## Travaux sélectionnés

## sur le prolongement holomorphe

## des fonctions CR,

sur leurs singularités éliminables,
et sur le phénomène de Hartogs
dans les espaces complexes

## ( $n-1$ )-complets

[493 pages, 53 figures en couleurs, 39 en noir et blanc]
Document 1, p. 3: Article de survol (publié) sur la géométrie CR locale, le prolongement holomorphe des fonctions CR et l'élimination de leurs singularités.
Document 2, p. 282: Article de recherche (publié) sur l'élimination des singularités de codimension 1 dans les variétés CR de dimension 1 et de codimension $\geqslant 1$.
Document 3, p. 407: Article (publié) fournissant une démonstration géométrique du phénomène classique de Hartogs.
Document 4, p. 448: Article (publié incessamment sous peu, voir le site web du Journal de Crelle) sur le phénomène de Hartogs pour les domaines non pseudoconvexes dans les espaces complexes $(n-1)$-complets.
Document 5, p. 467: Article (publié) sur le prolongement méromorphe des fonctions CR-méromorphes (question de Henkin).
[5 documents, travaux effectués entre fin 2002 et février 2007.]

# Holomorphic extension of CR functions, envelopes of holomorphy, and removable singularities 

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## Table of contents ( 7 main parts)

I. Introduction ..... 4.
II. Analytic vector field systems, formal CR mappings and local CR automorphism groups .....  8.
III. Sussman's orbit theorem, locally integrable systems of vector fields and CR functions .....  48.
IV. Hilbert transform and Bishop's equation in Hölder spaces ..... 106.
V. Holomorphic extension of CR functions ..... 148.
VI. Removable singularities ..... 202.

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## I: Introduction

1.1. CR extension theory. In the past decades, remarkable progress has been accomplished towards the understanding of compulsory extendability of holomorphic functions, of CR functions and of differential forms. These phenomena, whose exploration is still active in current research, originate from the seminal Hartogs-Bochner extension theorem.
In local CR extension theory, the most satisfactory achievement was the discovery that, on a smooth embedded generic submanifold $M \subset \mathbb{C}^{n}$, there is a precise correspondence between $C R$ orbits of $M$ and families of small Bishop discs attached to $M$. Such discs cover a substantial part of the polynomial hull of $M$, and in most cases, this part may be shown to constitute a global one-sided neighborhood $\mathscr{V}^{ \pm}(M)$ of $M$, if $M$ is a hypersurface, or else a wedgelike domain $\mathscr{W}$ attached to $M$, if $M$ has codimension $\geqslant 2$. A local polynomial approximation theorem, or a CR version of the Kontinuitätssatz (continuity principle) assures that CR functions automatically extend holomorphically to such domains $\mathscr{W}$, which are in addition contained in the envelope of holomorphy of arbitrarily thin neighborhoods of $M$ in $\mathbb{C}^{n}$.

Trépreau in the hypersurface (1986) case and slightly after Tumanov in arbitrary codimension (1988) established a nowadays celebrated extension theorem: if $M \subset \mathbb{C}^{n}$ is a sufficiently smooth ( $\mathscr{C}^{2}$ or $\mathscr{C}^{2, \alpha}$ suffices) generic submanifold, then at every point $p \in M$ whose local CR orbit $\mathscr{O}_{C R}^{\text {loc }}(M, p)$ has maximal dimension equal to dim $M$, there exists a local wedge $\mathscr{W}_{p}$ of edge $M$ at $p$ to which continuous CR functions extend holomorphically. Several reconstructions and applications of this groundbreaking result, together with surveys about the local Bishop equation have already appeared in the literature.

Propagational aspects of CR extension theory are less known by contemporary experts of several complex variables, but they lie deeper in the theory. Using FBI transform and concepts of microlocal analysis, Trépreau showed in 1990 that holomorphic extension to a wedge propagates along curves whose velocity vector is complex-tangential to $M$. His conjecture that extension to a wedge should hold at every point of a generic submanifold $M \subset \mathbb{C}^{n}$ consisting of a single global CR orbit has been answered independently by Jöricke and by the first author in 1994, using tools introduced previously by Tumanov. To the knowledge of the two authors, there is no survey of these global aspects in the literature.

The first main objective of the present survey is to expose the techniques underlying these results in a comprehensive and unified way, emphasizing propagational aspects of embedded CR geometry and discussing optimal smoothness assumptions. Thus, topics that are necessary to build the theory from scratch will be selected and accompanied with thorough proofs,
whereas other results that are nevertheless central in CR geometry will be presented in concise survey style, without any proof.

The theory of CR extension by means of analytic discs combines various concepts emanating mainly from three (wide) mathematical areas: Harmonic analysis, Partial differential equations and Complex analysis in several variables. As the project evolved, we felt the necessity of being conceptional, extensive and systematic in the restitution of (semi)known results, so that various contributions to the subject would recover a certain coherence and a certain unity. With the objective of adressing to a younger audience, we decided to adopt a style accessible to doctoral candidates working on a dissertation. Parts III, IV and V present elementarily general CR extension theory. Also, most sections of the text may be read independently by experts, as quanta of mathematical information.
1.2. Concise presentation of the contents. The survey text is organized in six main parts. Actually, the present brief introduction constitutes the first and shortest one. Although the reader will find a "conceptional summaryintroduction" at the beginning of each part, a few descriptive words explaining some of our options governing the reconstruction of CR extension theory (Parts III, IV and V) are welcome.

The next Part II is independent of the others and can be skipped in a first reading. It opens the text, because it is concerned with propagational aspects of analytic CR structures, better understood than the smooth ones.

- In Part III, exclusively concerned with the smooth category, Sussmann's orbit theorem and its consequences are first explained in length. Involutive structures and embedded CR manifolds, together with their elementary properties, are introduced. Structural properties of finite type structures, of CR orbits and of CR functions are presented without proofs. As a collection of background material, this part should be consulted first.
- In Part IV, fundamental results about singular integral operators in the complex plane are first surveyed. Explicit estimates of the norms of the Cauchy, of the Schwarz and of the Hilbert transforms in the Hölder spaces $\mathscr{C}^{\kappa, \alpha}$ are provided. They are useful to reconstruct the main Theorem 3.7(IV), due to Tumanov, which asserts the existence of unique solutions to a parametrized Bishop-type equation with an optimal loss of smoothness with respect to parameters. Following Bishop's constructive philosophy, the smallness of the constants insuring existence is precised explicitly, thanks to sharp norm inequalities in Hölder spaces. This part is meant to introduce interested readers to further reading of Tumanov's recent works about extremal (pseudoholomorphic) discs in higher codimension.
- In Part V, CR extension theory is first discussed in the hypersurface case. A simplified proof of wedge extendability that treats both locally minimal
and globally minimal generic submanifolds on the same footing constitutes the main Theorem 4.12(V): If $M$ is a globally minimal $\mathscr{C}^{2, \alpha}(0<\alpha<$ 1) generic submanifold of $\mathbb{C}^{n}$ of codimension $\geqslant 1$ and of $C R$ dimension $\geqslant 1$, there exists a wedgelike domain $\mathscr{W}$ attached to $M$ such that every continuous $C R$ function $f \in \mathscr{C}_{C R}^{0}(M)$ possesses a holomorphic extension $F \in \mathscr{O}(\mathscr{W}) \cap \mathscr{C}^{0}(M \cup \mathscr{W})$ with $\left.F\right|_{M}=f$. The figures are intended to share the geometric insight of experts in higher codimensional geometry.

In fact, throughout the text, diagrams (33 in sum) facilitating readability (especially of Part V) are included. Selected open questions and open problems (16 in sum) are formulated. They are systematically inserted in the right place of the architecture. The sign " $[*]$ " added after one or several bibliographical references in a statement (Problem, Definition, Theorem, Proposition, Lemma, Corollary, Example, Open question and Open problem, e.g. Theorem 1.11(I)) indicates that, compared to the existing literature, a slight modification or a slight improvement has been brought by the two authors. Statements containing no bibliographical reference are original and appear here for the first time.

We apologize for having not treated some central topics of CR geometry that also involve propagation of holomorphicity, exempli gratia the geometric reflection principle, in the sense of Pinchuk, Webster, Diederich, Fornæss, Shafikov and Verma. By lack of space, embeddability of abstract CR structures, polynomial hulls, Bishop discs growing at elliptic complex tangencies, filling by Levi-flat surfaces, Riemann-Hilbert boundary value problems, complex Plateau problem in Kähler manifolds, partial indices of analytic discs, pseudoholomorphic discs, etc. are not reviewed either. Certainly, better experts will fill this gap in the near future.

To conclude this introductory presentation, we believe that, although uneasy to build, surveys and syntheses play a decisive rôle in the evolution of mathematical subjects. For instance, in the last decades, the remarkable development of $\bar{\partial}$ techniques and of $L^{2}$ estimates has been regularly accompanied by monographs and panoramas, some of which became landmarks in the field. Certainly, the (local) method of analytic discs deserves to be known by a wider audience; in fact, its main contributors have brought it to the degree of achievement that opened the way to the present survey.
1.3. Further readings. Using the tools exposed and reconstructed in this survey, the research article [26] studies removable singularities on CR manifolds of CR dimension equal to 1 and solves a delicate remaining open problem in the field (see the Introduction there for motivations). Recently also, the authors built in [MP2006c] a new, rigorous proof of the classical

Hartogs extension theorem which relies only on the basic local Levi argument along analytic discs, hence avoids both multidimensional integral representation formulas and the Serre-Ehrenpreis argument about vanishing of $\bar{\partial}$ cohomology with compact support.

# II: Analytic vector field systems and formal CR mappings 

Table of contents<br>1. Analytic vector field systems and Nagano's theorem . 8.<br>2. Analytic CR manifolds, Segre chains and minimality ......................... 19.<br>3. Formal CR mappings, jets of Segre varieties and CR reflection mapping ... 28.

[3 diagrams]

According to the theorem of Frobenius, a system $\mathbb{L}$ of local vector fields having real or complex analytic coefficients enjoys the integral manifolds property, provided it is closed under Lie bracket. If the Lie brackets exceed $\mathbb{L}$, considering the smallest analytic system $\mathbb{L}^{\text {lie }}$ containing $\mathbb{L}$ which is closed under Lie bracket, Nagano showed that through every point, there passes a submanifold whose tangent space is spanned by $\mathbb{L}^{\text {lie }}$. Without considering Lie brackets, these submanifolds may also be constructed by means of compositions of local flows of elements of $\mathbb{L}$. Such a construction has applications in real analytic Cauchy-Riemann geometry, in the reflection principle, in formal CR mappings, in analytic hypoellipticity theorems and in the problem of local solvability and of local uniqueness for systems of first order linear partial differential operators (Part III).

For a generic set of $r \geqslant 2$ vector fields having analytic coefficients, $\mathbb{L}^{\text {lie }}$ has maximal rank equal to the dimension of the ambient space.

The extrinsic complexification $\mathscr{M}$ of a real algebraic or analytic CauchyRiemann submanifold $M$ of $\mathbb{C}^{n}$ carries two pairs of intrinsic foliations, obtained by complexifying the classical Segre varieties together with their conjugates. The Nagano leaves of this pair of foliations coincide with the extrinsic complexifications of local CR orbits. If $M$ is (Nash) algebraic, its CR orbits are algebraic too, because they are projections of complexified algebraic Nagano leaves.

A complexified formal CR mapping between two complexified generic submanifolds must respect the two pairs of intrinsic foliations that lie in the source and in the target. This constraint imposes strong rigidity properties, as for instance: convergence, analyticity or algebraicity of the formal CR mapping, according to the smoothness of the target and of the source. There is a combinatorics of various nondegeneracy conditions that entail versions of the so-called analytic reflection principle. The concept of $C R$ reflection mapping provides a unified synthesis of recent results of the literature.

## §1. Analytic vector field systems and Nagano's theorem

1.1. Formal, analytic and (Nash) algebraic power series. Let $n \in \mathbb{N}$ with $n \geqslant 1$ and let $\mathrm{x}=\left(\mathrm{x}_{1}, \ldots, \mathrm{x}_{n}\right) \in \mathbb{K}^{n}$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Let $\mathbb{K} \llbracket \mathrm{x} \rrbracket$ be the ring of formal power series in $\left(x_{1}, \ldots, x_{n}\right)$. An element $\varphi(x) \in \mathbb{K} \llbracket x \rrbracket$ writes
$\varphi(\mathrm{x})=\sum_{\alpha \in \mathbb{N}^{n}} \varphi_{\alpha} \mathrm{x}^{\alpha}$, with $\mathrm{x}^{\alpha}:=\mathrm{x}_{1}^{\alpha_{1}} \cdots \mathrm{x}_{n}^{\alpha_{n}}$ and with $\varphi_{\alpha} \in \mathbb{K}$, for every multiindex $\alpha:=\left(\alpha_{1}, \ldots, \alpha_{n}\right) \in \mathbb{N}^{n}$. We put $|\alpha|:=\alpha_{1}+\cdots+\alpha_{n}$.

On the vector space $\mathbb{K}^{n}$, we choose once for all the maximum norm $|x|:=$ $\max _{1 \leqslant i \leqslant n}\left|\mathrm{x}_{i}\right|$ and, for any "radius" $\rho_{1}$ satisfying $0<\rho_{1} \leqslant \infty$, we define the open cube

$$
\square_{\rho_{1}}^{n}:=\left\{\mathbf{x} \in \mathbb{K}^{n}:|x|<\rho_{1}\right\}
$$

as a fundamental, concrete open set. For $\rho_{1}=\infty$, we identify of course $\square_{\infty}^{n}$ with $\mathbb{K}^{n}$.

If the coefficients $\varphi_{\alpha}$ satisfy a Cauchy estimate of the form $\left|\varphi_{\alpha}\right| \leqslant C \rho_{2}^{-|\alpha|}$, $C>0$, for every $\rho_{2}$ satisfying $0<\rho_{2}<\rho_{1}$, the formal power series is $\mathbb{K}$-analytic ( $\mathscr{C}^{\omega}$ ) in $\square_{\rho_{1}}^{n}$. It then defines a true point map $\varphi: \square_{\rho_{1}}^{n} \rightarrow \mathbb{K}$. Such a $\mathbb{K}$-analytic function $\varphi$ is called (Nash) $\mathbb{K}$-algebraic if there exists a nonzero polynomial $P(\mathrm{X}, \Phi) \in \mathbb{K}[\mathrm{X}, \Phi]$ in $(n+1)$ variables such that the relation $P(\mathrm{x}, \varphi(\mathrm{x})) \equiv 0$ holds in $\mathbb{K} \llbracket \mathrm{x} \rrbracket$, hence for all x in $\square_{\rho_{1}}^{n}$. The category of $\mathbb{K}$-algebraic functions and maps is stable under elementary algebraic operations, under differentiation and under composition. Implicit solutions of $\mathbb{K}$-algebraic equations are $\mathbb{K}$-algebraic ([BER1999]).

### 1.2. Analytic vector field systems and their integral manifolds. Let

$$
\mathbb{L}^{0}:=\left\{L_{a}\right\}_{1 \leqslant a \leqslant r}, \quad r \in \mathbb{N}, \quad r \geqslant 1,
$$

be a finite set of vector fields $L_{a}=\sum_{i=1}^{n} \varphi_{a, i}(\mathrm{x}) \frac{\partial}{\partial \mathrm{x}_{i}}$, whose coefficients $\varphi_{a, i}$ are algebraic or analytic in $\square_{\rho_{1}}^{n}$. Let $\mathbb{A}_{\rho_{1}}$ denote the ring of algebraic or analytic functions in $\square_{\rho_{1}}^{n}$. The set of linear combinations of elements of $\mathbb{L}^{0}$ with coefficients in $\mathbb{A}_{\rho_{1}}$ will be denoted by $\mathbb{L}\left(\right.$ or $\left.\mathbb{L}^{1}\right)$ and will be called the $\mathbb{A}_{\rho_{1}}$-linear hull of $\mathbb{L}^{0}$.

If $p$ is a point of $\square_{\rho_{1}}^{n}$, denote by $L_{a}(p)$ the vector $\left.\sum_{i=1}^{n} \varphi_{a, i}(p) \frac{\partial}{\partial \mathrm{x}_{i}}\right|_{p}$. It is an element of $T_{p} \square_{\rho_{1}}^{n} \simeq \mathbb{K}^{n}$. Define the linear subspace

$$
\mathbb{L}(p):=\operatorname{Span}_{\mathbb{K}}\left\{L_{a}(p): 1 \leqslant a \leqslant r\right\}=\{L(p): L \in \mathbb{L}\} .
$$

No constancy of dimension, no linear independency assumption are made.
Problem 1.3. Find local submanifolds $\Lambda$ passing through the origin satisfying $T_{q} \Lambda \supset \mathbb{L}(q)$ for every $q \in \Lambda$.

By the theorem of Frobenius ([Stk2000]; original article: [Fr1877]), if the $L_{a}$ are linearly independent at every point of $\square_{\rho_{1}}^{n}$ and if the Lie brackets [ $L_{a}, L_{a^{\prime}}$ ] belong to $\mathbb{L}$, for all $a, a^{\prime}=1, \ldots, r$, then $\square_{\rho_{1}}^{n}$ is foliated by $r$ dimensional submanifolds $N$ satisfying $T_{q} N=\mathbb{L}(q)$ for every $q \in N$.

