|\right\}+ \\
&+y_{x} y_{x} \cdot\left\{-\left|\begin{array}{cc}
X_{x} & X_{y y} \\
Y_{x} & Y_{y y}
\end{array}\right|+2\left|\begin{array}{cc}
X_{x y} & X_{y} \\
Y_{x y} & Y_{y}
\end{array}\right|\right\}+ \\
&\left.+y_{x} y_{x} y_{x} \cdot\left\{\left|\begin{array}{cc}
X_{y y} & X_{y} \\
Y_{y y} & Y_{y}
\end{array}\right|\right\}\right) \\
&=: f\left(x, y, y_{x}\right) .
\end{align*}
$$

Open problem 3.6. Find general formulas expressing the $F_{\alpha}^{j}$ in terms of $F^{\prime j}{ }_{\alpha}$, $x^{\prime i}, y^{\prime j}$.

Conversely, given two such systems $(\mathscr{E})$ and $\left(\mathscr{E}^{\prime}\right)$, when do they transform to each other ? Let $\pi_{\kappa, p}^{\prime}$ denote the projection from $\mathscr{J}_{n, m}^{\prime \kappa+1}$ to $\Delta_{\mathscr{E}^{\prime}}$ defined by

$$
\begin{equation*}
\pi_{\kappa, p}^{\prime}\left(x^{\prime i}, y^{\prime j}, y_{i_{1}}^{\prime j}, \ldots, y_{i_{1}, \ldots, i_{\kappa+1}}^{\prime j}\right):=\left(x^{\prime i}, y^{\prime j}, y_{\beta(q)}^{\prime j(q)}\right) . \tag{3.7}
\end{equation*}
$$

Let $\varphi^{(\kappa+1)}$ be the $(\kappa+1)$-th prolongation of $\varphi$ (Section 1(II)).
Lemma 3.8. ([Ol1986, BK1989, Ol1995]) The following three conditions are equivalent:
(1) $\varphi$ transforms ( $\mathscr{E}$ ) to ( $\left.\mathscr{E}^{\circ}\right)$;
(2) its $(\kappa+1)$-th prolongation $\varphi^{(\kappa+1)}: \mathscr{J}_{n, m}^{\kappa+1} \rightarrow \mathscr{J}_{n, m}^{\prime \kappa+1}$ maps $\Delta_{\mathscr{E}}$ to $\Delta_{\mathscr{E}^{\prime}}$;
(3) $\varphi^{(\kappa+1)}: \mathscr{J}_{n, m}^{\kappa+1} \rightarrow \mathscr{J}_{n, m}^{\prime \kappa+1}$ maps $\Delta_{\mathscr{E}}$ to $\Delta_{\mathscr{E}^{\prime}}$ and the associated map

$$
\begin{equation*}
\Phi_{\mathscr{E}, \mathscr{E}^{\prime}}:=\pi_{\kappa, p}^{\prime} \circ\left(\left.\varphi^{(\kappa+1)}\right|_{\Delta_{\mathscr{E}}}\right) \tag{3.9}
\end{equation*}
$$

sends every leaf of $\mathrm{F}_{\Delta_{\mathscr{E}}}$ to some leaf of $\mathrm{F}_{\Delta_{\mathscr{E}^{\prime}}}$.

## Equivalence problem 3.10. Find an algorithm to decide whether two given

 ( $\mathscr{E})$ and ( $\mathscr{E}^{\prime}$ ) are equivalent.Élie Cartan's widely applicable method (not reviewed here; [Ca1937, Ste1983, G1989, HK1989, Fe1995, Ol1995]) provides an answer "in principle" to this question by reducing to an $\{e\}$-structure an initial G-structure associated to $(\mathscr{E})$. Due to the incredible size-length-complexity of the underlying computations, this approach almost never abutes: it is forced to incompleteness. But in fact, the main question is to classify.

Classification problem 3.11. Classify systems ( $\mathscr{E}$ ), namely provide complete lists of all possible such equations written in simplified "normal", easily recognizable forms.

Both problems are deeply linked to the classification of Lie algebras of local vector fields. For $n=1, m=1$ and $\kappa=1$, namely $\left(\mathscr{E}_{1}\right): y_{x x}=$ $F\left(x, y, y_{x}\right)$, Lie and Tresse solved the two problems ${ }^{31}$. Table 7 of [Ol1986], below reproduced, describes the results.

|  | Symmetry group | Dimension | Invariant equation |
| :--- | :--- | :--- | :--- |
| $\mathbf{( 1 )}$ |  | 0 | $y_{x x}=F\left(x, y, y_{x}\right)$ |
| $\mathbf{( 2 )}$ | $\partial_{y}$ | 1 | $y_{x x}=F\left(x, y_{x}\right)$ |
| (3) | $\partial_{x}, \partial_{y}$ | 2 | $y_{x x}=F\left(y_{x}\right)$ |
| (4) | $\partial_{x}, e^{x} \partial_{y}$ | 2 | $y_{x x}-y_{x}=F\left(y_{x}-y\right)$ |
| $\mathbf{( 5 )}$ | $\partial_{x}, \partial_{x}-y \partial_{y}, x^{2} \partial_{x}-2 x y \partial_{y}$ | 3 | $y_{x x}=\frac{3 y_{x}^{2}}{2 y}+c y^{3}$ |
| $\mathbf{( 6 )}$ | $\partial_{x}, x \partial_{x}-y \partial_{y}$, <br> $x^{2} \partial_{x}-(2 x y+1) \partial_{y}$ | 3 | $y_{x x}=6 y y_{x}-4 y^{3}+$ <br> $+c\left(y_{x}-y^{2}\right)^{3 / 2}$ |
| $\mathbf{( 7 )}$ | $\partial_{x}, \partial_{y}, x \partial_{x}+\alpha y \partial_{y}$, <br> $\alpha \neq 0, \frac{1}{2}, 1,2$ | 3 | $y_{x x}=c\left(y_{x}\right)^{\frac{\alpha-2}{\alpha-1}}$ |
| $\mathbf{( 8 )}$ | $\partial_{x}, \partial_{y}, x \partial_{x}+(x+y) \partial_{y}$ | 3 | $y_{x x}=c e^{-y_{x}}$ |
| $\mathbf{( 9 )}$ | $\partial_{x}, \partial_{y}, y \partial_{x}, x \partial_{y}, y \partial_{y}$, <br> $x^{2} \partial_{x}+x y \partial_{y}, x y \partial_{x}+y^{2} \partial_{y}$ | 8 | $y_{x x}=0$ |

Table 1.
However, the author knows no modern reference offering a complete proof of this classification, with precise insight on the assumptions (some normal forms hold true only at a generic point). In addition, the above LieTresse list is still slightly incomplete in the sense that it does not precise which are the conditions satisfied by $F$ (Table 7 in [Ol1986]) insuring in the first four lines that $\mathfrak{S Y M}\left(\mathscr{E}_{1}\right)$ is indeed of small dimension 0,1 or 2 .

Open question 3.12. Specify some precise nondegeneracy conditions upon $F$ in the first four lines of Table 1.

## §4. Punctual and infinitesimal Lie symmetries

4.1. Lie symmetries of $(\mathscr{E})$. Let $\varphi=(\phi, \psi)$ be a diffeomorphism of $\mathbb{K}_{x}^{n} \times$ $\mathbb{K}_{y}^{n}$ as in (1.7)(II).

[^22]Definition 4.2. ([Ol1986, Ol1995, BK1989]) $\varphi$ is a (local) Lie symmetry of $(\mathscr{E})$ if it transforms the graph of every solution of $(\mathscr{E})$ into the graph of another solution.

To explain, we must pass to jet spaces. Denote the components of the $(\kappa+1)$-th prolongation $\varphi^{(\kappa+1)}: \mathscr{J}_{n, m}^{\kappa+1} \rightarrow \mathscr{J}_{n, m}^{\kappa+1}$ by

$$
\begin{equation*}
\varphi^{(\kappa+1)}=\left(\phi^{i_{1}}, \psi^{j_{1}}, \Phi_{i_{1}}^{j}, \Phi_{i_{1}, i_{2}}^{j}, \ldots \ldots, \Phi_{i_{1}, i_{2}, \ldots, i_{\kappa+1}}^{j}\right) \tag{4.3}
\end{equation*}
$$

The restriction $\left.\varphi^{(\kappa+1)}\right|_{\Delta_{\mathscr{E}}}$ is obtained by replacing each jet variable $y_{\alpha}^{j}$ by $F_{\alpha}^{j}$, whenever $(j, \alpha) \neq(j, 0)$ and $\neq(j(q), \beta(q))$, and wherever it appears ${ }^{32}$ in the $\Phi_{i_{1}, \ldots, i_{\lambda}}^{j}$.

Let $\pi_{\kappa, p}$ denote the projection from $\mathscr{J}_{n, m}^{\kappa+1}$ to $\Delta_{\mathscr{E}} \simeq \mathbb{K}^{m+n+p}$ defined by

$$
\begin{equation*}
\pi_{\kappa, p}\left(x^{i}, y^{j}, y_{i_{1}}^{j}, \ldots, y_{i_{1}, \ldots, i_{\kappa+1}}^{j}\right):=\left(x^{i}, y^{j}, y_{\beta(q)}^{j(q)}\right), \tag{4.4}
\end{equation*}
$$

and introduce the map

$$
\begin{equation*}
\varphi_{\Delta_{\mathscr{E}}}:=\pi_{\kappa, p} \circ\left(\left.\varphi^{(\kappa+1)}\right|_{\Delta_{\mathscr{E}}}\right) \equiv\left(\varphi\left(x^{i}, y^{j}\right), \Phi_{\beta(q)}^{j(q)}\left(x^{i}, y^{j}, y_{\beta\left(q_{1}\right)}^{j\left(q_{1}\right)}\right)\right) . \tag{4.5}
\end{equation*}
$$

Lemma 4.6. ([Ol1986, Ol1995, BK1989], [*]) The following three conditions are equivalent:
(1) the diffeomorphism $\varphi$ is a Lie symmetry of $(\mathscr{E})$;
(2) $\left.\varphi^{(\kappa+1)}\right|_{\Delta_{\mathscr{E}}}$ sends $\Delta_{\mathscr{E}}$ to $\Delta_{\mathscr{E}}$;
(3) $\left.\varphi^{(\kappa+1)}\right|_{\Delta_{\mathscr{E}}}$ sends $\Delta_{\mathscr{E}}$ to $\Delta_{\mathscr{E}}$ and $\varphi_{\Delta_{\mathscr{E}}}=\pi_{\kappa, p}\left(\left.\varphi^{(\kappa+1)}\right|_{\Delta_{\mathscr{E}}}\right)$ is a symmetry of the foliation $\mathrm{F}_{\Delta_{\mathscr{E}}}$, namely it sends every leaf to some other leaf.
Then the set of Lie symmetries of ( $\mathscr{E}$ ) constitutes a local Lie (pseudo)group.

### 4.7. Infinitesimal Lie symmetries of ( $\mathscr{E}$ ). Let

$$
\begin{equation*}
\mathscr{L}=\sum_{i=1}^{n} \mathscr{X}^{i}(x, y) \frac{\partial}{\partial x^{i}}+\sum_{j=1}^{m} \mathscr{Y}^{j}(x, y) \frac{\partial}{\partial y^{j}}, \tag{4.8}
\end{equation*}
$$

be a (local) vector field on $\mathbb{K}^{n+m}$ having analytic coefficients. Denote its flow by $\varphi_{t}(x, y):=\exp (t \mathscr{L})(x, y), t \in \mathbb{K}$. As in Section 1(II), by differentiating the prolongation $\left(\varphi_{t}\right)^{(\kappa+1)}$ with respect to $t$ at $t=0$, we get the prolonged vector field $\mathscr{L}^{(\kappa+1)}$ on $\mathscr{J}_{n, m}^{\kappa+1}$, having the general form (Part II):

$$
\begin{equation*}
\mathscr{L}^{(\kappa+1)}=\mathscr{L}+\sum_{j=1}^{m} \sum_{i_{1}=1}^{n} \mathbf{Y}_{i_{1}}^{j} \frac{\partial}{\partial y_{i_{1}}^{j}}+\cdots+\sum_{j=1}^{m} \sum_{i_{1}, \ldots, i_{\kappa+1}=1}^{n} \mathbf{Y}_{i_{1}, \ldots, i_{\kappa+1}}^{j} \frac{\partial}{\partial y_{i_{1}, \ldots, i_{\kappa+1}}^{j}}, \tag{4.9}
\end{equation*}
$$

with known explicit expressions for the $\mathbf{Y}_{i_{1}, \ldots, i_{\lambda}}^{j}$.

[^23]Definition 4.10. $\mathscr{L}$ is an infinitesimal symmetry of $(\mathscr{E})$ if for every small $t$, its time- $t$ flow map $\varphi_{t}$ is a Lie symmetry of $(\mathscr{E})$.

The restriction $\left.\mathscr{L}^{(\kappa+1)}\right|_{\Delta_{\mathscr{E}}}$ is obtained by replacing every $y_{\alpha}^{j}$ by $F_{\alpha}^{j}$ in all coefficients $\mathbf{Y}_{i_{1}}^{j}, \ldots, \mathbf{Y}_{i_{1}, \ldots, i_{k+1}}^{j}$. Then the coefficients become functions of $\left(x^{i_{1}}, y^{j_{1}}, y_{\beta\left(q_{1}\right)}^{j\left(q_{1}\right)}\right)$ only.

Lemma 4.11. ([Ol1986, Ol1995, BK1989], [*]) The following three conditions are equivalent:
(1) the vector field $\mathscr{L}$ is an infinitesimal Lie symmetry of $(\mathscr{E})$;
(2) its $(\kappa+1)$-th prolongation $\mathscr{L}^{(\kappa+1)}$ is tangent to the skeleton $\Delta_{\mathscr{E}}$;
(3) $\mathscr{L}^{(\kappa+1)}$ is tangent to $\Delta_{\mathscr{E}}$ and the push-forward

$$
\begin{equation*}
\mathscr{L}_{\Delta_{\mathscr{E}}}:=\left(\pi_{\kappa, p}\right)_{*}\left(\left.\mathscr{L}^{(\kappa+1)}\right|_{\Delta_{\mathscr{E}}}\right) \tag{4.12}
\end{equation*}
$$

is an infinitesimal symmetry of the foliation $\mathrm{F}_{\Delta_{\mathscr{E}}}$, namely for every $i=1, \ldots, n$, the Lie bracket $\left[\mathscr{L}_{\Delta_{\mathscr{E}}}, \mathrm{D}_{i}\right]$ is a linear combination of $\left\{\mathrm{D}_{1}, \ldots, \mathrm{D}_{n}\right\}$.
According to [Ol1986, BK1989, Ol1995], the set of infinitesimal Lie symmetries constitutes a Lie algebra, with the property $\left[\mathscr{L}^{(\kappa+1)}, \mathscr{L}^{\prime(\kappa+1)}\right]=$ $\left[\mathscr{L}, \mathscr{L}^{\prime}\right]^{(\kappa+1)}$. We summarize by a diagram.

4.13. Sophus Lie's algorithm. We describe the general process. Its complexity will be exemplified in Section 5 (to be read simultaneously).

The tangency of $\mathscr{L}^{(\kappa+1)}$ to $\Delta_{\mathscr{E}}$ is expressed by applying $\mathscr{L}^{(\kappa+1)}$ to the equations $0=-y_{\alpha}^{j}+F_{\alpha}^{j}$, which yields:

$$
\begin{equation*}
0=-\mathbf{Y}_{\alpha}^{j}+\sum_{i=1}^{n} \mathscr{X}^{i} \frac{\partial F_{\alpha}^{j}}{\partial x^{i}}+\sum_{l=1}^{n} \mathscr{Y}^{l} \frac{\partial F_{\alpha}^{j}}{\partial y^{l}}+\sum_{q=1}^{p} \mathbf{Y}_{\beta(q)}^{j(q)} \frac{\partial F_{\alpha}^{j}}{\partial y_{\beta(q)}^{j(q)}}, \tag{4.14}
\end{equation*}
$$

for $(j, \alpha) \neq(j, 0)$ and $\neq(j(q), \beta(q))$. Restricting a coefficient $\mathbf{Y}_{i_{1}, \ldots, i_{\lambda}}^{j}$ to $\Delta_{\mathscr{E}}$, namely replacing everywhere in it each $y_{\alpha}^{j}$ by $F_{\alpha}^{j}$, provides a specialized
coefficient

$$
\begin{equation*}
\widehat{\mathbf{Y}}_{i_{1}, \ldots, i_{\lambda}}^{j}=\widehat{\mathbf{Y}}_{i_{1}, \ldots, i_{\lambda}}^{j}\left(x^{i_{1}}, y^{j_{1}}, y_{\beta\left(q_{1}\right)}^{j\left(q_{1}\right)}, J_{x, y}^{\lambda} \mathscr{X}^{i_{1}}, J_{x, y}^{\lambda} \mathscr{Y}^{j_{1}}\right), \tag{4.15}
\end{equation*}
$$

that depends linearly on the $\lambda$-th jet of the coefficients of $\mathscr{L}$, as confirmed by an inspection of Part II's formulas. Here, we use the jet notation $J_{x, y}^{\lambda} \mathscr{Z}:=\left(\partial_{x}^{\alpha_{1}} \partial_{y}^{\beta_{1}} \mathscr{Z}\right)_{\left|\alpha_{1}\right|+\left|\beta_{1}\right| \leqslant \lambda}$. We thus get equations

$$
\begin{equation*}
0 \equiv-\widehat{\mathbf{Y}}_{\alpha}^{j}+\sum_{i=1}^{n} \mathscr{X}^{i} \frac{\partial F_{\alpha}^{j}}{\partial x^{i}}+\sum_{l=1}^{n} \mathscr{Y}^{l} \frac{\partial F_{\alpha}^{j}}{\partial y^{l}}+\sum_{q=1}^{p} \widehat{\mathbf{Y}}_{\beta(q)}^{j(q)} \frac{\partial F_{\alpha}^{j}}{\partial y_{\beta(q)}^{j(q)}}, \tag{4.16}
\end{equation*}
$$

involving only the variables $\left(x^{i_{1}}, y^{j_{1}}, y_{\beta\left(q_{1}\right)}^{j\left(q_{1}\right)}\right)$.
Next, we develope every such equation with respect to the powers of $y_{\beta\left(q_{1}\right)}^{j\left(q_{1}\right)}$ :
$0 \equiv \sum_{\mu_{1}, \ldots, \mu_{p} \geqslant 0}\left(y_{\beta(1)}^{j(1)}\right)^{\mu_{1}} \cdots\left(y_{\beta(p)}^{j(p)}\right)^{\mu_{p}} \Psi_{\alpha, \mu_{1}, \ldots, \mu_{p}}^{j}\left(x^{i_{1}}, y^{j_{1}}, J_{x, y}^{\kappa+1} \mathscr{X}^{i_{1}}, J_{x, y}^{\kappa+1} \mathscr{Y}^{j_{1}}\right)$.
The $\Psi_{\alpha, \mu_{1}, \ldots, \mu_{p}}^{j}$ are linear with respect to $\left(J_{x, y}^{\kappa+1} \mathscr{X}^{i_{1}}, J_{x, y}^{\kappa+1} \mathscr{Y}^{j_{1}}\right)$, with certain coefficients analytic with respect to $(x, y)$, which depend intrinsically (but in a complex manner) on the right hand sides $F_{\alpha}^{j}$.

Proposition 4.18. The vector field $\mathscr{L}$ is an infinitesimal Lie symmetry of $(\mathscr{E})$ if and only if its coefficients $\mathscr{X}^{i_{1}}, \mathscr{Y}^{j_{1}}$ satisfy the linear PDE system:

$$
\begin{equation*}
0=\Psi_{\alpha, \mu_{1}, \ldots, \mu_{p}}^{j}\left(x^{i_{1}}, y^{j_{1}}, J_{x, y}^{\kappa+1} \mathscr{X}^{i_{1}}, J_{x, y}^{\kappa+1} \mathscr{Y}^{j_{1}}\right) \tag{4.19}
\end{equation*}
$$

for all $(j, \alpha) \neq(j, 0)$ and $\neq(j(q), \beta(q))$ and for all $\left(\mu_{1}, \ldots, \mu_{p}\right) \in \mathbb{N}^{p}$.
In all known instances, a finite number of these equations suffices.
Example 4.20. With $n=m=\kappa=1$, a second prolongation $\mathscr{L}^{(2)}=$ $\mathscr{X} \frac{\partial}{\partial x}+\mathscr{Y} \frac{\partial}{\partial y}+\mathbf{Y}_{1} \frac{\partial}{\partial y_{1}}+\mathbf{Y}_{2} \frac{\partial}{\partial y_{2}}$ is tangent to the skeleton $0=-y_{2}+$ $F\left(x, y, y_{1}\right)$ of $\left(\mathscr{E}_{1}\right)$ if and only if $0=-\mathbf{Y}_{2}+\mathscr{X} F_{x}+\mathscr{Y} F_{y}+\mathbf{Y}_{1} F_{y_{1}}$, or, developing:

## (4.21)

$$
\left\{\begin{aligned}
0= & -\mathscr{Y}_{x x}+\left[-2 \mathscr{Y}_{x y}+\mathscr{X}_{x x}\right] y_{1}+\left[-\mathscr{Y}_{y y}+2 \mathscr{X}_{x y}\right]\left(y_{1}\right)^{2}+\left[\mathscr{X}_{y y}\right]\left(y_{1}\right)^{3}+ \\
& +\left[-\mathscr{Y}_{y}+2 \mathscr{X}_{x}\right] F+\left[3 \mathscr{X}_{y}\right] y_{1} F+[\mathscr{X}] F_{x}+[\mathscr{Y}] F_{y}+ \\
& +\left[\mathscr{Y}_{x}\right] F_{y_{1}}+\left[\mathscr{Y}_{y}-\mathscr{X}_{x}\right] y_{1} F_{y_{1}}+\left[-\mathscr{X}_{y}\right]\left(y_{1}\right)^{2} F_{y_{1}} .
\end{aligned}\right.
$$

Developing $F=\sum_{k \geqslant 0}\left(y_{1}\right)^{k} F_{k}(x, y)$, we may obtain equations (4.19).

## §5. EXAMPLES

5.1. Second order ordinary differential equation. Pursuing the study of $\left(\mathscr{E}_{1}\right)$, according to Section 7 below, we may assume that $F=\mathrm{O}\left(y_{x}\right)$, or equivalently $F(x, y, 0) \equiv 0$.

Convention 5.2. The letters $\mathbf{R}$ will denote various functions of $\left(x, y, y_{1}\right)$, changing with the context. Similarly, $r=r(x, y)$, excluding the pure jet variable $y_{1}$. Hence, symbolically:

$$
\begin{equation*}
\mathrm{R}=\mathrm{r}+y_{1} \mathrm{r}+\left(y_{1}\right)^{2} \mathrm{r}+\left(y_{1}\right)^{3} \mathrm{r}+\cdots . \tag{5.3}
\end{equation*}
$$

So the skeleton is

$$
\begin{equation*}
y_{2}=F\left(x, y, y_{1}\right)=y_{1} \mathrm{R}=y_{1} \mathrm{r}+\left(y_{1}\right)^{2} \mathrm{r}+\left(y_{1}\right)^{3} \mathrm{r}+\cdots . \tag{5.4}
\end{equation*}
$$

Applying $\mathscr{L}^{(2)}$, see (2.3)(II) for its expression, we get:

$$
\begin{equation*}
0=-\mathbf{Y}_{2}+\mathscr{X} F_{x}+\mathscr{Y} F_{y}+\mathrm{Y}_{1} F_{y_{1}} . \tag{5.5}
\end{equation*}
$$

Observe that $F_{x}=\left(y_{1} \mathrm{R}\right)_{x}=\mathrm{r} y_{1}+\mathrm{r}\left(y_{1}\right)^{2}+\cdots$ and similarly for $F_{y}$, but that $\left(y_{1} R\right)_{y_{1}}=r+r y_{1}+r\left(y_{1}\right)^{2}+\cdots$. Inserting above $Y_{1}, Y_{2}$ given by (2.6)(II), replacing $y_{2}$ by $y_{1} \mathrm{R}$ and computing $\bmod \left(y_{1}\right)^{4}$, we get:

$$
\begin{align*}
0 \equiv & -\mathscr{Y}_{x x}+\left[-2 \mathscr{Y}_{x y}+\mathscr{X}_{x x}\right] y_{1}+\left[-\mathscr{Y}_{y y}+2 \mathscr{X}_{x y}\right]\left(y_{1}\right)^{2}+\left[\mathscr{X}_{y y}\right]\left(y_{1}\right)^{3}+  \tag{5.6}\\
& +\left[-\mathscr{Y}_{y}+2 \mathscr{X}_{x}\right]\left(y_{1} \mathrm{r}+\left(y_{1}\right)^{2} \mathrm{r}+\left(y_{1}\right)^{3} \mathrm{r}\right)+\left[3 \mathscr{X}_{y}\right]\left(\left(y_{1}\right)^{2} \mathrm{r}+\left(y_{1}\right)^{3} \mathrm{r}\right)+ \\
& +[\mathscr{X}]\left(y_{1} \mathrm{r}+\left(y_{1}\right)^{2} \mathrm{r}+\left(y_{1}\right)^{3} \mathrm{r}\right)+[\mathscr{Y}]\left(y_{1} \mathrm{r}+\left(y_{1}\right)^{2} \mathrm{r}+\left(y_{1}\right)^{3} \mathrm{r}\right)+ \\
& +\left[\mathscr{Y}_{x}\right]\left(\mathrm{r}+y_{1} \mathrm{r}+\left(y_{1}\right)^{2} \mathrm{r}+\left(y_{1}\right)^{3} \mathrm{r}\right)+ \\
& +\left[\mathscr{Y}_{y}-\mathscr{X}_{x}\right]\left(y_{1} \mathrm{r}+\left(y_{1}\right)^{2} \mathrm{r}+\left(y_{1}\right)^{3} \mathrm{r}\right)+\left[-\mathscr{X}_{y}\right]\left(\left(y_{1}\right)^{2} \mathrm{r}+\left(y_{1}\right)^{3} \mathrm{r}\right) .
\end{align*}
$$

We gather the powers cst., $y_{1},\left(y_{1}\right)^{2}$ and $\left(y_{1}\right)^{3}$, equating their coefficients to 0 :

$$
\begin{align*}
& 0=-\mathscr{Y}_{x x}+\mathrm{P}\left(\mathscr{Y}_{x}\right), \\
& 0=-2 \mathscr{Y}_{x y}+\mathscr{X}_{x x}+\mathrm{P}\left(\mathscr{Y}_{y}, \mathscr{X}_{x}, \mathscr{X}, \mathscr{Y}, \mathscr{Y}_{x}\right), \\
& 0=-\mathscr{Y}_{y y}+2 \mathscr{X}_{x y}+\mathrm{P}\left(\mathscr{Y}_{y}, \mathscr{X}_{x}, \mathscr{X}_{y}, \mathscr{X}, \mathscr{Y}, \mathscr{Y}_{x}\right),  \tag{5.7}\\
& 0=\mathscr{X}_{y y}+\mathrm{P}\left(\mathscr{Y}_{y}, \mathscr{X}_{x}, \mathscr{X}_{y}, \mathscr{X}^{\prime}, \mathscr{Y}_{x}, \mathscr{Y}_{x}\right)
\end{align*}
$$

Convention 5.8. The letter P will denote various linear combinations of some precise partial derivatives of $\mathscr{X}, \mathscr{Y}$ which have analytic coefficients in $(x, y)$.

By cross-differentiations and substitutions in the above system, all third, fourth, fifth, etc. order derivatives of $\mathscr{X}, \mathscr{Y}$ may be expressed as $\mathrm{P}\left(\mathscr{X}, \mathscr{Y}_{1}, \mathscr{X}_{x}, \mathscr{X}_{y}, \mathscr{Y}_{x}, \mathscr{\mathscr { Y }}_{y}, \mathscr{\mathscr { Y }}_{x y}, \mathscr{Y}_{y y}\right)$.

Proposition 5.9. An infinitesimal Lie symmetry $\mathscr{X} \frac{\partial}{\partial x}+\mathscr{Y} \frac{\partial}{\partial y}$ of $\left(\mathscr{E}_{1}\right)$ is uniquely determined by the eight initial Taylor coefficients:

$$
\begin{equation*}
\mathscr{X}(0), \mathscr{Y}_{(0)}, \mathscr{X}_{x}(0), \mathscr{X}_{y}(0), \mathscr{Y}_{x}(0), \mathscr{Y}_{y}(0), \mathscr{Y}_{x y}(0), \mathscr{Y}_{y y}(0) . \tag{5.10}
\end{equation*}
$$

The bound $\operatorname{dim} \mathfrak{S Y M}\left(\mathscr{E}_{1}\right) \leqslant 8$ is attained with $F=0$, whence all $\mathrm{P}=0$ and

$$
\begin{cases}A:=\partial_{y}, & E:=y \partial_{y},  \tag{5.11}\\ B:=\partial_{x}, & F:=y \partial_{x}, \\ C:=x \partial_{y}, & G:=x x \partial_{x}+x y \partial_{y} \\ D:=x \partial_{x}, & H:=x y \partial_{x}+y y \partial_{y}\end{cases}
$$

are infinitesimal generators of the group $\operatorname{PGL}_{3}(\mathbb{K})=\operatorname{Aut}\left(P_{2}(\mathbb{K})\right)$ of projective transformations

$$
\begin{equation*}
(x, y) \mapsto\left(\frac{\alpha x+\beta y+\gamma}{\lambda x+\mu y+\nu}, \frac{\delta x+\eta y+\epsilon}{\lambda x+\mu y+\nu},\right) \tag{5.12}
\end{equation*}
$$

stabilizing the collections of all affine lines of $\mathbb{K}^{2}$, namely the solutions of the model equation $y_{x x}=0$. The model Lie algebra $\operatorname{pgl}_{3}(\mathbb{K}) \simeq \mathfrak{s l}_{3}(\mathbb{K})$ is simple.

Theorem 5.13. The bound $\operatorname{dim} \mathfrak{S M M}\left(\mathscr{E}_{1}\right) \leqslant 8$ is attained if and only if $\left(\mathscr{E}_{1}\right)$ is equivalent, through a diffeomorphism $(x, y) \mapsto(X, Y)$, to $Y_{X X}=0$.

Proof. The statement is well known ([Lie1883, EL1890, Tr1896, Se1931, Ca1932a, Ol1986, HK1989, Ib1992, Ol1995, Sh1997, Su2001, N2003, Me2004]). We provide a (new?) proof which has the advantage to enjoy direct generalizations to all PDE systems whose model Lie algebras are semisimple, for instance $\left(\mathscr{E}_{2}\right),\left(\mathscr{E}_{3}\right)$ and $\left(\mathscr{E}_{5}\right)$.

The Lie brackets between the eight generators (5.11) are:

|  | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $A$ | 0 | 0 | 0 | 0 | $A$ | $B$ | $C$ | $D+2 E$ |
| $B$ | 0 | 0 | $A$ | $B$ | 0 | 0 | $E+2 D$ | $F$ |
| $C$ | 0 | $-A$ | 0 | $-C$ | $C$ | $D-E$ | 0 | $G$ |
| $D$ | 0 | $-B$ | $C$ | 0 | 0 | $-F$ | $G$ | 0 |
| $E$ | $-A$ | 0 | $-C$ | 0 | 0 | $F$ | 0 | $H$ |
| $F$ | $-B$ | 0 | $-D+E$ | $F$ | $-F$ | 0 | $H$ | 0 |
| $G$ | $-C$ | $-E-2 D$ | 0 | $-G$ | 0 | $H$ | 0 | 0 |
| $H$ | $-D-2 E$ | $-F$ | $-G$ | 0 | $-H$ | 0 | 0 | 0 |

Table 2.

Assuming that $\operatorname{dim} \mathfrak{S Y M}\left(\mathscr{E}_{1}\right)=8$, taking account of (5.7), after making some linear combinations, there must exist eight generators of the form

$$
\begin{cases}A^{\prime}:=\partial_{y}+\mathrm{O}(1), & E^{\prime}:=y \partial_{y}+\mathrm{O}(2),  \tag{5.14}\\ B^{\prime}:=\partial_{x}+\mathrm{O}(1), & F^{\prime}:=y \partial_{x}+\mathrm{O}(2), \\ C^{\prime}:=x \partial_{y}+\mathrm{O}(2), & G^{\prime}:=x x \partial_{x}+x y \partial_{y}+\mathrm{O}(3), \\ D^{\prime}:=x \partial_{x}+\mathrm{O}(2), & H^{\prime}:=x y \partial_{x}+y y \partial_{y}+\mathrm{O}(3)\end{cases}
$$

To insure that the Lie brackets between these vector fields are small perturbations of the model ones, we can in advance replace $(x, y)$ by $(\varepsilon x, \varepsilon y)$, so that $y_{x x}=\varepsilon F\left(\varepsilon x, \varepsilon y, y_{x}\right)$ is an $\mathrm{O}(\varepsilon)$, hence all the remainders $\mathrm{O}(1), \mathrm{O}(2)$ and $\mathrm{O}(3)$ above are also $\mathrm{O}(\varepsilon)$. It follows that the structure constants for $A^{\prime}, \ldots, H^{\prime}$ are $\varepsilon$-close to those of Table 2.

Theorem 5.15. ([OV1994]) Every semisimple Lie algebra over $\mathbb{R}$ or $\mathbb{C}$ is rigid: small deformations of the structure constants just give isomorphic Lie algebras.

Consequently, there exists a change of basis close to the identity leading to new generators $A^{\prime \prime}, B^{\prime \prime}, \ldots, G^{\prime \prime}, H^{\prime \prime}$ having exactly the same structure constants as in Table 2. Then $A^{\prime \prime}(0)$ and $B^{\prime \prime}(0)$ are still linearly independent. Since $\left[A^{\prime \prime}, B^{\prime \prime}\right]=[A, B]=0$, there exist local coordinates $(X, Y)$ centered at 0 in which $A^{\prime \prime}=\partial_{X}$ and $B^{\prime \prime}=\partial_{Y}$. Since $\left[A^{\prime \prime}, C^{\prime \prime}\right]=[A, C]=0$ and $\left[B^{\prime \prime}, C^{\prime \prime}\right]=[B, C]=A$, it follows that $C^{\prime \prime}=X \partial_{Y}$. The tangency to $0=-Y_{2}+F\left(X, Y, Y_{1}\right)($ with $F(0)=0)$ of $\left(\partial_{X}\right)^{(2)}=\partial_{X}$, of $\left(\partial_{Y}\right)^{(2)}=\partial_{Y}$ and of $\left(X \partial_{Y}\right)^{(2)}=X \partial_{Y}+\partial_{Y_{1}}$ yields $F=0$.

Open question 5.16. Does this proof generalize to $y_{x^{\kappa+1}}=$ $F\left(x, y, y_{x}, \ldots, y_{x^{\kappa}}\right)$ ?
5.17. Complete system of second order. We now summarize a generalization to $\left(\mathscr{E}_{2}\right)$. According to Section 7 below, one may assume that the submanifold of solutions is $y=b+\sum_{i=1}^{n} a^{i}\left[x^{i}+\mathrm{O}\left(|x|^{2}\right)+\mathrm{O}(a)+\mathrm{O}(b)\right]$, whence $y_{x^{i_{1} x^{i_{2}}}}=F_{i_{1}, i_{2}}\left(x^{i}, y, y_{x^{k}}\right)$ with $F(x, y, 0) \equiv 0$. Applying to the skeleton $0=-y_{i_{1}, i_{2}}+F_{i_{1}, i_{2}}\left(x^{i}, y, y_{k}\right)$ a second prolongation $\mathscr{L}^{(2)}$ having coefficients $\mathbf{Y}_{i_{1}}$ given by (3.9)(II) and $\mathbf{Y}_{i_{1}, i_{2}}$ given by (3.20)(II), we get

$$
\begin{equation*}
0=-\mathrm{Y}_{i_{1}, i_{2}}+\sum_{k=1}^{n}\left[\mathscr{X}^{k}\right] \frac{\partial F_{i_{1}, i_{2}}}{\partial x^{k}}+[\mathscr{Y}] \frac{\partial F_{i_{1}, i_{2}}}{\partial y}+\sum_{k=1}^{n}\left[\mathbf{Y}_{k}\right] \frac{\partial F_{i_{1}, i_{2}}}{\partial y_{k}} \tag{5.18}
\end{equation*}
$$

Replacing $y_{i_{1}, i_{2}}$ everywhere by $F_{i_{1}, i_{2}}=y_{1} \mathrm{R}+\cdots+y_{n} \mathrm{R}$, developping in powers of the pure jet variables $y_{l}$ and picking the coefficients of cst., of $y_{k}$,
of $\left(y_{k}\right)^{2}$ and of $\left(y_{k}\right)^{3}$, we get the linear system

$$
\left\{\begin{align*}
\mathscr{Y}_{x^{i_{1}} x^{i_{2}}} & =\mathrm{P}\left(\mathscr{Y}_{x^{l}}\right)  \tag{5.19}\\
\delta_{i_{1}}^{k} \mathscr{Y}_{x^{i_{2}} y}+\delta_{i_{2}}^{k} \mathscr{Y}_{x^{i_{1} y}}-\mathscr{X}_{x^{i_{1}} x^{i_{2}}}^{k} & =\mathrm{P}\left(\mathscr{Y}_{y}, \mathscr{X}_{x^{l_{1}}}^{l_{2}}, \mathscr{X}^{l}, \mathscr{Y}, \mathscr{Y}_{x^{l}}\right) \\
\delta_{i_{1}, i_{2}}^{k, k} \mathscr{Y}_{y y}-\delta_{i_{1}}^{k} \mathscr{X}_{x^{i_{2} y}}^{k}-\delta_{i_{2}}^{k} \mathscr{X}_{x^{i_{1} y}}^{k} & =\mathrm{P}\left(\mathscr{Y}_{y}, \mathscr{X}_{x^{l_{1}}}^{l_{1}}, \mathscr{X}_{y}^{l}, \mathscr{X}^{l}, \mathscr{Y}, \mathscr{Y}_{x^{l}}\right) \\
\delta_{i_{1}, i_{2}}^{k, k} \mathscr{X}_{y y}^{k} & =\mathrm{P}\left(\mathscr{Y}_{y}, \mathscr{X}_{x^{l_{1}}}^{l_{2}}, \mathscr{X}_{y}^{l}, \mathscr{X}^{l}, \mathscr{Y}, \mathscr{Y}_{x^{l}}\right),
\end{align*}\right.
$$

upon which obvious linear combinations yield a known generalization of Proposition 5.9.
Proposition 5.20. ([Su2001, GM2003a]) An infinitesimal Lie symmetry $\sum_{k=1}^{n} \mathscr{X}^{k} \frac{\partial}{\partial x^{k}}+\mathscr{Y} \frac{\partial}{\partial y}$ is uniquely determined by the $n^{2}+4 n+3$ initial Taylor coefficients:

$$
\begin{equation*}
\mathscr{X}^{l}(0), \mathscr{Y}_{( }(0), \mathscr{X}_{x^{l_{1}}}^{l_{2}}(0), \mathscr{X}_{y}^{l}(0), \mathscr{Y}_{x^{l}}(0), \mathscr{Y}_{y}(0), \mathscr{Y}_{x^{l} y}(0), \mathscr{Y}_{y y}(0) . \tag{5.21}
\end{equation*}
$$

The bound $\operatorname{dim} \mathfrak{S Y M}\left(\mathscr{E}_{2}\right) \leqslant n^{2}+4 n+3$ is attained with $F_{i_{1}, i_{2}}=0$, whence all $\mathrm{P}=0$ and

$$
\left\{\begin{array}{rlrl}
A & :=\partial_{y}, & & E:=y \partial_{y},  \tag{5.22}\\
B_{i} & :=\partial_{x^{i}}, & & F_{i}:=y \partial_{x^{i}}, \\
C_{i} & :=x^{i} \partial_{y}, & & G_{i} \\
:=x^{i}\left(x^{1} \partial_{x^{1}}+\cdots+x^{n} \partial_{x^{n}}+y \partial_{y}\right)+x y \partial_{y}, \\
D_{i, k} & :=x^{i} \partial_{x^{k}}, & & H
\end{array}:=y\left(x^{1} \partial_{x^{1}}+\cdots+x^{n} \partial_{x^{n}}+y \partial_{y}\right) .\right.
$$

are infinitesimal generators of the group $\operatorname{PGL}_{n+2}(\mathbb{K})=\operatorname{Aut}\left(P_{n+1}(\mathbb{K})\right)$ of projective transformations
$(x, y) \mapsto\left(\frac{\alpha_{1} x^{1}+\cdots+\alpha_{n} x^{n}+\beta y+\gamma}{\lambda_{1} x^{1}+\cdots+\lambda_{n} x^{n}+\mu y+\nu}, \frac{\delta_{1} x^{1}+\cdots+\delta_{n} x^{n}+\eta y+\epsilon}{\lambda_{1} x^{1}+\cdots+\lambda_{n} x^{n}+\mu y+\nu},\right)$
stabilizing the collections of all affine planes of $\mathbb{K}^{n+1}$, namely the solutions of the model equation $y_{x^{i_{1}} x^{i_{2}}}=0$. The model Lie algebra $\mathfrak{p g l}_{n+2}(\mathbb{K}) \simeq$ $\mathfrak{s l}_{n+2}(\mathbb{K})$ is simple, hence rigid.
Theorem 5.24. The bound $\operatorname{dim} \mathfrak{S M M}\left(\mathscr{E}_{2}\right) \leqslant n^{2}+4 n+3$ is attained if and only if $\left(\mathscr{E}_{2}\right)$ is equivalent, through a diffeomorphism $\left(x^{i}, y\right) \mapsto\left(X^{k}, Y\right)$, to $Y_{X^{k_{1}} X^{k_{2}}}=0$.

The proof, similar to that of Theorem 5.13, is skipped.
The study of $\left(\mathscr{E}_{3}\right)$ also leads to the model algebra $\mathfrak{p g l}_{n+2}(\mathbb{K}) \simeq \mathfrak{s l}_{n+2}(\mathbb{K})$ and an analog to Theorem 5.13 holds. Details are similar.

## §6. Transfer of Lie symmetries to the parameter space

6.1. Stabilization of foliations. As announced in $\S 2.38$, we now transfer the theory of Lie symmetries to submanifolds of solutions.

Restarting from §4.1, let $\varphi$ a Lie symmetry of $(\mathscr{E})$, namely $\varphi_{\Delta_{\mathscr{E}}}$ stabilizes $\mathrm{F}_{\Delta_{\mathscr{E}}}$. The diffeomorphism $A$ defined by (2.9) transforms $\mathrm{F}_{\mathrm{v}}$ to $\mathrm{F}_{\Delta_{\mathscr{E}}}$. Conjugating, we get the self-transformation $\mathrm{A}^{-1} \circ \varphi_{\Delta_{\mathbb{E}}} \circ \mathrm{A}$ of the ( $x, a, b$ )-space that must stabilize also the foliation $F_{v}$. Equivalently, it must have expression:

$$
\begin{equation*}
\left[\mathrm{A}^{-1} \circ \varphi_{\Delta_{g}} \circ \mathrm{~A}\right](x, a, b)=(\theta(x, a, b), f(a, b), g(a, b)) \in \mathbb{K}^{n} \times \mathbb{K}^{p} \times \mathbb{K}^{m} \tag{6.2}
\end{equation*}
$$

where, importantly, the last two components are independent of the coordinate $x$, because the leaves of $\mathrm{F}_{\mathrm{v}}$ are just $\{a=\operatorname{cst}$., $b=\operatorname{cst}\}$.

Lemma 6.3. To every Lie symmetry $\varphi$ of $(\mathscr{E})$, there corresponds a transformation of the parameters

$$
\begin{equation*}
(a, b) \longmapsto(f(a, b), g(a, b))=: h(a, b) \tag{6.4}
\end{equation*}
$$

meaning that $\varphi$ transforms the local solution $y_{a, b}(x):=\Pi(x, a, b)$ to the local solution $y_{h(a, b)}(x)=\Pi(x, h(a, b))$.

Unfortunately, the expression of $\mathrm{A}^{-1} \circ \varphi_{\Delta_{\mathscr{E}}} \circ \mathrm{A}$ does not clearly show that $f$ and $g$ are independent of $x$. Indeed, reminding the expressions of A and of $\Phi$, we have:
$\varphi_{\Delta_{\mathscr{E}}} \circ \mathrm{A}(x, a, b)=\left(\varphi(x, \Pi(x, a, b)), \Phi_{\beta(q)}^{j(q)}\left(x^{i_{1}}, \Pi^{j_{1}}(x, a, b), \Pi_{x^{\beta\left(q_{1}\right)}}^{j\left(q_{1}\right)}(x, a, b)\right)\right)$.
To compose with $\mathrm{A}^{-1}$ whose expression is given by (2.21), it is useful to split $\varphi=(\phi, \psi) \in \mathbb{K}^{n} \times \mathbb{K}^{m}$, so above we write

$$
\begin{equation*}
\varphi(x, \Pi(x, a, b))=(\phi(x, \Pi(x, a, b)), \psi(x, \Pi(x, a, b))) \tag{6.6}
\end{equation*}
$$

and finally, droping the arguments:

$$
\begin{equation*}
\left[\mathrm{A}^{-1} \circ \varphi_{\Delta_{\mathscr{E}}} \circ \mathrm{A}\right](x, a, b)=\left(\phi^{i}, A^{q}\left(\phi^{i_{1}}, \psi^{j_{1}}, \Phi_{\beta\left(q_{1}\right)}^{j\left(q_{1}\right)}\right), B^{j}\left(\phi^{i_{1}}, \psi^{j_{1}}, \Phi_{\beta\left(q_{1}\right)}^{j\left(q_{1}\right)}\right)\right) . \tag{6.7}
\end{equation*}
$$

In case $(\mathscr{E})=\left(\mathscr{E}_{1}\right)$, is an exercise to verify by computations that the $A^{q}(\cdot)$ and $B^{j}(\cdot)$ are independent of $x$. In general however, the explicit expression of $\Phi_{i_{1}, \ldots, i_{\lambda}}^{j}$ is unknown. Unfortunately also, nothing shows how $(f(a, b), g(a, b))$ is uniquely associated to $\varphi(x, y)$. Further explanations are needed.
6.8. Determination of parameter transformations. At first, we state a geometric reformulation of the preceding lemma.

Lemma 6.9. Every Lie symmetry $(x, y) \mapsto \varphi(x, y)$ of $(\mathscr{E})$ induces a local $\mathbb{K}$-analytic diffeomorphism

$$
\begin{equation*}
(x, y, a, b) \longmapsto(\varphi(x, y), h(a, b)) \tag{6.10}
\end{equation*}
$$

of $\mathbb{K}_{x}^{n} \times \mathbb{K}_{y}^{m} \times \mathbb{K}_{a}^{p} \times \mathbb{K}_{b}^{m}$ that maps to itself the associated submanifold of solutions

$$
\begin{equation*}
\mathscr{M}_{\mathscr{E}}=\{(x, y, a, b): y=\Pi(x, a, b)\} . \tag{6.11}
\end{equation*}
$$

Proof. In fact, we know that the $n$-dimensional leaf $\{(x, \Pi(x, a, b)): x \in$ $\left.\mathbb{K}^{n}\right\}$ is sent $\left\{(x, \Pi(x, h(a, b))): x \in \mathbb{K}^{n}\right\}$.

Equivalently, setting $c:=(a, b)$ and writing $(\varphi, h)=(\phi, \psi, h)$, we have $\psi=\Pi(\phi, h)$ when $y=\Pi(x, c)$, namely

$$
\begin{equation*}
\psi(x, \Pi(x, c)) \equiv \Pi(\phi(x, \Pi(x, c)), h(c)) \tag{6.12}
\end{equation*}
$$

Proposition 6.13. There exists a universal rational map $\widehat{\mathrm{H}}$ such that

$$
\begin{equation*}
h(c) \equiv \widehat{\mathrm{H}}\left(J_{x, a, b}^{\kappa+1} \Pi(x, c), J_{x, y}^{\kappa} \varphi(x, \Pi(x, c))\right) \tag{6.14}
\end{equation*}
$$

This shows unique determination of $h$ from $\varphi$, given $(\mathscr{E})$ or equivalently, given $\Pi$.

Proof. Differentiating a function $\chi(x, \Pi(x, c))$ with respect to $x^{k}, k=$ $1, \ldots, n$, corresponds to applying to $\chi$ the vector field

$$
\begin{equation*}
\mathrm{L}_{k}:=\frac{\partial}{\partial x_{k}}+\sum_{j=1}^{m} \frac{\partial \Pi^{j}}{\partial x_{k}}(x, c) \frac{\partial}{\partial y^{j}}, \quad k=1, \ldots, n . \tag{6.15}
\end{equation*}
$$

Thus, applying $\mathrm{L}_{k}$ to the $m$ scalar equations (6.12), we get

$$
\begin{equation*}
\mathrm{L}_{k} \psi^{j}=\sum_{l=1}^{n} \frac{\partial \Pi^{j}}{\partial x^{l}} \mathrm{~L}_{k} \phi^{l}, \tag{6.16}
\end{equation*}
$$

for $1 \leqslant k \leqslant n$ and $1 \leqslant j \leqslant m$. It follows from the assumption that $\varphi$ is a local diffeomorphism that $\operatorname{det}\left(\mathrm{L}_{k} \phi^{l}(0)\right)_{1 \leqslant k \leqslant n}^{1 \leqslant l \leqslant n} \neq 0$ also. So we may solve the first derivatives $\Pi_{x}$ above: there exist universal polynomials $\mathrm{S}_{l}^{j}$ such that

$$
\begin{equation*}
\frac{\partial \Pi^{j}}{\partial x^{l}}=\frac{\mathrm{S}_{l}^{j}\left(\left\{\mathrm{~L}_{k^{\prime}} \varphi^{i^{\prime}}\right\}_{1 \leqslant k^{\prime} \leqslant n}^{1 \leqslant i^{\prime} \leqslant n+m}\right)}{\operatorname{det}\left(\mathrm{L}_{k^{\prime}} \phi^{\prime}\right)_{\substack{1 \leqslant l^{\prime} \leqslant n \\ 1 \leqslant k^{\prime} \leqslant n}}^{1 l^{\prime}}} \tag{6.17}
\end{equation*}
$$

Again, we apply the $L_{k}$ to these equations, getting, thanks to the chain rule:

$$
\begin{equation*}
\sum_{l_{2}=1}^{n} \frac{\partial^{2} \Pi^{j}}{\partial x^{l_{1}} x^{l_{2}}} \mathrm{~L}_{k} \phi^{l_{2}}=\frac{\mathrm{R}_{l_{1}, k}^{j}\left(\left\{\mathrm{~L}_{k_{1}^{\prime}} \mathrm{L}_{k_{2}^{\prime}} \varphi^{i^{\prime}}\right\}_{1 \leqslant k_{1}^{\prime}, k_{2}^{\prime} \leqslant n}^{1 \leqslant i^{\prime} \leqslant n+m}\right)}{\left[\operatorname{det}\left(\mathrm{L}_{k^{\prime}} \phi^{l^{\prime}}\right)_{1 \leqslant l^{\prime} \leqslant n}^{1 \leqslant l^{\prime} \leqslant n}\right]^{2}} \tag{6.18}
\end{equation*}
$$

Here, $\mathrm{R}_{l_{1}, k}^{j}$ are universal polynomials. Solving the second derivatives $\Pi_{x^{l_{1} l^{2}}}^{j}$, we get

$$
\begin{equation*}
\frac{\partial^{2} \Pi^{j}}{\partial x^{l_{1}} x^{l_{2}}}=\frac{\mathrm{S}_{l_{1}, l_{2}}^{j}\left(\left\{\mathrm{~L}_{k_{1}^{\prime}} \mathrm{L}_{k_{2}^{\prime}} \varphi^{i^{\prime}}\right\}_{1 \leqslant k_{1}^{\prime}, k_{2}^{\prime} \leqslant n}^{1 \leqslant i^{\prime} \leqslant n+m}\right)}{\left[\operatorname{det}\left(\mathrm{L}_{k^{\prime}} \phi^{\prime}\right)_{1 \leqslant k^{\prime} \leqslant n}^{1 \leqslant l^{\prime} \leqslant n}\right]^{3}} \tag{6.19}
\end{equation*}
$$

By induction, for every $\beta \in \mathbb{N}^{n}$ :

$$
\begin{equation*}
\frac{\partial^{|\beta|} \Pi^{j}}{\partial x^{\beta}}=\frac{\mathrm{S}_{\beta}^{j}\left(\left\{\mathrm{~L}^{\beta^{\prime}} \varphi^{i^{\prime}}\right\}_{\left|\beta^{\prime}\right| \leqslant|\beta|}^{1 \leqslant i^{\prime} \leqslant n+m}\right)}{\left[\operatorname{det}\left(\mathrm{L}_{k^{\prime}} \phi^{\prime^{\prime}}\right)_{1 \leqslant k^{\prime} \leqslant n}^{1 \leqslant l^{\prime} \leqslant n}\right]^{2|\beta|+1}}, \tag{6.20}
\end{equation*}
$$

where $S_{\beta}^{j}$ are universal polynomials. Here, for $\beta^{\prime} \in \mathbb{N}^{n}$, we denote by $L^{\beta^{\prime}}$ the derivation of order $\left|\beta^{\prime}\right|$ defined by $\left(L_{1}\right)^{\beta_{1}^{\prime}} \cdots\left(L_{n}\right)^{\beta_{n}^{\prime}}$.

Next, thanks to the assumption that $\mathscr{M}$ is solvable with respect to the parameters, there exist integers $j(1), \ldots, j(p)$ with $1 \leqslant j(q) \leqslant m$ and multiindices $\beta(1), \ldots, \beta(p) \in \mathbb{N}^{n}$ with $|\beta(q)| \geqslant 1$ and $\max _{1 \leqslant q \leqslant p}|\beta(q)|=\kappa$ such that the local $\mathbb{K}$-analytic map

$$
\begin{equation*}
\mathbb{K}^{p+m} \ni c \longmapsto\left(\left(\Pi^{j}(0, c)\right)^{1 \leqslant j \leqslant m},\left(\frac{\partial^{|\beta(q)|} \Pi^{j(q)}}{\partial x^{\beta(q)}}(0, c)\right)_{1 \leqslant q \leqslant p}\right) \in \mathbb{K}^{p+m} \tag{6.21}
\end{equation*}
$$

has rank $p+m$ at $c=0$. We then consider in (6.20) only the $(p+m)$ equations written for $(j, 0),(j(q), \beta(q))$ and we solve $h(c)$ by means of the analytic implicit function theorem:

$$
\begin{equation*}
h=\widehat{H}\left(\phi, \frac{\mathrm{~S}_{\beta(1)}^{j(1)}\left(\left\{\mathrm{L}^{\beta^{\prime}} \varphi^{i^{\prime}}\right\}_{\left|\beta^{\prime}\right| \leqslant|\beta(1)|}^{1 \leqslant i^{\prime} \leqslant n+m}\right)}{\operatorname{det}\left[\left(\mathrm{L}_{k^{\prime}} \phi^{\phi^{\prime}}\right)_{1 \leqslant k^{\prime} \leqslant n}^{1 \leqslant l^{\prime} \leqslant n}\right]^{2|\beta(1)|+1}}, \ldots, \frac{\mathrm{~S}_{\beta(p)}^{j(p)}\left(\left\{\mathrm{L}^{\beta^{\prime}} \varphi^{i^{\prime}}\right\}_{\left|\beta^{\prime}\right| \leqslant|\beta(p)|}^{1 \leqslant i^{\prime} \leqslant n+m}\right)}{\operatorname{det}\left[\left(\mathrm{L}_{k^{\prime}} \phi^{l^{\prime}}\right)_{1 \leqslant l^{\prime} \leqslant n}^{1 \leqslant l^{\prime} \leqslant n}\right]^{2|\beta(p)|+1}}\right) . \tag{6.22}
\end{equation*}
$$

Finally, by developping every derivative $\mathrm{L}^{\beta^{\prime}} \varphi^{i^{\prime}}$ (including $\mathrm{L}_{k^{\prime}} \phi^{l^{\prime}}$ as a special case), taking account of the fact that the coefficients of the $\mathrm{L}_{k^{\prime}}$ depend directly on $\Pi$, we get some universal polynomial $\mathrm{P}_{\beta^{\prime}}\left(J_{x}^{\left|\beta^{\prime}\right|+1} \Pi, J_{x, y}^{\left|\beta^{\prime}\right|} \varphi^{i^{\prime}}\right)$. Inserting above, we get the map $\widehat{H}$.
6.23. Pseudogroup of twin transformations. The previous considerations lead to introducing the following.

Definition 6.24. $B y G_{v, p}$, we denote the infinite-dimensional (pseudo)group of local $\mathbb{K}$-analytic diffeomorphisms

$$
\begin{equation*}
(x, y, a, b) \longmapsto(\varphi(x, y), h(a, b)) \tag{6.25}
\end{equation*}
$$

that respect the separation between the variables and the parameters.

A converse to Lemma 6.3 holds.
Lemma 6.26. Let $\mathscr{M}$ be a submanifold $y=\Pi(x, a, b)$ that is solvable with respect to the parameters $(a, b)$. If a local $\mathbb{K}$-analytic diffeomorphism $(x, y, a, b) \longmapsto(\varphi(x, y), h(a, b))$ of $\mathbb{K}_{x}^{n} \times \mathbb{K}_{y}^{m} \times \mathbb{K}_{a}^{p} \times \mathbb{K}_{b}^{m}$ belonging to $\mathrm{G}_{\mathrm{v}, \mathrm{p}}$ sends $\mathscr{M}$ to $\mathscr{M}$, then $(x, y) \mapsto \varphi(x, y)$ is a Lie symmetry of the PDE system $\mathscr{E}_{\mathscr{M}}$ associated to $\mathscr{M}$.

Proof. In fact, since $(\varphi, h)$ respects the separation of variables and stabilizes $\mathscr{M}$, it respects the fundamental pair of foliations $\left(\mathrm{F}_{\mathrm{v}}, \mathrm{F}_{\mathrm{p}}\right)$, namely $\{(a, b)=$ $\left.\left(a_{0}, b_{0}\right)\right\} \cap \mathscr{M}$ is sent to $\left\{(a, b)=h\left(a_{0}, b_{0}\right)\right\} \cap \mathscr{M}$ and $\left\{(x, y)=\left(x_{0}, y_{0}\right)\right\} \cap$ $\mathscr{M}$ is sent to $\left\{(x, y)=\varphi\left(x_{0}, y_{0}\right)\right\} \cap \mathscr{M}$. Hence $\varphi_{\Delta_{\mathscr{E}_{\mathscr{M}}}}$ also stabilizes $\mathrm{F}_{\Delta_{\mathscr{E}}}$.

Corollary 6.27. Through the one-to-one correspondence $(\mathscr{E}) \longleftrightarrow \mathscr{M}$ of Proposition 2.17, Lie symmetries of $(\mathscr{E})$ correspond to elements of $\mathrm{G}_{\mathrm{v}, \mathrm{p}}$ which stabilize $\mathscr{M}$.

Definition 6.28. Let $\operatorname{Aut}_{\mathrm{v}, \mathrm{p}}(\mathscr{M})$ denote the local (pseudo)group of $(\varphi, h) \in$ $\mathrm{G}_{\mathrm{v}, \mathrm{p}}$ stabilizing $\mathscr{M}$. Let $\operatorname{Lie}(\mathscr{E})$ denote the local (pseudo)group of Lie symmetries of $(\mathscr{E})$.

In summary:

$$
\begin{equation*}
\operatorname{Lie}(\mathscr{E}) \simeq \operatorname{Aut}_{\mathrm{v}, \mathrm{p}}\left(\mathscr{M}_{(\mathscr{E})}\right) \quad \text { and } \quad \operatorname{Aut}_{\mathrm{v}, \mathrm{p}}(\mathscr{M}) \simeq \operatorname{Lie}\left(\mathscr{E}_{\mathscr{M}}\right) . \tag{6.29}
\end{equation*}
$$

6.30. Transfer of infinitesimal Lie symmetries. Let $\mathscr{L} \in \mathfrak{S Y M}(\mathscr{E})$, i.e. $\mathscr{L}_{\Delta_{\mathscr{E}}}$ is tangent to $\Delta_{\mathscr{E}}$. Through the diffeomorphism A, the push-forward of $\mathscr{L}_{\Delta_{\mathscr{E}}}$ must be of the form
$\mathrm{A}_{*}^{-1}\left(\mathscr{L}_{\Delta_{\mathcal{E}}}\right)=\sum_{k=1}^{n} \Theta^{i}(x, a, b) \frac{\partial}{\partial x^{i}}+\sum_{q=1}^{p} \mathscr{F}^{q}(a, b) \frac{\partial}{\partial a^{q}}+\sum_{j=1}^{m} \mathscr{G}^{j}(a, b) \frac{\partial}{\partial b^{j}}$,
where the last two families of $\mathbb{K}$-analytic coefficients $\mathscr{F}^{q}$ and $\mathscr{G}^{j}$ depend only on ( $a, b$ ).
Lemma 6.32. To every infinitesimal symmetry $\mathscr{L}$ of ( $\mathscr{E}$ ), we can associate an infinitesimal symmetry

$$
\begin{equation*}
\mathscr{L}^{*}:=\sum_{q=1}^{p} \mathscr{F}^{q}(a, b) \frac{\partial}{\partial a^{q}}+\sum_{j=1}^{m} \mathscr{G}^{j}(a, b) \frac{\partial}{\partial b^{j}} \tag{6.33}
\end{equation*}
$$

of the space of parameters which tells how the flow of $\mathscr{L}$ acts infinitesimally on the leaves of $\mathrm{F}_{\Delta_{\mathscr{E}}}$. Furthermore, $\mathscr{L}+\mathscr{L}^{*}$ is tangent to the submanifold of solutions $\mathscr{M}_{(\mathscr{E})}$.

Considering the flow of $\mathscr{L}+\mathscr{L}^{*}$ reduces these assertions and the next to the arguments of the preceding paragraphs. So we summarize.

Lemma 6.34. Let $\mathscr{M}$ be a submanifold $y=\Pi(x, a, b)$ that is solvable with respect to the parameters $(a, b)$. If a vector field that respects the separation between variables and parameters, namely of the form
$\mathscr{L}+\mathscr{L}^{*}=\sum_{i=1}^{n} \mathscr{X}^{i}(x, y) \frac{\partial}{\partial x^{i}}+\sum_{j=1}^{m} \mathscr{Y}^{j}(x, y) \frac{\partial}{\partial y^{j}}+\sum_{q=1}^{p} \mathscr{F}^{q}(a, b) \frac{\partial}{\partial a^{q}}+\sum_{j=1}^{m} \mathscr{G}^{j}(a, b) \frac{\partial}{\partial b^{j}}$
is tangent to $\mathscr{M}$, then $\mathscr{L}$ is an infinitesimal Lie symmetry of $\left(\mathscr{E}_{\mathscr{M}}\right)$
Corollary 6.36. Through the one-to-one correspondence $(\mathscr{E}) \longleftrightarrow \mathscr{M}$ of Proposition 2.17, infinitesimal Lie symmetries of $(\mathscr{E})$ correspond to vector fields $\mathscr{L}+\mathscr{L}^{*}$ tangent to $\mathscr{M}$.

Definition 6.37. Let $\mathfrak{S M M}(\mathscr{M})$ denote the Lie algebra of vector fields $\mathscr{L}+$ $\mathscr{L}^{*}$ tangent to $\mathscr{M}$. Let $\mathfrak{S Y M}(\mathscr{E})$ denote the Lie algebra of infinitesimal Lie symmetries of $(\mathscr{E})$.

In summary:
(6.38)
$\mathfrak{S Y M}(\mathscr{E}) \simeq \mathfrak{S Y M}\left(\mathscr{M}_{(\mathscr{E})}\right) \quad$ and $\quad \mathfrak{S Y M}(\mathscr{M}) \simeq \mathfrak{S Y M}\left(\mathscr{E}_{\mathscr{M}}\right)$.
6.39. Dual defining equations. As in $\S 2.10$, let $\mathscr{M} \subset \mathbb{K}_{x}^{n} \times \mathbb{K}_{y}^{m} \times \mathbb{K}_{a}^{p} \times \mathbb{K}_{b}^{m}$ given by $0=-y^{j}+\Pi^{j}(x, a, b)$ and assume if to be solvable with respect to the parameters. In particular, we can solve the $b^{j}$, obtaining dual defining equations

$$
\begin{equation*}
b^{j}=\Pi^{* j}(a, x, y), \quad j=1, \ldots, m \tag{6.40}
\end{equation*}
$$

for some local $\mathbb{K}$-analytic map map $\Pi^{*}=\left(\Pi^{* 1}, \ldots, \Pi^{* m}\right)$ satisfying

$$
\begin{equation*}
b \equiv \Pi^{*}(a, x, \Pi(x, a, b)) \quad \text { and } \quad y \equiv \Pi\left(x, a, \Pi^{*}(a, x, y)\right) \tag{6.41}
\end{equation*}
$$

6.42. An algorithm for the computation of $\mathfrak{S M M}(\mathscr{M})$. The tangency to $\mathscr{M}$ is expressed by applying the vector field (6.35) to $0=-y^{j}+\Pi^{j}(x, a, b)$, which yields:

$$
\begin{align*}
0=-\mathscr{Y}^{j}(x, y)+\sum_{i=1}^{n} \mathscr{X}^{i}(x, y) \Pi_{x^{i}}^{j}(x, a, b) & +\sum_{q=1}^{p} \mathscr{F}^{q}(a, b) \Pi_{a^{q}}^{j}(x, a, b)  \tag{6.43}\\
& +\sum_{l=1}^{m} \mathscr{G}^{l}(a, b) \Pi_{b^{l}}^{j}(x, a, b)
\end{align*}
$$

for $j=1, \ldots, m$ and for $(x, y, a, b) \in \mathscr{M}$. In fact, after replacing the variable $y$ by $\Pi(x, a, b)$, these equations should be interpreted as power series identities in $\mathbb{K}\{x, a, b\}$.

Denote by $\Delta(x, a, b)$ the determinant of the (invertible) matrix $\left(\Pi_{b^{l}}^{j}(x, a, b)\right)_{1 \leqslant l, j \leqslant m}$ and by $D(x, a, b)$ its matrix of cofactors, so that $\Pi_{b}^{-1}=[\Delta]^{-1} D$. Hence we can solve $\mathscr{G}$ from (6.43):
(6.44)

$$
\left\{\begin{aligned}
\mathscr{G}(a, b) \equiv \frac{D(x, a, b)}{\Delta(x, a, b)}[ & \mathscr{Y}(x, \Pi(x, a, b))-\sum_{i=1}^{n} \mathscr{X}^{i}(x, \Pi(x, a, b)) \Pi_{x^{i}}(x, a, b)- \\
& \left.-\sum_{q=1}^{p} \mathscr{F}^{q}(a, b) \Pi_{a^{q}}(x, a, b)\right]
\end{aligned}\right.
$$

Next, we aim to solve the $\mathscr{F}^{q}(a, b)$. Consequently, we gather all the other terms in the brackets as $\Psi_{0}\left(J_{x, a, b}^{1} \Pi, \mathscr{X}, \mathscr{Y}\right)$ :

$$
\begin{equation*}
\mathscr{G}(a, b) \equiv \frac{D(x, a, b)}{\Delta(x, a, b)}\left[-\sum_{q=1}^{p} \mathscr{F}^{q}(a, b) \Pi_{a^{q}}(x, a, b)\right]+\frac{\Psi_{0}\left(J_{x, a, b}^{1} \Pi, \mathscr{X}, \mathscr{Y}\right)}{\Delta(x, a, b)} . \tag{6.45}
\end{equation*}
$$

Here, $\Psi_{0}$ is linear with respect to $(\mathscr{X}, \mathscr{Y})$, with polynomial coefficients of degree one in $J_{x, a, b}^{1} \Pi$.

Next, for $k=1, \ldots, n$, we differentiate this identity with respect to $x_{k}$. Then $\mathscr{G}(a, b)$ disappears and we chase the denominator $\Delta^{2}$ :

$$
\left\{\begin{align*}
0 \equiv[\Delta D][ & \left.-\sum_{q=1}^{p} \mathscr{F}^{q}(a, b) \Pi_{a^{q} x^{k}}(x, a, b)\right]+  \tag{6.46}\\
& +\left[\Delta D_{x_{k}}-\Delta_{x_{k}} D\right]\left[-\sum_{q=1}^{p} \mathscr{F}^{q}(a, b) \Pi_{a^{q}}(x, a, b)\right]+ \\
& +\Psi_{k}\left(J_{x, a, b}^{2} \Pi, J_{x, y}^{1} \mathscr{X}, J_{x, y}^{1} \mathscr{Y}\right) .
\end{align*}\right.
$$

The $\Psi_{k}$ are linear with respect to $\left(J_{x, y}^{1} \mathscr{X}, J_{x, y}^{1} \mathscr{Y}\right)$, with polynomial coefficients in $J_{x, a, b}^{2} \Pi$. Then we further differentiate with respect to $x$ and by induction, for every $\beta \in \mathbb{N}^{n}$, we get:
(6.47)

$$
\left\{\begin{aligned}
0 \equiv & {[\Delta D]\left[-\sum_{q=1}^{p} \mathscr{F}^{q}(a, b) \Pi_{a^{q} x^{\beta}}(x, a, b)\right]+} \\
& +\sum_{\left|\beta_{1}\right|<|\beta|} \mathrm{D}_{\beta, \beta_{1}}\left(J^{\left|\beta_{1}\right|+1} \Pi\right)\left[-\sum_{q=1}^{p} \mathscr{F}^{q}(a, b) \Pi_{a^{q} x^{\beta_{1}}}(x, a, b)\right]+ \\
& +\Psi_{\beta}\left(J_{x, a, b}^{|\beta|+1} \Pi, J_{x, y}^{|\beta|} \mathscr{X}, J_{x, y}^{|\beta|} \mathscr{Y}\right),
\end{aligned}\right.
$$

where the expressions $\mathrm{D}_{\beta, \beta_{1}}$ are certain $m \times m$ matrices with polynomial coefficients in the jet $J_{x, a, b}^{\left|\beta_{1}\right|+1} \Pi$, and where the terms $\Psi_{\beta}\left(J_{x, a, b}^{|\beta|+1} \Pi, J_{x, y}^{|\beta|} \mathscr{X}, J_{x, y}^{|\beta|} \mathscr{Y}\right)$ are linear with respect to $\left(J_{x, y}^{|\beta|} \mathscr{X}, J_{x, y}^{|\beta|} \mathscr{Y}\right)$, with polynomial coefficients in $J_{x, a, b}^{|\beta|+1} \Pi$.

Writing these identity for $(j, \beta)=(j(q), \beta(q)), q=1, \ldots, p$, reminding $\max _{1 \leqslant q \leqslant p}|\beta(q)|=\kappa$, it follows from the assumption of solvability with respect to the parameters (a boring technical check is needed) that we may solve
$\mathscr{F}^{q}(a, b) \equiv \Phi^{q}\left(J_{x, a, b}^{\kappa+1} \Pi(x, a, b), J_{x, y}^{\kappa} \mathscr{X}(x, \Pi(x, a, b)), J_{x, y}^{\kappa} \mathscr{Y}(x, \Pi(x, a, b))\right)$,
for $q=1, \ldots, p$, where each local $\mathbb{K}$-analytic function $\Phi_{q}$ is linear with respect to $\left(J^{\kappa} \mathscr{X}, J^{\kappa} \mathscr{Y}\right)$ and rational with respect to $J^{\kappa+1} \Pi$, with denominator not vanishing at $(x, a, b):=(0,0,0)$.

Pursuing, we differentiate (6.48) with respect to $x^{l}$ for $l=1, \ldots, n$. Then $\mathscr{F}^{q}(a, b)$ disappears and we get:

$$
\begin{equation*}
0 \equiv \Phi_{q, l}\left(J_{x, a, b}^{\kappa+2} \Pi(x, a, b), J_{x, y}^{\kappa+1} \mathscr{X}(x, \Pi(x, a, b)), J_{x, y}^{\kappa+1} \mathscr{Y}(x, \Pi(x, a, b))\right) \tag{6.49}
\end{equation*}
$$

for $1 \leqslant q \leqslant p$ and $1 \leqslant l \leqslant n$. In (6.46), we then replace the functions $\mathscr{F}^{q}$ by their values $\Phi^{q}$ :

$$
\begin{equation*}
0 \equiv \Psi_{k, j}\left(J_{x, a, b}^{\kappa+1} \Pi(x, a, b), J_{x, y}^{\kappa} \mathscr{X}(x, \Pi(x, a, b)), J_{x, y}^{\kappa} \mathscr{Y}(x, \Pi(x, a, b))\right), \tag{6.50}
\end{equation*}
$$

for $1 \leqslant k \leqslant n$ and $1 \leqslant j \leqslant m$. Then we replace the variable $b$ by $\Pi^{*}(a, x, y)$ in the two obtained systems (6.49) and (6.50); taking account of the functional identity $y \equiv \Pi\left(x, a, \Pi^{*}(a, x, y)\right)$ written in (6.41), we get

$$
\left\{\begin{array}{l}
0 \equiv \Phi_{q, l}\left(J_{x, a, b}^{\kappa+2} \Pi\left(x, a, \Pi^{*}(a, x, y)\right), J_{x, y}^{\kappa+1} \mathscr{X}(x, y), J_{x, y}^{\kappa+1} \mathscr{Y}(x, y)\right),  \tag{6.51}\\
0 \equiv \Psi_{k, j}\left(J_{x, a, b}^{\kappa+1} \Pi\left(x, a, \Pi^{*}(a, x, y)\right), J_{x, y}^{\kappa} \mathscr{X}(x, y), J_{x, y}^{\kappa} \mathscr{Y}(x, y)\right) .
\end{array}\right.
$$

Finally, we develope these equations in power series with respect to $a$ :

$$
\left\{\begin{align*}
0 & \equiv \sum_{\gamma \in \mathbb{N}^{p}} a^{\gamma} \Phi_{q, l, \gamma}\left(x, y, J_{x, y}^{\kappa+1} \mathscr{X}(x, y), J_{x, y}^{\kappa+1} \mathscr{Y}(x, y)\right),  \tag{6.52}\\
0 & \equiv \sum_{\gamma \in \mathbb{N}^{p}} a^{\gamma} \Psi_{k, j, \gamma}\left(x, y, J_{x, y}^{\kappa} \mathscr{X}(x, y), J_{x, y}^{\kappa} \mathscr{Y}(x, y)\right),
\end{align*}\right.
$$

where the terms $\Phi_{q, l, \gamma}$ and $\Psi_{k, j, \gamma}$ are linear with respect to the jets of $\mathscr{X}, \mathscr{Y}$.
Proposition 6.53. A vector field (6.35) belongs to $\mathfrak{S Y M}(\mathscr{M})$ if and only if $\mathscr{X}^{i}, \mathscr{Y}^{j}$ satisfy the linear PDE system

$$
\left\{\begin{array}{l}
0 \equiv \Phi_{q, l, \gamma}\left(x, y, J_{x, y}^{\kappa+1} \mathscr{X}(x, y), J_{x, y}^{\kappa+1} \mathscr{Y}(x, y)\right),  \tag{6.54}\\
0 \equiv \Pi_{k, j, \gamma}\left(x, y, J_{x, y}^{\kappa} \mathscr{X}(x, y), J_{x, y}^{\kappa} \mathscr{Y}(x, y)\right),
\end{array}\right.
$$

where $1 \leqslant q \leqslant p, 1 \leqslant l \leqslant n, 1 \leqslant k \leqslant n$ and $\gamma \in \mathbb{N}^{p}$. Then $\mathscr{F}^{q}$ defined by (6.48) and $\mathscr{G}^{j}$ defined by (6.45) are independent of $x$.

This provides a second algorithm, essentially equivalent to Sophus Lie's.
Example 6.55. For $y_{x x}(x)=F\left(x, y(x), y_{x}(x)\right)$, the first line of (6.54) is (the second one is redundant):

$$
\left\{\begin{align*}
0 \equiv & \mathscr{X}\left[-\Pi_{x a} \Pi_{x x x} \Pi_{b}+\Pi_{a} \Pi_{x b} \Pi_{x x x}-\Pi_{x} \Pi_{x x a} \Pi_{x b}+\Pi_{x a} \Pi_{x} \Pi_{x x b}+\right.  \tag{6.56}\\
& \left.+\Pi_{x x a} \Pi_{x x} \Pi_{b}-\Pi_{a} \Pi_{x x b} \Pi_{x x}\right]+ \\
& +\mathscr{Y}^{[ }\left[-\Pi_{x a} \Pi_{x x b}+\Pi_{x x a} \Pi_{x b}\right]+ \\
& +\mathscr{X}_{x}\left[-2 \Pi_{x x} \Pi_{x a} \Pi_{b}+2 \Pi_{x x} \Pi_{a} \Pi_{x b}+\Pi_{x} \Pi_{b} \Pi_{x x a}-\Pi_{x} \Pi_{a} \Pi_{x x b}\right]+ \\
& +\mathscr{Y}_{x}\left[-\Pi_{b} \Pi_{x x a}+\Pi_{a} \Pi_{x x b}\right]+ \\
& +\mathscr{X}_{y}\left[-3 \Pi_{x} \Pi_{x x} \Pi_{x a} \Pi_{b}+3 \Pi_{x} \Pi_{a} \Pi_{x x} \Pi_{x b}+\left(\Pi_{x}\right)^{2} \Pi_{b} \Pi_{x x a}-\left(\Pi_{x}\right)^{2} \Pi_{a} \Pi_{x x b}\right]+ \\
& +\mathscr{Y}_{y}\left[\Pi_{x x} \Pi_{b} \Pi_{x a}-\Pi_{x x} \Pi_{a} \Pi_{x b}-\Pi_{x} \Pi_{b} \Pi_{x x a}+\Pi_{x} \Pi_{a} \Pi_{x x b}\right]+ \\
& +\mathscr{X}_{x x}\left[-\Pi_{x} \Pi_{b} \Pi_{x a}+\Pi_{x} \Pi_{a} \Pi_{x b}\right]+ \\
& +\mathscr{X}_{x y}\left[-2\left(\Pi_{x}\right)^{2} \Pi_{b} \Pi_{x a}+2\left(\Pi_{x}\right)^{2} \Pi_{a} \Pi_{x b}\right]+ \\
& +\mathscr{X}_{y^{2}}\left[-\left(\Pi_{x}\right)^{3} \Pi_{b} \Pi_{x a}+\left(\Pi_{x}\right)^{3} \Pi_{a} \Pi_{x b}\right]+ \\
& +\mathscr{Y}_{x x}\left[\Pi_{b} \Pi_{x a}-\Pi_{a} \Pi_{x b}\right]+ \\
& +\mathscr{Y}_{x y}\left[2 \Pi_{x} \Pi_{b} \Pi_{x a}-2 \Pi_{x} \Pi_{a} \Pi_{x b}\right]+ \\
& +\mathscr{Y}_{y^{2}}\left[\left(\Pi_{x}\right)^{2} \Pi_{b} \Pi_{x a}-\left(\Pi_{x}\right)^{2} \Pi_{a} \Pi_{x b}\right] .
\end{align*}\right.
$$

We observe the similarity with (4.19): the expression is linear in the partial derivatives of $\mathscr{X}, \mathscr{Y}$ of order $\leqslant 2$, but the coefficients in the equation above are more complicated. In fact, after dividing by $-\Pi_{b} \Pi_{x a}+\Pi_{a} \Pi_{x b}$, this equation coincides with (4.21), thanks to $\Pi_{x}=y_{1}$ and to the formulas (2.34) for $F_{x}, F_{y}, F_{y_{1}}$.
6.57. Infinitesimal CR automorphisms of generic submanifolds. If the system $(\mathscr{E})$ is associated to the complexification $\mathscr{M}=(M)^{c}$ of a generic $M \subset \mathbb{C}^{n+m}$ as in §1.16, then $a=(\bar{z})^{c}=\zeta, b=(\bar{w})^{c}=\xi$, and the vector field $\mathscr{L}^{*}$ associated to an infinitesimal Lie symmetry

$$
\begin{equation*}
\mathscr{L}=\sum_{i=1}^{n} \mathscr{X}^{i}(z, w) \frac{\partial}{\partial z^{i}}+\sum_{j=1}^{m} \mathscr{Y}^{j}(z, w) \frac{\partial}{\partial w^{j}} \tag{6.58}
\end{equation*}
$$

of $(\mathscr{E})$ is simply the complexification $\underline{\mathscr{L}}$ of its conjugate $\overline{\mathscr{L}}$, namely

$$
\begin{equation*}
\mathscr{L}^{*}=\underline{\mathscr{L}}=\sum_{i=1}^{n} \overline{\mathscr{X}}^{i}(\zeta, \xi) \frac{\partial}{\partial \zeta^{i}}+\sum_{j=1}^{m} \overline{\mathscr{Y}}^{j}(\zeta, \xi) \frac{\partial}{\partial \xi^{j}} . \tag{6.59}
\end{equation*}
$$

Then the sum $\mathscr{L}+\underline{L}$ is tangent to $\mathscr{M}$ and its flow stabilizes the two invariant foliations, obtained by intersecting $\mathscr{M}$ by $\{(z, w)=$ cst. $\}$ or by $\{(\zeta, \xi)=$ cst. $\}$. In [Me2005a, Me2005b], these two foliations, denoted $\mathscr{F}, \mathscr{F}$, are called (conjugate) Segre foliations, since its leaves are the complexifications of the (conjugate) classical Segre varieties ([Se1931, Pi1975, Pi1978, We1977, DW1980, BJT1985, DF1988, BER1999, Su2001, Su2002, Su2003, GM2003a]) associated to $M$, viewed in its ambient space $\mathbb{C}^{n+m}$. The next definition is also classical ([Be1979, Lo1981, EKV1985, Kr1987, KV1987, Be1988, Vi1990, St1996, Be1997, BER1999, Lo2002, FK2005a, FK2005b]):

Definition 6.60. By $\mathfrak{h o l}(M)$ is meant the Lie algebra of local holomorphic vector fields $\mathscr{L}=\sum_{i=1}^{n} \mathscr{X}^{i}(z, w) \frac{\partial}{\partial z^{i}}+\sum_{j=1}^{m} \mathscr{Y}^{j}(z, w) \frac{\partial}{\partial w^{j}}$ whose real flow $\exp (t \mathscr{L})(z, w)$ induces one-parameter families of local biholomorphic transformations of $\mathbb{C}^{n+m}$ stabilizing $M$. Equivalently,

$$
\begin{equation*}
2 \operatorname{Re} \mathscr{L}=\mathscr{L}+\overline{\mathscr{L}} \tag{6.61}
\end{equation*}
$$

is tangent to $M$. Again equivalently, $\mathscr{L}+\underline{\mathscr{L}}$ is tangent to $\mathscr{M}=M^{c}$.
Then obviously $\mathfrak{h o l}(M)$ is a real Lie algebra.
Theorem 6.62. ([Ca1932a, BER1999, GM2004]) The complexification $\mathfrak{h o l}(M) \otimes \mathbb{C}$ identifies with $\mathfrak{S M M}\left(\mathscr{E}\left(M^{c}\right)\right)$. Furthermore, if $M$ is finitely nondegenerate and minimal at the origin, both are finite-dimensional and $\mathfrak{h o l}(M)$ is totally real in $\mathfrak{S Y M}\left(\mathscr{E}\left(M^{c}\right)\right)$.