Lemma 1.4. If there exists a local submanifold $\Lambda$ passing through the origin and satisfying $T_{q} \Lambda \supset \mathbb{L}(q)$ for every $q \in \Lambda$, then for every two vector fields $L, L^{\prime} \in \mathbb{L}$, the restriction to $\Lambda$ of the Lie bracket $\left[L, L^{\prime}\right]$ is tangent to $\Lambda$.

Accordingly, set $\mathbb{L}^{1}:=\mathbb{L}$ and for $k \geqslant 2$, define $\mathbb{L}^{k}$ to be the $\mathbb{A}_{\rho_{1}}-$ linear hull of $\mathbb{L}^{k-1}+\left[\mathbb{L}^{1}, \mathbb{L}^{k-1}\right]$. Concretely, $\mathbb{L}^{k}$ is generated by $\mathbb{A}_{\rho_{1}}$-linear combinations of iterated Lie brackets $\left[L_{1},\left[L_{2}, \ldots,\left[L_{k-1}, L_{k}\right] \ldots\right]\right.$, where $L_{1}, L_{2}, \ldots, L_{k-1}, L_{k} \in \mathbb{L}^{1}$. The Jacobi identity insures (by induction) that $\left[\mathbb{L}^{k_{1}}, \mathbb{L}^{k_{2}}\right] \subset \mathbb{L}^{k_{1}+k_{2}}$. Define then $\mathbb{L}^{\text {lie }}:=\cup_{k \geqslant 1} \mathbb{L}^{k}$. Clearly, $\left[L, L^{\prime}\right] \in \mathbb{L}^{\text {lie }}$, for every two vector fields $L, L^{\prime} \in \mathbb{L}^{\text {lie }}$.

Theorem 1.5. (NAGANO [Na1966, Trv1992, BER1999, BCH2005]) There exists a unique local $\mathbb{K}$-analytic submanifold $\Lambda$ of $\mathbb{K}^{n}$ passing through the origin which satisfies $\mathbb{L}(q) \subset T_{q} \Lambda=\mathbb{L}^{\mathrm{lie}}(q)$, for every $q \in \Lambda$.

A discussion about what happens in the algebraic category is postponed to $\S 1.12$. In Frobenius' theorem, $\mathbb{L}^{\text {lie }}=\mathbb{L}$ and the dimension of $\mathbb{L}^{\text {lie }}(p)$ is constant. In the above theorem, the dimension of $\mathbb{L}^{\text {lie }}(q)$ is constant for $q$ belonging to $\Lambda$, but in general, not constant for $p \in \square_{\rho_{1}}^{n}$, the function $p \mapsto \operatorname{dim}_{\mathbb{K}} \mathbb{L}(p)$ being lower semi-continuous.

Nagano's theorem is stated at the origin; it also holds at every point $p \in \square_{\rho_{1}}^{n}$. The associated local submanifold $\Lambda_{p}$ passing through $p$ with the property that $T_{q} \Lambda=\mathbb{L}^{\text {lie }}(q)$ for every $q \in \Lambda_{p}$ is called a (local) Nagano leaf.

In the $\mathscr{C}^{\infty}$ category, the consideration of $\mathbb{L}^{\text {lie }}$ is insufficient. Part III handles smooth vector field systems, providing a different answer to the search of similar submanifolds $\Lambda_{p}$.

Example 1.6. In $\mathbb{R}^{2}$, take $\mathbb{L}^{0}=\left\{L_{1}, L_{2}\right\}$, where $L_{1}=\partial_{x_{1}}$ and $L_{2}=$ $e^{-1 / x_{1}^{2}} \partial_{x_{2}}$. Then $\mathbb{L}^{\text {lie }}(0)$ is the line $\left.\mathbb{R} \partial_{x_{1}}\right|_{0}$, while $\mathbb{L}^{\text {lie }}(p)=\left.\mathbb{R} \partial_{x_{1}}\right|_{p}+\left.\mathbb{R} \partial_{x_{2}}\right|_{p}$ at every point $p \notin \mathbb{R} \times\{0\}$. Hence, there cannot exist a $\mathscr{C}^{\infty}$ curve $\Lambda$ passing through 0 with $T_{0} \Lambda=\left.\mathbb{R} \partial_{x_{1}}\right|_{0}$ and $T_{q} \Lambda=\mathbb{L}^{\text {lie }}(q)$ for every $q \in \Lambda$.

Proof of Theorem 1.5. (May be skipped in a first reading.) If $n=1$, the statement is clear, depending on whether or not all vector fields in $\mathbb{L}^{\text {lie }}$ vanish at the origin. Let $n \geqslant 2$. Since $\mathbb{L}(q) \subset \mathbb{L}^{\text {lie }}(q)$, the condition $T_{q} \Lambda=\mathbb{L}^{\text {lie }}(q)$ implies the inclusion $\mathbb{L}(q) \subset T_{q} \Lambda$. Replacing $\mathbb{L}$ by $\mathbb{L}^{\text {lie }}$ if necessary, we may therefore assume that $\mathbb{L}^{\text {lie }}=\mathbb{L}$ and we then have to prove the existence of $\Lambda$ with $T_{q} \Lambda=\mathbb{L}^{\text {lie }}(q)=\mathbb{L}(q)$, for every $q \in \Lambda$.

We reason by induction, supposing that, in dimension $(n-1)$, for every $\mathbb{A}_{\rho_{1}}$-linear system $\mathbb{L}^{\prime}=\left(\mathbb{L}^{\prime}\right)^{\text {lie }}$ of vector fields locally defined in a neighborhood of the origin in $\mathbb{K}^{n-1}$, there exists a local $\mathbb{K}$-analytic submanifold $\Lambda^{\prime}$ passing through the origin and satisfying $T_{q^{\prime}} \Lambda^{\prime}=\mathbb{L}^{\prime}\left(q^{\prime}\right)$, for every $q^{\prime} \in \Lambda^{\prime}$.

If all vector fields in $\mathbb{L}=\mathbb{L}^{\text {lie }}$ vanish at 0 , we are done, trivially. Thus, assume there exists $L_{1} \in \mathbb{L}$ with $L_{1}(0) \neq 0$. After local straightening, $L_{1}=$ $\partial_{\mathrm{x}_{1}}$. Every $L \in \mathbb{L}$ writes uniquely $L=a(\mathrm{x}) \partial_{\mathrm{x}_{1}}+\widetilde{L}$, for some $a(\mathrm{x}) \in \mathbb{K}\{\mathrm{x}\}$, with $\widetilde{L}=\sum_{2 \leqslant i \leqslant n} a_{i}(\mathrm{x}) \partial_{\mathrm{x}_{i}}$. Introduce the space $\widetilde{\mathbb{L}}:=\{\widetilde{L}: L \in \mathbb{L}\}$ of such vector fields. As $\partial_{x_{1}}$ belongs to $\mathbb{L}$ and as $\mathbb{L}$ is $\mathbb{A}_{\rho_{1}}$-linear, $\widetilde{L}=L-a(x) \partial_{\times_{1}}$
belongs to $\mathbb{L}$. Since $[\mathbb{L}, \mathbb{L}] \subset \mathbb{L}$, we have $[\tilde{\mathbb{L}}, \widetilde{\mathbb{L}}] \subset \mathbb{L}$. On the other hand, we observe that the Lie bracket between two elements of $\widetilde{\mathbb{L}}$ does not involve $\partial_{x_{1}}$ :

$$
\begin{align*}
{\left[\widetilde{L}_{1}, \widetilde{L}_{2}\right] } & =\left[\sum_{2 \leqslant i_{2} \leqslant n} a_{i_{2}}^{1} \partial_{x_{i_{2}}}, \sum_{2 \leqslant i_{1} \leqslant n} a_{i_{1}}^{2} \partial_{x_{i_{1}}}\right] \\
& =\sum_{2 \leqslant i_{1} \leqslant n}\left(\sum_{2 \leqslant i_{2} \leqslant n}\left[a_{i_{2}}^{1} \frac{\partial a_{i_{1}}^{2}}{\partial x_{i_{2}}}-a_{i_{2}}^{2} \frac{\partial a_{i_{1}}^{1}}{\partial x_{i_{2}}}\right]\right) \partial_{x_{i_{1}}} \tag{1.7}
\end{align*}
$$

We deduce that $[\widetilde{\mathbb{L}}, \widetilde{\mathbb{L}}] \subset \widetilde{\mathbb{L}}$. In other words, $\widetilde{\mathbb{L}}^{\text {lie }}=\widetilde{\mathbb{L}}$. Next, we define the restriction

$$
\mathbb{L}^{\prime}:=\left\{L^{\prime}=\left.\widetilde{L}\right|_{\left\{x_{1}=0\right\}}: \widetilde{L} \in \widetilde{\mathbb{L}}\right\}
$$

and we claim that $\left(\mathbb{L}^{\prime}\right)^{\text {lie }}=\mathbb{L}^{\prime}$ also holds true. Indeed, restricting (1.7) above to $\left\{\mathrm{x}_{1}=0\right\}$, we observe that

$$
\left[\left.\widetilde{L}_{1}\right|_{\left\{x_{1}=0\right\}},\left.\widetilde{L}_{2}\right|_{\left\{x_{1}=0\right\}}\right]=\left.\left[\widetilde{L}_{1}, \widetilde{L}_{2}\right]\right|_{\left\{x_{1}=0\right\}},
$$

since neither $\widetilde{L}_{1}$ nor $\widetilde{L}_{2}$ involves $\partial_{x_{1}}$. This shows that $\left[\mathbb{L}^{\prime}, \mathbb{L}^{\prime}\right] \subset \mathbb{L}^{\prime}$, as claimed.

Since $\left(\mathbb{L}^{\prime}\right)^{\text {lie }}=\mathbb{L}^{\prime}$, the induction assumption applies: there exists a local $\mathbb{K}$-analytic submanifold $\Lambda^{\prime}$ of $\mathbb{K}^{n-1}$ passing through the origin such that $T_{q^{\prime}} \Lambda^{\prime}=\mathbb{L}^{\prime}\left(q^{\prime}\right)$, for every point $q^{\prime} \in \Lambda^{\prime}$. Let $d$ denote its codimension. If $d=0$, i.e. if $\Lambda^{\prime}$ coincides with an open neighborhood of the origin in $\mathbb{K}^{n-1}$, it suffices to chose for $\Lambda$ an open neighborhood of the origin in $\mathbb{K}^{n}$. Assuming $d \geqslant 1$, we split the coordinates $\mathrm{x}=\left(\mathrm{x}_{1}, \mathrm{x}^{\prime}\right) \in \mathbb{K} \times \mathbb{K}^{n-1}$ and we let $\rho_{j}\left(\mathrm{x}^{\prime}\right)=0, j=1, \ldots, d$, denote local $\mathbb{K}$-analytic defining equations for $\Lambda^{\prime}$. We claim that it suffices to choose for $\Lambda$ the local submanifold of $\mathbb{K}^{n}$ with the same equations, hence having the same codimension.

Indeed, since these equations are independent of $x_{1}$, it is first of all clear that the vector field $\partial_{x_{1}} \in \mathbb{L}$ is tangent to $\Lambda$. To conclude that every $L=$ $a \partial_{x_{1}}+\widetilde{L} \in \mathbb{L}$ is tangent to $\Lambda$, we thus have to prove that every $\widetilde{L} \in \widetilde{\mathbb{L}}$ is tangent to $\Lambda$.

Let $\widetilde{L}=\sum_{2 \leqslant i \leqslant n} a_{i}\left(\mathrm{x}, \mathrm{x}^{\prime}\right) \partial_{\mathrm{x}_{i}} \in \widetilde{\mathbb{L}}$. As a preliminary observation:

$$
\left(\operatorname{ad} \partial_{\mathrm{x}_{1}}\right) \widetilde{L}:=\left[\partial_{\mathrm{x}_{1}}, \widetilde{L}\right]=\sum_{2 \leqslant i \leqslant n} \frac{\partial a_{i}}{\partial \mathrm{x}_{1}}\left(\mathrm{x}_{1}, \mathrm{x}^{\prime}\right) \frac{\partial}{\partial \mathrm{x}_{i}},
$$

and more generally, for $\ell \in \mathbb{N}$ arbitrary:

$$
\left(\operatorname{ad} \partial_{x_{1}}\right)^{\ell} \widetilde{L}=\sum_{2 \leqslant i \leqslant n} \frac{\partial^{\ell} a_{i}}{\partial x_{1}^{\ell}}\left(\mathrm{x}_{1}, \mathrm{x}^{\prime}\right) \frac{\partial}{\partial \mathrm{x}_{i}}
$$

Since $\mathbb{L}$ is a Lie algebra, we have $\left(\operatorname{ad} \partial_{x_{1}}\right)^{\ell} \widetilde{L} \in \mathbb{L}$. Since $\left(\operatorname{ad} \partial_{x_{1}}\right)^{\ell} \widetilde{L}$ does not involve $\partial_{x_{1}}$, according to its expression above, it belongs in fact to $\widetilde{\mathbb{L}}$. Also, after restriction $\left.\left(\operatorname{ad} \partial_{x_{1}}\right)^{\ell} \widetilde{L}\right|_{x_{1}=0} \in \mathbb{L}^{\prime}$. By assumption, $\mathbb{L}^{\prime}$ is tangent to $\Lambda^{\prime}$. We deduce that, for every $\ell \in \mathbb{N}$, the vector field

$$
L_{\ell}^{\prime}:=\left.\left(\operatorname{ad} \partial_{x_{1}}\right)^{\ell} \widetilde{L}\right|_{\mathrm{x}_{1}=0}=\sum_{2 \leqslant i \leqslant n} \frac{\partial^{\ell} a_{i}}{\partial \mathrm{x}_{1}^{\ell}}\left(0, \mathrm{x}^{\prime}\right) \frac{\partial}{\partial \mathrm{x}_{i}}
$$

is tangent to $\Lambda^{\prime}$. Equivalently, $\left[L_{\ell}^{\prime} \rho_{j}\right]\left(x^{\prime}\right)=0$ for every $x^{\prime} \in \Lambda^{\prime}$. Letting $\left(\mathrm{x}_{1}, \mathrm{x}^{\prime}\right) \in \Lambda$, whence $\mathrm{x}^{\prime} \in \Lambda^{\prime}$, we compute:

$$
\begin{aligned}
{\left[\widetilde{L} \rho_{j}\right]\left(\mathrm{x}_{1}, \mathrm{x}^{\prime}\right) } & =\sum_{2 \leqslant i \leqslant n} a_{i}\left(\mathrm{x}_{1}, \mathrm{x}^{\prime}\right) \frac{\partial \rho_{j}}{\partial \mathrm{x}_{i}}\left(\mathrm{x}^{\prime}\right) \\
& =\sum_{2 \leqslant i \leqslant n} \sum_{\ell=0}^{\infty} \frac{\mathrm{x}_{1}^{\ell}}{\ell!} \frac{\partial^{\ell} a_{i}}{\partial \mathrm{x}_{1}^{\ell}}\left(0, \mathrm{x}^{\prime}\right) \frac{\partial \rho_{j}}{\partial \mathrm{x}_{i}}\left(\mathrm{x}^{\prime}\right) \quad \text { [Taylor development] } \\
& =\sum_{\ell=0}^{\infty} \frac{\mathrm{x}_{1}^{\ell}}{\ell!}\left[L_{\ell}^{\prime} \rho_{j}\right]\left(\mathrm{x}^{\prime}\right)=0
\end{aligned}
$$

so $\widetilde{L}$ is tangent to $\Lambda$. Finally, the property $T_{\times_{1}, x^{\prime}} \Lambda=\mathbb{L}\left(\mathrm{x}_{1}, \mathrm{x}^{\prime}\right)$ follows immediately from $T_{x^{\prime}} \Lambda^{\prime}=\mathbb{L}^{\prime}\left(\mathrm{x}^{\prime}\right)$ and the proof is complete (the Taylor development argument above was crucially used, and this enlightens why the theorem does not hold in the $\mathscr{C}^{\infty}$ category).
1.8. Free Lie algebras and generic sets of $\mathbb{K}$-analytic vector fields. For a generic set of $r \geqslant 2$ vector fields $\mathbb{L}^{0}=\left\{L_{a}\right\}_{1 \leqslant a \leqslant r}$, or after slightly perturbing any given set, one expects that $\mathbb{L}^{\text {lie }}(0)=T_{0} \mathbb{K}^{n}$. Then the Nagano leaf $\Lambda$ passing through 0 is just an open neighborhood of 0 in $\mathbb{K}^{n}$. Also, one expects that the dimensions of the intermediate spaces $\mathbb{L}^{k}(0)$ be maximal.

To realize this intuition, one has to count the maximal number of iterated Lie brackets that are linearly independent in $\mathbb{L}^{k}$, for $k=1,2,3, \ldots$, modulo antisymmetry and Jacobi identity.

Let $r \geqslant 2$ and let $h_{1}, h_{2}, \ldots, h_{r}$ be $r$ linearly independent elements of a vector space over $\mathbb{K}$. The free Lie algebra $\mathrm{F}(r)$ of rank $r$ is the smallest (noncommutative, non-associative) $\mathbb{K}$-algebra ([Re1993]) having $h_{1}, h_{2}, \ldots, h_{r}$ as elements, with multiplication $\left(h, h^{\prime}\right) \mapsto h h^{\prime}$ satisfying antisymmetry $0=h h^{\prime}+h^{\prime} h$ and Jacobi identity $0=h\left(h^{\prime} h^{\prime \prime}\right)+h^{\prime \prime}\left(h h^{\prime}\right)+h^{\prime}\left(h^{\prime \prime} h\right)$. It is unique up to isomorphism. The case $r=1$ yields only $\mathrm{F}(1)=\mathbb{K}$. The multiplication in $\mathrm{F}(r)$ plays the role of the Lie bracket in $\mathbb{L}^{\text {lie }}$. Importantly, no linear relation exists between iterated multiplications, i.e. between iterated Lie brackets, except those generated by antisymmetry and Jacobi identity. Thus, $\mathbf{F}(r)$ is infinite-dimensional. Every finite-dimensional Lie $\mathbb{K}$-algebra having $r$ generators embeds as a subalgebra of $\mathrm{F}(r)$, see [Re1993].

Since the bracket multiplication is not associative, one must carefully write some parentheses, for instance in $\left(h_{1} h_{2}\right) h_{3}$, or in $h_{1}\left(h_{2}\left(h_{1} h_{2}\right)\right.$ ), or in $\left(h_{1} h_{2}\right)\left(h_{3}\left(h_{5} h_{1}\right)\right)$. Writing all such words only with the alphabet $\left\{h_{1}, h_{2}, \ldots, h_{r}\right\}$, we define the length of a word $\mathbf{h}$ to be the number of elements $h_{i_{\alpha}}$ in it. For $\ell \in \mathbb{N}$ with $\ell \geqslant 1$, let $\mathrm{W}_{r}^{\ell}$ be the set of words of length equal to $\ell$ and let $\mathrm{W}_{r}=\bigcup_{\ell \geqslant 1} \mathrm{~W}_{r}^{\ell}$ be the set of all words.

Define $\mathrm{F}_{1}(r)$ to be the vector space generated by $h_{1}, h_{2}, \ldots, h_{r}$ and for $\ell \geqslant 2$, define $\mathrm{F}_{\ell}(r)$ to be the vector space generated by words of length $\leqslant \ell$. This corresponds to $\mathbb{L}^{\ell}$, except that in $\mathbb{L}^{\ell}$, there might exist special linear relations that are absent in the abstract case. Thus, $\mathrm{F}(r)$ is a graded Lie algebra. The Jacobi identity insures (by induction) that $\mathrm{F}_{\ell_{1}}(r) \mathrm{F}_{\ell_{2}}(r) \subset$ $\mathrm{F}_{\ell_{1}+\ell_{2}}(r)$, a property similar to $\left[\mathbb{L}^{k_{1}}, \mathbb{L}^{k_{2}}\right] \subset \mathbb{L}^{k_{1}+k_{2}}$. It follows that $\mathrm{F}_{\ell}(r)$ is generated by words of the form

$$
h_{i_{1}}\left(h_{i_{2}}\left(\ldots\left(h_{i_{\ell^{\prime}-1}} h_{i_{\ell^{\prime}}}\right) \ldots\right)\right),
$$

where $\ell^{\prime} \leqslant \ell$ and where $1 \leqslant i_{1}, i_{2}, \ldots, i_{\ell^{\prime}-1}, i_{\ell^{\prime}} \leqslant r$. For instance, $\left(h_{1} h_{2}\right)\left(h_{3}\left(h_{5} h_{1}\right)\right)$ may be written as a linear combination of such simple words whose length is $\leqslant 5$. Let us denote by

$$
\mathrm{SW}_{r}=\bigcup_{\ell \geqslant 1} \mathrm{SW}_{r}^{\ell}
$$

the set of these simple words, where $\mathrm{SW}_{r}^{\ell}$ denotes the set of simple words of length $\ell$. Although it generates $\mathrm{F}(r)$ as a vector space over $\mathbb{K}$, we point out that it is not a basis of $\mathrm{F}(r)$ : for instance, we have $h_{1}\left(h_{2}\left(h_{1} h_{2}\right)\right)=h_{2}\left(h_{1}\left(h_{1} h_{2}\right)\right)$, because of an obvious Jacobi identity in which $\left(h_{1} h_{2}\right)\left(h_{1} h_{2}\right)=0$ disappears. In fact, one verifies that this is the only Jacobi relation between simple words of length 4, that simple words of length 5 have no Jacobi relation, hence a basis of $F_{5}(2)$ is

$$
\begin{aligned}
& h_{1}, \quad h_{2}, \quad h_{1} h_{2}, \\
& h_{1}\left(h_{1} h_{2}\right), \quad h_{2}\left(h_{1} h_{2}\right), \\
& h_{1}\left(h_{1}\left(h_{1} h_{2}\right)\right), h_{1}\left(h_{2}\left(h_{1} h_{2}\right)\right), h_{2}\left(h_{2}\left(h_{2} h_{1}\right)\right), \\
& h_{1}\left(h_{1}\left(h_{1}\left(h_{1} h_{2}\right)\right)\right), h_{1}\left(h_{1}\left(h_{2}\left(h_{1} h_{2}\right)\right)\right), h_{1}\left(h_{2}\left(h_{2}\left(h_{2} h_{1}\right)\right)\right), \\
& \quad h_{2}\left(h_{1}\left(h_{1}\left(h_{1} h_{2}\right)\right)\right), h_{2}\left(h_{2}\left(h_{1}\left(h_{2} h_{1}\right)\right)\right), h_{2}\left(h_{2}\left(h_{2}\left(h_{2} h_{1}\right)\right)\right) .
\end{aligned}
$$

In general, what are the dimensions of the $\mathrm{F}_{\ell}(r)$ ? How to find bases for them, when considered as vector spaces?

Definition 1.9. A Hall-Witt basis of $\mathrm{F}(r)$ is a linearly ordered (infinite) subset $\mathrm{HW}_{r}=\bigcup_{\ell \geqslant 1} \mathrm{HW}_{r}^{\ell}$ of the set of simple words $\mathrm{SW}_{r}$ such that:

- if two simple words $\mathbf{h}$ and $\mathbf{h}^{\prime}$ satisfy length $(\mathbf{h})<$ length $\left(\mathbf{h}^{\prime}\right)$, then $\mathbf{h}<\mathbf{h}^{\prime}$;
- $\mathrm{HW}_{r}^{1}=\left\{h_{1}, h_{2}, \ldots, h_{r}\right\}$;
- $\mathrm{HW}_{r}^{2}=\left\{h_{i_{1}} h_{i_{2}}: 1 \leqslant i_{1}<i_{2} \leqslant r\right\}$;
- $\mathrm{HW}_{r} \backslash\left(\mathrm{HW}_{r}^{1} \cup \mathrm{HW}_{r}^{2}\right)=\left\{\mathbf{h}\left(\mathbf{h}^{\prime} \mathbf{h}^{\prime \prime}\right): \mathbf{h}, \mathbf{h}^{\prime}, \mathbf{h}^{\prime \prime} \in \mathrm{HW}_{r}, \mathbf{h}^{\prime}<\right.$ $\mathbf{h}^{\prime \prime}$ and $\left.\mathbf{h}^{\prime} \leqslant \mathbf{h}<\mathbf{h}^{\prime} \mathbf{h}^{\prime \prime}\right\}$.

A Hall-Witt basis essentially consists of the choice, for every $\ell \geqslant 1$, of some (among many possible) finite subset $\mathrm{HW}_{r}^{\ell}$ of $\mathrm{SW}_{r}^{\ell}$ that generates the finite-dimensional quotient vector space $\mathrm{F}_{\ell}(r) / \mathrm{F}_{\ell-1}(r)$. To fix ideas, an arbitrary linear ordering is added among the elements of the chosen basis $\mathrm{HW}_{r}^{\ell}$ of the vector space $\mathrm{F}_{\ell}(r) / \mathrm{F}_{\ell-1}(r)$. The last condition of the definition takes account of the Jacobi identity.
Theorem 1.10. ([Bo1972, Re1993]) Hall-Witt bases exist and are bases of the free Lie algebra $\mathrm{F}(r)$ of rank $r$, when considered as a vector space. The dimensions $\mathrm{n}_{\ell}(r)-\mathrm{n}_{\ell-1}(r)$ of $\mathrm{F}_{\ell}(r) / \mathrm{F}_{\ell-1}(r)$, or equivalently the cardinals of $\mathrm{HW}_{r}^{\ell}$, satisfy the induction relation

$$
\mathrm{n}_{\ell}(r)-\mathrm{n}_{\ell-1}(r)=\frac{1}{\ell} \sum_{d \text { divides } \ell} \mu(d) r^{\ell / d}
$$

where $\mu$ is the Möbius function.
Remind that
$\mu(d)=\left\{\begin{array}{l}1, \text { if } d=1 ; \\ 0, \text { if } d \text { contains square integer factors; } \\ (-1)^{\nu}, \text { if } d=p_{1} \cdots p_{\nu} \text { is the product of } \nu \text { distinct prime numbers. }\end{array}\right.$
Now, we come back to the system $\mathbb{L}^{0}=\left\{L_{a}\right\}_{1 \leqslant a \leqslant r}$ of local $\mathbb{K}$-analytic vector fields of $\S 1.1$, where $L_{a}=\sum_{i=1}^{n} \varphi_{a, i}(\mathrm{x}) \frac{\partial}{\partial \mathrm{x}_{i}}$. If the vector space $\mathbb{L}(0)$ has dimension $<r$, a slight perturbation of the coefficients $\varphi_{a, i}(\mathrm{x})$ of the $L_{a}$ yields a system $\mathbb{L}^{\prime 0}$ with $\mathbb{L}^{\prime}(0)$ of dimension $=r$. By an elementary computation with Lie brackets, one sees that a further slight perturbation yields a system $\mathbb{L}^{\prime \prime 0}$ with $\mathbb{L}^{\prime \prime}(0)$ of dimension $r+\frac{r(r-1)}{2}=\mathrm{n}_{2}(r)$.

To pursue, any simple iterated Lie bracket $\left[L_{a_{1}},\left[L_{a_{2}}, \ldots\left[L_{a_{\ell-1}}, L_{a_{\ell}}\right] \ldots\right]\right]$ of length $\ell$ is a vector field $\sum_{i=1}^{n} A_{a_{1}, a_{2}, \ldots, a_{\ell-1}, a_{\ell}}^{i} \frac{\partial}{\partial \mathrm{x}_{i}}$ having coefficients $A_{a_{1}, a_{2}, \ldots, a_{\ell-1}, a_{\ell}}^{i}$ that are universal polynomials in the jets

$$
J_{\mathrm{x}}^{\ell-1} \varphi(\mathrm{x}):=\left(\partial_{\mathrm{x}}^{\alpha} \varphi_{a, i}(\mathrm{x})\right)_{1 \leqslant a \leqslant r, 1 \leqslant i \leqslant n}^{\alpha \in \mathbb{N}^{n},|\alpha| \leqslant \ell-1} \in \mathbb{K}^{N_{r n, n, \ell-1}}
$$

of $\operatorname{order}(\ell-1)$ of the coefficients of $L_{1}, L_{2}, \ldots, L_{r}$. Here, $N_{r n, n, \ell-1}=$ $r n \frac{(n+\ell-1)!}{n!(\ell-1)!}$ denotes the number of such independent partial derivatives. A careful inspection of the polynomials $A_{a_{1}, a_{2}, \ldots, a_{\ell-1}, a_{\ell}}^{i}$ enables to get the following genericity statement, whose proof will appear elsewhere. It says in a
precise way that $\mathbb{L}^{\text {lie }}(0)=T_{0} \mathbb{K}^{n}$ with the maximal freedom, for generic sets of vector fields.

Theorem 1.11. ([GV1987, Ge1988], [*]) If $\ell_{0}$ denotes the smallest length $\ell$ such that $\mathrm{n}_{\ell}(r) \geqslant n$, there exists a proper $\mathbb{K}$-algebraic subset $\Sigma$ of the jet space $J_{0}^{\ell_{0}-1} \varphi=\mathbb{K}^{N_{r n, n, \ell_{0}-1}}$ such that for every collection $\mathbb{L}^{0}=\left\{L_{a}\right\}_{1 \leqslant a \leqslant r}$ of $r$ vector fields $L_{a}=\sum_{i=1}^{n} \varphi_{a, i}(\mathrm{x}) \frac{\partial}{\partial \mathrm{x}_{i}}$ such that $J_{0}^{\ell_{0}-1} \varphi(0)$ does not belong to $\Sigma$, the following two properties hold:

- $\operatorname{dim} \mathbb{L}^{\ell}(0)=\mathrm{n}_{\ell}(r)$, for every $\ell \leqslant \ell_{0}-1$,
- $\operatorname{dim} \mathbb{L}^{\ell_{0}}(0)=n$, hence $\mathbb{L}^{\text {lie }}(0)=T_{0} \mathbb{K}^{n}$.

The number of divisors of $\ell$ being an $\mathrm{O}\left(\frac{\log \ell}{\log 2}\right)$, one verifies that $\mathrm{n}_{\ell}(r)-$ $\mathrm{n}_{\ell-1}(r)=\frac{1}{\ell} r^{\ell}+\mathrm{O}\left(r^{\ell / 2} \frac{\log \ell}{\log 2}\right)$. It follows that, for $r$ fixed, the integer $\ell_{0}$ of the theorem is equivalent to $\frac{\log n}{\log r}$ as $n \rightarrow \infty$.
1.12. Local orbits of $\mathbb{K}$-analytic and of (Nash) $\mathbb{K}$-algebraic systems. We now describe a second, more concrete, simple and useful approach to the local Nagano Theorem 1.5. It is inspired by Sussmann's Theorem 1.21(III) and does not involve the consideration of any Lie bracket. Theorem 1.13 below will be applied in §2.11.

As above, consider a finite set

$$
\mathbb{L}^{0}:=\left\{L_{a}\right\}_{1 \leqslant a \leqslant r}, \quad r \in \mathbb{N}, \quad r \geqslant 1,
$$

of nonzero vector fields defined in the cube $\square_{\rho_{1}}^{n}$ and having $\mathbb{K}$-analytic coefficients. We shall neither consider its $\mathbb{A}_{\rho_{1}}$-linear hull $\mathbb{L}$, nor $\mathbb{L}^{\text {lie }}$. We will reconstruct the Nagano leaf passing through the origin only by means of the flows of $L_{1}, L_{2}, \ldots, L_{r}$.

Referring the reader to §1.3(III) for background, we denote the flow map of a vector field $L \in \mathbb{L}^{0}$ shortly by $(\mathrm{t}, \mathrm{x}) \mapsto L_{\mathrm{t}}(\mathrm{x})=\exp (\mathrm{t} L)(\mathrm{x})$. It is $\mathbb{K}$-analytic. What happens in the algebraic category?

So, assume that the coefficients of all vector fields $L \in \mathbb{L}^{0}$ are $\mathbb{K}$ algebraic. Unfortunately, algebraicity fails to be preserved under integration, so the flows are only $\mathbb{K}$-analytic, in general. To get algebraicity of Nagano leaves, there is nothing else than supposing that the flows are algebraic, which we will do (second phrase of (5) below).

Choose now $\rho_{2}$ with $0<\rho_{2}<\rho_{1}$. Let $k \in \mathbb{N}$ with $k \geqslant 1$, let $L=$ $\left(L^{1}, \ldots, L^{k}\right) \in\left(\mathbb{L}^{0}\right)^{k}$, let $\mathrm{t}=\left(\mathrm{t}_{1}, \ldots, \mathrm{t}_{k}\right) \in \mathbb{K}^{k}$ with $|\mathrm{t}|<\rho_{2}$, i.e. $\mathrm{t} \in \square_{\rho_{2}}^{k}$, and let $\mathrm{x} \in \square_{\rho_{2}}^{n}$. We shall adopt the contracted notation

$$
L_{\mathrm{t}}(\mathrm{x}):=L_{\mathrm{t}_{k}}^{k}\left(\cdots\left(L_{\mathrm{t}_{1}}^{1}(\mathrm{x})\right) \cdots\right)
$$

for the composition of flow maps, whenever it is defined. In fact, since $L_{0}(0)=\exp (0 L)(0)=0$, it is clear that if we bound the length $k \leqslant 2 n$,
then there exists $\rho_{2}>0$ sufficiently small such that all maps $(\mathrm{t}, \mathrm{x}) \mapsto L_{\mathrm{t}}(\mathrm{x})$ are well-defined, with $L_{\mathrm{t}}(\mathrm{x}) \in \square_{\rho_{1}}^{n}$, at least for all $\mathrm{t} \in \square_{\rho_{2}}^{k}$ and all $\mathrm{x} \in \square_{\rho_{2}}^{n}$. The reason why we may restrict to consider only compositions of length $k \leqslant 2 n$ will appear a posteriori in the proof of the theorem below. We shall be concerned with rank properties of $(\mathrm{t}, \mathrm{x}) \mapsto L_{\mathrm{t}}(\mathrm{x})$.

Let $n \geqslant 1, m \geqslant 1, \rho_{1}>0, \sigma_{1}>0$ and let $f: \square_{\rho_{1}}^{n} \rightarrow \square_{\sigma_{1}}^{m}, \mathrm{x} \mapsto f(\mathrm{x})$, be a $\mathbb{K}$-algebraic or $\mathbb{K}$-analytic map between two open cubes. Denote its Jacobian matrix by $\operatorname{Jac}(f)=\left(\frac{\partial f_{j}}{\partial \mathrm{x}_{i}}(\mathrm{x})\right)_{1 \leqslant i \leqslant n}^{1 \leqslant j \leqslant m}$. At a point $\mathrm{x} \in \square_{\rho_{1}}^{n}$, the map $f$ has rank $r$ if and only if $\operatorname{Jac} f$ has rank $r$ at x . Equivalently, by linear algebra, there is a $r \times r$ minor that does not vanish at $\times$ but all $s \times s$ minors with $r+1 \leqslant s \leqslant n$ do vanish at $\mathbf{x}$.