The minimality assumption is sometimes presented by saying that the Lie algebra generated by $T^{c} M$ generates $T M$ at the origin ([BER1999]). However, it is more natural to proceed with the fundamental pair of foliations associated to $\mathscr{M}$ ([Me2001, GM2004, Me2005a, Me2005b]). Anticipating Sections 10 and 11 to which the reader is referred, we set.

Definition 6.63. A real analytic generic submanifold $M \subset \mathbb{C}^{n+m}$ is minimal at one of its points $p$ if the fundamental pair of foliations of its complexification $\mathscr{M}$ is covering at $p$ (Definition 10.17).

Further informations may be found in Section 10. We conclude by formulating applications of Theorems 5.13 and 5.24.

Corollary 6.64. The bound $\operatorname{dim} \mathfrak{h o l}(M) \leqslant 8$ for a Levi nondegenerate hypersurface $M \subset \mathbb{C}^{2}$ is attained if and only if it is locally biholomorphic to the sphere $S^{3} \subset \mathbb{C}^{2}$.
Corollary 6.65. The bound $\operatorname{dim} \mathfrak{h o l}(M) \leqslant n^{2}+4 n+3$ for a Levi nondegenerate hypersurface $M \subset \mathbb{C}^{n+1}$ is attained if and only if it is locally biholomorphic to the sphere $S^{2 n+1} \subset \mathbb{C}^{n+1}$.

## §7. EQUIVALENCE PROBLEMS AND NORMAL FORMS

7.1. Equivalences of submanifolds of solutions. As in $\S 3.1$, let ( $\mathscr{E}$ ) and $\left(\mathscr{E}^{\prime}\right)$ be two PDE systems and assume that $\varphi$ transforms ( $\mathscr{E}$ ) to ( $\mathscr{E}^{\prime}$ ). Defining $\mathrm{A}^{\prime}$ similarly as A , it follows that
(7.2) $\mathrm{A}^{\prime-1} \circ \Phi_{\mathscr{E}, \mathscr{E}^{\prime}} \circ \mathrm{A}(x, a, b) \equiv(\theta(x, a, b), f(a, b), g(a, b))=:\left(x^{\prime}, a^{\prime}, b^{\prime}\right)$
transforms $\mathrm{F}_{\mathrm{v}}$ to $\mathrm{F}_{\mathrm{v}}^{\prime}$, hence induces a map $(a, b) \mapsto\left(a^{\prime}, b^{\prime}\right)$. The arguments of Section 6 apply here with minor modifications to provide two fundamental lemmas.

Lemma 7.3. Every equivalence $(x, y) \mapsto\left(x^{\prime}, y^{\prime}\right)$ between to PDE systems $(\mathscr{E})$ and $\left(\mathscr{E}^{\prime}\right)$ comes with an associated transformation $(a, b) \mapsto\left(a^{\prime}, b^{\prime}\right)$ of the parameter spaces such that

$$
\begin{equation*}
(x, y, a, b) \longmapsto\left(x^{\prime}, y^{\prime}, a^{\prime}, b^{\prime}\right) \tag{7.4}
\end{equation*}
$$

is an equivalence between the associated submanifolds of solutions $\mathscr{M}_{(\mathscr{E})} \rightarrow$ $\mathscr{M}_{\left(\mathscr{E}^{\prime}\right)}^{\prime}$.

Conversely, let $\mathscr{M}$ and $\mathscr{M}^{\prime}$ be two submanifolds of $\mathbb{K}_{x}^{n} \times \mathbb{K}_{y}^{m} \times \mathbb{K}_{a}^{p} \times \mathbb{K}_{b}^{m}$ and of $\mathbb{K}_{x^{\prime}}^{n} \times \mathbb{K}_{y^{\prime}}^{m} \times \mathbb{K}_{a^{\prime}}^{p} \times \mathbb{K}_{b^{\prime}}^{m}$ represented by $y=\Pi(x, a, b)$ and by $y^{\prime}=$ $\Pi^{\prime}\left(x^{\prime}, a^{\prime}, b^{\prime}\right)$, in the same dimensions. Assume both are solvable with respect to the parameters.

## Lemma 7.5. Every equivalence

$$
\begin{equation*}
(x, y, a, b) \longmapsto(\varphi(x, y), h(a, b)) \tag{7.6}
\end{equation*}
$$

between $\mathscr{M}$ and $\mathscr{M}^{\prime}$ belonging to $\mathrm{G}_{\mathrm{v}, \mathrm{p}}$ induces by projection the equivalence $(x, y) \mapsto \varphi(x, y)$ between the associated PDE systems $\left(\mathscr{E}_{\mathscr{M}}\right)$ and $\left(\mathscr{E}_{\mathscr{M}^{\prime}}^{\prime}\right)$.
7.7. Classification problems. Consequently, classifying PDE systems under point transformations (Section 3) is equivalent to the following.

Equivalence problem 7.8. Find an algorithm to decide whether two given submanifolds (of solutions) $\mathscr{M}$ and $\mathscr{M}^{\prime}$ are equivalent through an element of $G_{v, p}$.
Classification problem 7.9. Classify submanifolds (of solutions) $\mathscr{M}$, namely provide a complete list of all possible such equations, including their automorphism group Aut $_{\mathrm{v}, \mathrm{p}}(\mathscr{M}) \subset \mathrm{G}_{\mathrm{v}, \mathrm{p}}$.
7.10. Partial normal forms. Both problems above are of high complexity. At least as a preliminary step, it is useful to try to simplify somehow the defining equations of $\mathscr{M}$, by appropriate changes of coordinates belonging to $\mathrm{G}_{\mathrm{v}, \mathrm{p}}$. To begin with, the next lemma holds for $\mathscr{M}$ defined by $y=\Pi(x, a, b)$ with the only assumption that $b \mapsto \Pi(0,0, b)$ has rank $m$ at $b=0$.

Lemma 7.11. ([CM1974, BER1999, Me2005a], [*]) In coordinates $x^{\prime}=$ $\left(x^{\prime 1}, \ldots, x^{\prime n}\right)$ and $y^{\prime}=\left(y^{\prime 1}, \ldots, y^{\prime m}\right)$ an arbitrary submanifold $\mathscr{M}^{\prime}$ defined by $y^{\prime}=\Pi^{\prime}\left(x^{\prime}, a^{\prime}, b^{\prime}\right)$ or dually by $b^{\prime}=\Pi^{* *}\left(a^{\prime}, x^{\prime}, y^{\prime}\right)$ is equivalent to

$$
\begin{equation*}
y=\Pi(x, a, b) \quad \text { or dually to } \quad b=\Pi^{*}(a, x, y) \tag{7.12}
\end{equation*}
$$

with
(7.13)
$\Pi(0, a, b) \equiv \Pi(x, 0, b) \equiv b \quad$ or dually $\quad \Pi^{*}(0, x, y) \equiv \Pi^{*}(a, 0, y) \equiv y$,
namely $\Pi=b+\mathrm{O}(x a)$ and $\Pi^{*}=y+\mathrm{O}(a x)$.
Proof. We develope

$$
\begin{equation*}
y^{\prime}=\Pi^{\prime}\left(0, a^{\prime}, b^{\prime}\right)+\Lambda^{\prime}\left(x^{\prime}\right)+\mathrm{O}\left(x^{\prime} a^{\prime}\right) \tag{7.14}
\end{equation*}
$$

Since $b^{\prime} \mapsto \Pi^{\prime}\left(0, a^{\prime}, b^{\prime}\right)$ has rank $m$ at $b^{\prime}=0$, the coordinate change

$$
\begin{equation*}
b^{\prime \prime}:=\Pi^{\prime}\left(0, a^{\prime}, b^{\prime}\right), \quad a^{\prime \prime}:=a^{\prime}, \quad x^{\prime \prime}:=x^{\prime}, \quad y^{\prime \prime}:=y^{\prime} \tag{7.15}
\end{equation*}
$$

transforms $\mathscr{M}^{\prime}$ to $\mathscr{M}^{\prime \prime}$ defined by

$$
\begin{equation*}
y^{\prime \prime}=\Pi^{\prime \prime}\left(x^{\prime \prime}, a^{\prime \prime}, b^{\prime \prime}\right):=b^{\prime \prime}+\Lambda^{\prime}\left(x^{\prime \prime}\right)+\mathrm{O}\left(x^{\prime \prime} a^{\prime \prime}\right) \tag{7.16}
\end{equation*}
$$

Solving $b^{\prime \prime}$ by means of the implicit function theorem, we get

$$
\begin{equation*}
b^{\prime \prime}=\Pi^{\prime \prime *}\left(a^{\prime \prime}, x^{\prime \prime}, y^{\prime \prime}\right)=y^{\prime \prime}-\Lambda^{\prime}\left(x^{\prime \prime}\right)+\mathrm{O}\left(a^{\prime \prime} x^{\prime \prime}\right), \tag{7.17}
\end{equation*}
$$

and it suffices to set $y:=y^{\prime \prime}-\Lambda^{\prime}\left(x^{\prime \prime}\right), x:=x^{\prime \prime}$ and $a:=a^{\prime \prime}, b:=b^{\prime \prime}$.
Taking account of solvability with respect to the parameters, finer normalizations holds.

Lemma 7.18. With $n=m=\kappa=1$, every submanifold of solutions $y^{\prime}=$ $b^{\prime}+x^{\prime} a^{\prime}\left[1+\mathrm{O}_{1}\right]$ of $y_{x^{\prime} x^{\prime}}^{\prime}=F^{\prime}\left(x^{\prime}, y^{\prime}, y_{x^{\prime}}^{\prime}\right)$ is equivalent to

$$
\begin{equation*}
y_{x x}=b+x a+\mathrm{O}\left(x^{2} a^{2}\right) \tag{7.19}
\end{equation*}
$$

Proof. Writing $y^{\prime}=b^{\prime}+x^{\prime}\left[a^{\prime}+a^{\prime} \Lambda^{\prime}\left(a^{\prime}, b^{\prime}\right)+\mathrm{O}\left(x^{\prime} a^{\prime}\right)\right]$, where $\Lambda^{\prime}=\mathrm{O}_{1}$, we set $a^{\prime \prime}:=a^{\prime}+a^{\prime} \Lambda^{\prime}\left(a^{\prime}, b^{\prime}\right), b^{\prime \prime}:=b^{\prime}, x^{\prime \prime}:=x^{\prime}, y^{\prime \prime}:=y^{\prime}$, whence $y^{\prime \prime}=b^{\prime \prime}+$ $x^{\prime \prime}\left[a^{\prime \prime}+\mathrm{O}\left(x^{\prime \prime} a^{\prime \prime}\right)\right]$. Dually $b^{\prime \prime}=y^{\prime \prime}-a^{\prime \prime}\left[x^{\prime \prime}+x^{\prime \prime} x^{\prime \prime} \Lambda^{\prime \prime}\left(x^{\prime \prime}, y^{\prime \prime}\right)+\mathrm{O}\left(x^{\prime \prime} x^{\prime \prime} a^{\prime \prime}\right)\right]$, so we set $x:=x^{\prime \prime}+x^{\prime \prime} x^{\prime \prime} \Lambda^{\prime \prime}\left(x^{\prime \prime}, y^{\prime \prime}\right), y:=y^{\prime \prime}, a:=a^{\prime \prime}, b:=b^{\prime \prime}$.
Corollary 7.20. Every second order ordinary differential equation $y_{x^{\prime} x^{\prime}}^{\prime}=$ $F^{\prime}\left(x^{\prime}, y^{\prime}, y_{x^{\prime}}^{\prime}\right)$ is equivalent to

$$
\begin{equation*}
y_{x x}=\left(y_{x}\right)^{2} \mathrm{R}\left(x, y, y_{x}\right) \tag{7.21}
\end{equation*}
$$

7.22. Complete normal forms. The Moser theory of normal forms may be transferred with minor modifications to submanifolds of solutions associated to $\left(\mathscr{E}_{1}\right)$ and to $\left(\mathscr{E}_{2}\right)$.

Theorem 7.23. ([CM1974, Ja1990], [*]) A local $\mathbb{K}$-analytic submanifold of solutions associated to $\left(\mathscr{E}_{1}\right)$ :

$$
\begin{equation*}
y^{\prime}=b^{\prime}+x^{\prime} a^{\prime}+\mathrm{O}_{3}=\sum_{k^{\prime} \geqslant 0} \sum_{l^{\prime} \geqslant 0} \Pi_{k^{\prime}, l^{\prime}}^{\prime}\left(b^{\prime}\right) x^{\prime k^{\prime}} a^{l^{\prime}} \tag{7.24}
\end{equation*}
$$

can be mapped, by a transformation $\left(x^{\prime}, y^{\prime}, a^{\prime}, b^{\prime}\right) \mapsto(x, y, a, b)$ belonging to $\mathrm{G}_{\mathrm{v}, \mathrm{p}}$, to a submanifold of solutions of the specific form

$$
\begin{equation*}
y=b+x a+\Pi_{2,4}(b) x^{2} a^{4}+\Pi_{4,2}(b) a^{2} x^{4}+\sum_{k \geqslant 2} \sum_{l \geqslant 2} \sum_{k+l \geqslant 7} \Pi_{k, l}(b) x^{k} a^{l} . \tag{7.25}
\end{equation*}
$$

Solving $(a, b)$ from $y=\Pi$ and $y_{x}=\Pi_{x}$ with $\Pi$ as above, we deduce the following.

Corollary 7.26. Every $y_{x^{\prime} x^{\prime}}^{\prime}=F^{\prime}\left(x^{\prime}, y^{\prime}, y_{x^{\prime}}^{\prime}\right)$ is equivalent to

$$
\begin{align*}
y_{x x}= & \left(y_{x}\right)^{2}\left[x^{2} F_{2,2}(y)+x^{3} \mathrm{r}(x, y)\right]+\left(y_{x}\right)^{4}\left[F_{0,4}(y)+x \mathrm{r}(x, y)\right]+  \tag{7.27}\\
& +\sum_{k \geqslant 0} \sum_{l \geqslant 0} \sum_{k+l \geqslant 5} F_{k, l}(y) x^{k}\left(y_{x}\right)^{l} .
\end{align*}
$$

For the completely integrable system $\left(\mathscr{E}_{2}\right)$ having several dependent variables $\left(x^{1}, \ldots, x^{n}\right), n \geqslant 2$, we have the following.

Theorem 7.28. ([CM1974], [*]) A local $\mathbb{K}$-analytic submanifold of solutions associated to ( $\mathscr{E}_{2}$ ):

$$
\begin{equation*}
y^{\prime}=b^{\prime}+\sum_{1 \leqslant k \leqslant n} x^{\prime k} a^{\prime k}+\mathrm{O}_{3} \tag{7.29}
\end{equation*}
$$

can be mapped, by a transformation $\left(x^{\prime}, y^{\prime}, a^{\prime}, b^{\prime}\right) \mapsto(x, y, a, b)$ belonging to $\mathrm{G}_{\mathrm{v}, \mathrm{p}}$, to a submanifold of solutions of the specific form:

$$
\begin{equation*}
y=b+\sum_{1 \leqslant k \leqslant n} x^{k} a^{k}+\sum_{k \geqslant 2} \sum_{l \geqslant 2} \Pi_{k, l}(x, a, b) \tag{7.30}
\end{equation*}
$$

where
$\Pi_{k, l}(x, a, b):=\sum_{k_{1}+\cdots+k_{n}=k} \sum_{l_{1}+\cdots+l_{n}=l} \Pi_{k_{1}, \ldots, k_{n}, l_{1}, \ldots, l_{n}}(b)\left(x^{1}\right)^{k_{1}} \cdots\left(x^{n}\right)^{k_{n}}\left(a^{1}\right)^{l_{1}} \cdots\left(a^{n}\right)^{l_{n}}$
with the terms $\Pi_{2,2}, \Pi_{2,3}$ and $\Pi_{3,3}$ satisfying:

$$
\begin{equation*}
0=\Delta \Pi_{2,2}=\Delta \Delta \Pi_{2,3}=\Delta \Delta \Pi_{3,2}=\Delta \Delta \Delta \Pi_{3,3} \tag{7.32}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta:=\sum_{1 \leqslant k \leqslant n} \frac{\partial^{2}}{\partial x^{k} \partial a^{k}} . \tag{7.33}
\end{equation*}
$$

Exercise: solving $\left(a^{k}, b\right)$ from $y=\Pi$ and $y_{x^{l}}=\Pi_{x^{l}}$, with $\Pi$ as above, deduce a complete normal form for $\left(\mathscr{E}_{2}\right)$.

Open problem 7.34. Find complete normal forms for submanifolds of solutions associated to $\left(\mathscr{E}_{4}\right)$ and to $\left(\mathscr{E}_{5}\right)$.

## §8. Study of Two specific examples

8.1. Study of the Lie symmetries of $\left(\mathscr{E}_{4}\right)$. Its submanifold of solutions possesses two equations:

$$
\begin{equation*}
y^{1}=\Pi^{1}\left(x, a, b^{1}, b^{2}\right) \quad y^{2}=\Pi^{2}\left(x, a, b^{1}, b^{2}\right) \tag{8.2}
\end{equation*}
$$

For instance, a generic submanifold $M \subset \mathbb{C}^{3}$ of CR dimension 1 and of codimension 3 has equations of such a form.

Assuming $\mathscr{V}_{\mathscr{S}}\left(\mathscr{E}_{4}\right)$ to be twin solvable and having covering submanifold of solutions (see Definition 10.17), it may be verified (for $M \subset \mathbb{C}^{3}$, see [Be1997]) that at a Zariski-generic point, its equations are of the form:

$$
\begin{align*}
& y^{1}=b^{1}+x a+\mathrm{O}\left(x^{2}\right)+\mathrm{O}\left(b^{1}\right)+\mathrm{O}\left(b^{2}\right) \\
& y^{2}=b^{2}+x a(x+a)+\mathrm{O}\left(x^{3}\right)+\mathrm{O}\left(b^{1}\right)+\mathrm{O}\left(b^{2}\right) \tag{8.3}
\end{align*}
$$

The model has zero remainders with associated system

$$
\begin{equation*}
y_{1}^{2}=2 x y_{1}^{1}+\left(y_{1}^{1}\right)^{2}, \quad y_{2}^{1}=0 \tag{8.4}
\end{equation*}
$$

the third equation $y_{2}^{2}=2 y_{1}^{1}$ being obtained by differentiating the first.
We may put the submanifold in partial normal form. Proceeding as in [BES2005], some partial normalizations belonging to $G_{v, p}$ yield:
(8.5)
$y^{1}=b^{1}+a x+a^{2}\left[\Pi_{3,2}^{1}(b) x^{3}+\Pi_{4,2}^{1}(b) x^{4}+\cdots\right]+\mathrm{O}\left(a^{3} x^{2}\right)$,
$y^{2}=b^{2}+a\left[x^{2}+\Pi_{4,1}^{2}(b) x^{4}+\cdots\right]+a^{2}\left[x+\Pi_{3,2}^{2}(b) x^{3}+\cdots\right]+\mathrm{O}\left(a^{3} x^{2}\right)$.
Redifferentiating, we get an appropriate, partially normalized system ( $\mathscr{E}_{4}$ ):
(8.6)
$\left\{\begin{array}{l}y_{1}^{2}=y_{1}^{1}\left(2 x+\mathbf{g}^{1}\right)+\left(y_{1}^{1}\right)^{2}\left(1+\mathbf{g}^{2}\right)+\left(y_{1}^{1}\right)^{3} \mathbf{s}+\left(y_{1}^{1}\right)^{4} \mathbf{s}+\left(y_{1}^{1}\right)^{5} \mathbf{s}+\left(y_{1}^{1}\right)^{6} \mathbf{R}, \\ y_{2}^{1}=\left(y_{1}^{1}\right)^{2} \mathbf{h}+\left(y_{1}^{1}\right)^{3} \mathbf{R}, \\ y_{2}^{2}=y_{1}^{1}\left(2+\mathbf{g}_{x}^{1}\right)+\left(y_{1}^{1}\right)^{2}\left(\mathbf{g}_{x}^{2}+\left(2 x+\mathbf{g}^{1}\right) \mathbf{h}\right)+\left(y_{1}^{1}\right)^{3} \mathbf{r}+\left(y_{1}^{1}\right)^{4} \mathbf{r}+\left(y_{1}^{1}\right)^{5} \mathbf{r}+\left(y_{1}^{1}\right)^{6} \mathbf{R},\end{array}\right.$
where, precisely:

- $\mathbf{g}^{1}, \mathbf{g}^{2}$ and $\mathbf{h}$ are functions of $\left(x, y^{1}, y^{2}\right)$ satisfying $\mathbf{g}^{j}=\mathrm{O}(x x)+$ $\mathrm{O}\left(y^{1}\right)+\mathrm{O}\left(y^{2}\right), j=1,2$ and $\mathbf{h}=\mathrm{O}(x)+\mathrm{O}\left(y^{1}\right)+\mathrm{O}\left(y^{2}\right) ;$
- r and s are unspecified functions, varying in the context, of $\left(x, y^{1}, y^{2}\right)$ with $\mathrm{s}=\mathrm{O}(x)+\mathrm{O}\left(y^{1}\right)+\mathrm{O}\left(y^{2}\right)$, but possibly $\mathrm{r}(0) \neq 0$;
- R is a remainder function of all the variables $\left(x, y^{1}, y^{2}, y_{1}^{1}\right)$ parametrizing $\Delta_{\mathscr{E}_{4}}$.

Letting $\mathscr{L}=\mathscr{X} \frac{\partial}{\partial x}+\mathscr{Y}^{1} \frac{\partial}{\partial y^{1}}+\mathscr{Y}^{2} \frac{\partial}{\partial y^{2}}$ be a candidate infinitesimal Lie symmetry and applying $\mathscr{L}^{(2)}=\mathscr{L}+\mathbf{Y}_{1}^{1} \frac{\partial}{\partial y_{1}^{1}}+\mathbf{Y}_{1}^{2} \frac{\partial}{\partial y_{1}^{2}}+\mathbf{Y}_{2}^{1} \frac{\partial}{\partial y_{2}^{1}}+\mathbf{Y}_{2}^{2} \frac{\partial}{\partial y_{2}^{2}}$ to $\Delta_{\mathscr{E}_{4}}$, we obtain firstly, computing $\bmod \left(y_{1}^{1}\right)^{5}$ :

$$
\begin{align*}
0 \equiv & -\mathbf{Y}_{1}^{2}+[\mathscr{X}]\left(y_{1}^{1}\left(2+\mathbf{g}_{x}^{1}\right)+\left(y_{1}^{1}\right)^{2} \mathbf{g}_{x}^{2}+\left(y_{1}^{1}\right)^{3} \mathbf{r}+\left(y_{1}^{1}\right)^{4} \mathbf{r}\right)+ \\
& +\left[\mathscr{Y}^{1}\right]\left(y_{1}^{1} \mathbf{r}+\left(y_{1}^{1}\right)^{2} \mathbf{r}+\left(y_{1}^{1}\right)^{3} \mathbf{r}+\left(y_{1}^{1}\right)^{4} \mathbf{r}\right)+ \\
& +\left[\mathscr{Y}^{2}\right]\left(y_{1}^{1} \mathbf{r}+\left(y_{1}^{1}\right)^{2} \mathbf{r}+\left(y_{1}^{1}\right)^{3} \mathbf{r}+\left(y_{1}^{1}\right)^{4} \mathbf{r}\right)+  \tag{8.7}\\
& +\mathbf{Y}_{1}^{1}\left(2 x+\mathbf{g}^{1}+y_{1}^{1}\left(2+2 \mathbf{g}^{2}\right)+\left(y_{1}^{1}\right)^{2} \mathbf{s}+\left(y_{1}^{1}\right)^{3} \mathbf{s}+\left(y_{1}^{1}\right)^{4} \mathbf{s}\right),
\end{align*}
$$

and secondly, computing $\bmod \left(y_{1}^{1}\right)^{2}$ :

$$
\begin{equation*}
0 \equiv-\mathbf{Y}_{2}^{1}+2 y_{1}^{1} \mathbf{Y}_{1}^{1} h \tag{8.8}
\end{equation*}
$$

The third Lie equation involving $\mathbf{Y}_{2}^{2}$ will be superfluous. Specializing (4.6)(II) to $m=2$, we get $\mathbf{Y}_{1}^{1}$ and $\mathbf{Y}_{1}^{2}$ :

$$
\begin{align*}
\mathbf{Y}_{1}^{1} & =\mathscr{Y}_{x}^{1}+\left[\mathscr{Y}_{y^{1}}^{1}-\mathscr{X}_{x}\right] y_{1}^{1}+\left[\mathscr{Y}_{y^{2}}^{1}\right] y_{1}^{2}+\left[-\mathscr{X}_{y^{1}}\right]\left(y_{1}^{1}\right)^{2}+\left[-\mathscr{X}_{y^{2}}\right] y_{1}^{1} y_{1}^{2}  \tag{8.9}\\
\mathbf{Y}_{1}^{2} & =\mathscr{Y}_{x}^{2}+\left[\mathscr{Y}_{y^{1}}^{2}\right] y_{1}^{1}+\left[\mathscr{Y}_{y^{2}}^{2}-\mathscr{X}_{x}\right] y_{1}^{2}+\left[-\mathscr{X}_{y^{1}}\right] y_{1}^{2} y_{1}^{1}+\left[-\mathscr{X}_{y^{2}}\right]\left(y_{1}^{2}\right)^{2} .
\end{align*}
$$

and also $\mathbf{Y}_{2}^{1}$ and $\mathbf{Y}_{2}^{2}$ (in fact superfluous):
(8.10)

$$
\begin{aligned}
\mathbf{Y}_{2}^{1}= & \mathscr{Y}_{x x}^{1}+\left[2 \mathscr{Y}_{x y^{1}}^{1}-\mathscr{X}_{x x}\right] y_{1}^{1}+\left[2 \mathscr{Y}_{x y^{2}}^{1}\right] y_{1}^{2}+\left[\mathscr{Y}_{y^{1} y^{1}}^{1}-2 \mathscr{X}_{x y^{1}}\right]\left(y_{1}^{1}\right)^{2}+ \\
& +\left[2 \mathscr{Y}_{y^{1} y^{2}}^{1}-2 \mathscr{X}_{x y^{2}}\right] y_{1}^{1} y_{1}^{2}+\left[\mathscr{\mathscr { y }}_{y^{2} y^{2}}^{1}\right]\left(y_{1}^{2}\right)^{2}+\left[-\mathscr{X}_{y^{1} y^{1}}\right]\left(y_{1}^{1}\right)^{3}+ \\
& +\left[-2 \mathscr{X}_{y^{1} y^{2}}\right]\left(y_{1}^{1}\right)^{2} y_{1}^{2}+\left[-\mathscr{X}_{y^{2} y^{2}}\right] y_{1}^{1}\left(y_{1}^{2}\right)^{2}+\left[\mathscr{Y}_{y^{1}}^{1}-2 \mathscr{X}_{x}\right] y_{2}^{1}+ \\
& +\left[\mathscr{Y}_{y^{2}}^{1}\right] y_{2}^{2}+\left[-3 \mathscr{X}_{y^{1}}\right] y_{1}^{1} y_{2}^{1}+\left[-\mathscr{X}_{y^{2}}\right] y_{1}^{1} y_{2}^{2}+\left[-2 \mathscr{X}_{y^{2}}\right] y_{1}^{2} y_{2}^{1}, \\
\mathbf{Y}_{2}^{2}= & \mathscr{Y}_{x x}^{2}+\left[2 \mathscr{Y}_{x y^{1}}^{2}\right] y_{1}^{1}+\left[2 \mathscr{Y}_{x y^{2}}^{2}-\mathscr{X}_{x x}\right] y_{1}^{2}+\left[\mathscr{Y}_{y^{1} y^{1}}^{2}\right]\left(y_{1}^{1}\right)^{2}+ \\
& +\left[2 \mathscr{Y}_{y^{1} y^{2}}^{2}-2 \mathscr{X}_{x y^{1}}\right] y_{1}^{1} y_{1}^{2}+\left[\mathscr{Y}_{y^{2} y^{2}}^{2}-2 \mathscr{X}_{x y^{2}}\right]\left(y_{1}^{2}\right)^{2}+\left[-\mathscr{X}_{y^{1} y^{1}}\right]\left(y_{1}^{1}\right)^{2} y_{1}^{2}+ \\
& +\left[-2 \mathscr{X}_{y^{1} y^{2}}\right] y_{1}^{1}\left(y_{1}^{2}\right)^{2}+\left[-\mathscr{X}_{y^{2} y^{2}}\right]\left(y_{1}^{2}\right)^{3}+\left[\mathscr{Y}_{y^{1}}^{2}\right] y_{2}^{1}+ \\
& +\left[\mathscr{Y}_{y^{2}}^{2}-2 \mathscr{X}_{x}\right] y_{2}^{2}+\left[-2 \mathscr{X}_{y^{1}}\right] y_{1}^{1} y_{2}^{2}+\left[-\mathscr{X}_{y^{1}}\right] y_{1}^{2} y_{2}^{1}+\left[-3 \mathscr{X}_{y^{2}}\right] y_{1}^{2} y_{2}^{2} .
\end{aligned}
$$

Inserting $\mathbf{Y}_{1}^{2}$ and $\mathbf{Y}_{1}^{1}$ in the first Lie equation (8.7) in which $y_{1}^{2}$ is replaced by the value $(8.6)_{1}$ it has on $\Delta_{\mathscr{E}_{4}}$ and still computing $\bmod \left(y_{1}^{1}\right)^{5}$, we get, again
with r , s being unspecified functions of $\left(x, y^{1}, y^{2}\right)$ with $\mathbf{s}(0)=0$ :
(8.11)

$$
\begin{aligned}
0 \equiv & -\mathscr{Y}_{x}^{2}+\left[-\mathscr{Y}_{y^{1}}^{2}\right] y_{1}^{1}+ \\
& +\left[-\mathscr{Y}_{y^{2}}^{2}+\mathscr{X}_{x}\right]\left(y_{1}^{1}\left(2 x+\mathbf{g}^{1}\right)+\left(y_{1}^{1}\right)^{2}\left(1+\mathbf{g}^{2}\right)+\left(y_{1}^{1}\right)^{3} \mathbf{s}+\left(y_{1}^{1}\right)^{4} \mathbf{s}\right)+ \\
& +\left[\mathscr{X}_{y^{1}}\right]\left(\left(y_{1}^{1}\right)^{2}\left(2 x+\mathbf{g}^{1}\right)+\left(y_{1}^{1}\right)^{3}\left(1+\mathbf{g}^{2}\right)+\left(y_{1}^{1}\right)^{4} \mathbf{s}\right)+ \\
& +\left[\mathscr{X}_{y^{2}}\right]\left(\left(y_{1}^{1}\right)^{2}\left[2 x+\mathbf{g}^{1}\right]^{2}+\left(y_{1}^{1}\right)^{3}\left(4 x+2 \mathbf{g}^{1}\right)\left(1+\mathbf{g}^{2}\right)+\left(y_{1}^{1}\right)^{4}(1+\mathbf{s})\right)+ \\
& +\left[\mathscr{X}^{1}\right]\left(y_{1}^{1}\left(2+\mathbf{g}_{x}^{1}\right)+\left(y_{1}^{1}\right)^{2} \mathbf{g}_{x}^{2}+\left(y_{1}^{1}\right)^{3} \mathbf{r}+\left(y_{1}^{1}\right)^{4} \mathbf{r}\right)+ \\
& +\left[\mathscr{Y}^{1}\right]\left(y_{1}^{1} \mathbf{r}+\left(y_{1}^{1}\right)^{2} \mathbf{r}+\left(y_{1}^{1}\right)^{3} \mathbf{r}+\left(y_{1}^{1}\right)^{4} \mathbf{r}\right)+ \\
& +\left[\mathscr{Y}^{2}\right]\left(y_{1}^{1} \mathbf{r}+\left(y_{1}^{1}\right)^{2} \mathbf{r}+\left(y_{1}^{1}\right)^{3} \mathbf{r}+\left(y_{1}^{1}\right)^{4} \mathbf{r}\right)+ \\
& +\left[\mathscr{Y}_{x}^{1}\right]\left(2 x+\mathbf{g}^{1}+y_{1}^{1}\left(2+2 \mathbf{g}^{2}\right)+\left(y_{1}^{1}\right)^{2} \mathbf{s}+\left(y_{1}^{1}\right)^{3} \mathbf{s}+\left(y_{1}^{1}\right)^{4} \mathbf{s}\right)+ \\
& +\left[\mathscr{Y}_{y^{1}}^{1}-\mathscr{X}_{x}\right]\left(y_{1}^{1}\left(2 x+\mathbf{g}^{1}\right)+\left(y_{1}^{1}\right)^{2}\left(2+2 \mathbf{g}^{2}\right)+\left(y_{1}^{1}\right)^{3} \mathbf{s}+\left(y_{1}^{1}\right)^{4} \mathbf{s}\right)+ \\
& +\left[\mathscr{Y}_{y^{2}}^{1}\right]\left(y_{1}^{1}\left[2 x+\mathbf{g}^{1}\right]^{2}+\left(y_{1}^{1}\right)^{2}\left(2 x+\mathbf{g}^{1}\right)\left(3+3 \mathbf{g}^{2}\right)+\left(y_{1}^{1}\right)^{3}(2+\mathbf{s})+\left(y_{1}^{1}\right)^{4} \mathbf{s}\right)+ \\
& +\left[-\mathscr{X}_{y^{1}}\right]\left(\left(y_{1}^{1}\right)^{2}\left(2 x+\mathbf{g}^{1}\right)+\left(y_{1}^{1}\right)^{3}\left(2+2 \mathbf{g}^{2}\right)+\left(y_{1}^{1}\right)^{4} \mathbf{s}\right)+ \\
& +\left[-\mathscr{X}_{y^{2}}\right]\left(\left(y_{1}^{1}\right)^{2}\left[2 x+\mathbf{g}^{1}\right]^{2}+\left(y_{1}^{1}\right)^{3}\left(2 x+\mathbf{g}^{1}\right)\left(3+3 \mathbf{g}^{2}\right)+\left(y_{1}^{1}\right)^{4}(2+\mathbf{s})\right) .
\end{aligned}
$$

Collecting the coefficients of the monomials cst., $y_{1}^{1},\left(y_{1}^{1}\right)^{2},\left(y_{1}^{1}\right)^{3},\left(y_{1}^{1}\right)^{4}$, we get, after slight simplification (in the coefficient of $\left(y_{1}^{1}\right)^{2}$, the term $(2 x+$ $\left.\mathbf{g}^{1}\right) \mathscr{X}_{x}$ annihilates with its opposite; in the coefficient of $\left(y_{1}^{1}\right)^{3}$, two pairs annihilate and then, we divide by $\left[1+\mathbf{g}^{2}\right]$ ) a system of five linear PDE's:
(8.12)

$$
\begin{aligned}
& 0=-\mathscr{Y}_{x}^{2}+\left(2 x+\mathbf{g}^{1}\right) \mathscr{Y}_{x}^{1}, \\
& 0=-\mathscr{Y}_{y^{1}}^{2}-\left(2 x+\mathbf{g}^{1}\right) \mathscr{T}_{y^{2}}^{2}+\left(2+\mathbf{g}_{x}^{1}\right) \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+ \\
& +\left(2+2 \mathbf{g}^{2}\right) \mathscr{Y}_{x}^{1}+\left(2 x+\mathbf{g}^{1}\right) \mathscr{Y}_{y^{1}}^{1}+\left[2 x+\mathbf{g}^{1}\right]^{2} \mathscr{Y}_{y^{2}}^{1}, \\
& 0=-\mathscr{Y}_{y^{2}}^{2}+\mathscr{X}_{x}+\mathrm{g}_{x}^{2}\left[1+\mathrm{g}^{2}\right]^{-1} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+ \\
& +\mathrm{s} \mathscr{Y}_{x}^{1}+2 \mathscr{Y}_{y^{1}}^{1}-2 \mathscr{X}_{x}+\left(6 x+3 \mathbf{g}^{2}\right) \mathscr{Y}_{y^{2}}^{1}, \\
& 0=\mathbf{s} \mathscr{Y}_{y^{2}}^{2}+\mathbf{s} \mathscr{X}_{x}+\left(1+\mathbf{g}^{2}\right) \mathscr{X}_{y^{1}}+\left(2 x+\mathbf{g}^{1}\right)\left(2+2 \mathbf{g}^{2}\right) \mathscr{X}_{y^{2}}+ \\
& +\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{s} \mathscr{Y}_{x}^{1}+\mathrm{s} \mathscr{Y}_{y^{1}}^{1}+\mathrm{s} \mathscr{X}_{x}+(2+\mathrm{s}) \mathscr{Y}_{y^{2}}^{1}- \\
& -\left(2+2 \mathbf{g}^{2}\right) \mathscr{X}_{y^{1}}-\left(2 x+\mathbf{g}^{1}\right)\left(3+3 \mathbf{g}^{2}\right) \mathscr{X}_{y^{2}}, \\
& 0=\mathrm{s} \mathscr{Y}_{y^{2}}^{2}+\mathrm{s} \mathscr{X}_{x}+\mathrm{s} \mathscr{X}_{y^{1}}+(1+\mathrm{s}) \mathscr{X}_{y^{2}}+\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{s} \mathscr{Y}_{x}^{1}+\mathrm{s} \mathscr{Y}_{y^{1}}^{1}+ \\
& +\mathrm{s} \mathscr{X}_{x}+\mathrm{s} \mathscr{Y}_{y^{2}}^{1}+\mathrm{s} \mathscr{X}_{y^{1}}-(2+\mathrm{s}) \mathscr{X}_{y^{2}} .
\end{aligned}
$$

We then simplify the remainders using $s+s=s, r+s=r$ and $r+r=r$; we divide $(8.12)_{5}$ by $(1+\mathrm{s})$; we replace $\mathscr{X}_{y^{2}}$ obtained from (8.12) $)_{5}$ in $(8.12)_{4}$; we divide $(8.12)_{4}$ by $\left(1+\mathbf{g}^{2}\right)$; we then solve $\mathscr{X}_{y^{1}}$ from (8.12) $)_{4}$ and finally
we insert it in $(8.12)_{5}$; we get:
(8.13)

$$
\begin{aligned}
0= & -\mathscr{Y}_{x}^{2}+\left(2 x+\mathbf{g}^{1}\right) \mathscr{Y}_{x}^{1}, \\
0= & -\mathscr{Y}_{y^{1}}^{2}-\left(2 x+\mathbf{g}^{1}\right) \mathscr{Y}_{y^{2}}^{2}+\left(2+\mathbf{g}_{x}^{1}\right) \mathscr{X}+\left(2+2 \mathbf{g}^{2}\right) \mathscr{Y}_{x}^{1}+\left(2 x+\mathbf{g}^{1}\right) \mathscr{Y}_{y^{1}}^{1}+ \\
& \quad+\left[2 x+\mathbf{g}^{1}\right]^{2} \mathscr{Y}_{y^{2}}^{1}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}, \\
0= & -\mathscr{Y}_{y^{2}}^{2}-\mathscr{X}_{x}+2 \mathscr{Y}_{y^{1}}^{1}+\left(6 x+3 \mathbf{g}^{2}\right) \mathscr{Y}_{y^{2}}^{1}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{s} \mathscr{Y}_{x}^{1}+ \\
& \quad+\mathbf{g}_{x}^{2}\left[1+\mathbf{g}^{2}\right]^{-1} \mathscr{X}, \\
0= & -\mathscr{X}_{y^{1}}+(2+\mathrm{s}) \mathscr{Y}_{y^{2}}^{1}+\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{s} \mathscr{X}_{x}+\mathrm{s} \mathscr{Y}_{x}^{1}+\mathrm{s} \mathscr{Y}_{y^{1}}^{1}+\mathrm{s} \mathscr{Y}_{y^{2}}^{2}, \\
0= & -\mathscr{X}_{y^{2}}+\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{s} \mathscr{X}_{x}+\mathrm{s} \mathscr{Y}_{x}^{1}+\mathrm{s} \mathscr{Y}_{y^{1}}^{1}+\mathrm{s} \mathscr{Y}_{y^{2}}^{1}+\mathrm{s} \mathscr{Y}_{y^{2}}^{2} .
\end{aligned}
$$

Similarly, developing the second equation (8.8) and computing $\bmod \left(y_{1}^{1}\right)^{2}$, we get:

$$
\begin{align*}
0 \equiv-\mathscr{Y}_{x x}^{1} & +\left[-2 \mathscr{Y}_{x y^{1}}^{1}+\mathscr{X}_{x x}\right] y_{1}^{1}+\left[-\left(4 x+2 \mathbf{g}^{1}\right) \mathscr{Y}_{x y^{2}}-(2+\mathbf{h}) \mathscr{Y}_{y^{2}}^{1}\right]+  \tag{8.14}\\
& +\left[2 \mathbf{h} \mathscr{Y}_{x}^{1}\right] y_{1}^{1} .
\end{align*}
$$

Collecting the coefficients of the monomials cst., $y_{1}^{1}$, we get two more linear PDE's:

$$
\begin{align*}
& 0=-\mathscr{Y}_{x x}^{1}, \\
& 0=-2 \mathscr{Y}_{x y^{1}}^{1}+\mathscr{X}_{x x}-\left(4 x+2 \mathbf{g}^{1}\right) \mathscr{Y}_{x y^{2}}^{1}-(2+\mathbf{h}) \mathscr{Y}_{y^{2}}^{1}+2 \mathbf{h} \mathscr{Y}_{x}^{1} . \tag{8.15}
\end{align*}
$$

Proposition 8.16. Setting as initial conditions the five specific differential coefficients

$$
\begin{equation*}
\mathrm{P}:=\mathrm{P}\left(\mathscr{X}, \mathscr{Y}^{1}, \mathscr{Y}^{2}, \mathscr{Y}_{x}^{1}, \mathscr{X}_{x}\right)=\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{r} \mathscr{Y}_{x}^{1}+\mathrm{r} \mathscr{X}_{x}, \tag{8.17}
\end{equation*}
$$

it follows by cross differentiations and by linear substitutions from the seven equations $(8.13)_{i}, i=1,2,3,4,5,(8.15)_{j}, j=1,2$, that $\mathscr{X}_{y^{1}}, \mathscr{X}_{y^{2}}, \mathscr{Y}_{y^{1}}^{1}$, $\mathscr{Y}_{y^{2}}^{1}, \mathscr{Y}_{x}^{2}, \mathscr{Y}_{y^{1}}^{2}, \mathscr{Y}_{y^{2}}^{2}$ and $\mathscr{X}_{x x}, \mathscr{X}_{x y^{1}}, \mathscr{X}_{x y^{2}}, \mathscr{Y}_{x x}^{1}, \mathscr{Y}_{x y^{1}}^{1}, \mathscr{Y}_{x y^{2}}^{1}$ are uniquely determined as linear combinations of $\left(\mathscr{X}, \mathscr{Y}^{1}, \mathscr{Y}^{2}, \mathscr{Y}_{x}^{1}, \mathscr{X}_{x}\right)$, namely:
(8.18) $\left\{\begin{array}{lllll} & \mathscr{Y}_{x}^{2} \stackrel{1}{=} \mathrm{P}, & \mathscr{X}_{x x} \stackrel{2}{=} \mathrm{P}, & \mathscr{Y}_{x x}^{1} \stackrel{3}{=} \mathrm{P}, \\ \mathscr{X}_{y^{1}} \stackrel{4}{=} \mathrm{P}, & \mathscr{Y}_{y^{1}} \stackrel{5}{=} \mathrm{P}, & \mathscr{Y}_{y^{1}}^{2} \stackrel{6}{=} \mathrm{P}, & \mathscr{X}_{x y^{1}} \stackrel{7}{=} \mathrm{P}, & \mathscr{Y}_{x y^{1}}^{1} \stackrel{8}{=} \mathrm{P}, \\ \mathscr{X}_{y^{2}} \stackrel{9}{=} \mathrm{P}, & \mathscr{Y}_{y^{2}} \stackrel{10}{=} \mathrm{P}, & \mathscr{Y}_{y^{2}}^{2} \stackrel{11}{=} \mathrm{P}, & \mathscr{X}_{x y^{2}} \stackrel{12}{=} \mathrm{P}, & \mathscr{Y}_{x y^{2}}^{1} \stackrel{13}{=} \mathrm{P} .\end{array}\right.$

Then the expressions P are stable under differentiation:

$$
\begin{align*}
\mathrm{P}_{x} & =\mathrm{P}+\mathrm{r} \mathscr{Y}_{x}^{2}+\mathrm{r} \mathscr{Y}_{x x}^{1}+\mathrm{r} \mathscr{X}_{x x}=\mathrm{P}, \\
\mathrm{P}_{y^{1}} & =\mathrm{P}+\mathrm{r} \mathscr{X}_{y^{1}}+\mathrm{r} \mathscr{Y}_{y^{1}}^{1}+\mathrm{r} \mathscr{Y}_{y^{1}}^{2}+\mathrm{r} \mathscr{Y}_{x y^{1}}^{1}+\mathrm{r} \mathscr{X}_{x y^{1}}=\mathrm{P},  \tag{8.19}\\
\mathrm{P}_{y^{2}} & =\mathrm{P}+\mathrm{r} \mathscr{X}_{y^{2}}+\mathrm{r} \mathscr{Y}_{y^{2}}^{1}+\mathrm{r} \mathscr{Y}_{y^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x y^{2}}^{1}+\mathrm{r} \mathscr{X}_{x y^{2}}=\mathrm{P},
\end{align*}
$$

and moreover, all other, higher order partial derivatives of $\mathscr{X}$, of $\mathscr{Y}^{1}$ and of $\mathscr{Y}^{2}$ may be expressed as $\mathrm{P}\left(\mathscr{X}, \mathscr{Y}^{1}, \mathscr{Y}^{2}, \mathscr{Y}_{x}^{1}, \mathscr{X}_{x}\right)$.

Corollary 8.20. An infinitesimal Lie symmetry of $\left(\mathscr{E}_{4}\right)$ is uniquely determined by the five initial Taylor coefficients

$$
\begin{equation*}
\mathscr{X}(0), \mathscr{Y}^{1}(0), \mathscr{Y}^{2}(0), \mathscr{Y}_{x}^{1}(0), \mathscr{X}_{x}(0) . \tag{8.21}
\end{equation*}
$$

Proof of the proposition. We notice that $(8.18)_{1}$ and $(8.18)_{3}$ are given for free by $(8.13)_{1}$ and by $(8.15)_{1}$. Differentiating $(8.13)_{3}$ with respect to $x$, we get:

$$
\begin{align*}
0= & -\mathscr{Y}_{x y^{2}}-\mathscr{X}_{x x}+2 \mathscr{Y}_{x y^{1}}^{1}+\left(6+3 \mathbf{g}_{x}^{1}\right) \mathscr{Y}_{y^{2}}^{1}+\left(6 x+3 \mathbf{g}^{1}\right) \mathscr{Y}_{x y^{2}}^{1}+\mathrm{r} \mathscr{Y}^{1}+  \tag{8.22}\\
& +\mathrm{r} \mathscr{Y}_{x}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{r} \mathscr{Y}_{x}^{2}+\mathrm{r} \mathscr{Y}_{x}^{1}+\mathrm{s} \mathscr{Y}_{x x}^{1}+\mathrm{r} \mathscr{X}+\mathbf{g}_{x}^{2}\left[1+\mathbf{g}^{2}\right]^{-1} \mathscr{X}_{x} .
\end{align*}
$$

By $(8.15)_{1}, \mathrm{~s} \mathscr{Y}_{x x}^{1}$ vanishes. We replace $\mathscr{Y}_{x}^{2}$ thanks to $(8.13)_{1}$. Differentiating $(8.13)_{1}$ with respect to $y^{2}$, we may substract $0=-\mathscr{Y}_{x y^{2}}+\left(2 x+\mathbf{g}^{1}\right) \mathscr{Y}_{x y^{2}}^{1}+$ $\mathrm{r} \mathscr{Y}_{x}^{1}$. We get:

$$
\begin{align*}
0= & -\mathscr{X}_{x x}+2 \mathscr{Y}_{x y^{1}}^{1}+\left(4 x+2 \mathbf{g}^{1}\right) \mathscr{Y}_{x y^{2}}^{1}+  \tag{8.23}\\
& +\left(6+3 \mathbf{g}_{x}^{1}\right) \mathscr{Y}_{y^{2}}^{1}+\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{r} \mathscr{Y}_{x}^{1}+\mathrm{g}_{x}^{2}\left[1+\mathbf{g}^{2}\right]^{-1} \mathscr{X}_{x} .
\end{align*}
$$

By means of $(8.15)_{2}$, we replace the first three terms and then solve $\mathscr{Y}_{y^{2}}$ :

$$
\begin{equation*}
\mathscr{Y}_{y^{2}}^{1}=\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{r} \mathscr{Y}_{x}^{1}+\mathbf{k}^{*} \mathscr{X}_{x}, \tag{8.24}
\end{equation*}
$$

introducing a notation for a new function that should be recorded:

$$
\begin{equation*}
\mathbf{k}^{*}:=\mathbf{g}_{x}^{2}\left[1+\mathbf{g}^{2}\right]^{-1}\left[4+3 \mathbf{g}_{x}^{1}-\mathbf{h}\right]^{-1} \tag{8.25}
\end{equation*}
$$

This is $(8.18)_{10}$. Next, we differentiate the obtained equation with respect to $x$, getting:

$$
\begin{equation*}
\mathscr{Y}_{x y^{2}}^{1}=\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{r} \mathscr{Y}_{x}^{1}+\mathrm{r} \mathscr{X}_{x}+\mathbf{k}^{*} \mathscr{X}_{x x} . \tag{8.26}
\end{equation*}
$$

This is $(8.18)_{13}$. We replace the obtained value of $\mathscr{Y}_{y^{2}}^{1}$ in $(8.13)_{2},(8.13)_{3}$, $(8.15)_{2}$ and the obtained value of $\mathscr{Y}_{x y^{2}}^{1}$ in (8.15) $)_{2}$. This yields a new, simpler
system of seven equations:
(8.27)

$$
\begin{aligned}
0= & -\mathscr{Y}_{x}^{2}+\left(2 x+\mathbf{g}^{1}\right) \mathscr{Y}_{x}^{1}, \\
0= & -\mathscr{Y}_{y^{1}}^{2}-\left(2 x+\mathbf{g}^{1}\right) \mathscr{Y}_{y^{2}}^{2}+\left(2+\mathbf{g}_{x}^{1}\right) \mathscr{X}+\left(2+2 \mathbf{g}^{2}\right) \mathscr{Y}_{x}^{1}+\left(2 x+\mathbf{g}^{1}\right) \mathscr{Y}_{y^{1}}^{1}+ \\
& +\mathrm{s} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{s} \mathscr{Y}_{x}^{1}+\mathrm{k}^{*}\left[2 x+\mathbf{g}^{1}\right]^{2} \mathscr{X}_{x}, \\
0= & -\mathscr{Y}_{y^{2}}^{2}-\mathscr{X}_{x}+2 \mathscr{Y}_{y^{1}}^{1}+\mathrm{s} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{s} \mathscr{Y}_{x}^{1}+\mathrm{k}^{*}\left(6 x+3 \mathbf{g}^{1}\right) \mathscr{X}_{x}, \\
0= & -\mathscr{X}_{y^{1}}+\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{s} \mathscr{Y}_{x}^{1}+\mathrm{s} \mathscr{X}_{x}+\mathrm{s} \mathscr{Y}_{y^{1}}^{1}+\mathrm{s} \mathscr{Y}_{y^{2}}^{2}, \\
0= & -\mathscr{X}_{y^{2}}+\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{s} \mathscr{Y}_{x}^{1}+\mathrm{s} \mathscr{X}_{x}+\mathrm{s} \mathscr{Y}_{y^{1}}^{1}+\mathrm{s} \mathscr{Y}_{y^{2}}^{2}, \\
0= & -\mathscr{Y}_{x x}^{1}, \\
0= & -2 \mathscr{Y}_{x y^{1}}^{1}+\mathscr{X}_{x x}\left(1-\mathbf{k}^{*}\left(4 x+2 \mathbf{g}^{1}\right)\right)+\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{r} \mathscr{Y}_{x}^{1} .+\mathrm{s} \mathscr{X}_{x} .
\end{aligned}
$$

Restarting from this system, we differentiate $(8.27)_{3}$ with respect to $x$ :

$$
\begin{align*}
0= & -\mathscr{Y}_{x y^{2}}^{2}-\mathscr{X}_{x x}+2 \mathscr{Y}_{x y^{1}}^{1}+\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+  \tag{8.28}\\
& +\mathrm{r} \mathscr{X}_{x}+\mathrm{r} \mathscr{Y}_{x}^{1}+\mathrm{r} \mathscr{Y}_{x}^{2}+\mathrm{s} \mathscr{Y}_{x x}^{1}+\mathrm{k}^{*}\left(6 x+3 \mathbf{g}^{1}\right) \mathscr{X}_{x x} .
\end{align*}
$$

We replace $\mathscr{Y}_{x}^{2}$, we erase $\mathscr{Y}_{x x}^{1}$ and we add (8.27) ${ }_{7}$ :

$$
\text { (8.29) } 0=-\mathscr{Y}_{x y^{2}}^{2}+\mathrm{k}^{*}\left(2 x+\mathrm{g}^{1}\right) \mathscr{X}_{x x}+\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{r} \mathscr{Y}_{x}^{1}+\mathrm{r} \mathscr{X}_{x} .
$$

We differentiate $(8.27)_{2}$ with respect to $x$ :

$$
\begin{align*}
0= & -\mathscr{Y}_{x y^{1}}^{2}-\left(2+\mathbf{g}_{x}^{1}\right) \mathscr{Y}_{y^{2}}^{2}-\left(2 x+\mathbf{g}^{1}\right) \mathscr{Y}_{x y^{2}}^{2}+\mathrm{r} \mathscr{X}+\left(2+\mathbf{g}_{x}^{1}\right) \mathscr{X}_{x}+  \tag{8.30}\\
& +\mathbf{s} \mathscr{Y}_{x}^{1}+\left(2+2 \mathbf{g}^{2}\right) \mathscr{Y}_{x x}^{1}+\left(2+\mathbf{g}_{x}^{1}\right) \mathscr{Y}_{y^{1}}^{1}+\left(2 x+\mathbf{g}^{1}\right) \mathscr{Y}_{x y^{1}}^{1}+\mathrm{r} \mathscr{X}_{x}+ \\
& +\mathbf{s} \mathscr{X}_{x}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}_{x}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{r} \mathscr{Y}_{x}^{2}+\mathrm{r} \mathscr{Y}_{x}^{1}+\mathbf{s} \mathscr{Y}_{x x}^{1}+\mathbf{k}^{*}\left[2 x+\mathbf{g}^{1}\right]^{2} \mathscr{X}_{x x} .
\end{align*}
$$

Differentiating (8.27) $)_{1}$ with respect to $y^{1}$, we may substract $0=-\mathscr{Y}_{x y^{1}}^{2}+$ $\left(2 x+\mathrm{g}^{1}\right) \mathscr{Y}_{x y^{1}}^{1}+\mathrm{r} \mathscr{Y}_{x}^{1}$; we replace $\mathscr{Y}_{x}^{2}$ and erase $\mathscr{Y}_{x x}^{1}$; we substract (8.29) multiplied by $\left(2 x+\mathbf{g}^{1}\right)$; we get:

$$
\begin{equation*}
0=-\mathscr{Y}_{y^{2}}^{2}+(1+\mathrm{s}) \mathscr{X}_{x}+\mathscr{Y}_{y^{1}}^{1}+\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{r} \mathscr{Y}_{x}^{1} \tag{8.31}
\end{equation*}
$$

Comparing with $(8.27)_{3}$ yields:

$$
\begin{align*}
& \mathscr{Y}_{y^{1}}^{1}=(2+\mathrm{s}) \mathscr{X}_{x}+\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{r} \mathscr{Y}_{x}^{1}, \\
& \mathscr{Y}_{y^{2}}^{2}=(3+\mathrm{s}) \mathscr{X}_{x}+\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{r} \mathscr{Y}_{x}^{1} . \tag{8.32}
\end{align*}
$$

These are $(8.18)_{5}$ and $(8.18)_{11}$. Differentiating these two equations with respect to $x$, replacing $\mathscr{Y}_{x}^{2}$ and erasing $\mathscr{Y}_{x x}^{1}$, we get:

$$
\begin{align*}
& \mathscr{Y}_{x y^{1}}^{1}=(2+\mathrm{s}) \mathscr{X}_{x x}+\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{r} \mathscr{Y}_{x}^{1}+\mathrm{r} \mathscr{X}_{x},  \tag{8.33}\\
& \mathscr{Y}_{x y^{2}}^{2}=(3+\mathrm{s}) \mathscr{X}_{x x}+\mathrm{r} \mathscr{X}+\mathrm{r} \mathscr{Y}^{1}+\mathrm{r} \mathscr{Y}^{2}+\mathrm{r} \mathscr{Y}_{x}^{1}+\mathrm{r} \mathscr{X}_{x} .
\end{align*}
$$

We then replace this value of $\mathscr{Y}_{x y^{2}}^{2}$ in (8.29) and solve $\mathscr{X}_{x x}$ : this yields $(8.18)_{2}$.

To conclude, we replace $\mathscr{X}_{x x}$ so obtained in $(8.27)_{7}$ : this yields $(8.18)_{8}$. We replace $\mathscr{Y}_{y^{1}}^{1}$ and $\mathscr{Y}_{y^{2}}^{2}$ from (8.32) in (8.27) ${ }_{4}$ and in (8.27) $)_{5}$ : this yields $(8.18)_{4}$ and this yields $(8.18)_{9}$. Thanks to $(8.18)_{2}$ (got) we observe that

$$
\begin{equation*}
\mathrm{P}_{x}=\mathrm{P}+\mathrm{r} \mathscr{Y}_{x x}^{1}+\mathrm{r} \mathscr{Y}_{x}^{2}+\mathrm{r} \mathscr{X}_{x x}=\mathrm{P} . \tag{8.34}
\end{equation*}
$$

Differentiating $(8.18)_{4}$ (got) and (8.18) $)_{9}$ (got) with respect to $x$ then yields $(8.18)_{7}$ and $(8.18)_{12}$. We replace $\mathscr{Y}_{y^{1}}^{1}$ and $\mathscr{Y}_{y^{2}}^{2}$ from (8.18) (got) and (8.18) 11 (got) in $(8.27)_{2}$ : this yields $(8.18)_{6}$. Finally, to obtain the very last $(8.18)_{13}$, we differentiate $(8.18)_{10}$ (got) with respect to $x$.

The proof of Proposition 8.16 is complete.
We claim that the bound $\operatorname{dim} \mathfrak{S Y M}\left(\mathscr{E}_{4}\right) \leqslant 5$ is attained for the model (8.4). Indeed, with $0=r=s$ and $0=\mathrm{g}^{1}=\mathrm{g}^{2}=\mathrm{h}$ (whence $\left.\mathbf{k}^{*}=0\right)(8.24)$ is $\mathscr{Y}_{y^{2}}^{1}=0$ and then the seven equations (8.27) are:

$$
\left\{\begin{array}{l}
0=-\mathscr{Y}_{x}^{2}+2 x \mathscr{Y}_{x}^{1},  \tag{8.35}\\
0=-\mathscr{Y}_{y^{1}}^{2}-2 x \mathscr{Y}_{y^{2}}^{2}+2 \mathscr{X}+2 \mathscr{Y}_{x}^{1}+2 x \mathscr{Y}_{y^{1}}^{1}, \\
0=-\mathscr{Y}_{y^{2}}^{2}-\mathscr{X}_{x}+2 \mathscr{Y}_{y^{1}}^{1}, \\
0=-\mathscr{X}_{y^{1}}, \\
0=-\mathscr{X}_{y^{2}}, \\
0=-\mathscr{Y}_{x x}^{1}, \\
0=-2 \mathscr{Y}_{x y}^{1}+\mathscr{X}_{x x},
\end{array}\right.
$$

having the general solution

$$
\left\{\begin{align*}
\mathscr{X} & =a-d+e x,  \tag{8.36}\\
\mathscr{Y}^{1} & =b+d x+2 e y^{1}, \\
\mathscr{Y}^{2} & =c+2 a y^{1}+3 e y^{2}+d x x .
\end{align*}\right.
$$

depending on five parameters $a, b, c, d, e \in \mathbb{K}$. Five generators of $\mathfrak{S M M}\left(\mathscr{E}_{4}\right)$ are:

$$
\left\{\begin{align*}
\mathscr{D} & :=x \partial_{x}+2 y^{1} \partial_{y^{1}}+3 y^{2} \partial_{y^{2}}  \tag{8.37}\\
\mathscr{L}_{1} & :=-\partial_{x}+x \partial_{y^{1}}+x x \partial_{y^{2}} \\
\mathscr{L}_{1}^{\prime} & :=\partial_{x}+2 y^{1} \partial_{y^{2}} \\
\mathscr{L}_{2} & :=\partial_{y^{1}} \\
\mathscr{L}_{3} & :=\partial_{y^{2}}
\end{align*}\right.
$$

The commutator table

|  | $\mathscr{D}$ | $\mathscr{L}_{1}$ | $\mathscr{L}_{1}^{\prime}$ | $\mathscr{L}_{2}$ | $\mathscr{L}_{3}$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathscr{D}$ | 0 | $-\mathscr{L}_{1}$ | $-\mathscr{L}_{1}^{\prime}$ | $-2 \mathscr{L}_{2}$ | $-3 \mathscr{L}_{3}$ |
| $\mathscr{L}_{1}$ | $\mathscr{L}_{1}$ | 0 | $-\mathscr{L}_{2}$ | 0 | 0 |
| $\mathscr{L}_{1}^{\prime}$ | $\mathscr{L}_{1}^{\prime}$ | $\mathscr{L}_{2}$ | 0 | $-2 \mathscr{L}_{3}$ | 0 |
| $\mathscr{L}_{2}$ | $2 \mathscr{L}_{2}$ | 0 | $2 \mathscr{L}_{3}$ | 0 | 0 |
| $\mathscr{L}_{3}$ | $3 \mathscr{L}_{3}$ | 0 | 0 | 0 | 0 |

Table 3.
shows that the subalgebra spanned by $\mathscr{L}_{1}, \mathscr{L}_{1}^{\prime}, \mathscr{L}_{2}, \mathscr{L}_{3}$ is isomorphic to the unique irreducible 4 -dimensional nilpotent Lie algebra $\mathfrak{n}_{4}^{1}$ ([OV1994, BES2005]). Then $\mathfrak{S Y M}\left(\mathscr{E}_{4}\right)$ is a semidirect product of $\mathbb{K}$ with $\mathfrak{n}_{4}^{1}$. The author ignores whether it is rigid. The following accessible research will be pursued in a subsequent publication.

Open problem 8.38. Classify systems ( $\mathscr{E}_{4}$ ) up to point transformations. Deduce a complete classification, up to local biholomorphisms, of all real analytic generic submanifolds of codimension 2 in $\mathbb{C}^{3}$, valid at a Zariski-generic point.
8.39. Almost everywhere rigid hypersurfaces. When studying and classifying differential objects, it is essentially no restriction to assume their Lie symmetry groups to be of dimension $\geqslant 1$, the study of objects having no infinitesimal symmetries being an independent field of research. In particular, if $M \subset \mathbb{C}^{n+1}(n \geqslant 1)$ is a connected real analytic hypersurface, we may suppose that $\operatorname{dim} \mathfrak{h o l}(M) \geqslant 1$, at least. So let $\mathscr{L}$ be a nonzero holomorphic vector field with $\mathscr{L}+\overline{\mathscr{L}}$ tangent to $M$.

Lemma 8.40. ([Ca1932a, St1996, BER1999]) If in addition $M$ is finitely nondegenerate, then

$$
\begin{equation*}
\Sigma:=\left\{p \in M: \mathscr{L}(p) \in T_{p}^{c} M\right\} \tag{8.41}
\end{equation*}
$$

is a proper real analytic subset of $M$.
In other words, at every point $p$ belonging to the Zariski-dense subset $M \backslash \Sigma$, the real nonzero vector $\mathscr{L}(p)+\overline{\mathscr{L}(p)} \in T_{p} M$ supplements $T_{p}^{c} M$. Straightening $\mathscr{L}$ in a neighborhood of $p$, there exist local coordinates $t=$ $\left(z_{1}, \ldots, z_{n}, w\right)$ with $T_{0}^{c} M=\{w=0\}, T_{0} M=\{\operatorname{Im} w=0\}$, whence $M$ is given by $\operatorname{Im} w=h(z, \bar{z}, \operatorname{Re} w)$, and with $\mathscr{L}=\frac{\partial}{\partial w}$. The tangency of $\frac{\partial}{\partial w}+\frac{\partial}{\partial \bar{w}}=\frac{\partial}{\partial u}$ to $M$ entails that $h$ is indendepent of $u$. Then the complex equation of $M$ is of the precise form

$$
\begin{equation*}
w=\bar{w}+i \bar{\Theta}(z, \bar{z}) \tag{8.42}
\end{equation*}
$$

with $\bar{\Theta}=2 h$ simply. The reality of $h$ reads $\bar{\Theta}(z, \bar{z}) \equiv \Theta(\bar{z}, z)$.

Definition 8.43. A real analytic hypersurface $M \subset \mathbb{C}^{n+1}$ is called rigid at one of its points $p$ if there exists $\mathscr{L} \in \mathfrak{h o l}(M)$ with

$$
\begin{equation*}
T_{p} M=T_{p}^{c} M \oplus \mathbb{R}(\mathscr{L}(p)+\overline{\mathscr{L}}(p)) . \tag{8.44}
\end{equation*}
$$

Similar elementary facts hold for general submanifolds of solutions.
Lemma 8.45. With $n \geqslant 1$ and $m=1$, let $\mathscr{M}$ be a (connected) submanifold of solutions that is solvable with respect to the parameters. If there exists a nonzero $\mathscr{L}+\mathscr{L}^{*} \in \mathfrak{S M M}(\mathscr{M})$, then at Zariski-generic points $p \in \mathscr{M}$, we have $\mathscr{L}(p) \notin \mathrm{F}_{\mathrm{v}}(p)$ and there exist local coordinates centered at $p$ in which $\mathscr{L}=\frac{\partial}{\partial y}, \mathscr{L}^{*}=\frac{\partial}{\partial b}$, whence $\mathscr{M}$ has equation of the form

$$
\begin{equation*}
y=b+\Pi(x, a) \tag{8.46}
\end{equation*}
$$

with $\Pi$ independent of $b$.
The associated system $\left(\mathscr{E}_{\mathscr{M}}\right)$ has then equations $F_{\alpha}$ that are all independent of $y$.
8.47. Study of the Lie symmetries of $\left(\mathscr{E}_{5}\right)$. In Example 1.28, it is thus essentially no restriction to assume the hypersurface $M \subset \mathbb{C}^{3}$ to be rigid.

Theorem 8.48. ([GM2003b, FK2005a, FK2005b]) The model hypersurface $M_{0}$ of equation

$$
\begin{equation*}
w=\bar{w}+i \frac{2 z^{1} \bar{z}^{1}+z^{1} z^{1} \bar{z}^{2}+\bar{z}^{1} \bar{z}^{1} z^{2}}{1-z^{2} \bar{z}^{2}} \tag{8.49}
\end{equation*}
$$

has transitive Lie symmetry algebra $\mathfrak{h o l}\left(M_{0}\right)$ isomorphic to $\mathfrak{s o}(3,2)$ and is locally biholomorphic to a neighborhood of every geometrically smooth point of the tube

$$
\begin{equation*}
\left(\operatorname{Re} w^{\prime}\right)^{2}=\left(\operatorname{Re} z_{1}^{\prime}\right)^{2}+\left(\operatorname{Re} z_{1}^{\prime}\right)^{3} \tag{8.50}
\end{equation*}
$$

over the standard cone of $\mathbb{R}^{3}$. Both are Levi-degenerate with Levi form of rank 1 at every point and are 2-nondegenerate. The associated PDE system $\left(\mathscr{E}_{M_{0}}\right)$

$$
\begin{equation*}
y_{x^{2}}=\frac{1}{4}\left(y_{x^{1}}\right)^{2}, \quad y_{x^{1} x^{1} x^{1}}=0 \tag{8.51}
\end{equation*}
$$

(plus other equations obtained by cross differentiation) has infinitesimal Lie symmetry algebra isomorphic to $\mathfrak{s o}(5, \mathbb{C})$, the complexification $\mathfrak{s o}(3,2) \otimes \mathbb{C}$.

Through tentative issues ([Eb2006, GM2006]), it has been suspected that $M_{0}$ is the right model in the category of real analytic hypersurfaces $M \subset \mathbb{C}^{3}$ having Levi form of rank 1 that are 2-nondegenerate everywhere. Based on the rigidity of the simple Lie algebra $\mathfrak{s o}(5, \mathbb{C})$ (Theorem 5.15), Theorem 8.105 below will confirm this expectation.
8.52. Preparation. Thus, translating the considerations to the PDE language, with $n=2$ and $m=1$, consider a submanifold of solutions of the form

$$
\begin{align*}
y & =b+\Pi(x, a) \\
& =b+\frac{2 x^{1} a^{1}+x^{1} x^{1} a^{2}+a^{1} a^{1} x^{2}}{1-x^{2} a^{2}}+\mathrm{O}_{4}, \tag{8.53}
\end{align*}
$$

where $\mathrm{O}_{4}$ is a function of $(x, a)$ only. The term $2 x^{1} a^{1}$ corresponds to a Levi form of rank $\geqslant 1$ at every point. The term $x^{1} x^{1} a^{2}$ guarantees solvability with respect to the parameters (compare Definition 2.12). Let us develope

$$
\begin{equation*}
\Pi(x, a)=\sum_{k_{1}, k_{2} \geqslant 0} \sum_{l_{1}, l_{2} \geqslant 0} \Pi_{k_{1}, k_{2}, l_{1}, l_{2}}\left(x^{1}\right)^{k_{1}}\left(x^{2}\right)^{k_{2}}\left(a^{1}\right)^{l_{1}}\left(a^{2}\right)^{l_{2}}, \tag{8.54}
\end{equation*}
$$

with $\Pi_{k_{1}, k_{2}, l_{1}, l_{2}} \in \mathbb{K}$. Of course, $\Pi_{1,0,1,0}=2, \Pi_{2,0,0,1}=1$ and $\Pi_{0,1,2,0}=1$.
Lemma 8.55. A transformation belonging to $\mathrm{G}_{\mathrm{v}, \mathrm{p}}$ insures

$$
\begin{array}{llll}
\Pi_{k_{1}, k_{2}, 0,0}=0, & k_{1}+k_{2} \geqslant 0, & \Pi_{0,0, l_{1}, l_{2}}=0, & l_{1}+l_{2} \geqslant 0,  \tag{8.56}\\
\Pi_{k_{1}, k_{2}, 1,0}=0, & k_{1}+k_{2} \geqslant 2, & \Pi_{1,0, l_{1}, l_{2}}=0, & l_{1}+l_{2} \geqslant 2, \\
\Pi_{k_{1}, k_{2}, 2,0}=0, & k_{1}+k_{2} \geqslant 2, & \Pi_{2,0, l_{1}, l_{2}}=0, & l_{1}+l_{2} \geqslant 2 .
\end{array}
$$

Proof. Lemma 7.11 achieves the first line. The monomial $x^{1}$ being factored by $\left[a^{1}+\mathrm{O}_{2}(a)\right]$, we set $a^{1}:=a^{1}+\mathrm{O}_{2}(a)$ to achieve $\Pi_{1,0, l_{1}, l_{2}}=0, l_{1}+l_{2} \geqslant 2$. As in the proof of Lemma 7.18, we pass to the dual equation $b=y-\Pi(x, a)$ to complete $\Pi_{k_{1}, k_{2}, 1,0}=0, k_{1}+k_{2} \geqslant 2$. Finally, $x^{1} x^{1}$ is factored by $\left[a^{2}+\right.$ $\mathrm{O}_{2}(a)$ ], so we proceed similarly to achieve the third line.

Since $\Pi(x, a)$ is assumed to be independent of $b$, the assumption that the Levi form of $M \subset \mathbb{C}^{3}$ has exactly rank 1 at every point translates to:

$$
0 \equiv\left|\begin{array}{cc}
\Pi_{x^{1} a^{1}} & \Pi_{x^{1} a^{2}}  \tag{8.57}\\
\Pi_{x^{2} a^{1}} & \Pi_{x^{2} a^{2}}
\end{array}\right| .
$$

For later use, it is convenient to develope somehow $\Pi$ with respect to the powers of $\left(a^{1}, a^{2}\right)$ :

$$
\begin{align*}
y=b+ & \frac{2 x^{1} a^{1}+x^{1} x^{1} a^{2}+a^{1} a^{1} x^{2}}{1-x^{2} a^{2}}+  \tag{8.58}\\
& +a^{2} \mathbf{b}(x)+a^{1} a^{2} \mathbf{d}(x)+a^{2} a^{2} \mathbf{e}(x)+a^{1} a^{1} a^{1} \mathbf{f}(x)+a^{1} a^{1} a^{2} \mathbf{g}(x)+ \\
& +\left(a^{1}\right)^{4} \mathrm{R}+\left(a^{1}\right)^{3} a^{2} \mathrm{R}+a^{1}\left(a^{2}\right)^{2} \mathrm{R}+\left(a^{2}\right)^{3} \mathrm{R}
\end{align*}
$$

with $\mathrm{R}=\mathrm{R}(x, a)$ being an unspecified remainder. Thanks to the previous lemma, the coefficients a of $a^{1}$ and $\mathbf{c}$ of $a^{1} a^{1}$ must vanish. The function $\mathbf{b}$ is an $\mathrm{O}_{3}$.

Lemma 8.59. The function $\mathbf{b}$ depends only on $x^{1}$, is an $\mathrm{O}_{3}\left(x^{1}\right)$ and the function $\mathbf{g}$ satisfies $\mathbf{g}_{x^{2} x^{2}}(0)=0$.