For every $s \in \mathbb{N}$ with $1 \leqslant s \leqslant \min (n, m)$, compute all the possible $s \times s$ minors $\Delta_{1}^{s \times s}, \ldots, \Delta_{N(s)}^{s \times s}$ of $\operatorname{Jac}(f)$. They are universal (homogeneous of degree $s$ ) polynomials in the partial derivatives of $f$, hence are all $\mathbb{K}$ algebraic or $\mathbb{K}$-analytic functions. Let $e$ with $0 \leqslant e \leqslant \min (n, m)$ be the maximal integer $s$ with the property that there exists a minor $\Delta_{\mu}^{s \times s}(\mathrm{x}), 1 \leqslant$ $\mu \leqslant N(s)$, not vanishing identically. Then the set

$$
\mathscr{R}_{f}:=\left\{\mathrm{x} \in \square_{\rho_{1}}^{n}: \Delta_{\mu}^{s \times s}(\mathrm{x})=0, \mu=1, \ldots, N(s)\right\}
$$

is a proper $\mathbb{K}$-algebraic or analytic subset of $\square_{\rho_{1}}^{n}$. The principle of analytic continuation insures that $\square_{\rho_{1}}^{n} \backslash \mathscr{R}_{f}$ is open and dense.

The integer $e$ is called the generic rank of $f$. For every open, connected and nonempty subset $\Omega \subset \square_{\rho_{1}}^{n}$ the restriction $\left.f\right|_{\Omega}$ has the same generic rank $e$.

Theorem 1.13. ([Me1999, Me2001a, Me2004a]) There exists an integer e with $1 \leqslant e \leqslant n$ and an e-tuple of vector fields $L^{*}=\left(L^{* 1}, \ldots, L^{* e}\right) \in\left(\mathbb{L}^{0}\right)^{e}$ such that the following six properties hold true.
(1) For every $k=1, \ldots, e$, the $\operatorname{map}\left(\mathrm{t}_{1}, \ldots, \mathrm{t}_{k}\right) \mapsto$ $L_{\mathrm{t}_{k}}^{* k}\left(\cdots\left(L_{\mathrm{t}_{1}}^{* 1}(0)\right) \cdots\right)$ is of generic rank equal to $k$.
(2) For every arbitrary element $L^{\prime} \in \mathbb{L}^{0}$, the map $\left(\mathrm{t}_{1}, \ldots, \mathrm{t}_{e}, \mathrm{t}^{\prime}\right) \mapsto$ $L_{\mathbf{t}^{\prime}}^{\prime}\left(L_{\mathbf{t}_{e}}^{* e}\left(\cdots\left(L_{\mathrm{t}_{1}}^{* 1}(0)\right) \cdots\right)\right)$ is of generic rank e, hence e is the maximal possible generic rank.
(3) There exists an element $\mathrm{t}^{*} \in \square_{\rho_{2}}^{e}$ arbitrarily close to the origin which is of the special form $\left(\mathrm{t}_{1}^{*}, \ldots, \mathrm{t}_{e-1}^{*}, 0\right)$, namely with $\mathrm{t}_{e}^{*}=0$, and there exists an open connected neighborhood $\omega^{*}$ of $\mathrm{t}_{*}$ in $\square_{\rho_{2}}^{e}$ such that the map $\mathrm{t} \mapsto L_{\mathrm{t}_{e}}^{* e}\left(\cdots\left(L_{\mathrm{t}_{1}}^{* 1}(0)\right) \cdots\right)$ is of constant rank e in $\omega^{*}$.
(4) Setting $L^{*}:=\left(L^{* 1}, \ldots, L^{* e}\right), K^{*}:=\left(L^{* e-1}, \ldots, L^{* 1}\right)$ and $s^{*}:=$ $\left(-\mathrm{t}_{e-1}^{*}, \ldots,-\mathrm{t}_{1}^{*}\right)$, we have $K_{s^{*}}^{*} \circ L_{\mathrm{t}^{*}}^{*}(0)=0$ and the map $\psi: \omega^{*} \rightarrow$ $\square_{\rho_{1}}^{n}$ defined by $\psi: \mathrm{t} \mapsto K_{s^{*}}^{*} \circ L_{\mathrm{t}}^{*}(0)$ is also of constant rank equal to $e$ in $\omega^{*}$.
(5) The image $\Lambda:=\psi\left(\omega^{*}\right)$ is a piece of $\mathbb{K}$-analytic submanifold passing through the origin enjoying the most important property that every vector field $L^{\prime} \in \mathbb{L}^{0}$ is tangent to $\Lambda$. If the flows of all elements of $\mathbb{L}^{0}$ are algebraic, $\Lambda$ is $\mathbb{K}$-algebraic.
(6) Every local $\mathbb{K}$-algebraic or $\mathbb{K}$-analytic submanifold $\Lambda^{\prime}$ passing trough the origin to which all vector fields $L^{\prime} \in \mathbb{L}^{0}$ are tangent must contain $\Lambda$ in a neighborhood of 0 .
In conclusion, the dimension e of $\Lambda$ is characterized by the generic rank properties (1) and (2).

Previously, $\Lambda$ was called Nagano leaf. Since the above statement is superseded by Sussmann's Theorem 1.21 (III), we prefer to call it the local $\mathbb{L}$-orbit of 0 , introducing in advance the terminology of Part III and denoting it by $\mathscr{O}_{\mathbb{L}^{0}}^{\text {loc }}\left(\square_{\rho_{1}}^{n}, 0\right)$. The integer $e$ of the theorem is $\leqslant n$, just because the target of the maps $\left(\mathrm{t}_{1}, \ldots, \mathrm{t}_{k}\right) \mapsto L_{\mathrm{t}_{k}}^{* k}\left(\cdots\left(L_{\mathrm{t}_{1}}^{* 1}(0)\right) \cdots\right)$ is $\mathbb{K}^{n}$. It follows that in (4) and (5) we need $2 e-1 \leqslant 2 n-1$ compositions of flows to cover $\Lambda$.

We quickly mention an application about separate algebraicity. In [BM1949], it is shown that a local $\mathbb{K}$-analytic function $g: \square_{\rho_{1}}^{n} \rightarrow \mathbb{K}$ is $\mathbb{K}$-algebraic if and only if its restriction to every affine coordinate segment is $\mathbb{K}$-algebraic. Call the system $\mathbb{L}^{0}$ minimal at the origin if $\mathscr{O}_{\mathbb{L}^{0}}^{\text {loc }}\left(\square_{\rho_{1}}^{n}, 0\right)$ contains a neighborhood of the origin. Equivalently, the integer $e$ of Theorem 1.13 equals $n$.
Theorem 1.14. ([Me2001a]) If $\mathbb{L}^{0}=\left\{L_{a}\right\}_{1 \leqslant a \leqslant r}$ is minimal at 0 , a local $\mathbb{K}$-analytic function $g: \square_{\rho_{1}}^{n} \rightarrow \mathbb{K}$ is $\mathbb{K}$-algebraic if and only it its restriction to every integral curve of every $L_{a} \in \mathbb{L}^{0}$ is $\mathbb{K}$-algebraic.
Proof of Theorem 1.13. (May be skipped in a first reading.) If all vector fields of $\mathbb{L}^{0}$ vanish at the origin, $\Lambda=\{0\}$. We now exclude this possibility. Choose a vector field $L^{* 1} \in \mathbb{L}^{0}$ which does not vanish at 0 . The map $t_{1} \mapsto$ $L_{\mathrm{t}_{1}}^{* 1}(0)$ is of (generic) rank one at every $\mathrm{t}_{1} \in \square_{\rho_{2}}^{1}$. If there exists $L^{\prime} \in \mathbb{L}^{0}$ such that the map $\left(\mathrm{t}_{1}, \mathrm{t}^{\prime}\right) \mapsto L_{\mathrm{t}^{\prime}}^{\prime}\left(L_{\mathrm{t}_{1}}^{* 1}(0)\right)$ is of generic rank two, we choose one such $L^{\prime}$ and we denote it by $L^{* 2}$. Continuing in this way, we get vector fields $L^{* 1}, \ldots, L^{* e}$ satisfying properties (1) and (2), with $e \leqslant n$.

Since the generic rank of the map $\left(\mathrm{t}_{1}, \ldots, \mathrm{t}_{e}\right) \mapsto L_{\mathrm{t}_{e}}^{* e}\left(\cdots\left(L_{\mathrm{t}_{1}}^{* 1}(0)\right) \cdots\right)$ equals $e$, and since this map is $\mathbb{K}$-analytic, there exists a $\mathrm{t}^{*} \in \square_{\rho_{2}}^{e}$ arbitrarily close to the origin at which its rank equals $e$. We claim that we can moreover choose $\mathrm{t}^{*}$ to be of the special form $\left(\mathrm{t}_{1}^{*}, \ldots, \mathrm{t}_{e-1}^{*}, 0\right)$, i.e. with $t_{e}^{*}=0$. It suffices to apply the following lemma to $\varphi(\mathrm{t}):=L_{\mathrm{t}_{e-1}}^{* e-1}\left(\cdots\left(L_{\mathrm{t}_{1}}^{* 1}(0)\right) \cdots\right)$ and to $L^{\prime}:=L^{* e}$.

Lemma 1.15. Let $n \in \mathbb{N}, n \geqslant 1$, let $e \in \mathbb{N}, 1 \leqslant e \leqslant n$, let $t \in \square_{\rho_{2}}^{e-1}$ and let

$$
\square_{\rho_{2}}^{e-1} \ni \mathrm{t} \mapsto \varphi(\mathrm{t})=\left(\varphi_{1}(\mathrm{t}), \ldots, \varphi_{n}(\mathrm{t})\right) \in \square_{\rho_{1}}^{n}
$$

be a $\mathbb{K}$-analytic map whose generic rank equals $(e-1)$. Let $L^{\prime}$ be a $\mathbb{K}$ analytic vector field and assume that the map $\psi:\left(\mathrm{t}, \mathrm{t}^{\prime}\right) \mapsto L_{\mathrm{t}^{\prime}}^{\prime}(\varphi(\mathrm{t}))$ has generic rank e. Then there exists a point $\left(\mathrm{t}^{*}, 0\right)$ arbitrarily close to the origin at which the rank of $\psi$ is equal to $e$.

Proof. Choose $t^{\sharp} \in \square_{\rho_{2}}^{e-1}$ arbitrarily close to zero at which $\varphi$ has maximal rank, equal to $(e-1)$. Since the rank is lower semi-continuous, there exists a connected neighborhood $\omega^{\sharp}$ of $\mathrm{t}^{\sharp}$ in $\square_{\rho_{2}}^{e-1}$ such that $\varphi$ has rank $(e-1)$ at every point of $\omega^{\sharp}$. By the constant rank theorem, $\Pi:=\varphi\left(\omega^{\sharp}\right)$ is then a local $\mathbb{K}$-analytic submanifold of $\square_{\rho_{1}}^{n}$ passing through the point $\varphi\left(\mathrm{t}^{\sharp}\right)$. To complete the proof, we claim that there exists $t^{*} \in \omega^{\sharp}$ arbitrarily close to $t^{\sharp}$ such that the map $\left(\mathrm{t}, \mathrm{t}^{\prime}\right) \mapsto L_{\mathrm{t}^{\prime}}^{\prime}(\varphi(\mathrm{t}))$ has rank $e$ at $\left(\mathrm{t}^{*}, 0\right)$.

Let us reason by contradiction, supposing that at all points of the form $\left(\mathrm{t}^{*}, 0\right)$, for $\mathrm{t}^{*} \in \omega^{\sharp}$, the map $\psi:\left(\mathrm{t}, \mathrm{t}^{\prime}\right) \mapsto L_{\mathrm{t}^{\prime}}^{\prime}(\varphi(\mathrm{t}))$ has rank equal to $(e-1)$. Pick an arbitrary $\mathrm{t}^{*} \in \omega^{\sharp}$. Reminding that when $\mathrm{t}^{\prime}=0$, we have $L_{\mathrm{t}^{\prime}}^{\prime}=L_{0}^{\prime}=$ Id, we observe that $\psi(\mathrm{t}, 0) \equiv \varphi(\mathrm{t})$. Consequently, the partial derivatives of $\psi$ with respect to the variables $\mathrm{t}_{i}, i=1, \ldots, e-1$ at an arbitrary point ( $\mathrm{t}^{*}, 0$ ), with $\mathrm{t}^{*} \in \omega^{\sharp}$, coincide with the $(e-1)$ linearly independent vectors $\frac{\partial \varphi}{\partial t_{i}}\left(\mathrm{t}^{*}\right) \in \mathbb{K}^{n}, i=1, \ldots, e-1$. In fact, the tangent space to $\Pi$ at the point $\psi\left(\mathrm{t}^{*}, 0\right)=\varphi\left(\mathrm{t}^{*}\right)$ is generated by these $(e-1)$ vectors.

Reminding the fundamental property $\left.\frac{\partial}{\partial \mathrm{t}^{\prime}} L_{\mathrm{t}^{\prime}}^{\prime}(\mathrm{x})\right|_{\mathrm{t}^{\prime}=0}=L^{\prime}(\mathrm{x})$, we deduce [from our assumption that the map $\left(\mathrm{t}, \mathrm{t}^{\prime}\right) \mapsto L_{\mathrm{t}^{\prime}}^{\prime}(\varphi(\mathrm{t}))$ has rank equal to $(e-1)]$ that the vector

$$
\left.\frac{\partial}{\partial \mathrm{t}^{\prime}} L_{\mathrm{t}^{\prime}}^{\prime}(\varphi(\mathrm{t}))\right|_{\mathrm{t}^{\prime}=0}=L^{\prime}(\varphi(\mathrm{t}))
$$

must be linearly dependent with the $(e-1)$ vectors $\frac{\partial \varphi}{\partial t_{i}}(\mathrm{t}), i=1, \ldots, e-1$, for every $\mathrm{t} \in \omega^{\sharp}$. Equivalently, the vector field $L^{\prime}$ is tangent to the submanifold $\Pi$. It follows that the local flow of $L^{\prime}$ necessarily stabilizes $\Pi$ : if $\mathrm{x}=\varphi(\mathrm{t}) \in \Pi, \mathrm{t} \in \omega^{\sharp}$, then $L_{\mathrm{t}^{\prime}}^{\prime}(\mathrm{x}) \in \Pi$, for all $\mathrm{t}^{\prime} \in \square_{\rho(\mathrm{t})}^{1}$, where $\rho(\mathrm{t})>0$ is sufficiently small. Set $\Omega^{\sharp}:=\left\{\left(\mathrm{t}, \mathrm{t}^{\prime}\right): \mathrm{t} \in \omega^{\sharp}, \mathrm{t}^{\prime} \in \square_{\rho(\mathrm{t})^{1}}\right\}$. It is a nonempty connected open subset of $\square_{\rho_{2}}^{e}$. We have thus deduced that $\psi\left(\Omega^{\sharp}\right)$ is contained in the $(e-1)$-dimensional submanifold $\Pi$. This constraint entails that $\psi$ is of rank $\leqslant e-1$ at every point of $\Omega^{\sharp}$. However, $\left.\psi\right|_{\Omega^{\sharp}}$ being $\mathbb{K}$-analytic and of generic rank equal to $e$, by assumption, it should be of rank $e$ at every point of an open dense subset of $\Omega^{\sharp}$. This is the desired contradiction which proves the lemma.

Hence, there exists $\mathrm{t}^{*}=\left(\mathrm{t}_{1}^{*}, \ldots, \mathrm{t}_{e-1}^{*}, 0\right) \in \square_{\rho_{2}}^{e}$ arbitrarily close to the origin at which the rank of $\mathrm{t} \mapsto L_{\mathrm{t}_{e}}^{* e}\left(\cdots\left(L_{\mathrm{t}_{1}}^{* 1}(0)\right) \cdots\right)$ is maximal (hence locally constant) equal to $e$, so we get the constant rank property (3), for a sufficiently small neighborhood $\omega^{*}$ of $t^{*}$.

In (4), the property $K_{s^{*}}^{*} \circ L_{t^{*}}^{*}(0)=0$ is obvious, using $\mathrm{x} \equiv L_{0}(\mathrm{x}) \equiv$ $L_{-\mathrm{t}} \circ L_{\mathrm{t}}(\mathrm{x}):$

$$
L_{-\mathrm{t}_{1}^{*}}^{* 1} \circ \cdots \circ L_{-\mathrm{t}_{e-1}^{*}}^{*} \circ L_{0}^{* e} \circ L_{\mathrm{t}_{e-1}^{*}}^{*} \circ \cdots \circ L_{\mathrm{t}_{1}^{*}}^{*}(\mathrm{x}) \equiv \mathrm{x} .
$$

Since the map $\mathrm{x} \mapsto K_{s^{*}}^{*}(\mathrm{x})$ is a local diffeomorphism, it is clear that the map $\psi: \mathrm{t} \mapsto K_{\mathrm{s}^{*}}^{*} \circ L_{\mathrm{t}}^{*}(0)$ is also of constant rank $e$ in $\omega^{*}$. Thus, we obtain (4), and moreover, by the constant rank theorem, the image $\Lambda:=\psi\left(\omega^{*}\right)$ constitutes a local $\mathbb{K}$-analytic submanifold of $\mathbb{K}^{n}$ passing through the origin. If the flows of elements of $\mathbb{L}^{0}$ are all $\mathbb{K}$-algebraic, clearly $\psi$ and $\Lambda$ are also $\mathbb{K}$-algebraic.