Proof. Developing $\left[1-x^{2} a^{2}\right]^{-1}=1+x^{2} a^{2}+\left(x^{2} a^{2}\right)^{2}+\mathrm{O}_{3}\left(x^{2} a^{2}\right)$, inserting the right hand side of

$$
\begin{align*}
y-b= & a^{1}\left[2 x^{1}\right]+a^{2}\left[x^{1} x^{1}+\mathbf{b}(x)\right]+a^{1} a^{1}\left[x^{2}\right]+a^{1} a^{2}\left[2 x^{1} x^{2}+\mathbf{d}(x)\right]+  \tag{8.60}\\
& +a^{2} a^{2}\left[x^{1} x^{1} x^{2}+\mathbf{e}(x)\right]+a^{1} a^{1} a^{1}[\mathbf{f}(x)]+a^{1} a^{1} a^{2}\left[x^{2} x^{2}+\mathbf{g}(x)\right]+ \\
& +\left(a^{1}\right)^{4} \mathrm{R}+\left(a^{1}\right)^{3} a^{2} \mathrm{R}+a^{1}\left(a^{2}\right)^{2} \mathrm{R}+\left(a^{2}\right)^{3} \mathrm{R}
\end{align*}
$$

in the determinant (8.57) and selecting the coefficients of cst., of $a^{1}$, of $a^{2}$ and of $a^{1} a^{1}$, we get four PDEs:

$$
\begin{align*}
& 0=2 \mathbf{b}_{x^{2}}, \\
& 0=2 \mathbf{d}_{x^{2}}-2 \mathbf{b}_{x^{1}}, \\
& 0=4 \mathbf{e}_{x^{2}}-2 x^{1} \mathbf{d}_{x^{2}}-2 x^{1} \mathbf{b}_{x^{1}}-\mathbf{d}_{x^{2}} \mathbf{b}_{x^{1}},  \tag{8.61}\\
& 0=2 \mathbf{g}_{x^{2}}-2 \mathbf{d}_{x^{1}}-\left[6 x^{1}+3 \mathbf{b}_{x^{1}}\right] \mathbf{f}_{x^{2}} .
\end{align*}
$$

The first one yields $\mathbf{b}=\mathbf{b}\left(x^{1}\right)$, which must be an $\mathrm{O}_{3}\left(x^{1}\right)$, because the whole remainder is an $\mathrm{O}_{4}$. Differentiating the fourth with respect to $x^{2}$, it then follows that $\mathbf{g}_{x^{2} x^{2}}(0)=0$.
8.62. Associated PDE system $\left(\mathscr{E}_{5}\right)$. Next, differentiating (8.60) with respect to $x^{1}$, to $x^{1} x^{1}$ and to $x^{1} x^{1} x^{1}$, we compute $y_{x^{1}}$ and $y_{x^{1} x^{1}}$, we substitute $y_{1}$ and $y_{1,1}$ and we push the monomials $a^{2} a^{2}, a^{1} a^{1} a^{1}$ and $a^{1} a^{1} a^{2}$ in the remainder:

$$
\begin{align*}
y_{1} & =2 a^{1}+a^{2}\left[2 x^{1}+\mathbf{b}_{x^{1}}\right]+a^{1} a^{2}\left[2 x^{2}+\mathbf{d}_{x^{1}}\right]+\left(a^{2}\right)^{2} \mathrm{R}+\left(a^{1}\right)^{3} \mathrm{R}+\left(a^{1}\right)^{2} a^{2} \mathrm{R},  \tag{8.63}\\
y_{1,1} & =a^{2}\left[2+\mathbf{b}_{x^{1} x^{1}}\right]+a^{1} a^{2}\left[\mathbf{d}_{x^{1} x^{1}}\right]+\left(a^{2}\right)^{2} \mathrm{R}+\left(a^{1}\right)^{3} \mathrm{R}+\left(a^{1}\right)^{2} a^{2} \mathrm{R}, \\
y_{1,1,1} & =a^{2}\left[\mathbf{b}_{x^{1} x^{1} x^{1}}\right]+a^{1} a^{2}\left[\mathbf{d}_{x^{1} x^{1} x^{1}}\right]+\left(a^{2}\right)^{2} \mathrm{R}+\left(a^{1}\right)^{3} \mathrm{R}+\left(a^{1}\right)^{2} a^{2} \mathrm{R} .
\end{align*}
$$

Here, the written remainder cannot incorporate $a^{1} a^{1}$, so it is said that the coefficient of $a^{1} a^{1}$ does vanish in each equation above. Solving for $a^{1}$ and $a^{2}$ from the first two equations, we get

$$
\left\{\begin{align*}
a^{1}= & \frac{1}{2} y_{1}-y_{1,1}\left[\frac{2 x^{1}+\mathbf{b}_{x^{1}}}{4+2 \mathbf{b}_{x^{1} x^{1}}}\right]-y_{1} y_{1,1}\left[\frac{2 x^{2}+\mathbf{d}_{x^{1}}}{8+4 \mathbf{b}_{x^{1} x^{1}}}\right]+  \tag{8.64}\\
& +\left(y_{1,1}\right)^{2} \mathrm{R}+\left(y_{1}\right)^{3} \mathrm{R}+\left(y_{1}\right)^{2} y_{1,1} \mathbf{R}, \\
a^{2}= & y_{1,1}\left[\frac{1}{2+\mathbf{b}_{x^{1} x^{1}}}\right]-y_{1} y_{1,1}\left[\frac{\mathbf{d}_{x^{1} x^{1}}}{2\left(2+\mathbf{b}_{x^{1} x^{1}}\right)^{2}}\right]+\left(y_{1,1}\right)^{2} \mathrm{R}+\left(y_{1}\right)^{3} \mathrm{R}+\left(y_{1}\right)^{2} y_{1,1} \mathrm{R} .
\end{align*}\right.
$$

We then get (notice the change of remainder):

$$
\begin{align*}
a^{1} a^{1} & =\frac{1}{4}\left(y_{1}\right)^{2}-y_{1} y_{1,1}\left[\frac{2 x^{1}+\mathbf{b}_{x^{1}}}{4+2 \mathbf{b}_{x^{1} x^{1}}}\right]-\left(y_{1}\right)^{2} y_{1,1}\left[\frac{2 x^{2}+\mathbf{d}_{x^{1}}}{8+4 \mathbf{b}_{x^{1} x^{1}}}\right]+\left(y_{1}\right)^{3} \mathrm{R}+\left(y_{1,1}\right)^{2} \mathrm{R},  \tag{8.65}\\
a^{1} a^{2} & =y_{1} y_{1,1}\left[\frac{1}{4+2 \mathbf{b}_{x^{1} x^{1}}}\right]-\left(y_{1}\right)^{2} y_{1,1}\left[\frac{\mathbf{d}_{x^{1} x^{1}}}{\left(4+2 \mathbf{b}_{x^{1} x^{1}}\right)^{2}}\right]+\left(y_{1}\right)^{3} \mathrm{R}+\left(y_{1,1}\right)^{2} \mathrm{R}, \\
a^{2} a^{2} & =\left(y_{1}\right)^{3} \mathrm{R}+\left(y_{1,1}\right)^{2} \mathrm{R}, \\
a^{1} a^{1} a^{1} & =-\left(y_{1}\right)^{2} y_{1,1}\left[\frac{6 x^{1}+3 \mathbf{b}_{x^{1}}}{16+8 \mathbf{b}_{x^{1} x^{1}}}\right]\left(y_{1}\right)^{3} \mathrm{R}+\left(y_{1,1}\right)^{2} \mathrm{R}, \\
a^{1} a^{1} a^{2} & =\left(y_{1}\right)^{2} y_{1,1}\left[\frac{1}{8+4 \mathbf{b}_{x^{1} x^{1}}}\right]+\left(y_{1}\right)^{3} \mathrm{R}+\left(y_{1,1}\right)^{2} \mathrm{R} .
\end{align*}
$$

Differentiating (8.60) with respect to $x^{2}$, substituting $y_{2}$ for $y_{x^{2}}$ and replacing $\mathbf{d}_{x^{2}}$ by $\mathbf{b}_{x^{1}}$ thanks to $(8.61)_{2}$, we get

$$
\begin{align*}
y_{2}= & a^{1} a^{1}+a^{1} a^{2}\left[2 x^{1}+\mathbf{b}_{x^{1}}\right]+a^{2} a^{2}\left[x^{1} x^{1}+\mathbf{e}_{x^{2}}\right]+a^{1} a^{1} a^{1}\left[\mathbf{f}_{x^{2}}\right]+a^{1} a^{1} a^{2}\left[2 x^{2}+\mathbf{g}_{x^{2}}\right]+  \tag{8.66}\\
& +\left(a^{1}\right)^{4} \mathrm{R}+\left(a^{1}\right)^{3} a^{2} \mathrm{R}+a^{1}\left(a^{2}\right)^{2} \mathrm{R}+\left(a^{2}\right)^{3} \mathrm{R} .
\end{align*}
$$

Replacing the monomials (8.65), we finally obtain:
(8.67)

$$
\begin{aligned}
y_{2}=\frac{1}{4}\left(y_{1}\right)^{2}+ & \left(y_{1}\right)^{2} y_{1,1}\left[\frac{2 \mathbf{g}_{x^{2}}-2 \mathbf{d}_{x^{1}}-\left(6 x^{1}+3 \mathbf{b}_{x^{1}}\right) \mathbf{f}_{x^{2}}}{16+8 \mathbf{b}_{x^{1} x^{1}}}-\frac{\left(2 x^{1}+\mathbf{b}_{x^{1}}\right) \mathbf{d}_{x^{1} x^{1}}}{\left(4+2 \mathbf{b}_{x^{1} x^{1}}\right)^{2}}\right]+ \\
& +\left(y_{1}\right)^{3} \mathbf{R}+\left(y_{1,1}\right)^{2} \mathbf{R} .
\end{aligned}
$$

Thanks to $(8.61)_{4}$, the first (big) coefficient of $\left(y_{1}\right)^{2} y_{1,1}$ is zero; then the remainder coefficient is an $\mathrm{O}\left(x^{1}\right)$, hence vanishes at $x=0$, together with its partial first derivative with respect to $x^{2}$. Accordingly, by $\mathrm{s}^{*}=\mathrm{s}^{*}\left(x^{1}, x^{2}\right)$, we will denote an unspecified function satisfying

$$
\begin{equation*}
\mathrm{s}^{*}(0)=0 \quad \text { and } \quad \mathrm{s}_{x^{2}}^{*}(0)=0 \text {. } \tag{8.68}
\end{equation*}
$$

Lemma 8.69. The skeleton of the PDE system ( $\mathscr{E}_{5}$ ) associated to the submanifold (8.58) possesses three main equations of the form
( $\Delta_{\mathscr{C}_{5}}$ )

$$
\begin{aligned}
& \int_{2}=\frac{1}{4}\left(y_{1}\right)^{2}+\left(y_{1}\right)^{3} r+\left(y_{1}\right)^{4} r+\left(y_{1}\right)^{5} r+\left(y_{1}\right)^{6} \mathrm{R}+ \\
& +y_{1,1}\left[\left(y_{1}\right)^{2} \mathbf{s}^{*}+\left(y_{1}\right)^{3} \mathbf{r}+\left(y_{1}\right)^{4} \mathbf{r}+\left(y_{1}\right)^{5} \mathbf{r}\right]+\left(y_{1,1}\right)^{2} \mathbf{R}, \\
& y_{1,2}=\frac{1}{2} y_{1} y_{1,1}+\left(y_{1}\right)^{3} r+\left(y_{1}\right)^{4} r+\left(y_{1}\right)^{5} r+\left(y_{1}\right)^{6} R+ \\
& +y_{1,1}\left[\left(y_{1}\right)^{2} \mathrm{r}+\left(y_{1}\right)^{3} \mathrm{r}+\left(y_{1}\right)^{4} \mathrm{r}+\left(y_{1}\right)^{5} \mathrm{r}\right]+\left(y_{1}\right)^{6} \mathrm{R}, \\
& y_{1,1,1}=\left(y_{1}\right)^{3} r+\left(y_{1}\right)^{4} r+y_{1,1}\left[r+y_{1} r+\left(y_{1}\right)^{2} r+\left(y_{1}\right)^{3} r\right]+ \\
& +\left(y_{1,1}\right)^{2}\left[r+y_{1} r+\left(y_{1}\right)^{2} r+\left(y_{1}\right)^{3} r\right]+\left(y_{1,1}\right)^{3} R,
\end{aligned}
$$

where the letter r denotes an unspecified function of $\left(x^{1}, x^{2}\right)$, and where the coefficient $\mathrm{s}^{*}$ of $\left(y_{1}\right)^{2} y_{1,1}$ in the first equation satisfies (8.68).

Proof. To get the second equation, we compute:

$$
\begin{align*}
y_{1,2} & =a^{1} a^{2}\left[2+\mathbf{b}_{x^{1} x^{1}}\right]+a^{1} a^{1} a^{2}\left[\mathbf{g}_{x^{1} x^{2}}\right]+\left(a^{1}\right)^{3} \mathrm{R}+\left(a^{2}\right)^{2} \mathrm{R} \\
& =\frac{1}{2} y_{1} y_{1,1}+\left(y_{1}\right)^{2} y_{1,1} \mathbf{r}+\left(y_{1}\right)^{3} \mathrm{R}+\left(y_{1,1}\right)^{2} \mathrm{R} . \tag{8.70}
\end{align*}
$$

The third equation is got similarly from $(8.63)_{3}$. To conclude, we develope the first two equations $\bmod \left[\left(y_{1}\right)^{6},\left(y_{1,1}\right)^{2}\right]$ and the third one $\bmod \left[\left(y_{1}\right)^{4},\left(y_{1,1}\right)^{3}\right]$.

This precise skeleton will be referred to as $\Delta_{\mathscr{C}_{5}}$ in the sequel. With the letter $r$, the computation rules are cst. $r=r+r=r+s^{*}=r \cdot r=r$; sometimes, $\mathrm{s}^{*}$ may be replaced plainly by r .
8.71. Infinitesimal Lie symmetries of ( $\mathscr{E}_{5}$ ). Letting $\mathscr{L}=\mathscr{X}^{1} \frac{\partial}{\partial x^{1}}+$ $\mathscr{X}^{2} \frac{\partial}{\partial x^{2}}+\mathscr{Y} \frac{\partial}{\partial y}$ be a candidate infinitesimal Lie symmetry and applying

$$
\begin{align*}
\mathscr{L}^{(3)}= & \mathscr{X}^{1} \frac{\partial}{\partial x^{1}}+\mathscr{X}^{2} \frac{\partial}{\partial x^{2}}+\mathscr{Y} \frac{\partial}{\partial y}+\mathbf{Y}_{1} \frac{\partial}{\partial y_{1}}+\mathbf{Y}_{2} \frac{\partial}{\partial y_{2}}+ \\
& +\mathbf{Y}_{1,1} \frac{\partial}{\partial y_{1,1}}+\mathbf{Y}_{1,2} \frac{\partial}{\partial y_{1,2}}+\mathbf{Y}_{2,1} \frac{\partial}{\partial y_{2,1}}+\mathbf{Y}_{2,2} \frac{\partial}{\partial y_{2,2}}+  \tag{8.72}\\
& +\mathbf{Y}_{1,1,1} \frac{\partial}{\partial y_{1,1,1}}+\cdots+\mathbf{Y}_{2,2,2} \frac{\partial}{\partial y_{2,2,2}}
\end{align*}
$$

to the skeleton $\Delta_{\mathscr{C}_{5}}$, we obtain firstly, computing $\bmod \left[\left(y_{1}\right)^{5}, y_{1,1}\right]$ :

$$
\begin{align*}
0 \equiv & -\mathbf{Y}_{2}+\frac{1}{2} y_{1} \mathbf{Y}_{1}+ \\
& +\left(y_{1}\right)^{3} \mathrm{r} \mathscr{X}^{1}+\left(y_{1}\right)^{4} \mathrm{r} \mathscr{X}^{1}+\left(y_{1}\right)^{3} \mathrm{r} \mathscr{X}^{2}+\left(y_{1}\right)^{4} \mathrm{r} \mathscr{X}^{2}+  \tag{8.73}\\
& +\mathbf{Y}_{1}\left[\left(y_{1}\right)^{2} \mathrm{r}+\left(y_{1}\right)^{3} \mathrm{r}+\left(y_{1}\right)^{4} \mathrm{r}\right]+ \\
& +\mathbf{Y}_{1,1}\left[\left(y_{1}\right)^{2} \mathrm{~s}^{*}+\left(y_{1}\right)^{3} \mathrm{r}+\left(y_{1}\right)^{4} \mathrm{r}\right],
\end{align*}
$$

secondly, computing $\bmod \left[\left(y_{1}\right)^{5}, y_{1,1}\right]$ :

$$
\begin{align*}
0 \equiv & -\mathbf{Y}_{1,2}+\frac{1}{2} y_{1} \mathbf{Y}_{1,1}+ \\
& +\left(y_{1}\right)^{3} \mathrm{r} \mathscr{X}^{1}+\left(y_{1}\right)^{4} \mathrm{r} \mathscr{X}^{1}+\left(y_{1}\right)^{3} \mathrm{r} \mathscr{X}^{2}+\left(y_{1}\right)^{4} \mathrm{r} \mathscr{X}^{2}+  \tag{8.74}\\
& +\mathbf{Y}_{1}\left[\left(y_{1}\right)^{2} \mathrm{r}+\left(y_{1}\right)^{3} \mathrm{r}+\left(y_{1}\right)^{4} \mathrm{r}\right]+ \\
& +\mathbf{Y}_{1,1}\left[\left(y_{1}\right)^{2} \mathrm{r}+\left(y_{1}\right)^{3} \mathrm{r}+\left(y_{1}\right)^{4} \mathrm{r}\right],
\end{align*}
$$

and thirdly, computing $\bmod \left[\left(y_{1}\right)^{3},\left(y_{1,1}\right)^{2}\right]$ :
(8.75)

$$
\begin{aligned}
0 \equiv & -\mathbf{Y}_{1,1,1}+y_{1,1} \mathscr{X}^{1}+y_{1,1} \mathscr{X}^{2}+ \\
& +y_{1,1} y_{1} \mathscr{X}^{1}+y_{1,1} y_{1} \mathscr{X}^{2}+y_{1,1}\left(y_{1}\right)^{2} \mathscr{X}^{1}+y_{1,1}\left(y_{1}\right)^{2} \mathscr{X}^{2}+ \\
& +\mathbf{Y}_{1}\left[\left(y_{1}\right)^{2} \mathbf{r}\right]+\mathbf{Y}_{1,1}\left[\mathrm{r}+y_{1} \mathbf{r}+\left(y_{1}\right)^{2} \mathbf{r}\right]+y_{1,1} \mathbf{Y}_{1}\left[\mathrm{r}+y_{1} \mathbf{r}+\left(y_{1}\right)^{2} \mathrm{r}\right]+ \\
& \quad+y_{1,1} \mathbf{Y}_{1,1}\left[\mathbf{r}+y_{1} \mathbf{r}+\left(y_{1}\right)^{2} \mathbf{r}\right] .
\end{aligned}
$$

Specializing to $n=2$ the formulas (3.9)(II), (3.20)(II) and (3.24)(II), we get $\mathbf{Y}_{1}, \mathbf{Y}_{2}, \mathbf{Y}_{1,1}, \mathbf{Y}_{1,2}$ and $\mathbf{Y}_{1,1,1}$ :
(8.76)
$\mathbf{Y}_{1}=\mathscr{Y}_{x^{1}}+\left[\mathscr{Y}_{y}-\mathscr{X}_{x^{1}}^{1}\right] y_{1}+\left[-\mathscr{X}_{x^{1}}^{2}\right] y_{2}+\left[-\mathscr{X}_{y}^{1}\right]\left(y_{1}\right)^{2}+\left[-\mathscr{X}_{y}^{2}\right] y_{1} y_{2}$.
(8.77)
$\mathbf{Y}_{2}=\mathscr{Y}_{x^{2}}+\left[-\mathscr{X}_{x^{2}}^{1}\right] y_{1}+\left[\mathscr{Y}_{y}-\mathscr{X}_{x^{2}}^{2}\right] y_{2}+\left[-\mathscr{X}_{y}^{1}\right] y_{1} y_{2}+\left[-\mathscr{X}_{y}^{2}\right] y_{2} y_{2}$.
(8.78)

$$
\left\{\begin{aligned}
\mathbf{Y}_{1,1}= & \mathscr{Y}_{x^{1} x^{1}}+\left[2 \mathscr{Y}_{x^{1} y}-\mathscr{X}_{x^{1} x^{1}}^{1}\right] y_{1}+\left[-\mathscr{X}_{x^{1} x^{1}}^{2}\right] y_{2}+\left[\mathscr{Y}_{y y}-2 \mathscr{X}_{x^{1} y}^{1}\right]\left(y_{1}\right)^{2}+ \\
& +\left[-2 \mathscr{X}_{x^{1} y}^{2}\right] y_{1} y_{2}+\left[-\mathscr{X}_{y y}^{1}\right]\left(y_{1}\right)^{3}+\left[-\mathscr{X}_{y y}^{2}\right]\left(y_{1}\right)^{2} y_{2}+ \\
& +\left[\mathscr{Y}_{y}-2 \mathscr{X}_{x^{1}}^{1}\right] y_{1,1}+\left[-2 \mathscr{X}_{x^{1}}^{2}\right] y_{1,2}+\left[-3 \mathscr{X}_{y}^{1}\right] y_{1} y_{1,1}+ \\
& +\left[-\mathscr{X}_{y}^{2}\right] y_{2} y_{1,1}+\left[-2 \mathscr{X}_{y}^{2}\right] y_{1} y_{1,2} .
\end{aligned}\right.
$$

$$
\left\{\begin{align*}
\mathbf{Y}_{1,2}= & \mathscr{Y}_{x^{1} x^{2}}+\left[\mathscr{Y}_{x^{2} y}-\mathscr{X}_{x^{1} x^{2}}^{1}\right] y_{1}+\left[\mathscr{Y}_{x^{1} y}-\mathscr{X}_{x^{1} x^{2}}^{2}\right] y_{2}+  \tag{8.79}\\
& +\left[-\mathscr{X}_{x^{2} y}^{1}\right]\left(y_{1}\right)^{2}+\left[\mathscr{Y}_{y y}-\mathscr{X}_{x^{2} y}^{2}-\mathscr{X}_{x^{1} y}^{1}\right] y_{1} y_{2}+\left[-\mathscr{X}_{x^{1} y}^{2}\right] y_{2} y_{2}+ \\
& +\left[-\mathscr{X}_{y y}^{1}\right]\left(y_{1}\right)^{2} y_{2}+\left[-\mathscr{X}_{y y}^{2}\right] y_{1}\left(y_{2}\right)^{2}+ \\
& +\left[-\mathscr{X}_{x^{2}}^{1}\right] y_{1,1}+\left[\mathscr{Y}_{y}-\mathscr{X}_{x^{2}}^{2}-\mathscr{X}_{x^{1}}^{1}\right] y_{1,2}+\left[-\mathscr{X}_{x^{1}}^{2}\right] y_{2,2}+ \\
& +\left[-2 \mathscr{X}_{y}^{1}\right] y_{1} y_{1,2}+\left[-2 \mathscr{X}_{y}^{2}\right] y_{2} y_{1,2} .
\end{align*}\right.
$$

(8.80)

$$
\left\{\begin{aligned}
\mathbf{Y}_{1,1,1}= & \mathscr{Y}_{x^{1} x^{1} x^{1}}+\left[3 \mathscr{Y}_{x^{1} x^{1} y}-\mathscr{X}_{x^{1} x^{1} x^{1}}^{1}\right] y_{1}+\left[-\mathscr{X}_{x^{1} x^{1} x^{1}}^{2}\right] y_{2}+ \\
& +\left[3 \mathscr{Y}_{x^{1} y y}-3 \mathscr{X}_{x^{1} x^{1} y}^{1}\right]\left(y_{1}\right)^{2}+\left[-3 \mathscr{X}_{x^{1} x^{1} y}^{2}\right] y_{1} y_{2}+ \\
& +\left[\mathscr{Y}_{y y y}-3 \mathscr{X}_{x^{1} y y}^{1}\right]\left(y_{1}\right)^{3}+\left[-3 \mathscr{X}_{x^{1} y y}^{2}\right]\left(y_{1}\right)^{2} y_{2}+ \\
& +\left[-\mathscr{X}_{y y y}^{1}\right]\left(y_{1}\right)^{4}+\left[-\mathscr{X}_{y y y}^{2}\right]\left(y_{1}\right)^{3} y_{2}+ \\
& +\left[3 \mathscr{Y}_{x^{1} y}-3 \mathscr{X}_{x^{1} x^{1}}^{1}\right] y_{1,1}+\left[-3 \mathscr{X}_{x^{1} x^{1}}^{2}\right] y_{1,2}+ \\
& +\left[3 \mathscr{Y}_{y y}-9 \mathscr{X}_{x^{1} y}^{1}\right] y_{1} y_{1,1}+\left[-3 \mathscr{X}_{x^{1} y}^{2}\right] y_{2} y_{1,1}+ \\
& +\left[-6 \mathscr{X}_{x^{1} y}^{2}\right] y_{1} y_{1,2}+\left[-6 \mathscr{X}_{y y}^{1}\right]\left(y_{1}\right)^{2} y_{1,1}+\left[-3 \mathscr{X}_{y y}^{2}\right] y_{1} y_{2} y_{1,1}+ \\
& +\left[-3 \mathscr{X}_{y y}^{2}\right]\left(y_{1}\right)^{2} y_{1,2}+\left[-3 \mathscr{X}_{y}^{1}\right]\left(y_{1,1}\right)^{2}+\left[-3 \mathscr{X}_{y}^{2}\right] y_{1,1} y_{1,2}+ \\
& +\left[\mathscr{Y}_{y}-3 \mathscr{X}_{x^{1}}^{1}\right] y_{1,1,1}+\left[-3 \mathscr{X}_{x^{1}}^{2}\right] y_{1,1,2}+\left[-4 \mathscr{X}_{y}^{1}\right] y_{1} y_{1,1,1}+ \\
& +\left[-\mathscr{X}_{y}^{2}\right] y_{2} y_{1,1,1}+\left[-3 \mathscr{X}_{y}^{2}\right] y_{1} y_{1,1,2} .
\end{aligned}\right.
$$

Inserting $\mathbf{Y}_{2}, \mathbf{Y}_{1,2}, \mathbf{Y}_{1,1,1}, \mathbf{Y}_{1}, \mathbf{Y}_{1,1}$ in the three Lie equations (8.73), (8.74), (8.75), replacing $y_{2}, y_{1,2}, y_{1,1,1}$ by the values they have on $\Delta_{\mathscr{E}_{5}}$, we get firstly five linear PDEs by picking the coefficients of cst., of $y_{1}$, of $\left(y_{1}\right)^{2}$, of $\left(y_{1}\right)^{3}$, of $\left(y_{1}\right)^{4}$ in (8.73):
(8.81)
secondly, we get three more linear PDEs by picking the coefficients of $\left(y_{1}\right)^{2}$, of $\left(y_{1}\right)^{3}$, of $\left(y_{1}\right)^{4}$ in (8.74):
(8.82)

$$
\left\{\begin{array}{c}
0=3 \mathscr{Y}_{x^{1} y}+\mathscr{X}_{x^{1} x^{2}}^{2}+4 \mathscr{X}_{x^{2} y}^{1}-2 \mathscr{X}_{x^{1} x^{1}}^{1}+\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}} \\
0=2 \mathscr{Y}_{y y}+2 \mathscr{X}_{x^{2} y}^{2}-6 \mathscr{X}_{x^{1} y}^{1}-\mathscr{X}_{x^{1} x^{1}}^{2}+\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{X}_{x^{1}}^{1}+ \\
\quad+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{Y}_{x^{1} y}+\mathrm{r} \mathscr{X}_{x^{1} x^{1}}^{1}, \\
0=4 \mathscr{X}_{y y}^{1}+3 \mathscr{X}_{x^{1} y}^{2}+\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{X}_{x^{1}}^{1}+\mathrm{r} \mathscr{X}_{x^{1}}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+ \\
+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{Y}_{x^{1} y}+\mathrm{r} \mathscr{X}_{x^{1} x^{1}}^{1}+\mathrm{r} \mathscr{X}_{x^{1} x^{1}}^{2}+\mathrm{r} \mathscr{Y}_{y y}+\mathrm{r} \mathscr{X}_{x^{1} y}^{1} .
\end{array}\right.
$$

and thirdly, we get five more linear PDEs by picking the coefficients of cst., of $y_{1}$, of $y_{1,1}$, of $y_{1} y_{1,1}$, of $\left(y_{1}\right)^{2} y_{1,1}$ in (8.75) (8.83)

Proposition 8.84. Setting as initial conditions the ten specific differential coefficients
(8.85)
$\begin{aligned} \mathrm{P} & :=\mathrm{P}\left(\mathscr{X}^{1}, \mathscr{X}^{2}, \mathscr{Y}, \mathscr{X}_{y}^{1}, \mathscr{X}_{x^{2}}^{2}, \mathscr{Y}_{x^{1}}, \mathscr{Y}_{y}, \mathscr{X}_{x^{1} x^{2}}^{2}, \mathscr{Y}_{x^{1} x^{1}}, \mathscr{Y}_{y y}\right) \\ & =\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{Y}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{Y}_{y y},\end{aligned}$
it follows by cross differentiations and by linear substitutions from the Lie equations (8.81) $)_{i}, i=1,2,3,4,5,(8.82)_{j}, j=1,2,3,(8.83)_{i}, i=$ $1,2,3,4,5$, that $\mathscr{X}_{x^{1}}^{1}, \mathscr{X}_{x^{1}}^{2}, \mathscr{Y}_{x^{2}}, \mathscr{X}_{x^{2}}^{1}, \mathscr{X}_{y}^{2}, \mathscr{X}_{x^{1} y}^{1}, \mathscr{X}_{x^{2} x^{2}}^{2}, \mathscr{Y}_{x^{1} x^{2}}, \mathscr{X}_{x^{2} y^{1}}^{1}$, $\mathscr{X}_{x^{2} y}^{2}, \mathscr{Y}_{x^{1} y}, \mathscr{X}_{y y}^{1}, \mathscr{Y}_{x^{2} y}, \mathscr{X}_{x^{1} x^{1} x^{2}}^{2}, \mathscr{Y}_{x^{1} x^{1} x^{1}}, \mathscr{X}_{x^{1} x^{2} x^{2}}^{2} \mathscr{Y}_{x^{1} x^{1} x^{2}}, \mathscr{X}_{x^{1} x^{2} y^{2}}^{2}, \mathscr{Y}_{x^{1} x^{1} y}$, $\mathscr{Y}_{x^{1} y y}, \mathscr{Y}_{x^{2} y y}, \mathscr{Y}_{\text {yyy }}$ are uniquely determined as linear combinations of
$\left(\mathscr{X}^{1}, \mathscr{X}^{2}, \mathscr{Y}, \mathscr{X}_{y}^{1}, \mathscr{X}_{x^{2}}^{2}, \mathscr{Y}_{x^{1}}, \mathscr{Y}_{y}, \mathscr{X}_{x^{1} x^{1}}^{2}, \mathscr{Y}_{x^{1} x^{1}}, \mathscr{Y}_{y y}\right)$, namely:

$$
\left\{\begin{array}{lrr}
\mathscr{X}_{x^{1}}^{1} \stackrel{1}{=} \mathrm{P}, & \mathscr{X}_{x^{1}}^{2} \stackrel{2}{=} \mathrm{P}, & \mathscr{Y}_{x^{2}} \stackrel{3}{=} \mathrm{P}, \\
\mathscr{X}_{x^{2}}^{1} \stackrel{4}{=} \mathrm{P}, & \mathscr{X}_{y}^{2} \stackrel{5}{=} \mathrm{P}, & \\
\mathscr{X}_{x^{1} y}^{1} \stackrel{6}{=} \mathrm{P}, & \mathscr{X}_{x^{2} x^{2}}^{2} \stackrel{7}{=} \mathrm{P}, & \mathscr{Y}_{x^{1} x^{2}} \stackrel{8}{=} \mathrm{P} \\
\mathscr{X}_{x^{2} y}^{1} \stackrel{9}{=} \mathrm{P}, & \mathscr{X}_{x^{2} y}^{2} \stackrel{10}{=} \mathrm{P}, & \mathscr{Y}_{x^{1} y} \stackrel{11}{=} \mathrm{P}, \\
\mathscr{X}_{y y}^{1} \stackrel{12}{=} \mathrm{P}, & & \mathscr{Y}_{x^{2} y} \stackrel{13}{=} \mathrm{P}, \\
& \mathscr{X}_{x^{1} x^{1} x^{2}} \stackrel{14}{=} \mathrm{P}, & \mathscr{Y}_{x^{1} x^{1} x^{1}} \stackrel{15}{=} \mathrm{P}, \\
& \mathscr{X}_{x^{1} x^{2} x^{2}} \stackrel{16}{=} \mathrm{P}, & \mathscr{Y}_{x^{1} x^{1} x^{2}} \stackrel{17}{=} \mathrm{P} \\
& \mathscr{X}_{x^{1} x^{2} y}^{2} \stackrel{18}{=} \mathrm{P}, & \mathscr{Y}_{x^{1} x^{1} y} \stackrel{19}{=} \mathrm{P} \\
& & \mathscr{Y}_{x^{1} y y} \stackrel{20}{=} \mathrm{P} \\
& & \mathscr{Y}_{x^{2} y y} \stackrel{21}{=} \mathrm{P} \\
& & \mathscr{Y}_{y y y} \stackrel{22}{=} \mathrm{P}
\end{array}\right.
$$

Then the expressions $P$ are stable under differentiation with respect to $x^{1}$, to $x^{2}$, to $y$ and moreover, all other, higher order partial derivatives of $\mathscr{X}^{1}$, of $\mathscr{X}^{2}$, of $\mathscr{Y}$ may be expressed as $\mathrm{P}\left(\mathscr{X}^{1}, \mathscr{X}^{2}, \mathscr{Y}, \mathscr{X}_{y}^{1}, \mathscr{X}_{x^{2}}^{2}, \mathscr{Y}_{x^{1}}, \mathscr{Y}_{y}, \mathscr{X}_{x^{1} x^{2}}^{2}, \mathscr{Y}_{x^{1} x^{1}}, \mathscr{Y}_{y y}\right)$.

Corollary 8.87. Every infinitesimal Lie symmetry of the PDE system ( $\mathscr{E}_{5}$ ) is uniquely determined by the ten initial Taylor coefficients
$\mathscr{X}^{1}(0), \mathscr{X}^{2}(0), \mathscr{Y}(0), \mathscr{X}_{y}^{1}(0), \mathscr{X}_{x^{2}}^{2}(0), \mathscr{Y}_{x^{1}}(0), \mathscr{Y}_{y}(0), \mathscr{X}_{x^{1} x^{2}}^{2}(0), \mathscr{Y}_{x^{1} x^{1}}(0), \mathscr{Y}_{y y}(0)$.
Proof of the proposition. At first, (8.83) $)_{1}$ yields $(8.86)_{15} ;(8.81)_{1}$ yields $(8.86)_{3}$; differentiating $(8.81)_{1}$ with respect to $x^{1}$ yields $(8.86)_{8}$; differentiating $(8.81)_{1}$ with respect to $y$ yields $(8.86)_{13}$; differentiating $(8.81)_{1}$ with respect to $x^{1} x^{1}$ yields $(8.86)_{17}$; and differentiating $(8.81)_{1}$ with respect to $y y$ yields $(8.86)_{21}$. Also, rewriting $(8.81)_{2}$ as

$$
\begin{equation*}
\mathscr{X}_{x^{2}}^{1}=-\frac{1}{2} \mathscr{Y}_{x^{1}} \tag{8.89}
\end{equation*}
$$

we get $(8.86)_{4}$; and rewriting $(8.81)_{3}$ as

$$
\begin{equation*}
\mathscr{X}_{x^{1}}^{1}=\frac{1}{2} \mathscr{X}_{x^{2}}^{2}+\frac{1}{2} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{s}^{*} \mathscr{Y}_{x^{1} x^{1}} \tag{8.90}
\end{equation*}
$$

we get (8.86) ${ }_{1}$.
Next, differentiating $(8.81)_{2}$ with respect to $x^{1}$ and $(8.81)_{3}$ with respect to $x^{2}$, we get, taking account of $0=\mathscr{Y}_{x^{2} y}=\mathscr{Y}_{x^{1} x^{2}}=\mathscr{Y}_{x^{1} x^{1} x^{2}}$, replacing $\mathscr{X}_{x^{1} x^{2}}$
by $-\frac{1}{2} \mathscr{Y}_{x^{1} x^{1}}$ and solving for $\mathscr{X}_{x^{2} x^{2}}^{2}$ :

$$
\begin{align*}
0 & =\mathscr{X}_{x^{1} x^{2}}^{1}+\frac{1}{2} \mathscr{Y}_{x^{1} x^{1}},  \tag{8.91}\\
\mathscr{X}_{x^{2} x^{2}}^{2} & =-\left(1+\mathrm{s}_{x^{2}}^{*}\right) \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{Y}_{x^{1}} .
\end{align*}
$$

This is $(8.86)_{7}$. Differentiating $(8.91)_{2}$ with respect to $x^{1}$, taking account of $(8.83)_{1}$, we get $(8.86)_{16}$ :

$$
\begin{equation*}
\mathscr{X}_{x^{1} x^{2} x^{2}}^{2}=\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}} . \tag{8.92}
\end{equation*}
$$

We then replace $\mathscr{X}_{x^{1}}^{1}$ from (8.90) in (8.81) $)_{4}$ :

$$
\begin{gather*}
0=\mathscr{X}_{x^{1}}^{2}+2 \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+ \\
+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{s}^{*} \mathscr{Y}_{x^{1} y}+\mathrm{s}^{*} \mathscr{X}_{x^{1} x^{1}}^{1} . \tag{8.93}
\end{gather*}
$$

We differentiate this equation with respect to $x^{2}$, knowing $\mathscr{Y}_{x^{2}}=0$ :

$$
\begin{gather*}
0=\mathscr{X}_{x^{1} x^{2}}^{2}+2 \mathscr{X}_{x^{2} y}^{1}+\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{X}_{x^{2} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+  \tag{8.94}\\
+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{s}_{x^{2}}^{*} \mathscr{Y}_{x^{1} y}+\mathrm{s}_{x^{2}}^{*} \mathscr{X}_{x^{1} x^{1}}^{1}+\mathrm{s}^{*} \mathscr{X}_{x^{1} x^{1} x^{2}} .
\end{gather*}
$$

We replace: $\mathscr{X}_{x^{2}}^{1}$ from (8.89); $\mathscr{X}_{x^{2} x^{2}}^{2}$ from (8.91) ${ }_{2}$; we differentiate $(8.81)_{2}$ with respect to $x^{1} x^{1}$ to replace $\mathscr{X}_{x^{1} x^{1} x^{2}}^{1}$ by $\mathscr{Y}_{x^{1} x^{1}}$, thanks to (8.83) ; and we reorganize:
(8.95)
$2 \mathscr{X}_{x^{2} y}^{1}+\mathrm{s}_{x^{2}}^{*} \mathscr{Y}_{x^{1} y}+\mathrm{s}_{x^{2}}^{*} \mathscr{X}_{x^{1} x^{1}}^{1}=-\mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}$.
We differentiate $(8.81)_{2}$ with respect to $y$ and $(8.81)_{3}$ with respect to $x^{1}$ :

$$
\begin{align*}
\mathscr{X}_{x^{2} y}^{1}+\frac{1}{2} \mathscr{Y}_{x^{1} y} & =0  \tag{8.96}\\
\mathscr{Y}_{x^{1} y}-2 \mathscr{X}_{x^{1} x^{1}}^{1} & =-\mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}} .
\end{align*}
$$

For the three unknowns $\mathscr{X}_{x^{1} x^{1}}^{1}, \mathscr{Y}_{x^{1} y}, \mathscr{X}_{x^{2} y}^{1}$, we solve the three equations (8.95), (8.96) ${ }_{1},(8.96)_{2}$, reminding $s_{x^{2}}^{*}(0)=0$ :

$$
\begin{align*}
\mathscr{X}_{x^{1} x^{1}}^{1} & =\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2},  \tag{8.97}\\
\mathscr{Y}_{x^{1} y} & =\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}, \\
\mathscr{X}_{x^{2} y}^{1} & =\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{Y}_{x^{1}}^{2}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2} .
\end{align*}
$$

We get $(8.86)_{11}$ and $(8.86)_{9}$.
Thus, we may replace $\mathscr{X}_{x^{1} x^{1}}^{1}$ and $\mathscr{Y}_{x^{1} y}$ in (8.81) $)_{4}$ to get $(8.86)_{2}$ :
$\mathscr{X}_{x^{1}}^{2}=-2 \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}$.
Next, we differentiate $(8.83)_{3}$ with respect to $x^{1}$ and we replace: $\mathscr{X}_{x^{1}}^{1}$ from (8.90); $\mathscr{X}_{x^{1}}^{2}$ from (8.98); $\mathscr{Y}_{x^{1} y}$ from (8.97) $)_{2} ; \mathscr{X}_{x^{1} x^{1}}^{1}$ from (8.97) $)_{1} ; \mathscr{Y}_{x^{1} x^{1} x^{1}}$ from $(8.83)_{1}$; and we compare with $(8.83)_{2}$; we differentiate $(8.96)_{1}$ with
respect to $x^{1}$ and $(8.96)_{2}$ with respect to $x^{1}$; solving, we obtain four new relations:
(8.99)

$$
\begin{aligned}
\mathscr{X}_{x^{1} x^{1} x^{1}}^{1} & =\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}, \\
\mathscr{Y}_{x^{1} x^{1} y} & =\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2} \\
\mathscr{X}_{x^{1} x^{2} y}^{2} & =\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}, \\
\mathscr{X}_{x^{1} x^{1} x^{2}}^{2} & =\mathrm{X} \mathscr{X}_{x^{2}}^{1}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2} .
\end{aligned}
$$

We get $(8.86)_{19}$ and $(8.86)_{14}$.
Next, in $(8.81)_{5}$, we replace: $\mathscr{X}_{x^{1}}^{1}$ from (8.90); $\mathscr{X}_{x^{1}}^{2}$ from (8.98); $\mathscr{Y}_{x^{1} y}$ from (8.97) ${ }_{2}$; we get:

$$
\begin{align*}
\mathscr{X}_{y}^{2}= & \mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+ \\
& +\mathrm{s}^{*} \mathscr{X}_{x^{1} x^{1}}^{2}+\mathrm{s}^{*} \mathscr{Y}_{y y}+\mathrm{s}^{*} \mathscr{X}_{x^{1} y}^{1} . \tag{8.100}
\end{align*}
$$

We differentiate (8.98) with respect to $x^{1}$ and we replace: $\mathscr{X}_{x^{1}}^{1}$ from (8.90); $\mathscr{X}_{x^{1}}^{2}$ from (8.98); $\mathscr{Y}_{x^{1} y}$ from (8.97) $)_{2} ; \mathscr{Y}_{x^{1} x^{1} x^{1}}$ from (8.83) $)_{1} ; \mathscr{X}_{x^{1} x^{1} x^{2}}^{2}$ from (8.99) 4 ; we get:

$$
\begin{equation*}
\mathscr{X}_{x^{1} x^{1}}^{2}+2 \mathscr{X}_{x^{1} y}^{1}=\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2} . \tag{8.101}
\end{equation*}
$$

In $(8.82)_{2}$, we replace: $\mathscr{X}_{x^{1}}^{1}$ from (8.90); $\mathscr{X}_{x^{1} x^{1}}^{1}$ from (8.97) $)_{1} ; \mathscr{Y}_{x^{1} y}$ from $(8.97)_{2}$; and we reorganize:
$2 \mathscr{X}_{x^{2} y}^{2}-6 \mathscr{X}_{x^{1} y}^{1}-\mathscr{X}_{x^{1} x^{1}}^{2}=-2 \mathscr{Y}_{y y}+\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}$.
Differentiating $(8.81)_{3}$ with respect to $y$, we replace: $\mathscr{Y}_{x^{1} y}$ from $(8.97)_{2}$; $\mathscr{Y}_{x^{1} x^{1} y}$ from (8.99) ${ }_{2}$; and we reorganize:

## (8.103)

$$
\mathscr{X}_{x^{2} y}^{2}-2 \mathscr{X}_{x^{1} y}^{1}=-\mathscr{Y}_{y y}+\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2} .
$$

For the three unknowns $\mathscr{X}_{x^{1} x^{1}}^{2}, \mathscr{X}_{x^{1} y}^{1}, \mathscr{X}_{x^{2} y}^{2}$, we then solve the four equations (8.101), (8.102), (8.103), (8.83) (in which we replace: $\mathscr{X}_{x^{1}}^{1}$ from (8.90); $\mathscr{X}_{x^{1}}^{2}$ from (8.98); $\mathscr{Y}_{x^{1} y}$ from (8.97) $)_{2} ; \mathscr{X}_{x^{1} x^{1}}^{1}$ from (8.97) $)_{1}$ ):
(8.104)

$$
\begin{aligned}
& \mathscr{X}_{x^{1} x^{1}}^{2}=\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}+\mathrm{r} \mathscr{Y}_{y y}, \\
& \mathscr{X}_{x^{1} y}^{1}=\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{y y}, \\
& \mathscr{X}_{x^{2} y}^{2}=\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2} 2 \\
& \mathscr{Y}_{x^{1}} \\
& \mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{y y} .
\end{aligned}
$$

We get $(8.86)_{6}$ and $(8.86)_{10}$. Replacing then $\mathscr{X}_{x^{1} x^{1}}^{2}, \mathscr{X}_{x^{1} y}^{1}$ in (8.100) gives (8.105)

$$
\mathscr{X}_{y}^{2}=\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{y y} .
$$

This is $(8.86)_{5}$.

Next, we differentiate (8.103) with respect to $x^{1}$ and we replace: $\mathscr{X}_{x^{1}}^{1}$ from (8.90); $\mathscr{X}_{x^{1}}^{2}$ from (8.98); $\mathscr{Y}_{x^{1} y}$ from (8.97) ${ }_{2} ; \mathscr{Y}_{x^{1} x^{1} x^{1}}$ from (8.83) ${ }_{1}$; $\mathscr{X}_{x^{1} x^{1} x^{2}}^{2}$ from (8.99) ${ }_{4}$; we get:
(8.106)
$\mathscr{Y}_{x^{1} y y}+\mathscr{X}_{x^{1} x^{2} y}^{2}-2 \mathscr{X}_{x^{1} x^{1} y}^{1}=\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}$.
Also, we differentiate $(8.83)_{3}$ with respect to $y$ and we replace: $\mathscr{X}_{y}^{2}$ from (8.105); $\mathscr{Y}_{x^{1} y}$ from (8.97) $)_{2} ; \mathscr{X}_{x^{1} y}^{1}$ from (8.104) $)_{2} ; \mathscr{Y}_{x^{1} x^{1} y}$ from (8.99) $)_{2}$; we get:
(8.107)
$\mathscr{Y}_{x^{1} y y}-\mathscr{X}_{x^{1} x^{1} y}^{1}=\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{y y}$.
Also, we replace in (8.82) $)_{3}$ : $\mathscr{X}_{x^{1}}^{1}$ from (8.90); $\mathscr{X}_{x^{1}}^{2}$ from (8.98); $\mathscr{Y}_{x^{1} y}$ from $(8.97)_{2} ; \mathscr{X}_{x^{1} x^{1}}^{1}$ from (8.97) $; \mathscr{X}_{x^{1} x^{1}}^{2}$ from (8.104) $)_{1} ; \mathscr{X}_{x^{1} y}^{1}$ from (8.104) $)_{2}$; we get:
(8.108)

$$
4 \mathscr{X}_{y y}^{1}+3 \mathscr{X}_{x^{1} y}^{2}=\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{y y} .
$$

We differentiate this equation with respect to $x^{2}$ and we replace: $4 \mathscr{X}_{x^{2} y y}^{1}$ by $-2 \mathscr{\mathscr { Y }}_{x^{1} y y}$ from (8.89); $\mathscr{X}_{x^{2}}^{1}$ from (8.98); $\mathscr{X}_{x^{2} y}^{1}$ from (8.97) ${ }_{3}$; (notice $0=$ $\left.\mathscr{Y}_{x^{1} x^{2}}=\mathscr{Y}_{x^{2} y}\right) ; \mathscr{X}_{x^{2} x^{2}}^{2}$ from (8.91) $)_{2}$; $\mathscr{X}_{x^{1} x^{2} x^{2}}^{2}$ from (8.92); we get:
$-2 \mathscr{Y}_{x^{1} y y}+3 \mathscr{X}_{x^{1} x^{2} y}^{2}=\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{y y}$.
For the three unknowns $\mathscr{X}_{x^{1} x^{1} y}^{1}, \mathscr{Y}_{x^{1} y y}, \mathscr{X}_{x^{1} x^{2} y}^{2}$, we solve the three equations (8.106), (8.107), (8.108); we get:
(8.110)

$$
\begin{aligned}
\mathscr{X}_{x^{1} x^{1} y} & =\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{y y}, \\
\mathscr{Y}_{x^{1} y y} & =\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}+\mathrm{r} \mathscr{Y}_{y y}, \\
\mathscr{X}_{x^{1} x^{2} y}^{2} & =\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{y y} .
\end{aligned}
$$

We get $(8.86)_{20}$ and $(8.86)_{18}$.
Next, in (8.93), we replace: $\mathscr{Y}_{x^{1} y}$ from (8.97) $)_{2} ; \mathscr{X}_{x^{1} x^{1}}^{1}$ from (8.97) ${ }_{1}$; we get:
(8.111)
$\mathscr{X}_{x^{1}}^{2}+2 \mathscr{X}_{y}^{1}=\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}$.
We differentiate this equation with respect to $y$ and we replace: $\mathscr{X}_{y}^{2}$ from (8.105); $\mathscr{X}_{x^{2} y}^{2}$ from (8.104) $)_{3} ; \mathscr{Y}_{x^{1} y}$ from (8.97) $)_{2} ; \mathscr{Y}_{x^{1} x^{1} y}$ from (8.99) ${ }_{2} ; \mathscr{X}_{x^{1} x^{2} y}^{2}$ from (8.99) ${ }_{3}$; we get:
(8.112)

$$
\mathscr{X}_{x^{1} y}^{2}+2 \mathscr{X}_{y y}^{1}=\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{y y} .
$$

For the two unknowns $\mathscr{X}_{y y}^{1}$ and $\mathscr{X}_{x^{1} y}^{2}$, we solve the two equations (8.108) and (8.112); we get:

## (8.113)

$$
\begin{aligned}
\mathscr{X}_{y y}^{1} & =\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{y y}, \\
\mathscr{X}_{x^{1} y}^{2} & =\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{y y} .
\end{aligned}
$$

We get (8.86) ${ }_{12}$.
Next, we differentiate $(8.113)_{1}$ with respect to $x^{1}$ and we replace: $\mathscr{X}_{x^{1}}^{1}$, $\mathscr{X}_{x^{1}}^{2}, \mathscr{X}_{x^{1} y}^{1}, \mathscr{Y}_{x^{1} y}, \mathscr{Y}_{x^{1} x^{1} x^{1}}, \mathscr{X}_{x^{1} x^{1} x^{2}}^{2}, \mathscr{Y}_{x^{1} y y}$; we get:
(8.114)
$\mathscr{X}_{x^{1} y y}^{1}=\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{y y}$.

Also, we differentiate $(8.113)_{2}$ with respect to $x^{1}$ and we replace:
(8.115)
$\mathscr{X}_{x^{1} x^{1} y}^{2}=\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{y y}$.

Also, we differentiate $(8.83)_{4}$ with respect to $y$; we replace $\mathscr{X}_{x^{1} y y}^{1}$ from (8.114), we replace $\mathscr{X}_{x^{1} x^{1} y}^{2}$ from (8.115); and we achieve other evident replacements; we get:
(8.116)
$\mathscr{Y}_{y y y}=\mathrm{r} \mathscr{X}^{1}+\mathrm{r} \mathscr{X}^{2}+\mathrm{r} \mathscr{X}_{y}^{1}+\mathrm{r} \mathscr{X}_{x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{x^{1}}+\mathrm{r} \mathscr{Y}_{y}+\mathrm{r} \mathscr{Y}_{x^{1} x^{1}}+\mathrm{r} \mathscr{X}_{x^{1} x^{2}}^{2}+\mathrm{r} \mathscr{Y}_{y y}$.

This is $(8.96)_{22}$, which completes the proof.

Theorem 8.117. The bound $\operatorname{dim} \mathfrak{S Y M}\left(\mathscr{E}_{5}\right) \leqslant 10$ is attained if and only if $\left(\mathscr{E}_{5}\right)$ is equivalent, through a diffeomorphism $\left(x^{1}, x^{2}, y\right) \longmapsto\left(X^{1}, X^{2}, Y\right)$, to the model system

$$
\begin{equation*}
Y_{X^{2}}=0, \quad Y_{X^{1} X^{1} X^{1}}=0 \tag{8.118}
\end{equation*}
$$

Proof. Firstly, setting $\mathrm{r}=\mathrm{s}^{*}=0$ everywhere, the solution to (8.81), (8.82), (8.83) is

$$
\begin{align*}
\mathscr{X}^{1} & =k+(c+j) x^{1}-b x^{2}-h y+e x^{1} x^{1}-d x^{1} x^{2}+f x^{1} y-e x^{2} y,  \tag{8.119}\\
\mathscr{X}^{2} & =g+2 h x^{1}+2 j x^{2}-d x^{2} x^{2}+2 e x^{1} x^{2}-f x^{1} x^{1}, \\
\mathscr{Y} & =a+2 b x^{1}+2 c y+d x^{1} x^{1}+2 e x^{1} y+f y y,
\end{align*}
$$

where $a, b, c, d, e, f, g, h, j, k \in \mathbb{K}$ are arbitrary. Computing the third prolongations of the ten vector fields
(8.120)

$$
\begin{aligned}
& \frac{\partial}{\partial x^{1}}, \\
& -\frac{\partial}{\partial x^{2}}, \quad \frac{\partial}{\partial y}, \\
& -x^{2} \frac{\partial}{\partial x^{1}}+2 x^{1} \frac{\partial}{\partial y}, \quad x^{1} \frac{\partial}{\partial x^{1}}+2 y \frac{\partial}{\partial y}, \quad x^{1} \frac{\partial}{\partial x^{1}}+2 x^{2} \frac{\partial}{\partial x^{2}}, \quad-y \frac{\partial}{\partial x^{1}}+2 x^{1} \frac{\partial}{\partial x^{2}}, \\
& -x^{1} x^{2} \frac{\partial}{\partial x^{1}}-x^{2} x^{2} \frac{\partial}{\partial x^{2}}+x^{1} x^{1} \frac{\partial}{\partial y}, \quad\left(x^{1} x^{1}-x^{2} y\right) \frac{\partial}{\partial x^{1}}+2 x^{1} x^{2} \frac{\partial}{\partial x^{2}}+2 x^{1} y \frac{\partial}{\partial y}, \\
& x^{1} y \frac{\partial}{\partial x^{1}}-x^{1} x^{1} \frac{\partial}{\partial x^{2}}+y y \frac{\partial}{\partial y}
\end{aligned}
$$

one verifies that they all are tangent to the skeleton $y_{2}=\frac{1}{4}\left(y_{1}\right)^{2}, y_{1,1,1}=0$. Thus the bound is attained. One then verifies ([FK2005a]) that the spanned Lie algebra is isomorphic to $\mathfrak{s o}(5, \mathbb{C})$.

Lemma 8.121. Assuming the normalizations of Lemma 8.54, the remainder $\mathrm{O}_{4}$ in (8.53) is an $\mathrm{O}_{3}\left(x^{1}, a^{1}\right)$ :
$y=b+\frac{2 x^{1} a^{1}+x^{1} x^{1} a^{2}+a^{1} a^{1} x^{2}}{1-x^{2} a^{2}}+\left(x^{1}\right)^{3} \mathrm{R}+\left(x^{1}\right)^{2} a^{1} \mathrm{R}+x^{1}\left(a^{1}\right)^{2} \mathrm{R}+\left(a^{1}\right)^{3} \mathrm{R}$.
Proof. Indeed, writing
(8.123)
$y=b+x^{1} \Lambda^{1,0}+a^{1} \Lambda^{0,1}+x^{1} x^{1} \Lambda^{2,0}+x^{1} a^{1} \Lambda^{1,1}+a^{1} a^{1} \Lambda^{0,2}+\mathrm{O}_{3}\left(x^{1}, a^{1}\right)$,
with $\Lambda^{i, j}=\Lambda^{i, j}\left(x^{2}, a^{2}\right)$, and developing the determinant (8.54) with respect to the powers of $\left(x^{1}, a^{1}\right)$, the vanishing of the coefficients of cst., of $x^{1}$, of $a^{1}$ yields the system

$$
\left\{\begin{array}{l}
0 \equiv \Lambda_{a^{2}}^{1,0} \Lambda_{x^{2}}^{0,1}  \tag{8.124}\\
0 \equiv \Lambda^{1,1} \Lambda_{x^{2} a^{2}}^{1,0}-2 \Lambda_{a^{2}}^{2,0} \Lambda_{x^{2}}^{0,1}-\Lambda_{x^{2}}^{1,1} \Lambda_{a^{2}}^{1,0} \\
0 \equiv \Lambda^{1,1} \Lambda_{x^{2} a^{2}}^{0,1}-\Lambda_{a^{2}}^{1,1} \Lambda_{x^{2}}^{0,1}-2 \Lambda_{x^{2}}^{0,2} \Lambda_{a^{2}}^{1,0} .
\end{array}\right.
$$

If the first equation yields $\Lambda_{a^{2}}^{1,0} \equiv 0$, replacing in the second, using $\Lambda^{2,0}=$ $a^{2}+\mathrm{O}_{2}$, we deduce that $\Lambda_{x^{2}}^{0,1} \equiv 0$ also. Similarly, $\Lambda_{x^{2}}^{0,1} \equiv 0$ implies $\Lambda_{a^{2}}^{1,0} \equiv 0$. Since the coordinate system satisfies the normalization $\Pi(0, a) \equiv \Pi(x, 0) \equiv$ 0 , necessarily $\Lambda^{1,0}=\mathrm{O}\left(a^{2}\right)$ and $\Lambda^{0,1}=\mathrm{O}\left(x^{2}\right)$. We deduce:

$$
\begin{equation*}
0 \equiv \Lambda^{1,0} \equiv \Lambda^{0,1} \tag{8.125}
\end{equation*}
$$

Redeveloping the determinant, the vanishing of the coefficients of $x^{1} x^{1}$, of $x^{1} a^{1}$, of $a^{1} a^{1}$ yields the system

$$
\left\{\begin{array}{l}
0 \equiv \Lambda^{1,1} \Lambda_{x^{2} a^{2}}^{2,0}-2 \Lambda_{a^{2}}^{2,0} \Lambda_{x^{2}}^{1,1},  \tag{8.126}\\
0 \equiv \Lambda^{1,1} \Lambda_{x^{2} a^{2}}^{1,1}-\Lambda_{a^{2}}^{1,1} \Lambda_{x^{2}}^{1,1}-4 \Lambda_{a^{2}}^{2,0} \Lambda_{x^{2}}^{0,2}, \\
0
\end{array}\right.
$$

Since $\Lambda^{1,1}(0)=2 \neq 0$, we may divide by $\Lambda^{1,1}$, obtaining a PDE system with the three functions $\Lambda_{x^{2} a^{2}}^{2,0}, \Lambda_{x^{2} a^{2}}^{1,1}, \Lambda_{x^{2} a^{2}}^{0,2}$ in the left hand side. We observe that the normalizations of Lemma 8.55 entail
$\Lambda^{2,0}=a^{2}+\mathrm{O}\left(x^{2} a^{2}\right), \quad \Lambda^{1,1}=2+\mathrm{O}\left(x^{2} a^{2}\right), \quad \Lambda^{0,2}=x^{2}+\mathrm{O}\left(x^{2} a^{2}\right)$.
By cross differentiations in the mentioned PDE system, it follows that all the Taylor coefficients of $\Lambda^{2,0}, \Lambda^{1,1}, \Lambda^{0,2}$ are uniquely determined. As already discovered in [GM2003b], the unique solution

$$
\begin{equation*}
\Lambda^{2,0}=\frac{a^{2}}{1-x^{2} a^{2}}, \quad \Lambda^{1,1}=\frac{2}{1-x^{2} a^{2}}, \quad \Lambda^{0,2}=\frac{x^{2}}{1-x^{2} a^{2}} \tag{8.128}
\end{equation*}
$$

guarantees, when the remainder $\mathrm{O}_{3}\left(x^{1}, a^{1}\right)$ vanishes, that the determinant (8.45) indeed vanishes identically.

Conversely, suppose that $\operatorname{dim} \mathfrak{S Y M}\left(\mathscr{E}_{5}\right)=10$ is maximal.
With $\varepsilon \neq 0$ small, replacing $\left(x^{1}, x^{2}, y, a^{1}, a^{2}, b\right)$ by $\left(\varepsilon x^{1}, x^{2}, \varepsilon^{2} y, \varepsilon a^{1}, a^{2}, \varepsilon \varepsilon b\right)$ in (8.122) and dividing by $\varepsilon \varepsilon$, the remainder terms become small:

$$
\begin{equation*}
y=b+\frac{2 x^{1} a^{1}+x^{1} x^{1} a^{2}+a^{1} a^{1} x^{2}}{1-x^{2} a^{2}}+\mathrm{O}(\varepsilon) . \tag{8.129}
\end{equation*}
$$

Then all the remainders in the equations $\Delta_{\mathscr{E}_{5}}$ of the skeleton are $\mathrm{O}(\varepsilon)$. We get ten generators similar to (8.120), plus an $\mathrm{O}(\varepsilon)$ perturbation. Thanks to the rigidity of $\mathfrak{s o}(5, \mathbb{C})$, Theorem 5.15 provides a change of generators, close to the $10 \times 10$ identity matrix, leading to the same structure constants as those of the ten vector fields (8.120). As in the end of the proof of Theorem 5.13, we may then straighten some relevant vector fields (exercise) and finally check that their tangency to the skeleton implies that it is the model one. Theorem 8.117 is proved.

Corollary 8.130. Let $M \subset \mathbb{C}^{3}$ be a connected real analytic hypersurface whose Levi form has uniform rank 1 that is 2-nondegenerate at every point. Then

$$
\begin{equation*}
\operatorname{dim} \mathfrak{h o l}(M) \leqslant 10 \tag{8.131}
\end{equation*}
$$

and the bound is attained if and only if $M$ is locally, in a neighborhood of Zariski-generic points, biholomorphic to the model $M_{0}$.

## §9. DUAL SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

9.1. Solvability with respect to the variables. Let $\mathscr{M}$ be as in $\S 2.10$ defined by $y=\Pi(x, a, b)$ or dually by $b=\Pi^{*}(a, x, y)$.

Definition 9.2. $\mathscr{M}$ is solvable with respect to the variables if there exist an integer $\kappa^{*} \geqslant 1$ and multiindices $\delta(1), \ldots, \delta(n) \in \mathbb{N}^{p}$ with $|\delta(l)| \geqslant 1$ for $l=$ $1, \ldots, n$ and $\max _{1 \leqslant l \leqslant n}|\delta(l)|=\kappa^{*}$, together with integers $j(1), \ldots, j(n)$ with $1 \leqslant j(l) \leqslant m$ such that the local $\mathbb{K}$-analytic map
$\mathbb{K}^{n+m} \ni(x, y) \longmapsto\left(\left(\Pi^{* j}(0, x, y)\right)^{1 \leqslant j \leqslant m},\left(\Pi_{a}^{* j(l)}(0, x, y)\right)_{1 \leqslant l \leqslant n}\right) \in \mathbb{K}^{m+n}$
is of rank equal to $n+m$ at $(x, y)=(0,0)$
If $\mathscr{M}$ is a complexified generic submanifold, solvability with respect to the parameters is equivalent to solvability with respect to the variables, because $\Pi^{*}=\bar{\Pi}$. This is untrue in general: with $n=2, m=1$, consider the system $y_{x^{2}}=0, y_{x^{1} x^{1}}=0$, whose general solutions is $y(x)=b+x_{1} a$ with $x^{2}$ absent.

To characterize generally such a degeneration property, we develope both

$$
\begin{align*}
& y^{j}=\Pi^{j}(x, a, b)=\sum_{\beta \in \mathbb{N}^{n}} x^{\beta} \Pi_{\beta}^{j}(a, b) \quad \text { and } \\
& b^{j}=\Pi^{* j}(a, x, y)=\sum_{\delta \in \mathbb{N}^{p}} a^{\delta} \Pi_{\delta}^{* j}(x, y) \tag{9.4}
\end{align*}
$$

with analytic functions $\Pi_{\beta}^{j}(a, b), \Pi^{*}{ }_{\delta}^{j}(x, y)$ and we introduce two $\mathbb{K}^{\infty}$-valued maps

$$
\begin{array}{ll}
\mathscr{Q}_{\infty}: & (a, b) \longmapsto\left(\Pi_{\beta}^{j}(a, b)\right)_{\beta \in \mathbb{N}^{n}}^{1 \leqslant j \leqslant m} \quad \text { and }  \tag{9.5}\\
\mathscr{Q}_{\infty}^{*}: & (x, y) \longmapsto\left(\Pi_{\delta}^{* j}(x, y)\right)_{\delta \in j \leqslant m}^{1 \leqslant \mathbb{N}^{p}} .
\end{array}
$$

Since $b \mapsto\left(\Pi_{0}^{j}(0, b)\right)^{1 \leqslant j \leqslant m}$ and $y \mapsto\left(\Pi_{0}^{* j}(0, y)\right)^{1 \leqslant j \leqslant m}$ are already both of rank $m$ at the origin, the generic ranks of these two maps, defined by testing the nonvanishing of minors of their infinite Jacobian matrices, satisfy

$$
\begin{align*}
& \text { genrk } \mathscr{Q}_{\infty}=m+p_{\mathscr{M}} \quad \text { and } \\
& \text { genrk } \mathscr{Q}_{\infty}^{*}=m+n_{\mathscr{M}} \tag{9.6}
\end{align*}
$$

for some two integers $0 \leqslant p_{\mathscr{M}} \leqslant p$ and $0 \leqslant n_{\mathscr{M}} \leqslant n$. So at a Zariski-generic point, the ranks are equal to $m+p_{\mathscr{M}}$ and to $m+n_{\mathscr{M}}$.

Proposition 9.7. There exists a local proper $\mathbb{K}$-analytic subset $\Sigma_{\mathscr{M}}$ of $\mathbb{K}_{x}^{n} \times$ $\mathbb{K}_{y}^{m} \times \mathbb{K}_{a}^{p} \times \mathbb{K}_{b}^{m}$ whose equations, of the specific form

$$
\begin{equation*}
\Sigma_{\mathscr{M}}=\left\{r_{\nu}(a, b)=0, \nu \in \mathbb{N}, \quad r_{\mu}^{*}(x, y)=0 \quad \mu \in \mathbb{N}\right\} \tag{9.8}
\end{equation*}
$$

are obtained by equating to zero all $\left(m+p_{\mathscr{M}}\right) \times\left(m+p_{\mathscr{M}}\right)$ minors of Jac $\mathscr{Q}_{\infty}$ and all $\left(m+n_{\mathscr{M}}\right) \times\left(m+n_{\mathscr{M}}\right)$ minors of Jac $\mathscr{Q}_{\infty}^{*}$, such that for every point $p=\left(x_{p}, y_{p}, a_{p}, b_{p}\right) \notin \Sigma_{\mathscr{M}}$, there exists a local change of coordinates respecting the separation of the variables $(x, y)$ and $(a, b)$

$$
\begin{equation*}
(x, y, a, b) \mapsto(\varphi(x, y), h(a, b))=:\left(x^{\prime}, y^{\prime}, a^{\prime}, b^{\prime}\right) \tag{9.9}
\end{equation*}
$$

by which $\mathscr{M}$ is transformed to a submanifold $\mathscr{M}^{\prime}$ centered and localized at $p^{\prime}=p$ having equations

$$
\begin{equation*}
y^{\prime}=\Pi^{\prime}\left(x^{\prime}, a^{\prime}, b^{\prime}\right) \quad \text { and dually } \quad b^{\prime}=\Pi^{\prime *}\left(a^{\prime}, x^{\prime}, y^{\prime}\right) \tag{9.10}
\end{equation*}
$$

with $\Pi^{\prime}$ and $\Pi^{*}$ independent of

$$
\begin{equation*}
\left(x_{n_{\mathscr{M}}+1}^{\prime}, \ldots, x_{n}^{\prime}\right) \quad \text { and of } \quad\left(a_{p_{\mathscr{M}}+1}^{\prime}, \ldots, a_{p}^{\prime}\right) . \tag{9.11}
\end{equation*}
$$

So $\mathscr{M}^{\prime}$, may be considered to be living in $\mathbb{K}_{x^{\prime}}^{n / \mu} \times \mathbb{K}_{y^{\prime}}^{m} \times \mathbb{K}_{a^{\prime}}^{p} \neq \mathbb{K}_{b^{\prime}}^{m}$ and in such a smaller space, at $p^{\prime}=p$, it is solvable both with respect to the parameters and to the variables.

Interpretation: by forgetting some innocuous variables, at a Zariskigeneric point, any $\mathscr{M}$ is both solvable with respect to the parameters and to the variables. These two assumptions will be held up to the end of this Part I.
9.12. Dual system $\left(\mathscr{E}^{*}\right)$ and isomorphisms $\mathfrak{S Y M}(\mathscr{E}) \simeq$ $\mathfrak{S M M}\left(V_{\mathscr{S}}(\mathscr{E})\right)=\mathfrak{S Y M}\left(\mathscr{V}_{\mathscr{S}}\left(\mathscr{E}^{*}\right)\right) \simeq \mathfrak{S Y M}\left(\mathscr{E}^{*}\right)$. To a system $(\mathscr{E})$, we associate its submanifold of solutions $\mathscr{M}:=\mathscr{V}_{\mathscr{S}}(\mathscr{E})$. Assuming it to be solvable with respect to the variables and proceeding as in §2.10, we can derive a dual system of completely integrable partial differential equations of the form

$$
\begin{equation*}
b_{a^{\gamma}}^{j}(a)=G_{\gamma}^{j}\left(a, b(a),\left(b_{a^{\delta(l)}}^{j(l)}(a)\right)_{1 \leqslant l \leqslant n}\right), \tag{*}
\end{equation*}
$$

where $(j, \gamma) \neq(j, 0)$ and $\neq(j(l), \delta(l))$. Its submanifold of solutions $\mathscr{V}_{\mathscr{S}}\left(\mathscr{E}^{*}\right) \equiv \mathscr{V}_{\mathscr{S}}(\mathscr{E})$ has equations dual to those of $\mathscr{V}_{\mathscr{S}}(\mathscr{E})$.

Theorem 9.13. Under the assumption of twin solvability, we have:

$$
\begin{equation*}
\mathfrak{S M M}(\mathscr{E}) \simeq \mathfrak{S M M}\left(\mathscr{V}_{\mathscr{L}}(\mathscr{E})\right)=\mathfrak{S Y M}\left(\mathscr{V}_{\mathscr{S}}\left(\mathscr{E}^{*}\right)\right) \simeq \mathfrak{S M M}\left(\mathscr{E}^{*}\right) \tag{9.14}
\end{equation*}
$$

through $\mathscr{L} \longleftrightarrow \mathscr{L}+\mathscr{L}^{*}=\mathscr{L}^{*}+\mathscr{L} \longleftrightarrow \mathscr{L}^{*}$.

## §10. Fundamental pair of Foliations and covering property

10.1. Fundamental pair of foliations on $\mathscr{M}$. As in §2, let $(\mathscr{E})$ and $\mathscr{M}=$ $\mathscr{V}_{\mathscr{S}}(\mathscr{E})$ be defined by $y=\Pi(x, a, b)$ or dually by $b=\Pi^{*}(a, x, y)$. Abbreviate

$$
\begin{equation*}
z:=(x, y) \quad \text { and } \quad c:=(a, b) \tag{10.2}
\end{equation*}
$$

Every transformation $(z, c) \mapsto(\varphi(z), h(c))$ belonging to $\mathrm{G}_{\mathrm{v}, \mathrm{p}}$ stabilizes both $\{z=$ cst. $\}$ and $\{c=$ cst. $\}$. Accordingly, the two foliations of $\mathscr{M}$

$$
\begin{equation*}
\mathrm{F}_{\mathrm{v}}:=\bigcup_{c_{0}} \mathscr{M} \cap\left\{c=c_{0}\right\} \quad \text { and } \quad \mathrm{F}_{\mathrm{p}}:=\bigcup_{z_{0}} \mathscr{M} \cap\left\{z=z_{0}\right\} \tag{10.3}
\end{equation*}
$$

are invariant under changes of coordinates. We call $\left(F_{v}, F_{p}\right)$ the fundamental pair of foliations on $\mathscr{M}$. The leaves of the foliation by variables $\mathrm{F}_{\mathrm{v}}$ are $n$ dimensional:

$$
\begin{equation*}
\mathrm{F}_{\mathrm{v}}\left(c_{0}\right)=\left\{\left(z, c_{0}\right): y=\Pi\left(x, c_{0}\right)\right\} \tag{10.4}
\end{equation*}
$$

The leaves of the foliation by parameters $\mathrm{F}_{\mathrm{p}}$ are $p$-dimensional:

$$
\begin{equation*}
\mathbf{F}_{\mathbf{p}}\left(c_{0}\right)=\left\{\left(z_{0}, c\right): b=\Pi^{*}\left(a, z_{0}\right)\right\} \tag{10.5}
\end{equation*}
$$

We draw a diagram. In it, the positive codimension is invisible:

$$
\begin{equation*}
m=\operatorname{dim} \mathscr{M}-\operatorname{dim} \mathrm{F}_{\mathrm{v}}-\operatorname{dim} \mathrm{F}_{\mathrm{p}} \geqslant 1 \tag{10.6}
\end{equation*}
$$


10.7. Chains $\Gamma_{k}$ and dual chains $\Gamma_{k}^{*}$. Similarly as in [GM2004, Me2005a, Me2005b, MP2005] (in a CR context), we introduce two collections $\left(\mathrm{L}_{k}\right)_{1 \leqslant k \leqslant n}$ and $\left(\mathrm{L}_{q}^{*}\right)_{1 \leqslant q \leqslant p}$ of vector fields whose integral manifolds coincide with the leaves of $F_{v}$ and of $F_{p}$ :

$$
\left\{\begin{align*}
\mathrm{L}_{k}:=\frac{\partial}{\partial x_{k}}+\sum_{j=1}^{m} \frac{\partial \Pi^{j}}{\partial x_{k}}(x, a, b) \frac{\partial}{\partial y^{j}}, & k=1, \ldots, n,  \tag{10.8}\\
\mathrm{~L}_{q}^{*}:=\frac{\partial}{\partial a^{q}}+\sum_{j=1}^{m} \frac{\partial \Pi^{* j}}{\partial a^{q}}(a, x, y) \frac{\partial}{\partial b^{j}}, & q=1, \ldots, p .
\end{align*}\right.
$$

Let $\left(z_{0}, c_{0}\right)=\left(x_{0}, y_{0}, a_{0}, b_{0}\right) \in \mathscr{M}$ be a fixed point, let $x_{1}:=\left(x_{1}^{1}, \ldots, x_{1}^{n}\right) \in$ $\mathbb{K}^{n}$ and define the multiple flow map (10.9)
$\left\{\begin{aligned} \mathrm{L}_{x_{1}}\left(x_{0}, y_{0}, a_{0}, b_{0}\right) & :=\exp \left(x_{1} \mathrm{~L}\right)\left(p_{0}\right):=\exp \left(x_{1}^{n} \mathrm{~L}_{n}\left(\cdots\left(\exp \left(x_{1}^{1} \mathrm{~L}_{1}\left(z_{0}, c_{0}\right)\right)\right) \cdots\right)\right) \\ & :=\left(x_{0}+x_{1}, \Pi\left(x_{0}+x_{1}, a_{0}, b_{0}\right), a_{0}, b_{0}\right) .\end{aligned}\right.$
Similarly, for $a_{1}=\left(a_{1}^{1}, \ldots, a_{1}^{p}\right) \in \mathbb{K}^{p}$, define the multiple flow map

$$
\begin{equation*}
\mathrm{L}_{a_{1}}^{*}\left(x_{0}, y_{0}, a_{0}, b_{0}\right):=\left(x_{0}, y_{0}, a_{0}+a_{1}, \Pi^{*}\left(a_{0}+a_{1}, x_{0}, y_{0}\right)\right) \tag{10.10}
\end{equation*}
$$

Starting from the $\left(z_{0}, c_{0}\right)=(0,0)$ and moving alternately along $F_{v}, F_{p}, F_{v}$, etc., we obtain

$$
\left\{\begin{align*}
\Gamma_{1}\left(x_{1}\right) & :=\mathrm{L}_{x_{1}}(0)  \tag{10.11}\\
\Gamma_{2}\left(x_{1}, a_{1}\right) & :=\mathrm{L}_{a_{1}}^{*}\left(\mathrm{~L}_{x_{1}}(0)\right) \\
\Gamma_{3}\left(x_{1}, a_{1}, x_{2}\right) & :=\mathrm{L}_{x_{2}}\left(\mathrm{~L}_{a_{1}}^{*}\left(\mathrm{~L}_{x_{1}}(0)\right)\right) \\
\Gamma_{4}\left(x_{1}, a_{1}, x_{2}, a_{2}\right) & :=\mathrm{L}_{a_{2}}^{*}\left(\mathrm{~L}_{x_{2}}\left(\mathrm{~L}_{a_{1}}^{*}\left(\mathrm{~L}_{x_{1}}(0)\right)\right)\right)
\end{align*}\right.
$$

and so on. Generally, we get chains $\Gamma_{k}:=\Gamma_{k}\left([x a]_{k}\right)$, where $[x a]_{k}:=$ $\left(x_{1}, a_{1}, x_{2}, a_{2}, \ldots\right)$ with exactly $k$ terms, where each $x_{l} \in \mathbb{K}^{n}$ and each $a_{l} \in \mathbb{K}^{p}$.

If, instead, the first movement consists in moving along $\mathrm{F}_{\mathrm{p}}$, we start with $\Gamma_{1}^{*}\left(a_{1}\right):=\mathrm{L}_{a_{1}}^{*}(0), \Gamma_{2}^{*}\left(a_{1}, x_{1}\right):=\mathrm{L}_{x_{1}}\left(\mathrm{~L}_{a_{1}}^{*}(0)\right)$, etc., and generally we get dual chains $\Gamma_{k}^{*}\left([a x]_{k}\right)$, where $[a x]_{k}:=\left(a_{1}, x_{1}, a_{2}, x_{2}, \ldots\right)$, with exactly $k$ terms. Both $\Gamma_{k}$ and $\Gamma_{k}^{*}$ have range in $\mathscr{M}$.

For $k=1,2,3, \cdots$, integers $e_{k}$ and $e_{k}^{*}$ are defined inductively by

$$
\left\{\begin{array}{l}
e_{1}+e_{2}+e_{3}+\cdots+e_{k}=\operatorname{genrk}_{\mathbb{K}}\left(\Gamma_{k}\right),  \tag{10.12}\\
e_{1}^{*}+e_{2}^{*}+e_{3}^{*}+\cdots+e_{k}^{*}=\operatorname{genrk}_{\mathbb{K}}\left(\Gamma_{k}^{*}\right) .
\end{array}\right.
$$

By (10.9) and (10.10), it is clear that $e_{1}=n, e_{2}=p, e_{1}^{*}=p$, and $e_{2}^{*}=n$.
Example 10.13. For $y_{x x}=0$, the submanifold of solutions $\mathscr{M}$ is simply $y=b+x a$, whence

$$
\left\{\begin{align*}
\Gamma_{1}\left(x_{1}\right) & =\left(x_{1}, 0,0,0\right)  \tag{10.14}\\
\Gamma_{2}\left(x_{1}, a_{1}\right) & =\left(x_{1}, 0, a_{1},-x_{1} a_{1}\right) \\
\Gamma_{3}\left(x_{1}, a_{1}, x_{2}\right) & =\left(x_{1}+x_{2}, x_{2} a_{1}, a_{1},-x_{1} a_{1}\right)
\end{align*}\right.
$$

The rank at $(0,0,0)$ of $\Gamma_{3}$ is equal to two, not more. However, its generic rank is equal to three. Similar observations hold for the two submanifolds of solutions $y=b+x x a+x a a$ and $y=b+x a^{1}+x x a^{2}\left(\right.$ in $\left.\mathbb{K}^{5}\right)$.

Lemma 10.15. If $\operatorname{genrk}_{\mathbb{K}}\left(\Gamma_{k+1}\right)=$ genrk $_{\mathbb{K}}\left(\Gamma_{k}\right)$, then for each positive integer $l \geqslant 1$, we have $\operatorname{genrk}_{\mathbb{K}}\left(\Gamma_{k+l}\right)=\operatorname{genrk}_{\mathbb{K}}\left(\Gamma_{k}\right)$. The same stabilization property holds for $\Gamma_{k}^{*}$.
10.16. Covering property. We now formulate a central concept.

Definition 10.17. The pair of foliations $\left(F_{v}, F_{p}\right)$ is covering at the origin if there exists an integer $k$ such that the generic rank of $\Gamma_{k}$ is (maximal possible) equal to $\operatorname{dim}_{\mathbb{K}} \mathscr{M}$. Since for $a_{1}=0$, the dual $(k+1)$-th chain $\Gamma_{k+1}^{*}$ identifies with the $k$-th chain $\Gamma_{k}$, the same property holds for the dual chains.

Example 10.18. With $n=1, m=2$ and $p=1$ the submanifold defined by $y^{1}=b^{1}$ and $y^{2}=b^{2}+x a$ is twin solvable, but its pair of foliations is not covering at the origin. Then $\mathfrak{S Y M}(\mathscr{M})$ is infinite-dimensional, since for $a=a\left(y^{1}\right)$ an arbitrary function, it contains $a\left(y^{1}\right) \frac{\partial}{\partial y^{1}}+a\left(b^{1}\right) \frac{\partial}{\partial b^{1}}$.

Because we aim only to study finite-dimensional Lie symmetry groups of partial differential equations, in the remainder of this Part I, we will constantly assume the covering property to hold.

By Lemma 10.15, there exist two well defined integers $\mu$ and $\mu^{*}$ such that $e_{3}, e_{4}, \ldots, e_{\mu+1}>0$, but $e_{\mu+l}=0$ for all $l \geqslant 2$ and similarly, $e_{3}^{*}, e_{4}^{*}, \ldots, e_{\mu^{*}+1}^{*}>0$, but $e_{\mu^{*}+l}^{*}=0$ for all $l \geqslant 2$. Since the pair of foliations is covering, we have the two dimension equalities

$$
\left\{\begin{align*}
n+p+e_{3}+\cdots+e_{\mu+1} & =\operatorname{dim}_{\mathbb{K}} \mathscr{M}=n+m+p,  \tag{10.19}\\
p+n+e_{3}^{*}+\cdots+e_{\mu^{*}+1}^{*} & =\operatorname{dim}_{\mathbb{K}} \mathscr{M}=n+m+p .
\end{align*}\right.
$$

By definition, the ranges of $\Gamma_{\mu+1}$ and of $\Gamma_{\mu^{*}+1}^{*}$ cover (at least; more is true, see: Theorem 10.28) an open subset of $\mathscr{M}$. Also, it is elementary to verify the four inequalities

$$
\begin{array}{ll}
\mu \leqslant 1+m, & \mu^{*} \leqslant 1+m \\
\mu \leqslant \mu^{*}+1, & \mu^{*} \leqslant \mu+1 \tag{10.20}
\end{array}
$$

In fact, since $\Gamma_{k+1}$ with $x_{1}=0$ identifies with $\Gamma_{k}^{*}$, the second line follows.
Definition 10.21. The type of the covering pair of foliations $\left(F_{v}, F_{p}\right)$ is the pair of integers
(10.22) $\quad\left(\mu, \mu^{*}\right), \quad$ with $\quad \max \left(\mu, \mu^{*}\right) \leqslant 1+m$.

Example 10.23. (Continued) We write down the explicit expressions of $\Gamma_{4}$ and of $\Gamma_{5}$ :

$$
\left\{\begin{align*}
\Gamma_{4}\left(x_{1}, a_{1}, x_{2}, a_{2} ; 0\right) & =\left(x_{1}+x_{2}, x_{2} a_{1}, a_{1}+a_{2},-x_{1} a_{1}-x_{1} a_{2}-x_{2} a_{2},\right)  \tag{10.24}\\
\Gamma_{5}\left(x_{1}, a_{1}, x_{2}, a_{2}, x_{3} ; 0\right) & =\left(x_{1}+x_{2}+x_{3}, x_{2} a_{1}+x_{3} a_{1}+x_{3} a_{2}, a_{1}+a_{2}\right. \\
& \left.-x_{1} a_{1}-x_{1} a_{2}-x_{2} a_{2}\right) .
\end{align*}\right.
$$

Here, $\operatorname{dim} \mathscr{M}=3$. By computing its Jacobian matrix, $\Gamma_{5}$ is of rank 3 at every point $\left(x_{1}, a_{1}, 0,-a_{1},-x_{1}\right) \in \mathbb{K}^{5}$ with $a_{1} \neq 0$. Since (obviously)

$$
\begin{equation*}
\Gamma_{5}\left(x_{1}, a_{1}, 0,-a_{1},-x_{1}\right)=0 \in \mathscr{M}, \tag{10.25}
\end{equation*}
$$

we deduce that $\Gamma_{5}$ is submersive ("covering") from a small neighborhood of $\left(x_{1}, a_{1}, 0,-a_{1},-x_{1}\right)$ in $\mathbb{K}^{5}$ onto a neighborhood of the origin in $\mathscr{M}$.
10.26. Covering a neighborhood of the origin in $\mathscr{M}$. For $\left(z_{0}, c_{0}\right) \in \mathscr{M}$ fixed and close to the origin, we denote by $\Gamma_{k}\left([x a]_{k} ;\left(z_{0}, c_{0}\right)\right)$ and by $\Gamma_{k}^{*}\left([a x]_{k} ;\left(z_{0}, c_{0}\right)\right)$ the (dual) chains issued from $\left(z_{0}, c_{0}\right)$. For given parameters $[x a]_{k}=\left(x_{1}, a_{1}, x_{2}, \ldots\right)$, we denote by $[-x a]_{k}$ the collection $\left(\cdots,-x_{2},-a_{1},-x_{2}\right)$ with minus signs and reverse order; similarly, we introduce $[-a x]_{k}$. Notably, we have $\mathrm{L}_{-x_{1}}\left(\mathrm{~L}_{x_{1}}(0)\right)=0$ (because $\mathrm{L}_{-x_{1}+x_{1}}(\cdot)=$ $\left.\mathrm{L}_{0}(\cdot)=\mathrm{Id}\right)$, and also $\mathrm{L}_{-x_{1}}\left(\mathrm{~L}_{-a_{1}}^{*}\left(\mathrm{~L}_{a_{1}}^{*}\left(\mathrm{~L}_{x_{1}}(0)\right)\right)\right)=0$ and generally:

$$
\begin{equation*}
\Gamma_{k}\left([-x a]_{k} ; \Gamma_{k}\left([x a]_{k} ; 0\right)\right) \equiv 0 \tag{10.27}
\end{equation*}
$$

Geometrically speaking, by following backward the $k$-th chain $\Gamma_{k}$, we come back to 0 .