It remains to check that every vector field $L^{\prime} \in \mathbb{L}^{0}$ is tangent to $\Lambda$. As a preliminary, denote by $L_{\mathrm{t}^{\prime}}^{\prime}(\varphi(\mathrm{t})), \mathrm{t} \in \square_{\rho_{2}}^{e}, \mathrm{t}^{\prime} \in \square_{\rho_{2}}^{1}$, the map appearing in (2), where $L^{\prime} \in \mathbb{L}^{0}$ is arbitrary. Reasoning as in the lemma above, we see that $L^{\prime}$ is necessarily tangent to some local submanifold $\Pi$ obtained as the local image of an open connected set where $\varphi$ has maximal locally constant rank. It follows that the flows and the multiple flows of elements of $\mathbb{L}^{0}$ stabilize this submanifold. We deduce a generalization of (2): for $k \leqslant 2 n$, for $L^{\prime} \in\left(\mathbb{L}^{0}\right)^{k}$, for $\mathrm{t}^{\prime} \in \square_{\rho_{2}}^{k}$, the map $\left(\mathrm{t}, \mathrm{t}^{\prime}\right) \longmapsto L_{\mathrm{t}^{\prime}}^{\prime}\left(L_{\mathrm{t}_{e}}^{* e}\left(\cdots\left(L_{\mathrm{t}_{1}}^{* 1}(0)\right) \cdots\right)\right)$ is of generic rank $e$.

In particular, for every $L^{\prime} \in \mathbb{L}^{0}$, the map $\left(\mathrm{t}^{\prime}, \mathrm{s}, \mathrm{t}\right) \longmapsto L_{\mathrm{t}^{\prime}}^{\prime} \circ K_{\mathrm{s}}^{*} \circ L_{\mathrm{t}}^{*}(0)$ is of generic rank $e$. In fact, the restriction $\psi: \mathrm{t} \mapsto K_{\mathrm{s}^{*}}^{*} \circ L_{\mathrm{t}}^{*}(0)$ of this map to the open set $\left\{\left(0, s^{*}, \mathrm{t}\right): \mathrm{t} \in \omega^{*}\right\}$ is already of rank $e$ at every point and its image is the local submanifold $\Lambda$, by the above construction. So the map $\left(\mathrm{t}^{\prime}, \mathrm{t}\right) \longmapsto L_{\mathrm{t}^{\prime}}^{\prime} \circ K_{\mathrm{s}^{*}}^{*} \circ L_{\mathrm{t}}^{*}(0)$ must be of rank $e$ at every point. In particular, the vector

$$
\left.\frac{\partial}{\partial \mathrm{t}^{\prime}} L_{\mathrm{t}^{\prime}}^{\prime} \circ K_{\mathrm{s}^{*}}^{*} \circ L_{\mathrm{t}}^{*}(0)\right|_{\mathrm{t}^{\prime}=0}=L^{\prime}\left(K_{\mathrm{s}^{*}}^{*} \circ L_{\mathrm{t}}^{*}(0)\right)
$$

must necessarily be tangent to $\Lambda$ at the point $K_{\mathrm{s}^{*}}^{*} \circ L_{\mathrm{t}}^{*}(0) \in \Lambda$. Thus, (5) is proved.

Take $\Lambda^{\prime}$ as in (6). The local flows of all vector $L^{\prime} \in \mathbb{L}^{0}$ stabilize $\Lambda^{\prime}$. Shrinking $\rho_{2}$ if necessary, all the maps $(\mathrm{t}, \mathrm{x}) \longmapsto L_{\mathrm{t}}(\mathrm{x})$ have range in $\Lambda^{\prime}$. So $\Lambda \subset \Lambda^{\prime}$, proving (6).

## §2. Analytic CR manifolds, Segre chains and minimality

2.1. Local Cauchy-Riemann submanifolds of $\mathbb{C}^{n}$. Let $\left(z_{1}, \ldots, z_{n}\right)=$ $\left(x_{1}+i y_{1}, \ldots, x_{n}+i y_{n}\right)$ denote the canonical coordinates on $\mathbb{C}^{n}$. As before, we use the maximum norms $|x|=\max _{1 \leqslant k \leqslant n}\left|x_{k}\right|,|y|=\max _{1 \leqslant k \leqslant n}\left|y_{k}\right|$ and $|z|=\max _{1 \leqslant k \leqslant n}\left|z_{k}\right|$, where $\left|z_{k}\right|=\left(x_{k}^{2}+y_{k}^{2}\right)^{1 / 2}$. If $\rho>0$, we denote by $\Delta_{\rho}^{n}=\left\{z \in \mathbb{C}^{n}:|z|<\rho\right\}$ the open polydisc of radius $\rho$ centered at the origin, not to be confused with $\square_{\rho}^{2 n}=\left\{x+i y \in \mathbb{C}^{n}:|x|,|y|<\rho\right\}$.

Let $J$ denote the complex structure of $T \mathbb{C}^{n}$, acting on real vectors as if it were multiplication by $\sqrt{-1}$. Precisely, if $p$ is any point, $T_{p} \mathbb{C}^{n}$ is spanned by
the $2 n$ vectors $\left.\frac{\partial}{\partial x_{k}}\right|_{p},\left.\frac{\partial}{\partial y_{k}}\right|_{p}, k=1, \ldots, n$, and $J$ acts as follows: $\left.J \frac{\partial}{\partial x_{k}}\right|_{p}=$ $\left.\frac{\partial}{\partial y_{k}}\right|_{p} ;\left.J \frac{\partial}{\partial y_{k}}\right|_{p}=-\left.\frac{\partial}{\partial x_{k}}\right|_{p}$.
Choose the origin as a center point and consider a real $d$-codimensional local submanifold $M$ of $\mathbb{C}^{n} \simeq \mathbb{R}^{2 n}$ passing through the origin, defined by $d$ Cartesian equations $r_{1}(x, y)=\cdots=r_{d}(x, y)=0$, where the differentials $d r_{1}, \ldots, d r_{d}$ are linearly independent at the origin. The functions $r_{j}$ are assumed to be of class ${ }^{1} \mathscr{C}^{\mathscr{R}}$, where $\mathscr{R}=(\kappa, \alpha), \kappa \geqslant 1,0 \leqslant \alpha \leqslant 1, \mathscr{R}=\infty$, $\mathscr{R}=\omega$ or $\mathscr{R}=\mathscr{A} l g$. Accordingly, $M$ is said to be of class $\mathscr{C}^{\mathscr{A} l g}$ (real algebraic), $\mathscr{C}^{\omega}$ (real analytic), $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$.

For $p \in M$, the smallest $J$-invariant subspace of the tangent space $T_{p} M$ is given by $T_{p}^{c} M:=T_{p} M \cap J T_{p} M$ and is called the complex tangent space to $M$ at $p$.

Definition 2.2. The submanifold $M$ is called:

- holomorphic if $T_{p}^{c} M=T_{p} M$ at every point $p \in M$;
- totally real if $T_{p}^{c} M=\{0\}$ at every point $p \in M$;
- generic if $T_{p} M+J T_{p} M=T_{p} \mathbb{C}^{n}$ at every point $p \in M$;
- Cauchy-Riemann (CR for short) if the dimension of $T_{p}^{c} M$ is equal to a fixed constant at every point $p \in M$.

For fundamentals about Cauchy-Riemann (CR for short) structures, we refer the reader to [Ch1989, Ja1990, Ch1991, Bo1991, BER1999, Me2004a]. Here, we only summarize some elementary useful properties. The two $J$-invariant spaces $T_{p} M \cap J T_{p} M$ and $T_{p} M+J T_{p} M$ are of even real dimension. We denote by $m_{p}$ the integer $\frac{1}{2} \operatorname{dim}_{\mathbb{R}}\left(T_{p} M \cap J T_{p} M\right)$ and call it the $C R$ dimension of $M$ at $p$. If $M$ is $\mathrm{CR}, m_{p} \equiv m$ is constant. Holomorphic, totally real and generic submanifolds are CR, with $m=n-\frac{1}{2} d$, $m=0$ and $m=n-d$ respectively. If $M$ is totally real and generic, $\operatorname{dim}_{\mathbb{R}} M=n$ and $M$ is called maximally real. We denote by $c_{p}$ the integer $n-\frac{1}{2} \operatorname{dim}_{\mathbb{R}}\left(T_{p} M+J T_{p} M\right)$ and call it the holomorphic codimension of $M$ at $p$. It is constant if and only if $M$ is CR. Holomorphic, totally real, generic and Cauchy-Riemann submanifolds are all CR and have constant holomorphic codimensions $c=\frac{1}{2} d, c=d-n, c=0$ and $c=d-n+m$ respectively. Submanifolds of class $\mathscr{C}^{\kappa, \alpha}$ or $\mathscr{C}^{\infty}$ will be studied in Part III.

Let $M$ or be a real algebraic ( $\mathscr{C}^{\mathscr{A l g}}$ ) or analytic ( $\mathscr{C}^{\omega}$ ) submanifold of $\mathbb{C}^{n}$ of (real) codimension $d$ and let $p_{0} \in M$. There exist complex algebraic or analytic coordinates centered at $p_{0}$ and $\rho_{1}>0$ such that $M$ is locally represented as follows.
Theorem 2.3. ([Ch1989, Bo1991, BER1999, Me2004a])

[^0]- If $M$ is holomorphic, letting $m=n-\frac{1}{2} d \geqslant 0$ and $c:=\frac{1}{2} d$, then $m+c=n$ and $M=\left\{\left(z, w_{1}\right) \in \Delta_{\rho_{1}}^{m} \times \Delta_{\rho_{1}}^{c}: w_{1}=0\right\}$.
- If $M$ is totally real, letting $d_{1}=2 n-d \geqslant 0$ and $c=d-n \geqslant 0$, then $d_{1}+c=n$ and $M=$ $\left\{\left(w_{1}, w_{2}\right) \in \square_{\rho_{1}}^{2 d_{1}} \times \Delta_{\rho_{1}}^{c}: \operatorname{Im} w_{1}=0, w_{2}=0\right\}$.
- If $M$ is generic, letting $m=d-n$, then $m+d=n$ and

$$
M=\left\{(z, w) \in \Delta_{\rho_{1}}^{m} \times\left(\square_{\rho_{1}}^{d}+i \mathbb{R}^{d}\right): \operatorname{Im} w=\varphi(z, \bar{z}, \operatorname{Re} w)\right\},
$$

for some $\mathbb{R}^{d}$-valued algebraic or analytic map $\varphi$ satisfying $\varphi(0)=0$ whose power series converges normally in $\Delta_{2 \rho_{1}}^{m} \times \Delta_{2 \rho_{1}}^{m} \times \square_{2 \rho_{1}}^{d}$.

- If $M$ is Cauchy-Riemann, letting $m=$ CRdim $M, c=d-n+m \geqslant$ 0 , and $d_{1}=2 n-2 m-d \geqslant 0$, then $m+d_{1}+c=n$ and

$$
\begin{aligned}
& M=\left\{\left(z, w_{1}, w_{2}\right) \in \Delta_{\rho_{1}}^{m} \times\left(\square_{\rho_{1}}^{d_{1}}+i \mathbb{R}^{d_{1}}\right) \times \Delta_{\rho_{1}}^{c}:\right. \\
& \left.\operatorname{Im} w_{1}=\varphi_{1}\left(z, \bar{z}, \operatorname{Re} w_{1}\right), w_{2}=0\right\},
\end{aligned}
$$

for some $\mathbb{R}^{d_{1}}$-valued algebraic or analytic map $\varphi_{1}$ satisfying $\varphi_{1}(0)=0$ whose power series converges normally in $\Delta_{2 \rho_{1}}^{m} \times \Delta_{2 \rho_{1}}^{m} \times \square_{2 \rho_{1}}^{d_{1}}$.
A further linear change of coordinates may yield $d \varphi(0)=0$ and $d \varphi_{1}(0)=$ 0.

A CR algebraic or analytic manifold being generic in some local complex manifold of (smaller) dimension $n-c$, called its intrinsic complexification, in most occasions, questions, results and articles, one deals with generic manifolds. In this chapter, all generic submanifolds will be of positive codimension $d \geqslant 1$ and of positive CR dimension $m \geqslant 1$.
2.4. Algebraic and analytic generic submanifolds and their extrinsic complexification. Let $M$ be generic, represented by $\operatorname{Im} w=\varphi(z, \bar{z}, \operatorname{Re} w)$. The implicit function theorem applied to the vectorial equation $\frac{w-\bar{w}}{2 i}=$ $\varphi\left(z, \bar{z}, \frac{w+\bar{w}}{2}\right)$, enables to solve the variables $\bar{w} \in \mathbb{C}^{d}$, or the variables $w \in \mathbb{C}^{d}$. This yields the so-called complex defining equations for $M$, most useful in applications, as stated just below. Recall that, given a power series $\Phi(t)=\sum_{\gamma \in \mathbb{N}^{n}} \Phi_{\gamma} t^{\gamma}, t \in \mathbb{C}^{n}, \Phi_{\gamma} \in \mathbb{C}, \gamma \in \mathbb{N}^{n}$, one defines the series $\bar{\Phi}(t):=\sum_{\gamma \in \mathbb{N}^{n}} \bar{\Phi}_{\gamma} t^{\gamma}$ by conjugating only its complex coefficients. Then $\overline{\Phi(t)} \equiv \bar{\Phi}(\bar{t})$, a frequently used property.

Theorem 2.5. ([BER1999, Me2004a]) A local generic real algebraic or analytic d-codimensional generic submanifold $M \cap \Delta_{\rho_{1}}^{n}$ may be represented by $\bar{w}=\Theta(\bar{z}, z, w)$, or equivalently by $w=\bar{\Theta}(z, \bar{z}, \bar{w})$, for some complex algebraic or analytic $\mathbb{C}^{d}$-valued map $\Theta$ whose power series converges normally
in $\Delta_{2 \rho_{1}}^{m} \times \Delta_{2 \rho_{1}}^{m} \times \Delta_{2 \rho_{1}}^{d}$, with $\rho_{1}>0$. Here, $\Theta$ and $\bar{\Theta}$ satisfy the two (equivalent by conjugation) vectorial functional equations:

$$
\left\{\begin{array}{l}
\bar{w} \equiv \Theta(\bar{z}, z, \bar{\Theta}(z, \bar{z}, \bar{w})) \\
w \equiv \bar{\Theta}(z, \bar{z}, \Theta(\bar{z}, z, w))
\end{array}\right.
$$

Conversely, if such $a \mathbb{C}^{d}$-valued map $\Theta$ satisfies the above, the set $M:=$ $\left\{(z, w) \in \Delta_{\rho_{1}}^{n}: \bar{w}=\Theta(\bar{z}, z, w)\right\}$ is a real local generic submanifold of codimension $d$.

The coordinates $(z, w) \in \mathbb{C}^{m} \times \mathbb{C}^{d}$ will also be denoted by $t \in \mathbb{C}^{n}$. Let $\tau=(\zeta, \xi) \in \mathbb{C}^{m} \times \mathbb{C}^{d}$ be new independent complex variables. Define the extrinsic complexification $\mathscr{M}=(M)^{c}$ of $M$ to be the complex algebraic or analytic $d$-codimensional submanifold of $\mathbb{C}^{n} \times \mathbb{C}^{n}$ defined by the vectorial equation $\xi-\Theta(\zeta, t)=0$ (the map $\Theta$ being analytic, we may indeed substitute $\zeta$ for $\bar{z}$ in its power series). We also write $\tau=(\bar{t})^{c}$. Observe that $M$ identifies with the intersection $\mathscr{M} \cap\{\tau=\bar{t}\}$.

Lemma 2.6. ([Me2004a, Me2005]) There exists an invertible $d \times d$ matrix $a(t, \tau)$ of algebraic or analytic power series converging normally in $\Delta_{2 \rho_{1}}^{n} \times$ $\Delta_{2 \rho_{1}}^{n}$ such that $w-\bar{\Theta}(z, \tau) \equiv a(t, \tau)[\xi-\Theta(\zeta, t)]$.

Thus, $\mathscr{M}$ is equivalently defined by $w-\bar{\Theta}(z, \tau)=0$.
2.7. Complexified Segre varieties and complexified CR vector fields. Let $\tau_{p}, t_{p} \in \Delta_{\rho_{1}}^{n}$ be fixed and define the complexified Segre varieties $\mathscr{T}_{\tau_{p}}$ and the complexified conjugate Segre varieties $\mathscr{S}_{t_{p}}$ by:

$$
\left\{\begin{array}{l}
\mathscr{S}_{\tau_{p}}:=\left\{(t, \tau) \in \Delta_{\rho_{1}}^{n} \times \Delta_{\rho_{1}}^{n}: \tau=\tau_{p}, w=\bar{\Theta}\left(z, \tau_{p}\right)\right\} \quad \text { and } \\
\underline{\mathscr{S}}_{t_{p}}:=\left\{(t, \tau) \in \Delta_{\rho_{1}}^{n} \times \Delta_{\rho_{1}}^{n}: t=t_{p}, \xi=\Theta\left(\zeta, t_{p}\right)\right\} .
\end{array}\right.
$$

Geometrically, $\mathscr{S}_{\tau_{p}}=\mathscr{M} \cap\left\{\tau=\tau_{p}\right\}$ and $\mathscr{S}_{t_{p}}=\mathscr{M} \cap\left\{t=t_{p}\right\}$. We draw a diagram.


The complexification of a real analytic generic submanifold carries a pair of invariant foliations which are the integral submanifolds of the complexified $(1,0)$ and $(0,1)$ vector fields and which identify also with the complexified Segre varieties

Geometry of the extrinsic complexification $\mathscr{M}$

We warn the reader that

$$
\operatorname{dim}_{\mathbb{C}} \mathscr{M}-\operatorname{dim}_{\mathbb{C}} \mathscr{S}_{\tau_{p}}-\operatorname{dim}_{\mathbb{C}} \mathscr{\mathscr { S }}_{t_{p}}=d \geqslant 1
$$

so that the ambient codimension $d$ of the unions of $\mathscr{S}_{\tau_{p}}$ and of $\mathscr{S}_{t_{p}}$ is invisible in this picture; one should imagine for instance that $\mathscr{M}$ is the threedimensional physical space equipped with a pair of foliations by horizontal orthogonal real lines.

Next, define two collections of complex vector fields:

$$
\left\{\begin{aligned}
\mathscr{L}_{k}:=\frac{\partial}{\partial z_{k}}+\sum_{j=1}^{d} \frac{\partial \bar{\Theta}_{j}}{\partial z_{k}}(z, \zeta, \xi) \frac{\partial}{\partial w_{j}}, & k=1, \ldots, m, \quad \text { and } \\
\mathscr{L}_{k}:=\frac{\partial}{\partial \zeta_{k}}+\sum_{j=1}^{d} \frac{\partial \Theta_{j}}{\partial \zeta_{k}}(\zeta, z, w) \frac{\partial}{\partial \xi_{j}}, & k=1, \ldots, m
\end{aligned}\right.
$$

One verifies that $\mathscr{L}_{k}\left(w_{j}-\bar{\Theta}_{j}(z, \zeta, \xi)\right) \equiv 0$, which shows that the $\mathscr{L}_{k}$ are tangent to $\mathscr{M}$. Similarly, $\mathscr{L}_{k}\left(\xi_{j}-\Theta_{j}(\zeta, z, w)\right) \equiv 0$, so the $\mathscr{L}_{k}$ are also tangent to $\mathscr{M}$. In addition, $\left[\mathscr{L}_{k}, \mathscr{L}_{k^{\prime}}\right]=0$ and $\left[\mathscr{L}_{k}, \mathscr{L}_{k^{\prime}}\right]=0$ for $k, k^{\prime}=$ $1, \ldots, m$, so the theorem of Frobenius applies. In fact, the $m$-dimensional integral submanifolds of the two collections $\left\{\mathscr{L}_{k}\right\}_{1 \leqslant k \leqslant m}$ and $\left\{\mathscr{L}_{k}\right\}_{1 \leqslant k \leqslant m}$ are the $\mathscr{S}_{\tau_{p}}$ and the $\mathscr{S}_{t_{p}}$. In summary, $\mathscr{M}$ carries a fundamental pair of foliations.

Observe that the vector fields $\mathscr{L}_{k}$ are the complexifications of the vector fields $L_{k}:=\frac{\partial}{\partial z_{k}}+\sum_{j=1}^{d} \frac{\partial \bar{\Theta}_{j}}{\partial z_{k}}(z, \bar{z}, \bar{w}) \frac{\partial}{\partial w_{j}}, k=1, \ldots, m$, that generate the holomorphic tangent bundle $T^{1,0} M$. A similar observation applies to the vector fields $\mathscr{L}_{k}$.

In general (unless $M$ is Levi-flat), the total collection $\left\{\mathscr{L}_{k}, \mathscr{L}_{k}\right\}_{1 \leqslant k \leqslant m}$ does not enjoy the Frobenius property. In fact, the noncommutativity of this system of $2 m$ vector fields is at the very core of Cauchy-Riemann geometry.

To apply Theorem 1.13, introduce the "multiple" flows of the two collections $\left\{\mathscr{L}_{k}\right\}_{1 \leqslant k \leqslant m}$ and $\left\{\mathscr{L}_{k}\right\}_{1 \leqslant k \leqslant m}$. If $p \in \mathscr{M}$ has coordinates $\left(z_{p}, w_{p}, \zeta_{p}, \xi_{p}\right) \in \Delta_{\rho_{1}}^{m} \times \Delta_{\rho_{1}}^{d} \times \Delta_{\rho_{1}}^{m} \times \Delta_{\rho_{1}}^{d}$ satisfying $w_{p}=\bar{\Theta}\left(z_{p}, \zeta_{p}, \xi_{p}\right)$ and $\xi_{p}=\Theta\left(\zeta_{p}, z_{p}, w_{p}\right)$ and if $z_{1}:=\left(z_{1,1}, \ldots, z_{1, m}\right) \in \mathbb{C}^{m}$ is a small "multitime" parameter, define the "multiple" flow of $\mathscr{L}$ by:

$$
\begin{align*}
\mathscr{L}_{z_{1}}\left(z_{p}, w_{p}, \zeta_{p}, \xi_{p}\right) & :=\exp \left(z_{1} \mathscr{L}\right)(p) \\
& :=\exp \left(z_{1,1} \mathscr{L}_{1}\left(\cdots\left(\exp \left(z_{1, m} \mathscr{L}_{m}(p)\right)\right) \cdots\right)\right)  \tag{2.8}\\
& :=\left(z_{p}+z_{1}, \bar{\Theta}\left(z_{p}+z_{1}, \zeta_{p}, \xi_{p}\right), \zeta_{p}, \xi_{p}\right) .
\end{align*}
$$

Of course, $\mathscr{L}_{z_{1}}(p) \in \mathscr{M}$. Similarly, for $p \in \mathscr{M}$ and $\zeta_{1} \in \mathbb{C}^{m}$, defining:

$$
\begin{equation*}
\mathscr{L}_{\zeta_{1}}\left(z_{p}, w_{p}, \zeta_{p}, \xi_{p}\right):=\left(z_{p}, w_{p}, \zeta_{p}+\zeta_{1}, \Theta\left(\zeta_{p}+\zeta_{1}, z_{p}, w_{p}\right)\right) \tag{2.9}
\end{equation*}
$$

we have $\mathscr{L}_{\zeta_{1}}(p) \in \mathscr{M}$. Clearly, $\left(p, z_{1}\right) \mapsto \mathscr{L}_{z_{1}}(p)$ and $\left(p, \zeta_{1}\right) \mapsto \mathscr{L}_{\zeta_{1}}(p)$ are complex algebraic or analytic local maps.
2.10. Segre chains. Let us start from $p=0$ being the origin and move vertically along the complexified conjugate Segre variety $\mathscr{\mathscr { S }}_{0}$ of a height $z_{1} \in \mathbb{C}^{m}$, namely let us consider the point $\mathscr{L}_{z_{1}}(0)$, which we shall also denote by $\underline{\Gamma}_{1}\left(z_{1}\right)$. We have $\underline{\Gamma}_{1}(0)=0$. Let $z_{2} \in \mathbb{C}^{m}$. Starting from the point $\underline{\Gamma}_{1}\left(z_{1}\right)$, let us move horizontally along the complexified Segre variety of a length $z_{2} \in \mathbb{C}^{m}$, namely let us consider the point

$$
\underline{\Gamma}_{2}\left(z_{1}, z_{2}\right):=\mathscr{L}_{z_{2}}\left(\underline{\mathscr{L}}_{z_{1}}(0)\right) .
$$

Next, define $\underline{\Gamma}_{3}\left(z_{1}, z_{2}, z_{3}\right):=\mathscr{L}_{z_{3}}\left(\mathscr{L}_{z_{2}}\left(\underline{\mathscr{L}}_{z_{1}}(0)\right)\right)$, and then

$$
\underline{\Gamma}_{4}\left(z_{1}, z_{2}, z_{3}, z_{4}\right):=\mathscr{L}_{z_{4}}\left(\mathscr{L}_{z_{3}}\left(\mathscr{L}_{z_{2}}\left(\underline{\mathscr{L}}_{z_{1}}(0)\right)\right)\right)
$$

and so on. We draw a diagram:


By induction, for every $k \in \mathbb{N}, k \geqslant 1$, we obtain a local complex algebraic or analytic map $\underline{\Gamma}_{k}\left(z_{1}, \ldots, z_{k}\right)$, valued in $\mathscr{M}$, defined for sufficiently small $z_{1}, \ldots, z_{k} \in \mathbb{C}^{m}$ which satisfies $\underline{\Gamma}_{k}(0, \ldots, 0)=0$. The abbreviated notation $z_{(k)}:=\left(z_{1}, \ldots, z_{k}\right) \in \mathbb{C}^{m k}$ will be used. The map $\underline{\Gamma}_{k}$ is called the $k$-th conjugate Segre chain ([Me2004a, Me2005]).

If we had conducted this procedure by starting with $\mathscr{L}$ instead of starting with $\mathscr{L}$, we would have obtained maps $\Gamma_{1}\left(z_{1}\right):=\mathscr{L}_{z_{1}}(0), \Gamma_{2}\left(z_{(2)}\right):=$ $\mathscr{L}_{z_{2}}\left(\mathscr{L}_{z_{1}}(0)\right)$, etc., and generally $\Gamma_{k}\left(z_{(k)}\right)$. The map $\Gamma_{k}$ is called the $k$-th Segre chain.

There is a symmetry relation between $\Gamma_{k}$ and $\underline{\Gamma}_{k}$. Indeed, let $\bar{\sigma}$ be the antiholomorphic involution of $\mathbb{C}^{n} \times \mathbb{C}^{n}$ defined by $\bar{\sigma}(t, \tau):=(\bar{\tau}, \bar{t})$. Since we have $w=\bar{\Theta}(z, \zeta, \xi)$ if and only if $\xi=\Theta(\zeta, z, w)$, this involution is a bijection of $\mathscr{M}$. Applying $\bar{\sigma}$ to the definitions (2.8) and (2.9) of the flows of $\mathscr{L}$ and of $\mathscr{L}$, one may verify that $\bar{\sigma}\left(\mathscr{L}_{z_{1}}(p)\right)=\underline{\mathscr{L}}_{\bar{z}_{1}}(\bar{\sigma}(p))$. It follows the general symmetry relation $\bar{\sigma}\left(\Gamma_{k}\left(z_{(k)}\right)\right)=\underline{\Gamma}_{k}\left(\overline{z_{(k)}}\right)$. Thus, $\Gamma_{k}$ and $\underline{\Gamma}_{k}$ have the same behavior.
2.11. Minimality of $\mathscr{M}$ at the origin and complexified local CR orbits. Since $\Gamma_{k}(0)=\underline{\Gamma}_{k}(0)=0$, for every integer $k \geqslant 1$, there exists $\delta_{k}>0$ sufficiently small such that $\Gamma_{k}\left(z_{(k)}\right)$ and $\underline{\Gamma}_{k}\left(z_{(k)}\right)$ are well defined and belong to $\mathscr{M}$, at least for all $z_{(k)} \in \Delta_{\delta_{k}}^{m k}$. To fiw ideas, it will be convenient to consider that $\Delta_{\delta_{k}}^{m k}$ is the precise domain of definition of $\Gamma_{k}$ and of $\underline{\Gamma}_{k}$. We aim to apply the procedure of Theorem 1.13 to the system $\mathbb{L}^{0}:=\left\{\mathscr{L}_{1}, \ldots, \mathscr{L}_{m}, \mathscr{L}_{1}, \ldots, \mathscr{L}_{m}\right\}$.

However, there is a slight (innocuous) difference: each multitime $\mathrm{t}=$ $\left(\mathrm{t}_{1}, \ldots, \mathrm{t}\right) \in \mathbb{K}^{k}$ had scalar components $\mathrm{t}_{i} \in \mathbb{K}$, whereas now each $z_{(k)}=$ $\left(z_{1}, \ldots, z_{k}\right) \in \mathbb{C}^{m k}$ has vectorial components $z_{i} \in \mathbb{C}^{m}$. It is easy to see that both $\Gamma_{1}$ and $\Gamma_{1}$ are of constant rank $m$. Also, both $\Gamma_{2}$ and $\underline{\Gamma}_{2}$ are of constant rank $2 m$, since $\mathscr{L}_{1}, \ldots, \mathscr{L}_{m}, \underline{\mathscr{L}}_{1}, \ldots, \mathscr{L}_{m}$ are linearly independent at the origin. However, when passing to (conjugate) Segre chains of length $\geqslant 3$, it is necessary to speak of generic ranks and to introduce some combinatorial integers $e_{k} \geqslant 1$. Justifying examples may be found in [Me1999, Me2004a].

Theorem 2.12. ([BER1996, BER1999, Me1999, Me2001a, Me2004a]) There exists an integer $\nu_{0}$ with $1 \leqslant \nu_{0} \leqslant d$ and, for $k=3, \ldots, \nu_{0}+1$, integers $e_{k}$ with $1 \leqslant e_{k} \leqslant m$ such that the following nine properties hold true.
(1) For every $k=3, \ldots, \nu_{0}+1$, the two maps $\Gamma_{k}$ and $\underline{\Gamma}_{k}$ are of generic rank equal to $2 m+e_{3}+\cdots+e_{k}$. In the special case $\nu_{0}=1$, the $e_{k}$ are inexistent ${ }^{2}$ and nothing is stated.
(2) For every $k \geqslant \nu_{0}+1$, both $\Gamma_{k}$ and $\underline{\Gamma}_{k}$ are of fixed, stabilized generic rank equal to $2 m+e$, where

$$
e:=e_{3}+\cdots+e_{\nu_{0}} \leqslant d
$$

(3) Setting $\mu_{0}:=2 \nu_{0}+1$, there exist two points $z_{\left(\mu_{0}\right)}^{*} \in \Delta_{\delta_{\mu_{0}}}^{m \mu_{0}}$ and $\underline{z}_{\left(\mu_{0}\right)}^{*} \in \Delta_{\delta_{\mu_{0}}}^{m \mu_{0}}$ satisfying $\Gamma_{\mu_{0}}\left(z_{\left(\mu_{0}\right)}^{*}\right)=0$ and $\underline{\Gamma}_{\mu_{0}}\left(\underline{z}_{\left(\mu_{0}\right)}^{*}\right)=0$ which are arbitrarily close to the origin in $\Delta_{\delta_{\mu_{0}}}^{m \mu_{0}}$ such that $\Gamma_{\mu_{0}}$ and $\underline{\Gamma}_{\mu_{0}}$ are of constant rank $2 m+e$ in neighborhoods $\omega^{*}$ and $\underline{\omega}^{*}$ of $z_{\left(\mu_{0}\right)}^{*}$ and of $\underline{z}_{\left(\mu_{0}\right)}^{*}$. The images $\Gamma_{\mu_{0}}\left(\omega^{*}\right)$ and $\underline{\Gamma}_{\mu_{0}}\left(\underline{\omega}^{*}\right)$ then constitute two pieces of local $\mathbb{K}$-algebraic or analytic submanifold of dimension $2 m+e$ contained in $\mathscr{M}$.
(4) Both $\Gamma_{\mu_{0}}\left(\omega^{*}\right)$ and $\underline{\Gamma}_{\mu_{0}}\left(\underline{\omega}^{*}\right)$ enjoy the most important property that all vector fields $\mathscr{L}_{1}, \ldots, \mathscr{L}_{m}, \mathscr{L}_{1}, \ldots, \mathscr{L}_{m}$ are tangent to $\Gamma_{\mu_{0}}\left(\omega^{*}\right)$ and to $\underline{\Gamma}_{\mu_{0}}\left(\underline{\omega}^{*}\right)$.
(5) $\Gamma_{\mu_{0}}\left(\omega^{*}\right)$ and $\underline{\Gamma}_{\mu_{0}}\left(\underline{\omega}^{*}\right)$ coincide together in a neighborhood of 0 in $\mathscr{M}$.

[^1](6) Denoting by
$$
\mathscr{O}_{\mathscr{L}, \underline{\mathscr{L}}}(\mathscr{M}, 0)
$$
this common local piece of complex analytic submanifold of $\mathscr{M}$, it is algebraic provided that the flows of $\left\{\mathscr{L}_{1}, \ldots, \mathscr{L}_{m}, \underline{\mathscr{L}}_{1}, \ldots, \mathscr{L}_{m}\right\}$ are themselves algebraic.
(7) Every local complex analytic or algebraic submanifold $\mathscr{N} \subset \mathscr{M}$ passing through the origin to which $\mathscr{L}_{1}, \ldots, \mathscr{L}_{m}, \mathscr{L}_{1}, \ldots, \mathscr{L}_{m}$ are all tangent must contain $\mathscr{O}_{\mathscr{L}, \mathscr{\mathscr { L }}}(\mathscr{M}, 0)$ in a neighborhood of the origin.
(8) The integers $\nu_{0}, e_{3}, \ldots, e_{\mu_{0}}$ and $e$ are biholomorphic invariants of $\mathscr{M}$.
(9) $\Gamma_{\mu_{0}}\left(\omega^{*}\right)$ and $\underline{\Gamma}_{\mu_{0}}\left(\underline{\omega}^{*}\right)$ also coincide (in a neighborhood of the origin) with the Nagano leaf of the system $\left\{\mathscr{L}_{1}, \ldots, \mathscr{L}_{m}, \mathscr{L}_{1}, \ldots, \mathscr{L}_{m}\right\}$, as it was constructed in Theorem 1.5.

As in [Me2004a, Me2005] (with different notations), the integer $\nu_{0}$ will be called the Segre type of M.

The "orbit notation" $\mathscr{O}_{\mathscr{L}, \mathscr{L}}(\mathscr{M}, 0)$ anticipates the presentation and the notation of Section 1(III). We will abandon Lie brackets and Nagano leaves.