Theorem 10.28. ([Me2005a, Me2005b], [*]) The two maps $\Gamma_{2 \mu+1}$ and $\Gamma_{2 \mu^{*}+1}^{*}$ are submersive onto a neighborhood of the origin in $\mathscr{M}$. Precisely, there exist two points $[x a]_{2 \mu+1}^{0} \in \mathbb{K}^{(\mu+1) n+\mu p}$ and $[a x]_{2 \mu^{*}+1}^{0} \in$ $\mathbb{K}^{\mu^{*} n+\left(\mu^{*}+1\right) p}$ arbitrarily close to the origin with $\Gamma_{2 \mu+1}\left([x a]_{2 \mu+1}^{0}\right)=0$ and $\Gamma_{2 \mu^{*}+1}^{*}\left([a x]_{2 \mu^{*}+1}^{0}\right)=0$ such that the two maps

$$
\left\{\begin{array}{c}
\mathbb{K}^{(\mu+1) n+\mu p} \ni[x a]_{2 \mu+1} \longmapsto \Gamma_{2 \mu+1}\left([x a]_{2 \mu+1}\right) \in \mathscr{M} \quad \text { and }  \tag{10.29}\\
\mathbb{K}^{\mu^{*} n+\left(\mu^{*}+1\right) p} \ni[a x]_{2 \mu^{*}+1} \longmapsto \Gamma_{2 \mu^{*}+1}^{*}\left([a x]_{2 \mu^{*}+1}\right) \in \mathscr{M}
\end{array}\right.
$$

are of rank $n+m+p=\operatorname{dim}_{\mathbb{K}} \mathscr{M}$ at the points $[x a]_{2 \mu}^{0}$ and $[a x]_{2 \mu^{*}}^{0}$ respectively. In particular, the ranges of the two maps $\Gamma_{2 \mu+1}$ and $\Gamma_{2 \mu^{*}+1}^{*}$ cover a neighborhood of the origin in $\mathscr{M}$.

Let $\pi_{z}(z, c):=z$ and $\pi_{c}(z, c):=c$ be the two canonical projections. The next corollary will be useful in Section 12. In the example above, it also follows that the map

$$
\begin{equation*}
[x a]_{4} \mapsto \pi_{c}\left(\Gamma_{4}\left(\left[x a_{4}\right]\right)\right)=\left(a_{1}+a_{2},-x_{1} a_{1}-x_{1} a_{2}-x_{2} a_{2}\right) \in \mathbb{K}^{2} \tag{10.30}
\end{equation*}
$$

is of rank two at all points $[x a]_{4}^{0}:=\left(x_{1}^{0}, a_{1}^{0}, 0,-a_{1}^{0}\right)$ with $a_{1}^{0} \neq 0$.
Corollary 10.31. ([Me2005a, Me2005b], [*]) There exist two points $[x a]_{2 \mu}^{0} \in \mathbb{K}^{\mu(n+p)}$ and $[a x]_{2 \mu^{*}}^{0} \in \mathbb{K}^{\mu^{*}(n+p)}$ arbitrarily close to the origin with $\pi_{c}\left(\Gamma_{2 \mu}\left([x a]_{2 \mu}^{0}\right)\right)=0$ and $\pi_{z}\left(\Gamma_{2 \mu^{*}}^{*}\left([a x]_{2 \mu^{*}}^{0}\right)\right)=0$ such that the two maps

$$
\left\{\begin{align*}
\mathbb{K}^{\mu(n+p)} \ni[x a]_{2 \mu} & \longmapsto \pi_{c}\left(\Gamma_{2 \mu}\left([x a]_{2 \mu}\right)\right) \in \mathbb{K}^{m+p} \quad \text { and }  \tag{10.32}\\
\mathbb{K}^{\mu^{*}(n+p)} \ni[a x]_{2 \mu^{*}} & \longmapsto \pi_{z}\left(\Gamma_{2 \mu^{*}}^{*}\left([a x]_{2 \mu^{*}}\right)\right) \in \mathbb{K}^{n+m}
\end{align*}\right.
$$

are of rank $m+p$ at the point $[x a]_{2 \mu}^{0} \in \mathbb{K}^{\mu(n+p)}$ and of rank $n+m$ at the point $[a x]_{2 \mu^{*}}^{0} \in \mathbb{K}^{\mu^{*}(n+p)}$.

In the case $m=1$ (single dependent variable $y \in \mathbb{K}$ ), the covering property always hold with $\mu=\mu^{*}=2$.

## §11. Formal and smooth equivalences between submanifolds of SOLUTIONS

11.1. Transformations of submanifolds of solutions. Lemma 7.3 shows that every equivalence $\varphi$ between two PDE systems $(\mathscr{E})$ and $\left(\mathscr{E}^{\prime}\right)$ lifts as a transformation which respects the separation between variables and parameters of the form
$(x, y, a, b) \longmapsto(\phi(x, y), \psi(x, y), f(a, b), g(a, b))=(\varphi(x, y), h(a, b))=:\left(x^{\prime}, y^{\prime}, a^{\prime}, b^{\prime}\right)$
from the source submanifolds of solutions $\mathscr{M}:=\mathscr{V}_{\mathscr{S}}(\mathscr{E})$ to the target $\mathscr{M}^{\prime}:=\mathscr{V}_{\mathscr{L}}\left(\mathscr{E}^{\circ}\right)$, whose equations are

$$
\begin{align*}
y & =\Pi(x, c) & & \text { or dually } \\
y^{\prime} & =\Pi^{\prime}\left(x^{\prime}, c^{\prime}\right) & & \text { or dually } \tag{11.3}
\end{align*} \quad b^{\prime}=\Pi^{*}(a, z) \quad \text { and }\left(a^{\prime}, z^{\prime}\right) . ~ \$
$$

The study of transformations between submanifolds of solutions possesses strong similarities with the study of CR mappings between CR manifolds ([Pi1975, We1977, DW1980, BJT1985, DF1988, BER1999, Me2005a, Me2005b]). In fact, one may transfer the whole theory of the analytic reflection principle to this more general context. In the present $\S 10$ and in the next $\S 11$, we select and establish some of the results that are useful to the Lie theory. Some accessible open questions will also be formulated.

Maps of the form (11.2) send leaves of $F_{v}$ and of $F_{p}$ to leaves of $F_{v}^{\prime}$, and of $F_{p}^{\prime}$, respectively.

11.4. Regularity and jet parametrization. Some strong rigidity properties underly the above diagram. Especially, the smoothness of the two pairs $\left(F_{v}, F_{p}\right)$ and $\left(F_{v}^{\prime}, F_{p}^{\prime}\right)$ governs the smoothness of $(\varphi, h)$.

We shall study the regularity of a purely formal map $\left(z^{\prime}, c^{\prime}\right)=$ $(\varphi(z), h(c))$, namely $\varphi(z) \in \mathbb{K} \llbracket z \rrbracket^{n+m}$ and $h(c) \in \mathbb{K} \llbracket c \rrbracket^{p+m}$, assuming ( $\left.\mathscr{E}\right)$ and $\left(\mathscr{E}^{\prime}\right)$ to be analytic. Concretely, the assumption that $(\varphi, h)$ maps $\mathscr{M}$ to $\mathscr{M}^{\prime}$ reads as one of the four equivalent identities:

$$
\left\{\begin{align*}
\psi(x, \Pi(x, c)) & \equiv \Pi^{\prime}(\phi(x, \Pi(x, c)), h(c))  \tag{11.5}\\
\psi(z) & \equiv \Pi^{\prime}\left(\phi(z), h\left(a, \Pi^{*}(a, z)\right)\right. \\
g\left(a, \Pi^{*}(a, z)\right) & \equiv \Pi^{\prime *}\left(f\left(a, \Pi^{*}(a, z)\right), \varphi(z)\right) \\
g(c) & \equiv \Pi^{\prime *}(f(c), \varphi(x, \Pi(x, c)))
\end{align*}\right.
$$

in $\mathbb{K} \llbracket x, c \rrbracket^{m}$ and in $\mathbb{K} \llbracket a, z \rrbracket^{m}$.
Theorem 11.6. Let $(\varphi, h):=\mathscr{M} \rightarrow \mathscr{M}^{\prime}$ be a purely formal equivalence between two local $\mathbb{K}$-analytic submanifolds of solutions. Assume that the fundamental pair of foliations $\left(\mathrm{F}_{\mathrm{v}}, \mathrm{F}_{\mathrm{p}}\right)$ is covering at the origin, with type $\left(\mu, \mu^{*}\right)$ at the origin. Assume that $\mathscr{M}^{\prime}$ is both $\kappa$-solvable with respect to the parameters and $\kappa^{*}$-solvable with respect to the variables. Set $\ell:=\mu^{*}(\kappa+$ $\left.\kappa^{*}\right)$ and $\ell^{*}:=\mu\left(\kappa^{*}+\kappa\right)$. Then there exist two $\mathbb{K}^{n+m}$-valued and $\mathbb{K}^{p+m}$ valued local $\mathbb{K}$-analytic maps $\Phi_{\ell}$ and $H_{\ell^{*}}$, constructible only by means of $\Pi, \Pi^{*}, \Pi^{\prime}, \Pi^{\prime *}$, such that the following two formal power series identities hold:

$$
\left\{\begin{align*}
\varphi(z) & \equiv \Phi_{\ell}\left(z, J_{z}^{\ell} \varphi(0)\right)  \tag{11.7}\\
h(c) & \equiv H_{\ell^{*}}\left(c, J_{c}^{\ell^{*}} h(0)\right)
\end{align*}\right.
$$

in $\mathbb{K} \llbracket z \rrbracket^{n+m}$ and in $\mathbb{K} \llbracket c \rrbracket^{p+m}$, where $J_{z}^{\ell} \varphi(0)$ denotes the $\ell$-th jet of $h$ at the origin and similarly for $J_{c}^{\ell^{*}} h(0)$. In particular, as a corollary, we have the following two automatic regularity properties:

- $\varphi(z) \in \mathbb{K}\{z\}^{n+m}$ and $h(c) \in \mathbb{K}\{c\}^{p+m}$ are in fact convergent;
- if in addition $\mathscr{M}$ and $\mathscr{M}^{\prime}$ are $\mathbb{K}$-algebraic in the sense of Nash, then $\Phi_{\ell}$ and $H_{\ell^{*}}$ are also $\mathbb{K}$-algebraic, whence $\varphi(z) \in \mathscr{A}_{\mathbb{K}}\{z\}^{n+m}$ and $h(c) \in \mathscr{A}_{\mathbb{K}}\{c\}^{p+m}$ are in fact $\mathbb{K}$-algebraic.

Proof. We remind the explicit expressions of the two collections of vector fields spanning the leaves of the two foliations $F_{v}$ and $F_{p}$ :

$$
\left\{\begin{align*}
\mathrm{L}_{k}:=\frac{\partial}{\partial x_{k}}+\sum_{j=1}^{m} \frac{\partial \Pi^{j}}{\partial x_{k}}(x, c) \frac{\partial}{\partial y^{j}}, & k=1, \ldots, n  \tag{11.8}\\
\mathrm{~L}_{q}^{*}:=\frac{\partial}{\partial a^{q}}+\sum_{j=1}^{m} \frac{\partial \Pi^{* j}}{\partial a^{q}}(a, z) \frac{\partial}{\partial b^{j}}, & q=1, \ldots, p
\end{align*}\right.
$$

Observe that differentiating the first line of (11.5) with respect to $x^{k}$ amounts to applying the derivation $\mathrm{L}_{k}$. Similarly, differentiating the third line of (11.5) with respect to $a^{q}$ amounts to applying $\mathrm{L}_{q}^{*}$. We thus get for $(z, c) \in \mathscr{M}$

$$
\left\{\begin{align*}
\mathrm{L}_{k} \psi(z) & =\sum_{l=1}^{n} \frac{\partial \Pi^{\prime}}{\partial{x^{\prime}}^{\prime}}(\phi(z), h(c)) \mathrm{L}_{k} \phi^{l}(z) \quad \text { and }  \tag{11.9}\\
\mathrm{L}_{q}^{*} g(c) & =\sum_{r=1}^{p} \frac{\partial \Pi^{\prime *}}{\partial{a^{\prime r}}^{r}}(f(c), \varphi(z)) \mathrm{L}_{q}^{*} f^{r}(c)
\end{align*}\right.
$$

It follows from $\operatorname{det}\left(\frac{\partial \varphi^{k}}{\partial z^{l}}\right)(0) \neq 0$ and $\operatorname{det}\left(\frac{\partial h^{k}}{\partial c^{l}}\right)(0) \neq 0$ that the two formal determinants

$$
\begin{equation*}
\operatorname{det}\left(\mathrm{L}_{k} \phi^{l}(z)\right)_{1 \leqslant k \leqslant n}^{1 \leqslant l \leqslant n} \quad \text { and } \quad \operatorname{det}\left(\mathrm{L}_{q}^{*} f^{r}(c)\right)_{1 \leqslant q \leqslant p}^{1 \leqslant r \leqslant p} \tag{11.10}
\end{equation*}
$$

have nonvanishing constant term. Consequently, these two matrices are invertible in $\mathbb{K} \llbracket z \rrbracket$ and in $\mathbb{K} \llbracket c \rrbracket$. So there exist universal polynomials $\mathrm{S}_{l}^{j}$ and $S^{* j}$ such that

$$
\left\{\begin{array}{l}
\frac{\partial \Pi^{\prime j}}{\partial x^{\prime l}}(\varphi(z), h(c))=\frac{\mathrm{S}_{l}^{j}\left(\left\{\mathrm{~L}_{k^{\prime}} \varphi^{i^{\prime}}(z)\right\}_{1 \leqslant k^{\prime} \leqslant n}^{1 \leqslant i^{\prime} \leqslant n+m}\right)}{\operatorname{det}\left(\mathrm{L}_{k^{\prime}} \phi^{l^{\prime}}(z)\right)_{1 \leqslant k^{\prime} \leqslant n}^{1 \leqslant l^{\prime} \leqslant n}}  \tag{11.11}\\
\frac{\partial \Pi^{\prime * j}}{\partial a^{\prime r}}(f(c), \varphi(z))=\frac{\mathrm{S}^{*}{ }_{r}^{* j}\left(\left\{\mathrm{~L}_{q^{\prime}}^{*} h^{i^{\prime}}(c)\right\}_{1 \leqslant q^{\prime} \leqslant p}^{1 \leqslant i^{\prime} \leqslant p+m}\right)}{\operatorname{det}\left(\mathrm{L}_{q^{\prime}}^{*} f^{r^{\prime}}(c)\right)_{1 \leqslant q^{\prime}}^{1 \leqslant r^{\prime} \leqslant p}},
\end{array}\right. \text { and }
$$

for $1 \leqslant j \leqslant m$, for $1 \leqslant l \leqslant n$, for $1 \leqslant r \leqslant p$ and for $(z, c) \in \mathscr{M}$.
Again, we apply the vector fields $L_{k}$ to the obtained first line and the vector fields $L_{q}^{*}$ to the obtained second line, getting, thanks to the chain rule: (11.12)

$$
\left\{\begin{array}{l}
\sum_{l_{2}=1}^{n} \frac{\partial^{2} \Pi^{\prime j}}{\partial x^{\prime l_{1}} x^{l_{2}}}(\phi(z), h(c)) \mathrm{L}_{k} \phi^{l_{2}}(z)=\frac{\mathrm{R}_{l_{1}, k}^{j}\left(\left\{\mathrm{~L}_{k_{1}^{\prime}} \mathrm{L}_{k_{2}^{\prime}} \varphi^{i^{\prime}}(z)\right\}_{1 \leqslant k_{1}^{\prime}, k_{2}^{\prime} \leqslant n}^{1 \leqslant i^{\prime} \leqslant n+m}\right)}{\left[\operatorname{det}\left(\mathrm{L}_{k^{\prime}} \phi^{\prime}(z)\right)_{1 \leqslant k^{\prime} \leqslant n}^{1 \leqslant l^{\prime} \leqslant n}\right]^{2}}
\end{array} \text { and },\left\{\begin{array}{l}
\sum_{r_{2}=1}^{p} \frac{\partial^{2} \Pi^{* * j}}{\partial a^{r_{1}} a^{\prime r_{2}}}(f(c), \varphi(z)) \mathrm{L}_{q}^{*} f^{r_{2}}(c)=\frac{\mathrm{R}_{r_{1}, q}^{* j}\left(\left\{\mathrm{~L}_{q_{1}^{\prime}}^{*} \mathrm{~L}_{q_{2}^{\prime}}^{*}, i^{i^{\prime}}(c)\right\}_{1 \leqslant i_{1}^{\prime} \leqslant p+m}^{1 \leqslant p}\right)}{\left[\operatorname{det}\left(\mathrm{L}_{q^{\prime}}^{*} f^{r^{\prime}}(c)\right)_{1 \leqslant q^{\prime} \leqslant p}^{1 \leqslant r^{\prime} \leqslant p}\right]^{2}},
\end{array}\right.\right.
$$

for $1 \leqslant j \leqslant m$, for $1 \leqslant l_{1}, l_{2} \leqslant n$, for $1 \leqslant r_{1}, r_{2} \leqslant p$ and for $(z, c) \in \mathscr{M}$. Here, $\mathrm{R}_{l_{1}, k}^{j}$ and $\mathrm{R}_{r_{1}, q}^{* j}$ are universal polynomials. Then applying once more

Cramer's rule, we get
(11.13)

$$
\left\{\begin{array}{c}
\frac{\partial^{2} \Pi^{\prime j}}{\partial x^{\prime l_{1}} x^{\prime l_{2}}}(\phi(z), h(c))=\frac{\mathrm{S}_{l_{1}, l_{2}}^{j}\left(\left\{\mathrm{~L}_{k_{1}^{\prime}} \mathrm{L}_{k_{2}^{\prime}} \varphi^{i^{\prime}}(z)\right\}_{1 \leqslant k_{1}^{\prime}, k_{2}^{\prime} \leqslant n}^{1 \leqslant i^{\prime}}\right)}{\left[\operatorname{det}\left(\mathrm{L}_{k^{\prime}} \phi^{l^{\prime}}(z)\right)_{1 \leqslant l^{\prime} \leqslant n}^{1 \leqslant k^{\prime} \leqslant n}\right]^{3}} \text { and } \\
\frac{\partial^{2} \Pi^{\prime * j}}{\partial a^{\prime r_{1}} a^{\prime r_{2}}}(f(c), \varphi(z))=\frac{\mathrm{S}_{r_{1}, r_{2}}^{* j}\left(\left\{\mathrm{~L}_{q_{1}^{\prime}}^{*} \mathrm{~L}_{q_{2}^{\prime}}^{*} h^{i^{\prime}}(c)\right\}_{1 \leqslant i_{1}^{\prime}, q_{2}^{\prime} \leqslant p}^{1 \leqslant i^{\prime}}\right)}{\left[\operatorname{det}\left(\mathrm{L}_{q^{\prime}}^{*} f^{r^{\prime}}(c)\right)_{1 \leqslant r^{\prime} \leqslant q^{\prime} \leqslant p}^{1 \leqslant p}\right]^{3}}
\end{array}\right.
$$

By induction, for every $j$ with $1 \leqslant j \leqslant m$ and every two multiindices $\beta \in \mathbb{N}^{n}$ and $\delta \in \mathbb{N}^{p}$, there exists two universal polynomials $\mathrm{S}_{\beta}^{j}$ and $\mathrm{S}_{\delta}^{* j}$ such that

$$
\left\{\begin{array}{l}
\frac{\partial^{|\beta|} \Pi^{\prime j}}{\partial x^{\prime \beta}}(\phi(z), h(c))=\frac{\mathrm{S}_{\beta}^{j}\left(\left\{\mathrm{~L}^{\beta^{\prime}} \varphi^{i^{\prime}}(z)\right\}_{\left|\beta^{\prime}\right| \leqslant|\beta|}^{1 \leqslant i^{\prime} \leqslant n+m}\right)}{\left[\operatorname{det}\left(\mathrm{L}_{k^{\prime}} \phi^{l^{\prime}}(z)\right)_{1 \leqslant l^{\prime} \leqslant n}^{1 \leqslant k^{\prime} \leqslant n}\right]^{2|\beta|+1}} \quad \text { and }  \tag{11.14}\\
\frac{\partial^{|\gamma|} \Pi^{\prime * j}}{\partial a^{\prime \delta}}(f(c), \varphi(z))=\frac{\mathrm{S}_{\delta}^{* j}\left(\left\{\mathrm{~L}^{* \delta^{\prime}} h^{i^{\prime}}(c)\right\}_{\left|\delta^{\prime}\right| \leqslant p+m}^{1 \leqslant i^{\prime} \leqslant p \mid}\right)}{\left[\operatorname{det}\left(\mathrm{L}_{q^{\prime}}^{*} f^{r^{\prime}}(c)\right)_{1 \leqslant q^{\prime} \leqslant p}^{1 \leqslant r^{\prime} \leqslant p}\right]^{2|\delta|+1}}
\end{array}\right.
$$

Here, for $\beta^{\prime} \in \mathbb{N}^{n}$, we denote by $\mathrm{L}^{\beta^{\prime}}$ the derivation of order $\left|\beta^{\prime}\right|$ defined by $\left(\mathrm{L}_{1}\right)^{\beta_{1}^{\prime}} \cdots\left(\mathrm{L}_{n}\right)^{\beta_{n}^{\prime}}$. Similarly, for $\delta^{\prime} \in \mathbb{N}^{p}, \mathrm{~L}^{* \delta^{\prime}}$ denotes the derivation of order $\left|\delta^{\prime}\right|$ defined by $\left(L_{1}^{*}\right)^{\delta_{1}^{\prime}} \cdots\left(L_{p}^{*}\right)^{\delta_{p}^{\prime}}$.

Next, by the assumption that $\mathscr{M}^{\prime}$ is solvable with respect to the parameters, there exist integers $j(1), \ldots, j(p)$ with $1 \leqslant j(q) \leqslant m$ and multiindices $\beta(1), \ldots, \beta(p) \in \mathbb{N}^{n}$ with $|\beta(q)| \geqslant 1$ and $\max _{1 \leqslant q \leqslant p}|\beta(q)|=\kappa$ such that the local $\mathbb{K}$-analytic map
(11.15)
$\mathbb{K}^{p+m} \ni c^{\prime} \longmapsto\left(\left(\Pi^{\prime j}\left(0, c^{\prime}\right)\right)^{1 \leqslant j \leqslant m},\left(\frac{\partial^{|\beta(q)|} \Pi^{\prime j(q)}}{\partial x^{\prime \beta(q)}}\left(0, c^{\prime}\right)\right)_{1 \leqslant q \leqslant p}\right) \in \mathbb{K}^{p+m}$
is of rank $p+m$ at $c^{\prime}=0$. Similarly, by the assumption that $\mathscr{M}^{\prime}$ is solvable with respect to the variables, there exist integers $j^{\sim}(1), \ldots, j^{\sim}(n)$ with $1 \leqslant j^{\sim}(l) \leqslant m$ and multiindices $\delta(1), \ldots, \delta(p) \in \mathbb{N}^{n}$ with $|\delta(q)| \geqslant 1$ and $\max _{1 \leqslant q \leqslant p}|\delta(q)|=\kappa^{*}$ such that the local $\mathbb{K}$-analytic map
$\mathbb{K}^{n+m} \ni z^{\prime} \longmapsto\left(\left(\Pi^{\prime * j}\left(0, z^{\prime}\right)\right)^{1 \leqslant j \leqslant m},\left(\frac{\partial^{|\delta(l)|} \Pi^{* * j^{\sim}(l)}}{\partial{a^{\delta(l)}}^{\prime}}\left(0, z^{\prime}\right)\right)_{1 \leqslant l \leqslant n}\right) \in \mathbb{K}^{n+m}$
is of rank $n+m$ at $z^{\prime}=0$. We then consider from the first line of (11.14) only the $(p+m)$ equations written for $(j, 0),(j(q), \beta(q))$ and we solve $h(c)$
by means of the analytic implicit function theorem; also, in the second line of (11.14), we consider the $(n+m)$ equations written for $(j, 0),\left(j^{\sim}(l), \delta(l)\right)$ and we solve $\varphi(z)$. We get:

$$
\left\{\begin{array}{l}
h(c)=\widehat{H}\left(\phi(z), \frac{\mathrm{S}_{\beta(1)}^{j(1)}\left(\left\{\mathrm{L}^{\beta^{\prime}} \varphi^{i^{\prime}}(z)\right\}_{\left|\beta^{\prime}\right| \leqslant|\beta(1)|}^{1 \leqslant i^{\prime} \leqslant n+m}\right)}{\operatorname{det}\left[\left(\mathrm{L}_{k^{\prime}} \phi^{l^{\prime}}(z)\right)_{1 \leqslant k^{\prime} \leqslant n}^{1 \leqslant l^{\prime} \leqslant n}\right]^{2|\beta(1)|+1}}, \ldots\right. \\
\left.\ldots, \frac{\mathrm{S}_{\beta(p)}^{j(p)}\left(\left\{\mathrm{L}^{\beta^{\prime}} \varphi^{i^{\prime}}(z)\right\}_{\left|\beta^{\prime}\right| \leqslant|\beta(p)|}^{1 \leqslant i^{\prime} \leqslant n+m}\right)}{\operatorname{det}\left[\left(\mathrm{L}_{k^{\prime}} \phi^{\prime}(z)\right)_{1 \leqslant k^{\prime} \leqslant n}^{1 \leqslant l^{\prime} \leqslant n}\right]^{2|\beta(p)|+1}}\right) \tag{11.17}
\end{array},\right.
$$

for $(z, c) \in \mathscr{M}$. The maps $\widehat{H}$ and $\widehat{\Phi}$ depend only on $\Pi^{\prime}, \Pi^{\prime *}$.
Lemma 11.18. For every $\beta^{\prime} \in \mathbb{N}^{n}$, there exists a universal polynomial $\mathrm{P}_{\beta^{\prime}}$ in the jet variables $J_{z}^{\left|\beta^{\prime}\right|}$ having $\mathbb{K}$-analytic coefficients in $(z, c)$ which depends only on $\Pi, \Pi^{*}$ such that, for $i^{\prime}=1, \ldots, n+m$ :

$$
\begin{equation*}
\mathrm{L}^{\beta^{\prime}} \varphi^{i^{\prime}}(z) \equiv \mathrm{P}_{\beta^{\prime}}\left(z, c, J_{z}^{\left|\beta^{\prime}\right|} \varphi^{i^{\prime}}(z)\right) \tag{11.19}
\end{equation*}
$$

A similar property holds for $\mathrm{L}^{* \delta^{\prime}} h^{i^{\prime}}(c)$.
We deduce that there exist two local $\mathbb{K}$-analytic mappings $\Phi_{0}^{0}$ and $H_{0}^{0}$ such that we can write

$$
\left\{\begin{array}{l}
\varphi(z)=\Phi_{0}^{0}\left(z, c, J_{c}^{\kappa^{*}} h(c)\right)  \tag{11.20}\\
h(c)=H_{0}^{0}\left(z, c, J_{z}^{\kappa} \varphi(z)\right)
\end{array}\right.
$$

for $(z, c) \in \mathscr{M}$. Concretely, this means that we have two equivalent pairs of formal identities

$$
\begin{align*}
\varphi(z) & \equiv \Phi_{0}^{0}\left(z, a, \Pi^{*}(a, z), J_{c}^{\kappa^{*}} h\left(a, \Pi^{*}(a, z)\right)\right) \\
\varphi(x, \Pi(x, c)) & \equiv \Phi_{0}^{0}\left(x, \Pi(x, c), c, J_{c}^{\kappa^{*}} h(c)\right) \\
h(c) & \equiv H_{0}^{0}\left(x, \Pi(x, c), c, J_{z}^{\kappa} \varphi(x, \Pi(x, c))\right)  \tag{11.21}\\
h\left(a, \Pi^{*}(a, z)\right) & \equiv H_{0}^{0}\left(z, a, \Pi^{*}(a, z), J_{z}^{\kappa} \varphi(z)\right)
\end{align*}
$$

in $\mathbb{K} \llbracket a, z \rrbracket^{n+m}$ and in $\mathbb{K} \llbracket x, c \rrbracket^{p+m}$. We notice that, whereas $\varphi$ and $h$ are $a$ priori only purely formal, by construction, $\Phi_{0}^{0}$ and $H_{0}^{0}$ are $\mathbb{K}$-analytic near $\left(0,0, J_{c}^{\kappa^{*}} h(0)\right)$ and near $\left(0,0, J_{z}^{\kappa} \varphi(0)\right)$.

Next, we introduce the following vector fields with $\mathbb{K}$-analytic coefficients tangent to $\mathscr{M}$ :

$$
\begin{cases}\mathrm{V}_{j}:=\frac{\partial}{\partial y^{j}}+\sum_{l=1}^{m} \frac{\partial \Pi^{* l}}{\partial y^{j}}(a, z) \frac{\partial}{\partial b^{l}}, \quad j=1, \ldots, m \quad \text { and }  \tag{11.22}\\ \mathrm{V}_{j}^{*}:=\frac{\partial}{\partial b^{j}}+\sum_{l=1}^{m} \frac{\partial \Pi^{l}}{\partial b^{j}}(x, c) \frac{\partial}{\partial y^{l}}, \quad j=1, \ldots, m .\end{cases}
$$

Indeed, we check that $\mathrm{V}_{j_{1}}\left[b^{j_{2}}-\Pi^{* j_{2}}(a, z)\right] \equiv 0$ and that $\mathrm{V}_{j_{1}}^{*}\left[y^{j_{2}}-\right.$ $\left.\Pi^{j 2}(x, c)\right] \equiv 0$.

For $\delta^{\prime} \in \mathbb{N}^{m}$, we observe that $\mathrm{V}^{\delta^{\prime}} \varphi=\frac{\partial^{\left|\delta^{\prime}\right|} \varphi}{\partial y^{\delta^{\prime}}}$. Applying then $\mathrm{L}^{\beta^{\prime}}$ with $\beta^{\prime} \in \mathbb{N}^{n}$, we get for $i=1, \ldots, n+m$ :

$$
\begin{equation*}
\mathrm{L}^{\beta^{\prime}} \mathrm{V}^{\delta^{\prime}} \varphi^{i}(z)=\mathrm{Q}_{\beta^{\prime}, \delta^{\prime}}\left(z, c, J_{z}^{\left.\mid \beta^{\beta^{\prime}\left|+\left|\delta^{\prime}\right|\right.} \varphi^{i}(z)\right), ~, ~}\right. \tag{11.23}
\end{equation*}
$$

with $\mathrm{Q}_{\beta^{\prime}, \delta^{\prime}}$ universal. Since the $n+m$ vector fields $\mathrm{L}_{k}$ and $\mathrm{V}_{j}$, having coefficients depending on $(z, c)$, span the tangent space to $\mathbb{K}_{x}^{n} \times \mathbb{K}_{y}^{m}$, the change of basis of derivations yields, by induction, the following.
Lemma 11.24. For every $\alpha \in \mathbb{N}^{n+m}$, there exists a universal polynomial $\mathrm{P}_{\alpha}$ in its last variables with coefficients being $\mathbb{K}$-analytic in $(z, c)$ and depending only on $\Pi, \Pi^{*}$ such that, for $i=1, \ldots, n+m$ :

$$
\begin{equation*}
\partial_{z}^{\alpha} \varphi^{i}(z) \equiv \mathrm{P}_{\alpha}\left(z, c,\left(\mathrm{~L}^{\beta^{\prime}} \mathrm{V}^{\delta^{\prime}} \varphi^{i}(z)\right)_{\left|\beta^{\prime}\right|+\left|\delta^{\prime}\right| \leqslant|\alpha|}\right) . \tag{11.25}
\end{equation*}
$$

We are now in position to state and to prove the first fundamental technical lemma which generalizes the two formulas $(11.20)$ to arbitrary jets.
Lemma 11.26. For every $\lambda \in \mathbb{N}$, there exist two local $\mathbb{K}$-analytic maps, $\Phi_{0}^{\lambda}$ valued in $\mathbb{K}^{(n+m) C_{n+m+\lambda}^{\lambda}}$, and $H_{0}^{\lambda}$ valued in $\mathbb{K}^{(p+m) C_{p+m+\lambda}^{\lambda}}$, such that:

$$
\left\{\begin{align*}
J_{z}^{\lambda} \varphi(z) & \equiv \Phi_{0}^{\lambda}\left(z, c, J_{c}^{\kappa^{*}+\lambda} h(c)\right),  \tag{11.27}\\
J_{c}^{\lambda} h(c) & \equiv H_{0}^{\lambda}\left(z, c, J_{z}^{\kappa+\lambda} \varphi(z)\right)
\end{align*}\right.
$$

Proof. Consider for instance the first line. To obtain it, it suffices to apply the derivations $\mathrm{L}^{\beta^{\prime}} \mathrm{V}^{\delta^{\prime}}$ with $\left|\beta^{\prime}\right|+\left|\delta^{\prime}\right| \leqslant \lambda$ to the first line of (11.20), to use the chain rule and to apply Lemma 11.24.

Let $\theta \in \mathbb{K}^{l}, l \in \mathbb{N}$, let $Q(\theta)=\left(Q_{1}(\theta), \ldots, Q_{n+2 m+p}(\theta)\right) \in \mathbb{K} \llbracket \theta \rrbracket^{n+2 m+p}$ and let $a_{1} \in \mathbb{K}^{p}$. As the multiple flow of $\mathrm{L}^{*}$ given by (10.10) does not act on the variables $(x, y)$, we have the trivial but crucial property:

$$
\begin{equation*}
\varphi\left(\mathrm{L}_{a_{1}}^{*}(Q(\theta))\right) \equiv \varphi\left(\pi_{z}\left(\mathrm{~L}_{a_{1}}^{*}(Q(\theta))\right)\right) \equiv \varphi\left(\pi_{z}(Q(\theta))\right) \equiv \varphi(Q(\theta)) . \tag{11.28}
\end{equation*}
$$

At the end, we allow to suppress the projection $\pi_{z}$ : this slight abuse of notation will lighten slightly the writting of further formulas. More generally, for $\lambda \in \mathbb{N}$, $a_{1} \in \mathbb{K}^{p}, x_{1} \in \mathbb{K}^{n}$ :

$$
\begin{align*}
J_{z}^{\lambda} \varphi\left(\mathrm{L}_{a_{1}}^{*}(Q(\theta))\right) & \equiv J_{z}^{\lambda} \varphi(Q(\theta)) \quad \text { and } \\
J_{c}^{\lambda} h\left(\mathrm{~L}_{x_{1}}(Q(\theta))\right) & \equiv J_{c}^{\lambda} h(Q(\theta)) . \tag{11.29}
\end{align*}
$$

As a consequence, for $2 k$ even and for $2 k+1$ odd, we have the following four cancellation relations, useful below (we drop $\pi_{z}$ and $\pi_{c}$ after $J_{z}^{\lambda} \varphi$ and after $J_{c}^{\lambda} h$ ):

$$
\left\{\begin{align*}
J_{z}^{\lambda} \varphi\left(\Gamma_{2 k}\left([x a]_{2 k}\right)\right) & \equiv J_{z}^{\lambda} \varphi\left(\Gamma_{2 k-1}\left([x a]_{2 k-1}\right)\right),  \tag{11.30}\\
J_{c}^{\lambda} h\left(\Gamma_{2 k}^{*}\left([a x]_{2 k}\right)\right) & \equiv J_{c}^{\lambda} h\left(\Gamma_{2 k-1}^{*}\left([a x]_{2 k-1}\right)\right), \\
J_{z}^{\lambda} \varphi\left(\Gamma_{2 k+1}^{*}\left([a x]_{2 k+1}\right)\right) & \equiv J_{z}^{\lambda} \varphi\left(\Gamma_{2 k}^{*}\left([a x]_{2 k}\right)\right), \\
J_{c}^{\lambda} h\left(\Gamma_{2 k+1}\left([x a]_{2 k+1}\right)\right) & \equiv J_{c}^{\lambda} h\left(\Gamma_{2 k}\left([x a]_{2 k}\right)\right) .
\end{align*}\right.
$$

We are now in position to state and to prove the second main technical proposition.

Proposition 11.31. For every even chain-length $2 k$ and for every jet-height $\lambda$, there exist two local $\mathbb{K}$-analytic maps, $\Phi_{2 k}^{\lambda}$ valued in $\mathbb{K}^{(n+m) C_{n+m+\lambda}^{\lambda} \text {, and }}$ $H_{2 k}^{\lambda}$ valued in $\mathbb{K}^{(p+m) C_{p+m+\lambda}^{\lambda}}$ such that:

$$
\left\{\begin{align*}
J_{z}^{\lambda} \varphi\left(\Gamma_{2 k}^{*}\left([a x]_{2 k}\right)\right) & \equiv \Phi_{2 k}^{\lambda}\left([a x]_{2 k}, J_{z}^{k\left(\kappa+\kappa^{*}\right)+\lambda} \varphi(0)\right) \quad \text { and }  \tag{11.32}\\
J_{c}^{\lambda} h\left(\Gamma_{2 k}\left([x a]_{2 k}\right)\right) & \equiv H_{2 k}^{\lambda}\left([x a]_{2 k}, J_{c}^{k\left(\kappa+\kappa^{*}\right)+\lambda} \varphi(0)\right) .
\end{align*}\right.
$$

Similarly, for every odd chain length $2 k+1$ and for every jet eight $\lambda$, there exist two local $\mathbb{K}$-analytic maps, $\Phi_{2 k+1}^{\lambda}$ valued in $\mathbb{K}^{(n+m) C_{n+m+\lambda}^{\lambda}}$ and $H_{2 k+1}^{\lambda}$ valued in $\mathbb{K}^{(p+m) C_{p+m+\lambda}^{\lambda}}$, such that:

$$
\left\{\begin{align*}
& J_{z}^{\lambda} \varphi\left(\Gamma_{2 k+1}\left([x a]_{2 k+1}\right)\right) \equiv \Phi_{2 k+1}^{\lambda}\left([x a]_{2 k+1}, J_{c}^{k \kappa+(k+1) \kappa^{*}+\lambda} h(0)\right),  \tag{11.33}\\
& J_{c}^{\lambda} h\left(\Gamma_{2 k+1}^{*}\left([a x]_{2 k+1}\right)\right) \equiv H_{2 k+1}^{\lambda}\left([a x]_{2 k+1}, J_{z}^{(k+1) \kappa+k \kappa^{*}+\lambda} \varphi(0)\right)
\end{align*}\right.
$$

These maps depend only on $\Pi, \Pi^{*}, \Pi^{\prime}, \Pi^{*}$.
Proof. For $2 k+1=1$, we replace $(z, c)$ by $\Gamma_{1}\left([x a]_{1}\right)$ in the first line of (11.27) and by $\Gamma_{1}^{*}\left([a x]_{1}\right)$ in the second line. Taking crucially account
of the cancellation properties (11.29), we get:

$$
\left\{\begin{align*}
J_{z}^{\lambda} \varphi\left(\Gamma_{1}\left([x a]_{1}\right)\right) & \equiv \Phi_{0}^{\lambda}\left(\Gamma_{1}\left([x a]_{1}\right), J_{c}^{\kappa^{*}+\lambda} h\left(\Gamma_{1}\left([x a]_{1}\right)\right)\right)  \tag{11.34}\\
& \left.\equiv \Phi_{0}^{\lambda}\left(\Gamma_{1}\left([x a]_{1}\right), J_{c}^{\kappa^{*}+\lambda} h(0)\right)\right) \\
& =: \Phi_{1}^{\lambda}\left([x a]_{1}, J_{c}^{\kappa^{*}+\lambda} h(0)\right) \\
J_{c}^{\lambda} h\left(\Gamma_{1}^{*}\left([a x]_{1}\right)\right) & \equiv H_{0}^{\lambda}\left(\Gamma_{1}^{*}\left([a x]_{1}\right), J_{z}^{\kappa+\lambda} \varphi\left(\Gamma_{1}^{*}\left([a x]_{1}\right)\right)\right) \\
& \equiv H_{0}^{\lambda}\left(\Gamma_{1}^{*}\left([a x]_{1}\right), J_{z}^{\kappa+\lambda} \varphi(0)\right) \\
& =: H_{1}^{\lambda}\left([a x]_{1}, J_{z}^{\kappa+\lambda} \varphi(0)\right)
\end{align*}\right.
$$

Here, the third line defines $\Phi_{1}^{\lambda}$ and the sixth line defines $H_{1}^{\lambda}$. Thus, the proposition holds for $2 k+1=1$.

The rest of the proof proceeds by induction. We treat only the induction step from an odd chain-length $2 k+1$ to an even chain-length $2 k+2$, the other induction step being similar.

To this aim, we replace the variables $(z, c)$ in the first line of (11.27) by $\Gamma_{2 k+2}^{*}\left([a x]_{2 k+2}\right)$. Taking account of the cancellation property and of the induction assumption:
(11.35)

$$
\begin{aligned}
J_{z}^{\lambda} \varphi\left(\Gamma_{2 k+2}^{*}\left([a x]_{2 k+2}\right)\right) & \equiv \Phi_{0}^{\lambda}\left(\Gamma_{2 k+2}^{*}\left([a x]_{2 k+2}\right), J_{c}^{\kappa^{*}+\lambda} h\left(\Gamma_{2 k+2}^{*}\left([a x]_{2 k+2}\right)\right)\right) \\
& \equiv \Phi_{0}^{\lambda}\left(\Gamma_{2 k+2}^{*}\left([a x]_{2 k+2}\right), J_{c}^{\kappa^{*}+\lambda} h\left(\Gamma_{2 k+1}^{*}\left([a x]_{2 k+1}\right)\right)\right) \\
& \equiv \Phi_{0}^{\lambda}\left(\Gamma_{2 k+2}^{*}\left([a x]_{2 k+2}\right), H_{2 k+1}^{\kappa^{*}+\lambda}\left([a x]_{2 k+1}, J_{c}^{(k+1)\left(\kappa+\kappa^{*}\right)+\lambda} \varphi(0)\right)\right) \\
& =: \Phi_{2 k+2}^{\lambda}\left([a x]_{2 k+2}, J_{c}^{(k+1)\left(\kappa+\kappa^{*}\right)+\lambda} \varphi(0)\right),
\end{aligned}
$$

The last line defines $\Phi_{2 k+2}^{\lambda}$. Similarly, we replace $(z, c)$ in the second line of (11.27) by $\Gamma_{2 k+2}\left([x a]_{2 k+2}\right)$. Taking account of the cancellation property and of the induction assumption:
(11.36)

$$
\begin{aligned}
J_{c}^{\lambda} h\left(\Gamma_{2 k+2}\left([x a]_{2 k+2}\right)\right) & \equiv H_{0}^{\lambda}\left(\Gamma_{2 k+2}\left([x a]_{2 k+2}\right), J_{c}^{\kappa+\lambda} \varphi\left(\Gamma_{2 k+2}\left([x a]_{2 k+2}\right)\right)\right) \\
& \equiv H_{0}^{\lambda}\left(\Gamma_{2 k+2}\left([x a]_{2 k+2}\right), J_{c}^{\kappa+\lambda} \varphi\left(\Gamma_{2 k+1}\left([x a]_{2 k+1}\right)\right)\right) \\
& \equiv H_{0}^{\lambda}\left(\Gamma_{2 k+2}\left([x a]_{2 k+2}\right), \Phi_{2 k+1}^{\kappa+\lambda}\left([x a]_{2 k+1}, J_{c}^{(k+1)\left(\kappa+\kappa^{*}\right)+\lambda} h(0)\right)\right) \\
& =: H_{2 k+2}^{\lambda}\left([x a]_{2 k+2}, J_{c}^{(k+1)\left(\kappa+\kappa^{*}\right)+\lambda} h(0)\right) .
\end{aligned}
$$

This completes the proof.

End of the proof of Theorem 11.6. With $\left(\mu, \mu^{*}\right)$ being the type of $\left(\mathrm{F}_{\mathrm{v}}, \mathrm{F}_{\mathrm{p}}\right)$ and with $[a x]_{2 \mu^{*}}^{0}$ given by Corollary 10.31 , the rank property (10.32) insures the existence of an affine $(n+m)$-dimensional space $H \subset \mathbb{K}^{\mu^{*}(p+n)}$ passing
through $[a x]_{2 \mu^{*}}^{0}$ and equipped with a local parametrization

$$
\begin{equation*}
\mathbb{K}^{n+m} \ni s \mapsto[a x]_{2 \mu^{*}}(s) \in H \tag{11.37}
\end{equation*}
$$

satisfying $[a x]_{2 \mu^{*}}(0)=[a x]_{2 \mu^{*}}^{0}$, such that the map

$$
\begin{equation*}
\mathbb{K}^{n+m} \ni s \longmapsto \pi_{z}\left(\Gamma_{2 \mu^{*}}^{*}\left([a x]_{2 \mu^{*}}(s)\right)\right)=: z(s) \in \mathbb{K}^{n+m} \tag{11.38}
\end{equation*}
$$

is a local diffeomorphism fixing $0 \in \mathbb{K}^{n+m}$. Replacing $z$ by $z(s)$ in $\varphi(z)$ and applying the formula in the first line of (11.32) with $\lambda=0$ and with $k=2 \mu^{*}$, we obtain

$$
\begin{align*}
\varphi(z(s)) & =\varphi\left(\pi_{z}\left(\Gamma_{2 \mu^{*}}^{*}\left([a x]_{2 \mu^{*}}(s)\right)\right)\right. \\
& =\varphi\left(\Gamma_{2 \mu^{*}}^{*}\left([a x]_{2 \mu^{*}}(s)\right)\right)  \tag{11.39}\\
& \equiv \Phi_{2 \mu^{*}}^{0}\left([a x]_{2 \mu^{*}}(s), J_{z}^{\mu^{*}\left(\kappa+\kappa^{*}\right)} \varphi(0)\right) .
\end{align*}
$$

Inverting $s \mapsto z=z(s)$ as $z \mapsto s=s(z)$, we finally get

$$
\begin{align*}
\varphi(z)=\varphi(z(s(z))) & \equiv \Phi_{2 \mu^{*}}^{0}\left([a x]_{2 \mu^{*}}(s(z)), J_{z}^{\mu^{*}\left(\kappa+\kappa^{*}\right)} \varphi(0)\right) \\
& =: \Phi_{\ell}\left(z, J_{z}^{\mu^{*}\left(\kappa+\kappa^{*}\right)} \varphi(0)\right), \tag{11.40}
\end{align*}
$$

with $\ell:=\mu^{*}\left(\kappa+\kappa^{*}\right)$, where the last line defines $\Phi_{\ell}$. In conclusion, we have derived the first line of (11.7). The second one is obtained similarly.

If $\Pi, \Pi^{*}, \Pi^{\prime}, \Pi^{* *}$ are algebraic, so are $\Gamma_{k}, \Gamma_{k}^{*}, \widehat{H}, \widehat{\Phi}, \Phi_{0}^{\lambda}, H_{0}^{\lambda}, \Phi_{k}^{\lambda}, H_{k}^{\lambda}$ and $\Phi_{\ell}, H_{\ell^{*}}$.

The proof of Theorem 11.6 is complete.

# II: Explicit prolongations of infinitesimal Lie symmetries 

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## §1. Jet spaces and prolongations

1.1. Choice of notations for the jet space variables. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Let $n \geqslant 1$ and $m \geqslant 1$ be two positive integers and consider two sets of variables $x=\left(x^{1}, \ldots, x^{n}\right) \in \mathbb{K}^{n}$ and $y=\left(y^{1}, \ldots, y^{m}\right)$. In the classical theory of Lie symmetries of partial differential equations, one considers certain differential systems whose (local) solutions should be mappings of the form $y=y(x)$. We refer to [O11986] and to [BK1989] for an exposition of the fundamentals of the theory. Accordingly, the variables $x$ are usually called independent, whereas the variables $y$ are called dependent. Not to enter in subtle regularity considerations (as in [Me2005b]), we shall assume $\mathscr{C}^{\infty}$ smoothness of all functions throughout this paper.

Let $\kappa \geqslant 1$ be a positive integer. For us, in a very concrete way (without fiber bundles), the $\kappa$-th jet space $\mathscr{J}_{n, m}^{\kappa}$ consists of the space $\mathbb{K}^{n+m+m \frac{(n+m)!}{n!m!}}$ equipped with the affine coordinates

$$
\begin{equation*}
\left(x^{i}, y^{j}, y_{i_{1}}^{j}, y_{i_{1}, i_{2}}^{j}, \ldots \ldots, y_{i_{1}, i_{2}, \ldots, i_{k}}^{j}\right), \tag{1.2}
\end{equation*}
$$

having the symmetries

$$
\begin{equation*}
y_{i_{1}, i_{2}, \ldots, i_{\lambda}}^{j}=y_{i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(\lambda)}}^{j}, \tag{1.3}
\end{equation*}
$$

for every $\lambda$ with $1 \leqslant \lambda \leqslant \kappa$ and for every permutation $\sigma$ of the set $\{1, \ldots, \lambda\}$. The variable $y_{i_{1}, i_{2}, \ldots, i_{\lambda}}^{j}$ is an independent coordinate corresponding to the $\lambda$-th partial derivative $\frac{\partial^{\lambda} y^{j}}{\partial x^{i_{1}} \partial x^{i_{2}} \ldots \partial x^{i} \lambda}$. So the symmetries (1.3) are natural.

In the classical Lie theory ([OL1979], [Ol1986], [BK1989]), all the geometric objects: point transformations, vector fields, etc., are local, defined in a neighborhood of some point lying in some affine space $\mathbb{K}^{N}$. However, in this paper, the original geometric motivations are rapidly forgotten in order to focus on combinatorial considerations. Thus, to simplify the presentation, we shall not introduce any special notation to speak of certain local
 will always work in global affine spaces $\mathbb{K}^{N}$.

### 1.4. Prolongation $\varphi^{(\kappa)}$ of a local diffeomorphism $\varphi$ to the $\kappa$-th jet space.

 In this paragraph, we recall how the prolongation of a diffeomorphism to the $\kappa$-th jet space is defined ([OL1979], [Ol1986], [BK1989]).Let $x_{*} \in \mathbb{K}^{n}$ be a central fixed point and let $\varphi: \mathbb{K}^{n+m} \rightarrow \mathbb{K}^{n+m}$ be a diffeomorphism whose Jacobian matrix is close to the identity matrix, at least in a small neighborhood of $x_{*}$. Let

$$
\begin{equation*}
J_{x_{*}}^{\kappa}:=\left.\left(x_{*}^{i}, y_{* i_{1}}^{j}, y_{* i_{1}, i_{2}}^{j}, \ldots ., y_{* i_{1}, i_{2}, \ldots, i \kappa}^{j}\right) \in \mathscr{J}_{n, m}^{\kappa}\right|_{x_{*}} \tag{1.5}
\end{equation*}
$$

be an arbitrary $\kappa$-jet based at $x_{*}$. The goal is to defined its transformation $\varphi^{(\kappa)}\left(J_{x_{*}}^{\kappa}\right)$ by $\varphi$.

To this aim, choose an arbitrary mapping $\mathbb{K}^{n} \ni x \mapsto g(x) \in \mathbb{K}^{m}$ defined at least in a neighborhood of $x_{*}$ and representing this $\kappa$-th jet, i.e. satisfying

$$
\begin{equation*}
y_{* i_{1}, \ldots, i_{\lambda}}^{j}=\frac{\partial^{\lambda} g^{j}}{\partial x^{i_{1}} \cdots \partial x^{i_{\lambda}}}\left(x_{*}\right), \tag{1.6}
\end{equation*}
$$

for every $\lambda \in \mathbb{N}$ with $0 \leqslant \lambda \leqslant \kappa$, for all indices $i_{1}, \ldots, i_{\lambda}$ with $1 \leqslant$ $i_{1}, \ldots, i_{\lambda} \leqslant n$ and for every $j \in \mathbb{N}$ with $1 \leqslant j \leqslant m$. In accordance with the splitting $(x, y) \in \mathbb{K}^{n} \times \mathbb{K}^{m}$ of coordinates, split the components of the diffeomorphism $\varphi$ as $\varphi=(\phi, \psi) \in \mathbb{K}^{n} \times \mathbb{K}^{m}$. Write $(\bar{x}, \bar{y})$ the coordinates in the target space, so that the diffeomorphism $\varphi$ is:

$$
\begin{equation*}
\mathbb{K}^{n+m} \ni(x, y) \longmapsto(\bar{x}, \bar{y})=(\phi(x, y), \psi(x, y)) \in \mathbb{K}^{n+m} . \tag{1.7}
\end{equation*}
$$

Restrict the variables $(x, y)$ to belong to the graph of $g$, namely put $y:=g(x)$ above, which yields

$$
\left\{\begin{array}{l}
\bar{x}=\phi(x, g(x)),  \tag{1.8}\\
\bar{y}=\psi(x, g(x)) .
\end{array}\right.
$$

As the differential of $\varphi$ at $x_{*}$ is close to the identity, the first family of $n$ scalar equations may be solved with respect to $x$, by means of the implicit function theorem. Denote $x=\bar{\chi}(\bar{x})$ the resulting mapping, satisfying by definition

$$
\begin{equation*}
\bar{x} \equiv \phi(\bar{\chi}(\bar{x}), g(\bar{\chi}(\bar{x}))) . \tag{1.9}
\end{equation*}
$$

Replace $x$ by $\bar{\chi}(\bar{x})$ in the second family of $m$ scalar equations (1.8) above, which yields:

$$
\begin{equation*}
\bar{y}=\psi(\bar{\chi}(\bar{x}), g(\bar{\chi}(\bar{x}))) . \tag{1.10}
\end{equation*}
$$

Denote simply by $\bar{y}=\bar{g}(\bar{x})$ this last relation, where $\bar{g}(\cdot):=$ $\psi(\bar{\chi}(\cdot), g(\bar{\chi}(\cdot)))$.

In summary, the graph $y=g(x)$ has been transformed to the graph $\bar{y}=$ $\bar{g}(\bar{x})$ by the diffeomorphism $\varphi$.

Define then the transformed jet $\varphi^{(\kappa)}\left(J_{x_{*}}^{\kappa}\right)$ to be the $\kappa$-th jet of $\bar{g}$ at the point $\bar{x}_{*}:=\phi\left(x_{*}\right)$, namely:

$$
\begin{equation*}
\varphi^{(\kappa)}\left(J_{x_{*}}^{\kappa}\right):=\left.\left(\frac{\partial^{\lambda} \bar{g}^{j}}{\partial \bar{x}^{i_{1}} \cdots \partial \bar{x}_{\lambda}^{i_{\lambda}}}\left(\bar{x}_{*}\right)\right)_{1 \leqslant i_{1}, \ldots, i_{\lambda} \leqslant n, 0 \leqslant \lambda \leqslant \kappa}^{1 \leqslant j \leqslant m} \in \mathscr{J}_{n, m}^{\kappa}\right|_{\bar{x}_{*}} . \tag{1.11}
\end{equation*}
$$

It may be shown that this jet does not depend on the choice of a local graph $y=g(x)$ representing the $\kappa$-th jet $J_{x_{*}}^{\kappa}$ at $x_{*}$. Furthermore, if $\pi_{\kappa}:=\mathscr{J}_{n, m}^{\kappa} \rightarrow \mathbb{K}^{m}$ denotes the canonical projection onto the first factor, the following diagram commutes:

1.12. Inductive formulas for the $\kappa$-th prolongation $\varphi^{(\kappa)}$. To present them, we change our notations. Instead of $(\bar{x}, \bar{y})$, as coordinates in the target space $\mathbb{K}^{n} \times \mathbb{K}^{m}$, we shall use capital letters:

$$
\begin{equation*}
\left(X^{1}, \ldots, X^{n}, Y^{1}, \ldots, Y^{m}\right) \tag{1.13}
\end{equation*}
$$

In the source space $\mathbb{K}^{n+m}$ equipped with the coordinates $(x, y)$, we use the jet coordinates (1.2) on the associated $\kappa$-th jet space. In the target space $\mathbb{K}^{n+m}$ equipped with the coordinates $(X, Y)$, we use the coordinates

$$
\begin{equation*}
\left(X^{i}, Y^{j}, Y_{X^{i_{1}}}^{j}, Y_{X^{i_{1}} X^{i_{2}}}^{j}, \ldots \ldots, Y_{X^{i_{1}} X^{i_{2}} \ldots X^{i_{k}}}^{j}\right) \tag{1.14}
\end{equation*}
$$

on the associated $\kappa$-th jet space; to avoid confusion with $y_{i_{1}}, y_{i_{1}, i_{2}}, \ldots$ in subsequent formulas, we do not write $Y_{i_{1}}, Y_{i_{1}, i_{2}}, \ldots$. In these notations, the diffeomorphism $\varphi$ whose first order approximation is close to the identity mapping in a neighborhood of $x_{*}$ may be written under the form:

$$
\begin{equation*}
\varphi:\left(x^{i^{\prime}}, y^{j^{\prime}}\right) \mapsto\left(X^{i}, Y^{j}\right)=\left(X^{i}\left(x^{i^{\prime}}, y^{j^{\prime}}\right), Y^{j}\left(x^{i^{\prime}}, y^{j^{\prime}}\right)\right), \tag{1.15}
\end{equation*}
$$

for some $\mathscr{C}^{\infty}$-smooth functions $X^{i}\left(x^{i^{\prime}}, y^{j^{\prime}}\right), i=1, \ldots, n$, and $Y^{j}\left(x^{i^{\prime}}, y^{j^{\prime}}\right)$, $j=1, \ldots, m$. The first prolongation $\varphi^{(1)}$ of $\varphi$ may be written under the form:

$$
\begin{equation*}
\varphi^{(1)}:\left(x^{i^{\prime}}, y^{j^{\prime}}, y_{i_{1}^{j^{\prime}}}^{j^{\prime}}\right) \longmapsto\left(X^{i}\left(x^{i^{\prime}}, y^{j^{\prime}}\right), Y^{j}\left(x^{i^{\prime}}, y^{j^{\prime}}\right), Y_{X^{i_{1}}}^{j}\left(x^{i^{\prime}}, y^{j^{\prime}}, y_{i_{1}^{\prime}}^{j^{\prime}}\right)\right) \tag{1.16}
\end{equation*}
$$

for some functions $Y_{X^{i_{1}}}^{j}\left(x^{i^{\prime}}, y^{j^{\prime}}, y_{i_{1}^{\prime}}^{j^{\prime}}\right)$ which depend on the pure first jet variables $y_{i_{1}^{\prime}}^{j^{\prime}}$. The way how these functions depend on the first order partial derivatives functions $X_{x^{i^{i}}}^{i}, X_{y j^{\prime}}^{i}, Y_{x^{i^{\prime}}}^{j}, Y_{y j^{\prime}}^{j}$ and on the pure first jet
variables $y_{i_{1}^{\prime}}^{j^{\prime}}$ is provided (in principle) by the following compact formulas ([BK1989]):

$$
\left(\begin{array}{c}
Y_{X^{1}}^{j}  \tag{1.17}\\
\vdots \\
Y_{X^{n}}^{j}
\end{array}\right)=\left(\begin{array}{ccc}
D_{1}^{1} X^{1} & \cdots & D_{1}^{1} X^{n} \\
\vdots & \cdots & \vdots \\
D_{n}^{1} X^{1} & \cdots & D_{n}^{1} X^{n}
\end{array}\right)^{-1}\left(\begin{array}{c}
D_{1}^{1} Y^{j} \\
\vdots \\
D_{n}^{1} Y^{j}
\end{array}\right)
$$

where, for $i^{\prime}=1, \ldots, n$, the symbol $D_{i^{\prime}}^{1}$ denotes the $i^{\prime}$-th first order total differentiation operator:

$$
\begin{equation*}
D_{i^{\prime}}^{1}:=\frac{\partial}{\partial x^{i^{\prime}}}+\sum_{j^{\prime}=1}^{m} y_{i^{\prime}}^{j^{\prime}} \frac{\partial}{\partial y^{j^{\prime}}} . \tag{1.18}
\end{equation*}
$$

Striclty speaking, these formulas (1.17) are not explicit, because an inverse matrix is involved and because the terms $D_{i^{\prime}}^{1} X^{i}, D_{i^{\prime}}^{1} Y^{j}$ are not developed. However, it would be feasible and elementary to write down the corresponding totally explicit complete formulas for the functions $Y_{X^{i_{1}}}^{j}=$ $Y_{X^{i_{1}}}^{j}\left(x^{i^{\prime}}, y^{j^{\prime}}, y_{i_{1}^{\prime}}^{j^{\prime}}\right)$.

Next, the second prolongation $\varphi^{(2)}$ is of the form
$\varphi^{(2)}:\left(x^{i^{\prime}}, y^{j^{\prime}}, y_{i_{1}^{\prime}}^{j^{\prime}}, y_{i_{1}^{\prime}, i_{2}^{\prime}}^{j^{\prime}}\right) \longmapsto\left(\varphi^{(1)}\left(x^{i^{\prime}}, y^{j^{\prime}}, y_{i_{1}^{\prime}}^{j^{\prime}}\right), Y_{X^{i_{1}} X^{i_{2}}}^{j}\left(x^{i^{\prime}}, y^{j^{\prime}}, y_{i_{1}^{\prime}}^{j^{\prime}}, y_{i_{1}^{\prime}, i_{2}^{\prime}}^{j^{\prime}}\right)\right)$,
for some functions $Y_{X^{i_{1}} X^{i_{2}}}^{j}\left(x^{i^{\prime}}, y^{j^{\prime}}, y_{i_{1}^{\prime}}^{j^{\prime}}, y_{i_{1}^{\prime}, i_{2}^{\prime}}^{j^{\prime}}\right)$ which depend on the pure first and second jet variables. For $i=1, \ldots, n$, the expressions of $Y_{X^{i_{1}} X^{i}}^{j}$ are given by the following compact formulas (again [BK1989]):

$$
\left(\begin{array}{c}
Y_{X^{i_{1} X^{1}}}^{j}  \tag{1.20}\\
\vdots \\
Y_{X^{i_{1} X^{n}}}^{j}
\end{array}\right)=\left(\begin{array}{ccc}
D_{1}^{1} X^{1} & \cdots & D_{1}^{1} X^{n} \\
\vdots & \cdots & \vdots \\
D_{n}^{1} X^{1} & \cdots & D_{n}^{1} X^{n}
\end{array}\right)^{-1}\left(\begin{array}{c}
D_{1}^{2} Y_{X^{i_{1}}}^{j} \\
\vdots \\
D_{n}^{2} Y_{X^{i_{1}}}^{j}
\end{array}\right)
$$

where, for $i^{\prime}=1, \ldots, n$, the symbol $D_{i^{\prime}}^{2}$ denotes the $i^{\prime}$-th second order total differentiation operator:

$$
\begin{equation*}
D_{i^{\prime}}^{2}:=\frac{\partial}{\partial x^{i^{\prime}}}+\sum_{j^{\prime}=1}^{m} y_{i^{\prime}}^{j^{\prime}} \frac{\partial}{\partial y^{j^{\prime}}}+\sum_{j^{\prime}=1}^{m} \sum_{i_{1}^{\prime}=1}^{n} y_{i^{\prime}, i_{1}^{\prime}}^{j^{\prime}} \frac{\partial}{\partial y_{i_{1}^{\prime}}^{j^{\prime}}} \tag{1.21}
\end{equation*}
$$

Again, these formulas (1.20) are not explicit in the sense that an inverse matrix is involved and that the terms $D_{i^{\prime}}^{1} X^{i}, D_{i^{\prime}}^{2} Y_{X^{i_{1}}}^{j}$ are not developed. It would already be a nontrivial computational task to develope these expressions and to find out some nice satisfying combinatorial formulas.

In order to present the general inductive non-explicit formulas for the computation of the $\kappa$-th prolongation $\varphi^{(\kappa)}$, we need some more notation.

Let $\lambda \in \mathbb{N}$ be an arbitrary integer. For $i^{\prime}=1, \ldots, n$, let $D_{i^{\prime}}^{\lambda}$ denotes the $i^{\prime}$-th $\lambda$-th order total differentiation operators, defined precisely by:

$$
\left\{\begin{align*}
D_{i^{\prime}}^{\lambda}:= & \frac{\partial}{\partial x^{i^{\prime}}}+\sum_{j^{\prime}=1}^{m} y_{i^{\prime}}^{j^{\prime}} \frac{\partial}{\partial y^{j^{\prime}}}+\sum_{j^{\prime}=1}^{m} \sum_{i_{1}^{\prime}=1}^{n} y_{i^{\prime}, i_{1}^{\prime}}^{j^{\prime}} \frac{\partial}{\partial y_{i_{1}^{\prime}}^{j^{\prime}}}+\sum_{j^{\prime}=1}^{m} \sum_{i_{1}^{\prime}, i_{2}^{\prime}=1}^{n} y_{i^{\prime}, i_{1}^{\prime}, i_{2}^{\prime}}^{j^{\prime}} \frac{\partial}{\partial y_{i_{1}^{\prime}, i_{2}^{\prime}}^{j^{\prime}}}+  \tag{1.22}\\
& +\cdots+\sum_{i_{1}^{\prime}, i_{2}^{\prime}, \ldots, i_{\lambda-1}^{\prime}=1}^{m} y_{i^{\prime}, i_{1}^{\prime}, i_{2}^{\prime}, \ldots, i_{\lambda-1}^{\prime}}^{j^{\prime}} \frac{\partial}{\partial y_{i_{1}^{\prime}, i_{2}^{\prime}, \ldots, i_{\lambda-1}^{\prime}}^{j^{\prime}}} .
\end{align*}\right.
$$

Then, for $i=1, \ldots, n$, the expressions of $Y_{X^{i_{1}} \ldots X^{i}{ }^{i}-1 X^{i}}^{j}$ are given by the following compact formulas (again [BK1989]):

$$
\left(\begin{array}{c}
Y_{X^{i_{1} \ldots X^{i_{\lambda-1}} X^{1}}}^{j}  \tag{1.23}\\
\vdots \\
Y_{X^{i_{1} \cdots X^{i}{ }^{2}-1} X^{n}}^{j}
\end{array}\right)=\left(\begin{array}{ccc}
D_{1}^{1} X^{1} & \cdots & D_{1}^{1} X^{n} \\
\vdots & \cdots & \vdots \\
D_{n}^{1} X^{1} & \cdots & D_{n}^{1} X^{n}
\end{array}\right)^{-1}\left(\begin{array}{c}
D_{1}^{\lambda} Y_{X^{i_{1} \ldots X^{i} \lambda^{1} 1}}^{j} \\
\vdots \\
D_{n}^{\lambda} Y_{X^{i_{1} \ldots X^{i}{ }^{i}-1}}^{j}
\end{array}\right)
$$

Again, these inductive formulas are incomplete and unsatisfactory.
Problem 1.24. Find totally explicit complete formulas for the $\kappa$-th prolongation $\varphi^{(\kappa)}$.

Except in the cases $\kappa=1,2$, we have not been able to solve this problem. The case $\kappa=1$ is elementary. Complete formulas in the particular cases $\kappa=2, n=1, m \geqslant 1$ and $n \geqslant 1, m=1$ are implicitely provided in [Me2004] and in Section ?(?), where one observes the appearance of some modifications of the Jacobian determinant of the diffeomorphism $\varphi$, inserted in a clearly understandable combinatorics. In fact, there is a nice dictionary between the formulas for $\varphi^{(2)}$ and the formulas for the second prolongation $\mathscr{L}^{(2)}$ of a vector field $\mathscr{L}$ which were written in equation (43) of [GM2003a] (see also equations (2.6), (3.20), (4.6) and (5.3) in the next paragraphs). In the passage from $\varphi^{(2)}$ to $\mathscr{L}^{(2)}$, a sort of formal first order linearization may be observed and the reverse passage may be easily guessed. However, for $\kappa \geqslant 3$, the formulas for $\varphi^{(\kappa)}$ explode faster than the formulas for the $\kappa$-th prolongation $\mathscr{L}^{(\kappa)}$ of a vector field $\mathscr{L}$. Also, the dictionary between $\varphi^{(\kappa)}$ and $\mathscr{L}^{(\kappa)}$ disappears. In fact, to elaborate an appropriate dictionary, we believe that one should introduce before a sort of formal $(\kappa-1)$-th order linearizations of $\varphi^{(\kappa)}$, finer than the first order linearization $\mathscr{L}^{(\kappa)}$. To be optimistic, we believe that the final answer to Problem 1.24 is, nevertheless, accessible after hard work.

The present article is devoted to present totally explicit complete formulas for the $\kappa$-th prolongation $\mathscr{L}^{(\kappa)}$ of a vector field $\mathscr{L}$ to $\mathscr{J}_{n, m}^{\kappa}$, for $n \geqslant 1$ arbitrary, for $m \geqslant 1$ arbitrary and for $\kappa \geqslant 1$ arbitrary.
1.25. Prolongation of a vector field to the $\kappa$-th jet space. Consider a vector field

$$
\begin{equation*}
\mathscr{L}=\sum_{i=1}^{n} \mathscr{X}^{i}(x, y) \frac{\partial}{\partial x^{i}}+\sum_{j=1}^{m} \mathscr{Y}^{j}(x, y) \frac{\partial}{\partial y^{j}}, \tag{1.26}
\end{equation*}
$$

defined in $\mathbb{K}^{n+m}$. Its flow:

$$
\begin{equation*}
\varphi_{t}(x, y):=\exp (t \mathscr{L})(x, y) \tag{1.27}
\end{equation*}
$$

constitutes a one-parameter family of diffeomorphisms of $\mathbb{K}^{n+m}$ close to the identity. The lift $\left(\varphi_{t}\right)^{(\kappa)}$ to the $\kappa$-th jet space constitutes a one-parameter family of diffeomorphisms of $\mathscr{J}_{n, m}^{\kappa}$. By definition, the $\kappa$-th prolongation $\mathscr{L}^{(\kappa)}$ of $\mathscr{L}$ to the jet space $\mathscr{J}_{n, m}^{\kappa}$ is the infinitesimal generator of $\left(\varphi_{t}\right)^{(\kappa)}$, namely:

$$
\begin{equation*}
\mathscr{L}^{(\kappa)}:=\left.\frac{d}{d t}\right|_{t=0}\left[\left(\varphi_{t}\right)^{(\kappa)}\right] . \tag{1.28}
\end{equation*}
$$

1.29. Inductive formulas for the $\kappa$-th prolongation $\mathscr{L}^{(\kappa)}$. As a vector field
 the general form:

$$
\left\{\begin{align*}
\mathscr{L}^{(\kappa)}= & \sum_{i=1}^{n} \mathscr{X}^{i} \frac{\partial}{\partial x^{i}}+\sum_{j=1}^{m} \mathscr{Y}^{j} \frac{\partial}{\partial y^{j}}+  \tag{1.30}\\
& +\sum_{j=1}^{m} \sum_{i_{1}=1}^{n} \mathbf{Y}_{i_{1}}^{j} \frac{\partial}{\partial y_{i_{1}}^{j}}+\sum_{j=1}^{m} \sum_{i_{1}, i_{2}=1}^{n} \mathbf{Y}_{i_{1}, i_{2}}^{j} \frac{\partial}{\partial y_{i_{1}, i_{2}}^{j}}+\cdots+ \\
& +\sum_{j=1}^{m} \sum_{i_{1}, \ldots, i_{\kappa}=1}^{n} \mathbf{Y}_{i_{1}, \ldots, i_{\kappa}}^{j} \frac{\partial}{\partial y_{i_{1}, \ldots, i_{\kappa}}^{j}} .
\end{align*}\right.
$$

Here, the coefficients $\mathbf{Y}_{i_{1}}^{j}, \mathbf{Y}_{i_{1}, i_{2}}^{j}, \ldots, \mathbf{Y}_{i_{1}, i_{2}, \ldots, i_{\kappa}}^{j}$ are uniquely determined in terms of partial derivatives of the coefficients $\mathscr{X}^{i}$ and $\mathscr{Y}^{j}$ of the original vector field $\mathscr{L}$, together with the pure jet variables $\left(y_{i_{1}}^{j}, \ldots, y_{i_{1}, \ldots, i_{\kappa}}^{j}\right)$, by means of the following fundamental inductive formulas ([OL1979],
[O11986], [BK1989]):
(1.31)
where, for every $\lambda \in \mathbb{N}$ with $0 \leqslant \lambda \leqslant \kappa$, and for every $i \in \mathbb{N}$ with $1 \leqslant i^{\prime} \leqslant$ $n$, the $i^{\prime}$-th $\lambda$-th order total differentiation operator $D_{i^{\prime}}^{\lambda}$ was defined in (1.22) above.

Problem 1.32. Applying these inductive formulas, find totally explicit complete formulas for the $\kappa$-th prolongation $\mathscr{L}^{(\kappa)}$.

The present article is devoted to provide all the desired formulas.
1.33. Methodology of induction. We have the intention of presenting our results in a purely inductive style, based on several thorough visual comparisons between massive formulas which will be written and commented in four different cases:
(i) $n=1$ and $m=1$; $\kappa \geqslant 1$ arbitrary;
(ii) $n \geqslant 1$ and $m=1 ; \kappa \geqslant 1$ arbitrary;
(iii) $n=1$ and $m \geqslant 1 ; \kappa \geqslant 1$ arbitrary;
(iv) general case: $n \geqslant 1$ and $m \geqslant 1$; $\kappa \geqslant 1$ arbitrary.

Accordingly, we shall particularize and slightly lighten our notations in each of the three (preliminary) cases (i) [Section 2], (ii) [Section 3] and (iii) [Section 4].

## §2. OnE INDEPENDENT VARIABLE AND ONE DEPENDENT VARIABLE

2.1. Simplified adapted notations. Assume $n=1$ and $m=1$, let $\kappa \in \mathbb{N}$ with $\kappa \geqslant 1$ and simply denote the jet variables by:

$$
\begin{equation*}
\left(x, y, y_{1}, y_{2}, \ldots, y_{\kappa}\right) \in \mathscr{J}_{1,1}^{\kappa} . \tag{2.2}
\end{equation*}
$$

The $\kappa$-th prolongation of a vector field $\mathscr{L}=\mathscr{X} \frac{\partial}{\partial x}+\mathscr{Y} \frac{\partial}{\partial y}$ will be denoted by:

$$
\begin{equation*}
\mathscr{L}^{(\kappa)}=\mathscr{X} \frac{\partial}{\partial x}+\mathscr{Y} \frac{\partial}{\partial y}+\mathbf{Y}_{1} \frac{\partial}{\partial y_{1}}+\mathbf{Y}_{2} \frac{\partial}{\partial y_{2}}+\cdots+\mathbf{Y}_{\kappa} \frac{\partial}{\partial y_{\kappa}} . \tag{2.3}
\end{equation*}
$$

The coefficients $\mathbf{Y}_{1}, \mathbf{Y}_{2}, \ldots, \mathbf{Y}_{\kappa}$ are computed by means of the inductive formulas:

$$
\left\{\begin{array}{l}
\mathbf{Y}_{1}:=D^{1}(\mathscr{Y})-D^{1}(\mathscr{X}) y_{1}  \tag{2.4}\\
\mathbf{Y}_{2}:=D^{2}\left(\mathbf{Y}_{1}\right)-D^{1}(\mathscr{X}) y_{2} \\
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
\mathbf{Y}_{\kappa}:=D^{\kappa}\left(\mathbf{Y}_{\kappa-1}\right)-D^{1}(\mathscr{X}) y_{\kappa}
\end{array}\right.
$$

where, for $1 \leqslant \lambda \leqslant \kappa$ :

$$
\begin{equation*}
D^{\lambda}:=\frac{\partial}{\partial x}+y_{1} \frac{\partial}{\partial y}+y_{2} \frac{\partial}{\partial y_{1}}+\cdots+y_{\lambda} \frac{\partial}{\partial y_{\lambda-1}} \tag{2.5}
\end{equation*}
$$

By direct elementary computations, for $\kappa=1$ and for $\kappa=2$, we obtain the following two very classical formulas :

$$
\left\{\begin{align*}
\mathbf{Y}_{1}= & \mathscr{Y}_{x}+\left[\mathscr{Y}_{y}-\mathscr{X}_{x}\right] y_{1}+\left[-\mathscr{X}_{y}\right]\left(y_{1}\right)^{2}  \tag{2.6}\\
\mathbf{Y}_{2}= & \mathscr{Y}_{x^{2}}+\left[2 \mathscr{Y}_{x y}-\mathscr{X}_{x^{2}}\right] y_{1}+\left[\mathscr{Y}_{y^{2}}-2 \mathscr{X}_{x y}\right]\left(y_{1}\right)^{2}+\left[-\mathscr{X}_{y^{2}}\right]\left(y_{1}\right)^{3}+ \\
& +\left[\mathscr{Y}_{y}-2 \mathscr{X}_{x}\right] y_{2}+\left[-3 \mathscr{X}_{y}\right] y_{1} y_{2} .
\end{align*}\right.
$$

Our main objective is to devise the general combinatorics. Thus, to attain this aim, we have to achieve patiently formal computations of the next coefficients $\mathbf{Y}_{3}, \mathbf{Y}_{4}$ and $\mathbf{Y}_{5}$. We systematically use parentheses $[\cdot]$ to single out every coefficient of the polynomials $\mathbf{Y}_{3}, \mathbf{Y}_{4}$ and $\mathbf{Y}_{5}$ in the pure jet variables $y_{1}, y_{2}, y_{3}, y_{4}$ and $y_{5}$, putting every sign inside these parentheses. We always put the monomials in the pure jet variables $y_{1}, y_{2}, y_{3}, y_{4}$ and $y_{5}$ after the parentheses. For completeness, let us provide the intermediate computation of the third coefficient $\mathbf{Y}_{3}$. In detail:

$$
\begin{aligned}
\mathbf{Y}_{3}= & D^{3}\left(\mathbf{Y}_{2}\right)-D^{1}(\mathscr{X}) y_{3} \\
= & \left(\frac{\partial}{\partial x}+y_{1} \frac{\partial}{\partial y}+y_{2} \frac{\partial}{\partial y_{1}}+y_{3} \frac{\partial}{\partial y_{2}}\right)\left(\mathscr{Y}_{x^{2}}+\left[2 \mathscr{Y}_{x y}-\mathscr{X}_{x^{2}}\right] y_{1}+\right. \\
& +\left[\mathscr{Y}_{y^{2}}-2 \mathscr{X}_{x y}\right]\left(y_{1}\right)^{2}+\left[-\mathscr{X}_{y^{2}}\right]\left(y_{1}\right)^{3}+ \\
+ & {\left.\left[\mathscr{Y}_{y}-2 \mathscr{X}_{x}\right] y_{2}+\left[-3 \mathscr{X}_{y}\right] y_{1} y_{2}\right) }
\end{aligned}
$$

$$
\begin{align*}
& \left.=\underline{\mathscr{Y}_{x^{3}}} \underline{1}+\underline{\left[2 \mathscr{Y}_{x^{2} y}-\mathscr{X}_{x^{3}}\right] y_{1}} \underline{\square}+\underline{\left[\mathscr{Y}_{x y^{2}}-2 \mathscr{X}_{x^{2} y}\right]\left(y_{1}\right)^{2}} \underline{3}+\underline{\left[-\mathscr{X}_{x y^{2}}\right]\left(y_{1}\right)^{3}}\right]^{+}  \tag{2.7}\\
& \left.\left.+\underline{\left[\mathscr{Y}_{x y}-2 \mathscr{X}_{x^{2}}\right] y_{2}} \underline{6}^{+\underline{\left[-3 \mathscr{X}_{x y}\right] y_{1} y_{2}}}\right]_{7}+\underline{\left[\mathscr{Y}_{x^{2} y}\right] y_{1}}\right]^{+} \\
& +\underline{\left[2 \mathscr{Y}_{x y^{2}}-\mathscr{X}_{x^{2} y}\right]\left(y_{1}\right)^{2}} \underline{3}+\underline{\left[\mathscr{Y}_{y^{3}}-2 \mathscr{X}_{x y^{2}}\right]\left(y_{1}\right)^{3}} \underline{4}+\underline{\left[-\mathscr{X}_{y^{3}}\right]\left(y_{1}\right)^{4}} \underline{5}^{+}
\end{align*}
$$

$$
\begin{aligned}
& \left.\left.\left.+\underline{\left[\mathscr{Y}_{y^{2}}-2 \mathscr{X}_{x y}\right] y_{1} y_{2}} \underline{7}+\underline{\left[-3 \mathscr{X}_{y^{2}}\right]\left(y_{1}\right)^{2} y_{2}}\right]+\underline{8}+\mathscr{\mathscr { Y }}_{x y}-\mathscr{X}_{x^{2}}\right] y_{2}\right]^{+} \\
& +\underline{\left[\mathscr{Y}_{y^{2}}-2 \mathscr{X}_{x y}\right] 2 y_{1} y_{2}} \underline{7}+\underline{\left[-\mathscr{X}_{y^{2}}\right] 3\left(y_{1}\right)^{2} y_{2}} \underline{8}+\underline{\left[-3 \mathscr{X}_{y}\right]\left(y_{2}\right)^{2}} \underline{9}+ \\
& +\underline{\left[\mathscr{Y}_{y}-2 \mathscr{X}_{x}\right] y_{3}}+\underline{\left[-3 \mathscr{X}_{y}\right] y_{1} y_{3}} \underline{11}^{-} \\
& -\underline{\left[\mathscr{X}_{x}\right] y_{3}}{ }_{10}-\underline{\left[\mathscr{X}_{y}\right] y_{1} y_{3}}{ }_{11} .
\end{aligned}
$$

We have underlined all the terms with a number appended. Each number refers to the order of appearance of the terms in the final simplified expression of $\mathbf{Y}_{3}$, also written in [BK1989] with different notations:

$$
\left\{\begin{align*}
\mathbf{Y}_{3}= & \mathscr{Y}_{x^{3}}+\left[3 \mathscr{Y}_{x^{2} y}-\mathscr{X}_{x^{3}}\right] y_{1}+\left[3 \mathscr{Y}_{x y^{2}}-3 \mathscr{X}_{x^{2} y}\right]\left(y_{1}\right)^{2}+  \tag{2.8}\\
& +\left[\mathscr{Y}_{y^{3}}-3 \mathscr{X}_{x y^{2}}\right]\left(y_{1}\right)^{3}+\left[-\mathscr{X}_{y^{3}}\right]\left(y_{1}\right)^{4}+\left[3 \mathscr{Y}_{x y}-3 \mathscr{X}_{x^{2}}\right] y_{2}+ \\
& +\left[3 \mathscr{Y}_{y^{2}}-9 \mathscr{X}_{x y}\right] y_{1} y_{2}+\left[-6 \mathscr{X}_{y^{2}}\right]\left(y_{1}\right)^{2} y_{2}+\left[-3 \mathscr{X}_{y}\right]\left(y_{2}\right)^{2}+ \\
& +\left[\mathscr{Y}_{y}-3 \mathscr{X}_{x}\right] y_{3}+\left[-4 \mathscr{X}_{y}\right] y_{1} y_{3} .
\end{align*}\right.
$$

After similar manual computations, the intermediate details of which we will not copy in this Latex file, we get the desired expressions of $\mathbf{Y}_{4}$ and of $Y_{5}$. Firstly:

$$
\left\{\begin{align*}
\mathbf{Y}_{4}= & \mathscr{Y}_{x^{4}}+\left[4 \mathscr{Y}_{x^{3} y}-\mathscr{X}_{x^{4}}\right] y_{1}+\left[6 \mathscr{Y}_{x^{2} y^{2}}-4 \mathscr{X}_{x^{3} y}\right]\left(y_{1}\right)^{2}+  \tag{2.9}\\
& +\left[4 \mathscr{Y}_{x y^{3}}-6 \mathscr{X}_{x^{2} y^{2}}\right]\left(y_{1}\right)^{3}+\left[\mathscr{Y}_{y^{4}}-4 \mathscr{X}_{x y^{3}}\right]\left(y_{1}\right)^{4}+\left[-\mathscr{X}_{y^{4}}\right]\left(y_{1}\right)^{5}+ \\
& +\left[6 \mathscr{Y}_{x^{2} y}-4 \mathscr{X}_{x^{3}}\right] y_{2}+\left[12 \mathscr{\mathscr { X }}_{x y^{2}}-18 \mathscr{X}_{x^{2} y}\right] y_{1} y_{2}+ \\
& +\left[6 \mathscr{Y}_{y^{3}}-24 \mathscr{X}_{x y^{2}}\right]\left(y_{1}\right)^{2} y_{2}+\left[-10 \mathscr{X}_{y^{3}}\right]\left(y_{1}\right)^{3} y_{2}+ \\
& +\left[3 \mathscr{Y}_{y^{2}}-12 \mathscr{X}_{x y}\right]\left(y_{2}\right)^{2}+\left[-15 \mathscr{X}_{y^{2}}\right] y_{1}\left(y_{2}\right)^{2}+ \\
& +\left[4 \mathscr{Y}_{x y}-6 \mathscr{X}_{x^{2}}\right] y_{3}+\left[4 \mathscr{Y}_{y^{2}}-16 \mathscr{X}_{x y}\right] y_{1} y_{3}+\left[-10 \mathscr{X}_{y^{2}}\right]\left(y_{1}\right)^{2} y_{3}+ \\
& +\left[-10 \mathscr{X}_{y}\right] y_{2} y_{3}+\left[\mathscr{Y}_{y}-4 \mathscr{X}_{x}\right] y_{4}+\left[-5 \mathscr{X}_{y}\right] y_{1} y_{4} .
\end{align*}\right.
$$

Secondly:

$$
\begin{aligned}
\left(\mathbf{Y}_{5}=\right. & \mathscr{Y}_{x^{5}}+\left[5 \mathscr{Y}_{x^{4} y}-\mathscr{X}_{x^{5}}\right] y_{1}+\left[10 \mathscr{Y}_{x^{3} y^{2}}-5 \mathscr{X}_{x^{4} y}\right]\left(y_{1}\right)^{2}+ \\
& +\left[10 \mathscr{Y}_{x^{2} y^{3}}-10 \mathscr{X}_{x^{3} y^{2}}\right]\left(y_{1}\right)^{3}+\left[5 \mathscr{Y}_{x y^{4}}-10 \mathscr{X}_{x^{2} y^{3}}\right]\left(y_{1}\right)^{4}+ \\
& +\left[\mathscr{y}_{y^{5}}-5 \mathscr{X}_{x y^{4}}\right]\left(y_{1}\right)^{5}+\left[-\mathscr{X}_{y^{5}}\right]\left(y_{1}\right)^{6}+\left[10 \mathscr{Y}_{x^{3} y}-5 \mathscr{X}_{x^{4}}\right] y_{2}+ \\
& +\left[30 \mathscr{Y}_{x^{2} y^{2}}-30 \mathscr{X}_{\left.x^{3}{ }^{3}\right]}\right] y_{1} y_{2}+\left[30 \mathscr{Y}_{x y^{3}}-60 \mathscr{X}_{x^{2} y^{2}}\right]\left(y_{1}\right)^{2} y_{2}+ \\
& +\left[10 \mathscr{Y}_{y^{4}}-50 \mathscr{X}_{x y^{3}}\right]\left(y_{1}\right)^{3} y_{2}+\left[-15 \mathscr{X}_{y^{4}}\right]\left(y_{1}\right)^{4} y_{2}+ \\
& +\left[15 \mathscr{Y}_{x y^{2}}-30 \mathscr{X}_{x^{2} y}\right]\left(y_{2}\right)^{2}+\left[15 \mathscr{Y}_{y^{3}}-75 \mathscr{X}_{x y^{2}}\right] y_{1}\left(y_{2}\right)^{2}+ \\
& +\left[-45 \mathscr{X}_{y^{3}}\right]\left(y_{1}\right)^{2}\left(y_{2}\right)^{2}+\left[-15 \mathscr{X}_{y^{2}}\right]\left(y_{2}\right)^{3}+ \\
& +\left[10 \mathscr{Y}_{x^{2} y}-10 \mathscr{X}_{x^{3}}\right] y_{3}+\left[20 \mathscr{Y}_{x y^{2}}-40 \mathscr{X}_{x^{2} y}\right] y_{1} y_{3}+ \\
& +\left[10 \mathscr{Y}_{y^{3}}-50 \mathscr{X}_{x y^{2}}\right]\left(y_{1}\right)^{2} y_{3}+\left[-20 \mathscr{X}_{y^{3}}\right]\left(y_{1}\right)^{3} y_{3}+ \\
& +\left[10 \mathscr{Y}_{y^{2}}-50 \mathscr{X}_{x y}\right] y_{2} y_{3}+\left[-60 \mathscr{X}_{y^{2}}\right] y_{1} y_{2} y_{3}+\left[-10 \mathscr{X}_{y}\right]\left(y_{3}\right)^{2}+ \\
& +\left[5 \mathscr{Y}_{x y}-10 \mathscr{X}_{x^{2}}\right] y_{4}+\left[5 \mathscr{Y}_{y^{2}}-25 \mathscr{X}_{x y}\right] y_{1} y_{4}+\left[-15 \mathscr{X}_{y^{2}}\right]\left(y_{1}\right)^{2} y_{4}+ \\
& +\left[-15 \mathscr{X}_{y}\right] y_{2} y_{4}+\left[\mathscr{Y}_{y}-5 \mathscr{X}_{y}\right] y_{5}+\left[-6 \mathscr{X}_{y}\right] y_{1} y_{5} .
\end{aligned}
$$

2.11. Formal inspection, formal intuition and formal induction. Now, we have to comment these formulas. We have written in length the five polynomials $\mathbf{Y}_{1}, \mathbf{Y}_{2}, \mathbf{Y}_{3}, \mathbf{Y}_{4}$ and $\mathbf{Y}_{5}$ in the pure jet variables $y_{1}, y_{2}, y_{3}, y_{4}$ and $y_{5}$. Except the first "constant" term $\mathscr{Y}_{x^{\kappa}}$, all the monomials in the expression of $\mathbf{Y}_{\kappa}$ are of the general form

$$
\begin{equation*}
\left(y_{\lambda_{1}}\right)^{\mu_{1}}\left(y_{\lambda_{2}}\right)^{\mu_{2}} \cdots\left(y_{\lambda_{d}}\right)^{\mu_{d}}, \tag{2.12}
\end{equation*}
$$

for some positive integer $d \geqslant 1$, for some collection of strictly increasing jet indices:

$$
\begin{equation*}
1 \leqslant \lambda_{1}<\lambda_{2}<\cdots<\lambda_{d} \leqslant \kappa \tag{2.13}
\end{equation*}
$$

and for some positive integers $\mu_{1}, \ldots, \mu_{d} \geqslant 1$. This and the next combinatorial facts may be confirmed by reading the formulas giving $\mathbf{Y}_{1}, \mathbf{Y}_{2}, \mathbf{Y}_{3}$, $\mathbf{Y}_{4}$ and $\mathbf{Y}_{5}$. It follows that the integer $d$ satisfies the inequality $d \leqslant \kappa+1$. To include the first "constant" term $\mathscr{Y}_{x^{\kappa}}$, we shall make the convention that putting $d=0$ in the monomial (2.12) yields the constant term 1 .