The complex vector fields $L_{k}:=\frac{\partial}{\partial z_{k}}+\sum_{j=1}^{d} \frac{\partial \bar{\Theta}_{j}}{\partial z_{k}}(z, \bar{z}, \bar{w}) \frac{\partial}{\partial w_{j}}, k=$ $1, \ldots, m$, are tangent to $M$ of equations $w_{j}=\bar{\Theta}_{j}(z, \bar{z}, \bar{w}), j=1, \ldots, d$; their conjugates $\bar{L}_{k}$ are also tangent to $M$; it follows that the real and imaginary parts $\operatorname{Re} L_{k}$ and $\operatorname{Im} L_{k}$ are also tangent to $M$. We may then apply Theorem 1.13 to the system $\left\{\operatorname{Re} L_{k}, \operatorname{Im} L_{k}\right\}_{1 \leqslant k \leqslant m}$, getting a certain real analytic local submanifold $\mathscr{O}_{L, \bar{L}}(M, 0)$ of $M$ passing through the origin. It will be called the local CR orbit of the origin in $M$ (terminology of Part III).

The relation between $\mathscr{O}_{\mathscr{L}, \mathscr{L}}(\mathscr{M}, 0)$ and $\mathscr{O}_{L, \bar{L}}(M, 0)$ is as follows ([BER1996, Me1999, Me2001a, Me2004a]). Let $\pi_{t}(t, \tau):=t$ and $\pi_{\tau}(t, \tau):=\tau$ denote the two canonical projections associated to the product $\Delta_{\rho_{1}}^{n} \times \Delta_{\rho_{1}}^{n}$. Let $\underline{A}:=\left\{(t, \tau) \in \Delta_{\rho_{1}}^{n} \times \Delta_{\rho_{1}}^{n}: \tau=\bar{t}\right\}$ be the antiholomorphic diagonal. Observe that $\pi_{t}(\underline{A} \cap \mathscr{M})=M$.

- The extrinsic complexification $\left[\mathscr{O}_{L, \bar{L}}(M, 0)\right]^{c}=\mathscr{O}_{\mathscr{L}, \underline{\mathscr{L}}}(\mathscr{M}, 0)$.
- The projection $\pi_{t}\left(\underline{A} \cap \mathscr{O}_{\mathscr{L}, \underline{\mathscr{L}}}(\mathscr{M}, 0)\right)=\mathscr{O}_{L, \bar{L}}(M, 0)$.

Concerning smoothness, a striking subtelty happens: if $M$ is real algebraic, although the local multiple flows of $\mathscr{L}$ and of $\mathscr{L}$ are complex algebraic (thanks to their definitions (2.8) and (2.9)), the flows of $\operatorname{Re} L_{k}$ and of $\operatorname{Im} L_{k}$ are only real analytic in general.

Example 2.13. ([Me2004a]) For the real algebraic hypersurface of $\mathbb{C}^{2}$ defined by $\operatorname{Im} w=\sqrt{1+z \bar{z}}-1$, the vector field $L:=\frac{\partial}{\partial z}+i \bar{z} \sqrt{1+z \bar{z}} \frac{\partial}{\partial w}$
generates $T^{1,0} M$ and the flow of $2 \operatorname{Re} L$ involves the transcendent function Arcsh.

Theorem 2.14. ([BER1996, Me2001a]) The local CR orbit $\mathscr{O}_{L, \bar{L}}(M, 0)$ is real algebraic if $M$ is.

For the proof, assuming $M$ to be real algebraic, it is impossible, because of the example, to apply the second phrase of Theorem 1.13 (5) to the system $\left\{\operatorname{Re} L_{k}, \operatorname{Im} L_{k}\right\}_{1 \leqslant k \leqslant m}$. Fortunately, this phrase applies to the complexified system $\left\{\mathscr{L}_{k}, \mathscr{L}_{k}\right\}_{1 \leqslant k \leqslant m}$, whence $\mathscr{O}_{\mathscr{L}, \mathscr{L}}(\mathscr{M}, 0)$ is algebraic, and then the local CR orbit $\mathscr{O}_{L, \bar{L}}(M, 0)=\pi_{t}\left(\underline{A} \cap \mathscr{O}_{\mathscr{L}, \mathscr{L}}(\mathscr{M}, 0)\right)$ is real algebraic.
Definition 2.15. The generic submanifold $M$ or its extrinsic complexification $\mathscr{M}$ is said to be minimal at the origin if $\mathscr{O}_{L, \bar{L}}(M, 0)$ contains a neighborhood of 0 in $M$, or equivalently if $\mathscr{O}_{\mathscr{L}, \mathscr{L}}(\mathscr{M}, 0)$ contains a neighborhood of 0 in $\mathscr{M}$.

The minimality at the origin of the algebraic or analytic complexified local generic submanifold $\mathscr{M}=(M)^{c}$ is a biholomorphically invariant property; it neither depends on the choice of defining equations nor on the choice of a conjugate pair of systems of complex vector fields $\left\{\mathscr{L}_{k}\right\}_{1 \leqslant k \leqslant m}$ and $\left\{\mathscr{L}_{k}\right\}_{1 \leqslant k \leqslant m}$ spanning the tangent space to the two foliations.

Minimality at 0 reads $e=d$ in Theorem 2.12. For a hypersurface $M$, namely with $d=1$, minimality at 0 is equivalent to $\nu_{0}=2$.
2.16. Projections of the submersions $\Gamma_{\mu_{0}}$ and $\underline{\Gamma}_{\mu_{0}}$. Let $\mu_{0}=2 \nu_{0}+1$ as in Theorem 2.12. If $\mathscr{M}$ is minimal at the origin, the two local holomorphic maps

$$
\Gamma_{\mu_{0}} \text { and } \underline{\Gamma}_{\mu_{0}}: \quad \Delta_{\delta_{\mu_{0}}}^{m \mu_{0}} \longrightarrow \mathscr{M}
$$

satisfy $\Gamma_{\mu_{0}}\left(z_{\left(\mu_{0}\right)}^{*}\right)=0$ and $\underline{\Gamma}_{\mu_{0}}\left(\underline{z}_{\left(\mu_{0}\right)}^{*}\right)=0$ and they are submersive at $z_{\left(\mu_{0}\right)}^{*}$ and at $\underline{z}_{\left(\mu_{0}\right)}^{*}$.

Consider the two projections $\pi_{t}(t, \tau):=t$ and $\pi_{\tau}(t, \tau):=\tau$ and four compositions $\pi_{t}\left(\Gamma_{\mu_{0}}\left(z_{\left(\mu_{0}\right)}\right)\right)$, $\pi_{t}\left(\underline{\Gamma}_{\mu_{0}}\left(z_{\left(\mu_{0}\right)}\right)\right)$ and $\pi_{\tau}\left(\Gamma_{\mu_{0}}\left(z_{\left(\mu_{0}\right)}\right)\right)$, $\pi_{\tau}\left(\underline{\Gamma}_{\mu_{0}}\left(z_{\left(\mu_{0}\right)}\right)\right)$. Since $\mu_{0}=2 \nu_{0}+1$ is odd, observe that the composition $\underline{\Gamma}_{2 \nu_{0}+1}=\mathscr{L}(\cdots)$ ends with a $\underline{\mathscr{L}}$ and that $\Gamma_{2 \nu_{0}+1}=\mathscr{L}(\cdots)$ ends with a $\mathscr{L}$. According to the two definitions of the flow maps, the coordinates $\left(\zeta_{p}, \zeta_{p}\right)$ are untouched in (2.8) and the coordinates $\left(z_{p}, w_{p}\right)$ are untouched in (2.9). It follows that

$$
\left\{\begin{aligned}
\pi_{t}\left(\underline{\Gamma}_{2 \nu_{0}+1}\left(z_{\left(2 \nu_{0}+1\right)}\right)\right) & \equiv \pi_{t}\left(\underline{\Gamma}_{2 \nu_{0}}\left(z_{\left(2 \nu_{0}\right)}\right)\right) \\
\pi_{\tau}\left(\Gamma_{2 \nu_{0}+1}\left(z_{\left(2 \nu_{0}+1\right)}\right)\right) & \equiv \pi_{\tau}\left(\Gamma_{2 \nu_{0}}\left(z_{\left(2 \nu_{0}\right)}\right)\right)
\end{aligned}\right.
$$

Corollary 2.17. ([Me1999, BER1999, Me2004a]) If $M$ is minimal at the origin, there exists a integer $\nu_{0} \leqslant d+1$ (the Segre type of $M$ at the origin)
and there exist points $\underline{z}_{\left(2 \nu_{0}\right)}^{*} \in \mathbb{C}^{2 m \nu_{0}}$ and $z_{\left(2 \nu_{0}\right)}^{*} \in \mathbb{C}^{2 m \nu_{0}}$ arbitrarily close to the origin, such that the two maps

$$
\left\{\begin{array}{l}
\Delta_{\delta_{2 \nu_{0}}}^{m 2 \nu_{0}} \ni z_{\left(2 \nu_{0}\right)} \longmapsto \pi_{t}\left(\underline{\Gamma}_{2 \nu_{0}}\left(z_{\left(2 \nu_{0}\right)}\right)\right) \in \mathbb{C}^{n} \quad \text { and } \\
\Delta_{\delta_{2 \nu_{0}}}^{m 2 \nu_{0}} \ni z_{\left(2 \nu_{0}\right)} \longmapsto \pi_{\tau}\left(\Gamma_{2 \nu_{0}}\left(z_{\left(2 \nu_{0}\right)}\right)\right) \in \mathbb{C}^{n}
\end{array}\right.
$$

are of rank $n$ and send $\underline{z}_{\left(2 \nu_{0}\right)}^{*}$ and $z_{\left(2 \nu_{0}\right)}^{*}$ to the origin.

## §3. Formal CR mappings, jets of Segre varieties and CR REFLECTION MAPPING

3.1. Complexified CR mappings respect pairs of foliations. Let $n^{\prime} \in \mathbb{N}$ with $n^{\prime} \geqslant 1$ and let $M^{\prime} \subset \mathbb{C}^{n^{\prime}}$ be a second algebraic or analytic generic submanifold of codimension $d^{\prime} \geqslant 1$ and of CR dimension $m^{\prime}=n^{\prime}-d^{\prime} \geqslant 1$. Let $p^{\prime} \in M^{\prime}$. There exist local coordinates $t^{\prime}=\left(z^{\prime}, w^{\prime}\right) \in \mathbb{C}^{m^{\prime}} \times \mathbb{C}^{d^{\prime}}$ centered at $p^{\prime}$ in which $M^{\prime}$ is represented by $\bar{w}^{\prime}=\Theta^{\prime}\left(\bar{z}^{\prime}, t^{\prime}\right)$, or equivalently by $w^{\prime}=$ $\bar{\Theta}^{\prime}\left(z^{\prime}, \bar{t}^{\prime}\right)$. If $\left(\bar{t}^{\prime}\right)^{c}=\tau^{\prime}=\left(\zeta^{\prime}, \xi^{\prime}\right) \in \mathbb{C}^{m^{\prime}} \times \mathbb{C}^{d^{\prime}}$, the extrinsic complexification is represented by $\xi^{\prime}=\Theta^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)$, or equivalently by $w^{\prime}=\bar{\Theta}^{\prime}\left(z^{\prime}, \tau^{\prime}\right)$. We shall denote by $0^{\prime}$ the origin of $\mathbb{C}^{n^{\prime}}$.

Let $t \in \mathbb{C}^{n}$ and let $h(t)=\left(h_{1}(t), \ldots, h_{n^{\prime}}(t)\right) \in \mathbb{C} \llbracket t \rrbracket^{n^{\prime}}$ be a formal power series mapping with no constant term, i.e. $h(0)=0^{\prime}$; it may also be holomorphic namely $h(t) \in \mathbb{C}\{t\}^{n^{\prime}}$, or even (Nash) algebraic. We have $(\overline{h(t)})^{c}=\bar{h}\left((\bar{t})^{c}\right)=\bar{h}(\tau)$. Define $h^{c}(t, \tau):=(h(t), \bar{h}(\tau))$.

Set $r(t, \tau):=\xi-\Theta(\zeta, t)$, set $\bar{r}(\tau, t):=w-\bar{\Theta}(z, \tau)$, set $r^{\prime}\left(t^{\prime}, \tau^{\prime}\right):=$ $\xi^{\prime}-\Theta^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)$ and set $\bar{r}^{\prime}\left(\tau^{\prime}, t^{\prime}\right):=w^{\prime}-\bar{\Theta}^{\prime}\left(z^{\prime}, \tau^{\prime}\right)$. We say that the power series mapping $h$ is a formal CR mapping from $(M, 0)$ to $\left(M^{\prime}, 0^{\prime}\right)$ if there exists a $d^{\prime} \times d$ matrix of formal power series $b(t, \bar{t})$ such that

$$
r^{\prime}(h(t), \bar{h}(\bar{t})) \equiv b(t, \bar{t}) r(t, \bar{t})
$$

in $\mathbb{C} \llbracket t, \not t \rrbracket^{d^{\prime}}$. By complexification, it follows that $r^{\prime}(h(t), \bar{h}(\tau)) \equiv$ $b(t, \tau) r(t, \tau)$ in $\mathbb{C} \llbracket t, \tau \rrbracket^{d^{\prime}}$, namely $h^{c}(t, \tau)=(h(t), \bar{h}(\tau)) \operatorname{maps}(\mathscr{M}, 0)$ formally to $\left(\mathscr{M}^{\prime}, 0^{\prime}\right)$. By Lemma 2.6 , there exist two complex analytic invertible matrices $a(t, \tau)$ and $a^{\prime}\left(t^{\prime}, \tau^{\prime}\right)$ satisfying :

$$
\begin{cases}r(t, \tau) \equiv a(t, \tau) \bar{r}(\tau, t), & r^{\prime}\left(t^{\prime}, \tau^{\prime}\right) \equiv a^{\prime}\left(t^{\prime}, \tau^{\prime}\right) \bar{r}^{\prime}\left(\tau^{\prime}, t^{\prime}\right) \\ \bar{r}(\tau, t) \equiv \bar{a}(\tau, t) r(t, \tau), & \bar{r}^{\prime}\left(\tau^{\prime}, t^{\prime}\right) \equiv \bar{a}^{\prime}\left(\tau^{\prime}, t^{\prime}\right) r^{\prime}\left(t^{\prime}, \tau^{\prime}\right)\end{cases}
$$

in $\mathbb{C} \llbracket t, \tau \rrbracket^{d}$ and in $\mathbb{C} \llbracket t^{\prime}, \tau^{\prime} \rrbracket^{d^{\prime}}$. So, to define a complexified formal CR mapping $h^{c}:(\mathscr{M}, 0) \mapsto \mathscr{\mathscr { Y }}\left(\mathscr{M}^{\prime}, 0^{\prime}\right)$, we get four vectorial formal identities, each one implying the remaining three:

$$
\left\{\begin{aligned}
r^{\prime}(h(t), \bar{h}(\tau)) \equiv b(t, \tau) r(t, \tau), & & r^{\prime}(h(t), \bar{h}(\tau)) \equiv \bar{c}(\tau, t) \bar{r}(\tau, t), \\
\bar{r}^{\prime}(\bar{h}(\tau), h(t)) \equiv \bar{b}(\tau, t) \bar{r}(\tau, t), & & \bar{r}^{\prime}(\bar{h}(\tau), h(t)) \equiv c(t, \tau) r(t, \tau) .
\end{aligned}\right.
$$

Here, we have set $c(t, \tau):=\bar{b}(\tau, t) a(t, \tau)$.
These identities are independent of the choice of local coordinates and of local complex defining equations for $(M, 0)$ and for $\left(M^{\prime}, 0^{\prime}\right)$. Since $h$ is not a true point-map, we write $h:(M, 0) \rightarrow_{\mathscr{F}}\left(M^{\prime}, 0^{\prime}\right)$, the index $\mathscr{F}$ being the initial of Formal. If $h$ is convergent, it is a true point-map from a neighborhood of 0 in $M$ to a neighborhood of $0^{\prime}$ in $M^{\prime}$.


If $h$ is holomorphic in a polydisc $\Delta_{\rho_{1}}^{n}, \rho_{1}>0$, its extrinsic complexification $h^{c}$ sends both the $n$-dimensional coordinate spaces $\{t=$ cst. $\}$ and $\{\tau=$ cst. $\}$ to the $n^{\prime}$-dimensional coordinate spaces $\left\{t^{\prime}=\operatorname{cst}.\right\}$ and $\left\{\tau^{\prime}=\right.$ cst. $\}$.

Equivalently, $h^{c}$ maps complexified (conjugate) Segre varieties of the source to complexified (conjugate) Segre varieties of the target. Some strong rigidity properties are due to the fact that $h^{c}=(h, \bar{h})$ must respect the two pairs of Segre foliations.

The most important rigidity feature, called the reflection principle ${ }^{3}$, says that the smoothness of $M, M^{\prime}$ governs the smoothness of $h$ :

- suppose that $M$ and $M^{\prime}$ are real analytic and that $h(t) \in \mathbb{C} \llbracket t \rrbracket^{n^{\prime}}$ is only formal; statement: under suitable assumptions, $h(t) \in \mathbb{C}\{t\}^{n^{\prime}}$ is in fact convergent.
- suppose that $M$ and $M^{\prime}$ are real algebraic and that $h(t) \in \mathbb{C} \llbracket t \rrbracket^{n^{\prime}}$ is only formal; statement: under suitable assumptions, $h(t)$ is complex algebraic.