Furthermore, by inspecting the formulas giving $\mathbf{Y}_{1}, \mathbf{Y}_{2}, \mathbf{Y}_{3}, \mathbf{Y}_{4}$ and $\mathbf{Y}_{5}$, we see that the following inequality should be satisfied:

$$
\begin{equation*}
\mu_{1} \lambda_{1}+\mu_{2} \lambda_{2}+\cdots+\mu_{d} \lambda_{d} \leqslant \kappa+1 . \tag{2.14}
\end{equation*}
$$

For instance, in the expression of $\mathbf{Y}_{4}$, the two monomials $\left(y_{1}\right)^{3} y_{2}$ and $y_{1}\left(y_{2}\right)^{2}$ do appear, but the two monomials $\left(y_{1}\right)^{4} y_{2}$ and $\left(y_{1}\right)^{2}\left(y_{2}\right)^{2}$ cannot appear. All coefficients of the pure jet monomials are of the general form:

$$
\begin{equation*}
\left[A \mathscr{Y}_{x^{\alpha} y^{\beta+1}}-B \mathscr{X}_{x^{\alpha+1} y^{\beta}}\right], \tag{2.15}
\end{equation*}
$$

for some nonnegative integers $A, B, \alpha, \beta \in \mathbb{N}$. Sometimes $A$ is zero, but $B$ is zero only for the (constant, with respect to pure jet variables) term $\mathscr{Y}_{x^{\kappa}}$. Importantly, $\mathscr{X}$ is differentiated once more with respect to $x$ and $\mathscr{Y}$ is differentiated once more with respect to $y$. Again, this may be confirmed by reading all the terms in the formulas for $\mathbf{Y}_{1}, \mathbf{Y}_{2}, \mathbf{Y}_{3}, \mathbf{Y}_{4}$ and $\mathbf{Y}_{5}$.

In addition, we claim that there is a link between the couple $(\alpha, \beta)$ and the collection $\left\{\mu_{1}, \lambda_{1}, \ldots, \mu_{d}, \lambda_{d}\right\}$. To discover it, let us write some of the monomials appearing in the expressions of $\mathbf{Y}_{4}$ (first column) and of $\mathbf{Y}_{5}$ (second column), for instance:

$$
\left\{\begin{array}{lr}
{\left[6 \mathscr{Y}_{x^{2} y^{2}}-4 \mathscr{X}_{x^{3} y}\right]\left(y_{1}\right)^{2},} & {\left[5 \mathscr{Y}_{x y^{4}}-10 \mathscr{X}_{x^{2} y^{3}}\right]\left(y_{1}\right)^{4},}  \tag{2.16}\\
{\left[12 \mathscr{Y}_{x y^{2}}-18 \mathscr{X}_{x^{2} y}\right] y_{1} y_{2},} & {\left[30 \mathscr{Y}_{x y^{3}}-60 \mathscr{X}_{x^{2} y^{2}}\right]\left(y_{1}\right)^{2} y_{2},} \\
{\left[-10 \mathscr{X}_{y^{3}}\right]\left(y_{1}\right)^{3} y_{2},} & {\left[-15 \mathscr{X}_{y^{4}}\right]\left(y_{1}\right)^{4} y_{2},} \\
{\left[4 \mathscr{Y}_{y^{2}}-16 \mathscr{X}_{x y}\right] y_{1} y_{3},} & {\left[10 \mathscr{Y}_{y^{2}}-50 \mathscr{X}_{x y}\right] y_{2} y_{3},} \\
{\left[-10 \mathscr{X}_{y^{2}}\right]\left(y_{1}\right)^{2} y_{3},} & {\left[-60 \mathscr{X}_{y^{2}}\right] y_{1} y_{2} y_{3} .}
\end{array}\right.
$$

After some reflection, we discover the hidden intuitive rule: the partial derivatives of $\mathscr{Y}$ and of $\mathscr{X}$ associated with the monomial $\left(y_{\lambda_{1}}\right)^{\mu_{1}} \cdots\left(y_{\lambda_{d}}\right)^{\mu_{d}}$ are, respectively:

$$
\left\{\begin{array}{l}
\mathscr{Y}_{x^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}} y^{\mu_{1}+\cdots+\mu_{d}}}  \tag{2.17}\\
\mathscr{X}_{x^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+1} y^{\mu_{1}+\cdots+\mu_{d}-1}} .
\end{array}\right.
$$

This may be checked on each of the 10 examples (2.16) above.
Now that we have explored and discovered the combinatorics of the pure jet monomials, of the partial derivatives and of the complete sum giving $\mathbf{Y}_{\kappa}$, we may express that it is of the following general form:

$$
\left\{\begin{array}{c}
\mathbf{Y}_{\kappa}=\mathscr{Y}_{x^{\kappa}}+\sum_{d=1}^{\kappa+1} \sum_{1 \leqslant \lambda_{1}<\cdots<\lambda_{d} \leqslant \kappa} \sum_{\mu_{1} \geqslant 1, \ldots, \mu_{d} \geqslant 1} \sum_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d} \leqslant \kappa+1}  \tag{2.18}\\
{\left[A_{\kappa}^{\left(\mu_{1}, \lambda_{1}\right), \ldots,\left(\mu_{d}, \lambda_{d}\right) \cdot \mathscr{Y}_{x^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}} y^{\mu_{1}+\cdots+\mu_{d}}-}} \begin{array}{r}
-B_{\kappa}^{\left(\mu_{1}, \lambda_{1}\right), \ldots,\left(\mu_{d}, \lambda_{d}\right)} \cdot \mathscr{X}_{\left.x^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+1} y^{\mu_{1}+\cdots+\mu_{d}-1}\right]} \\
\cdot\left(y_{\lambda_{1}}\right)^{\mu_{1}} \cdots\left(y_{\lambda_{d}}\right)^{\mu_{d}}
\end{array} .\right.}
\end{array}\right.
$$

Here, we separate the first term $\mathscr{Y}_{x^{\kappa}}$ from the general sum; it is the constant term in $\mathbf{Y}_{\kappa}$, which itself is a polynomial with respect to the jet variables $y_{\lambda}$. In this general formula, the only remaining unknowns are the nonnegative integer coefficients $A_{\kappa}^{\left(\mu_{1}, \lambda_{1}\right), \ldots,\left(\mu_{d}, \lambda_{d}\right)} \in \mathbb{N}$ and $B_{\kappa}^{\left(\mu_{1}, \lambda_{1}\right), \ldots,\left(\mu_{d}, \lambda_{d}\right)} \in \mathbb{N}$. In Section 3 below, we shall explain how we have discovered their exact value.

At present, even if we are unable to devise their explicit expression, we may observe that the value of the special integer coefficients $A_{\mu_{1}}^{\left(\mu_{1}, 1\right)}$ and $B_{\mu_{1}}^{\left(\mu_{1}, 1\right)}$ which are attached to the monomials ct., $y_{1},\left(y_{1}\right)^{2},\left(y_{1}\right)^{3},\left(y_{1}\right)^{4}$ and
$\left(y_{1}\right)^{5}$ are simple. Indeed, by inspecting the first terms in the expressions of $\mathbf{Y}_{1}, \mathbf{Y}_{2}, \mathbf{Y}_{3}, \mathbf{Y}_{4}$ and $\mathbf{Y}_{5}$, we of course recognize the binomial coefficients. In general:
Lemma 2.19. For $\kappa \geqslant 1$,

$$
\left\{\begin{align*}
\mathbf{Y}_{\kappa}= & \mathscr{Y}_{x^{\kappa}}+\sum_{\lambda=1}^{\kappa}\left[\binom{\kappa}{\lambda} \mathscr{Y}_{x^{\kappa-\lambda} y^{\lambda}}-\binom{\kappa}{\lambda-1} \mathscr{X}_{x^{\kappa-\lambda+1} y^{\lambda-1}}\right]\left(y_{1}\right)^{\lambda}+  \tag{2.20}\\
& +\left[-\mathscr{X}_{y^{\kappa}}\right]\left(y_{1}\right)^{\kappa}+\text { remainder },
\end{align*}\right.
$$

where the term remainder collects all remaining monomials in the pure jet variables.

In addition, let us remind what we have observed and used in a previous co-signed work.
Lemma 2.21. ([GM2003a], p. 536) For $\kappa \geqslant 4$, nine among the monomials of $\mathbf{Y}_{\kappa}$ are of the following general form:

$$
\left\{\begin{align*}
\mathbf{Y}_{\kappa}= & \mathscr{Y}_{x^{\kappa}}+\left[C_{\kappa}^{1} \mathscr{Y}_{x^{\kappa-1} y}-\mathscr{X}_{x^{\kappa}}\right] y_{1}+\left[C_{\kappa}^{2} \mathscr{Y}_{x^{\kappa-2} y}-C_{\kappa}^{1} \mathscr{X}_{x^{\kappa-1}}\right] y_{2}+  \tag{2.22}\\
& +\left[C_{\kappa}^{2} \mathscr{Y}_{x^{2} y}-C_{\kappa}^{3} \mathscr{X}_{x^{3}}\right] y_{\kappa-2}+\left[C_{\kappa}^{1} \mathscr{Y}_{x y}-C_{\kappa}^{2} \mathscr{X}_{x^{2}}\right] y_{\kappa-1}+ \\
& +\left[C_{\kappa}^{1} \mathscr{Y}_{y^{2}}-\kappa^{2} \mathscr{X}_{x y}\right] y_{1} y_{\kappa-1}+\left[-C_{\kappa}^{2} \mathscr{X}_{y}\right] y_{2} y_{\kappa-1}+ \\
& +\left[\mathscr{Y}_{y}-C_{\kappa}^{1} \mathscr{X}_{x}\right]+\left[-C_{\kappa+1}^{1} \mathscr{X}_{y}\right] y_{1} y_{\kappa}+\text { remainder },
\end{align*}\right.
$$

where the term remainder denotes all the remaining monomials, and where $C_{\kappa}^{\lambda}:=\frac{\kappa!}{(\kappa-\lambda)!\lambda!}$ is a notation for the binomial coefficient which occupies less space in Latex "equation mode" than the classical notation

$$
\begin{equation*}
\binom{\kappa}{\lambda} . \tag{2.23}
\end{equation*}
$$

Now, we state directly the final theorem, without further inductive or intuitive information.
Theorem 2.24. For $\kappa \geqslant 1$, we have:
(2.25)

$$
\begin{array}{|l}
\mathbf{Y}_{\kappa}=\mathscr{Y}_{x^{\kappa}}+\sum_{d=1}^{\kappa+1} \sum_{1 \leqslant \lambda_{1}<\cdots<\lambda_{d} \leqslant \kappa} \sum_{\mu_{1} \geqslant 1, \ldots, \mu_{d} \geqslant 1} \sum_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d} \leqslant \kappa+1} \\
{\left[\frac{\kappa \cdots\left(\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+1\right)}{\left(\lambda_{1}!\right)^{\mu_{1}} \mu_{1}!\cdots\left(\lambda_{d}!\right)^{\mu_{d}} \mu_{d}!} \cdot \mathscr{Y}_{x^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}} y^{\mu_{1}+\cdots+\mu_{d}}-}-\frac{\kappa \cdots\left(\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+2\right)\left(\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}\right)}{\left(\lambda_{1}!\right)^{\mu_{1}} \mu_{1}!\cdots\left(\lambda_{d}!\right)^{\mu_{d}} \mu_{d}!} .\right.} \\
\cdot \mathscr{X}_{\left.x^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+1} y^{\mu_{1}+\cdots+\mu_{d}-1}\right]\left(y_{\lambda_{1}}\right)^{\mu_{1}} \cdots\left(y_{\lambda_{d}}\right)^{\mu_{d}} .}
\end{array}
$$

Once the correct theorem is formulated, its proof follows by accessible induction arguments which will not be developed here. It is better to continue through and to examine thorougly the case of several variables, since it will help us considerably to explain how we discovered the exact values of the integer coefficients $A_{\kappa}^{\left(\mu_{1}, \lambda_{1}\right), \ldots,\left(\mu_{d}, \lambda_{d}\right)}$ and $B_{\kappa}^{\left(\mu_{1}, \lambda_{1}\right), \ldots,\left(\mu_{d}, \lambda_{d}\right)}$.
2.26. Verification and application. Before proceeding further, let us rapidly verify that the above general formula (2.25) is correct by inspecting two instances extracted from $\mathbf{Y}_{5}$.

Firstly, the coefficient of $\left(y_{1}\right)^{3} y_{3}$ in $\mathbf{Y}_{5}$ is obtained by putting $\kappa=5$, $d=2, \lambda_{1}=1, \mu_{1}=3, \lambda_{2}=3$ and $\mu_{2}=1$ in the general formula (2.25), which yields:

$$
\begin{equation*}
\left[0-\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 6}{(1!)^{3} 3!(3!)^{1} 1!} \mathscr{X}_{y^{3}}\right]=\left[-20 \mathscr{X}_{y^{3}}\right] . \tag{2.27}
\end{equation*}
$$

This value is the same as in the original formula (2.10): confirmation.
Secondly, the coefficient of $y_{1}\left(y_{2}\right)^{2}$ in $\mathbf{Y}_{5}$ is obtained by $\kappa=5, d=2$, $\lambda_{1}=1, \mu_{1}=1, \lambda_{2}=2$ and $\mu_{2}=2$ in the general formula (2.25), which yields:
(2.28)

$$
\left[\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(1!)^{1} 1!(2!)^{2} 2!} \mathscr{Y}_{y^{3}}-\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 5}{(1!)^{1} 1!(2!)^{2} 2!} \mathscr{X}_{x y^{2}}\right]=\left[15 \mathscr{Y}_{y^{3}}-75 \mathscr{X}_{x y^{2}}\right] .
$$

This value is the same as in the original formula (2.10); again: confirmation.
Finally, applying our general formula (2.25), we deduce the value of $\mathbf{Y}_{6}$ without having to use $\mathbf{Y}_{5}$ and the induction formulas (2.4), which shortens substantially the computations.

For the pleasure, we obtain:
(2.29)

$$
\begin{aligned}
\mathbf{Y}_{6}= & \mathscr{Y}_{x^{6}}+\left[6 \mathscr{Y}_{x^{5} y}-\mathscr{X}_{x^{6}}\right] y_{1}+\left[15 \mathscr{Y}_{x^{4} y^{2}}-6 \mathscr{X}_{x^{5} y}\right]\left(y_{1}\right)^{2}+ \\
& +\left[20 \mathscr{Y}_{x^{3} y^{3}}-15 \mathscr{X}_{x^{4} y^{2}}\right]\left(y_{1}\right)^{3}+\left[15 \mathscr{Y}_{x^{2} y^{4}}-20 \mathscr{X}_{x^{3} y^{3}}\right]\left(y_{1}\right)^{4}+ \\
& +\left[6 \mathscr{Y}_{x y^{5}}-15 \mathscr{X}_{x^{2} y^{4}}\right]\left(y_{1}\right)^{5}+\left[\mathscr{Y}_{y^{6}}-6 \mathscr{X}_{x y^{5}}\right]\left(y_{1}\right)^{6}+\left[-\mathscr{X}_{y^{6}}\right]\left(y_{1}\right)^{7}+ \\
& +\left[15 \mathscr{Y}_{x^{4} y}-6 \mathscr{X}_{x^{5}}\right] y_{2}+\left[60 \mathscr{Y}_{x^{3} y^{2}}-45 \mathscr{X}_{x^{4} y}\right] y_{1} y_{2}+ \\
& +\left[90 \mathscr{Y}_{x^{2} y^{3}}-120 \mathscr{X}_{x^{3} y^{2}}\right]\left(y_{1}\right)^{2} y_{2}+\left[60 \mathscr{Y}_{x y^{4}}-150 \mathscr{X}_{x^{2} y^{3}}\right]\left(y_{1}\right)^{3} y_{2}+ \\
& +\left[15 \mathscr{Y}_{y^{5}}-90 \mathscr{X}_{x y^{4}}\right]\left(y_{1}\right)^{4} y_{2}+\left[-21 \mathscr{\mathscr { X }}_{y^{5}}\right]\left(y_{1}\right)^{5} y_{2}+ \\
& +\left[45 \mathscr{Y}_{x^{2} y^{2}}-60 \mathscr{X}_{x^{3} y}\right]\left(y_{2}\right)^{2}+\left[90 \mathscr{y}_{x y^{3}}-225 \mathscr{X}_{x^{2} y^{2}}\right] y_{1}\left(y_{2}\right)^{2}+ \\
& +\left[45 \mathscr{Y}_{y^{4}}-270 \mathscr{X}_{x y^{3}}\right]\left(y_{1}\right)^{2}\left(y_{2}\right)^{2}+\left[-210 \mathscr{X}_{y^{4}}\right]\left(y_{1}\right)^{3}\left(y_{2}\right)^{2}+ \\
& +\left[15 \mathscr{Y}_{y^{3}}-90 \mathscr{X}_{x y^{2}}\right]\left(y_{2}\right)^{3}+\left[-105 \mathscr{X}_{y^{3}}\right] y_{1}\left(y_{2}\right)^{3}+ \\
& +\left[20 \mathscr{Y}_{x^{3} y}-15 \mathscr{X}_{x^{4}}\right] y_{3}+\left[60 \mathscr{Y}_{x^{2} y^{2}}-80 \mathscr{X}_{x^{3} y}\right] y_{1} y_{3}+ \\
& +\left[60 \mathscr{Y}_{x y^{3}}-150 \mathscr{X}_{x^{2} y^{2}}\right]\left(y_{1}\right)^{2} y_{3}+\left[20 \mathscr{Y}_{y^{4}}-120 \mathscr{X}_{x y^{3}}\right]\left(y_{1}\right)^{3} y_{3}+ \\
& +\left[-35 \mathscr{X}_{y^{4}}\right]\left(y_{1}\right)^{4} y_{3}+\left[60 \mathscr{Y}_{x y^{2}}-150 \mathscr{X}_{x^{2} y}\right] y_{2} y_{3}+ \\
& +\left[60 \mathscr{Y}_{y^{3}}-360 \mathscr{X}_{x y^{2}}\right] y_{1} y_{2} y_{3}+\left[-210 \mathscr{X}_{y^{3}}\right]\left(y_{1}\right)^{2} y_{2} y_{3}+ \\
& +\left[-105 \mathscr{X}_{y^{2}}\right]\left(y_{2}\right)^{2} y_{3}+\left[10 \mathscr{Y}_{y^{2}}-60 \mathscr{X}_{x y}\right]\left(y_{3}\right)^{2}+ \\
& +\left[-70 \mathscr{X}_{y^{2}}\right] y_{1}\left(y_{3}\right)^{2}+\left[15 \mathscr{Y}_{x^{2} y}-20{\mathscr{X} x^{3}}\right] y_{4}+ \\
& +\left[30 \mathscr{Y}_{x y^{2}}-75 \mathscr{X}_{x^{2} y}\right] y_{1} y_{4}+\left[15 \mathscr{Y}_{y^{3}}-90 \mathscr{X}_{x y^{2}}\right]\left(y_{1}\right)^{2} y_{4}+ \\
& +\left[-35 \mathscr{\mathscr { X }}_{y^{3}}\right]\left(y_{1}\right)^{3} y_{4}+\left[15 \mathscr{Y}_{y^{2}}-90 \mathscr{X}_{x y}\right] y_{2} y_{4}+ \\
& +\left[-105 \mathscr{\mathscr { X }}_{y^{2}}\right] y_{1} y_{2} y_{4}+\left[-35 \mathscr{X}_{y}\right] y_{3} y_{4}+\left[6 \mathscr{Y}_{x y}-15 \mathscr{X}_{x^{2}}\right] y_{5}+ \\
& +\left[6 \mathscr{Y}_{y^{2}}-36 \mathscr{X}_{x y}\right] y_{1} y_{5}+\left[-21 \mathscr{X}_{y^{2}}\right]\left(y_{1}\right)^{2} y_{5}+\left[-21 \mathscr{X}_{y}\right] y_{2} y_{5}+ \\
& +\left[\mathscr{Y}_{y}-6 \mathscr{X}_{y}\right] y_{6}+\left[-7 \mathscr{X}_{y}\right] y_{1} y_{6} .
\end{aligned}
$$

2.30. Deduction of the classical Faà di Bruno formula. Let $x, y \in \mathbb{K}$ and let $g=g(x), f=f(y)$ be two $\mathscr{C}^{\infty}$-smooth functions $\mathbb{K} \rightarrow \mathbb{K}$. Consider the composition $h:=f \circ g$, namely $h(x)=f(g(x))$. For $\lambda \in \mathbb{N}$ with $\lambda \geqslant 1$, simply denote by $g_{\lambda}$ the $\lambda$-th derivative $\frac{d^{\lambda} g}{d x^{\lambda}}$ and similarly for $h_{\lambda}$. Also, abbreviate $f_{\lambda}:=\frac{d^{\lambda} f}{d y^{\lambda}}$.

By the classical formula for the derivative of a composite function, we have $h_{1}=f_{1} g_{1}$. Further computations provide the following list of subsequent derivatives of $h$ :
(2.31)

$$
\left\{\begin{aligned}
h_{1}= & f_{1} g_{1}, \\
h_{2}= & f_{2}\left(g_{1}\right)^{2}+f_{1} g_{2}, \\
h_{3}= & f_{3}\left(g_{1}\right)^{3}+3 f_{2} g_{1} g_{2}+f_{1} g_{3} \\
h_{4}= & f_{4}\left(g_{1}\right)^{4}+6 f_{3}\left(g_{1}\right)^{2} g_{2}+3 f_{2}\left(g_{2}\right)^{2}+4 f_{2} g_{1} g_{3}+f_{1} g_{4}, \\
h_{5}= & f_{5}\left(g_{1}\right)^{5}+10 f_{4}\left(g_{1}\right)^{3} g_{2}+15 f_{3}\left(g_{1}\right)^{2} g_{3}+10 f_{3} g_{1}\left(g_{2}\right)^{2}+ \\
& +10 f_{2} g_{2} g_{3}+5 f_{2} g_{1} g_{4}+f_{1} g_{5}, \\
h_{6}= & f_{6}\left(g_{1}\right)^{6}+15 f_{5}\left(g_{1}\right)^{4} g_{2}+45 f_{4}\left(g_{1}\right)^{2}\left(g_{2}\right)^{2}+15 f_{3}\left(g_{2}\right)^{3}+ \\
& +20 f_{4}\left(g_{1}\right)^{3} g_{3}+60 f_{3} g_{1} g_{2} g_{3}+10 f_{2}\left(g_{3}\right)^{2}+15 f_{3}\left(g_{1}\right)^{2} g_{4}+ \\
& +15 f_{2} g_{2} g_{4}+6 f_{2} g_{1} g_{5}+f_{1} g_{6} .
\end{aligned}\right.
$$

Theorem 2.32. For every integer $\kappa \geqslant 1$, the $\kappa$-th derivative of the composite function $h=f \circ g$ may be expressed as an explicit polynomial in the partial derivatives of $f$ and of $g$ having integer coefficients:
(2.33)

$$
\begin{aligned}
\frac{d^{\kappa} h}{d x^{\kappa}}= & \sum_{d=1}^{\kappa} \sum_{1 \leqslant \lambda_{1}<\cdots<\lambda_{d} \leqslant \kappa} \sum_{\mu_{1} \geqslant 1, \ldots, \mu_{d} \geqslant 1} \sum_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}=\kappa} \\
& \frac{\kappa!}{\left(\lambda_{1}!\right)^{\mu_{1}}} \mu_{1}!\cdots\left(\lambda_{d}!\right)^{\mu_{d}} \mu_{d}!
\end{aligned} \frac{d^{\mu_{1}+\cdots+\mu_{d}} f}{d y^{\mu_{1}+\cdots+\mu_{d}}}\left(\frac{d^{\lambda_{1}} g}{d x^{\lambda_{1}}}\right)^{\mu_{1}} \cdots \cdots\left(\frac{d^{\lambda_{d}} g}{d x^{\lambda_{d}}}\right)^{\mu_{d}} . \mid .
$$

This is the classical Faà di Bruno formula. Interestingly, we observe that this formula is included as a subpart of the general formula for $\mathbf{Y}_{\kappa}$, after a suitable translation. Indeed, in the formulas for $\mathbf{Y}_{1}, \mathbf{Y}_{2}, \mathbf{Y}_{3}, \mathbf{Y}_{4}, \mathbf{Y}_{5}, \mathbf{Y}_{6}$ and in the general sum for $\mathbf{Y}_{\kappa}$, pick only the terms for which $\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}=$ $\kappa$ and drop $\mathscr{X}$, which yields:

$$
\begin{align*}
& \sum_{d=1}^{\kappa} \sum_{1 \leqslant \lambda_{1}<\cdots<\lambda_{d} \leqslant \kappa} \sum_{\mu_{1} \geqslant 1, \ldots, \mu_{d} \geqslant 1} \sum_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}=\kappa}  \tag{2.34}\\
& {\left[\frac{\kappa!}{\mu_{1}!\left(\lambda_{1}!\right)^{\mu_{1}} \cdots \mu_{d}!\left(\lambda_{d}!\right)^{\mu_{d}}} \mathscr{Y}_{\left.y^{\mu_{1}+\cdots+\mu_{d}}\right]}\left(y_{\lambda_{1}}\right)^{\mu_{1}} \cdots\left(y_{\lambda_{d}}\right)^{\mu_{d}} .\right.}
\end{align*}
$$

The similarity between the two formulas (2.33) and (2.34) is now clearly visible.

The Faà di Bruno formula may be established by means of substitutions of power series ([F1969], p. 222), by means of the umbral calculus ([CS1996]), or by means of some induction formulas, which we write for completeness.

Define the differential operators

$$
\begin{align*}
& F_{2}:=g_{2} \frac{\partial}{\partial g_{1}}+g_{1}\left(f_{2} \frac{\partial}{\partial f_{1}}\right),  \tag{2.35}\\
& F_{3}:=g_{2} \frac{\partial}{\partial g_{1}}+g_{3} \frac{\partial}{\partial g_{2}}+g_{1}\left(f_{2} \frac{\partial}{\partial f_{1}}+f_{3} \frac{\partial}{\partial f_{2}}\right),
\end{align*}
$$

$$
F_{\lambda}:=g_{2} \frac{\partial}{\partial g_{1}}+g_{3} \frac{\partial}{\partial g_{2}}+\cdots+g_{\lambda} \frac{\partial}{\partial g_{\lambda-1}}+g_{1}\left(f_{2} \frac{\partial}{\partial f_{1}}+f_{3} \frac{\partial}{\partial f_{2}}+\cdots+f_{\lambda} \frac{\partial}{\partial f_{\lambda-1}}\right) .
$$

Then we have

$$
\begin{align*}
h_{2} & =F^{2}\left(h_{1}\right), \\
h_{3} & =F^{3}\left(h_{2}\right),  \tag{2.36}\\
\ldots & \cdots \cdots \cdots \\
h_{\lambda} & =F^{\lambda}\left(h_{\lambda-1}\right) .
\end{align*}
$$

## §3. SEVERAL INDEPENDENT VARIABLES AND ONE DEPENDENT VARIABLE

3.1. Simplified adapted notations. As announced after the statement of Theorem 2.24, it is only after we have treated the case of several independent variables that we will understand perfectly the general formula (2.25), valid in the case of one independent variable and one dependent variable. We will discover massive formal computations, exciting our computational intuition.

Thus, assume $n \geqslant 1$ and $m=1$, let $\kappa \in \mathbb{N}$ with $\kappa \geqslant 1$ and simply denote (instead of (1.2)) the jet variables by:

$$
\begin{equation*}
\left(x^{i}, y, y_{i_{1}}, y_{i_{1}, i_{2}}, \ldots, y_{i_{1}, i_{2}, \ldots, i_{k}}\right) . \tag{3.2}
\end{equation*}
$$

Also, instead of (1.30), denote the $\kappa$-th prolongation of a vector field by:

$$
\left\{\begin{array}{c}
\mathscr{L}^{(\kappa)}=\sum_{i=1}^{n} \mathscr{X}^{i} \frac{\partial}{\partial x^{i}}+\mathscr{Y} \frac{\partial}{\partial y}+\sum_{i_{1}=1}^{n} \mathbf{Y}_{i_{1}} \frac{\partial}{\partial y_{i_{1}}}+\sum_{i_{1}, i_{2}=1}^{n} \mathbf{Y}_{i_{1}, i_{2}} \frac{\partial}{\partial y_{i_{1}, i_{2}}}+  \tag{3.3}\\
+\cdots+\sum_{i_{1}, i_{2}, \ldots, i_{\kappa}=1}^{n} \mathbf{Y}_{i_{1}, i_{2}, \ldots, i_{\kappa}} \frac{\partial}{\partial y_{i_{1}, i_{2}, \ldots, i_{\kappa}}}
\end{array}\right.
$$

The induction formulas are
where the total differentiation operators $D_{i^{\prime}}^{\lambda}$ are defined as in (1.22), dropping the sums $\sum_{j^{\prime}=1}^{m}$ and the indices $j^{\prime}$.
3.5. Two instructing explicit computations. To begin with, let us compute $\mathbf{Y}_{i_{1}}$. With $D_{i_{1}}^{1}=\frac{\partial}{\partial x^{i_{1}}}+y_{i_{1}} \frac{\partial}{\partial y}$, we have:

$$
\begin{align*}
\mathbf{Y}_{i_{1}} & =D_{i_{1}}(\mathscr{Y})-\sum_{k_{1}=1}^{n} D_{i_{1}}^{1}\left(\mathscr{X}^{k_{1}}\right) y_{k_{1}} \\
& =\mathscr{Y}_{x^{i_{1}}}+\mathscr{Y}_{y} y_{i_{1}}-\sum_{k_{1}=1}^{n} \mathscr{X}_{x^{i_{1}}}^{k_{1}} y_{k_{1}}-\sum_{k_{1}=1}^{n} \mathscr{X}_{y}^{k_{1}} y_{i_{1}} y_{k_{1}} . \tag{3.6}
\end{align*}
$$

Searching for formal harmony and for coherence with the formula $(2.6)_{1}$, we must include the term $\mathscr{Y}_{y} y_{i_{1}}$ inside the sum $\sum_{k_{1}=1}^{n}[\cdot] y_{k_{1}}$. Using the Kronecker symbol, we may write:

$$
\begin{equation*}
\mathscr{Y}_{y} y_{i_{1}} \equiv \sum_{k_{1}=1}^{n}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{y}\right] y_{k_{1}} \tag{3.7}
\end{equation*}
$$

Also, we may rewrite the last term of (3.6) with a double sum:

$$
\begin{equation*}
-\sum_{k_{1}=1}^{n} \mathscr{X}_{y}^{k_{1}} y_{i_{1}} y_{k_{1}} \equiv \sum_{k_{1}, k_{2}=1}^{n}\left[-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}} \tag{3.8}
\end{equation*}
$$

From now on and up to equation (3.39), we shall abbreviate any sum $\sum_{k=1}^{n}$ from 1 to $n$ as $\sum_{k}$. Putting everything together, we get the final desired perfect expression of $\mathbf{Y}_{i_{1}}$ :

$$
\begin{equation*}
\mathbf{Y}_{i_{1}}=\mathscr{Y}_{x^{i_{1}}}+\sum_{k_{1}}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{y}-\mathscr{X}_{x^{i_{1}}}^{k_{1}}\right] y_{k_{1}}+\sum_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}} \tag{3.9}
\end{equation*}
$$

This completes the first explicit computation.
The second one is about $\mathbf{Y}_{i_{1}, i_{2}}$. It becomes more delicate, because several algebraic transformations must be achieved until the final satisfying formula is obtained. Our goal is to present each step very carefully, explaining every
tiny detail. Without such a care, it would be impossible to claim that some of our subsequent computations, for which we will not provide the intermediate steps, may be redone and verified. Consequently, we will expose our rules of formal computation thoroughly.

Replacing the value of $\mathbf{Y}_{1}$ just obtained in the induction formula $(3.4)_{2}$ and developing, we may conduct the very first steps of the computation:

$$
\begin{aligned}
& \mathbf{Y}_{i_{1}, i_{2}}= D_{i_{2}}^{2}\left(\mathbf{Y}_{i_{1}}\right)- \\
&=\left(\frac{\partial}{\partial x^{i_{2}}}+y_{i_{2}} \frac{\partial}{\partial y}+\sum_{i_{2}}^{1}\left(\mathscr{X}^{k_{1}}\right) y_{i_{1}, k_{1}}\right. \\
&\left.y_{i_{2}, k_{1}} \frac{\partial}{\partial y_{k_{1}}}\right)\left(\mathscr{Y}_{x^{i_{1}}}+\sum_{k_{1}}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{y}-\mathscr{X}_{x^{i_{1}}}^{k_{1}}\right] y_{k_{1}}+\right. \\
&\left.+\sum_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}}\right)-\sum_{k_{1}}\left[\mathscr{X}_{x^{i_{2}}}^{k_{1}}+y_{i_{2}} \mathscr{X}_{y}^{k_{1}}\right] y_{i_{1}, k_{1}}
\end{aligned}
$$

$$
\begin{align*}
= & \left(\frac{\partial}{\partial x^{i_{2}}}\right)\left(\mathscr{Y}_{x^{i_{1}}}+\sum_{k_{1}}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{y}-\mathscr{X}_{x^{i_{1}}}^{k_{1}}\right] y_{k_{1}}+\sum_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}}\right)+  \tag{3.10}\\
& +\left(y_{i_{2}} \frac{\partial}{\partial y}\right)\left(\mathscr{Y}_{x^{i_{1}}}+\sum_{k_{1}}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{y}-\mathscr{X}_{x^{i_{1}}}^{k_{1}}\right] y_{k_{1}}+\sum_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}}\right)+ \\
& +\left(\sum_{k_{1}} y_{i_{2}, k_{1}} \frac{\partial}{\partial y_{k_{1}}}\right)\left(\mathscr{Y}_{x^{i_{1}}}+\sum_{k_{1}}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{y}-\mathscr{X}_{x^{i_{1}}}^{k_{1}}\right] y_{k_{1}}+\sum_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}}\right)+ \\
& +\sum_{k_{1}}\left[-\mathscr{X}_{x^{2}}^{k_{1}}\right] y_{k_{1}, i_{1}}+\sum_{k_{1}}\left[-\mathscr{X}_{y}^{k_{1}}\right] y_{i_{2}} y_{i_{1}, k_{1}}
\end{align*}
$$

$$
=\mathscr{Y}_{x^{i_{1}} x^{i_{2}}}+\sum_{k_{1}}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{x^{i_{2}} y}-\mathscr{X}_{x^{i_{1}} x^{i_{2}}}^{k_{1}}\right] y_{k_{1}}+\sum_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{x^{i_{2} y}}^{k_{2}}\right] y_{k_{1}} y_{k_{2}}+
$$

$$
+\mathscr{Y}_{x^{i_{1}} y} y_{i_{2}}+\sum_{k_{1}}\left[\delta_{i_{1}}^{k_{1}} \mathscr{y}_{y y}-\mathscr{X}_{x^{i_{1}} y}^{k_{1}}\right] y_{k_{1}} y_{i_{2}}+\sum_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{y y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}} y_{i_{2}}+
$$

$$
+\sum_{k_{1}}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{y}-\mathscr{X}_{x^{i_{1}}}^{k_{1}}\right] y_{i_{2}, k_{1}}+\sum_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{2}} y_{i_{2}, k_{1}}+\sum_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{1}} y_{i_{2}, k_{2}}+
$$

$$
+\sum_{k_{1}}\left[-\mathscr{X}_{x^{i_{2}}}^{k_{1}}\right] y_{k_{1}, i_{1}}+\sum_{k_{1}}\left[-\mathscr{X}_{y}^{k_{1}}\right] y_{i_{2}} y_{i_{1}, k_{1}} .
$$

Some explanations are needed about the computation of the last two terms of line 11 , i.e. about the passage from line 7 of (3.10) just above to line 11.

We have to compute:

$$
\begin{equation*}
\left(\sum_{k_{1}} y_{i_{2}, k_{1}} \frac{\partial}{\partial y_{k_{1}}}\right)\left(\sum_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}}\right) . \tag{3.11}
\end{equation*}
$$

This term is of the form

$$
\begin{equation*}
\left(\sum_{k_{1}} A_{k_{1}} \frac{\partial}{\partial y_{k_{1}}}\right)\left(\sum_{k_{1}, k_{2}}\left[B_{k_{1}, k_{2}}\right] y_{k_{1}} y_{k_{2}}\right), \tag{3.12}
\end{equation*}
$$

where the terms $B_{k_{1}, k_{2}}$ are independent of the pure first jet variables $y_{x^{k}}$. By the rule of Leibniz for the differentiation of a product, we may write

$$
\begin{align*}
& \left(\sum_{k_{1}} A_{k_{1}} \frac{\partial}{\partial y_{k_{1}}}\right)\left(\sum_{k_{1}, k_{2}}\left[B_{k_{1}, k_{2}}\right] y_{k_{1}} y_{k_{2}}\right)=  \tag{3.13}\\
& =\sum_{k_{1}, k_{2}}\left[B_{k_{1}, k_{2}}\right] y_{k_{2}}\left(\sum_{k_{1}^{\prime}} A_{k_{1}^{\prime}} \frac{\partial}{\partial y_{k_{1}^{\prime}}^{\prime}}\left(y_{k_{1}}\right)\right)+\sum_{k_{1}, k_{2}}\left[B_{k_{1}, k_{2}}\right] y_{k_{1}}\left(\sum_{k_{2}^{\prime}} A_{k_{2}^{\prime}} \frac{\partial}{\partial y_{k_{2}^{\prime}}}\left(y_{k_{2}}\right)\right) \\
& =\sum_{k_{1}, k_{2}}\left[B_{k_{1}, k_{2}}\right] y_{k_{2}} A_{k_{1}}+\sum_{k_{1}, k_{2}}\left[B_{k_{1}, k_{2}}\right] y_{k_{1}} A_{k_{2}} .
\end{align*}
$$

This is how we have written line 11 of (3.10).
Next, the first term $\mathscr{Y}_{x^{i_{1}} y} y_{i_{2}}$ in line 10 of (3.10) is not in a suitable shape. For reasons of harmony and coherence, we must insert it inside a sum of the form $\sum_{k_{1}}[\cdot] y_{k_{1}}$. Hence, using the Kronecker symbol, we transform:

$$
\begin{equation*}
\mathscr{Y}_{x^{i_{1}} y} y_{i_{2}} \equiv \sum_{k_{1}}\left[\delta_{i_{2}}^{k_{1}} \mathscr{Y}_{x^{i_{1}} y}\right] y_{k_{1}} . \tag{3.14}
\end{equation*}
$$

Also, we must "summify" the seven other terms, remaining in lines 10,11 and 12 of (3.10). Sometimes, we use the symmetry $y_{i_{2}, k_{1}} \equiv y_{k_{1}, i_{2}}$ without mention. Similarly, we get:

$$
\begin{align*}
& \sum_{k_{1}}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{y y}-\mathscr{X}_{x^{1_{1}} y}^{k_{1}}\right] y_{k_{1}} y_{i_{2}} \equiv \sum_{k_{1}, k_{2}}\left[\delta_{i_{1}}^{k_{1}} \delta_{i_{2}}^{k_{2}} \mathscr{Y}_{y y}-\delta_{i_{2}}^{k_{2}} \mathscr{X}_{x^{i_{1}} y}^{k_{1}}\right] y_{k_{1}} y_{k_{2}}, \\
& \sum_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{y y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}} y_{i_{2}} \equiv \sum_{k_{1}, k_{2}, k_{3}}\left[-\delta_{i_{1}}^{k_{1}} \delta_{i_{2}}^{k_{3}} \mathscr{X}_{y y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}} y_{k_{3}}, \\
& \sum_{k_{1}}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{y}-\mathscr{X}_{x_{1}}^{k_{1}}\right] y_{k_{1}, i_{2}} \equiv \sum_{k_{1}, k_{2}}\left[\delta_{i_{1}}^{k_{1}} \delta_{i_{2}}^{k_{2}} \mathscr{Y}_{y}-\delta_{i_{2}}^{k_{2}} \mathscr{X}_{x_{1} k_{1}}^{k_{1}}\right] y_{k_{1}, k_{2}}, \\
& \sum_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{2}} y_{k_{1}, i_{2}}=\sum_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{2}} \mathscr{X}_{y}^{k_{1}}\right] y_{k_{1}} y_{k_{2}, i_{2}}  \tag{3.15}\\
& \equiv \sum_{k_{1}, k_{2}, k_{3}}\left[-\delta_{i_{1}}^{k_{2}} \delta_{i_{2}}^{k_{3}} \mathscr{X}_{y}^{k_{1}}\right] y_{k_{1}} y_{k_{2}, k_{3}},
\end{align*}
$$

$$
\begin{aligned}
\sum_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}, i_{2}} & \equiv \sum_{k_{1}, k_{2}, k_{3}}\left[-\delta_{i_{1}}^{k_{1}} \delta_{i_{2}}^{k_{3}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}, k_{3}}, \\
\sum_{k_{1}}\left[-\mathscr{X}_{x^{2}}^{k_{1}}\right] y_{k_{1}, i_{1}} & \equiv \sum_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{2}} \mathscr{X}_{x^{i_{2}}}^{k_{1}}\right] y_{k_{1}, k_{2}}, \\
\sum_{k_{1}}\left[-\mathscr{X}_{y}^{k_{1}}\right] y_{i_{2}} y_{k_{1}, i_{1}} & =\sum_{k_{2}}\left[-\mathscr{X}_{y}^{k_{2}}\right] y_{i_{2}} y_{k_{2}, i_{1}} \\
& \equiv \sum_{k_{1}, k_{2}, k_{3}}\left[-\delta_{i_{2}}^{k_{1}} \delta_{i_{1}}^{k_{3}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}, k_{3}}
\end{aligned}
$$

In the sequel, for products of Kronecker symbols, it will be convenient to adopt the following self-evident contracted notation:

$$
\begin{equation*}
\delta_{i_{1}}^{k_{1}} \delta_{i_{2}}^{k_{2}} \equiv \delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} ; \quad \text { generally : } \delta_{i_{1}}^{k_{1}} \delta_{i_{2}}^{k_{2}} \cdots \delta_{i_{\lambda}}^{k_{\lambda}} \equiv \delta_{i_{1}, i_{2}, \cdots, i_{\lambda}}^{k_{1}, k_{2}, \cdots, k_{\lambda}} . \tag{3.16}
\end{equation*}
$$

Re-inserting plainly these eight summified terms (3.14), (3.15) in the last expression (3.10) of $\mathbf{Y}_{i_{1}, i_{2}}$ (lines 10, 11 and 12), we get: (3.17)

$$
\begin{aligned}
& \mathbf{Y}_{i_{1}, i_{2}}=\underline{\mathscr{Y}_{x_{1} x^{i_{2}}}} \underline{1}+\underset{\sum_{k_{1}}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{x^{i_{2}} y}-\mathscr{X}_{x^{i_{1}} x^{i_{2}}}^{k_{1}}\right] y_{k_{1}}+\sum_{k_{k_{1}, k_{2}}}\left[-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{x^{i_{2} y}}^{k_{2}}\right] y_{k_{1}} y_{k_{2}}+}{3}+ \\
& +\sum_{k_{1}}^{\sum_{k_{1}}\left[\delta_{i_{2}}^{k_{1}} \mathscr{Y}_{x^{i_{1}} y}\right] y_{k_{1}}+\sum_{k_{1, k_{2}}}\left[\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{Y}_{y y}-\delta_{i_{2}}^{k_{2}} \mathscr{X}_{x^{i_{1}} y}^{k_{1}}\right] y_{k_{1}} y_{k_{2}}}+ \\
& +\sum_{\underline{k_{1}, k_{2}, k_{3}}}\left[-\delta_{i_{1}, i_{2}}^{k_{1}, k_{3}} \mathscr{X}_{y y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}} y_{k_{3}}+\sum_{k_{k_{1}, k_{2}}}\left[\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{Y}_{y}-\delta_{i_{2}}^{k_{2}} \mathscr{X}_{x^{i_{1}}}^{k_{1}}\right] y_{k_{1}, k_{2}}+ \\
& +\sum_{k_{1}, k_{2}, k_{3}}\left[-\delta_{i_{1}, i_{2}}^{k_{2}, k_{3}} \mathscr{X}_{y}^{k_{1}}\right] y_{k_{1}} y_{k_{2}, k_{3}}+\sum_{6} \xrightarrow[k_{1}, k_{2}, k_{3}]{ }\left[-\delta_{i_{1}, i_{2}}^{k_{1}, k_{3}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}, k_{3}}+ \\
& +\underbrace{6}_{k_{k_{1}, k_{2}}\left[-\delta_{i_{1}}^{k_{2}} \mathscr{X}_{x^{2}}^{k_{1}}\right] y_{k_{1}, k_{2}}+\sum_{5}^{\sum_{k_{1}, k_{2}, k_{3}}\left[-\delta_{i_{2}, i_{1}}^{k_{1}, k_{3}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}, k_{3}}} .}
\end{aligned}
$$

Next, we gather the underlined terms, ordering them according to their number. This yields 6 collections of sums of monomials in the pure jet variables: (3.18)

$$
\begin{aligned}
\mathbf{Y}_{i_{1}, i_{2}}= & \mathscr{Y}_{x^{i_{1} x^{i_{2}}}}+\sum_{k_{1}}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{x^{i_{2}} y}+\delta_{i_{2}}^{k_{1}} \mathscr{Y}_{x^{i_{1}} y}-\mathscr{X}_{x^{i_{1}} x^{i_{2}}}^{k_{1}}\right] y_{k_{1}}+ \\
& +\sum_{k_{1}, k_{2}}\left[\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{Y}_{y y}-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{x^{i_{2}} y}^{k_{2}}-\delta_{i_{2}}^{k_{2}} \mathscr{X}_{x^{i_{1} y}}^{k_{1}}\right] y_{k_{1}} y_{k_{2}}+ \\
& +\sum_{k_{1}, k_{2}, k_{3}}\left[-\delta_{i_{1}, i_{2}}^{k_{1}, k_{3}} \mathscr{X}_{y y}^{k_{2}}\right] y_{k_{1} y_{k_{2}} y_{k_{3}}+} \\
& +\sum_{k_{1}, k_{2}}\left[\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{Y}_{y}-\delta_{i_{2}}^{k_{2}} \mathscr{X}_{x^{i_{1}}}^{k_{1}}-\delta_{i_{1}}^{k_{2}} \mathscr{X}_{x^{i_{2}}}^{k_{1}}\right] y_{k_{1}, k_{2}}+ \\
& +\sum_{k_{1}, k_{2}, k_{3}}\left[-\delta_{i_{1}, i_{2}}^{k_{2}, k_{3}} \mathscr{X}_{y}^{k_{1}}-\delta_{i_{1}, i_{2}}^{k_{1}, k_{3}} \mathscr{X}_{y}^{k_{2}}-\delta_{i_{2}, i_{1}}^{k_{1}, k_{3}} \mathscr{X}_{y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}, k_{3}} .
\end{aligned}
$$

To attain the real perfect harmony, this last expression has still to be worked out a little bit.

Lemma 3.19. The final expression of $\mathbf{Y}_{i_{1}, i_{2}}$ is as follows:
(3.20)

$$
\left\{\begin{aligned}
\mathbf{Y}_{i_{1}, i_{2}}= & \mathscr{Y}_{x^{i_{1}} x^{i_{2}}}+\sum_{k_{1}}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{x^{i_{2}} y}+\delta_{i_{2}}^{k_{1}} \mathscr{Y}_{x^{i_{1}} y}-\mathscr{X}_{x^{1_{1}} x^{i_{2}}}^{k_{1}}\right] y_{k_{1}}+ \\
& +\sum_{k_{1}, k_{2}}\left[\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{Y}_{y y}-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{x^{i_{2}} y}^{k_{2}}-\delta_{i_{2}}^{k_{1}} \mathscr{X}_{x^{i_{1}} y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}}+ \\
& +\sum_{k_{1}, k_{2}, k_{3}}\left[-\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{X}_{y y}^{k_{3}}\right] y_{k_{1}} y_{k_{2}} y_{k_{3}}+ \\
& +\sum_{k_{1}, k_{2}}\left[\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{Y}_{y}-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{x^{2_{2}}}^{k_{2}}-\delta_{i_{2}}^{k_{1}} \mathscr{X}_{x^{i_{1}}}^{k_{2}}\right] y_{k_{1}, k_{2}}+ \\
& +\sum_{k_{1}, k_{2}, k_{3}}\left[-\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{X}_{y}^{k_{3}}-\delta_{i_{1}, i_{2}}^{k_{3}, k_{1}} \mathscr{X}_{y}^{k_{2}}-\delta_{i_{1}, i_{2}}^{k_{2}, k_{3}} \mathscr{X}_{y}^{k_{1}}\right] y_{k_{1}} y_{k_{2}, k_{3}} .
\end{aligned}\right.
$$

Proof. As promised, we explain every tiny detail.
The first lines of (3.18) and of (3.20) are exactly the same. For the transformations of terms in the second, in the third and in the fourth lines, we use the following device. Let $\Upsilon_{k_{1}, k_{2}}$ be an indexed quantity which is symmetric: $\Upsilon_{k_{1}, k_{2}}=\Upsilon_{k_{2}, k_{1}}$. Let $A_{k_{1}, k_{2}}$ be an arbitrary indexed quantity. Then obviously:

$$
\begin{equation*}
\sum_{k_{1}, k_{2}} A_{k_{1}, k_{2}} \Upsilon_{k_{1}, k_{2}}=\sum_{k_{1}, k_{2}} A_{k_{2}, k_{1}} \Upsilon_{k_{1}, k_{2}} . \tag{3.21}
\end{equation*}
$$

Similar relations hold with a quantity $\Upsilon_{i_{1}, i_{2}, \ldots, i_{\lambda}}$ which is symmetric with respect to its $\lambda$ indices. Consequently, in the second, in the third and in
the fourth lines of (3.18), we may permute freely certain indices in some of the terms inside the brackets. This yields the passage from lines 2,3 and 4 of (3.18) to lines 2, 3 and 4 of (3.20).

It remains to explain how we pass from the fifth (last) line of (3.18) to the fifth (last) line of (3.20). The bracket in the fifth line of (3.18) contains three terms: $\left[-T_{1}-T_{2}-T_{3}\right]$. The term $T_{3}$ involves the product $\delta_{i_{2}, i_{1}}^{k_{1}, k_{3}}$, which we rewrite as $\delta_{i_{1}, i_{2}}^{k_{3}, k_{1}}$, in order that $i_{1}$ appears before $i_{2}$. Then, we rewrite the three terms in the new order $\left[-T_{2}-T_{3}-T_{1}\right]$, which yields:

$$
\begin{equation*}
\sum_{k_{1}, k_{2}, k_{3}}\left[-\delta_{i_{1}, i_{2}}^{k_{1}, k_{3}} \mathscr{X}_{y}^{k_{2}}-\delta_{i_{1}, i_{2}}^{k_{3}, k_{1}} \mathscr{X}_{y}^{k_{2}}-\delta_{i_{1}, i_{2}}^{k_{2}, k_{3}} \mathscr{X}_{y}^{k_{1}}\right] y_{k_{1}} y_{k_{2}, k_{3}} . \tag{3.22}
\end{equation*}
$$

It remains to observe that we can permute $k_{2}$ and $k_{3}$ in the first term $-T_{2}$, which yields the last line of (3.20). The detailed proof is complete.
3.23. Final perfect expression of $\mathbf{Y}_{i_{1}, i_{2}, i_{3}}$. Thanks to similar (longer) computations, we have obtained an expression of $\mathbf{Y}_{i_{1}, i_{2}, i_{3}}$ which we consider to be in final harmonious shape. Without copying the intermediate steps, let us write down the result. The comments which are necessary to read it and to interpret it start just below.

$$
\begin{align*}
& \mathbf{Y}_{i_{1}, i_{2}, i_{3}}=\mathscr{Y}_{x^{i_{1}} x^{i_{2}} x^{i_{3}}}+\sum_{k_{1}}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{x^{i_{2}} x^{i_{3}} y}+\delta_{i_{2}}^{k_{1}} \mathscr{Y}_{x^{i_{1}} x^{i_{3}} y}+\delta_{i_{3}}^{k_{1}} \mathscr{Y}_{x^{i_{1}} x^{i_{2}} y}-\mathscr{X}_{x^{1_{1}} x^{i_{2}} x^{i_{3}}}^{k_{1}}\right] y_{k_{1}}+ \\
& +\sum_{k_{1}, k_{2}}\left[\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{3}} y^{2}}+\delta_{i_{3}, i_{1}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{2}} y^{2}}+\delta_{i_{2}, i_{3}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{1}} y^{2}}-\right. \\
& \left.-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{x^{2} x^{i} 3 y}^{k_{2}}-\delta_{i_{2}}^{k_{1}} \mathscr{X}_{x^{i^{1}} x^{i} 3 y}^{k_{2}}-\delta_{i_{3}}^{k_{1}} \mathscr{X}_{x^{i_{1}} x^{i_{2}} y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}}+ \\
& +\sum_{k_{1}, k_{2}, k_{3}}\left[\delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{2}, k_{3}} \mathscr{Y}_{y^{3}}-\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{3}} y^{2}}^{k_{3}}-\delta_{i_{1}, i_{3}}^{k_{1}, k_{2}} \mathscr{X}_{x^{2} y^{2}}^{k_{3}}-\delta_{i_{2}, i_{3}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{1}} y^{2}}^{k_{3}}\right] y_{k_{1}} y_{k_{2}} y_{k_{3}}+ \\
& +\sum_{k_{1}, k_{2}, k_{3}, k_{4}}\left[-\delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{2}, k_{3}} \mathscr{X}_{y^{3}}^{k_{4}}\right] y_{k_{1}} y_{k_{2}} y_{k_{3}} y_{k_{4}}+ \\
& +\sum_{k_{1}, k_{2}}\left[\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{O}_{x^{i_{3}} y}+\delta_{i_{3}, i_{1}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{2}} y}+\delta_{i_{2}, i_{3}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{1}} y}-\right.  \tag{3.24}\\
& \left.-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{x^{i_{2}} x^{i_{3}}}^{k_{2}}-\delta_{i_{2}}^{k_{1}} \mathscr{X}_{x^{i_{1}} x^{i_{3}}}^{k_{2}}-\delta_{i_{3}}^{k_{1}} \mathscr{X}_{x^{i_{1}} x^{i_{2}}}^{k_{2}}\right] y_{k_{1}, k_{2}}+ \\
& +\sum_{k_{1}, k_{2}, k_{3}}\left[\delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{2}, k_{3}} \mathscr{Y}_{y^{2}}+\delta_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{1}, k_{2}} \mathscr{Y}_{y^{2}}+\delta_{i_{1}, i_{2}, i_{3}}^{k_{2}, k_{3}, k_{1}} \mathscr{Y}_{y^{2}}-\right. \\
& -\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i} 3 y}^{k_{3}}-\delta_{i_{1}, i_{2}}^{k_{3}, k_{1}} \mathscr{X}_{x^{i} 3}^{k_{2}}-\delta_{i_{1}, i_{2}}^{k_{2}, k_{3}} \mathscr{X}_{x^{i} 3}^{k_{1}}- \\
& -\delta_{i_{1}, i_{3}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{2}} y}^{k_{3}}-\delta_{i_{1}, i_{3}}^{k_{3}, k_{1}} \mathscr{X}_{x^{i_{2}} y}^{k_{2}}-\delta_{i_{1}, i_{3}}^{k_{2}, k_{3}} \mathscr{X}_{x^{i} 2}^{k_{1}}- \\
& \left.-\delta_{i_{2}, i_{3}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{1}} y}^{k_{3}}-\delta_{i_{2}, i_{3}}^{k_{3}, k_{1}} \mathscr{X}_{x^{i_{1}} y}^{k_{2}}-\delta_{i_{2}, i_{3}}^{k_{2}, k_{3}} \mathscr{X}_{x^{i_{1}} y}^{k_{1}}\right] y_{k_{1}} y_{k_{2}, k_{3}}+
\end{align*}
$$

$$
\begin{aligned}
& +\sum_{k_{1}, k_{2}, k_{3}, k_{4}}\left[-\delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{2}, k_{3}} \mathscr{X}_{y^{2}}^{k_{4}}-\delta_{i_{1}, i_{2}, i_{3}}^{k_{2}, k_{3}, k_{1}} \mathscr{X}_{y^{2}}^{k_{4}}-\delta_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{2}, k_{1}} \mathscr{X}_{y^{2}}^{k_{4}-}\right. \\
& \left.+\sum_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{4}, k_{1}} \mathscr{X}_{y^{2}}^{k_{2}}-\delta_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{1}, k_{4}} \mathscr{X}_{y^{2}}^{k_{2}}-\delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{3}, k_{4}} \mathscr{X}_{y^{2}}^{k_{2}}\right] y_{k_{1}} y_{k_{2},} y_{k_{3}, k_{4}}+ \\
& +\sum_{k_{1}, k_{2}, k_{3}, k_{4}}\left[-\delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{2}, k_{3}} \mathscr{X}_{y}^{k_{4}}-\delta_{i_{1}, i_{2}, i_{3}}^{k_{2}, k_{3}, k_{1}} \mathscr{X}_{y}^{k_{4}}-\delta_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{1}, k_{2}} \mathscr{X}_{y}^{k_{4}}\right] y_{k_{1}, k_{2} y_{k_{3}, k_{4}}+}+\sum_{k_{1}, k_{2}, k_{3}}\left[\delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{2}, k_{3}} \mathscr{Y}_{y}-\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{3}}}^{k_{3}}-\delta_{i_{1}, i_{3}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{2}}}^{k_{3}}-\delta_{i_{2}, i_{3}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{1}}}^{k_{3}}\right] y_{k_{1}, k_{2}, k_{3}}+ \\
& +\sum_{k_{1}, k_{2}, k_{3}, k_{4}}\left[-\delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{2}, k_{3}}{\left.\mathscr{X} y_{y}^{k_{4}}-\delta_{i_{1}, i_{2}, i_{3}}^{k_{4}, k_{1}, k_{2}} \mathscr{X}_{y}^{k_{3}}-\delta_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{4}, k_{1}} \mathscr{X}_{y}^{k_{2}}-\delta_{i_{1}, i_{2}, i_{3}}^{k_{2}, k_{3}, k_{4}} \mathscr{X}_{y}^{k_{1}}\right] y_{k_{1}} y_{k_{2}, k_{3}, k_{4}} .}^{+}\right.
\end{aligned}
$$

3.25. Comments, analysis and induction. First of all, by comparing this expression of $\mathbf{Y}_{i_{1}, i_{2}, i_{3}}$ with the expression (2.8) of $\mathbf{Y}_{3}$, we easily guess a part of the (inductional) dictionary beween the cases $n=1$ and the case $n \geqslant 1$. For instance, the three monomials $[\cdot]\left(y_{1}\right)^{3},[\cdot] y_{1} y_{2}$ and $[\cdot]\left(y_{1}\right)^{2} y_{2}$ in $\mathbf{Y}_{3}$ are replaced in $\mathbf{Y}_{i_{1}, i_{2}, i_{3}}$ by the following three sums:
(3.26)
$\sum_{k_{1}, k_{2}, k_{3}}[\cdot] y_{k_{1}} y_{k_{2}} y_{k_{3}}, \quad \sum_{k_{1}, k_{2}, k_{3}}[\cdot] y_{k_{1}} y_{k_{2}, k_{3}}, \quad$ and $\quad \sum_{k_{1}, k_{2}, k_{3}, k_{4}}[\cdot] y_{k_{1}} y_{k_{2}} y_{k_{3}, k_{4}}$.
Similar formal correspondences may be observed for all the monomials of $\mathbf{Y}_{1}, \mathbf{Y}_{i_{1}}$, of $\mathbf{Y}_{2}, \mathbf{Y}_{i_{1}, i_{2}}$ and of $\mathbf{Y}_{3}, \mathbf{Y}_{i_{1}, i_{2}, i_{3}}$. Generally and inductively speaking, the monomial

$$
\begin{equation*}
[\cdot]\left(y_{\lambda_{1}}\right)^{\mu_{1}} \cdots\left(y_{\lambda_{d}}\right)^{\mu_{d}} \tag{3.27}
\end{equation*}
$$

appearing in the expression (2.25) of $\mathbf{Y}_{\kappa}$ should be replaced by a certain multiple sum generalizing (3.26). However, it is necessary to think, to pause and to search for an appropriate formalism before writing down the desired multiple sum.

The jet variable $y_{\lambda_{1}}$ should be replaced by a jet variable corresponding to a $\lambda_{1}$-th partial derivative, say $y_{k_{1}, \ldots, k_{\lambda_{1}}}$, where $k_{1}, \ldots, k_{\lambda_{1}}=1, \ldots, n$. For the moment, to simplify the discussion, we leave out the presence of a sum of the form $\sum_{k_{1}, \ldots, k_{\lambda_{1}}}$. The $\mu_{1}$-th power $\left(y_{\lambda_{1}}\right)^{\mu_{1}}$ should be replaced not by $\left(y_{k_{1}, \ldots, k_{\lambda_{1}}}\right)^{\mu_{1}}$, but by a product of $\mu_{1}$ different jet variables $y_{k_{1}, \ldots, k_{\lambda_{1}}}$ of length $\lambda$, with all indices $k_{\alpha}=1, \ldots, n$ being distinct. This rule may be confirmed by inspecting the expressions of $\mathbf{Y}_{i_{1}}$, of $\mathbf{Y}_{i_{1}, i_{2}}$ and of $\mathbf{Y}_{i_{1}, i_{2}, i_{3}}$. So $y_{k_{1}, \ldots, k_{\lambda_{1}}}$ should be developed as a product of the form

$$
\begin{equation*}
y_{k_{1}, \ldots, k_{\lambda_{1}}} y_{k_{\lambda_{1}+1}, \ldots, k_{2 \lambda_{1}}} \cdots y_{k_{\left(\mu_{1}-1\right) \lambda_{1}+1}, \ldots, k_{\mu_{1} \lambda_{1}}}, \tag{3.28}
\end{equation*}
$$

where

$$
\begin{equation*}
k_{1}, \ldots, k_{\lambda_{1}}, \ldots, k_{\mu_{1} \lambda_{1}}=1, \ldots, n \tag{3.29}
\end{equation*}
$$

Consider now the product $\left(y_{\lambda_{1}}\right)^{\mu_{1}}\left(y_{\lambda_{2}}\right)^{\mu_{2}}$. How should it develope in the case of several independent variables? For instance, in the expression of $\mathbf{Y}_{i_{1}, i_{2}, i_{3}}$, we have developed the product $\left(y_{1}\right)^{2} y_{2}$ as $y_{k_{1}} y_{k_{2}} y_{k_{3}, k_{4}}$. Thus, a reasonable proposal of formalism would be that the product $\left(y_{\lambda_{1}}\right)^{\mu_{1}}\left(y_{\lambda_{2}}\right)^{\mu_{2}}$ should be developed as a product of the form

$$
\begin{align*}
& y_{k_{1}, \ldots, k_{\lambda_{1}}} y_{k_{\lambda_{1}+1}, \ldots, k_{2 \lambda_{1}}} \cdots y_{k_{\left(\mu_{1}-1\right) \lambda_{1}+1}, \ldots, k_{\mu_{1} \lambda_{1}}}  \tag{3.30}\\
& y_{k_{\mu_{1} \lambda_{1}+1}, \ldots, k_{\mu_{1} \lambda_{1}+\lambda_{2}}} \cdots y_{k_{\mu_{1} \lambda_{1}+\left(\mu_{2}-1\right) \lambda_{2}+1}, \ldots, k_{\mu_{1} \lambda_{1}+\mu_{2} \lambda_{2}}},
\end{align*}
$$

where

$$
\begin{equation*}
k_{1}, \ldots, k_{\lambda_{1}}, \ldots, k_{\mu_{1} \lambda_{1}}, \ldots, k_{\mu_{1} \lambda_{1}+\mu_{2} \lambda_{2}}=1, \ldots, n . \tag{3.31}
\end{equation*}
$$

However, when trying to write down the development of the general monomial $\left(y_{\lambda_{1}}\right)^{\mu_{1}}\left(y_{\lambda_{2}}\right)^{\mu_{2}} \cdots\left(y_{\lambda_{d}}\right)^{\mu_{d}}$, we would obtain the complicated product

$$
\begin{align*}
& y_{k_{1}, \ldots, k_{\lambda_{1}}} y_{k_{\lambda_{1}+1}, \ldots, k_{2 \lambda_{1}}} \cdots y_{k_{\left(\mu_{1}-1\right) \lambda_{1}+1}, \ldots, k_{\mu_{1} \lambda_{1}}} \\
& y_{k_{\mu_{1} \lambda_{1}+1}, \ldots, k_{\mu_{1} \lambda_{1}+\lambda_{2}}} \cdots y_{k_{\mu_{1} \lambda_{1}+\left(\mu_{2}-1\right) \lambda_{2}+1}, \ldots, k_{\mu_{1} \lambda_{1}+\mu_{2} \lambda_{2}}}  \tag{3.32}\\
& y_{k_{\mu_{1} \lambda_{1}+\cdots+\mu_{d-1} \lambda_{d-1}+1}, \ldots, k_{\mu_{1} \lambda_{1}+\cdots+\mu_{d-1} \lambda_{d-1}+\lambda_{d}} \cdots} \\
& \cdots y_{k_{\mu_{1} \lambda_{1}+\cdots+\mu_{d-1} \lambda_{d-1}+\left(\mu_{d}-1\right) \lambda_{d}+1}, \cdots, k_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}} .} .
\end{align*}
$$

Essentially, this product is still readable. However, in it, some of the integers $k_{\alpha}$ have a too long index $\alpha$, often involving a sum. Such a length of $\alpha$ would be very inconvenient in writing down and in reading the general Kronecker symbols $\delta_{i_{1}, \ldots \ldots, i_{\lambda}}^{k_{\alpha_{1}}, \ldots, k_{\alpha_{\lambda}}}$ which should appear in the final expression of $\mathbf{Y}_{i_{1}, \ldots, i_{k}}$. One should read in advance Theorem 3.73 below to observe the presence of such multiple Kronecker symbols. Consequently, for $\alpha=1, \ldots, \mu_{1} \lambda_{1}, \ldots, \mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}$, we have to denote the indices $k_{\alpha}$ differently.

Notational convention 3.33. We denote $d$ collection of $\mu_{d}$ groups of $\lambda_{d}$ (a priori distinct) integers $k_{\alpha}=1, \ldots, n$ by


Correspondingly, we identify the set

$$
\begin{equation*}
\left\{1, \ldots, \lambda_{1}, \ldots, \mu_{1} \lambda_{1}, \ldots \ldots, \mu_{1} \lambda_{1}+\mu_{2} \lambda_{2}, \ldots \ldots, \mu_{1} \lambda_{1}+\mu_{2} \lambda_{2}+\cdots+\mu_{d} \lambda_{d}\right\} \tag{3.35}
\end{equation*}
$$

of all integers $\alpha$ from 1 to $\mu_{1} \lambda_{1}+\mu_{2} \lambda_{2}+\cdots+\mu_{d} \lambda_{d}$ with the following specific set

written in a lexicographic way which emphasizes clearly the subdivision in $d$ collections of $\mu_{d}$ groups of $\lambda_{d}$ integers.

With this notation at hand, we see that the development, in several independent variables, of the general monomial $\left(y_{\lambda_{1}}\right)^{\mu_{1}} \cdots\left(y_{\lambda_{d}}\right)^{\mu_{d}}$, may be written as follows:

$$
\begin{equation*}
y_{k_{1: 1: 1}, \ldots, k_{1: 1: \lambda_{1}}} \cdots y_{k_{1: \mu_{1}: 1, \ldots, \ldots, k_{1: \mu_{1}: \lambda_{1}}} \cdots y_{k_{d: 1: 1}, \ldots, k_{d: 1: \lambda_{d}}} \cdots \cdots y_{k_{d: \mu_{d}: 1}, \ldots, k_{d: \mu_{d}: \lambda_{d}}} . . . . ~} . \tag{3.37}
\end{equation*}
$$

Formally speaking, this expression is better than (3.32). Using product symbols, we may even write it under the slightly more compact form

$$
\begin{equation*}
\prod_{1 \leqslant \nu_{1} \leqslant \mu_{1}} y_{k_{1: \nu_{1}: 1}, \ldots, k_{1: \nu_{1}: \lambda_{1}}} \cdots \prod_{1 \leqslant \nu_{d} \leqslant \mu_{d}} y_{k_{d: \nu_{d}: 1}, \ldots, k_{d: \nu_{d}: \lambda_{d}}} . \tag{3.38}
\end{equation*}
$$

Now that we have translated the monomial, we may add all the summation symbols: the general expression of $\mathbf{Y}_{\kappa}$ (which generalizes our three previous examples (3.26)) will be of the form:

$$
\text { [?] } \prod_{1 \leqslant \nu_{1} \leqslant \mu_{1}} y_{k_{1: \nu_{1}: 1}, \ldots, k_{1: \nu_{1}: \lambda_{1}}} \cdots \prod_{1 \leqslant \nu_{d} \leqslant \mu_{d}} y_{k_{d: \nu_{d}: 1}, \ldots, k_{d: \nu_{d}: \lambda_{d}}} .
$$

From now on, up to the end of the article, to be very precise, we will restitute the bounds $\sum_{k=1}^{n}$ of all the previously abbreviated sums $\sum_{k}$. This is justified by the fact that, since we shall deal in Section 5 below simultaneously with several independent variables $\left(x^{1}, \ldots, x^{n}\right)$ and with several dependent variables $\left(y^{1}, \ldots, y^{m}\right)$, we shall encounter sums $\sum_{l=1}^{m}$, not to be confused with sums $\sum_{k=1}^{n}$.

$$
\begin{align*}
& \mathbf{Y}_{\kappa}=\mathscr{Y}_{x^{i_{1}} \ldots x^{i_{\kappa}}}+\sum_{d=1}^{\kappa+1} \sum_{1 \leqslant \lambda_{1}<\cdots<\lambda_{d} \leqslant \kappa} \sum_{\mu_{1} \geqslant 1, \ldots, \mu_{d} \geqslant 1} \sum_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d} \leqslant \kappa+1} \tag{3.39}
\end{align*}
$$

3.40. Combinatorics of the Kronecker symbols. Our next task is to determine what appears inside the brackets [?] of the above equation. We will treat this rather delicate question very progressively. Inductively, we have to guess how we may pass from the bracketed term of (2.25), namely from

$$
\begin{gather*}
{\left[\frac{\kappa \cdots\left(\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+1\right)}{\left(\lambda_{1}!\right)^{\mu_{1}} \mu_{1}!\cdots\left(\lambda_{d}!\right)^{\mu_{d}} \mu_{d}!} \cdot \mathscr{Y}_{x^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}} y^{\mu_{1}+\cdots+\mu_{d}}-} \quad-\frac{\kappa \cdots\left(\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+2\right)\left(\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}\right)}{\left(\lambda_{1}!\right)^{\mu_{1}} \mu_{1}!\cdots\left(\lambda_{d}!\right)^{\mu_{d}} \mu_{d}!} .\right.} \\
\left.\quad \cdot \mathscr{X}_{x^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+1} y^{\mu_{1}+\cdots+\mu_{d}-1}}\right], \tag{3.41}
\end{gather*}
$$

to the corresponding (still unknown) bracketed term [?].
First of all, we examine the following term, extracted from the complete expression of $\mathbf{Y}_{i_{1}, i_{2}, i_{3}}($ first line of (3.24)):

$$
\begin{equation*}
\sum_{k_{1}=1}^{n}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{x^{i_{2}} x^{i_{3}} y}+\delta_{i_{2}}^{k_{1}} \mathscr{Y}_{x^{i_{1}} x^{i_{3}} y}+\delta_{i_{3}}^{k_{1}} \mathscr{Y}_{x^{i_{1}} x^{i_{2}} y}-\mathscr{X}_{x^{i_{1} x^{i} 2 x^{i_{3}}}}^{k_{1}}\right] y_{k_{1}} \tag{3.42}
\end{equation*}
$$

Here, the coefficient $\left[3 \mathscr{\mathscr { x }}_{x^{2} y}-\mathscr{X}_{x^{3}}\right]$ of the monomial $y_{1}$ in $\mathbf{Y}_{3}$ is replaced by the above bracketed terms.

Let us precisely analyze the combinatorics. Here, $\mathscr{X}_{x^{3}}$ is replaced by $\mathscr{X}_{x^{i_{1}} x^{i_{2}} x^{i_{3}}}^{k_{1}}$, where the lower indices $i_{1}, i_{2}, i_{3}$ come from $\mathbf{Y}_{i_{1}, i_{2}, i_{3}}$ and where the upper index $k_{1}$ is the summation index. Also, the integer 3 in $3 \mathscr{Y}_{x^{2} y}$ is replaced by a sum of exactly three terms, each involving a single Kronecker symbol $\delta_{i}^{k}$, in which the lower index is always an index $i=i_{1}, i_{2}, i_{3}$ and in which the upper index is always equal to the summation index $k_{1}$. By the way, more generally, we immediately observe that all the successive positive integers

$$
\begin{equation*}
1,3,1,3,3,1,3,1,3,3,3,9,6,3,1,3,4 \tag{3.43}
\end{equation*}
$$

appearing in the formula (2.8) for $\mathbf{Y}_{3}$ are replaced, in the formula (3.24) for $\mathbf{Y}_{i_{1}, i_{2}, i_{3}}$, by sums of exactly the same number of terms involving Kronecker symbols. This observation will be a precious guide. Finally, in the symbol $\delta_{i}^{k_{1}}$, if $i$ is chosen among the set $\left\{i_{1}, i_{2}, i_{3}\right\}$, for instance if $i=i_{1}$, it follows that the development of $\mathscr{Y}_{x^{2} y}$ necessarily involves the remaining indices, for instance $\mathscr{Y}_{x^{i_{2}} x^{i_{3}} y}$. Since there are three choices for $i=i_{1}, i_{2}, i_{3}$, we recover the number 3 .

Next, comparing $\left[\mathscr{Y}_{y y}-2 \mathscr{X}_{x y}\right]\left(y_{1}\right)^{2}$ with the term

$$
\begin{equation*}
\sum_{k_{1}, k_{2}=1}^{n}\left[\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{Y}_{y y}-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{x^{i_{2}} y}^{k_{1}}-\delta_{i_{2}}^{k_{1}} \mathscr{X}_{x^{i_{1}} y}^{k_{1}}\right] y_{k_{1}} y_{k_{2}}, \tag{3.44}
\end{equation*}
$$

extracted from the complete expression of $\mathbf{Y}_{i_{1}, i_{2}}$ (second line of (3.18)), we learn and we guess that the number of Kronecker symbols before $\mathscr{Y}_{x^{\gamma} y^{\delta}}$
must be equal to the number of indices $k_{\alpha}$ minus $\gamma$. This rule is confirmed by examining the term (second and third line of (3.24))

$$
\begin{align*}
& \sum_{k_{1}, k_{2}}\left[\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{3}} y^{2}}+\delta_{i_{3}, i_{1}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{2}} y^{2}}+\delta_{i_{2}, i_{3}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{1}} y^{2}}-\right.  \tag{3.45}\\
& \left.\quad-\delta_{i_{1}}^{k_{1}} \mathscr{X}_{x^{i_{2}} x^{i_{3}} y}^{k_{2}}-\delta_{i_{2}}^{k_{1}} \mathscr{X}_{x^{i_{1} x^{i} y}}^{k_{2}}-\delta_{i_{3}}^{k_{1}} \mathscr{X}_{x^{1_{1}} x^{i_{2} y}}^{k_{2}}\right] y_{k_{1}} y_{k_{2}},
\end{align*}
$$

developing $\left[3 \mathscr{Y}_{x y^{2}}-3 \mathscr{X}_{x^{2} y}\right]\left(y_{1}\right)^{2}$.
Also, we may examine the following term

$$
\begin{align*}
& \sum_{k_{1}, k_{2}=1}^{n}\left[\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{3}} x^{i_{4}} y^{2}}+\delta_{i_{1}, i_{3}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{2}} x^{i_{4}} y^{2}}+\delta_{i_{1}, i_{4}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{2}} x^{i_{3}} y^{2}}+\right. \\
& +\delta_{i_{2}, i_{3}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{1}} x^{i_{4}} y^{2}}+\delta_{i_{2}, i_{4}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{1}} x^{i_{3}} y^{2}}+\delta_{i_{3}, i_{4}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{1}} x^{i_{2}} y^{2}}-  \tag{3.46}\\
& -\delta_{i_{1}}^{k_{1}} \mathscr{X}_{x^{i_{2}} x^{i_{3}} x^{i_{4} y}}^{k_{1}}-\delta_{i_{2}}^{k_{1}} \mathscr{X}_{x^{1} 1 x^{i_{2} x^{i} 3 y}}^{k_{1}}-\delta_{i_{3}}^{k_{1}} \mathscr{X}_{x^{i_{1}} x^{i_{2}} x^{i_{4} y}}^{k_{1}}- \\
& \left.-\delta_{i_{4}}^{k_{1}} \mathscr{X}_{x^{i_{1}} x^{i_{2}} x^{i_{3}} y}^{k_{1}}\right] y_{k_{1}} y_{k_{2}},
\end{align*}
$$

extracted from $\mathbf{Y}_{i_{1}, i_{2}, i_{3}, i_{4}}$ and developing $\left[6 \mathscr{Y}_{x^{2} y^{2}}-4 \mathscr{X}_{x^{3} y}\right]\left(y_{1}\right)^{2}$. We would like to mention that we have not written the complete expression of $\mathbf{Y}_{i_{1}, i_{2}, i_{3}, i_{4}}$, because it would cover two and a half printed pages.

By inspecting the way how the indices are permuted in the multiple Kronecker symbols of the first two lines of this expression (3.46), we observe that the six terms correspond exactly to the six possible choices of two complementary ordered couples of integers in the set $\{1,2,3,4\}$, namely

$$
\begin{array}{lll}
\{1,2\} \cup\{3,4\}, & \{1,3\} \cup\{2,4\}, & \{1,4\} \cup\{2,3\}, \\
\{2,3\} \cup\{1,4\}, & \{2,4\} \cup\{1,3\}, & \{3,4\} \cup\{1,2\} . \tag{3.47}
\end{array}
$$

At this point, we start to devise the general combinatorics. Before proceeding further, we need some notation.
3.48. Permutation groups. For every $p \in \mathbb{N}$ with $p \geqslant 1$, we denote by $\mathfrak{S}_{p}$ the full permutation group of the set $\{1,2, \ldots, p-1, p\}$. Its cardinal equals $p!$. The letters $\sigma$ and $\tau$ will be used to denote an element of $\mathfrak{S}_{p}$. If $p \geqslant 2$, and if $q \in \mathbb{N}$ satisfies $1 \leqslant q \leqslant p-1$, we denote by $\mathfrak{S}_{p}^{q}$ the subset of permutations $\sigma \in \mathfrak{S}_{p}$ satisfying the two collections of inequalities
$\sigma(1)<\sigma(2)<\cdots<\sigma(q) \quad$ and $\quad \sigma(q+1)<\sigma(q+2)<\cdots<\sigma(p)$.
The cardinal of $\mathfrak{S}_{p}^{q}$ equals $C_{p}^{q}=\frac{p!}{q!(p-q)!}$.

Lemma 3.50. For $\kappa \geqslant 1$, the development of (2.20) to several independent variables $\left(x^{1}, \ldots, x^{n}\right)$ is:

$$
\begin{align*}
& \mathbf{Y}_{i_{1}, i_{2}, \ldots, i_{\kappa}}=\mathscr{Y}_{x^{i_{1}} x^{i_{2} \ldots} \ldots x^{i_{\kappa}}}+\sum_{k_{1}=1}^{n}\left[\sum_{\tau \in \mathfrak{S}_{\kappa}^{1}} \delta_{i_{\tau(1)}}^{k_{1}} \mathscr{Y}_{x^{i} \tau(2) \ldots x^{i} \tau(\kappa)}-\mathscr{X}_{x^{i_{1}} x^{i_{2} \ldots} x^{i_{\kappa}}}^{k_{1}}\right] y_{k_{1}}+  \tag{3.51}\\
& +\sum_{k_{1}, k_{2}=1}^{n}\left[\sum_{\tau \in \mathfrak{S}_{\kappa}^{2}} \delta_{i_{\tau(1)}, i_{\tau(2)}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{\tau}(3) \ldots} x^{i_{\tau(\kappa)}} y^{2}}-\sum_{\tau \in \mathfrak{S}_{\kappa}^{1}} \delta_{i_{\tau(1)}}^{k_{1}} \mathscr{X}_{x^{i \tau(2)} \ldots x^{i_{\tau(k)}} y}^{k_{2}}\right] y_{k_{1}} y_{k_{2}}+ \\
& +\sum_{k_{1}, k_{2}, k_{3}=1}^{n}\left[\sum_{\tau \in \mathfrak{G}_{\kappa}^{3}} \delta_{i_{\tau(1)}, i_{\tau(2)}, i_{\tau(3)}}^{k_{1}, k_{2}, k_{3}} \mathscr{Y}_{x^{i} \tau(4) \ldots x^{i}{ }^{i_{\tau(\kappa)}} y^{3}-}\right. \\
& \left.-\sum_{\tau \in \mathfrak{S}_{k}^{2}} \delta_{i_{\tau(1), i}^{i_{\tau(2)}}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i} \tau(3) \ldots x^{i_{\tau(\kappa)}} y^{2}}^{k_{3}}\right] y_{k_{1}} y_{k_{2}} y_{k_{3}}+ \\
& +\cdots \cdots+ \\
& +\sum_{k_{1}, \ldots, k_{k}=1}^{n}\left[\delta_{i_{1}, \ldots, i_{\kappa}}^{k_{1}, \ldots, k_{\kappa}} \mathscr{Y}_{y^{\kappa}}-\sum_{\tau \in \mathfrak{S}_{\kappa}^{\kappa-1}} \delta_{i_{\tau(1)}, \ldots, i_{\tau(k-1)}}^{k_{1}, \ldots \ldots, k_{\kappa-1}} \mathscr{X}_{x^{i} \tau(\kappa)}^{k_{\kappa}} y^{\kappa-1}\right] y_{k_{1}} \cdots y_{k_{\kappa}}+ \\
& +\sum_{k_{1}, \ldots, k_{\kappa}, k_{\kappa+1}=1}^{n}\left[-\delta_{i_{1}, \ldots, i_{\kappa}}^{k_{1}, \ldots, k_{\kappa}} \mathscr{X}_{y^{\kappa^{\kappa}}}^{k_{\kappa+1}}\right] y_{k_{1}} \cdots y_{k_{\kappa}} y_{k_{\kappa+1}}+\text { remainder } .
\end{align*}
$$

Here, the term remainder collects all remaining monomials in the pure jet variables $y_{k_{1}, \ldots, k_{\lambda}}$.
3.52. Continuation. Thus, we have devised how the part of $\mathbf{Y}_{i_{1}, \ldots, i_{\kappa}}$ which involves only the jet variables $y_{k_{\alpha}}$ must be written. To proceed further, we shall examine the following term, extracted from $\mathbf{Y}_{i_{1}, i_{2}, i_{3}}$ (lines 12 and 13 of (3.24))

$$
\begin{align*}
& \sum_{k_{1}, k_{2}, k_{3}, k_{4}}\left[-\delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{2}, k_{3}} \mathscr{X}_{y^{2}}^{k_{4}}-\delta_{i_{1}, i_{2}, i_{3}}^{k_{2}, k_{3}, k_{1}} \mathscr{X}_{y^{2}}^{k_{4}}-\delta_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{2}, k_{1}} \mathscr{X}_{y^{2}}^{k_{4}}-\right.  \tag{3.53}\\
&\left.\quad-\delta_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{4}, k_{1}} \mathscr{X}_{y^{2}}^{k_{2}}-\delta_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{1}, k_{4}} \mathscr{X}_{y^{2}}^{k_{2}}-\delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{3}, k_{4}} \mathscr{X}_{y^{2}}^{k_{2}}\right] y_{k_{1}} y_{k_{2}} y_{k_{3}, k_{4},},
\end{align*}
$$

which developes the term $\left[-6 \mathscr{X}_{y^{2}}\right]\left(y_{1}\right)^{2} y_{2}$ of $\mathbf{Y}_{3}$ (third line of (2.8)). During the computation which led us to the final expression (3.24), we organized the formula in order that, in the six Kronecker symbols, the lower indices $i_{1}, i_{2}, i_{3}$ are all written in the same order. But then, what is the rule for the appearance of the four upper indices $k_{1}, k_{2}, k_{3}, k_{4}$ ?

In April 2001, we discovered the rule by inspecting both (3.53) and the following complicated term, extracted from the complete expression of
$\mathbf{Y}_{i_{1}, i_{2}, i_{3}, i_{4}}$ written in one of our manuscripts:

$$
\begin{align*}
& \sum_{k_{1}, k_{2}, k_{3}}\left[\delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{2}, k_{3}} \mathscr{Y}_{x^{i_{4}} y^{2}}+\delta_{i_{1}, i_{2}, i_{3}}^{k_{2}, k_{1}, k_{3}} \mathscr{Y}_{x^{i} y^{2}}+\delta_{i_{1}, i_{2}, i_{3}}^{k_{2}, k_{3}, k_{1}} \mathscr{Y}_{x^{i_{4}} y^{2}}+\right. \\
& +\delta_{i_{1}, i_{2}, i_{4}}^{k_{1}, k_{2}, k_{3}} \mathscr{Y}_{x^{i} 3 y^{2}}+\delta_{i_{1}, i_{2}, i_{4}}^{k_{2}, k_{1}, k_{3}} \mathscr{Y}_{x^{i} 3 y^{2}}+\delta_{i_{1}, i_{2}, i_{4}}^{k_{2}, k_{3}, k_{1}} \mathscr{Y}_{x^{i} y^{2}}+ \\
& +\delta_{i_{1}, i_{3}, i_{4}}^{k_{1}, k_{2}, k_{3}} \mathscr{Y}_{x^{i} 2} y^{2}+\delta_{i_{1}, i_{3}, i_{4}}^{k_{2}, k_{1}, k_{3}} \mathscr{Y}_{x^{i} 2} y^{2}+\delta_{i_{1}, i_{3}, i_{4}}^{k_{2}, k_{3}, k_{1}} \mathscr{Y}_{x^{i} y^{2}}+  \tag{3.54}\\
& +\delta_{i_{2}, i_{3}, i_{4}}^{k_{1}, k_{2}, k_{3}} \mathscr{Y}_{x^{i_{1}} y^{2}}+\delta_{i_{2}, i_{3}, i_{4}}^{k_{2}, k_{1}, k_{3}} \mathscr{Y}_{x^{i} 1} y^{2}+\delta_{i_{2}, i_{3}, i_{4}}^{k_{2}, k_{3}, k_{1}} \mathscr{Y}_{x^{i_{1}} y^{2}}- \\
& -\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{3}} x^{i_{4} y}}^{k_{3}}-\delta_{i_{1}, i_{2}}^{k_{2}, k_{1}} \mathscr{X}_{x^{i_{3} x^{i} 4}}^{k_{3}}-\delta_{i_{1}, i_{2}}^{k_{2}, k_{3}} \mathscr{X}_{x^{3} x^{i_{4}} y}^{k_{1}}- \\
& -\delta_{i_{1}, i_{3}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{2}} x^{i_{4}}}^{k_{3}}-\delta_{i_{1}, i_{3}}^{k_{2}, k_{1}} \mathscr{X}_{x^{i_{2}} x^{i_{4}} y}^{k_{3}}-\delta_{i_{1}, i_{3}}^{k_{2}, k_{3}} \mathscr{X}_{x^{i_{2}} x^{i_{4}} y}^{k_{1}}- \\
& -\delta_{i_{1}, i_{4}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{2}} x^{i_{3}}}^{k_{3}}-\delta_{i_{1}, i_{4}}^{k_{2}, k_{1}} \mathscr{X}_{x^{x_{2} x^{i} y}}^{k_{3}}-\delta_{i_{1}, i_{4}}^{k_{2}, k_{3}} \mathscr{X}_{x^{2} x^{i_{3}} y}^{k_{1}}- \\
& -\delta_{i_{2}, i_{3}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{1}} x^{i_{4}} y}^{k_{3}}-\delta_{i_{2}, i_{3}}^{k_{2}, k_{1}} \mathscr{X}_{x^{1_{1}{ }^{i_{4}} y}}^{k_{k_{3}}}-\delta_{i_{2}, i_{3}}^{k_{2}, k_{3}} \mathscr{X}_{x^{i_{1} x^{i} 4}{ }^{k_{1}}}^{k_{1}}- \\
& -\delta_{i_{2}, i_{4}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{1}} x^{i} 3}^{k_{3}}-\delta_{i_{2}, i_{4}}^{k_{2}, k_{1}} \mathscr{X}_{x^{i_{1}} x^{i} 3 y}^{k_{3}}-\delta_{i_{2}, i_{4}}^{k_{2}, k_{3}} \mathscr{X}_{x^{11} x^{i} 3 y}^{k_{1}}- \\
& \left.-\delta_{i_{3}, i_{4}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{1}} x^{i_{2}} y}^{k_{3}}-\delta_{i_{3}, i_{4}}^{k_{2}, k_{1}} \mathscr{X}_{x^{i_{1}} x^{i_{2}} y}^{k_{3}}-\delta_{i_{3}, i_{4}}^{k_{2}, k_{3}} \mathscr{X}_{x^{1_{1}} x^{i_{2}} y}^{k_{1}}\right] y_{k_{1}} y_{k_{2}, k_{3}} .
\end{align*}
$$

This sum developes the term $\left[12 \mathscr{Y}_{x y^{2}}-18 \mathscr{X}_{x^{2} y}\right] y_{1} y_{2}$ of $\mathbf{Y}_{3}$ (third line of (2.9)). Let us explain what are the formal rules.

In the bracketed terms of (3.53), there are no permutation of the indices $i_{1}, i_{2}, i_{3}$, but there is a certain unknown subset of all the permutations of the four indices $k_{1}, k_{2}, k_{3}, k_{4}$. In the bracketed terms of (3.54), two combinatorics are present:

- there are some permutations of the indices $i_{1}, i_{2}, i_{3}, i_{4}$ and
- there are some permutations of the indices $k_{1}, k_{2}, k_{3}$.

Here, the permutations of the indices $i_{1}, i_{2}, i_{3}, i_{4}$ are easily guessed, since they are the same as the permutations which were introduced in $\S 3.48$ above. Indeed, in the first four lines of (3.54), we see the four decompositions (3.55)
$\left\{i_{1}, i_{2}, i_{3}\right\} \cup\left\{i_{4}\right\}, \quad\left\{i_{1}, i_{2}, i_{4}\right\} \cup\left\{i_{3}\right\}, \quad\left\{i_{1}, i_{3}, i_{4}\right\} \cup\left\{i_{2}\right\}, \quad\left\{i_{2}, i_{3}, i_{4}\right\} \cup\left\{i_{1}\right\}$,
of the set $\left\{i_{1}, i_{2}, i_{3}, i_{4}\right\}$, and in the last six lines of (3.54), we see the six decompositions

$$
\begin{array}{lll}
\left\{i_{1}, i_{2}\right\} \cup\left\{i_{3}, i_{4}\right\}, & \left\{i_{1}, i_{3}\right\} \cup\left\{i_{2}, i_{4}\right\}, & \left\{i_{1}, i_{4}\right\} \cup\left\{i_{2}, i_{3}\right\},  \tag{3.56}\\
\left\{i_{2}, i_{3}\right\} \cup\left\{i_{1}, i_{4}\right\}, & \left\{i_{2}, i_{4}\right\} \cup\left\{i_{1}, i_{3}\right\}, & \left\{i_{3}, i_{4}\right\} \cup\left\{i_{1}, i_{2}\right\},
\end{array}
$$

so that (3.54) may be written under the form

$$
\begin{equation*}
\sum_{k_{1}, k_{2}, k_{3}}\left[\sum_{\tau \in \mathfrak{S}_{4}^{3}} \sum_{\sigma \in ?} \delta_{i_{\tau(1)}, i_{\tau(2)}, i_{\tau(3)}}^{k_{\tau(1)}, k_{\tau(2)}, k_{\tau(3)}} \mathscr{Y}_{x^{i_{\tau(4)}} y^{2}}-\sum_{\tau \in \mathfrak{S}_{4}^{2}} \sum_{\sigma \in ?} \delta_{i_{\tau(1)}, i_{\tau(2)}}^{k_{\tau(1)}, k_{\tau(2)}} \mathscr{X}_{x^{i}(3)}^{k_{\tau(3)} x^{i}(4) y}\right] y_{k_{1}} y_{k_{2}, k_{3}}, \tag{3.57}
\end{equation*}
$$

where in the two above sums $\sum_{\sigma \in ?}$, the letter $\sigma$ denotes a permutation of the set $\{1,2,3\}$ and where the sign ? refers to two (still unknown) subset of the full permutation group $\mathfrak{S}_{3}$. The only remaining question is to determine how the indices $k_{\alpha}$ are permuted in (3.53) and in (3.54).

The answer may be guessed by looking at the permutations of the set $\left\{k_{1}, k_{2}, k_{3}, k_{4}\right\}$ which stabilize the monomial $y_{k_{1}} y_{k_{2}} y_{k_{3}, k_{4}}$ in (3.53): we clearly have the following four symmetry relations between monomials:

$$
\begin{equation*}
y_{k_{1}} y_{k_{2}} y_{k_{3}, k_{4}} \equiv y_{k_{2}} y_{k_{1}} y_{k_{3}, k_{4}} \equiv y_{k_{1}} y_{k_{2}} y_{k_{4}, k_{3}} \equiv y_{k_{2}} y_{k_{1}} y_{k_{4}, k_{3}} \tag{3.58}
\end{equation*}
$$

and nothing more. Then the number 6 of bracketed terms in (3.53) is exactly equal to the cardinal $24=4$ ! of the full permutation group of the set $\left\{k_{1}, k_{2}, k_{3}, k_{4}\right\}$ divided by the number 4 of these symmetry relations. The set of permutations $\sigma$ of $\{1,2,3,4\}$ satisfying these symmetry relations

$$
\begin{equation*}
y_{k_{\sigma(1)}} y_{k_{\sigma(2)}} y_{k_{\sigma(3)}, k_{\sigma(4)}} \equiv y_{k_{1}} y_{k_{2}} y_{k_{3}, k_{4}} \tag{3.59}
\end{equation*}
$$

consitutes a subgroup of $\mathfrak{S}_{4}$ which we will denote by $\mathfrak{H}_{4}^{(2,1),(1,2)}$. Furthermore, the coset

$$
\begin{equation*}
\mathfrak{F}_{4}^{(2,1),(1,2)}:=\mathfrak{S}_{4} / \mathfrak{H}_{4}^{(2,1),(1,2)} \tag{3.60}
\end{equation*}
$$

possesses the six representatives

$$
\begin{array}{lll}
\left(\begin{array}{llll}
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4
\end{array}\right), & \left(\begin{array}{llll}
1 & 2 & 3 & 4 \\
2 & 3 & 1 & 4
\end{array}\right), & \left(\begin{array}{llll}
1 & 2 & 3 & 4 \\
3 & 2 & 1 & 4
\end{array}\right),  \tag{3.61}\\
\left(\begin{array}{llll}
1 & 2 & 3 & 4 \\
3 & 4 & 1 & 2
\end{array}\right), & \left(\begin{array}{llll}
1 & 2 & 3 & 4 \\
3 & 1 & 4 & 2
\end{array}\right), & \left(\begin{array}{llll}
1 & 2 & 3 & 4 \\
1 & 3 & 4 & 2
\end{array}\right),
\end{array}
$$

which exactly appear as the permutations of the upper indices of our example (3.53). Of course, the question arises whether the choice of such six representatives in the quotient $\mathfrak{S}_{4} / \mathfrak{H}_{4}^{(2,1),(1,2)}$ is legitimate.

Fortunately, we observe that after conjugation by any permutation $\sigma \in$ $\mathfrak{H}_{4}^{(2,1),(1,2)}$, we do not perturb any of the six terms of (3.53), for instance the third term of (3.53) is not perturbed, as shown by the following computation

$$
\begin{align*}
& \sum_{k_{1}, k_{2}, k_{3}, k_{4}}\left[-\delta_{i_{1},,}^{k_{\sigma(3)}, k_{\sigma(2)}, i_{3}} k_{i_{\sigma(1)}} \mathscr{X}_{y^{2}}^{k_{\sigma(4)}}\right] y_{k_{1}} y_{k_{2}} y_{k_{3}, k_{4}}= \\
& =\sum_{k_{1}, k_{2}, k_{3}, k_{4}}\left[-\delta_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{2}, k_{1}} \mathscr{X}_{y^{2}}^{k_{\sigma(4)}}\right] y_{k_{\sigma^{-1}(1)}} y_{k_{\sigma-1(2)}} y_{k_{\sigma^{-1}(3)}, k_{\sigma-1}(4)}  \tag{3.62}\\
& =\sum_{k_{1}, k_{2}, k_{3}, k_{4}}\left[-\delta_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{2}, k_{1}} \mathscr{X}_{y^{2}}^{k_{\sigma(4)}}\right] y_{k_{1}} y_{k_{2}} y_{k_{3}, k_{4}}
\end{align*}
$$

thanks to the symmetry (3.59). Thus, as expected, the choice of 6 arbitrary representatives $\sigma \in \mathfrak{F}_{4}^{(2,1),(1,2)}$ in the bracketed terms of (3.53) is free. In
conclusion, we have shown that (3.53) may be written under the form:

$$
\begin{equation*}
\sum_{k_{1}, k_{2}, k_{3}, k_{4}}\left[-\sum_{\sigma \in \mathfrak{F}_{4}^{(2,1),(1,2)}} \delta_{i_{1}, \quad, i\left(i_{2}, i_{3}\right.}^{k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)}} \mathscr{X}_{y^{2}}^{k_{\sigma(4)}}\right] y_{k_{1}} y_{k_{2}} y_{k_{3}, k_{4}} \tag{3.63}
\end{equation*}
$$

This rule is confirmed by inspecting (3.54) (as well as all the other terms of $\mathbf{Y}_{i_{1}, i_{2}, i_{3}}$ and of $\mathbf{Y}_{i_{1}, i_{2}, i_{3}, i_{4}}$ ). Indeed, the permutations $\sigma$ of the set $\left\{k_{1}, k_{2}, k_{3}\right\}$ which stabilize the monomial $y_{k_{1}} y_{k_{2}, k_{3}}$ consist just of the identity permutation and the transposition of $k_{2}$ and $k_{3}$. The coset $\mathfrak{S}_{3} / \mathfrak{H}_{3}^{(1,1),(1,2)}$ has the three representatives

$$
\left(\begin{array}{lll}
1 & 2 & 3  \tag{3.64}\\
1 & 2 & 3
\end{array}\right), \quad\left(\begin{array}{lll}
1 & 2 & 3 \\
2 & 1 & 3
\end{array}\right), \quad\left(\begin{array}{lll}
1 & 2 & 3 \\
2 & 3 & 1
\end{array}\right),
$$

which appear in the upper index position of each of the ten lines of (3.54). It follows that (3.54) may be written under the form

$$
\begin{align*}
\sum_{k_{1}, k_{2}, k_{3}} & {\left[\sum_{\tau \in \mathfrak{S}_{4}^{3}} \sum_{\sigma \in \mathfrak{F}_{3}^{(1,1),(1,2)}} \delta_{i_{\tau(1), i_{\tau(2)}, i_{\tau(3)}}^{k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3}}} \mathscr{Y}_{x^{i_{\tau(4)}} y^{2}}-\right.}  \tag{3.65}\\
& \left.-\sum_{\sigma \in \mathfrak{S}_{4}^{2}} \sum_{\tau \in \mathfrak{F}_{3}^{(1,1),(1,2)}} \delta_{i_{\tau(1)}, i_{\tau(2)}}^{k_{\sigma(1)}, k_{\sigma(2)}} \mathscr{X}_{x^{i(3)} x^{i} \tau(4) y}^{k_{\sigma(3)}}\right] y_{k_{1}} y_{k_{2}, k_{3}} .
\end{align*}
$$

3.66. General complete expression of $\mathbf{Y}_{i_{1}, \ldots, i_{k}}$. As in the incomplete expression (3.39) of $\mathbf{Y}_{i_{1}, \ldots, i_{\kappa}}$, consider integers $1 \leqslant \lambda_{1}<\cdots<\lambda_{d} \leqslant \kappa$ and $\mu_{1} \geqslant 1, \ldots, \mu_{d} \geqslant 1$ satisfying $\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d} \leqslant \kappa+1$. By $\mathfrak{H}_{\mu_{1} \lambda_{1}+\cdots+\mathfrak{H}_{\mu_{d} \lambda_{d}}}$, we denote the subgroup of permutations $\tau \in \mathfrak{S}_{\mu_{1} \lambda_{1}+\cdots+\mathfrak{H}_{\mu_{d} \lambda_{d}}}$ that leave unchanged the general monomial (3.38), namely that satisfy

$$
\begin{align*}
& \prod_{1 \leqslant \nu_{1} \leqslant \mu_{1}} y_{k_{\sigma\left(1: \nu_{1}: 1\right)}, \ldots, k_{\sigma\left(1: \nu_{1}: \lambda_{1}\right)}} \cdots \prod_{1 \leqslant \nu_{d} \leqslant \mu_{d}} y_{k_{\sigma\left(d: \nu_{d}: 1\right)}, \ldots, k_{\sigma\left(d: \nu_{d}: \lambda_{d}\right)}}=  \tag{3.67}\\
& =\prod_{1 \leqslant \nu_{1} \leqslant \mu_{1}} y_{k_{1: \nu_{1}: 1}, \ldots, k_{1: \nu_{1}: \lambda_{1}}} \cdots \prod_{1 \leqslant \nu_{d} \leqslant \mu_{d}} y_{k_{d: \nu_{d}: 1}, \ldots, k_{d: \nu_{d}: \lambda_{d}} .}
\end{align*}
$$

The structure of this group may be described as follows. For every $e=$ $1, \ldots, d$, an arbitrary permutation $\sigma$ of the set (3.68)

which leaves unchanged the monomial

$$
\begin{equation*}
\prod_{1 \leqslant \nu_{e} \leqslant \mu_{e}} y_{k_{\sigma\left(e: \nu_{e}: 1\right)}, \ldots, k_{\sigma\left(e: \nu_{e}: \lambda_{e}\right)}}=\prod_{1 \leqslant \nu_{e} \leqslant \mu_{e}} y_{k_{e: \nu_{e}: 1}, \ldots, k_{e: \nu_{e}: \lambda_{e}}} \tag{3.69}
\end{equation*}
$$

uniquely decomposes as the composition of

- $\mu_{e}$ arbitrary permutations of the $\mu_{e}$ groups of $\lambda_{e}$ integers $\left\{e: \nu_{e}\right.$ : $\left.1, \ldots, e: \nu_{e}: \lambda_{e}\right\}$, of total cardinal $\left(\lambda_{e}!\right)^{\mu_{e}}$;
- an arbitrary permutation between these $\mu_{e}$ groups, of total cardinal $\mu_{e}$ !.


## Consequently

$$
\begin{equation*}
\operatorname{Card}\left(\mathfrak{H}_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}}^{\left(\mu_{1}, \lambda_{1}\right), \ldots,\left(\mu_{d}, \lambda_{d}\right)}\right)=\mu_{1}!\left(\lambda_{1}!\right)^{\mu_{1}} \cdots \mu_{d}!\left(\lambda_{d}!\right)^{\mu_{d}} . \tag{3.70}
\end{equation*}
$$

Finally, define the coset

$$
\begin{equation*}
\mathfrak{F}_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}}^{\left(\mu_{1}, \lambda_{1}\right), \ldots,\left(\mu_{d}, \lambda_{d}\right)}:=\mathfrak{S}_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}} / \mathfrak{H}_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}}^{\left(\mu_{1}, \lambda_{1}\right), \ldots,\left(\mu_{d}, \lambda_{d}\right)} \tag{3.71}
\end{equation*}
$$

with

$$
\begin{align*}
\operatorname{Card}\left(\mathfrak{F}_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}}^{\left(\mu_{1}, \lambda_{1}\right), \ldots,\left(\mu_{d}, \lambda_{d}\right)}\right) & =\frac{\operatorname{Card}\left(\mathfrak{S}_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}}\right)}{\operatorname{Card}\left(\mathfrak{H}_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}}^{\left(\mu_{1}, \lambda_{1}\right), \ldots,\left(\mu_{d}\right)}\right)}  \tag{3.72}\\
& =\frac{\left(\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}\right)!}{\mu_{1}!\left(\lambda_{1}!\right)^{\mu_{1}} \cdots \mu_{d}!\left(\lambda_{d}!\right)^{\mu_{d}}} .
\end{align*}
$$

In conclusion, by means of this formalism, we may write down the complete generalization of Theorem 2.24 to several independent variables.

Theorem 3.73. For every $\kappa \geqslant 1$ and for every choice of $\kappa$ indices $i_{1}, \ldots, i_{\kappa}$ in the set $\{1,2, \ldots, n\}$, the general expression of $\mathbf{Y}_{i_{1}, \ldots, i_{\kappa}}$ is as follows: (3.74)

$$
\begin{aligned}
& \mathbf{Y}_{i_{1}, \ldots, i_{\kappa}}=\mathscr{Y}_{x^{i_{1} \ldots x^{i_{\kappa}}}}+\sum_{d=1}^{\kappa+1} \sum_{1 \leqslant \lambda_{1}<\cdots<\lambda_{d} \leqslant \kappa} \sum_{\mu_{1} \geqslant 1, \ldots, \mu_{d} \geqslant 1} \sum_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d} \leqslant \kappa+1} \\
& \sum_{k_{1: 1: 1}, \ldots, k_{1: 1: \lambda_{1}}=1}^{n} \ldots \sum_{k_{1: \mu_{1}: 1, \ldots, k_{1: \mu_{1}: \lambda_{1}}=1}^{n} \ldots \ldots \sum_{k_{d: 1: 1}, \ldots, k_{d: 1: \lambda_{d}}=1}^{n} \ldots \sum_{k_{d: \mu_{d}: 1, \ldots, k_{d: \mu_{d}: \lambda}}=1}^{n}, ~}^{n}
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{\sigma \in \mathfrak{F}_{\left.\mu_{1} \lambda_{1}+\ldots+\lambda_{1}\right), \ldots,\left(\mu_{d} \lambda_{d}, \lambda_{d}\right)}} \sum_{\tau \in \mathfrak{S}_{\kappa}^{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}-1}} \\
& \left.\delta_{i_{\tau(1), \ldots, i^{2}}^{\left.\left.\left.\left.k_{\sigma(1: 1)}\right), \ldots, k_{\sigma(1)} \mu_{1}\right), \ldots, \mu_{\tau}: \lambda_{1}\right), \ldots, k_{\sigma\left(d: \mu_{d}\right.} \lambda_{d} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}-1\right)}} \frac{\partial^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+\mu_{1}+\cdots+\mu_{d}} \mathscr{X}^{k_{\sigma\left(d: \mu_{d}: \lambda_{d}\right)}}}{\partial x^{i_{\tau\left(\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}\right)} \cdots \partial x^{i_{\tau(k)}}(\partial y)^{\mu_{1}+\cdots+\mu_{d}-1}}}\right] \\
& \cdot \prod_{1 \leqslant \nu_{1} \leqslant \mu_{1}} y_{k_{1: \nu_{1}}: 1, \ldots, k_{1: \nu_{1}: \lambda_{1}}} \cdots \prod_{1 \leqslant \nu_{d} \leqslant \mu_{d}} y_{k_{d: \nu_{d}: 1}, \ldots, k_{d: \nu_{d}: \lambda_{d}}} .
\end{aligned}
$$

Since the fundamental monomials appearing in the last line of (3.74) just above are not independent of each other, this formula has still to be modified a little bit. We refer to Section 6 for details.
3.75. Deduction of a multivariate Faà di Bruno formula. Let $n \in \mathbb{N}$ with $n \geqslant 1$, let $x=\left(x^{1}, \ldots, x^{n}\right) \in \mathbb{K}^{n}$, let $g=g\left(x^{1}, \ldots, x^{n}\right)$ be a $\mathscr{C}^{\infty}$-smooth function from $\mathbb{K}^{n}$ to $\mathbb{K}$, let $y \in \mathbb{K}$ and let $f=f(y)$ be a $\mathscr{C}^{\infty}$ function from $\mathbb{K}$ to $\mathbb{K}$. The goal is to obtain an explicit formula for the partial derivatives of the composition $h:=f \circ g$, namely $h\left(x^{1}, \ldots, x^{n}\right):=f\left(g\left(x^{1}, \ldots, x^{n}\right)\right)$. For $\lambda \in \mathbb{N}$ with $\lambda \geqslant 1$ and for arbitrary indices $i_{1}, \ldots, i_{\lambda}=1, \ldots, n$, we shall abbreviate the partial derivative $\frac{\partial^{\lambda} g}{\partial x^{i_{1}} \ldots \partial x^{i_{\lambda}}}$ by $g_{i_{1}, \ldots, i_{\lambda}}$ and similarly for $h_{i_{1}, \ldots, i_{\lambda}}$. The derivative $\frac{d^{\lambda} f}{d y^{\lambda}}$ will be abbreviated by $f_{\lambda}$.

Appying the chain rule, we may compute:

$$
\begin{align*}
h_{i_{1}}= & f_{1}\left[g_{i_{1}}\right],  \tag{3.76}\\
h_{i_{1}, i_{2}}= & f_{2}\left[g_{i_{1}} g_{i_{2}}\right]+f_{1}\left[g_{i_{1}, i_{2}}\right], \\
h_{i_{1}, i_{2}, i_{3}}= & f_{3}\left[g_{i_{1}} g_{i_{2}} g_{i_{3}}\right]+f_{2}\left[g_{i_{1}} g_{i_{2}, i_{3}}+g_{i_{2}} g_{i_{1}, i_{3}}+g_{i_{3}} g_{i_{1}, i_{2}}\right]+f_{1}\left[g_{i_{1}, i_{2}, i_{3}}\right], \\
h_{i_{1}, i_{2}, i_{3}, i_{4}}= & f_{4}\left[g_{i_{1}} g_{i_{2}} g_{i_{3}} g_{i_{4}}\right]+f_{3}\left[g_{i_{2}} g_{i_{3}} g_{i_{1}, i_{4}}+g_{i_{3}} g_{i_{1}} g_{i_{2}, i_{4}}+g_{i_{1}} g_{i_{2}} g_{i_{3}, i_{4}}+\right. \\
& \left.+g_{i_{1}} g_{i_{4}} g_{i_{2}, i_{3}}+g_{i_{2}} g_{i_{4}} g_{i_{1}, i_{3}}+g_{i_{3}} g_{i_{4}} g_{i_{1}, i_{2}}\right]+ \\
& +f_{2}\left[g_{i_{1}, i_{2}} g_{i_{3}, i_{4}}+g_{i_{1}, i_{3}} g_{i_{2}, i_{4}}+g_{i_{1}, i_{4}} g_{i_{2}, i_{3}}\right]+ \\
& +f_{2}\left[g_{i_{1}} g_{i_{2}, i_{3}, i_{4}}+g_{i_{2}} g_{i_{1}, i_{3}, i_{4}}+g_{i_{3}} g_{i_{1}, i_{2}, i_{4}}+g_{i_{4}} g_{i_{1}, i_{2}, i_{3}}\right]+ \\
& +f_{1}\left[g_{i_{1}, i_{2}, i_{3}, i_{4}}\right] .
\end{align*}
$$

Introducing the derivations
(3.77)

$$
\begin{aligned}
& F_{i}^{2}:=\sum_{k_{1}=1}^{n} g_{k_{1}, i} \frac{\partial}{\partial g_{k_{1}}}+g_{i}\left(f_{2} \frac{\partial}{\partial f_{1}}\right), \\
& F_{i}^{3}:=\sum_{k_{1}=1}^{n} g_{k_{1}, i} \frac{\partial}{\partial g_{k_{1}}}+\sum_{k_{1}, k_{2}=1}^{n} g_{k_{1}, k_{2}, i} \frac{\partial}{\partial g_{k_{1}, k_{2}}}+g_{i}\left(f_{2} \frac{\partial}{\partial f_{1}}+f_{3} \frac{\partial}{\partial f_{2}}\right),
\end{aligned}
$$

$$
\begin{aligned}
& F_{i}^{\lambda}:=\sum_{k_{1}=1}^{n} g_{k_{1}, i} \frac{\partial}{\partial g_{k_{1}}}+\sum_{k_{1}, k_{2}=1}^{n} g_{k_{1}, k_{2}, i} \frac{\partial}{\partial g_{k_{1}, k_{2}}}+\cdots+ \\
&+\sum_{k_{1}, \ldots, k_{\lambda-1}=1}^{n} g_{k_{1}, \ldots, k_{\lambda-1}, i} \frac{\partial}{\partial g_{k_{1}, \ldots, k_{\lambda-1}}}+ \\
&+g_{i}\left(f_{2} \frac{\partial}{\partial f_{1}}+f_{3} \frac{\partial}{\partial f_{2}}+\cdots+f_{\lambda} \frac{\partial}{\partial f_{\lambda-1}}\right)
\end{aligned}
$$

we observe that the following induction relations hold:

$$
\begin{align*}
h_{i_{1}, i_{2}} & =F_{i_{2}}^{2}\left(h_{i_{1}}\right), \\
h_{i_{1}, i_{2}, i_{3}} & =F_{i_{3}}^{3}\left(h_{i_{1}, i_{2}}\right),  \tag{3.78}\\
\ldots \ldots & \cdots \cdots \cdots \cdots \\
h_{i_{1}, i_{2}, \ldots, i_{\lambda}}= & F_{i_{\lambda}}^{\lambda}\left(h_{i_{1}, i_{2}, \ldots, i_{\lambda-1}}\right) .
\end{align*}
$$

To obtain the explicit version of the Faà di Bruno in the case of several variables $\left(x^{1}, \ldots, x^{n}\right)$ and one variable $y$, it suffices to extract from the expression of $\mathbf{Y}_{i_{1}, \ldots, i_{\kappa}}$ provided by Theorem 3.73 only the terms corresponding to $\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}=\kappa$, dropping all the $\mathscr{X}$ terms. After some simplifications and after a translation by means of an elementary dictionary, we obtain a statement.

Theorem 3.79. For every integer $\kappa \geqslant 1$ and for every choice of indices $i_{1}, \ldots, i_{\kappa}$ in the set $\{1,2, \ldots, n\}$, the $\kappa$-th partial derivative of the composite function $h=h\left(x^{1}, \ldots, x^{n}\right)=f\left(g\left(x^{1}, \ldots, x^{n}\right)\right)$ with respect to the variables $x^{i_{1}}, \ldots, x^{i_{\kappa}}$ may be expressed as an explicit polynomial depending on the derivatives of $f$, on the partial derivatives of $g$ and having integer coefficients:
(3.80)

$$
\begin{aligned}
& \frac{\partial^{\kappa} h}{\partial x^{i_{1}} \cdots \partial x^{i_{\kappa}}}=\sum_{d=1}^{\kappa} \sum_{1 \leqslant \lambda_{1}<\cdots<\lambda_{d} \leqslant \kappa} \sum_{\mu_{1} \geqslant 1, \ldots, \mu_{d} \geqslant 1} \sum_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}=\kappa} \frac{d^{\mu_{1}+\cdots+\mu_{d}} f}{d y^{\mu_{1}+\cdots+\mu_{d}}} \\
& {\left[\begin{array}{c}
\sum_{\sigma \in \mathfrak{F}_{\hbar}^{\left(\mu_{1}, \lambda_{1}\right), \ldots,\left(\mu_{d}, \lambda_{d}\right)}} \prod_{1 \leqslant \nu_{1} \leqslant \mu_{1}} \frac{\partial^{\lambda_{1}} g}{\partial x^{i_{\sigma\left(1: \nu_{1}: 1\right)} \cdots \partial x^{i} i_{\sigma\left(1: \nu_{1}: \lambda_{1}\right)}}} \cdots \\
\cdots \\
\left.\prod_{1 \leqslant \nu_{d} \leqslant \mu_{d}} \frac{\partial^{\lambda_{d}} g}{\partial x^{i_{\sigma\left(d: \nu_{d}: 1\right)} \cdots \partial x^{i_{\sigma\left(d: \nu_{d}: \lambda_{d}\right)}}}}\right]
\end{array}\right]}
\end{aligned}
$$

In this formula, the coset $\mathfrak{F}_{\kappa}^{\left(\mu_{1}, \lambda_{1}\right), \ldots,\left(\mu_{d}, \lambda_{d}\right)}$ was defined in equation (3.71); we have made the identification:

$$
\begin{equation*}
\{1, \ldots, \kappa\} \equiv\left\{1: 1: 1, \ldots, 1: \mu_{1}: \lambda_{1}, \ldots \ldots, d: 1: 1, \ldots, d: \mu_{d}: \lambda_{d}\right\} \tag{3.81}
\end{equation*}
$$

and also, for the sake of clarity, we have restituted the complete (not abbreviated) notation for the (partial) derivatives of $f$ and of $g$.

We refer to Section 6 for the final writing of the above formula (3.80).

## §4. One independent variable and several dependent VARIABLES

4.1. Simplified adapted notations. Assume $n=1$ and $m \geqslant 1$, let $\kappa \in \mathbb{N}$ with $\kappa \geqslant 1$ and simply denote the jet variables by (instead of (1.2)):

$$
\begin{equation*}
\left(x, y^{j}, y_{1}^{j}, y_{2}^{j}, \ldots, y_{\kappa}^{j}\right) \in \mathscr{J}_{1, m}^{\kappa} . \tag{4.2}
\end{equation*}
$$

Instead of (1.30), denote the $\kappa$-th prolongation of a vector field by:

$$
\left\{\begin{align*}
\mathscr{L}^{(\kappa)}= & \mathscr{X} \frac{\partial}{\partial x}+\sum_{j=1}^{m} \mathscr{Y}^{j} \frac{\partial}{\partial y^{j}}+\sum_{j=1}^{m} \mathbf{Y}_{1}^{j} \frac{\partial}{\partial y_{1}^{j}}+\sum_{j=1}^{m} \mathbf{Y}_{2}^{j} \frac{\partial}{\partial y_{2}^{j}}+  \tag{4.3}\\
& +\cdots+\sum_{j=1}^{m} \mathbf{Y}_{\kappa}^{j} \frac{\partial}{\partial y_{\kappa}^{j}} .
\end{align*}\right.
$$

The induction formulas are:

$$
\left\{\begin{align*}
\mathbf{Y}_{1}^{j} & :=D^{1}\left(\mathscr{Y}^{j}\right)-D^{1}(\mathscr{X}) y_{1}^{j}  \tag{4.4}\\
\mathbf{Y}_{2}^{j} & :=D^{2}\left(\mathbf{Y}_{1}^{j}\right)-D^{1}(\mathscr{X}) y_{2}^{j} \\
\cdots \cdots \cdots & \cdots \\
\mathbf{Y}_{\lambda}^{j} & :=D^{\lambda}\left(\mathbf{Y}_{\lambda-1}^{j}\right)-D^{1}(\mathscr{X}) y_{\lambda}^{j}
\end{align*}\right.
$$

where the total differentiation operators $D^{\lambda}$ are denoted by (instead of (1.22)):

$$
\begin{equation*}
D^{\lambda}:=\frac{\partial}{\partial x}+\sum_{l=1}^{m} y_{1}^{l} \frac{\partial}{\partial y^{l}}+\sum_{l=1}^{m} y_{2}^{l} \frac{\partial}{\partial y_{1}^{l}}+\cdots+\sum_{l=1}^{m} y_{\lambda}^{l} \frac{\partial}{\partial y_{\lambda-1}^{l}} . \tag{4.5}
\end{equation*}
$$

Applying the definitions in the first two lines of (4.4), we compute, we simplify and we organize the results in a harmonious way, using in an essential way the Kronecker symbol. Here, the computations are more elementary than the computations of $\mathbf{Y}_{i_{1}}$ and of $\mathbf{Y}_{i_{1}, i_{2}}$ achieved thoroughly in the previous Section 3, so that we do not provide a Latex track of the details. Firstly and secondly:

$$
\left\{\begin{align*}
& \mathbf{Y}_{1}^{j}= \mathscr{Y}_{x}^{j}+\sum_{l_{1}=1}^{m}\left[\mathscr{Y}_{y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} \mathscr{X}_{x}\right] y_{1}^{l_{1}}+\sum_{l_{1}, l_{2}=1}^{m}\left[-\delta_{l_{1}}^{j} \mathscr{X}_{y^{l_{2}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}},  \tag{4.6}\\
& \mathbf{Y}_{x^{2}}^{j}+\sum_{l_{1}=1}^{j}\left[2 \mathscr{Y}_{x y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} \mathscr{X}_{x^{2}}\right] y_{1}^{l_{1}}+\sum_{l_{1}, l_{2}=1}^{m}\left[\mathscr{Y}_{y^{l_{1}} y^{l_{2}}}^{j}-\delta_{l_{1}}^{j} 2 \mathscr{X}_{x y^{l_{2}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}}+ \\
&+\sum_{l_{1}, l_{2}, l_{3}}\left[-\delta_{l_{1}}^{j} \mathscr{X}_{y^{l_{2}} y^{l_{3}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3}}+\sum_{l_{1}}\left[\mathscr{Y}_{y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} 2 \mathscr{X}_{x}\right] y_{2}^{l_{1}}+ \\
&+\sum_{l_{1}, l_{2}=1}^{m}\left[-\delta_{l_{1}}^{j} \mathscr{X}_{y^{l_{2}}}-\delta_{l_{2}}^{j} 2 \mathscr{X}_{y^{l_{1}}}\right] y_{1}^{l_{1}} y_{2}^{l_{2}} .
\end{align*}\right.
$$

Thirdly:

$$
\begin{align*}
\mathbf{Y}_{3}^{j}= & \mathscr{Y}_{x^{3}}^{j}+\sum_{l_{1}=1}^{m}\left[3 \mathscr{Y}_{x^{2} y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} \mathscr{X}_{x^{3}}\right] y_{1}^{l_{1}}+\sum_{l_{1}, l_{2}=1}^{m}\left[3 \mathscr{Y}_{x y^{l_{1}} y^{l_{2}}}^{j}-\delta_{l_{1}}^{j} 3 \mathscr{X}_{x^{2} y^{l_{2}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}}+  \tag{4.7}\\
& +\sum_{l_{1}, l_{2}, l_{3}}\left[\mathscr{Y}_{y^{l_{1}} y^{l_{2}} y^{l_{3}}}^{j}-\delta_{l_{1}}^{j} 3 \mathscr{X}_{x y^{l_{2}} y^{l_{3}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3}}+ \\
& +\sum_{l_{1}, l_{2}, l_{3}, l_{4}}\left[-\delta_{l_{1}}^{j} \mathscr{X}_{y^{l_{2}} y_{3} y^{l_{4}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3}} y_{1}^{l_{4}}+\sum_{l_{1}=1}^{m}\left[3 \mathscr{Y}_{x y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} 3 \mathscr{X}_{x^{2}}\right] y_{2}^{l_{1}}+
\end{align*}
$$

$$
\begin{aligned}
& +\sum_{l_{1}, l_{2}=1}^{m}\left[3 \mathscr{Y}_{y^{l_{1}} y^{l_{2}}}^{j}-\delta_{l_{1}}^{j} 3 \mathscr{X}_{x y^{l_{2}}}-\delta_{l_{2}}^{j} 6 \mathscr{X}_{x y^{l_{1}}}\right] y_{1}^{l_{1}} y_{2}^{l_{2}}+ \\
& +\sum_{l_{1}, l_{2} l_{3}=1}^{m}\left[-\delta_{l_{1}}^{j} 3 \mathscr{X}_{y^{l_{2}} y^{l_{3}}}-\delta_{l_{3}}^{j} 3 \mathscr{X}_{y^{l_{1}} y^{l_{2}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{2}^{l_{3}}+\sum_{l_{1}, l_{2}=1}^{m}\left[-\delta_{l_{3}}^{j} 3 \mathscr{X}_{y^{l_{2}}}\right] y_{2}^{l_{1}} y_{2}^{l_{2}}+ \\
& +\sum_{l_{1}=1}^{m}\left[\mathscr{Y}_{y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} 3 \mathscr{X}_{x}\right] y_{3}^{l_{1}}+\sum_{l_{1}, l_{2}=1}^{m}\left[-\delta_{l_{1}}^{j} \mathscr{X}_{y^{l_{2}}}-\delta_{l_{2}}^{j} 3 \mathscr{X}_{y^{l_{1}}}\right] y_{1}^{l_{1}} y_{3}^{l_{2}} .
\end{aligned}
$$

Fourthly:

$$
\begin{aligned}
\mathbf{Y}_{4}^{j}= & \mathscr{Y}_{x^{4}}^{j}+\sum_{l_{1}=1}^{m}\left[4 \mathscr{Y}_{x^{3} y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} \mathscr{X}_{x^{4}}\right] y_{1}^{l_{1}}+\sum_{l_{1}, l_{2}=1}^{m}\left[6 \mathscr{Y}_{x^{2} y^{l_{1}} y^{l_{2}}}^{j}-\delta_{l_{1}}^{j} 4 \mathscr{X}_{x^{3} y^{l_{2}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}}+ \\
& +\sum_{l_{1}, l_{2}, l_{3}=1}^{m}\left[4 \mathscr{Y}_{x y^{l_{1}} y^{l_{2}} y^{l_{3}}}^{j}-\delta_{l_{1}}^{j} 6 \mathscr{X}_{x^{2} y^{l_{2}} y^{l_{3}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3}}+ \\
& +\sum_{l_{1}, l_{2}, l_{3}, l_{4}=1}^{m}\left[\mathscr{Y}_{x y^{l_{1}} y^{l_{2}} y^{l_{3}} y^{l_{4}}}^{j}-\delta_{l_{1}}^{j} 4 \mathscr{X}_{x y^{l_{2}} y^{l_{3}} y^{l_{4}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3}} y_{1}^{l_{4}}+ \\
& +\sum_{l_{1}, l_{2}, l_{3}, l_{4}, l_{5}=1}^{m}\left[-\delta_{l_{1}}^{j} \mathscr{X}_{y^{l_{2}} y^{l_{3}} y^{l_{4} y^{l_{5}}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3}} y_{1}^{l_{4}} y_{1}^{l_{5}}+\sum_{l_{1}=1}^{m}\left[6 \mathscr{Y}_{x^{2} y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} 4 \mathscr{X}_{x^{3}}\right] y_{2}^{l_{1}}+
\end{aligned}
$$

$$
\begin{align*}
& +\sum_{l_{1}, l_{2}=1}^{m}\left[12 \mathscr{Y}_{x y^{l_{1} y^{l_{2}}}}^{j}-\delta_{l_{1}}^{j} 6 \mathscr{X}_{x^{2} y^{l_{2}}}-\delta_{l_{2}}^{j} 12 \mathscr{X}_{x^{2} y^{l_{1}}}\right] y_{1}^{l_{1}} y_{2}^{l_{2}}+  \tag{4.8}\\
& +\sum_{l_{1}, l_{2}, l_{3}=1}^{m}\left[6 \mathscr{Y}_{y^{l_{1}} y^{l_{2}} y^{l_{3}}}^{j}-\delta_{l_{1}}^{j} 12 \mathscr{X}_{x y^{l_{2}} y^{l_{3}}}-\delta_{l_{3}}^{j} 12 \mathscr{X}_{x y^{l_{1} y^{l_{2}}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{2}^{l_{3}}+ \\
& +\sum_{l_{1}, l_{2}, l_{3}, l_{4}=1}^{m}\left[-\delta_{l_{1}}^{j} 6 \mathscr{X}_{y^{l_{2}} y^{l_{3} y^{l_{4}}}}-\delta_{l_{4}}^{j} 4 \mathscr{X}_{y^{l_{1} y^{2} y^{l_{3}}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3}} y_{2}^{l_{4}}+
\end{align*}
$$

$$
\begin{aligned}
& +\sum_{l_{1}, l_{2}=1}^{m}\left[3 \mathscr{Y}_{y^{l_{1}} y^{l_{2}}}^{j}-\delta_{l_{1}}^{j} 12 \mathscr{X}_{x y^{l_{2}}}\right] y_{2}^{l_{1}} y_{2}^{l_{2}}+ \\
& +\sum_{l_{1}, l_{2}, l_{3}=1}^{m}\left[-\delta_{l_{1}}^{j} 3 \mathscr{X}_{y^{l_{2}} y^{l_{3}}}-\delta_{l_{2}}^{j} 12 \mathscr{X}_{y^{l_{1}} y^{l_{3}}}\right] y_{1}^{l_{1}} y_{2}^{l_{2}} y_{2}^{l_{3}}+\sum_{l_{1}=1}^{m}\left[4 \mathscr{Y}_{x y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} 6 \mathscr{X}_{x^{2}}\right] y_{3}^{l_{1}}+ \\
& +\sum_{l_{1}, l_{2}=1}^{m}\left[4 \mathscr{Y}_{y^{l_{1}} y^{l_{2}}}^{j}-\delta_{l_{1}}^{j} 4 \mathscr{X}_{x y^{l_{2}}}-\delta_{l_{2}}^{j} 12 \mathscr{X}_{x y^{l_{1}}}\right] y_{1}^{l_{1}} y_{3}^{l_{2}}+ \\
& +\sum_{l_{1}, l_{2}, l_{3}=1}^{m}\left[-\delta_{l_{1}}^{j} 4 \mathscr{X}_{y^{l_{2}} y^{l_{3}}}-\delta_{l_{3}}^{j} 6 \mathscr{X}_{y^{l_{1}} y^{l_{2}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{3}^{l_{3}}+ \\
& +\sum_{l_{1}, l_{2}=1}^{m}\left[-\delta_{l_{1}}^{j} 4 \mathscr{X}_{y^{l_{2}}}-\delta_{l_{2}}^{j} 6 \mathscr{X}_{y^{l_{1}}}\right] y_{2}^{l_{1}} y_{3}^{l_{2}}+ \\
& +\sum_{l_{1}=1}^{m}\left[\mathscr{Y}_{y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} 4 \mathscr{X}_{x}\right] y_{4}^{l_{1}}+\sum_{l_{1}, l_{2}=1}^{m}\left[-\delta_{l_{1}}^{j} \mathscr{X}_{y^{l_{2}}}-\delta_{l_{2}}^{j} 4 \mathscr{X}_{y^{l_{1}}}\right] y_{1}^{l_{1}} y_{4}^{l_{2}} .
\end{aligned}
$$

4.9. Inductive elaboration of the general formula. Now we compare the formula (2.9) for $\mathbf{Y}_{4}$ with the above formula (4.8) for $\mathbf{Y}_{4}^{j}$. The goal is to find the rules of transformation and of development by inspecting several instances, in order to devise how to transform and to develope the formula (2.25) to several dependent variables.

First of all, we have to develope the general monomial $\left(y_{\lambda_{1}}\right)^{\mu_{1}} \cdots\left(y_{\lambda_{d}}\right)^{\mu_{d}}$. In every monomial present in the expressions of $\mathbf{Y}_{1}^{j}$, of $\mathbf{Y}_{2}^{j}$, of $\mathbf{Y}_{3}^{j}$ and of $\mathbf{Y}_{4}^{j}$ above, we see that the number $\alpha$ of indices $l_{\beta}$ appearing in all the sums $\sum_{l_{1}, \ldots, l_{\alpha}=1}^{m}$ is exactly equal to $\mu_{1}+\cdots+\mu_{d}$. To denote these $\mu_{1}+\cdots+\mu_{d}$ indices $l_{\beta}$, we shall use the notation:

inspired by Convention 3.33. With such a choice of notation, we may avoid long subscripts in the indices $l_{\beta}$, like $l_{\mu_{1}+\cdots+\mu_{d}}$. It follows that the development of the general monomial $\left(y_{\lambda_{1}}\right)^{\mu_{1}} \cdots\left(y_{\lambda_{d}}\right)^{\mu_{d}}$ to several dependent variables yields $m^{\mu_{1}+\cdots+\mu_{d}}$ possible choices:

$$
\begin{equation*}
\prod_{1 \leqslant \nu_{1} \leqslant \mu_{1}} y_{\lambda_{1}}^{l_{1: \nu_{1}}} \cdots \cdots \prod_{1 \leqslant \nu_{d} \leqslant \mu_{d}} y_{\lambda_{d}}^{l_{d: \nu_{d}}} \tag{4.11}
\end{equation*}
$$

where the indices $l_{1: 1}, \ldots, l_{1: \mu_{1}}, \ldots, l_{d: 1}, \ldots, l_{d: \mu_{d}}$ take their values in the set $\{1,2, \ldots, m\}$. Consequently, the general expression of $\mathbf{Y}_{\kappa}^{j}$ must be of the
form:

$$
\begin{gather*}
\mathbf{Y}_{\kappa}^{j}=\mathscr{Y}_{x^{\kappa}}^{j}+\sum_{d=1}^{\kappa+1} \sum_{1 \leqslant \lambda_{1}<\cdots<\lambda_{d} \leqslant \kappa} \sum_{\mu_{1} \geqslant 1, \ldots, \mu_{d} \geqslant 1} \sum_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d} \leqslant \kappa+1} \\
\sum_{l_{1: 1}=1}^{m} \cdots \sum_{l_{1: \mu_{1}=1}^{m}}^{m} \cdots \cdot \sum_{l_{d: 1}=1}^{m} \cdots \sum_{l_{l_{d: \mu_{d}}=1}^{m}}[?]  \tag{4.12}\\
\prod_{1 \leqslant \nu_{1} \leqslant \mu_{1}} y_{\lambda_{1}}^{l_{1: \nu_{1}}} \cdots \cdots \prod_{1 \leqslant \nu_{d} \leqslant \mu_{d}} y_{\lambda_{d}}^{l_{d: \nu_{d}}},
\end{gather*}
$$

where the term in brackets [?] is still unknown. To determine it, let us examine a few instances.

According to (4.8) (fourth line), the term $\left[6 \mathscr{Y}_{x^{2} y}-4 \mathscr{X}_{x^{3}}\right] y_{2}$ of $\mathbf{Y}_{4}$ developes as $\sum_{l_{1}=1}^{m}\left[6 \mathscr{Y}_{x^{2} y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} 4 \mathscr{X}_{x^{3}}\right] y_{2}^{l_{1}}$ in $\mathbf{Y}_{4}^{j}$. Here, $6 \mathscr{Y}_{x^{2} y}$ just becomes $6 \mathscr{Y}_{x^{2} y^{l_{1}}}^{j}$. Thus, we suspect that the term $\frac{\kappa \cdots\left(\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+1\right)}{\left(\lambda_{1}!\right)^{\mu_{1}} \mu_{1}!\cdots\left(\lambda_{d}!\right)^{\mu_{d}} \mu_{d}!}$. $\mathscr{Y}_{x^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}} y^{\mu_{1}+\cdots+\mu_{d}}}$ of the second line of (2.25) should simply be developed as

$$
\begin{align*}
& \frac{\kappa(\kappa-1) \cdots\left(\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+1\right)}{\left(\lambda_{1}!\right)^{\mu_{1}} \mu_{1}!\cdots\left(\lambda_{d}!\right)^{\mu_{d}} \mu_{d}!}  \tag{4.13}\\
& \quad \cdot \frac{\partial^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+\mu_{1}+\cdots+\mu_{d} \mathscr{Y}^{j}}}{(\partial x)^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}} \partial y^{l_{1: 1}} \cdots \partial y^{l_{1: \mu_{1}}} \cdots \partial y^{l_{d: 1}} \cdots \partial y^{l_{d: \mu_{d}}}}
\end{align*}
$$

This rule is confirmed by inspecting all the other monomials of $\mathbf{Y}_{1}^{j}$, of $\mathbf{Y}_{2}^{j}$, of $\mathbf{Y}_{3}^{j}$ and of $\mathbf{Y}_{4}^{j}$.

It remains to determine how we must develope the term in $\mathscr{X}$ appearing in the last two lines of (2.25). To begin with, let us rewrite in advance this term in the slightly different shape, emphasizing a factorization:

$$
\begin{equation*}
\frac{\kappa \cdots\left(\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+2\right)}{\left(\lambda_{1}!\right)^{\mu_{1}} \mu_{1}!\cdots\left(\lambda_{d}!\right)^{\mu_{d}} \mu_{d}!}\left[\left(\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}\right) \mathscr{X}_{x^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+1} y^{\mu_{1}+\cdots+\mu_{d}-1}}\right] . \tag{4.14}
\end{equation*}
$$

Then we examine four instances extracted from the complete expression of $\mathbf{Y}_{4}^{j}$ :

$$
\left\{\begin{array}{l}
\sum_{l_{1}, l_{2}, l_{3}=1}^{m}\left[4 \mathscr{Y}_{x y^{l_{1}} y^{l_{2}} y^{l_{3}}}^{j}-\delta_{l_{1}}^{j} 6 \mathscr{X}_{x^{2} y^{l_{2}} y^{l_{3}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3}},  \tag{4.15}\\
\sum_{l_{1}, l_{2}=1}^{m}\left[12 \mathscr{Y}_{x y^{l_{1} y^{l_{2}}}}^{j}-\delta_{l_{1}}^{j} 6 \mathscr{X}_{x^{2} y^{l_{2}}}-\delta_{l_{2}}^{j} 12 \mathscr{X}_{x^{2} y^{l_{1}}}\right] y_{1}^{l_{1}} y_{2}^{l_{2}}, \\
\sum_{l_{1}, l_{2}, l_{3}, l_{4}=1}^{m}\left[-\delta_{l_{1}}^{j} 6 \mathscr{X}_{y^{l_{2}} y^{l_{3} y^{l_{4}}}}-\delta_{l_{4}}^{j} 4 \mathscr{X}_{y^{1_{1}} y^{l_{2}} y^{l_{3}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3} y_{2}^{l_{4}},} \\
\sum_{l_{1}, l_{2}, l_{3}=1}^{m}\left[-\delta_{l_{1}}^{j} 4 \mathscr{X}_{y^{l_{2}} y_{3}}-\delta_{l_{3}}^{j} 6 \mathscr{X}_{y_{1} y^{l_{2}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{3}^{l_{3}},
\end{array}\right.
$$

and we compare them to the corresponding terms of $\mathbf{Y}_{4}$ :

$$
\left\{\begin{array}{l}
{\left[4 \mathscr{Y}_{x y^{3}}-6 \mathscr{X}_{x^{2} y^{2}}\right]\left(y_{1}\right)^{3},}  \tag{4.16}\\
{\left[12 \mathscr{Y}_{x y^{2}}-18 \mathscr{X}_{x^{2} y}\right] y_{1} y_{2},} \\
{\left[-10 \mathscr{X}_{y^{3}}\right]\left(y_{1}\right)^{3} y_{2},} \\
{\left[-10 \mathscr{X}_{y^{2}}\right]\left(y_{1}\right)^{2} y_{3} .}
\end{array}\right.
$$

In the development from (4.16) to (4.15), we see that the four integers just before $\mathscr{X}$, namely $6=6,18=6+12,10=6+4$ and $10=4+6$, are split in a certain manner. Also, a single Kronecker symbol $\delta_{l_{\alpha}}^{j}$ is added as a factor. What are the rules?

In the second splitting $18=6+12$, we see that the relative weight of 6 and of 12 is the same as the relative weight of 1 and 2 in the splitting $3=1+2$ issued from the lower indices of the corresponding monomial $y_{1}^{l_{1}} y_{2}^{l_{2}}$. Similarly, in the third splitting $10=6+4$, the relative weight of 6 and of 4 is the same as the relative weight of $1+1+1$ and of 2 issued from the lower indices of the corresponding monomial $y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3}} y_{2}^{l_{4}}$. This rule may be confirmed by inspecting all the other monomials of $\mathbf{Y}_{2}, \mathbf{Y}_{2}^{j}$, of $\mathbf{Y}_{3}, \mathbf{Y}_{3}^{j}$ and of $\mathbf{Y}_{4}, \mathbf{Y}_{4}^{j}$. For a general $\kappa \geqslant 1$, the splitting of integers just amounts to decompose the sum appearing inside the brackets of (4.14) as $\mu_{1} \lambda_{1}, \mu_{2} \lambda_{2}, \ldots, \mu_{d} \lambda_{d}$. In fact, when we wrote (4.14), we emphasized in advance the decomposable factor $\left(\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}\right)$.

Next, we have to determine what is the subscript $\alpha$ in the Kronecker symbol $\delta_{l_{\alpha}}^{j}$. We claim that in the four instances (4.15), the subscript $\alpha$ is intrinsically related to weight splitting. Indeed, recall that in the second line of (4.15), the number 6 of the splitting $18=6+12$ is related to the number 1 in the splitting $3=1+2$ of the lower indices of the monomial $y_{1}^{l_{1}} y_{2}^{l_{2}}$. It follows that the index $l_{\alpha}$ must be the index $l_{1}$ of the monomial $y_{1}^{l_{1}}$. Similarly, also in the second line of (4.15), the number 12 of the splitting $18=6+12$
being related to the number 2 in the splitting $3=1+2$ of the lower indices of the monomial $y_{1}^{l_{1}} y_{2}^{l_{2}}$, it follows that the index $l_{\alpha}$ attached to the second $\mathscr{X}$ term must be the index $l_{2}$ of the monomial $y_{2}^{l_{2}}$.

This rule is still ambiguous. Indeed, let us examine the third line of (4.15). We have the splitting $10=6+4$, homologous to the splitting of relative weights $5=(1+1+1)+2$ in the monomial $y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3}} y_{2}^{l_{4}}$. Of course, it is clear that we must choose the index $l_{4}$ for the Kronecker symbol associated to the second $\mathscr{X}$ term $-4 \mathscr{X}_{y^{3}}$, thus obtaining $-\delta_{l_{4}}^{j} 4 \mathscr{X}_{y^{l_{1}} y^{l^{2} y^{l_{3}}}}$. However, since the monomial $y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3}}$ has three indices $l_{1}, l_{2}$ and $l_{3}$, there arises a question: what index $l_{\alpha}$ must we choose for the Kronecker symbol $\delta_{l_{\alpha}}^{j}$ attached to the first $\mathscr{X}$ term $6 \mathscr{X}_{y^{3}}$ : the index $l_{1}$, the index $l_{2}$ or the index $l_{3}$ ?

The answer is simple: any of the three indices $l_{1}, l_{2}$ or $l_{3}$ works. Indeed, since the monomial $y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3}}$ is symmetric with respect to all permutations of the set of three indices $\left\{l_{1}, l_{2}, l_{3}\right\}$, we have

$$
\begin{align*}
\sum_{l_{1}, l_{2}, l_{3}, l_{4}=1}^{m}\left[-\delta_{l_{1}}^{j} 6 \mathscr{X}_{y^{l_{2}} y^{l_{3} y_{4}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3}} y_{2}^{l_{4}} & =\sum_{l_{1}, l_{2}, l_{3}, l_{4}=1}^{m}\left[-\delta_{l_{2}}^{j} 6 \mathscr{X}_{y^{l_{1}} y^{l_{3}} y^{l_{4}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3}} y_{2}^{l_{4}}=  \tag{4.17}\\
& =\sum_{l_{1}, l_{2}, l_{3}, l_{4}=1}^{m}\left[-\delta_{l_{3}}^{j} 6 \mathscr{X}_{y^{l_{1}} y^{l_{2}} y^{l_{4}}}\right] y_{1}^{l_{1}} y_{1}^{l_{2}} y_{1}^{l_{3}} y_{2}^{l_{4}} .
\end{align*}
$$

In fact, we have systematically used such symmetries during the intermediate computations (not exposed here) which we achieved manually to obtain the final expressions of $\mathbf{Y}_{1}^{j}$, of $\mathbf{Y}_{2}^{j}$, of $\mathbf{Y}_{3}^{j}$ and of $\mathbf{Y}_{4}^{j}$. To fix ideas, we have always choosen the first index. Here, the first index is $l_{1}$; in the first sum of line 9 of (4.8), the first index $l_{\alpha}$ for the second weight 12 is $l_{2}$.

This rule may be confirmed by inspecting all the monomials of $\mathbf{Y}_{2}^{j}$, of $\mathbf{Y}_{3}^{j}$, of $\mathbf{Y}_{4}^{j}$ (and also of $\mathbf{Y}_{5}^{j}$, which we have computed in a manuscript, but not copied in this Latex file).

From these considerations, we deduce that for the general formula, the weight decomposition is simply $\mu_{1} \lambda_{1}, \ldots, \mu_{d} \lambda_{d}$ and that the Kronecker symbol $\delta_{\alpha}^{j}$ is intrinsically associated to the weights. In conclusion, building on inductive reasonings, we have formulated the following statement.

Theorem 4.18. For one independent variable $x$, for several dependent variables $\left(y^{1}, \ldots, y^{m}\right)$ and for $\kappa \geqslant 1$, the general expression of the coefficient
$\mathbf{Y}_{\kappa}^{j}$ of the prolongation (4.3) of a vector field is:
(4.19)

$$
\begin{aligned}
& \mathbf{Y}_{\kappa}^{j}=\mathscr{Y}_{x^{\kappa}}^{j}+\sum_{d=1}^{\kappa+1} \sum_{1 \leqslant \lambda_{1}<\cdots<\lambda_{d} \leqslant \kappa} \sum_{\mu_{1} \geqslant 1, \ldots, \mu_{d} \geqslant 1} \sum_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d} \leqslant \kappa+1} \\
& \sum_{l_{1: 1}=1}^{m} \cdots \sum_{l_{1: \mu_{1}}=1}^{m} \cdots \cdots \sum_{l_{d: 1}=1}^{m} \cdots \sum_{l_{d: \mu_{d}}=1}^{m} \frac{\kappa(\kappa-1) \cdots\left(\kappa-\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}+2\right)}{\left(\lambda_{1}!\right)^{\mu_{1}} \mu_{1}!\cdots\left(\lambda_{d}!\right)^{\mu_{d}} \mu_{d}!} \\
& {\left[\left(\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+1\right) \frac{\partial^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+\mu_{1}+\cdots+\mu_{d} \not \mathscr{Y}^{j}}}{(\partial x)^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}} \partial y^{l_{1: 1}} \cdots \partial y^{l_{1: \mu}} \cdots \partial y^{l_{d: 1}} \cdots \partial y^{l_{d: \mu_{d}}}}-\right]} \\
& -\delta_{l_{1: 1}}^{j} \mu_{1} \lambda_{1} \frac{\partial^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+\mu_{1}+\cdots+\mu_{d}} \mathscr{X}}{(\partial x)^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+1} \widehat{\partial y^{l_{1: 1}} \cdots \partial y^{l_{1: \mu}} \cdots \partial y^{l_{d: 1}} \cdots \partial y^{l_{d: \mu_{d}}}}--~-~-~} \\
& \text { - } \cdots \text { - } \\
& -\delta_{l_{d: 1}}^{j} \mu_{d} \lambda_{d} \frac{\partial^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+\mu_{1}+\cdots+\mu_{d}} \mathscr{X}}{(\partial x)^{\kappa-\mu_{1} \lambda_{1}-\cdots-\mu_{d} \lambda_{d}+1} \partial y^{l_{1: 1}} \cdots \partial y^{l_{1: \mu_{1}}} \cdots \widehat{\partial y^{l_{d: 1}}} \cdots \partial y^{l_{d: \mu_{d}}}} \\
& \cdot \prod_{1 \leqslant \nu_{1} \leqslant \mu_{1}} y_{\lambda_{1}}^{l_{1: \nu_{1}}} \cdots \cdots \prod_{1 \leqslant \nu_{d} \leqslant \mu_{d}} y_{\lambda_{d}}^{l_{d: \nu_{d}}} .
\end{aligned}
$$

Here, the notation $\widehat{\partial y^{l}}$ means that the partial derivative is dropped.

Since the fundamental monomials appearing in the last line of (4.19) just above are not independent of each other, this formula has still to be modified a little bit. We refer to Section 6 for details.
4.20. Deduction of a multivariate Faà di Bruno formula. Let $m \in \mathbb{N}$ with $m \geqslant 1$, let $y=\left(y^{1}, \ldots, y^{m}\right) \in \mathbb{K}^{m}$, let $f=f\left(y^{1}, \ldots, y^{m}\right)$ be a $\mathscr{C}^{\infty}$-smooth function from $\mathbb{K}^{m}$ to $\mathbb{K}$, let $x \in \mathbb{K}$ and let $g^{1}=g^{1}(x), \ldots, g^{m}=g^{m}(x)$ be $\mathscr{C}^{\infty}$ functions from $\mathbb{K}$ to $\mathbb{K}$. The goal is to obtain an explicit formula for the derivatives, with respect to $x$, of the composition $h:=f \circ g$, namely $h(x):=f\left(g^{1}(x), \ldots, g^{m}(x)\right)$. For $\lambda \in \mathbb{N}$ with $\lambda \geqslant 1$, and for $j=1, \ldots, m$, we shall abbreviate the derivative $\frac{d^{\lambda} g^{j}}{d x^{\lambda}}$ by $g_{\lambda}^{j}$ and similarly for $h_{\lambda}$. The partial derivatives $\frac{\partial^{\lambda} f}{\partial y^{l_{1}} \ldots \partial y^{l_{\lambda}}}$ will be abbreviated by $f_{l_{1}, \ldots, l_{\lambda}}$.

Appying the chain rule, we may compute:

$$
\begin{align*}
& h_{1}= \sum_{l_{1}=1}^{m} f_{l_{1}} g_{1}^{l_{1}},  \tag{4.21}\\
& h_{2}= \sum_{l_{1}, l_{2}=1}^{m} f_{l_{1}, l_{2}} g_{1}^{l_{1}} g_{1}^{l_{2}}+\sum_{l_{1}=1}^{m} f_{l_{1}} g_{2}^{l_{1}}, \\
& h_{3}= \sum_{l_{1}, l_{2}, l_{3}=1}^{m} f_{l_{1}, l_{2}, l_{3}} g_{1}^{l_{1}} g_{1}^{l_{2}} g_{1}^{l_{3}}+3 \sum_{l_{1}, l_{2}=1}^{m} f_{l_{1}, l_{2}} g_{1}^{l_{1}} g_{2}^{l_{2}}+\sum_{l_{1}=1}^{m} f_{l_{1}} g_{3}^{l_{1}}, \\
& h_{4}= \sum_{l_{1}, l_{2}, l_{3}, l_{4}=1}^{m} f_{l_{1}, l_{2}, l_{3}, l_{4}} g_{1}^{l_{1}} g_{1}^{l_{2}} g_{1}^{l_{3}} g_{1}^{l_{4}}+6 \sum_{l_{1}, l_{2}, l_{3}=1}^{m} f_{l_{1}, l_{2}, l_{3}} g_{1}^{l_{1}} g_{1}^{l_{2}} g_{2}^{l_{3}}+ \\
&+3 \sum_{l_{1}, l_{2}=1}^{m} f_{l_{1}, l_{2}} g_{2}^{l_{1}} g_{2}^{l_{2}}+4 \sum_{l_{1}, l_{2}=1}^{m} f_{l_{1}, l_{2}} g_{1}^{l_{1}} g_{3}^{l_{2}}+\sum_{l_{1}=1}^{m} f_{l_{1}} g_{4}^{l_{1}}, \\
& h_{5}= \sum_{l_{1}, l_{2}, l_{3}, l_{4}, l_{5}=1}^{m} f_{l_{1}, l_{2}, l_{3}, l_{4}, l_{5}}^{g_{1}^{l_{1}} g_{1}^{l_{2}} g_{1}^{l_{3}} g_{1}^{l_{4}} g_{1}^{l_{5}}+10} \sum_{l_{1}, l_{2}, l_{3}, l_{4}=1}^{m} f_{l_{1}, l_{2}, l_{3}, l_{4}} g_{1}^{l_{1}} g_{1}^{l_{2}} g_{1}^{l_{3}} g_{2}^{l_{4}}+ \\
&+15 \sum_{l_{1}, l_{2}, l_{3}=1}^{m} f_{l_{1}, l_{2}, l_{3}} g_{1}^{l_{1}} g_{2}^{l_{2}} g_{2}^{l_{3}}+10 \\
& \sum_{l_{1}, l_{2}, l_{3}=1}^{m} f_{l_{1}, l_{2}, l_{3}}^{l_{1}} g_{1}^{l_{2}} g_{1}^{l_{3}} g_{3}+ \\
&+10 \sum_{l_{1}, l_{2}=1}^{m} f_{l_{1}, l_{2}} g_{2}^{l_{1}} g_{3}^{l_{2}}+5 \sum_{l_{1}, l_{2}=1}^{m} f_{l_{1}, l_{2}}^{l_{1}} g_{1}^{l_{2}} g_{4}^{l_{2}}+\sum_{l_{1}=1}^{m} f_{l_{1}} g_{5}^{l_{1}} .
\end{align*}
$$

Introducing the derivations

$$
\begin{align*}
& F^{2}:=\sum_{l_{1}=1}^{m} g_{2}^{l_{1}} \frac{\partial}{\partial g_{1}^{l_{1}}}+\sum_{l_{1}=1}^{m} g_{1}^{l_{1}}\left(\sum_{l_{2}=1}^{m} f_{l_{1}, l_{2}} \frac{\partial}{\partial f_{l_{2}}}\right)  \tag{4.22}\\
& F^{3}:=\sum_{l_{1}=1}^{m} g_{2}^{l_{1}} \frac{\partial}{\partial g_{1}^{l_{1}}}+\sum_{l_{1}=1}^{m} g_{3}^{l_{1}} \frac{\partial}{\partial g_{2}^{l_{1}}}+\sum_{l_{1}=1}^{m} g_{1}^{l_{1}}\left(\sum_{l_{2}=1}^{m} f_{l_{1}, l_{2}} \frac{\partial}{\partial f_{l_{2}}}+\sum_{l_{2}, l_{3}=1}^{m} f_{l_{1}, l_{2}, l_{3}} \frac{\partial}{\partial f_{l_{2}, l_{3}}}\right),
\end{align*}
$$

$$
F^{\lambda}:=\sum_{l_{1}=1}^{m} g_{2}^{l_{1}} \frac{\partial}{\partial g_{1}^{l_{1}}}+\sum_{l_{1}=1}^{m} g_{3}^{l_{1}} \frac{\partial}{\partial g_{2}^{l_{1}}}+\cdots+\sum_{l_{1}=1}^{m} g_{\lambda}^{l_{1}} \frac{\partial}{\partial g_{\lambda-1}^{l_{1}}}+
$$

$$
+\sum_{l_{1}=1}^{m} g_{1}^{l_{1}}\left(\sum_{l_{2}=1}^{m} f_{l_{1}, l_{2}} \frac{\partial}{\partial f_{l_{2}}}+\sum_{l_{2}, l_{3}=1}^{m} f_{l_{1}, l_{2}, l_{3}} \frac{\partial}{\partial f_{l_{2}, l_{3}}}+\cdots+\sum_{l_{2}, \ldots, l_{\lambda}=1}^{m} f_{l_{1}, l_{2}, \ldots, l_{\lambda}} \frac{\partial}{\partial f_{l_{2}, \ldots, l_{\lambda}}}\right)
$$

we observe that the following induction relations hold:

$$
\begin{align*}
h_{2}= & F^{2}\left(h_{1}\right), \\
h_{3} & =F^{3}\left(h_{2}\right),  \tag{4.23}\\
\cdots & \quad \cdots \cdots \cdots \\
h_{\lambda}= & F^{\lambda}\left(h_{\lambda-1}\right) .
\end{align*}
$$

To obtain the explicit version of the Faà di Bruno in the case of one variable $x$ and several variables $\left(y^{1}, \ldots, y^{m}\right)$, it suffices to extract from the expression of $\mathbf{Y}_{\kappa}^{j}$ provided by Theorem 4.18 only the terms corresponding to $\mu_{1} \lambda_{1}+$ $\cdots+\mu_{d} \lambda_{d}=\kappa$, dropping all the $\mathscr{X}$ terms. After some simplifications and after a translation by means of an elementary dictionary, we may formulate a statement.

Theorem 4.24. For every integer $\kappa \geqslant 1$, the $\kappa$-th partial derivative of the composite function $h=h(x)=f\left(g^{1}(x), \ldots, g^{m}(x)\right)$ with respect to $x$ may be expressed as an explicit polynomial depending on the partial derivatives of $f$, on the derivatives of $g$ and having integer coefficients:
(4.25)

$$
\begin{aligned}
& \frac{d^{\kappa} h}{d x^{\kappa}}= \sum_{d=1}^{\kappa} \sum_{1 \leqslant \lambda_{1}<\cdots<\lambda_{d} \leqslant \kappa} \sum_{\mu_{1} \geqslant 1, \ldots, \mu_{d} \geqslant 1} \sum_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}=\kappa} \frac{\kappa!}{\left(\lambda_{1}!\right)^{\mu_{1}} \mu_{1}!\cdots\left(\lambda_{d}!\right)^{\mu_{d}} \mu_{d}!} \\
& \sum_{l_{1: 1}, \ldots, l_{1: \mu_{1}}=1}^{m} \cdots \sum_{l_{d: 1}, \ldots, l_{d: \mu_{d}}=1}^{m} \\
& \frac{\partial^{\mu_{1}+\cdots+\mu_{d}} f}{\partial y^{l_{1: 1}} \cdots \partial y^{l_{1: \mu_{1}}} \cdots \partial y^{l_{d: 1}} \cdots \partial y^{l_{d: \mu_{d}}}} \prod_{1 \leqslant \nu_{1} \leqslant \mu_{1}} \frac{d^{\lambda_{1}} g^{l_{1: \nu_{1}}}}{d x^{\lambda_{1}}} \cdots \prod_{1 \leqslant \nu_{d} \leqslant \mu_{d}} \frac{d^{\lambda_{d}} g^{l_{d: \nu_{d}}}}{d x^{\lambda_{d}}} .
\end{aligned}
$$

We refer to Section 6 for the final writing of the above formula (4.25).

## §5. SEVERAL INDEPENDENT VARIABLES AND SEVERAL DEPENDENT VARIABLES

5.1. Expression of $\mathbf{Y}_{i_{1}}^{j}$, of $\mathbf{Y}_{i_{1}, i_{2}}^{j}$ and of $\mathbf{Y}_{i_{1}, i_{2}, i_{3}}^{j}$. Applying the induction (1.31) and working out the obtained formulas until they take a perfect shape, we obtain firstly:

$$
\begin{equation*}
\mathbf{Y}_{i_{1}}^{j}=\mathscr{Y}_{x^{i_{1}}}^{j}+\sum_{l_{1}=1}^{m} \sum_{k_{1}=1}^{n}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} \mathscr{X}_{x^{i_{1}}}^{k_{1}}\right] y_{k_{1}}^{l_{1}}+\sum_{l_{1}, l_{2}=1}^{m} \sum_{k_{1}, k_{2}=1}^{n}\left[-\delta_{l_{2}}^{j} \delta_{i_{1}}^{k_{1}} \mathscr{X}_{y^{l_{1}}}^{k_{2}}\right] y_{k_{1}}^{l_{1}} y_{k_{2}}^{l_{2}} . \tag{5.2}
\end{equation*}
$$

## Secondly:

$$
\begin{align*}
\mathbf{Y}_{i_{1}, i_{2}}^{j}= & \mathscr{Y}_{x^{i_{1}} x^{i_{2}}}^{j}+\sum_{l_{1}=1}^{m} \sum_{k_{1}=1}^{n}\left[\delta_{i_{1}}^{k_{1}} \mathscr{O}_{x^{i_{2}} y^{l_{1}}}^{j}+\delta_{i_{2}}^{k_{1}} \mathscr{Y}_{x^{i_{1}} y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} \mathscr{X}_{x^{1_{1} x^{i_{2}}}}^{k_{1}}\right] y_{k_{1}}^{l_{1}}+  \tag{5.3}\\
& +\sum_{l_{1}, l_{2}=1}^{m} \sum_{k_{1}, k_{2}=1}^{n}\left[\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{Y}_{y^{l_{1}} y^{l_{2}}}^{j}-\delta_{l_{2}}^{j} \delta_{i_{1}}^{k_{1}} \mathscr{X}_{x^{i_{2}} y^{l_{1}}}^{k_{2}}-\delta_{l_{2}}^{j} \delta_{i_{2}}^{k_{1}} \mathscr{X}_{x^{i_{1}} y_{1}^{l_{1}}}^{k_{2}}\right] y_{k_{1}}^{l_{1}} y_{k_{2}}^{l_{2}}+ \\
& +\sum_{l_{1}, l_{2}, l_{3}=1}^{m} \sum_{k_{1}, k_{2}, k_{3}=1}^{n}\left[-\delta_{l_{3}}^{j} \delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{X}_{y^{1_{1}} y^{l_{2}}}^{k_{3}}\right] y_{k_{1}}^{l_{1}} y_{k_{2}}^{l_{2}} y_{k_{3}}^{l_{3}}+ \\
& +\sum_{l_{1}=1}^{m} \sum_{k_{1}, k_{2}=1}^{n}\left[\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{Y}_{y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} \delta_{i_{1}}^{k_{1}} \mathscr{X}_{x^{i_{2}}}^{k_{2}}-\delta_{l_{1}}^{j} \delta_{i_{2}}^{k_{1}} \mathscr{X}_{x^{1}}^{k_{2}}\right] y_{k_{1}, k_{2}}^{l_{1}}+ \\
& +\sum_{l_{1}, l_{2}=1}^{m} \sum_{k_{1}, k_{2}, k_{3}=1}^{n}\left[-\delta_{l_{1}}^{j} \delta_{i_{1}, i_{2}}^{k_{2}, k_{3}} \mathscr{X}_{y^{2}}^{k_{1}}-\delta_{l_{2}}^{j} \delta_{i_{1}, i_{2}}^{k_{3}, k_{1}} \mathscr{X}_{y_{1}}^{k_{2}}-\delta_{l_{2}}^{j} \delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{X}_{\left.y_{1}\right]}^{k_{3}}\right] y_{k_{1}}^{l_{1}} y_{k_{2}}^{l_{2}} y_{k_{3}}^{l_{3}} .
\end{align*}
$$

Thirdly:

$$
\begin{aligned}
& \mathbf{Y}_{i_{1}, i_{2}, i_{3}}^{j}=\mathscr{Y}_{x^{i_{1}} x^{i_{2}} x^{i_{3}}}^{j}+\sum_{l_{1}=1}^{m} \sum_{k_{1}=1}^{n}\left[\delta_{i_{1}}^{k_{1}} \mathscr{Y}_{x^{i_{2}} x^{i_{3}} y_{1}^{l_{1}}}^{j}+\delta_{i_{2}}^{k_{1}} \mathscr{Y}_{x^{i_{1}} x^{i_{3}} y^{l_{1}}}^{j}+\delta_{i_{3}}^{k_{1}} \mathscr{Y}_{x^{i_{1}} x^{i_{2}} y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} \mathscr{X}_{x^{i_{1}} x^{i_{2}} x^{i_{3}}}^{k_{1}}\right] y_{k_{1}}^{l_{1}}+ \\
& +\sum_{l_{1}, l_{2}=1}^{m} \sum_{k_{1}, k_{2}=1}^{n}\left[\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i} 3 y^{l_{1}} y^{l_{2}}}^{j}+\delta_{i_{3}, i_{1}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{2}} y^{l_{1}} y^{l_{2}}}^{j}+\delta_{i_{2}, i_{3}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{1}} y_{1} y^{l_{2}}}^{j}-\right. \\
& \left.-\delta_{l_{2}}^{j} \delta_{i_{1}}^{k_{1}} \mathscr{X}_{x^{i_{2}} x^{i} 3}^{k_{2}} y^{l_{1}}-\delta_{l_{2}}^{j} \delta_{i_{2}}^{k_{1}} \mathscr{X}_{x^{i_{1}} x^{i} y^{l_{1}}}^{k_{2}}-\delta_{l_{2}}^{j} \delta_{i_{3}}^{k_{1}} \mathscr{X}_{x^{i_{1}} x^{i_{2}} y^{l_{1}}}^{k_{2}}\right] y_{k_{1}}^{l_{1}} y_{k_{2}}^{l_{2}}+ \\
& +\sum_{l_{1}, l_{2}, l_{3}=1}^{m} \sum_{k_{1}, k_{2}, k_{3}=1}^{n}\left[\delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{2}, k_{3}} \mathscr{Y}_{y^{l_{1}} y^{l_{2}} y^{l_{3}}}^{j}-\delta_{l_{3}}^{j} \delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{X}_{x^{3} 3 y_{1} y^{l_{2}}}^{k_{3}}-\right. \\
& \left.-\delta_{l_{3}}^{j} \delta_{i_{1}, i_{3}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{2}} y^{l_{1}} y^{l_{2}}}^{k_{3}}-\delta_{l_{3}}^{j} \delta_{i_{2}, i_{3}}^{k_{1}, k_{2}} \mathscr{X}_{x^{1_{1}} y^{l_{1}} y^{l_{2}}}^{k_{3}}\right] y_{k_{1}}^{l_{1}} y_{k_{2}}^{l_{2}} y_{k_{3}}^{l_{3}}+ \\
& +\sum_{l_{1}, l_{2}, l_{3}, l_{4}=1}^{m} \sum_{k_{1}, k_{2}, k_{3}, k_{4}=1}^{n}\left[-\delta_{l_{4}}^{j} \delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{2}, k_{3}} \mathscr{X}_{y^{l_{1}} y^{l_{2}} y^{l_{3}}}^{k_{4}}\right] y_{k_{1}}^{l_{1}} y_{k_{2}}^{l_{2}} y_{k_{3}}^{l_{3}} y_{k_{4}}^{l_{4}}+ \\
& +\sum_{l_{1}=1}^{m} \sum_{k_{1}, k_{2}=1}^{n}\left[\delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{3}} y^{l_{1}}}^{j}+\delta_{i_{3}, i_{1}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{2}} y^{l_{1}}}^{j}+\delta_{i_{2}, i_{3}}^{k_{1}, k_{2}} \mathscr{Y}_{x^{i_{1}} y^{l_{1}}}^{j}-\right. \\
& \left.-\delta_{l_{1}}^{j} \delta_{i_{1}}^{k_{1}} \mathscr{X}_{x^{i_{2}} x^{i_{3}}}^{k_{2}}-\delta_{l_{1}}^{j} \delta_{i_{2}}^{k_{1}} \mathscr{X}_{x^{i_{1}} x^{i_{3}}}^{k_{2}}-\delta_{l_{1}}^{j} \delta_{i_{3}}^{k_{1}} \mathscr{X}_{x^{i_{1}} x^{i_{2}}}^{k_{2}}\right] y_{k_{1}, k_{2}}^{l_{1}}+
\end{aligned}
$$

$$
\begin{align*}
& +\sum_{l_{1}, l_{2}=1}^{m} \sum_{k_{1}, k_{2}, k_{3}=1}^{n}\left[\delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{2}, k_{3}} \mathscr{Y}_{y^{l_{1}} y^{l_{2}}}^{j}+\delta_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{1}, k_{2}} \mathscr{Y}_{y^{l_{1}} y^{l_{2}}}^{j}+\delta_{i_{1}, i_{2}, i_{3}}^{k_{2}, k_{3}, k_{1}} \mathscr{Y}_{y^{l_{1}} y^{l_{2}}}^{j}-\right.  \tag{5.4}\\
& -\delta_{l_{1}}^{j} \delta_{i_{1}, i_{2}}^{k_{2}, k_{3}} \mathscr{X}_{x^{i} 3}^{k_{1} y_{2}}-\delta_{l_{1}}^{j} \delta_{i_{1}, i_{3}}^{k_{2}, k_{3}} \mathscr{X}_{x^{i} 2}^{k_{1} y_{2}}-\delta_{l_{1}}^{j} \delta_{i_{2}, i_{3}}^{k_{2}, k_{3}} \mathscr{X}_{x^{i_{1}} y^{l_{2}}}^{k_{1}}- \\
& -\delta_{l_{2}}^{j} \delta_{i_{1}, i_{2}}^{k_{3}, k_{1}} \mathscr{X}_{x^{i} 3 y_{1}}^{k_{2}}-\delta_{l_{2}}^{j} \delta_{i_{1}, i_{3}}^{k_{3}, k_{1}} \mathscr{X}_{x^{i_{2}} y^{l_{1}}}^{k_{2}}-\delta_{l_{2}}^{j} \delta_{i_{2}, i_{3}}^{k_{3}, k_{1}} \mathscr{X}_{x^{1_{1}} y^{l_{1}}}^{k_{2}}- \\
& \left.-\delta_{l_{2}}^{j} \delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{3}} y_{1}}^{k_{3}}-\delta_{l_{2}}^{j} \delta_{i_{1}, i_{3}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{2}} y^{l_{1}}}^{k_{3}}-\delta_{l_{2}}^{j} \delta_{i_{2}, i_{3}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{1}} y_{1}^{l_{1}}}^{k_{3}}\right] y_{k_{1}}^{l_{1}} y_{k_{2}, k_{3}}^{l_{2}}+ \\
& +\sum_{l_{1}, l_{2}, l_{3}=1}^{m} \sum_{k_{1}, k_{2}, k_{3}, k_{4}=1}^{n}\left[-\delta_{l_{3}}^{j} \delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{2}, k_{3}} \mathscr{X}_{y^{1} y^{l_{2}}}^{k_{4}}-\delta_{l_{3}}^{j} \delta_{i_{1}, i_{2}, i_{3}}^{k_{2}, k_{3}, k_{1}} \mathscr{X}_{y^{l_{1}} y^{l_{2}}}^{k_{4}}-\delta_{l_{3}}^{j} \delta_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{2}, k_{1}} \mathscr{X}_{y_{1} y^{l_{2}}}^{k_{4}}-\right. \\
& -\delta_{l_{2}}^{j} \delta_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{4}, k_{1}} \mathscr{X}_{y^{l_{1}} y^{l_{3}}}^{k_{2}}-\delta_{l_{2}}^{j} \delta_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{1}, k_{4}} \mathscr{X}_{y^{1_{1}} y^{l_{3}}}^{k_{2}}- \\
& \left.-\delta_{l_{2}}^{j} \delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{3}, k_{4}} \mathscr{X}_{y^{l_{1}} y^{l_{3}}}^{k_{2}}\right] y_{k_{1}}^{l_{1}} y_{k_{2}}^{l_{2}} y_{k_{3}, k_{4}}^{l_{3}}+ \\
& +\sum_{l_{1}, l_{2}=1}^{m} \sum_{k_{1}, k_{2}, k_{3}, k_{4}=1}^{n}\left[-\delta_{l_{2}}^{j} \delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{2}, k_{3}} \mathscr{X}_{y^{l_{1}}}^{k_{3}}-\delta_{l_{2}}^{j} \delta_{i_{1}, i_{2}, i_{3}}^{k_{2}, k_{4}, k_{1}} \mathscr{X}_{y_{1}^{l}}^{k_{3}}-\delta_{l_{2}}^{j} \delta_{i_{1}, i_{2}, i_{3}}^{k_{4}, k_{1}, k_{2}} \mathscr{X}_{y_{1}}^{k_{3}}\right] y_{k_{1}, k_{2}}^{l_{1}} y_{k_{3}, k_{4}}^{l_{2}}+ \\
& +\sum_{l_{1}=1}^{m} \sum_{k_{1}, k_{2}, k_{3}=1}^{n}\left[\delta_{i_{1}, i_{2}, \beta_{3}}^{k_{1}, k_{2}, k_{3}} \mathscr{Y}_{y^{l_{1}}}^{j}-\delta_{l_{1}}^{j} \delta_{i_{1}, i_{2}}^{k_{1}, k_{2}} \mathscr{X}_{x^{i_{3}}}^{k_{3}}-\delta_{l_{1}}^{j} \delta_{i_{1}, i_{3}}^{k_{1}, k_{2}} \mathscr{X}_{x^{2}}^{k_{3}}-\delta_{l_{1}}^{j} \delta_{i_{2}, i_{3}}^{k_{1}, k_{2}} \mathscr{X}_{x^{1_{1}}}^{k_{3}}\right] y_{k_{1}, k_{2}, k_{3}}^{l_{1}}+ \\
& +\sum_{l_{1}, l_{2}=1}^{m} \sum_{k_{1}, k_{2}, k_{3}, k_{4}=1}^{n}\left[-\delta_{l_{2}}^{j} \delta_{i_{1}, i_{2}, i_{3}}^{k_{1}, k_{2}, k_{3}} \mathscr{X}_{y^{1}}^{k_{4}}-\delta_{l_{2}}^{j} \delta_{i_{1}, i_{2}, i_{3}}^{k_{4}, k_{1}, k_{2}} \mathscr{X}_{y^{1}}^{k_{3}}-\delta_{l_{2}}^{j} \delta_{i_{1}, i_{2}, i_{3}}^{k_{3}, k_{4}, k_{1}} \mathscr{X}_{y^{1}}^{k_{2}}-\right. \\
& \left.-\delta_{l_{1}}^{j} \delta_{i_{1}, i_{2}, i_{3}}^{k_{2}, k_{3}, k_{4}} \mathscr{X}_{y^{2}}^{k_{1}}\right] y_{k_{1}}^{l_{1}} y_{k_{2}, k_{3}, k_{4}}^{l_{2}} .
\end{align*}
$$

5.5. Final synthesis. To obtain the general formula for $\mathbf{Y}_{i_{1}, \ldots, i_{k}}^{j}$, we have to achieve the synthesis between the two formulas (3.74) and (4.19). We start with (3.74) and we modify it until we reach the final formula for $\mathbf{Y}_{i_{1}, \ldots, i_{\kappa}}^{j}$.

We have to add the $\mu_{1}+\cdots+\mu_{d}$ sums $\sum_{l_{1: 1}=1}^{m} \cdots \sum_{l_{1: \mu_{1}}=1}^{m} \cdots \cdots \sum_{l_{d: 1}=1}^{m} \cdots \sum_{l_{d: \mu_{d}}=1}^{m}$, together with various indices $l_{\alpha}$. About these indices, the only point which is not obvious may be analyzed as follows.

A permutation $\sigma \in \mathfrak{F}_{\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}}^{\left(\mu_{1}, \lambda_{1}\right) \ldots,\left(\mu_{d}, \lambda_{d}\right)}$ yields the list:

$$
\begin{align*}
& \sigma(1: 1: 1), \ldots, \sigma\left(1: 1: \lambda_{1}\right), \ldots \sigma\left(1: \mu_{1}: 1\right), \ldots, \sigma\left(1: \mu_{1}: \lambda_{1}\right), \ldots  \tag{5.6}\\
& \quad \ldots, \sigma(d: 1: 1), \ldots, \sigma\left(1: 1: \lambda_{d}\right), \ldots \sigma\left(d: \mu_{d}: 1\right), \ldots, \sigma\left(d: \mu_{d}: \lambda_{d}\right),
\end{align*}
$$

In the sixth line of (3.74), the last term $\sigma\left(d: \mu_{d}: \lambda_{d}\right)$ of the above list appears as the subscript of the upper index $k_{\sigma\left(d: \mu_{d}: \lambda_{d}\right)}$ of the term $\mathscr{X}^{k_{\sigma\left(d: \mu_{d}: \lambda_{d}\right)}}$. According to the formal rules of Theorem 4.19, we have to multiply the partial derivative of $\mathscr{X}^{k_{\sigma\left(d: \mu_{d}: \lambda_{d}\right)}}$ by a certain Kronecker symbol $\delta_{l_{\alpha}}^{j}$. The question is: what is the subscript $\alpha$ and how to denote it?

As explained before the statement of Theorem 4.19, the subscript $\alpha$ is obtained as follows. The term $\sigma\left(d: \mu_{d}: \lambda_{d}\right)$ is of the form $\left(e: \nu_{d}: \gamma_{e}\right)$, for some $e$ with $1 \leqslant e \leqslant d$, for some $\nu_{e}$ with $1 \leqslant \nu_{e} \leqslant \mu_{e}$ and for some $\gamma_{e}$ with $1 \leqslant \gamma_{e} \leqslant \lambda_{e}$. The single pure jet variable

$$
\begin{equation*}
y_{k_{e: \nu_{e}: 1}, \ldots, k_{e: v_{e}: v_{e}}, \ldots, k_{e: \nu_{e}: \lambda_{e}}^{l_{e}}}^{l_{e}} \tag{5.7}
\end{equation*}
$$

appears inside the total monomial

$$
\begin{equation*}
\prod_{1 \leqslant \nu_{1} \leqslant \mu_{1}} y_{k_{1: \nu_{1}: 1}, \ldots, k_{1: \nu_{1}: \lambda_{1}}}^{l_{1: \nu_{1}}} \cdots \prod_{1 \leqslant \nu_{d} \leqslant \mu_{d}} y_{k_{d: \nu_{d}: 1}, \ldots, k_{d: \nu_{d}: \lambda_{d}}}^{l_{: \nu_{d}}}, \tag{5.8}
\end{equation*}
$$

placed at the end of the formula for $\mathbf{Y}_{i_{1}, \ldots, i_{\kappa}}^{j}$ (see in advance formula (5.13) below; this total monomial generalizes to several dependent variables the total monomial appearing in the last line of (3.74)). According to the rule explained before the statement of Theorem 4.18, the index $l_{\alpha}$ must be equal to $l_{e: \nu_{e}}$, since $l_{e: \nu_{e}}$ is attached to the monomial (5.7). Coming back to the term $\sigma\left(d: \mu_{d}: \lambda_{d}\right)$, we shall denote this index by

$$
\begin{equation*}
l_{e: \nu_{e}}=: l_{\pi\left(e: \nu_{e}: \gamma_{e}\right)}=: l_{\pi \sigma\left(d: \mu_{d}: \lambda_{d}\right)}, \tag{5.9}
\end{equation*}
$$

where the symbol $\pi$ denotes the projection from the set

$$
\begin{equation*}
\left\{1: 1: 1, \ldots, 1: \mu_{1}: \lambda_{1}, \ldots \ldots, d: 1: 1, \ldots, d: \mu_{d}: \lambda_{d}\right\} \tag{5.10}
\end{equation*}
$$

to the set

$$
\begin{equation*}
\left\{1: 1, \ldots, 1: \mu_{1}, \ldots, d: 1, \ldots, d: \mu_{d}\right\} \tag{5.11}
\end{equation*}
$$

simply defined by $\pi\left(e: \nu_{e}: \gamma_{e}\right):=\left(e: \nu_{e}\right)$.
In conclusion, by means of this formalism, we may write down the complete generalization of Theorems 2.24, 3.73 and 4.18 to several independent variables and several dependent variables

Theorem 5.12. For $j=1, \ldots, m$, for every $\kappa \geqslant 1$ and for every choice of $\kappa$ indices $i_{1}, \ldots, i_{\kappa}$ in the set $\{1,2, \ldots, n\}$, the general expression of $\mathbf{Y}_{i_{1}, \ldots, i_{\kappa}}^{j}$
is as follows:
(5.13)

$$
\begin{aligned}
& \mathbf{Y}_{i_{1}, \ldots, i_{\kappa}}^{j}=\mathscr{g}_{x^{i_{1}} \ldots x^{i_{k}}}^{j}+\sum_{d=1}^{\kappa+1} \sum_{1 \leqslant \lambda_{1}<\ldots<\lambda_{d} \leqslant \kappa} \sum_{\mu_{1} \geqslant 1, \ldots, \mu_{d} \geqslant 1} \sum_{\mu_{1} \lambda_{1}+\ldots+\mu_{d} \lambda_{d} \leqslant \kappa+1} \\
& \sum_{l_{1: 1}=1}^{m} \cdots \sum_{l_{1: \mu}=1}^{m} \cdots \cdots \sum_{l_{d: 1}=1}^{m} \cdots \sum_{l_{d: \mu_{d}}=1}^{m}
\end{aligned}
$$

$$
\begin{aligned}
& \cdot \prod_{1 \leqslant \nu_{1} \leqslant \mu_{1}} y_{k_{1: v_{1}}: 1, \ldots, k_{1: v_{1}: \lambda_{1}}}^{l_{1: \nu_{1}}} \cdots \prod_{1 \leqslant \nu_{d} \leqslant \mu_{d}} y_{k_{d, v_{d}: 1}, \ldots, k_{d: \nu} \lambda_{d} \lambda_{d}}^{l_{d: v_{d}}} .
\end{aligned}
$$

In this formula, the coset $\mathfrak{F}_{\mu_{1} \lambda_{1}+\ldots+\mu_{d} \lambda_{d}}^{\left(\mu_{1}, \lambda_{1}\right), \ldots,\left(\mu_{d}\right)}$ was defined in equation (3.71); as in Theorem 3.73, we have made the identification:

$$
\begin{equation*}
\{1, \ldots, \kappa\} \equiv\left\{1: 1: 1, \ldots, 1: \mu_{1}: \lambda_{1}, \ldots \ldots, d: 1: 1, \ldots, d: \mu_{d}: \lambda_{d}\right\} \tag{5.14}
\end{equation*}
$$

Since the fundamental monomials appearing in the last line of (4.19) just above are not independent of each other, this formula has still to be modified a little bit. We refer to Section 6 for details.

### 5.15. Deduction of the most general multivariate Faà di Bruno formula.

Let $n \in \mathbb{N}$ with $n \geqslant 1$, let $x=\left(x^{1}, \ldots, x^{n}\right) \in \mathbb{K}^{n}$, let $m \in \mathbb{N}$ with $m \geqslant 1$, let $g^{j}=g^{j}\left(x^{1}, \ldots, x^{n}\right), j=1, \ldots, m$, be $\mathscr{C}^{\infty}$-smooth functions from $\mathbb{K}^{n}$ to $\mathbb{K}^{m}$, let $y=\left(y^{1}, \ldots, y^{m}\right) \in \mathbb{K}^{m}$ and let $f=f\left(y^{1}, \ldots, y^{m}\right)$ be a $\mathscr{C}^{\infty}$ function from $\mathbb{K}^{m}$ to $\mathbb{K}$. The goal is to obtain an explicit formula for the partial derivatives of the composition $h:=f \circ g$, namely

$$
\begin{equation*}
h\left(x^{1}, \ldots, x^{n}\right):=f\left(g^{1}\left(x^{1}, \ldots, x^{n}\right), \ldots, g^{m}\left(x^{1}, \ldots, x^{n}\right)\right) . \tag{5.16}
\end{equation*}
$$

For $j=1, \ldots, m$, for $\lambda \in \mathbb{N}$ with $\lambda \geqslant 1$ and for arbitrary indices $i_{1}, \ldots, i_{\lambda}=1, \ldots, n$, we shall abbreviate the partial derivative $\frac{\partial^{\lambda} g^{j}}{\partial x^{i_{1}} \partial \partial x^{i} \lambda}$ by
$g_{i_{1}, \ldots, i_{\lambda}}^{j}$ and similarly for $h_{i_{1}, \ldots, i_{\lambda}}$. For arbitrary indices $l_{1}, \ldots, l_{\lambda}=1, \ldots, m$, the partial derivative $\frac{\partial^{\lambda} f}{\partial y^{l_{1} \ldots \partial y^{l_{\lambda}} \lambda}}$ will be abbreviated by $f_{l_{1}, \ldots, l_{\lambda}}$.

Appying the chain rule, we may compute:

$$
\begin{align*}
h_{i_{1}}= & \sum_{l_{1}=1}^{m} f_{l_{1}}\left[g_{i_{1}}^{l_{1}}\right],  \tag{5.17}\\
h_{i_{1}, i_{2}}= & \sum_{l_{1}, l_{2}=1}^{m} f_{l_{1}, l_{2}}\left[g_{i_{1}}^{l_{1}} g_{i_{2}}^{l_{2}}\right]+\sum_{l_{1}=1}^{m} f_{l_{1}}\left[g_{i_{1}, i_{2}}^{l_{1}}\right], \\
h_{i_{1}, i_{2}, i_{3}}= & \sum_{l_{1}, l_{2}, l_{3}=1}^{m} f_{l_{1}, l_{2}, l_{3}}\left[g_{i_{1}}^{l_{1}} g_{i_{2}}^{l_{2}} g_{i_{3}}^{l_{3}}\right]+\sum_{l_{1}, l_{2}=1}^{m} f_{l_{1}, l_{2}}\left[g_{i_{1}}^{l_{1}} g_{i_{2}, i_{3}}^{l_{2}}+g_{i_{2}}^{l_{1}} g_{i_{1}, i_{3}}^{l_{2}}+g_{i_{3}}^{l_{1}} g_{i_{1}, i_{2}}^{l_{2}}\right]+ \\
& +\sum_{l_{1}=1}^{m} f_{l_{1}}\left[g_{\left.i_{1}, i_{2}, i_{3}\right]}^{l_{1}}\right], \\
h_{i_{1}, i_{2}, i_{3}, i_{4}}= & \sum_{l_{1}, l_{2}, l_{3}, l_{4}=1}^{m} f_{l_{1}, l_{2}, l_{3}, l_{4}}\left[g_{i_{1}}^{l_{1}} g_{i_{2}}^{l_{2}} g_{i_{3}}^{l_{3}} g_{i_{4}}^{l_{4}}\right]+ \\
& \sum_{l_{1}, l_{2}, l_{3}=1}^{m} f_{l_{1}, l_{2}, l_{3}}\left[g_{i_{2}}^{l_{1}} g_{i_{3}}^{l_{2}} g_{i_{1}, i_{4}}^{l_{3}}+g_{i_{3}}^{l_{1}} g_{i_{1}}^{l_{2}} g_{i_{2}, i_{4}}^{l_{3}}+g_{i_{1}}^{l_{1}} g_{i_{2}}^{l_{2}} g_{i_{3}, i_{4}}^{l_{3}}+\right. \\
& \left.+g_{i_{1}}^{l_{1}} g_{i_{4}}^{l_{2}} g_{i_{2}, i_{3}}^{l_{3}}+g_{i_{2}}^{l_{1}} g_{i_{4}}^{l_{2}} g_{i_{3}, i_{1}}^{l_{3}}+g_{i_{3}}^{l_{1}} g_{i_{4}}^{l_{2}} g_{i_{1}, i_{2}}^{l_{3}}\right]+ \\
& +\sum_{l_{1}, l_{2}=1}^{m} f_{l_{1}, l_{2}}\left[g_{i_{1}, i_{2}}^{l_{1}} g_{i_{3}, i_{4}}^{l_{2}}+g_{i_{1}, i_{3}}^{l_{1}} g_{i_{2}, i_{4}}^{l_{2}}+g_{i_{1}, i_{4}}^{l_{1}} g_{i_{2}, i_{3}}^{l_{2}}\right]+ \\
& +\sum_{l_{1}, l_{2}=1}^{m} f_{l_{1}, l_{2}}\left[g_{i_{1}}^{l_{1}} g_{i_{2}, i_{3}, i_{4}}^{l_{2}}+g_{i_{2}}^{l_{1}} g_{i_{1}, i_{3}, i_{4}}^{l_{2}}+g_{i_{3}}^{l_{1}} g_{i_{1}, i_{2}, i_{4}}^{l_{2}}+g_{i_{4}}^{l_{1}} g_{\left.i_{1}, i_{2}, i_{3}\right]}^{l_{2}}\right]+ \\
& +\sum_{l_{1}=1}^{m} f_{l_{1}}\left[g_{\left.i_{1}, i_{2}, i_{3}, i_{4}\right]}^{l_{1}}\right]
\end{align*}
$$

Introducing the derivations

$$
\begin{align*}
F_{i}^{2}:= & \sum_{k_{1}=1}^{n} \sum_{l_{1}=1}^{m} g_{k_{1}, i}^{l_{1}} \frac{\partial}{\partial g_{k_{1}}^{l_{1}}}+\sum_{l_{1}=1}^{m} g_{i}^{l_{1}}\left(\sum_{l_{2}=1}^{m} f_{l_{1}, l_{2}} \frac{\partial}{\partial f_{l_{2}}}\right), \\
F_{i}^{3}:= & \sum_{k_{1}=1}^{n} \sum_{l_{1}=1}^{m} g_{k_{1}, i}^{l_{1}} \frac{\partial}{\partial g_{k_{1}}^{l_{1}}}+\sum_{k_{1}, k_{2}=1}^{n} \sum_{l_{1}=1}^{m} g_{k_{1}, k_{2}, i}^{l_{1}} \frac{\partial}{\partial g_{k_{1}, k_{2}}^{l_{1}}}+ \\
& +\sum_{l_{1}=1}^{m} g_{i}^{l_{1}}\left(\sum_{l_{2}=1}^{m} f_{l_{1}, l_{2}} \frac{\partial}{\partial f_{l_{2}}}+\sum_{l_{2}, l_{3}=1}^{m} f_{l_{1}, l_{2}, l_{3}} \frac{\partial}{\partial f_{l_{2}, l_{3}}}\right), \tag{5.18}
\end{align*}
$$

$$
\begin{aligned}
F_{i}^{\lambda}:= & \sum_{k_{1}=1}^{n} \sum_{l_{1}=1}^{m} g_{k_{1}, i}^{l_{1}} \frac{\partial}{\partial g_{k_{1}}^{l_{1}}}+\sum_{k_{1}, k_{2}=1}^{n} \sum_{l_{1}=1}^{m} g_{k_{1}, k_{2}, i}^{l_{1}} \frac{\partial}{\partial g_{k_{1}, k_{2}}^{l_{1}}}+\cdots+ \\
& +\sum_{k_{1}, k_{2}, \ldots, k_{\lambda-1}=1}^{n} \sum_{l_{1}=1}^{m} g_{k_{1}, k_{2}, \ldots, k_{\lambda-1}, i} \frac{\partial}{\partial g_{k_{1}, \ldots, k_{\lambda-1}}^{l_{1}}}+ \\
& +\sum_{l_{1}=1}^{m} g_{i}^{l_{1}}\left(\sum_{l_{2}=1}^{m} f_{l_{1}, l_{2}} \frac{\partial}{\partial f_{l_{2}}}+\sum_{l_{2}, l_{3}=1}^{m} f_{l_{1}, l_{2}, l_{3}} \frac{\partial}{\partial f_{l_{2}, l_{3}}}+\right. \\
& \left.+\cdots+\sum_{l_{2}, l_{3}, \ldots, l_{\lambda}} f_{l_{1}, l_{2}, l_{3}, \ldots, l_{\lambda}} \frac{\partial}{\partial f_{l_{2}, l_{3}, \ldots, l_{\lambda}}}\right),
\end{aligned}
$$

we observe that the following induction relations hold:

$$
\begin{align*}
h_{i_{1}, i_{2}} & =F_{i_{2}}^{2}\left(h_{i_{1}}\right), \\
h_{i_{1}, i_{2}, i_{3}}= & F_{i_{3}}^{3}\left(h_{i_{1}, i_{2}}\right),  \tag{5.19}\\
\ldots \ldots \ldots & \cdots \cdots \cdots \ldots \ldots \\
h_{i_{1}, i_{2}, \ldots, i_{\lambda}}= & F_{i_{\lambda}}^{\lambda}\left(h_{i_{1}, i_{2}, \ldots, i_{\lambda-1}}\right) .
\end{align*}
$$

To obtain the explicit version of the Faà di Bruno in the case of several variables $\left(x^{1}, \ldots, x^{n}\right)$ and several variables $\left(y^{1}, \ldots, y^{m}\right)$, it suffices to extract from the expression of $\mathbf{Y}_{i_{1}, \ldots, i_{\kappa}}^{j}$ provided by Theorem 5.12 only the terms corresponding to $\mu_{1} \lambda_{1}+\cdots+\mu_{d} \lambda_{d}=\kappa$, dropping all the $\mathscr{X}$ terms. After some simplifications and after a translation by means of an elementary dictionary, we obtain the fourth and the most general multivariate Faà di Bruno formula.

Theorem 5.20. For every integer $\kappa \geqslant 1$ and for every choice of indices $i_{1}, \ldots, i_{\kappa}$ in the set $\{1,2, \ldots, n\}$, the $\kappa$-th partial derivative of the composite function

$$
\begin{equation*}
h=h\left(x^{1}, \ldots, x^{n}\right)=f\left(g^{1}\left(x^{1}, \ldots, x^{n}\right), \ldots, g^{m}\left(x^{1}, \ldots, x^{n}\right)\right) \tag{5.21}
\end{equation*}
$$

with respect to the variables $x^{i_{1}}, \ldots, x^{i_{\kappa}}$ may be expressed as an explicit polynomial depending on the partial derivatives of $f$, on the partial derivatives of the $g^{j}$ and having integer coefficients:
(5.22)


# III: Systems of second order 

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## §1. EXPLICIT CHARACTERIZATIONS OF FLATNESS

In 1883, S. Lie obtained the following explicit characterization of the local equivalence of a second order ordinary differential equation $\left(\mathscr{E}_{1}\right)$ : $y_{x x}=F\left(x, y, y_{x}\right)$ to the Newtonian free particle equation with one degree of freedom $Y_{X X}=0$. All the functions are assumed to be analytic.

Theorem 1.1. ([Lie1883], pp. 362-365) Let $\mathbb{K}=\mathbb{R}$ of $\mathbb{C}$. Let $x \in \mathbb{K}$ and $y \in \mathbb{K}$. A local second order ordinary differential equation $y_{x x}=$ $F\left(x, y, y_{x}\right)$ is equivalent under an invertible point transformation $(x, y) \mapsto$ $(X(x, y), Y(x, y))$ to the free particle equation $Y_{X X}=0$ if and only if the following two conditions are satisfied:
(i) $F_{y_{x} y_{x} y_{x} y_{x}}=0$, or equivalently $F$ is a degree three polynomial in $y_{x}$, namely there exist four functions $G, H, L$ and $M$ of $(x, y)$ such that $F$ can be written as
(1.2) $F\left(x, y, y_{x}\right)=G(x, y)+y_{x} \cdot H(x, y)+\left(y_{x}\right)^{2} \cdot L(x, y)+\left(y_{x}\right)^{3} \cdot M(x, y)$;
(ii) the four functions $G, H, L$ and $M$ satisfy the following system of two second order quasi-linear partial differential equations:
(1.3)

$$
\left\{\begin{aligned}
0=-2 G_{y y} & +\frac{4}{3} H_{x y}-\frac{2}{3} L_{x x}+ \\
& +2(G L) y-2 G_{x} M-4 G M_{x}+\frac{2}{3} H L_{x}-\frac{4}{3} H H_{y} \\
0=-\frac{2}{3} H_{y y} & +\frac{4}{3} L_{x y}-2 M_{x x}+ \\
& +2 G M_{y}+4 G_{y} M-2(H M)_{x}-\frac{2}{3} H_{y} L+\frac{4}{3} L L_{x}
\end{aligned}\right.
$$

Open question 1.4. Deduce an explicit necessary and sufficient condition for the associated submanifold of solutions $y=\Pi(x, a, b)$ to be locally equivalent to $Y=B+X A$.

Assuming $F=F\left(x, y_{x}\right)$ to be independent of $y$, or equivalently assuming $\mathscr{M}_{\left(\mathscr{E}_{1}\right)}$ to be:

$$
\begin{equation*}
y=b+\Pi(x, a) \tag{1.5}
\end{equation*}
$$

the author has checked that equivalence to $Y=B+X A$ holds if and only if two differential rational expressions annihilate:

$$
\begin{align*}
0= & \frac{\Pi_{x^{2} a^{4}}}{\left(\Pi_{x a}\right)^{4}}-6 \frac{\Pi_{x^{2} a^{3}} \Pi_{x a^{2}}}{\left(\Pi_{x a}\right)^{5}}+15 \frac{\Pi_{x^{2} a^{2}}\left(\Pi_{x a^{2}}\right)^{2}}{\left(\Pi_{x a}\right)^{6}}-4 \frac{\Pi_{x^{2} a^{2}} \Pi_{x a^{3}}}{\left(\Pi_{x a}\right)^{5}} \\
& -\frac{\Pi_{x^{2} a} \Pi_{x a^{4}}}{\left(\Pi_{x a}\right)^{5}}+10 \frac{\Pi_{x a^{3}} \Pi_{x^{2} a} \Pi_{x a^{2}}}{\left(\Pi_{x a}\right)^{6}}-15 \frac{\Pi_{x^{2} a}\left(\Pi_{x a^{2}}\right)^{3}}{\left(\Pi_{x a}\right)^{7}} \text { and }  \tag{1.6}\\
0= & \frac{\Pi_{x^{4} a^{2}}}{\left(\Pi_{x a}\right)^{2}}-6 \frac{\Pi_{x^{3} a^{2}} \Pi_{x^{2} a}}{\left(\Pi_{x a}\right)^{3}}-4 \frac{\Pi_{x^{3} a} \Pi_{x^{2} a^{2}}}{\left(\Pi_{x a}\right)^{3}}-\frac{\Pi_{x^{4} a} \Pi_{x a^{2}}}{\left(\Pi_{x a}\right)^{3}}+ \\
& +15 \frac{\Pi_{x^{2} a^{2}}\left(\Pi_{x^{2} a}\right)^{2}}{\left(\Pi_{x a}\right)^{4}}+10 \frac{\Pi_{x^{3} a} \Pi_{x^{2} a} \Pi_{x a^{2}}}{\left(\Pi_{x a}\right)^{4}}-15 \frac{\left(\Pi_{x^{2} a}\right)^{3} \Pi_{x a^{2}}}{\left(\Pi_{x a}\right)^{5}} .
\end{align*}
$$

As an application, this characterizes local sphericity of a rigid hypersurface $w=\bar{w}+i \Theta(z, \bar{z})$ of $\mathbb{C}^{2}$. The answer for a general $y=\Pi(x, a, b)$, together with a proof, will appear elsewhere.

A modern restitution of Lie's original proof of Theorem 1.1 may be found in [Me2004]. In this reference, we generalize Theorem 1.1 to several dependent variables $y=\left(y^{1}, y^{2}, \ldots, y^{m}\right)$. In the present Part III, we will instead pass to several independent variables $x=\left(x^{1}, x^{2}, \ldots, x^{n}\right)$.

Theorem 1.7. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, let $n \in \mathbb{N}$, suppose $n \geqslant 2$ and consider a system of completely integrable partial differential equations in $n$ independent variables $x=\left(x^{1}, \ldots, x^{n}\right) \in \mathbb{K}^{n}$ and in one dependent variable $y \in \mathbb{K}$ of the form:

$$
\begin{equation*}
y_{x^{j_{1}} x^{j_{2}}}(x)=F^{j_{1}, j_{2}}\left(x, y(x), y_{x^{1}}(x), \ldots, y_{x^{n}}(x)\right), \quad 1 \leqslant j_{1}, j_{2} \leqslant n \tag{1.8}
\end{equation*}
$$

where $F^{j_{1}, j_{2}}=F^{j_{2}, j_{1}}$. Under a local change of coordinates $(x, y) \mapsto$ $(X, Y)=(X(x, y), Y(x, y))$, this system (1.8) is equivalent to the simplest "flat" system

$$
\begin{equation*}
Y_{X^{j_{1} X^{j_{2}}}}=0, \quad 1 \leqslant j_{1}, j_{2} \leqslant n \tag{1.9}
\end{equation*}
$$

if and only if there exist arbitrary functions $G_{j_{1}, j_{2}}, H_{j_{1}, j_{2}}^{k_{1}}, L_{j_{1}}^{k_{1}}$ and $M^{k_{1}}$ of the variables $(x, y)$, for $1 \leqslant j_{1}, j_{2}, k_{1} \leqslant n$, satisfying the two symmetry conditions $G_{j_{1}, j_{2}}=G_{j_{2}, j_{1}}$ and $H_{j_{1}, j_{2}}^{k_{1}}=H_{j_{2}, j_{1}}^{k_{1}}$, such that the equation (1.8)
is of the specific cubic polynomial form:
(1.10)
$y_{x^{j_{1}} x^{j_{2}}}=G_{j_{1}, j_{2}}+\sum_{k_{1}=1}^{n} y_{x^{k_{1}}}\left(H_{j_{1}, j_{2}}^{k_{1}}+\frac{1}{2} y_{x^{j_{1}}} L_{j_{2}}^{k_{1}}+\frac{1}{2} y_{x^{j_{2}}} L_{j_{1}}^{k_{1}}+y_{x^{j_{1}}} y_{x^{j_{2}}} M^{k_{1}}\right)$,
for $j_{1}, j_{2}=1, \ldots, n$.
It may seem quite paradoxical and counter-intuitive (or even false?) that every system (1.10), for arbitrary choices of functions $G_{j_{1}, j_{2}}, H_{j_{1}, j_{2}}^{k_{1}}, L_{j_{1}}^{k_{1}}$ and $M^{k_{1}}$, is automatically equivalent to $Y_{X^{j_{1} X^{j_{2}}}}=0$. However, a strong hidden assumption holds: that of complete integrability. Shortly, this crucial condition amounts to say that

$$
\begin{equation*}
D_{j_{3}}\left(F^{j_{1}, j_{2}}\right)=D_{j_{2}}\left(F^{j_{1}, j_{3}}\right), \tag{1.11}
\end{equation*}
$$

for all $j_{1}, j_{2}, j_{3}=1, \ldots, n$, where, for $j=1, \ldots, n$, the $D_{j}$ are the total differentiation operators defined by

$$
\begin{equation*}
D_{j}:=\frac{\partial}{\partial x^{j}}+y_{x^{j}} \frac{\partial}{\partial y}+\sum_{l=1}^{n} F^{j, l} \frac{\partial}{\partial y_{x^{l}}} \tag{1.12}
\end{equation*}
$$

These conditions are non-void precisely when $n \geqslant 2$. More concretely, developing out (1.11) when the $F^{j_{1}, j_{2}}$ are of the specific cubic polynomial form (1.10), after some nontrivial manual computation, we obtain the complicated cubic differential polynomial in the variables $y_{x^{k}}$. Equating to zero all the coefficients of this cubic polynomial, we obtain four familes (I'), (II'), (III') and (IV') of first order partial differential equations satisfied by $G_{j_{1}, j_{2}}$, $H_{j_{1}, j_{2}}^{k_{1}}, L_{j_{1}}^{k_{1}}$ and $M^{k_{1}}$ :
(I') $\left\{0=G_{j_{1}, j_{2}, x^{j_{3}}}-G_{j_{1}, j_{3}, x^{j_{2}}}+\sum_{k_{1}=1}^{n} H_{j_{1}, j_{2}}^{k_{1}} G_{k_{1}, j_{3}}-\sum_{k_{1}=1}^{n} H_{j_{1}, j_{3}}^{k_{1}} G_{k_{1}, j_{2}}\right.$.

$$
\left(\text { II' }^{\prime}\right)\left\{\begin{aligned}
0= & \delta_{j_{3}}^{k_{1}} G_{j_{1}, j_{2}, y}-\delta_{j_{2}}^{k_{1}} G_{j_{1}, j_{3}, y}+H_{j_{1}, j_{2}, x^{j_{3}}}^{k_{1}}-H_{j_{1}, j_{3}, x^{j_{2}}}^{k_{1}}+ \\
& +\frac{1}{2} G_{j_{1}, j_{3}} L_{j_{2}}^{k_{1}}-\frac{1}{2} G_{j_{1}, j_{2}} L_{j_{3}}^{k_{1}}+ \\
& +\frac{1}{2} \delta_{j_{1}}^{k_{1}} \sum_{k_{2}=1}^{n} G_{k_{2}, j_{3}} L_{j_{2}}^{k_{2}}-\frac{1}{2} \delta_{j_{1}}^{k_{1}} \sum_{k_{2}=1}^{n} G_{k_{2}, j_{2}} L_{j_{3}}^{k_{2}}+ \\
& +\frac{1}{2} \delta_{j_{2}}^{k_{1}} \sum_{k_{2}=1}^{n} G_{k_{2}, j_{3}} L_{j_{1}}^{k_{2}}-\frac{1}{2} \delta_{j_{3}}^{k_{1}} \sum_{k_{2}=1}^{n} G_{k_{2}, j_{2}} L_{j_{1}}^{k_{2}}+ \\
& +\sum_{k_{2}=1}^{n} H_{k_{2}, j_{3}}^{k_{1}} H_{j_{1}, j_{2}}^{k_{2}}-\sum_{k_{2}=1}^{n} H_{k_{2}, j_{2}}^{k_{1}} H_{j_{1}, j_{3}}^{k_{2}} .
\end{aligned}\right.
$$

(III')

$$
\left\{\begin{aligned}
& 0=\sum_{\sigma \in \mathfrak{S}_{2}}\left(\delta_{j_{3}}^{k_{2}} H_{j_{1}, j_{2}, y}^{k_{\sigma(1)}}-\delta_{j_{2}}^{k_{\sigma(2)}} H_{j_{1}, j_{3}, y}^{k_{\sigma \sigma}}+\right. \\
&+\frac{1}{2} \delta_{j_{2}}^{k_{\sigma(2)}} L_{j_{1}, x^{3}}^{k_{\sigma(1)}}-\frac{1}{2} \delta_{j_{3}}^{k_{\sigma(2)}} L_{j_{1}, x^{j_{2}}}^{k_{\sigma(1)}}+ \\
&+\frac{1}{2} \delta_{j_{1}}^{k_{\sigma(2)}} L_{j_{2}, x^{j_{3}}}^{k_{\sigma(1)}}-\frac{1}{2} \delta_{j_{1}}^{k_{\sigma_{(2)}}} L_{j_{3}, x^{j_{2}}}^{k_{\sigma(1)}}+ \\
&+\delta_{j_{2}}^{k_{\sigma(2)}} G_{j_{1}, j_{3}} M^{k_{\sigma(1)}}-\delta_{j_{3}}^{k_{\sigma(2)}} G_{j_{1}, j_{2}} M^{k_{\sigma(1)}}+ \\
&+\delta_{j_{1},{ }_{2}}^{k_{\sigma(1)}, k_{\sigma(2)}} \sum_{j_{3}=1}^{n} G_{k_{3}, j_{3}} M^{k_{3}}-\delta_{j_{1},}^{k_{\sigma(1)}, k_{\sigma(2)}} \sum_{j_{3}}^{n} G_{k_{3}=1}^{n} G_{k_{3}, j_{2}} M^{k_{3}}+ \\
&+\frac{1}{2} \delta_{j_{1}}^{k_{\sigma(1)}} \sum_{k_{3}=1}^{n} H_{k_{3}, j_{3}}^{k_{\sigma(2)}} L_{j_{2}}^{k_{3}}-\frac{1}{2} \delta_{j_{1}}^{k_{\sigma(1)}} \sum_{k_{3}=1}^{n} H_{k_{3}, j_{2}}^{k_{\sigma(2)}} L_{j_{3}}^{k_{3}}+ \\
&+\frac{1}{2} \delta_{j_{2}}^{k_{\sigma(1)}} \sum_{k_{3}=1}^{n} H_{k_{3}, j_{3}}^{k_{\sigma(2)}} L_{j_{1}}^{k_{3}}-\frac{1}{2} \delta_{j_{3}}^{k_{\sigma(1)}} \sum_{k_{3}=1}^{n} H_{k_{3}, j_{2}}^{k_{\sigma(2)}} L_{j_{1}}^{k_{3}}+ \\
&\left.+\frac{1}{2} \delta_{j_{3}}^{k_{\sigma(1)}} \sum_{k_{3}=1}^{n} H_{j_{1}, j_{2}}^{k_{3}} L_{k_{3}}^{k_{\sigma(2)}}-\frac{1}{2} \delta_{j_{2}}^{k_{\sigma(1)}} \sum_{k_{3}=1}^{n} H_{j_{1}, j_{3}}^{k_{3}} L_{k_{3}}^{k_{\sigma(2)}}\right) .
\end{aligned}\right.
$$

(IV')
(These systems (I'), (II'), (III') and (IV') should be distinguished from the systems (I), (II), (III) and (IV) of Theorem 1.7 in [Me2004], although they are quite similar.) Here, the indices $j_{1}, j_{2}, j_{3}, k_{1}, k_{2}, k_{3}$ vary in $\{1,2, \ldots, n\}$. By $\mathfrak{S}_{2}$ and by $\mathfrak{S}_{3}$, we denote the permutation group of $\{1,2\}$ and of $\{1,2,3\}$. To facilitate hand- and Latex-writing, partial derivatives are denoted as indices after a comma; for instance, $G_{j_{1}, j_{2}, x^{j_{3}}}$ is an abreviation for $\partial G_{j_{1}, j_{2}} / \partial x^{j_{3}}$. To deduce (I'), (II'), (III') and (IV') from equation (1.11), we
use the fact that every cubic polynomial equation of the form

$$
\begin{align*}
0 \equiv A & +\sum_{k_{1}=1}^{n} B_{k_{1}} \cdot y_{x^{k_{1}}}+\sum_{k_{1}=1}^{n} \sum_{k_{2}=1}^{n} C_{k_{1}, k_{2}} \cdot y_{x^{k_{1}}} y_{x^{k_{2}}}+ \\
& +\sum_{k_{1}=1}^{n} \sum_{k_{2}=1}^{n} \sum_{k_{2}=1}^{n} D_{k_{1}, k_{2}, k_{3}} \cdot y_{x^{k_{1}}} y_{x^{k_{2}}} y_{x^{k_{3}}} \tag{1.13}
\end{align*}
$$

is equivalent to the annihilation of the following symmetric sums of its coefficients:
(1.14)

$$
\left\{\begin{array}{l}
0=A \\
0=B_{k_{1}} \\
0=C_{k_{1}, k_{2}}+C_{k_{2}, k_{1}} \\
0=D_{k_{1}, k_{2}, k_{3}}+D_{k_{3}, k_{1}, k_{2}}+D_{k_{2}, k_{3}, k_{1}}+D_{k_{2}, k_{1}, k_{3}}+D_{k_{3}, k_{2}, k_{1}}+D_{k_{1}, k_{3}, k_{2}}
\end{array}\right.
$$

for all $k_{1}, k_{2}, k_{3}=1, \ldots, n$.
In conclusion, the functions $G_{j_{1}, j_{2}}, H_{j_{1}, j_{2}}^{k_{1}}, L_{j_{1}}^{k_{1}}$ and $M^{k_{1}}$ in the statement of Theorem 1.7 are far from being arbitrary: they satisfy the complicated system of first order partial differential equations (I'), (II'), (III') and (IV') above.

Our proof of Theorem 1.7 is similar to the one provided in [Me2004], in the case of systems of second order ordinary differential equations, so that most steps of the proof will be summarized.

In the end of this paper, we will delineate a complicated system of second order partial differential equations satisfied by $G_{j_{1}, j_{2}}, H_{j_{1}, j_{2}}^{k_{1}}, L_{j_{1}}^{k_{1}}$ and $M^{k_{1}}$ which is the exact analog of the system described in the abstract. The main technical part of the proof of Theorem 1.7 will be to establish that this second order system is a consequence, by linear combinations and by differentiations, of the first order system (I'), (II'), (III') and (IV').

Open question 1.15. Are Theorems 1.1 and 1.7 true under weaker smoothness assumptions, namely with a $\mathscr{C}^{2}$ or a $W_{\text {loc }}^{1, \infty}$ right-hand side ?

We refer to [Ma2003] for inspiration and appropriate tools.
Open question 1.16. Deduce from Theorem 1.7 an explicit necessary and sufficient condition for the associated submanifold of solutions $y=b+$ $\Pi\left(x^{i}, a^{k}, b\right)$ to be locally equivalent to $Y=B+X^{1} A^{1}+\cdots+X^{n} A^{n}$.

As an application, this would characterize local sphericity of a Levi nondegenerate hypersurface $M \subset \mathbb{C}^{n+1}$ with $n \geqslant 2$.

Generalizing the Lie-Tresse classification would be a great achievement.

Open problem 1.17. For $n=2$ establish a complete list of normal forms of all possible systems (1.?) according to their Lie symmetry group. In case of success, classify Levi nondegenerate real analytic hypersurfaces of $\mathbb{C}^{3}$ up to biholomorphisms.

## §2. COMPLETELY INTEGRABLE SYSTEMS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

2.1. Prolongation of a point transformation to the second order jet space. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, let $n \in \mathbb{N}$, suppose $n \geqslant 2$, let $x=\left(x^{1}, \ldots, x^{n}\right) \in$ $\mathbb{K}^{n}$ and let $y \in \mathbb{K}$. According to the main assumption of Theorem 1.7, we have to consider a local $\mathbb{K}$-analytic diffeomorphism of the form

$$
\begin{equation*}
\left(x^{j_{1}}, y\right) \longmapsto\left(X^{j}\left(x^{j_{1}}, y\right), Y\left(x^{j_{1}}, y\right)\right), \tag{2.2}
\end{equation*}
$$

which transforms the system (1.8) to the system $Y_{X^{i_{1} X^{i_{2}}}}=0,1 \leqslant j_{1}, j_{2} \leqslant$ $n$. Without loss of generality, we shall assume that this transformation is close to the identity. To obtain the precise expression (2.35) of the transformed system (1.8), we have to prolong the above diffeomorphism to the second order jet space. We introduce the coordinates $\left(x^{j}, y, y_{x^{j_{1}}}, y_{x^{j_{1}} x^{j_{2}}}\right)$ on the second order jet space. Let

$$
\begin{equation*}
D_{k}:=\frac{\partial}{\partial x^{k}}+y_{x^{k}} \frac{\partial}{\partial y}+\sum_{l=1}^{n} y_{x^{k} x^{l}} \frac{\partial}{\partial y_{x^{l}}} \tag{2.3}
\end{equation*}
$$

be the $k$-th total differentiation operator. According to [Ol1986, BK1989, Ol1995], for the first order partial derivatives, one has the (implicit, compact) expression:

$$
\left(\begin{array}{c}
Y_{X^{1}}  \tag{2.4}\\
\vdots \\
Y_{X^{n}}
\end{array}\right)=\left(\begin{array}{ccc}
D_{1} X^{1} & \cdots & D_{1} X^{n} \\
\vdots & \cdots & \vdots \\
D_{n} X^{1} & \cdots & D_{n} X^{n}
\end{array}\right)^{-1}\left(\begin{array}{c}
D_{1} Y \\
\vdots \\
D_{n} Y
\end{array}\right)
$$

where $(\cdot)^{-1}$ denotes the inverse matrix, which exists, since the transformation (2.2) is close to the identity. For the second order partial derivatives, again according to [Ol1986, BK1989, Ol1995], one has the (implicit, compact) expressions:

$$
\left(\begin{array}{c}
Y_{X^{j} X^{1}}  \tag{2.5}\\
\vdots \\
Y_{X^{j} X^{n}}
\end{array}\right)=\left(\begin{array}{ccc}
D_{1} X^{1} & \cdots & D_{1} X^{n} \\
\vdots & \cdots & \vdots \\
D_{n} X^{1} & \cdots & D_{n} X^{n}
\end{array}\right)^{-1}\left(\begin{array}{c}
D_{1} Y_{X^{j}} \\
\vdots \\
D_{n} Y_{X^{j}}
\end{array}\right)
$$

for $j=1, \ldots, n$. Let $D X$ denote the matrix $\left(D_{i} X^{j}\right)_{1 \leqslant i \leqslant n}^{1 \leqslant i \leqslant n}$, where $i$ is the index of lines and $j$ the index of columns, let $Y_{X}$ denote the column matrix $\left(Y_{X^{i}}\right)_{1 \leqslant i \leqslant n}$ and let $D Y$ be the column matrix $\left(D_{i} Y\right)_{1 \leqslant i \leqslant n}$.

By inspecting (2.5) above, we see that the equivalence between (i), (ii) and (iii) just below is obvious:

Lemma 2.6. The following conditions are equivalent:
(i) the differential equations $Y_{X^{j} X^{k}}=0$ hold for $1 \leqslant j, k \leqslant n$;
(ii) the matrix equations $D_{k}\left(Y_{X}\right)=0$ hold for $1 \leqslant k \leqslant n$;
(iii) the matrix equations $D X \cdot D_{k}\left(Y_{X}\right)=0$ hold for $1 \leqslant k \leqslant n$;
(iv) the matrix equations $0=D_{k}(D X) \cdot Y_{X}-D_{k}(D Y)$ hold for $1 \leqslant k \leqslant n$.

Formally, in the sequel, it will be more convenient to achieve the explicit computations starting from condition (iv), since no matrix inversion at all is involved in it.

Proof. Indeed, applying the total differentiation operator $D_{k}$ to the matrix equation (2.4) written under the equivalent form $0=D X \cdot Y_{X}-D Y$, we get:

$$
\begin{equation*}
0=D_{k}(D X) \cdot Y_{X}+D X \cdot D_{k}\left(Y_{X}\right)-D_{k}(D Y) \tag{2.7}
\end{equation*}
$$

so that the equivalence between (iii) and (iv) is now clear.
2.8. An explicit formula in the case $n=2$. Thus, we can start to develope explicitely the matrix equations

$$
\begin{equation*}
0=D_{k}(D X) \cdot Y_{X}-D_{k}(D Y) \tag{2.9}
\end{equation*}
$$

In it, some huge formal expressions are hidden behind the symbol $D_{k}$. Proceeding inductively, we start by examinating the case $n=2$ thoroughly. By direct computations which require to be clever, we reconstitute some $3 \times 3$ determinants in the four (in fact three) developed equations (2.9). After some work, the first equation is:

$$
\begin{align*}
& 0=y_{x^{1} x^{1}} \cdot\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{y}^{1} \\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{y}
\end{array}\right|+\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{x^{1} x^{1}}^{1} \\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{x^{1} x^{1}}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{x^{1} x^{1}}
\end{array}\right|+  \tag{2.10}\\
& +y_{x^{1}} \cdot\left\{2\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{x^{1} y}^{1} \\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{x^{1} y}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{x^{1} y}
\end{array}\right|-\left|\begin{array}{ccc}
X_{x^{1} x^{1}}^{1} & X_{x^{2}}^{1} & X_{y}^{1} \\
X_{x^{1} 1^{1}}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{1} x^{1}} & Y_{x^{2}} & Y_{y}
\end{array}\right|\right\}+ \\
& +y_{x^{2}} \cdot\left\{-\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{1} x^{1}}^{1} & X_{y}^{1} \\
X_{x^{1}}^{2} & X_{x^{1} x^{1}}^{2} & X_{y}^{2} \\
Y_{x^{1}} & Y_{x^{1} x^{1}} & Y_{y}
\end{array}\right|\right\}+
\end{align*}
$$

$$
\begin{aligned}
& +y_{x^{1}} y_{x^{1}} \cdot\left\{\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{y y}^{1} \\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{y y}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{y y}
\end{array}\right|-2\left|\begin{array}{ccc}
X_{x^{1} y}^{1} & X_{x^{2}}^{1} & X_{y}^{1} \\
X_{x^{1} y}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{1} y} & Y_{x^{2}} & Y_{y}
\end{array}\right|\right\}+ \\
& +y_{x^{1}} y_{x^{2}} \cdot\left\{-2\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{1} y}^{1} & X_{y}^{1} \\
X_{x^{1}}^{2} & X_{x^{1} y}^{2} & X_{y}^{2} \\
Y_{x^{1}} & Y_{x^{1} y} & Y_{y}
\end{array}\right|\right\}+ \\
& +y_{x^{1}} y_{x^{1}} y_{x^{1}} \cdot\left\{-\left|\begin{array}{ccc}
X_{y y}^{1} & X_{x^{2}}^{1} & X_{y}^{1} \\
X_{y y}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{y y} & Y_{x^{2}} & Y_{y}
\end{array}\right|\right\}+ \\
& +y_{x^{1}} y_{x^{1}} y_{x^{2}} \cdot\left\{-\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{y y}^{1} & X_{y}^{1} \\
X_{x^{1}}^{2} & X_{y y}^{2} & X_{y}^{2} \\
Y_{x^{1}} & Y_{y y} & Y_{y}
\end{array}\right|\right\} .
\end{aligned}
$$

This formula and the two next (2.22), (2.23) have been checked by Sylvain Neut and Michel Petitot with the help of Maple.
2.11. Comparison with the coefficients of the second prolongation of a vector field. At present, it is useful to make an illuminating digression which will help us to devise what is the general form of the development of the equations (2.9). Consider an arbitrary vector field of the form

$$
\begin{equation*}
\mathscr{L}:=\sum_{k=1}^{n} \mathscr{X}^{k} \frac{\partial}{\partial x^{k}}+\mathscr{Y} \frac{\partial}{\partial y}, \tag{2.12}
\end{equation*}
$$

where the coefficients $\mathscr{X}^{k}$ and $\mathscr{Y}$ are functions of $\left(x^{i}, y\right)$. According to [Ol1986, BK1989, Ol1995], there exists a unique prolongation $\mathscr{L}^{(2)}$ of this vector field to the second order jet space, of the form

$$
\begin{equation*}
\mathscr{L}^{(2)}:=\mathscr{L}+\sum_{j_{1}=1}^{n} \mathbf{Y}_{j_{1}} \frac{\partial}{\partial y_{x^{j_{1}}}}+\sum_{j_{1}=1}^{n} \sum_{j_{2}=1}^{n} \mathbf{Y}_{j_{1}, j_{2}} \frac{\partial}{\partial y_{x^{j_{1} x^{j_{2}}}}}, \tag{2.13}
\end{equation*}
$$

where the coefficients $\mathbf{Y}_{j_{1}}, \mathbf{Y}_{j_{1}, j_{2}}$ may be computed by means of formulas (3.4) of Section 3(II). In Part II, we obtained the following perfect formulas:
(2.14)

$$
\left\{\begin{aligned}
\mathbf{Y}_{j_{1}, j_{2}}= & \mathscr{Y}_{x^{j_{1}} x^{j_{2}}}+\sum_{k_{1}=1}^{n} y_{x^{k_{1}}} \cdot\left\{\delta_{j_{1}}^{k_{1}} \mathscr{Y}_{x^{j_{2}} y}+\delta_{j_{2}}^{k_{1}} \mathscr{Y}_{x^{j_{1}} y}-\mathscr{X}_{x^{j_{1}} x^{j_{2}}}^{k_{1}}\right\}+ \\
& +\sum_{k_{1}=1}^{n} \sum_{k_{2}=1}^{n} y_{x^{k_{1}}} y_{x^{k_{2}}} \cdot\left\{\delta_{j_{1}, j_{2}}^{k_{1}, k_{2}} \mathscr{Y}_{y y}-\delta_{j_{1}}^{k_{1}} \mathscr{X}_{x^{j_{2}} y}^{k_{2}}-\delta_{j_{2}}^{k_{1}} \mathscr{X}_{\left.x^{j_{1}}\right\}}^{k_{2}}\right\}+ \\
& +\sum_{k_{1}=1}^{n} \sum_{k_{2}=1}^{n} \sum_{k_{3}=1}^{n} y_{x^{k_{1}}} y_{x^{k_{2}}} y_{x^{k_{3}}} \cdot\left\{-\delta_{j_{1}, j_{2}}^{k_{1}, k_{2}} \mathscr{X}_{y y}^{k_{3}}\right\},
\end{aligned}\right.
$$

for $j_{1}, j_{2}=1, \ldots, n$. The expression of $\mathbf{Y}_{j_{1}}$ does not matter for us here. Specifying this formula to the the case $n=2$ and taking account of the symmetry $\mathbf{Y}_{1,2}=\mathbf{Y}_{2,1}$ we get the following three second order coefficients: (2.15)

$$
\left\{\begin{aligned}
\mathbf{Y}_{1,1}= & \mathscr{Y}_{x^{1} x^{1}}+y_{x^{1}} \cdot\left\{2 \mathscr{Y}_{x^{1} y}-\mathscr{X}_{x^{1} x^{1}}^{1}\right\}+y_{x^{2}} \cdot\left\{-\mathscr{X}_{x^{1} x^{1}}^{2}\right\}+ \\
& +y_{x^{1}} y_{x^{1}} \cdot\left\{\mathscr{Y}_{y y}-2 \mathscr{X}_{x^{1} y}^{1}\right\}+y_{x^{1}} y_{x^{2}} \cdot\left\{-2 \mathscr{X}_{x^{1} y}^{2}\right\}+ \\
& +y_{x^{1}} y_{x^{1}} y_{x^{1}} \cdot\left\{-\mathscr{X}_{y y}^{1}\right\}+y_{x^{1}} y_{x^{1}} y_{x^{2}} \cdot\left\{-\mathscr{X}_{y y}^{2}\right\}, \\
\mathbf{Y}_{1,2}= & \mathscr{Y}_{x^{1} x^{2}}+y_{x^{1}} \cdot\left\{\mathscr{Y}_{x^{2} y}-\mathscr{X}_{x^{1} x^{2}}^{1}\right\}+y_{x^{2}} \cdot\left\{\mathscr{Y}_{x^{1} y}-\mathscr{X}_{x^{1} x^{2}}^{2}\right\}+ \\
& +y_{x^{1}} y_{x^{1}} \cdot\left\{-\mathscr{X}_{x^{2} y}^{1}\right\}+y_{x^{1}} y_{x^{2}} \cdot\left\{\mathscr{Y}_{y y}-\mathscr{X}_{x^{1} y}^{1}-\mathscr{X}_{x^{2} y}^{2}\right\}+ \\
& +y_{x^{2}} y_{x^{2}} \cdot\left\{-\mathscr{X}_{x^{1} y}^{2}\right\}+ \\
& +y_{x^{1}} y_{x^{1}} y_{x^{2}} \cdot\left\{-\mathscr{X}_{y y}^{1}\right\}+y_{x^{1}} y_{x^{2}} y_{x^{2}} \cdot\left\{-\mathscr{X}_{y y}^{2}\right\}, \\
\mathbf{Y}_{2,2}= & \mathscr{Y}_{x^{2} x^{2}}+y_{x^{1}} \cdot\left\{-\mathscr{X}_{x^{2} x^{2}}^{1}\right\}+y_{x^{2}} \cdot\left\{2 \mathscr{Y}_{x^{2} y}-\mathscr{X}_{x^{2} x^{2}}^{2}\right\}+ \\
& +y_{x^{1}} y_{x^{2}} \cdot\left\{-2 \mathscr{X}_{x^{2} y}^{1}\right\}+y_{x^{2}} y_{x^{2}} \cdot\left\{\mathscr{Y}_{y y}-2 \mathscr{X}_{x^{2} y}^{2}\right\}+ \\
& +y_{x^{1}} y_{x^{2}} y_{x^{2}} \cdot\left\{-\mathscr{X}_{y y}^{1}\right\}+y_{x^{2}} y_{x^{2}} y_{x^{2}} \cdot\left\{-\mathscr{X}_{y y}^{2}\right\} .
\end{aligned}\right.
$$

We would like to mention that the computation of $\mathbf{Y}_{j_{1}, j_{2}}, 1 \leqslant j_{1}, j_{2} \leqslant 2$, above is easier than the verification of (2.10). Based on the three formulas (2.15), we claim that we can guess the second and the third equations, which would be obtained by developing and by simplifying (2.9), namely with $y_{x^{1} x^{2}}$ and with $y_{x^{2} x^{2}}$ instead of $y_{x^{1} x^{2}}$ in (2.10). Our dictionary to translate from the first formula (2.15) to (2.10) may be described as follows. Begin with the Jacobian determinant

$$
\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{y}^{1}  \tag{2.16}\\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{y}
\end{array}\right|
$$

of the change of coordinates (2.2). Since this change of coordinates is close to the identity, we may consider that the following Jacobian matrix approximation holds:

$$
\left(\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{y}^{1}  \tag{2.17}\\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{y}
\end{array}\right) \cong\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

The jacobian matrix has three columns. There are six possible second order derivatives with respect to the variables $\left(x^{1}, x^{2}, y\right)$, namely

$$
\begin{equation*}
(\cdot)_{x^{1} x^{1}}, \quad(\cdot)_{x^{1} x^{2}}, \quad(\cdot)_{x^{2} x^{2}}, \quad(\cdot)_{x^{1} y}, \quad(\cdot)_{x^{2} y}, \quad(\cdot)_{y y} \tag{2.18}
\end{equation*}
$$

In the Jacobian determinant (2.16), by replacing any one of the three columns of first order derivatives with a column of second order derivatives, we obtain exactly $3 \times 6=18$ possible determinants. For instance, by
replacing the third column by the second order derivative $(\cdot)_{x^{1} y}$ or the first column by the second order derivative $(\cdot)_{x^{1} x^{1}}$, we get:

$$
\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{x^{1} y}^{1}  \tag{2.19}\\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{x^{1} y}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{x^{1} y}
\end{array}\right| \quad \text { or } \quad\left|\begin{array}{ccc}
X_{x^{1} x^{1}}^{1} & X_{x^{2}}^{1} & X_{y}^{1} \\
X_{x^{1} x^{1}}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{1} x^{1}} & Y_{x^{2}} & Y_{y}
\end{array}\right| .
$$

We recover the two determinants appearing in the second line of (2.10). On the other hand, according to the approximation (2.17), these two determinants are essentially equal to
(2.20) $\left|\begin{array}{ccc}1 & 0 & X_{x^{1} y}^{1} \\ 0 & 1 & X_{x^{1} y}^{2} \\ 0 & 0 & Y_{x^{1} y}\end{array}\right|=Y_{x^{1} y} \quad$ or to $\quad\left|\begin{array}{ccc}X_{x^{1} x^{1}}^{1} & 0 & 0 \\ X_{x^{1} x^{1}}^{2} & 1 & 0 \\ Y_{x^{1} x^{1}} & 0 & 1\end{array}\right|=X_{x^{1} x^{1}}^{1}$.

Consequently, in the second line of (2.10), up to a change to calligraphic letters, we recover the coefficient

$$
\begin{equation*}
2 \mathscr{Y}_{x^{1} y}-\mathscr{X}_{x^{1} x^{1}}^{1} \tag{2.21}
\end{equation*}
$$

of $y_{x_{1}}$ in the expression of $\mathbf{Y}_{1,1}$ in (2.15). In conclusion, we have discovered how to pass symbolically from the first equation (2.15) to the equation (2.10) and conversely.

Translating the second equation (2.15), we deduce, without any further computation, that the second equation which would be obtained by developing (2.9) in length, is:

$$
\begin{aligned}
& 0=y_{x^{1} x^{2}} \cdot\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{y}^{1} \\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{y}
\end{array}\right|+\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{x^{1}}^{1} x^{2} \\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{x^{1} x^{2}}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{x^{1} x^{2}}
\end{array}\right|+ \\
& +y_{x^{1}} \cdot\left\{\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{x^{2} y}^{1} \\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{x^{2} y}^{2} \\
Y_{x^{1}}^{1} & Y_{x^{2}} & Y_{x^{2} y}
\end{array}\right|-\left|\begin{array}{ccc}
X_{x^{1} x^{2}}^{1} & X_{x^{2}}^{1} & X_{y}^{1} \\
X_{x^{1} x^{2}}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{1} x^{2}} & Y_{x^{2}} & Y_{y}
\end{array}\right|\right\}+ \\
& +y_{x^{2}} \cdot\left\{\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{x^{1} y}^{1} \\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{x^{1} y}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{x^{1} y}
\end{array}\right|-\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{1} x^{2}}^{1} & X_{y}^{1} \\
X_{x^{1}}^{2} & X_{x^{1} x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{1}} & Y_{x^{1} x^{2}} & Y_{y}
\end{array}\right|\right\}+
\end{aligned}
$$

$$
\begin{aligned}
& +y_{x^{1}} y_{x^{1}} \cdot\left\{-\left|\begin{array}{ccc}
X_{x^{2} y}^{1} & X_{x^{2}}^{1} & X_{y}^{1} \\
X_{x^{2} y}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{2} y} & Y_{x^{2}} & Y_{y}
\end{array}\right|\right\}+ \\
& +y_{x^{1}} y_{x^{2}} \cdot\left\{\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{y y}^{1} \\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{y y}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{y y}
\end{array}\right|-\left|\begin{array}{ccc}
X_{x^{1} y}^{1} & X_{x^{2}}^{1} & X_{y}^{1} \\
X_{x^{1} y}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{1} y} & Y_{x^{2}} & Y_{y}
\end{array}\right|-\right. \\
& \left.-\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2} y}^{1} & X_{y}^{1} \\
X_{x^{1}}^{2} & X_{x^{2} y}^{2} & X_{y}^{2} \\
Y_{x^{1}} & Y_{x^{2} y} & Y_{y}
\end{array}\right|\right\}+y_{x^{2}} y_{x^{2}}\left\{-\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{1} y}^{1} & X_{y}^{1} \\
X_{x^{1}}^{2} & X_{x^{1} y}^{2} & X_{y}^{2} \\
Y_{x^{1}}^{1} & Y_{x^{1} y} & Y_{y}
\end{array}\right|\right\}+ \\
& +y_{x^{1}} y_{x^{1}} y_{x^{2}} \cdot\left\{-\left\{\left.\begin{array}{ccc}
X_{y y}^{1} & X_{x^{2}}^{1} & X_{y}^{1} \\
X_{y y}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{y y} & Y_{x^{2}} & Y_{y}
\end{array} \right\rvert\,\right\}+\right. \\
& +y_{x^{1}} y_{x^{2}} y_{x^{2}} \cdot\left\{-\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{y y}^{1} & X_{y}^{1} \\
X_{x^{1}}^{2} & X_{y y}^{2} & X_{y}^{2} \\
Y_{x^{1}} & Y_{y y} & Y_{y}
\end{array}\right|\right\} .
\end{aligned}
$$

Using the third equation (2.15), we also deduce, without any further computation, that the third equation which would be obtained by developing (2.9) in length, is:

$$
\begin{align*}
0=y_{x^{2} x^{2}} \cdot & \left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{y}^{1} \\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{y}
\end{array}\right|+\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{x^{2} x^{2}}^{1} \\
X_{x^{1}}^{2} & X_{x^{2}} & X_{x^{2} x^{2}}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{x^{2} x^{2}}
\end{array}\right|+  \tag{2.23}\\
+y_{x^{1}} \cdot & \left\{-\left|\begin{array}{ccc}
X_{x^{2} x^{2}}^{1} & X_{x^{2}}^{1} & X_{y}^{1} \\
X_{x^{1} x^{2}}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{2} x^{2}} & Y_{x^{2}} & Y_{y}
\end{array}\right|\right\}+ \\
& +y_{x^{2}} \cdot\left\{2\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{x^{2} y}^{1} \\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{x^{2}}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{x^{2} y}
\end{array}\right|-\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2} x^{2}}^{1} & X_{y}^{1} \\
X_{x^{1}}^{2} & X_{x^{2} x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{1}}^{1} & Y_{x^{2} x^{2}} & Y_{y}
\end{array}\right|\right\}+
\end{align*}
$$

$$
\begin{aligned}
& +y_{x^{1}} y_{x^{2}} \cdot\left\{-2\left|\begin{array}{ccc}
X_{x^{2} y}^{1} & X_{x^{2}}^{1} & X_{y}^{1} \\
X_{x^{2} y}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{2} y} & Y_{x^{2}} & Y_{y}
\end{array}\right|\right\}+ \\
& +y_{x^{2}} y_{x^{2}} \cdot\left\{\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{y y}^{1} \\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{y y}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{y y}
\end{array}\right|-2\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2} y}^{1} & X_{y}^{1} \\
X_{x^{1}}^{2} & X_{x^{2} y}^{2} & X_{y}^{2} \\
Y_{x^{1}} & Y_{x^{2} y} & Y_{y}
\end{array}\right|\right\}+ \\
& +y_{x^{1}} y_{x^{2}} y_{x^{2}} \cdot\left\{-\left|\begin{array}{ccc}
X_{y y}^{1} & X_{x^{2}}^{1} & X_{y}^{1} \\
X_{y y}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{y y} & Y_{x^{2}} & Y_{y}
\end{array}\right|\right\}+ \\
& +y_{x^{2}} y_{x^{2}} y_{x^{2}} \cdot\left\{-\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{y y}^{1} & X_{y}^{1} \\
X_{x^{1}}^{2} & X_{y y}^{2} & X_{y}^{2} \\
Y_{x^{1}} & Y_{y y} & Y_{y}
\end{array}\right|\right\} .
\end{aligned}
$$

2.24. Appropriate formalism. To describe the combinatorics underlying formulas (2.10), (2.22) and (2.23), as in [Me2004], let us introduce the following notation for the Jacobian determinant:

$$
\Delta\left(x^{1}\left|x^{2}\right| y\right):=\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{y}^{1}  \tag{2.25}\\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{y}
\end{array}\right| .
$$

Here, in the notation $\Delta\left(x^{1}\left|x^{2}\right| y\right)$, the three spaces between the two vertical lines $\mid$ refer to the three columns of the Jacobian determinant, and the terms $x^{1}, x^{2}, y$ in $\left(x^{1}\left|x^{2}\right| y\right)$ designate the partial derivatives appearing in each column. Accordingly, in the following two examples of modified Jacobian determinants:

$$
\left\{\begin{align*}
\Delta\left(\underline{x^{1} x^{2}}\left|x^{2}\right| y\right) & :=\left|\begin{array}{ccc}
X_{x^{1} x^{2}}^{1} & X_{x^{2}}^{1} & X_{y}^{1} \\
X_{\underline{x^{1} x^{2}}}^{2} & X_{x^{2}}^{2} & X_{y}^{2} \\
Y_{\underline{x^{1} x^{2}}}^{2} & Y_{x^{2}} & Y_{y}
\end{array}\right| \quad \text { and }  \tag{2.26}\\
\Delta\left(x^{1}\left|x^{2}\right| \underline{x^{1} y}\right) & :=\left|\begin{array}{ccc}
X_{x^{1}}^{1} & X_{x^{2}}^{1} & X_{x^{1} y}^{1} \\
X_{x^{1}}^{2} & X_{x^{2}}^{2} & X_{x^{1} y}^{2} \\
Y_{x^{1}} & Y_{x^{2}} & Y_{\underline{x^{1} y}}
\end{array}\right|
\end{align*}\right.
$$

we simply mean which column of first order derivatives is replaced by a column of second order derivatives in the original Jacobian determinant.

As there are 6 possible second order derivatives $(\cdot)_{x^{1} x^{1}},(\cdot)_{x^{1} x^{2}},(\cdot)_{x^{1} x^{y}}$, $(\cdot)_{x^{2} x^{2}},(\cdot)_{x^{2} y}$ and $(\cdot)_{y y}$ together with 3 columns, we obtain $3 \times 6=18$
possible modified Jacobian determinants:

$$
\left\{\begin{array}{lrl}
\Delta\left(x^{1} x^{1}\left|x^{2}\right| y\right) & \Delta\left(x^{1}\left|x^{1} x^{1}\right| y\right) & \Delta\left(x^{1}\left|x^{2}\right| x^{1} x^{1}\right)  \tag{2.27}\\
\Delta\left(x^{1} x^{2}\left|x^{2}\right| y\right) & \Delta\left(x^{1}\left|x^{1} x^{2}\right| y\right) & \Delta\left(x^{1}\left|x^{2}\right| x^{1} x^{2}\right) \\
\Delta\left(x^{1} y\left|x^{2}\right| y\right) & \Delta\left(x^{1}\left|x^{1} y\right| y\right) & \Delta\left(x^{1}\left|x^{2}\right| x^{1} y\right) \\
\Delta\left(x^{2} x^{2}\left|x^{2}\right| y\right) & \Delta\left(x^{1}\left|x^{2} x^{2}\right| y\right) & \Delta\left(x^{1}\left|x^{2}\right| x^{2} x^{2}\right) \\
\Delta\left(x^{2} y\left|x^{2}\right| y\right) & \Delta\left(x^{1}\left|x^{2} y\right| y\right) & \Delta\left(x^{1}\left|x^{2}\right| x^{2} y\right) \\
\Delta\left(y y\left|x^{2}\right| y\right) & \Delta\left(x^{1}|y y| y\right) & \Delta\left(x^{1}\left|x^{2}\right| y y\right) .
\end{array}\right.
$$

Next, we observe that if we want to solve with respect to $y_{x^{1} x^{1}}$ in (2.10), with respect to $y_{x^{1} x^{2}}$ in (2.22) and with respect to $y_{x^{2} x^{2}}$ in (2.23), we have to divide by the Jacobian determinant $\Delta\left(x^{1}\left|x^{2}\right| y\right)$. Consequently, we introduce 18 new square functions as follows:

$$
\left\{\begin{array}{lll}
\square_{x^{1} x^{1}}^{1}:=\frac{\Delta\left(x^{1} x^{1}\left|x^{2}\right| y\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} & \square_{x^{1} x^{2}}^{1}:=\frac{\Delta\left(x^{1} x^{2}\left|x^{2}\right| y\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} & \square_{x^{1} y}^{1}:=\frac{\Delta\left(x^{1} y\left|x^{2}\right| y\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)}  \tag{2.28}\\
\square_{x^{2} x^{2}}^{1}:=\frac{\Delta\left(x^{2} x^{2}\left|x^{2}\right| y\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} & \square_{x^{2} y}^{1}:=\frac{\Delta\left(x^{2} y\left|x^{2}\right| y\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} & \square_{y y}^{1}:=\frac{\Delta\left(y y\left|x^{2}\right| y\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} \\
\square_{x^{1} x^{1}}^{2}:=\frac{\Delta\left(x^{1}\left|x^{1} x^{1}\right| y\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} & \square_{x^{1} x^{2}}^{2}:=\frac{\Delta\left(x^{1}\left|x^{1} x^{2}\right| y\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} & \square_{x^{1} y}^{2}:=\frac{\Delta\left(x^{1}\left|x^{1} y\right| y\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} \\
\square_{x^{2} x^{2}}^{2}:=\frac{\Delta\left(x^{1}\left|x^{2} x^{2}\right| y\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} & \square_{x^{2} y}^{2}:=\frac{\Delta\left(x^{1}\left|x^{2}\right| y\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} & \square_{y y}^{2}:=\frac{\Delta\left(x^{1}|y y| y\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} \\
\square_{x^{1} x^{1}}^{3}:=\frac{\Delta\left(x^{1}\left|x^{2}\right| x^{1} x^{1}\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} & \square_{x^{1} x^{2}}^{3}:=\frac{\Delta\left(x^{1}\left|x^{2}\right| x^{1} x^{2}\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} & \square_{x^{1} y}^{3}:=\frac{\Delta\left(x^{1}\left|x^{2}\right| x^{1} y\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} \\
\square_{x^{2} x^{2}}^{3}:=\frac{\Delta\left(x^{1}\left|x^{2}\right| x^{2} x^{2}\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} & \square_{x^{2} y}^{3}:=\frac{\Delta\left(x^{1}\left|x^{2}\right| x^{2} y\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} & \square_{y y}^{3}:=\frac{\Delta\left(x^{1}\left|x^{2}\right| y y\right)}{\Delta\left(x^{1}\left|x^{2}\right| y\right)} .
\end{array}\right.
$$

Thanks to these notations, we can rewrite the three equations (2.10), (2.22) and (2.23) in a more compact style.

Lemma 2.29. A completely integrable system of three second order partial differential equations

$$
\left\{\begin{array}{l}
y_{x^{1} x^{1}}(x)=F^{1,1}\left(x^{1}, x^{2}, y(x), y_{x^{1}}(x), y_{x^{2}}(x)\right)  \tag{2.30}\\
y_{x^{1} x^{2}}(x)=F^{1,2}\left(x^{1}, x^{2}, y(x), y_{x^{1}}(x), y_{x^{2}}(x)\right) \\
y_{x^{2} x^{2}}(x)=F^{2,2}\left(x^{1}, x^{2}, y(x), y_{x^{1}}(x), y_{x^{2}}(x)\right)
\end{array}\right.
$$

is equivalent to the simplest system $Y_{X^{1} X^{1}}=0, Y_{X^{1} X^{2}}=0, Y_{X^{2}, X^{2}}=0$, if and only if there exist local $\mathbb{K}$-analytic functions $X^{1}, X^{2}, Y$ such that it may
be written under the specific form:
(2.31)

$$
\left\{\begin{aligned}
y_{x^{1} x^{1}}= & -\square_{x^{1} x^{1}}^{3}+y_{x^{1}} \cdot\left(-2 \square_{x^{1} y}^{3}+\square_{x^{1} x^{1}}^{1}\right)+y_{x^{2}} \cdot\left(\square_{x^{1} x^{1}}^{2}\right)+ \\
& +y_{x^{1}} y_{x^{1}} \cdot\left(-\square_{y y}^{3}+2 \square_{x^{1} y}^{1}\right)+y_{x^{1}} y_{x^{2}} \cdot\left(2 \square_{x^{1} y}^{2}\right)+ \\
& +y_{x^{1}} y_{x^{1}} y_{x^{1}} \cdot\left(\square_{y y}^{1}\right)+y_{x^{1}} y_{x^{1}} y_{x^{2}} \cdot\left(\square_{y y}^{2}\right), \\
y_{x^{1} x^{2}}= & -\square_{x^{1} x^{2}}^{3}+y_{x^{1}} \cdot\left(-\square_{x^{2} y}^{3}+\square_{x^{1} x^{2}}^{1}\right)+y_{x^{2}} \cdot\left(-\square_{x^{1} y}^{3}+\square_{x^{1} x^{2}}^{2}\right)+ \\
& +y_{x^{1}} y_{x^{1}} \cdot\left(\square_{x^{2} y}^{1}\right)+y_{x^{1}} y_{x^{2}} \cdot\left(-\square_{y y}^{3}+\square_{x^{1} y}^{1}+\square_{x^{2} y}^{2}\right)+ \\
& +y_{x^{2}} y_{x^{2}} \cdot\left(\square_{x^{1} y}^{2}\right)+y_{x^{1}} y_{x^{1}} y_{x^{2}} \cdot\left(\square_{y y}^{1}\right)+y_{x^{1}} y_{x^{2}} y_{x^{2}} \cdot\left(\square_{y y}^{2}\right) \\
y_{x^{2} x^{2}}= & -\square_{x^{2} x^{2}}^{3}+y_{x^{1}} \cdot\left(\square_{x^{2} x^{2}}^{1}\right)+y_{x^{2}} \cdot\left(-2 \square_{x^{2} y}^{3}+\square_{x^{2} x^{2}}^{2}\right)+ \\
& +y_{x^{1}} y_{x^{2}} \cdot\left(2 \square_{x^{2} y}^{1}\right)+y_{x^{2}} y_{x^{2}} \cdot\left(-\square_{y y}^{3}+2 \square_{x^{2} y}^{2}\right)+ \\
& +y_{x^{1}} y_{x^{2}} y_{x^{2}} \cdot\left(\square_{y y}^{1}\right)+y_{x^{2}} y_{x^{2}} y_{x^{2}} \cdot\left(\square_{y y}^{2}\right) .
\end{aligned}\right.
$$

2.32. General formulas. The formal dictionary between the original determinantial formulas (2.10), (2.22), (2.23), between the coefficients (2.15) of the second order prolongation of a vector field and between the new square formulas (2.31) above is evident. Consequently, without any computation, just by translating the family of formulas (2.14), we may deduce the exact formulation of the desired generalization of Lemma 2.29 above.

Lemma 2.33. A completely integrable system of second order partial differential equations of the form

$$
\begin{equation*}
y_{x^{j_{1}} x^{j_{2}}}(x)=F^{j_{1}, j_{2}}\left(x, y(x), y_{x^{1}}(x), \ldots, y_{x^{n}}(x)\right), \quad j_{1}, j_{2}=1, \ldots n \tag{2.34}
\end{equation*}
$$

is equivalent to the simplest system $Y_{X^{j_{1} X^{j}}}=0, j_{1}, j_{2}=1, \ldots, n$, if and only if there exist local $\mathbb{K}$-analytic functions $X^{l}$, Y such that it may be written under the specific form:
(2.35)

$$
\left\{\begin{aligned}
y_{x^{j_{1}} x^{j_{2}}}= & -\square_{x^{j_{1}} x^{j_{2}}}^{n+1}+\sum_{k_{1}=1}^{n} y_{x^{k_{1}}} \cdot\left\{\left(\square_{x^{j_{1}} x^{j_{2}}}^{k_{1}}-\delta_{j_{1}}^{k_{1}} \square_{x^{j_{2}} y}^{n+1}-\delta_{j_{2}}^{k_{1}} \square_{x^{j_{1}} y}^{n+1}\right)+\right. \\
& +y_{x^{j_{1}}} \cdot\left(\square_{x^{j_{2}} y}^{k_{1}}-\frac{1}{2} \delta_{j_{2}}^{k_{1}} \square_{y y}^{n+1}\right)+y_{x^{j_{2}}} \cdot\left(\square_{x^{j_{1}} y}^{k_{1}}-\frac{1}{2} \delta_{j_{1}}^{k_{1}} \square_{y y}^{n+1}\right)+ \\
& \left.+y_{x^{j_{1}}} y_{x^{j_{2}}} \cdot\left(\square_{y y}^{k_{1}}\right)\right\} .
\end{aligned}\right.
$$

Of course, to define the square functions in the context of $n \geqslant 2$ independent variables $\left(x^{1}, x^{2}, \ldots, x^{n}\right)$, we introduce the Jacobian determinant

$$
\Delta\left(x^{1}\left|x^{2}\right| \cdots\left|x^{n}\right| y\right):=\left|\begin{array}{cccc}
X_{x^{1}}^{1} & \cdots & X_{x^{n}}^{1} & X_{y}^{1}  \tag{2.36}\\
\vdots & \cdots & \vdots & \vdots \\
X_{x^{1}}^{n} & \cdots & X_{x^{n}}^{n} & X_{y}^{n} \\
Y_{x^{1}} & \cdots & Y_{x^{n}} & Y_{y}
\end{array}\right|
$$

together with its modifications

$$
\begin{equation*}
\Delta\left(x^{1}|\cdots|^{k_{1}} x^{j_{1}} x^{j_{2}}|\cdots| y\right) \tag{2.37}
\end{equation*}
$$

in which the $k_{1}$-th column of partial first order derivatives $\left.\right|^{k_{1}} x^{k_{1}} \mid$ is replaced by the column $\left|{ }^{k_{1}} x^{j_{1}} x^{j_{2}}\right|$ of partial derivatives. Here, the indices $k_{1}, j_{1}$, $j_{2}$ satisfy $1 \leqslant k_{1}, j_{1}, j_{2} \leqslant n+1$, with the convention that we adopt the notational equivalence

$$
\begin{equation*}
x^{n+1} \equiv y \tag{2.38}
\end{equation*}
$$

This convention will be convenient to write some of our general formulas in the sequel.

As we promised to only summarize the proof of Theorem 1.7 in this paper, we will not develope the proof of Lemma 2.33: it is similar to the proof of Lemma 3.32 in [Me2004].

## §3. First and second auxiliary system

3.1. Functions $G_{j_{1}, j_{2}}, H_{j_{1}, j_{2}}^{k_{1}}, L_{j_{1}}^{k_{1}}$ and $M^{k_{1}}$. To discover the four families of functions appearing in the statement of Theorem 1.7, by comparing (2.35) and (1.10), it suffices (of course) to set:

$$
\left\{\begin{align*}
& G_{j_{1}, j_{2}}:=-\square_{x^{j_{1}} x^{j_{2}}}^{n+},  \tag{3.2}\\
& H_{j_{1}, j_{2}}^{k_{1}}:=\square_{x^{j_{1} x^{j_{2}}}-\delta_{j_{1}}^{k_{1}} \square_{x^{j_{2}} y}^{n+1}-\delta_{j_{2}}^{k_{1}} \square_{x^{j_{1}} y}^{n+1}}^{L_{j_{1}}^{k_{1}}}: \\
&:=2 \square_{x^{j_{1}} y}^{k_{1}}-\delta_{j_{1}}^{k_{1}} \square_{y y}^{n+1}, \\
& M^{k_{1}}:=\square_{y y}^{k_{1}} .
\end{align*}\right.
$$

Consequently, we have shown the "only if" part of Theorem 1.7, which is the easiest implication.

To establish the "if" part, by far the most difficult implication, the very main lemma can be stated as follows.

Lemma 3.3. The partial differerential relations (I'), (II'), (III') and (IV') which express in length the compatibility conditions (1.11) are necessary and sufficient for the existence of functions $X^{l}, Y$ of $\left(x^{l_{1}}, y\right)$ satisfying the second order nonlinear system of partial differential equations (3.2) above.

Indeed, the collection of equations (3.2) is a system of partial differential equations with unknowns $X^{l}, Y$, by virtue of the definition of the square functions.
3.4. First auxiliary system. To proceed further, we observe that there are $(m+1)$ more square functions than functions $G_{j_{1}, j_{2}}, H_{j_{1}, j_{2}}^{k_{1}}, L_{j_{1}}^{k_{1}}$ and $M^{k_{1}}$. Indeed, a simple counting yields:

$$
\left\{\begin{array}{lr}
\#\left\{\square_{x^{j_{1} x^{j_{2}}}}^{k_{1}}\right\}=\frac{n^{2}(n+1)}{2}, & \#\left\{\square_{x^{j_{1}} y}^{k_{1}}\right\}=n^{2},  \tag{3.5}\\
\#\left\{\square_{y y}^{k_{1}}\right\}=n, & \#\left\{\square_{x^{j_{1} x^{j_{2}}}}^{n+1}\right\}=\frac{n(n+1)}{2}, \\
\#\left\{\square_{x^{j_{1}} y}^{n+1}\right\}=n, & \#\left\{\square_{y y}^{n+1}\right\}=1,
\end{array}\right.
$$

whereas

$$
\left\{\begin{array}{lr}
\#\left\{G^{j_{1}, j_{2}}\right\}=\frac{n(n+1)}{2}, & \#\left\{H_{j_{1}, j_{2}}^{k_{1}}\right\}=\frac{n^{2}(n+1)}{2},  \tag{3.6}\\
\#\left\{L_{j_{1}}^{k_{1}}\right\}=n^{2}, & \#\left\{M^{k_{1}}\right\}=n .
\end{array}\right.
$$

Here, the indices $j_{1}, j_{2}, k_{1}$ satisfy $1 \leqslant j_{1}, j_{2}, k_{1} \leqslant n$. Similarly as in [Me2004], to transform the system (3.2) in a true complete system, let us introduce functions $\Pi_{j_{1}, j_{2}}^{k_{1}}$ of $\left(x^{l_{1}}, y\right)$, where $1 \leqslant j_{1}, j_{2}, k_{1} \leqslant n+1$, which satisfy the symmetry $\Pi_{j_{1}, j_{2}}^{k_{1}}=\Pi_{j_{1}, j_{1}}^{k_{1}}$, and let us introduce the following first auxiliary system:

$$
\left\{\begin{array}{lll}
\square_{x^{j_{1}} x^{j_{2}}}^{k_{1}}=\Pi_{j_{1}, j_{2}}^{k_{1}}, & \square_{x^{j_{1}} y}^{k_{1}}=\Pi_{j_{1}, n+1}^{k_{1}}, & \square_{y y}^{k_{1}}=\Pi_{n+1, n+1}^{k_{1}},  \tag{3.7}\\
\square_{x^{j_{1}} x^{j_{2}}}^{n+1} \Pi_{j_{1}, j_{2}}^{n+1}, & \square_{x^{j_{1}} y}^{n+1}=\Pi_{j_{1}, n+1}^{n+1}, & \square_{y y}^{n+1}=\Pi_{n+1, n+1}^{n+1} .
\end{array}\right.
$$

It is complete. The necessary and sufficient conditions for the existence of solutions $X^{l}, Y$ follow by cross differentiations.
Lemma 3.8. For all $j_{1}, j_{2}, j_{3}, k_{1}=1,2, \ldots, n+1$, we have the cross differentiation relations

$$
\begin{equation*}
\left(\square_{x^{j_{1}} x^{j_{2}}}^{k_{1}}\right)_{x^{j_{3}}}-\left(\square_{x^{j_{1}} x^{j_{3}}}^{k_{1}}\right)_{x^{j_{2}}}=-\sum_{k_{2}=1}^{n+1} \square_{x^{j_{1}} x^{j_{2}}}^{k_{2}} \square_{x^{j_{3}} x^{k_{2}}}^{k_{1}}+\sum_{k_{2}=1}^{n+1} \square_{x^{j_{1}} x^{j_{3}}}^{k_{2}} \square_{x^{j_{2}} x^{k_{2}}}^{k_{1}} . \tag{3.9}
\end{equation*}
$$

The proof of this lemma is exactly the same as the proof of Lemma 3.40 in [Me2004].

As a direct consequence, we deduce that a necessary and sufficient condition for the existence of solutions $\Pi_{j_{1}, j_{2}}^{k_{1}}$ to the first auxiliary system is that they satisfy the following compatibility partial differential relations:

$$
\begin{equation*}
\frac{\partial \Pi_{j_{1}, j_{2}}^{k_{1}}}{\partial x^{j_{3}}}-\frac{\partial \Pi_{j_{1}, j_{3}}^{k_{1}}}{\partial x^{j_{2}}}=-\sum_{k_{2}=1}^{n=1} \Pi_{j_{1}, j_{2}}^{k_{2}} \cdot \Pi_{j_{3}, k_{2}}^{k_{1}}+\sum_{k_{2}=1}^{n=1} \Pi_{j_{1}, j_{3}}^{k_{2}} \cdot \Pi_{j_{2}, k_{2}}^{k_{1}}, \tag{3.10}
\end{equation*}
$$

for all $j_{1}, j_{2}, j_{3}, k_{1}=1, \ldots, n+1$.
We shall have to specify this system in length according to the splitting $\{1,2, \ldots, n\}$ and $\{n+1\}$ of the indices of coordinates. We obtain six families of equations equivalent to (3.10) just above:

$$
\begin{align*}
\left(\Pi_{j_{1}, j_{2}}^{n+1}\right)_{x^{j_{3}}}-\left(\Pi_{j_{1}, j_{3}}^{n+1}\right)_{x^{j_{2}}}= & -\sum_{k_{2}=1}^{n} \Pi_{j_{1}, j_{2}}^{k_{2}} \Pi_{j_{3}, k_{2}}^{n+1}-\Pi_{j_{1}, j_{2}}^{n+1} \Pi_{j_{3}, n+1}^{n+1}+  \tag{3.11}\\
& +\sum_{k_{2}=1}^{n} \Pi_{j_{1}, j_{3}}^{k_{2}} \Pi_{j_{2}, k_{2}}^{n+1}+\Pi_{j_{1}, j_{3}}^{n+1} \Pi_{j_{2}, n+1}^{n+1}, \\
\left(\Pi_{j_{1}, j_{2}}^{n+1}\right)_{y}-\left(\Pi_{j_{1}, n+1}^{n+1}\right)_{x^{j_{2}}}=- & \sum_{k_{2}=1}^{n} \Pi_{j_{1}, j_{2}}^{k_{2}} \Pi_{n+1, k_{2}}^{n+1}-\Pi_{j_{1}, j_{2}}^{n+1} \Pi_{n+1, n+1}^{n+1}+ \\
& +\sum_{k_{2}=1}^{n} \Pi_{j_{1}, n+1}^{k_{2}} \Pi_{j_{2}, k_{2}}^{n+1}+\Pi_{j_{1}, n+1}^{n+1} \Pi_{j_{2}, n+1}^{n+1}, \\
\left(\Pi_{j_{1}, n+1}^{n+1}\right)_{y}-\left(\Pi_{n+1, n+1}^{n+1}\right)_{x^{j_{1}}}=- & \sum_{k_{2}=1}^{n} \Pi_{j_{1}, n+1}^{k_{2}} \Pi_{n+1, k_{2}}^{n+1}-\underline{\Pi}_{j_{1}, n+1}^{n+1} \Pi_{n+1, n+1}^{n+1}+ \\
& +\sum_{k_{2}=1}^{n} \Pi_{n+1, n+1}^{k_{2}} \Pi_{j_{1}, k_{2}}^{n+1}+\underline{\Pi_{n+1, n+1}^{n+1} \Pi_{j_{1}, n+2_{2}}^{n+1}}, \\
\left(\Pi_{j_{1}, j_{2}}^{k_{1}}\right)_{x^{j_{3}}}-\left(\Pi_{j_{1}, j_{3}}^{k_{1}}\right)_{x^{j_{2}}}=- & \sum_{k_{2}=1}^{n} \Pi_{j_{1}, j_{2}}^{k_{2}} \Pi_{j_{3}, k_{2}}^{k_{1}}-\Pi_{j_{1}, j_{2}}^{n+1} \Pi_{j_{3}, n+1}^{k_{1}}+ \\
& +\sum_{k_{2}=1}^{n} \Pi_{j_{1}, j_{3}}^{k_{2}} \Pi_{j_{2}, k_{2}}^{k_{1}}+\Pi_{j_{1}, j_{3}}^{n+1} \Pi_{j_{2}, n+1}^{k_{1}}, \\
& \quad+\sum_{k_{2}=1}^{n} \Pi_{j_{1}, n+1}^{k_{2}} \Pi_{j_{2}, k_{2}}^{k_{1}}+\Pi_{j_{1}, n+1}^{n+1} \Pi_{j_{2}, n+1}^{k_{1}}, \\
\left(\Pi_{j_{1}, j_{2}}^{k_{1}}\right)_{y}-\left(\Pi_{j_{1}, n+1}^{k_{1}}\right)_{x^{j_{2}}}=- & \sum_{k_{2}=1}^{n} \Pi_{j_{1}, j_{2}}^{k_{2}} \Pi_{n+1, k_{2}}^{k_{1}}-\Pi_{j_{1}, j_{2}}^{n+1} \Pi_{n+1, n+1}^{k_{1}}+ \\
& +\sum_{k_{2}=1}^{n} \Pi_{n+1, n+1}^{k_{2}} \Pi_{j_{1}, k_{2}}^{k_{1}}+\Pi_{n+1, n+1}^{n+1} \Pi_{j_{1}, n+1}^{k_{1}} .
\end{align*}
$$

where the indices $j_{1}, j_{2}, j_{3}, k_{1}$ vary in the set $\{1,2,1, \ldots, n\}$.
3.12. Principal unknowns. As there are $(m+1)$ more square (or Pi ) functions than the functions $G_{j_{1}, j_{2}}, H_{j_{1}, j_{2}}^{k_{1}}, L_{j_{1}}^{k_{1}}$ and $M^{k_{1}}$, we cannot invert directly the linear system (3.2). To quasi-inverse it, we choose the $(m+1)$ specific square functions

$$
\begin{equation*}
\Theta^{1}:=\square_{x^{1} x^{1}}^{1}, \quad \Theta^{2}:=\square_{x^{2} x^{2}}^{2}, \cdots \cdots, \Theta^{n+1}:=\square_{x^{n+1} x^{n+1}}^{n+1} \tag{3.13}
\end{equation*}
$$

calling them principal unknowns, and we get the quasi-inversion:
(3.14)

$$
\left\{\begin{aligned}
\Pi_{j_{1}, j_{2}}^{k_{1}} & =\square_{x^{j_{1}} x^{j_{2}}}^{k_{1}}=H_{j_{1}, j_{2}}^{k_{1}}-\frac{1}{2} \delta_{j_{1}}^{k_{1}} H_{j_{2}, j_{2}}^{j_{2}}-\frac{1}{2} \delta_{j_{2}}^{k_{1}} H_{j_{1}, j_{1}}^{j_{1}}+\frac{1}{2} \delta_{j_{1}}^{k_{1}} \Theta^{j_{2}}+\frac{1}{2} \delta_{j_{2}}^{k_{1}} \Theta^{j_{1}} \\
\Pi_{j_{1}, n+1}^{k_{1}} & =\square_{x^{j_{1}} y}^{k_{1}}=\frac{1}{2} L_{j_{1}}^{k_{1}}+\frac{1}{2} \delta_{j_{1}}^{k_{1}} \Theta^{n+1} \\
\Pi_{n+1, n+1}^{k_{1}} & =\square_{y y}^{k_{1}}=M^{k_{1}} \\
\Pi_{j_{1}, j_{2}}^{n+1} & =\square_{x^{j_{1}} x^{j_{2}}}^{n+1}=-G_{j_{1}, j_{2}}, \\
\Pi_{j_{1}, n+1}^{n+1} & =\square_{x^{j_{1} y}}^{n+1}=-\frac{1}{2} H_{j_{1}, j_{1}}^{j_{1}}+\frac{1}{2} \Theta^{j_{1}} .
\end{aligned}\right.
$$

3.15. Second auxiliary system. Replacing the five families of functions $\Pi_{j_{1}, j_{2}}^{k_{1}}, \Pi_{j_{1}, n+1}^{k_{1}}, \Pi_{n+1, n+1}^{k_{1}}, \Pi_{j_{1}, j_{2}}^{n+1}, \Pi_{j_{1}, n+1}^{n+1}$ by their values obtained in (3.14) just above together with the principal unknowns

$$
\begin{equation*}
\Pi_{j_{1}, j_{1}}^{j_{1}}=\Theta^{j_{1}}, \quad \Pi_{n+1, n+1}^{n+1}=\Theta^{n+1} \tag{3.16}
\end{equation*}
$$

in the six equations $(3.11)_{1},(3.11)_{2},(3.11)_{3},(3.11)_{4},(3.11)_{5}$ and $(3.11)_{6}$, after hard computations that we will not reproduce here, we obtain six families of equations. From now on, we abbreviate every sum $\sum_{k=1}^{n}$ as $\sum_{k_{1}}$.

Firstly:

$$
\begin{equation*}
0=\underline{G_{j_{1}, j_{2}, x^{j} 3}}-_{j_{1}, j_{3}, x^{j_{2}}}+\sum_{k_{1}} G_{j_{3}, k_{1}} H_{j_{1}, j_{2}}^{k_{1}}-\sum_{k_{1}} G_{j_{2}, k_{1}} H_{j_{1}, j_{3}}^{k_{1}} . \tag{3.17}
\end{equation*}
$$

This is (I') of Theorem 1.7. Just above and below, we plainly underline the monomials involving a first order derivative. Secondly:
(3.18)

$$
\left\{\begin{aligned}
\Theta_{x^{j_{2}}}^{j_{1}}= & -2 G_{j_{1}, j_{2}, y}+H_{j_{1}, j_{1}, x^{j_{2}}}^{j_{1}}+ \\
& +\sum_{k_{1}} G_{j_{2}, k_{1}} L_{j_{1}}^{k_{1}}+\frac{1}{2} H_{j_{1}, j_{1}}^{j_{1}} H_{j_{2}, j_{2}}^{j_{2}}-\sum_{k_{1}} H_{j_{1}, j_{2}}^{k_{1}} H_{k_{1}, k_{1}}^{k_{1}}- \\
& -G_{j_{1}, j_{2}} \Theta^{n+1}-\frac{1}{2} H_{j_{1}, j_{1}}^{j_{1}} \Theta^{j_{2}}-\frac{1}{2} H_{j_{2}, j_{2}}^{j_{2}} \Theta^{j_{1}}+\sum_{k_{1}} H_{j_{1}, j_{2}}^{k_{1}} \Theta^{k_{1}}+ \\
& +\frac{1}{2} \Theta^{j_{1}} \Theta^{j_{2}}
\end{aligned}\right.
$$

Thirdly:

$$
\left\{\begin{aligned}
-\Theta_{x^{j_{1}}}^{n+1}+\frac{1}{2} \Theta_{y}^{j_{1}}= & \frac{1}{2} H_{j_{1}, j_{1}, y}^{j_{1}}- \\
& -\sum_{k_{1}} G_{j_{1}, k_{1}} M^{k_{1}}+\frac{1}{4} \sum_{k_{1}} H_{k_{1}, k_{1}}^{k_{1}} L_{j_{1}}^{k_{1}}+ \\
& +\frac{1}{4} H_{j_{1}, j_{1}}^{j_{1}} \Theta^{n+1}-\frac{1}{4} \sum_{k_{1}} L_{j_{1}}^{k_{1}} \Theta^{k_{1}}-\frac{1}{4} \Theta^{j_{1}} \Theta^{n+1}
\end{aligned}\right.
$$

Fourtly:
(3.20)

$$
\left\{\begin{aligned}
\frac{1}{2} \delta_{j_{1}}^{k_{1}} & \Theta_{x^{j_{3}}}^{j_{2}}-\frac{1}{2} \delta_{j_{1}}^{k_{1}} \Theta_{x^{j_{2}}}^{j_{3}}+\frac{1}{2} \delta_{j_{2}}^{k_{1}} \Theta_{x^{j_{3}}}^{j_{1}}-\frac{1}{2} \delta_{j_{3}}^{k_{1}} \Theta_{x^{j_{2}}}^{j_{1}}= \\
= & -\frac{H_{j_{1}, j_{2}, x^{j_{3}}}^{k_{1}}+H_{j_{1}, j_{3}, x^{j_{2}}}^{k_{1}}-\frac{1}{2} \delta_{j_{1}}^{k_{1}} H_{j_{3}, j_{3}, x^{j_{2}}}^{j_{3}}+\frac{1}{2} \delta_{j_{1}}^{k_{1}} H_{j_{2}, j_{2}, x^{j_{3}}}^{j_{2}}-}{} \\
& -\frac{1}{2} \delta_{j_{3}}^{k_{1}} H_{j_{1}, j_{1}, x^{j_{2}}}^{j_{1}}+\frac{1}{2} \delta_{j_{2}}^{k_{1}} H_{j_{1}, j_{1}, x^{j_{3}}}^{j_{1}}- \\
& -\frac{1}{2} G_{j_{1}, j_{2}} L_{j_{3}}^{k_{1}}+\frac{1}{2} G_{j_{1}, j_{3}} L_{j_{2}}^{k_{1}}-\frac{1}{4} \delta_{j_{3}}^{k_{1}} H_{j_{1}, j_{1}}^{j_{1}} H_{j_{2}, j_{2}}^{j_{2}}+\frac{1}{4} \delta_{j_{2}}^{k_{1}} H_{j_{1}, j_{1}}^{j_{1}} H_{j_{3}, j_{3}}^{j_{3}}- \\
& -\sum_{k_{2}} H_{j_{1}, j_{2}}^{k_{2}} H_{j_{3}, k_{2}}^{k_{1}}+\sum_{k_{2}} H_{j_{1}, j_{3}}^{k_{2}} H_{j_{2}, k_{2}}^{k_{1}}-\frac{1}{2} \delta_{j_{2}}^{k_{1}} H_{j_{1}, j_{3}}^{k_{2}} H_{k_{2}, k_{2}}^{k_{2}}+\frac{1}{2} \delta_{j_{3}}^{k_{1}} H_{j_{1}, j_{2}}^{k_{2}} H_{k_{2}, k_{2}}^{k_{2}}- \\
& -\frac{1}{2} \delta_{j_{2}}^{k_{1}} G_{j_{1}, j_{3}} \Theta^{n+1}+\frac{1}{2} \delta_{j_{3}}^{k_{1}} G_{j_{1}, j_{2}} \Theta^{n+1}- \\
& -\frac{1}{4} \delta_{j_{2}}^{k_{1}} H_{j_{1}, j_{1}}^{j_{1}} \Theta^{j_{3}}+\frac{1}{4} \delta_{j_{3}}^{k_{1}} H_{j_{1}, j_{1}}^{j_{1}} \Theta^{j_{2}}-\frac{1}{4} \delta_{j_{2}}^{k_{1}} H_{j_{3}, j_{3}}^{j_{3}} \Theta^{j_{1}}+\frac{1}{4} \delta_{j_{3}}^{k_{1}} H_{j_{2}, j_{2}}^{j_{2}} \Theta^{j_{1}}- \\
& -\frac{1}{2} \delta_{j_{3}}^{k_{1}} \sum_{k_{1}} H_{j_{1}, j_{2}}^{k_{2}} \Theta^{k_{2}}+\frac{1}{2} \delta_{j_{2}}^{k_{1}} \sum_{k_{1}} H_{j_{1}, j_{3}}^{k_{2}} \Theta^{k_{2}}- \\
& -\frac{1}{4} \delta_{j_{3}}^{k_{1}} \Theta^{j_{1}} \Theta^{j_{2}}+\frac{1}{4} \delta_{j_{2}}^{k_{1}} \Theta^{j_{1}} \Theta^{j_{3}} .
\end{aligned}\right.
$$

Fifthly:
(3.21)

$$
\left\{\begin{array}{l}
\frac{1}{2} \delta_{j_{1}}^{k_{1}} \Theta_{y}^{j_{2}}+\frac{1}{2} \delta_{j_{2}}^{k_{1}} \Theta_{y}^{j_{1}}-\frac{1}{2} \delta_{j_{1}}^{k_{1}} \Theta_{x^{j_{2}}}^{n+1}= \\
= \\
=G_{j_{1}, j_{2}} M^{k_{1}}+\frac{1}{2} \sum_{k_{2}} H_{j_{1}, k_{2}}^{k_{1}} L_{j_{1}}^{k_{2}}-\frac{1}{2} \sum_{k_{2}} H_{j_{1}, j_{2}}^{k_{2}} L_{k_{2}}^{k_{1}}-\frac{1}{4} \delta_{j_{2}}^{k_{1}} \sum_{k_{2}} H_{k_{2}, k_{2}}^{k_{2}} L_{j_{1}}^{k_{2}}- \\
\quad-\frac{1}{4} \delta_{j_{2}}^{k_{1}} H_{j_{1}, j_{1}}^{j_{1}} \Theta^{n+1}+\frac{1}{4} \delta_{j_{2}}^{k_{1}} \sum_{k_{2}} L_{j_{1}}^{k_{2}} \Theta^{k_{2}}+\frac{1}{4} \delta_{j_{2}}^{k_{1}} \Theta^{j_{1}} \Theta^{n+1} .
\end{array}\right.
$$

Sixthly:
(3.22)

$$
\left\{\begin{aligned}
\delta_{j_{1}}^{k_{1}} \Theta_{y}^{n+1}= & -\frac{L_{j_{1}, y}^{k_{1}}+2 M_{x^{j_{1}}}^{k_{1}}}{}+ \\
& +2 \sum_{k_{2}} H_{j_{1}, k_{2}}^{k_{1}} M^{k_{2}}-\delta_{j_{1}}^{k_{1}} \sum_{k_{2}} H_{k_{2}, k_{2}}^{k_{2}} M^{k_{2}}-\frac{1}{2} \sum_{k_{2}} L_{j_{1}}^{k_{2}} L_{k_{2}}^{k_{1}}+ \\
& +\delta_{j_{1}}^{k_{1}} \sum_{k_{2}} M^{k_{2}} \Theta^{k_{2}}+\frac{1}{2} \delta_{j_{1}}^{k_{1}} \Theta^{n+1} \Theta^{n+1} .
\end{aligned}\right.
$$

3.23. Solving $\Theta_{x^{j_{2}}}^{j_{1}}, \Theta_{y}^{j_{1}}, \Theta_{x^{j_{1}}}^{n+1}$ and $\Theta_{y}^{n+1}$. From the six families of equations (3.17), (3.18), (3.19), (3.20), (3.21) and (3.22), we can solve $\Theta_{x^{j_{2}}}^{j_{1}}$, $\Theta_{y}^{j_{1}}, \Theta_{x^{j_{1}}}^{n+1}$ and $\Theta_{y}^{n+1}$. Not mentioning the (hard) intermediate computations, we obtain firstly:

$$
\left\{\begin{array}{rl}
\Theta_{x^{j_{2}}}^{j_{1}}= & -2 G_{j_{1}, j_{2}, y}+H_{j_{1}, j_{1}, x^{j_{2}}}^{j_{1}} \tag{3.24}
\end{array}+\sum_{l} G_{j_{2}, l} L_{j_{1}}^{l}+\frac{1}{2} H_{j_{1}, j_{1}}^{j_{1}} H_{j_{2}, j_{2}}^{j_{2}}-\sum_{l} H_{j_{1}, j_{2}}^{l} H_{l, l}^{l}-\right]
$$

Secondly:
(3.25)

$$
\left\{\begin{aligned}
\Theta_{y}^{j_{1}}= & -\frac{1}{3} H_{j_{1}, j_{1}, y}^{j_{1}}+\frac{2}{3} L_{j_{1}, x^{j_{1}}}^{j_{1}}+\frac{4}{3} G_{j_{1}, j_{1}} M^{j_{1}}+\frac{2}{3} \sum_{l} G_{j_{1}, l} M^{l}-\frac{1}{2} \sum_{l} H_{l, l}^{l} L_{j_{1}}^{l}+ \\
& +\frac{2}{3} \sum_{l} H_{j_{1}, l}^{j_{1}} L_{j_{1}}^{l}-\frac{2}{3} \sum_{l} H_{j_{1}, j_{1}}^{l} L_{l}^{j_{1}}-\frac{1}{2} H_{j_{1}, j_{1}}^{j_{1}} \Theta^{n+1}+\frac{1}{2} \sum_{l} L_{j_{1}}^{l} \Theta^{l}+ \\
& +\frac{1}{2} \Theta^{j_{1}} \Theta^{n+1} .
\end{aligned}\right.
$$

Thirdly:

$$
\left\{\begin{aligned}
\Theta_{x^{j_{1}}}^{n+1}= & -\frac{2}{3} H_{j_{1}, j_{1}, y}^{j_{1}}+\frac{1}{3} L_{j_{1}, x^{j_{1}}}^{j_{1}}
\end{aligned}\right) \frac{2}{3} G_{j_{1}, j_{1}} M^{j_{1}}+\frac{4}{3} \sum_{l} G_{j_{1}, l} M^{l}-\frac{1}{2} \sum_{l} H_{l, l}^{l} L_{j_{1}}^{l}+{ }_{l}+H_{l}^{3} \sum_{l} H_{j_{1}, l}^{j_{1}} L_{j_{1}}^{l}-\frac{1}{3} \sum_{l} H_{j_{1}, j_{1}}^{l} L_{l}^{j_{1}}-\frac{1}{2} H_{j_{1}, j_{1}}^{j_{1}} \Theta^{n+1}+\frac{1}{2} \sum_{l} L_{j_{1}}^{l} \Theta^{l}+,
$$

Fourtly:
(3.27)

$$
\left\{\begin{aligned}
\Theta_{y}^{n+1}= & -\underline{L_{j_{1}, y}^{j_{1}}+2 M_{x^{j_{1}}}^{j_{1}}}+2 \sum_{l} H_{j_{1}, l}^{j_{1}} M^{l}-\sum_{l} H_{l, l}^{l} M^{l}-\frac{1}{2} \sum_{l} L_{j_{1}}^{l} L_{l}^{j_{1}}+ \\
& +\sum_{l} M^{l} \Theta^{l}+\frac{1}{2} \Theta^{n+1} \Theta^{n+1} .
\end{aligned}\right.
$$

These four families of partial differential equations constitute the second auxiliary system. By replacing these solutions in the three remaining families of equations (3.20), (3.21) and (3.22), we obtain supplementary equations (which we do not copy) that are direct consequences of (I'), (II'), (III'), (IV').

To complete the proof of the main Lemma 3.3 above, it suffices now to establish the first implication of the following list, since the other three have been already established.

- Some given functions $G_{j_{1}, j_{2}}, H_{j_{1}, j_{2}}^{k_{1}}, L_{j_{1}}^{k_{1}}$ and $M^{k_{1}}$ of $\left(x^{l_{1}}, y\right)$ satisfy the four families of partial differential equations (I'), (II'), (III') and (IV') of Theorem 1.7.
$\Downarrow$
- There exist functions $\Theta^{j_{1}}, \Theta^{n+1}$ satisfying the second auxiliary system (3.24), (3.25), (3.26) and (3.27).
$\Downarrow$
- These solution functions $\Theta^{j_{1}}, \Theta^{n+1}$ satisfy the six families of partial differential equations (3.17), (3.18), (3.19), (3.20), (3.21) and (3.22).
$\Downarrow$
- There exist functions $\Pi_{j_{1}, j_{2}}^{k_{1}}$ of $\left(x^{l_{1}}, y\right), 1 \leqslant j_{1}, j_{2}, k_{1} \leqslant m+1$, satisfying the first auxiliary system (3.7) of partial differential equations.
$\Downarrow$
- There exist functions $X^{l_{2}}, Y$ of $\left(x^{l_{1}}, y\right)$ transforming the system $y_{x^{j_{1}} x^{j_{2}}}=F^{j_{1}, j_{2}}\left(x^{l_{1}}, y, y_{x^{l_{2}}}\right), j_{1}, j_{2}=1, \ldots, n$, to the simplest system $Y_{X^{j_{1}}{ }^{j_{2}}}=0, j_{1}, j_{1}=1, \ldots, n$.
3.28. Compatibility conditions for the second auxiliary system. We notice that the second auxiliary system is also a complete system. Thus, to establish the first above implication, it suffices to show that the four families of compatibility conditions:

$$
\left\{\begin{array}{l}
0=\left(\Theta_{x^{j_{2}}}^{j_{1}}\right)_{x^{j_{3}}}-\left(\Theta_{x^{j_{3}}}^{j_{1}}\right)_{x^{j_{2}}},  \tag{3.29}\\
0=\left(\Theta_{x^{j_{2}}}^{)_{y}}-\left(\Theta_{y}^{j_{1}}\right)_{x^{j_{2}},},\right. \\
0=\left(\Theta_{x^{j_{1}}}^{n+}\right)_{x^{j_{2}}}-\left(\Theta_{x^{j_{2}}}^{n+1}\right)_{x^{j_{1}}}, \\
0=\left(\Theta_{x^{j_{2}}}^{n+1}\right)_{y}-\left(\Theta_{y}^{n+1}\right)_{x^{j_{2}}},
\end{array}\right.
$$

are a consequence of (I'), (I'), (III'), (IV').
For instance, in $(3.29)_{1}$, replacing $\Theta_{x^{j_{2}}}^{j_{1}}$ by its expression (3.24), differentiating it with respect to $x^{j_{3}}$, replacing $\Theta_{x^{j_{3}}}^{j_{1}}$ by its expression (3.24), differentiating it with respect to $x^{j_{2}}$ and substracting, we get:

$$
\begin{align*}
& \left(0=-2 G_{j_{1}, j_{2}, y x^{j_{3}}}+2 G_{j_{1}, j_{3}, y x^{j_{2}}}+\underline{H_{j_{1}, j_{1}, x^{j_{2}} x^{j_{3}}}^{j_{1}}} \mathrm{a}-H_{j_{1}, j_{1}, x^{j_{3} x^{j_{2}}}}{ }^{j_{1}}+\right.  \tag{3.30}\\
& +\frac{1}{2} \underline{\Theta_{x^{j_{3}}}^{j_{1}}} \Theta^{j_{2}}+\frac{1}{2} \Theta^{j_{1}} \underline{\Theta_{x^{j_{3}}}^{j_{2}}}-\frac{1}{2} \underline{\Theta_{x^{j_{2}}}^{j_{1}}} \Theta^{j_{3}}-\frac{1}{2} \Theta^{j_{1}} \underline{\Theta_{x^{j_{2}}}^{j_{3}}-} \\
& -\frac{1}{2} H_{j_{1}, j_{1}, x^{j_{3}}}^{j_{1}} \Theta^{j_{2}}-\frac{1}{2} H_{j_{1}, j_{1}}^{j_{1}} \underline{\Theta_{x^{j_{3}}}^{j_{2}}}+\frac{1}{2} H_{j_{1}, j_{1}, x^{j_{2}}}^{j_{1}} \Theta^{j_{3}}+\frac{1}{2} H_{j_{1}, j_{1}}^{j_{1}} \Theta_{x^{j_{2}}}^{j_{3}}- \\
& -\frac{1}{2} H_{j_{2}, j_{2}, x^{j_{3}}}^{j_{2}} \Theta^{j_{1}}-\frac{1}{2} H_{j_{2}, j_{2}}^{j_{2}} \underline{\Theta_{x^{j_{3}}}^{j_{1}}}+\frac{1}{2} H_{j_{3}, j_{3}, x^{j_{2}}}^{j_{3}} \Theta^{j_{1}}+\frac{1}{2} H_{j_{3}, j_{3}}^{j_{3}} \underline{\Theta_{x^{j_{2}}}^{j_{1}}-} \\
& -G_{j_{1}, j_{2}, x^{j_{3}}} \Theta^{n+1}-G_{j_{1}, j_{2}} \underline{\Theta_{x^{j_{3}}}^{n+1}}+G_{j_{1}, j_{3}, x^{j_{2}}} \Theta^{n+1}+G_{j_{1}, j_{3}} \underline{\Theta_{x^{j_{2}}}^{n+1}+} \\
& +\sum_{l} H_{j_{1}, j_{2}, x^{j_{3}}}^{l} \Theta^{l}+\sum_{l} H_{j_{1}, j_{2}}^{l} \underline{\Theta_{x^{j_{3}}}^{l}}-\sum_{l} H_{j_{1}, j_{3}, x^{j_{2}}}^{l} \Theta^{l}-\sum_{l} H_{j_{1}, j_{3}}^{l} \underline{\Theta_{x^{j_{2}}}^{l}}+ \\
& +\frac{1}{2} H_{j_{1}, j_{1}, x^{j_{3}}}^{j_{1}} H_{j_{2}, j_{2}}^{j_{2}}+\frac{1}{2} H_{j_{1}, j_{1}}^{j_{1}} H_{j_{2}, j_{2}, x^{j_{3}}}^{j_{2}}-\frac{1}{2} H_{j_{1}, j_{1}, x^{j_{2}}}^{j_{1}} H_{j_{3}, j_{3}}^{j_{3}}-\frac{1}{2} H_{j_{1}, j_{1}}^{j_{1}} H_{j_{3}, j_{3}, x^{j_{2}}}^{j_{3}}- \\
& -\sum_{l} H_{j_{1}, j_{2}, x^{j_{3}}}^{l} H_{l, l}^{l}-\sum_{l} H_{j_{1}, j_{2}}^{l} H_{l, l, x^{j_{3}}}^{l}+\sum_{l} H_{j_{1}, j_{3}, x^{j_{2}}}^{l} H_{l, l}^{l}+\sum_{l} H_{j_{1}, j_{3}}^{l} H_{l, l, x^{j_{2}}}^{l}+ \\
& +\sum_{l} G_{j_{2}, l, x^{j_{3}}} L_{j_{1}}^{l}+\sum_{l} G_{j_{2}, l} L_{j_{1}, x^{j_{3}}}^{l}-\sum_{l} G_{j_{3}, l, x^{j_{2}}} L_{j_{1}}^{l}-\sum_{l} G_{j_{3}, l} L_{j_{1}, x^{j_{2}}}^{l} .
\end{align*}
$$

Next, replacing the twelve first order partial derivatives underlined just above:
by their values issued from (3.24), (3.26) and adapting the summation indices, we get the explicit developed form of the first family of compatibility
conditions (3.29) ${ }_{1}$ :

$$
\begin{aligned}
& \text { (3.32) } \\
& \left(\begin{array}{rl}
0=?= & -\underline{\underline{2} G_{j_{1}, j_{2}, x^{j_{3}} y}+2 G_{j_{1}, j_{3}, x^{j_{2}} y}}- \\
& -\underline{\sum_{l} G_{j_{3}, l, x^{j_{2}}} L_{j_{1}}^{l}+\sum_{l} G_{j_{2}, l, x^{j_{3}}} L_{j_{1}}^{l}-G_{j_{1}, j_{2}, y} H_{j_{3}, j_{3}}^{j_{3}}+G_{j_{1}, j_{3}, y} H_{j_{2}, j_{2}}^{j_{2}}-}
\end{array}\right. \\
& -2 \sum_{l} G_{l, j_{3}} H_{j_{1}, j_{2}}^{l}+2 \sum_{l} G_{l, j_{2}} H_{j_{1}, j_{3}}^{l}-\sum_{l} H_{j_{1}, j_{2}, x^{j_{3}}}^{l} H_{l, l}^{l}+\sum_{l} H_{j_{1}, j_{3}, x^{j_{2}}}^{l} H_{l, l}^{l}- \\
& -\frac{2}{3} H_{j_{2}, j_{2}, y}^{j_{2}} G_{j_{1}, j_{3}}+\frac{2}{3} H_{j_{3}, j_{3}, y}^{j_{3}} G_{j_{1}, j_{2}}-\frac{2}{3} L_{j_{3}, x^{j_{3}}}^{j_{3}} G_{j_{1}, j_{2}}+\frac{2}{3} L_{j_{2}, x^{j_{2}}}^{j_{2}} G_{j_{1}, j_{3}}- \\
& -\sum_{l} L_{j_{1}, x^{j_{2}}}^{l} G_{j_{3}, l}+\sum_{l} L_{j_{1}, x^{j_{3}}}^{l} G_{j_{2}, l}- \\
& -\frac{2}{3} G_{j_{1}, j_{2}} G_{j_{3}, j_{3}} M^{j_{3}}+\frac{2}{3} G_{j_{1}, j_{3}} G_{j_{2}, j_{2}} M^{j_{2}}-\frac{4}{3} \sum_{l} G_{j_{1}, j_{2}} G_{j_{3}, l} M^{l}+ \\
& +\frac{4}{3} \sum_{l} G_{j_{1}, j_{3}} G_{j_{2}, l} M^{l}-\frac{1}{2} \sum_{l} G_{j_{3}, l} H_{j_{1}, j_{1}}^{j_{1}} L_{j_{2}}^{l}+\frac{1}{2} \sum_{l} G_{j_{2}, l} H_{j_{1}, j_{1}}^{j_{1}} L_{j_{3}}^{l}- \\
& -\frac{1}{2} \sum_{l} G_{j_{3}, l} H_{j_{2}, j_{2}}^{j_{2}} L_{j_{1}}^{l}+\frac{1}{2} \sum_{l} G_{j_{2}, l} H_{j_{3}, j_{3}}^{j_{3}} L_{j_{1}}^{l}-\frac{1}{2} \sum_{l} G_{j_{1}, j_{3}} H_{l, l}^{l} L_{j_{2}}^{l}+ \\
& +\frac{1}{2} \sum_{l} G_{j_{1}, j_{2}} H_{l, l}^{l} L_{j_{3}}^{l}-\frac{1}{3} \sum_{l} G_{j_{1}, j_{2}} H_{j_{3}, l}^{j_{3}} L_{j_{3}}^{l}+\frac{1}{3} \sum_{l} G_{j_{1}, j_{3}} H_{j_{2}, l}^{j_{2}} L_{j_{2}}^{l}- \\
& -\frac{1}{3} G_{j_{1}, j_{3}} H_{j_{2}, j_{2}}^{l} L_{l}^{j_{2}}+\frac{1}{3} G_{j_{1}, j_{2}} H_{j_{3}, j_{3}}^{l} L_{l}^{j_{3}}- \\
& -\sum_{l} \sum_{p} G_{j_{2}, p} H_{j_{1}, j_{3}}^{l} L_{l}^{p}+\sum_{l} \sum_{p} G_{j_{3}, p} H_{j_{1}, j_{2}}^{l} L_{l}^{p}- \\
& -\sum_{l} \sum_{p} H_{j_{1}, j_{2}}^{l} H_{l, j_{3}}^{p} H_{p, p}^{p}+\sum_{l} \sum_{p} H_{j_{1}, j_{3}}^{l} H_{l, j_{2}}^{p} H_{p, p}^{p} .
\end{aligned}
$$

Lemma 3.33. ([Me2003, Me2004]) This first family of compatibility conditions for the second auxiliary system obtained by developing $(3.29)_{1}$ in length, together with the three remaining families obtained by developing $(3.29)_{2},(3.29)_{3},(3.29)_{4}$ in length, are consequences, by linear combinations and by differentiations, of (I'), (II'), (III'), (IV'), of Theorem 1.7.

The summarized proof of Theorem 1.7 is complete.

## IV: Bibliography

## References

[Ar1988] Arnol'd, V.I.: Dynamical systems. I. Ordinary differential equations and smooth dynamical systems, Translated from the Russian. Edited by D. V. Anosov and V. I. Arnol'd. Encyclopaedia of Mathematical Sciences, vol. 1. Springer-Verlag, Berlin, 1988. x+233 pp.
[Ar1968] Artin, M.: On the solutions of analytic equations, Invent. Math. 5 (1968), 277-291.
[BER1999] Baouendi, M.S.; Ebenfelt, P.; Rothschild, L.P.: Real submanifolds in complex space and their mappings. Princeton Mathematical Series, vol. 47, Princeton University Press, Princeton, NJ, 1999, xii+404 pp.
[BJT1985] Baouendi, M.S.; Jacobowitz, H.; Treves, F.: On the analyticity of CR mappings, Ann. of Math. 122 (1985), no. 2, 365-400.
[Bel1996] BellaïChe, A.: SubRiemannian Geometry, Progress in Mathematics, vol. 144, Birkhäuser Verlag, Basel/Switzerland, 1996, 1-78.
[Be1979] Beloshapka, V.K.: On the dimension of the group of automorphisms of an analytic hypersurface, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), 243-266; English transl. in Math. USSR-Izv. 14 (1980), 223-245.
[Be1988] BELOSHAPKA, V.K.: Finite-dimensionality of the group of automorphisms of a real-analytic surface, Izv. Akad. Nauk SSSR Ser. Mat. 52 (1988), 437-442; English transl. in Math. USSR-Izv. 32 (1989), 443-448.
[Be1997] BELOSHAPKA, V.K.: CR-varieties of the type (1,2) as varieties of 'super-high' codimension, Russian J. Math. Phys. 5 (1997), 399-404.
[Be2002] BELOSHAPKA, V.K.: Real submanifolds in complex space: polynomial models, automorphisms, and classification problems, Uspekhi Mat. Nauk 57 (2002), no. 1, 3-44; English transl. in Russian Math. Surveys 57 (2002), no. 1, 1-41.
[BES2005] Beloshapka, V.K.; Ezhov, V.; Schmalz, G.: Canonical Cartan connection and holomorphic invariants of Engel CR manifolds, arxiv.org/abs/math.CV/0508084.
[BK1989] Bluman, G.W.; Kumei, S.: Symmetries and differential equations, Springer Verlag, Berlin, 1989.
[Bo1991] Boggess, A.: CR manifolds and the tangential Cauchy-Riemann complex. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1991, xviii+364 pp.
[Bo1972] Bourbaki, N.: Groupes at algèbres de Lie, chapitre 2, Hermann, Paris, 1972.
[BSW1978] Burns, D.Jr.; Shnider, S.; Wells, R.O.Jr.: Deformations of strictly pseudoconvex domains Invent. Math. 46 (1978), no. 3, 237-253.
[Ca1922] CARTAN, É.: Sur les équations de la gravitation d'Einstein, J. Math. pures et appl. 1 (1922), 141-203.
[Ca1924] Cartan, É.: Sur les variétés à connexion projective, Bull. Soc. Math. France 52 (1924), 205-241.
[Ca1932a] CARTAN, É.: Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes, I, Ann. Math. Pura Appl. 11 (1932), 17-90.
[Ca1932b] CARTAN, É.: Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes, II, Ann. Scuola Norm. Sup. Pisa 1 (1932), 333-354.
[Ca1937] Cartan, É.: Les problèmes d'équivalence (Séminaire de Math., 4 e année, 1936-37), Euvres complètes, II, 1311-1334, Gauthier-Villars, Paris, 1953.
[CM1974] ChERN, S.S.; MOSER, J.K.: Real hypersurfaces in complex manifolds, Acta Math. 133 (1974), no. 2, 219-271.
[Ch1975] CHERN, S.-S.: On the projective structure of a real hypersurface in $\mathbb{C}^{n+1}$, Math. Scand. 36 (1975), 74-82.
[Ch1989] Chirka, E.M.: Complex analytic sets, Mathematics and its applications (Soviet Series), vol. 46. Kluwer Academic Publishers Group, Dordrecht, 1989. $\mathrm{xx}+372 \mathrm{pp}$.
[Ch1991] Chirka, E.M.: An introduction to the geometry of CR manifolds (Russian), Uspekhi Mat. Nauk 46 (1991), no. 1(277), 81-164, 240; translation in Russian Math. Surveys 46 (1991), no. 1, 95-197
[CS1996] Constantine, G.M.; SAvits, T.H.: A multivariate Faà di Bruno formula with applications, Trans. Amer. Math. Soc. 348 (1996), no. 2, 503-520.
[DF1988] Diederich, K.; Forness, J.E.: Proper holomorphic mappings between real-analytic pseudoconvex domains in $\mathbb{C}^{n}$, Math. Ann 282 (1988), no. 4, 681700.
[DP2003] DIEDERICH, K.; PINCHUK, S.: Regularity of continuous CR-maps in arbitrary dimension, Michigan Math. J. 51 (2003), 111-140. Erratum: ib., no. 3, 667668.
[DW1980] DIEDERICH, K.; WEBSTER, S.M.: A reflection principle for degenerate real hypersurfaces, Duke Math. J. 47 (1980), no. 4, 835-843.
[Eb1998] EbENFELT, P.: Normal forms and biholomorphic equivalences of real hypersurfaces in $\mathbb{C}^{3}$, Indiana Univ. Math. J. 47 (1998), 311-366.
[Eb2006] Ebenfelt, P.: Correction to "Uniformly Levi degenerate CR manifolds: the 5-dimensional case, Duke Math. J. 131 (2006), 589-591.
[EL1890] Engel, F.; Lie, S.: Theorie der Transformationsgruppen, I, II, II, Teubner, Leipzig, 1889, 1891, 1893.
[EKV1985] Ezhov, V.V.; Kruzhilin, N.G.; Vitushkin, A.G.: Continuation of holomorphic mappings along real-analytic hypersurfaces (Russian). Current problems in mathematics. Mathematical analysis, algebra, topology. Trudy Mat. Inst. Steklov 167 (1985), 60-95, 276.
[Fa1980] Faran, J.: Segre families and real hypersurfaces, Invent. Math. 60 (1980), no. 2, 135-172.
[F1969] Federer, H.: Geometric measure theory, Die Grundlehren der Mathematischen Wissenschaften, Band 153, Springer Verlag, New York, 1969, xiv+676 pp. Springer-Verlag, Berlin, 1969.
[Fe1995] FELS, M.: The equivalence problem for systems of second-order ordinary differential equations, Proc. London Math. Soc. 71 (1995), no. 2, 221-240.
[FK2005a] Fels, G.; Kaup, W.: CR-manifolds of dimension 5: A Lie algebra approach, J. Reine Angew. Math., to appear. arxiv.org/abs/math.DG/050811.
[FK2005b] FEls, G.; Kaup, W.: Homogeneous Levi degenerate CR-manifolds in dimension 5, to appear.
[Fr1877] Frobenius, G.: Ueber das Pfaffsche Problem, J. Reine Angew. Math. 82 (1877), 230-315.
[G1989] GARDNER, R.B.: The method of equivalence and its applications, CBMSNSF Regional Conference Series in Applied Mathematics, vol. 58 (SIAM, Philadelphia, 1989), 127 pp.
[GM2003a] Gaussier, H.; Merker, J.: Symmetries of partial differential equations, J. Korean Math. Soc. 40 (2003), no. 3, 517-561; e-print: http://fr.arxiv.org/abs/math.CV/0404127.
[GM2003b] GAUSSIER, H.; MERKER, J.: A new example of uniformly Levi nondegenerate hypersurface in $\mathbb{C}^{3}$, Ark. Mat. 41 (2003), no. 1, 85-94.
[GM2004] GAUSSIER, H.; MERKER, J.: Nonalgebraizable real analytic tubes in $\mathbb{C}^{n}$, Math. Z. 247 (2004), no. 2, 337-383.
[GM2006] GAUSSIER, H.; MERKER, J.: Erratum to "A new example of a uniformly Levi degenerate hypersurface in $\mathbb{C}^{3 \prime}, 2006$, to appear.
[GV1987] Gershkovich, V.Ya.; Vershik, A.M.: Nonholonomic dynamical systems. Geometry of distributions and variational problems. Dynamical Systems VII, Encyclopædia of mathematical sciences, vol. 16, V.I. Arnol'd and S.P. Novikov (Eds.), 1-81, Springer-Verlag, Berlin, 1994.
[Gr2005] DE GraAF, W.A.: Classification of solvable Lie algebras, Experiment. Math. 14 (2005), no. 1, 15-25.
[GTW1989] Grissom, C.; Thompson, G.; Wilkens, G.: Linearization of second order ordinary differential equations via Cartan's equivalence method, J. Diff. Eq. 77 (1989), no. 1, 1-15.
[Gr2000] Grossman, D.A.: Torsion-free path geometries and integrable second order ode systems, Selecta Math. (N.S.) 6 (2000), no. 4, 399-442.
[Ha1937] Hachtroudi, M.: Les espaces d'éléments à connexion projective normale, Actualités Scientifiques et Industrielles, vol. 565, Paris, Hermann, 1937.
[Ha1982] Hartman, P.: Ordinary Differential Equations. Birkhäuser, Boston 1982.
[Ha2003] HaUSER, H.: The Hironaka theorem on resolution of singularities (or: A proof we always wanted to understand), Bull. Amer. Math. Soc. (N.S.) 40 (2003), no. 3, 323-403.
[Hi1976] Hirsch, M.W.: Differential topology, Graduate Texts in Mathematics, 33, Springer-Verlag, Berlin, 1976, x+222 pp.
[HK1989] Hsu, L.; KAmRAN, N.: Classification of second order ordinary differential equations admitting Lie groups of fibre-preserving point symmetries, Proc. London Math. Soc. 58 (1989), no. 3, 387-416.
[Ib1992] Ibragimov, N.H.: Group analysis of ordinary differential equations and the invariance principle in mathematical physics, Russian Math. Surveys 47:4 (1992), 89-156.
[Ib1999] IbRAGIMOV, N.H.: Elementary Lie group analysis and ordinary differential equations, Wiley Series in Mathematical Methods in Practice, 4. John Wiley \& Sons, Ltd., Chichester, 1999. xviii+347 pp.
[IL2003] IVEY, J.A.; LANDSBERG, J.M.: Cartan for beginners: differential geometry via moving frames and exterior differential systems, Graduate Studies in Mathematics, 61. American Mathematical Society, Providence, RI, 2003. xiv+378 pp.
[Ja1990] Jacobowitz, An introduction to CR structures, Math. Surveys and Monographs, 32. Amer. Math. Soc., Providence, 1990. x+237 pp.
[Ji2002] JI, S.: Algebraicity of real analytic hypersurfaces with maximal rank, Amer. J. Math. 124 (2002), no. 6, 1083-1102.
[JoPf2000] De Jong, T.; Pfister, G.: Local analytic geometry. Basic theory and applications, Advanced Lectures in Mathematics. Friedr. Vieweg \& Sohn, Braunschweig, 2000. xii+382 pp.
[Kn2004] KnAPP, A.W.: Lie groups beyond an introduction, Progress in Mathematics, 140, Birkhäuser, Basel, third edition, 2004, xviii+812 pp.
[KN1963] Kobayashi, S.; Nomizu, K.m: Foundations of differential geometry, I, Interscience publishers, John Wiley \& Sons, New York, 1963. xi+329 pp.
[Kr1985] KruZhilin, N.G.: Local automorphisms and mappings of smooth strictly pseudoconvex hypersurfaces (Russian). Izv. Akad. Nauk SSSR Ser. Mat. 49 (1985), no. 3, 566-591, 672.
[Kr1987] Kruzhilin, N.G.: Description of the local automorphism groups of real hypersurfaces, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 749-758, Amer. Math. Soc., Providence, RI, 1987.
[KV1987] Kruzhilin, N.G.; Vitushkin, A.G.: Description of the automorphism groups of real hypersurfaces in complex space (Russian). Investigations in the theory of the approximation of functions, 26-69, Akad. Nauk SSSR Bashkir Filial, Otdel. Fiz. Mat., Ufa, 1987.
[Lie1880] LIE, S.: Theorie der Transformationsgruppen, Math. Ann. 16 (1880), 441528.
[Lie1883] LIE, S.: Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen $x, y$, die eine Gruppe von Transformationen gestaten I-IV. In: Gesammelte Abhandlungen, Vol. 5, B.G. Teubner, Leipzig, 1924, pp. 240310; 362-427, 432-448.
[LS1893] LIE, S.; SChEFFERS, G.: Vorlesungen b̈er continuierliche Gruppen mit Geometrischen und anderen Anwendungen. (German) Nachdruck der Auflage des Jahres 1893. Chelsea Publishing Co., Bronx, New York, 1971. xii+810 pp.
[Lo1981] LOBODA, A.V.: Local automorphisms of real-analytic hypersurfaces (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981), no. 3, 620-645.
[Lo2001] LOBODA, A.V.: Homogeneous strictly pseudoconvex hypersurfaces in $\mathbb{C}^{3}$ with two-dimensional isotropy groups (Russian) Mat. Sb. 192 (2001), no. 12, 3-24; translation in Sb. Math. 192 (2001), no. 11-12, 1741-1761.
[Lo2002] Loboda, A.V.: Homogeneous nondegenerate surfaces in $\mathbb{C}^{3}$ with twodimensional isotropy groups (Russian) Mat. Sb. 192 (2001), no. 12, 3-24; translation in Sb. Math. 192 (2001), no. 11-12, 1741-1761.
[Lo2003] Loboda, A.V.: On the determination of a homogeneous strictly pseudoconvex hypersurface from the coefficients of its normal form (Russian) Mat. Zametki 73 (2003), no. 3, 453-456; translation in Math. Notes 73 (2003), no. 3-4, 419423.
[Ma2003] Mardare, S.: On isometric immersions of a Riemannian space under a weak regularity assumption, C. R. Acad. Sci. Paris, Sér. I 337 (2003), 785-790.
[Me2001] MERKER, J.: On the partial algebraicity of holomorphic mappings between two real algebraic sets in the complex euclidean spaces of different dimensions, Bull. Soc. Math. France 129 (2001), no. 4, 547-591.
[Me2003] MERKER, J.: hand manuscript I, 212 pp., May - July 2003; hand manuscript II, 114 pp., August 2003.
[Me2004] MERKER, J.: Explicit differential characterization of the Newtonian free particle system in $m \geqslant 2$ dependent variables, Acta Mathematicæ Applicandæ, to appear, 73 pp ; e-print: arxiv.org/abs/math.DG/0411165.
[Me2005a] MERKER, J.: On the local geometry of generic submanifolds of $\mathbb{C}^{n}$ and the analytic reflection principle (Part I), Journal of Mathematical Sciences (N.Y.) 125 (2005), no. 6, 751-824.
[Me2005b] MERKER, J.: Étude de la régularité analytique de l'application de réflexion CR formelle, Annales Fac. Sci. Toulouse, XIV (2005), no. 2, 215-330.
[MP2005] MERKER, J.; Porten, E.: Holomorphic extension of CR functions, envelopes of holomorphy and removable singularities, 432 pp., to appear. Downloadable at: www.cmi.univ-mrs/~merker/index.html.
[N2003] NeUt, S.: Implantation et nouvelles applications de la méthode d'équivalence d'Élie Cartan, Thèse, Université Lille 1, October 2003.
[OL1979] OlVER, P.J.: Symmetry groups and group invariant solutions of partial differential equations, J. Diff. Geom. 14 (1979), 497-542.
[Ol1986] OlVER, P.J.: Applications of Lie groups to differential equations. Springer Verlag, New York, 1986. xxvi+497 pp.
[Ol1995] Olver, P.J.: Equivalence, Invariance and Symmetries. Cambridge, Cambridge University Press, 1995, xvi+525 pp.
[OV1994] Onishchik, A.L.; Vinberg, E.B.: Lie groups and Lie algebras, III. Encyclopædia of mathematical sciences, 41. Springer Verlag, Berlin, 248 pp.
[Pi1975] Pinchuk, S.: On the analytic continuation of holomorphic mappings (Russian), Mat. Sb. (N.S.) 98(140) (1975) no.3(11), 375-392, 416-435, 495-496.
[Pi1978] Pinchuk, S.: Holomorphic mappings of real-analytic hypersurfaces (Russian), Mat. Sb. (N.S.) 105(147) (1978), no. 4, 574-593, 640; English translation in Math. USSR Sbornik 34 (1978), 503-519.
[Re1993] Reutenauer, C.: Free Lie algebras, London Mathematical Society Monograph, New Series, 7. Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993. xviii+269 pp.
[Se1931] SEGRE, B.: Intorno al problema di Poincaré della rappresentazione pseudoconforme, Rend. Acc. Lincei, VI, Ser. 13 (1931), 676-683.
[Se1932] SEgRE, B.: Questioni geometriche legate colla teoria delle funzioni di due variabili complesse, Rendiconti del Seminario di Matematici di Roma, II, Ser. 7 (1932), no. 2, 59-107.
[Sh1997] SHARPE, R.W.: Differential geometry; Cartan's generalization of Klein's Erlangen program, Graduate texts in mathematics, vol. 166, Springer Verlag, Berlin, 1997, xix+421 pp.
[Sp1970] Spivak, M.: A comprehensive introduction to differential geometry, vols. 1 and 2. Published by M. Spivak, Brandeis Univ., Waltham, Mass. 1970.
[St1996] Stanton, N.: Infinitesimal CR automorphisms of real hypersurfaces, Amer. J. Math. 118 (1996), no. 1, 209-233.
[Ste1983] Sternberg, S.: Lectures in differential geometry, Second edition, Chelsea publishing co., New York, 1983, xviii+442 pp.
[Stk1982] Stormark, O.: On the theorem of Frobenius for complex vector fields, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 9 (1982), no. 1, 57-90.
[Stk2000] Stormark, O.: Lie's structural approach to PDE systems, Encyclopædia of mathematics and its applications, vol. 80, Cambridge University Press, Cambridge, 2000, xv+572 pp.
[Su2001] Sukhov, A.: Segre varieties and Lie symmetries, Math. Z. 238 (2001), no. 3, 483-492.
[Su2002] SuKhov, A.: CR maps and point Lie transformations, Michigan Math. J. 50 (2002), 369-379.
[Su2003] SUKhov, A.: On transformations of analytic structures (Russian). Izv. Ross. Akad. Nauk Ser. Mat. 67 (2003), no. 2, 101-132; transl. in Izv. Math. 67 (2003), no. 2, 303-332.
[Su1973] SUSSMANN, H.J.: Orbits of families of vector fields and integrability of distributions, Trans. Amer. Math. Soc. 180 (1973), no. 1, 171-188.
[Tr1896] Tresse, A.: Détermination des invariants ponctuels de l'équation différentielle du second ordre $y^{\prime \prime}=\omega\left(x, y, y^{\prime}\right)$, Hirzel, Leipzig, 1896.
[Ve1924] Vessiot, E.: Sur une théorie nouvelle des problèmes généraux d'intégration, Bull. Soc. Math. France 52 (1924), 336-395.
[Vi1990] Vitushkin, A.G.: Holomorphic mappings and the Geometry of Hypersurfaces, Encyclopædia of Mathematical Sciences, Volume 7, Several Complex Variables, I, Springer-Verlag, Berlin, 1990, pp. 159-214.
[We1977] WEBSTER, S.M.: On the mapping problem for algebraic real hypersurfaces, Invent. Math. 43 (1977), no.1, 53-68.


[^0]:    ${ }^{1}$ By borrowing techniques developed in [Ma2003], this theorem as well as the next both hold under weaker smoothness assumptions, namely with a $\mathscr{C}^{2}$ or a $W_{\text {loc }}^{1, \infty}$ right-hand side.

[^1]:    ${ }^{2}$ We note that in these references, the already substantial computations are stopped just after the reduction to an $\{e\}$-structure on an eight-dimensional (local) principal bundle over the three-dimensional first order jet space. The vanishing of two (among four) fundamental tensors in the structure equations of the obtained $\{e\}$-structure yields two partial differential equations satisfied by the right-hand side $F\left(x, y, y_{x}\right)$, which are equivalent to (4) of Theorem 1.2. We mention that with the help of Maple programming, the complete reduction to an $\{e\}$-structure on the base (not only on the principal bundle) is achieved in [HK1989], in the simpler case of so-called fiber-preserving transformations, namely point transformations leaving invariant the "vertical" foliation $\{x=c t$.$\} . To the author's knowledge, the$ complete confirmation of Tresse's results by means of É. Cartan's method has never been achieved.

[^2]:    ${ }^{3}$ Throughout the article, we do not adopt the summation convention, because in several subsequent equations, some repeated indices shall appear that will not be summed. Also, we always put commas between the indices. For instance $L_{l_{1}, l_{3}, y^{l_{2}}}^{j}$ denotes $\partial L_{l_{1}, l_{3}}^{j} / \partial y^{l_{2}}$ shortly. As usual, $\delta_{i}^{j}$ is the Kronecker symbol.

[^3]:    ${ }^{4}$ Similar rotation formulas are known in the much simpler case of (pseudo-) Riemannian metrics, see Chapter 12 of Olver [Ol1995]. It would be interesting to write a program, finer and more efficient than [N2003], which would systematically recognize such rotation formulas in any application of É. Cartan's equivalence method.

[^4]:    ${ }^{5}$ For a presentation of these concepts, the reader is referred to the extensive introductions of [14, Me2005b] and also to [19] for more about why dealing only with complex defining equations is natural and unavoidable when one wants to insert CR geometry in the wider universe of completely integrable systems of real or complex analytic partial differential equations.
    ${ }^{6}$ More will be said shortly in Section 2 below.

[^5]:    ${ }^{7}$ See [3] and also [23], where the tight analogy with second-order ordinary differential equations is well explained.

[^6]:    ${ }^{8}$ To our knowledge, the only existing reference where this strategy is seriously endeavoured in order to classify second-order ordinary differential equations $y_{x x}(x)=$ $F\left(x, y(x), y_{x}(x)\right)$ is [HK1989], but only for certain point transformations - called there "fiber-preserwing" - of the special form $(x, y) \mapsto\left(x^{\prime}, y^{\prime}\right)=\left(x^{\prime}(x), y^{\prime}(x, y)\right)$, the first component of which is independent of $y$.
    ${ }^{9}$ Three decades earlier, Christoffel in his famous memoir [4] of 1869 devoted to the equivalence problem for Riemannian metrics discovered that the covariant differentiations

[^7]:    10 The reader might for instance consult the survey [18], pp. 5-44 or the memoirs [Me2005a, Me2005b], and look also at some of the concerned references therein.
    ${ }^{11}$ Thanks to $d \varphi(0)=0$, the holomorphic implicit function theorem readily applies.
    ${ }^{12}$ Notice that since $d \varphi(0)=0$, one has $\Theta=-\bar{w}+$ order 2 terms.

[^8]:    ${ }^{13}$ Compared to [18], we denote here by $\Theta$ the function denoted there by $\bar{\Theta}$.
    14 According to a general, common convention, given a power series $\Phi(t)=$ $\sum_{\gamma \in \mathbb{N}^{n}} \Phi_{\gamma} t^{\gamma}, t \in \mathbb{C}^{n}, \Phi_{\gamma} \in \mathbb{C}$, one defines the series $\bar{\Phi}(t):=\sum_{\gamma \in \mathbb{N}^{n}} \bar{\Phi}_{\gamma} t^{\gamma}$ by conjugating only its complex coefficients. Then the complex conjugation operator distributes oneself simultaneously on functions and on variables: $\overline{\Phi(t)} \equiv \bar{\Phi}(\bar{t})$, a trivial property which is nonetheless frequently used in the formal CR reflection principle ([Me2005a, Me2005b]).

[^9]:    ${ }^{15}$ A presentation of the general theory, valuable for generic CR manifolds of arbitrary codimension $d \geqslant 1$ and of arbitrary CR dimension $m \geqslant 1$ in $\mathbb{C}^{m+d}$ enjoying no specific nondegeneracy condition, may be found in [Me2005a, Me2005b, 18].

[^10]:    ${ }^{16}$ This idea, usually attributed by contemporary CR geometers to B. Segre, dates in fact back (at least) to Chapter 10 of Volume 1 of the 2100 pages long Theorie der Transformationsgruppen written by Sophus Lie and Friedrich Engel between 1884 and 1893, where it is even presented in the uppermost general context.
    ${ }^{17}$ In fact, such a normalization was made in advance just in order to make things concrete and clear, but thanks to what the Lemma on p. 192 expresses in a biholomorphically invariant way, everything which follows next holds in an arbitrary system of coordinates.

    18 Justification: by our preliminary normalization, the $2 \times 2$ Jacobian determinant $\frac{\partial\left(\Theta, \Theta_{z}\right)}{\partial(\bar{z}, \bar{w})}$ computed at the origin equals $\left|\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right|$, hence is nonzero. Without the preliminary normalization, the condition of the Lemma on p. 192 also applies in any case.

[^11]:    19 - indicated already as the accessible Open Question 2.35 in [19] -

[^12]:    ${ }^{20}$ At this point, the reader is referred to [19] for more about how one can develope the whole theory of Lie symmetries of partial differential equations intrinsically within submanifolds of solutions only; the theory of Cartan connections associated to certain exterior differential systems could (and should also) be transferred to submanifolds of solutions.

[^13]:    ${ }^{21}$ Be careful not to write $\{(z, w, \widetilde{z}, \widetilde{w}): \widetilde{w}=\bar{\Theta}(\widetilde{z}, z, w)\}$, because this would regive the same subset $\mathscr{M}$ of $\mathbb{C}^{2} \times \mathbb{C}^{2}$, due to the reality identities (7.28).

[^14]:    ${ }^{22}$ To be precise, both factors of multiplicity ([5], p. 165) are nonvanishing in a neighborood of the origin, but for our purposes, it suffices just that they are not identically zero power series.

[^15]:    ${ }^{24}$ At the conference Cauchy-Riemann Analysis and Geometry organized by Ingo Lieb and Gerd Schmalz at the Max-Planck Institut of Bonn, 22-27 September 2003, the author gave a talk the title of which was "Explicit Chern-Moser tensors".

[^16]:    ${ }^{25}$ We will be very brief here, the reader being referred to [19, 21] for the general theoretical considerations.

[^17]:    ${ }^{26}$ We put a minus sign in front of $y(0)$ so as to match up with our choice of complex defining equation $w=-\bar{w}+\mathrm{O}(2)$.

[^18]:    ${ }^{27}$ Much more theoretical information is provided in [19].

[^19]:    ${ }^{28}$ Remind that, in order to differentiate a determinant, one should differentiate separately each column once and then sum all the obtained terms.

[^20]:    ${ }^{29}$ Part II of [Me2005a] already appeared as [Me2005b].

[^21]:    ${ }^{30}$ Fundamentals about Cauchy-Riemann geometry may be found in [Bo1991, BER1999, Me2005a, Me2005b, MP2005].

[^22]:    ${ }^{31}$ The author knows no complete confirmation of the Lie-Tresse classification by means of É. Cartan's method of equivalence.

[^23]:    ${ }^{32}$ Remind from Section 1(II) that we have not (open problem) provided a complete explicit expression of $\Phi_{i_{1}, \ldots, i_{\lambda}}^{j}$ for general $n \geqslant 1, m \geqslant 1$ and $\lambda \geqslant 1$.