After a mathematical phenomenon has been observed in a special, well understood situation, the research has to focus attention on the finest, the

[^2]most adequate, the necessary and sufficient conditions insuring it to hold true.

In this section, we aim to expose various possible assumptions for the reflection principle to hold. Our goal is to provide a synthesis by gathering various nondegeneracy assumptions which imply reflection. For more about history, for other results, for complements and for different points of view we refer to [Pi1975, Le1977, We1977, We1978, Pi1978, DF1978, DW1980, DF1988, BR1988, BR1990, DP1993, DP1995, DP1998, BER1999, Sh2000, BER2000, Me2001a, Me2002, Hu2001, Sh2003, DP2003, Ro2003, MMZ2003b, ER2004, Me2005].

The main theorems will be presented in $\S 3.19$ and in $\S 3.22$ below, after a long preliminary. In these results, $M$ will always be assumed to be minimal at the origin. Corollary 2.17 says already how to use concretely this assumption: to show the convergence or the algebraicity of a formal CR mapping $h:(M, 0) \mapsto_{\mathscr{F}}\left(M^{\prime}, 0^{\prime}\right)$, it suffices to establish that for every $k \in \mathbb{N}$, the formal maps $z_{(k)} \longmapsto \mathscr{F} h\left(\pi_{t}\left(\underline{\Gamma}_{k}\left(z_{(k)}\right)\right)\right)$ are convergent or algebraic.

Before surveying recent results about the reflection principle (without any indication of proof), we have to analyze thoroughly the geometry of the target $\mathscr{M}^{\prime}$ and to present the nondegeneracy conditions both on $\mathscr{M}^{\prime}$ and on $h$. Of course, everything will also be meaningful for sufficiently smooth ( $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa}$ ) local CR mappings, by considering Taylor series.

These conditions are classical in local analytic geometry and they may already be illustrated here with a plain formal map $h(t) \in \mathbb{C} \llbracket t \rrbracket^{n^{\prime}}$, not necessarily being CR.

Definition 3.2. A formal power series mapping $h:\left(\mathbb{C}^{n}, 0\right) \mapsto_{\mathscr{F}}\left(\mathbb{C}^{n^{\prime}}, 0^{\prime}\right)$ with components $h_{i^{\prime}}(t) \in \mathbb{C} \llbracket t \rrbracket, i^{\prime}=1, \ldots, n^{\prime}$, is called
(1) invertible if $n^{\prime}=n$ and $\operatorname{det}\left(\left[\partial h_{i_{1}} / \partial t_{i_{2}}\right](0)\right)_{1 \leqslant i_{1}, i_{2} \leqslant n} \neq 0$;
(2) submersive if $n \geqslant n^{\prime}$ and there exist integers $1 \leqslant i(1)<\cdots<$ $i\left(n^{\prime}\right) \leqslant n$ such that $\operatorname{det}\left(\left[\partial h_{i_{1}^{\prime}} / \partial t_{i\left(i_{2}^{\prime}\right)}\right](0)\right)_{1 \leqslant i_{1}^{\prime}, i_{2}^{\prime} \leqslant n^{\prime}} \neq 0$;
(3) finite if the ideal generated by the components $h_{1}(t), \ldots, h_{n^{\prime}}(t)$ is of finite codimension in $\mathbb{C} \llbracket t \rrbracket$; this implies $n^{\prime} \geqslant n$;
(4) dominating if $n \geqslant n^{\prime}$ and there exist integers $1 \leqslant i(1)<\cdots<$ $i\left(n^{\prime}\right) \leqslant n$ such that $\operatorname{det}\left(\left[\partial h_{i_{1}^{\prime}} / \partial t_{i\left(i_{2}^{\prime}\right)}\right](t)\right)_{1 \leqslant i_{1}^{\prime}, i_{2}^{\prime} \leqslant n^{\prime}} \neq 0$ in $\mathbb{C} \llbracket t \rrbracket$;
(5) transversal if there does not exist a nonzero power series $G\left(t_{1}^{\prime}, \ldots, t_{n^{\prime}}^{\prime}\right) \in \mathbb{C} \llbracket t_{1}^{\prime}, \ldots, t_{n^{\prime}}^{\prime} \rrbracket$ such that $G\left(h_{1}(t), \ldots, h_{n^{\prime}}(t)\right) \equiv 0$ in $\mathbb{C} \llbracket t \rrbracket$.

It is elementary to see that invertibility implies submersiveness which implies domination. Furthermore, if a formal power series is either invertible,
submersive or dominating, then it is transversal. Philosophically, the "distance" between finite and dominating or transversal is large, whereas the "distance" between invertible and submersive or finite is "small".
3.3. Jets of Segre varieties and Segre mapping. The target $M^{\prime}$ concentrates all geometric conditions that are central for the reflection principle. With respect to $\mathscr{M}^{\prime}$, the complexified conjugate Segre variety associated to a fixed $t^{\prime}$ is $\mathscr{\mathscr { L }}_{t^{\prime}}^{\prime}:=\left\{\left(\zeta^{\prime}, \xi^{\prime}\right) \in \mathbb{C}^{n^{\prime}}: \xi^{\prime}=\Theta^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)\right\}$. Here, $\zeta^{\prime}$ is a parametrizing variable. For $k^{\prime} \in \mathbb{N}$, define the morphism of $k^{\prime}$-th jets of complexified conjugate Segre varieties by:

$$
\varphi_{k^{\prime}}^{\prime}\left(\zeta^{\prime}, t^{\prime}\right):=J_{\tau^{\prime}}^{k^{\prime}} \underline{\mathscr{S}}_{t^{\prime}}^{\prime}:=\left(\zeta^{\prime},\left(\frac{1}{\beta^{\prime}!} \partial_{\zeta^{\prime}}^{\beta^{\prime}} \Theta_{j^{\prime}}^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)\right)_{1 \leqslant j^{\prime} \leqslant d^{\prime}, \beta^{\prime} \in \mathbb{N}^{m^{\prime}}, \mid \beta^{\prime} \leqslant k^{\prime}}\right) .
$$

It takes values in $\mathbb{C}^{m^{\prime}+N_{d^{\prime}}, m^{\prime}, k^{\prime}}$, with $N_{d^{\prime}, m^{\prime}, k^{\prime}}:=d^{\prime} \frac{\left(m^{\prime}+k^{\prime}\right)!}{m^{\prime}!k^{!}!}$. If $k_{1}^{\prime} \leqslant k_{2}^{\prime}$, we have of course $\pi_{k_{2}^{\prime}, k_{1}^{\prime}} \circ \varphi_{k_{2}^{\prime}}^{\prime}=\varphi_{k_{1}^{\prime}}^{\prime}$.

As observed in [DW1980], the properties of this morphism govern the various reflection principles. We shall say ([Me2004a, Me2005]) that $M^{\prime}$ (or equivalently $\mathscr{M}^{\prime}$ ) is:
(nd1) Levi non-degenerate at the origin if $\varphi_{1}^{\prime}$ is of rank $m^{\prime}+n^{\prime}$ at $\left(\zeta^{\prime}, t^{\prime}\right)=$ $\left(0^{\prime}, 0^{\prime}\right)$;
(nd2) finitely nondegenerate at the origin if there exists an integer $\ell_{0}^{\prime}$ such that $\varphi_{k^{\prime}}^{\prime}$ is of rank $n^{\prime}+m^{\prime}$ at $\left(\zeta^{\prime}, t^{\prime}\right)=\left(0^{\prime}, 0^{\prime}\right)$, for $k^{\prime}=\ell_{0}^{\prime}$, hence for all $k^{\prime} \geqslant \ell_{0}^{\prime}$;
(nd3) essentially finite at the origin if there exists an integer $\ell_{0}^{\prime}$ such that $\varphi_{k^{\prime}}^{\prime}$ is a finite holomorphic map at $\left(\zeta^{\prime}, t^{\prime}\right)=\left(0^{\prime}, 0^{\prime}\right)$, for $k^{\prime}=\ell_{0}^{\prime}$, hence for all $k^{\prime} \geqslant \ell_{0}^{\prime}$;
(nd4) Segre nondegenerate at the origin if there exists an integer $\ell_{0}^{\prime}$ such that the restriction of $\varphi_{k^{\prime}}^{\prime}$ to the complexified Segre variety $\mathscr{S}_{0}^{\prime}$ (of complex dimension $m^{\prime}$ ) is of generic rank $m^{\prime}$, for $k^{\prime}=\ell_{0}^{\prime}$, hence for all $k^{\prime} \geqslant \ell_{0}^{\prime}$;
(nd5) holomorphically nondegenerate if there exists an integer $\ell_{0}^{\prime}$ such that the map $\varphi_{k^{\prime}}^{\prime}$ is of maximal possible generic rank, equal to $m^{\prime}+n^{\prime}$, for $k^{\prime}=\ell_{0}^{\prime}$, hence for all $k^{\prime} \geqslant \ell_{0}^{\prime}$.

Theorem 3.4. ([Me2004a]) These five conditions are biholomorphically invariant and: (nd1) $\Rightarrow$ (nd2) $\Rightarrow$ (nd3) $\Rightarrow$ (nd4) $\Rightarrow$ (nd5).

Being not punctual, the last condition (nd5) is the finest: as every condition of maximal generic rank, it propagates from any small open subet to big connected open sets, thanks to the principle of analytic continuation. Notably, if a connected real analytic $M^{\prime}$ is holomorphically nondegenerate
"at" a point, it is automatically holomorphically nondegenerate "at" every point ([St1996, BER1999, Me2004a]).

To explain the (crucial) biholomorphic invariance of the jet map $\varphi_{k^{\prime}}^{\prime}$, consider a local biholomorphism $t^{\prime} \mapsto h^{\prime}\left(t^{\prime}\right)=t^{\prime \prime}$, where $t^{\prime}, t^{\prime \prime} \in \mathbb{C}^{n^{\prime}}$, that fixes the origin, $h_{i^{\prime}}^{\prime}\left(t^{\prime}\right) \in \mathbb{C}\left\{t^{\prime}\right\}, h_{i^{\prime}}^{\prime}\left(0^{\prime}\right)=0^{\prime}$, for $i^{\prime}=1, \ldots, n^{\prime}$. Splitting the coordinates $t^{\prime \prime}=\left(z^{\prime \prime}, w^{\prime \prime}\right) \in \mathbb{C}^{m^{\prime}} \times \mathbb{C}^{d^{\prime}}$, the image $M^{\prime \prime}$ may be similarly represented by $\bar{w}^{\prime \prime}=\Theta^{\prime \prime}\left(\bar{z}^{\prime \prime}, t^{\prime \prime}\right)$ and there exists a $d^{\prime} \times d^{\prime}$ matrix $b^{\prime}\left(t^{\prime}, \tau^{\prime}\right)$ of local holomorphic functions such that

$$
r^{\prime \prime}\left(h^{\prime}\left(t^{\prime}\right), \bar{h}^{\prime}\left(\tau^{\prime}\right)\right) \equiv b^{\prime}\left(t^{\prime}, \tau^{\prime}\right) r^{\prime}\left(t^{\prime}, \tau^{\prime}\right)
$$

in $\mathbb{C}\left\{t^{\prime}, \tau^{\prime}\right\}^{d^{\prime}}$, where $r_{j^{\prime}}^{\prime}\left(t^{\prime}, \tau^{\prime}\right):=\xi_{j^{\prime}}^{\prime}-\Theta_{j^{\prime}}^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)$ and $r_{j^{\prime}}^{\prime \prime}\left(t^{\prime \prime}, \tau^{\prime \prime}\right):=\xi_{j^{\prime}}^{\prime \prime}-$ $\Theta_{j^{\prime}}^{\prime \prime}\left(\zeta^{\prime \prime}, t^{\prime \prime}\right)$, for $j^{\prime}=1, \ldots, d^{\prime}$. Setting $h^{\prime}\left(t^{\prime}\right):=\left(f^{\prime}\left(t^{\prime}\right), g^{\prime}\left(t^{\prime}\right)\right) \in \mathbb{C}\left\{t^{\prime}\right\}^{m^{\prime}} \times$ $\mathbb{C}\left\{t^{\prime}\right\}^{d^{\prime}}$ and replacing $\xi^{\prime}$ by $\Theta^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)$ in the above equation, the right hand side vanishes identically (since $r^{\prime}\left(t^{\prime}, \tau^{\prime}\right)=\xi^{\prime}-\Theta^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)$ by definition) and we obtain the following formal identity in $\mathbb{C}\left\{\zeta^{\prime}, t^{\prime}\right\}^{d^{\prime}}$ :

$$
\bar{g}^{\prime}\left(\zeta^{\prime}, \Theta^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)\right) \equiv \Theta^{\prime \prime}\left(\bar{f}^{\prime}\left(\zeta^{\prime}, \Theta^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)\right), h^{\prime}\left(t^{\prime}\right)\right)
$$

Some algebraic manipulations conduct to the following.
Lemma 3.5. ([Me2004a, Me2005]) For every $j^{\prime}=1, \ldots, d^{\prime}$ and every $\beta^{\prime} \in$ $\mathbb{N}^{m^{\prime}}$, there exists a universal rational map $Q_{j^{\prime}, \beta^{\prime}}^{\prime}$ whose expression depends neither on $\mathscr{M}^{\prime}$, nor on $h^{\prime}$, nor on $\mathscr{M}^{\prime \prime}$, such that the following identities in $\mathbb{C}\left\{\zeta^{\prime}, t^{\prime}\right\}$ hold true :

$$
\begin{aligned}
& \frac{1}{\beta^{\prime}!} \frac{\partial^{\left|\beta^{\prime}\right|} \mid \Theta_{j^{\prime}}^{\prime \prime}}{\partial\left(\zeta^{\prime \prime}\right)^{\beta^{\prime}}}\left(\bar{f}^{\prime}\left(\zeta^{\prime}, \Theta^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)\right), h^{\prime}\left(t^{\prime}\right)\right) \equiv \\
& \equiv Q_{j^{\prime}, \beta^{\prime}}^{\prime}\left(\left(\partial_{\zeta^{\prime}}^{\beta_{1}^{\prime}} \Theta_{j_{1}^{\prime}}^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)\right)_{1 \leqslant j_{1}^{\prime} \leqslant d^{\prime},\left|\beta_{1}^{\prime}\right| \leqslant\left|\beta^{\prime}\right|},\left(\partial_{\tau^{\prime}}^{\alpha_{1}^{\prime}} \bar{h}_{i_{1}^{\prime}}^{\prime}\left(\zeta^{\prime}, \Theta^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)\right)\right)_{1 \leqslant i_{1}^{\prime} \leqslant n^{\prime},\left|\alpha_{1}^{\prime}\right| \leqslant\left|\beta^{\prime}\right|}\right) \\
& =: R_{j^{\prime}, \beta^{\prime}}^{\prime}\left(\zeta^{\prime},\left(\partial_{\zeta^{\prime}}^{\beta_{1}^{\prime}} \Theta_{j_{1}^{\prime}}\left(\zeta^{\prime}, t^{\prime}\right)\right)_{1 \leqslant j_{1}^{\prime} \leqslant d^{\prime},\left|\beta_{1}^{\prime}\right| \leqslant\left|\beta^{\prime}\right|}\right),
\end{aligned}
$$

where the last line defines $R_{j^{\prime}, \beta^{\prime}}^{\prime}$ by forgetting the jets of $\bar{h}^{\prime}$. Here, the $Q_{j^{\prime}, \beta^{\prime}}^{\prime}$ are holomorphic in a neighborhood of the constant jet

$$
\left(\left(\partial_{\zeta^{\prime}}^{\beta_{1}^{\prime}} \Theta_{j_{1}^{\prime}}^{\prime}(0,0)\right)_{1 \leqslant j_{1}^{\prime} \leqslant d^{\prime},\left|\beta_{1}^{\prime}\right| \leqslant\left|\beta^{\prime}\right|},\left(\partial_{\tau^{\prime}}^{\alpha_{1}^{\prime}} \bar{h}_{i_{1}^{\prime}}^{\prime}(0,0)\right)_{1 \leqslant i_{1}^{\prime} \leqslant n,\left|\alpha_{1}^{\prime}\right| \leqslant\left|\beta^{\prime}\right|}\right) .
$$

Some symmetric relations hold after replacing $\Theta^{\prime}, \Theta^{\prime \prime}, \zeta^{\prime}, t^{\prime}, \bar{f}^{\prime}, h^{\prime}$ by $\bar{\Theta}^{\prime}$, $\bar{\Theta}^{\prime \prime}, z^{\prime}, \tau^{\prime}, f^{\prime}, \bar{h}^{\prime}$.

The existence of $R_{j^{\prime}, \beta^{\prime}}^{\prime}$ says that the following diagram is commutative :
where the biholomorphic map $R_{k^{\prime}}^{\prime}\left(\left(h^{\prime}\right)^{c}\right)$, which depends on $\left(h^{\prime}\right)^{c}$, is defined by its components $R_{j^{\prime}, \beta^{\prime}}^{\prime}$ for $j^{\prime}=1, \ldots, d^{\prime}$ and $\left|\beta^{\prime}\right| \leqslant k^{\prime}$. Thanks to the invertibility of $h^{\prime}$, the map $R_{k^{\prime}}^{\prime}\left(\left(h^{\prime}\right)^{c}\right)$ is also checked to be invertible, and then the invariance of the five nondegeneracy conditions (nd1), (nd2), (nd3), (nd4) and (nd5) is easily established ([Me2004a]).

We now present the Segre mapping of $M^{\prime}$. By developing the series $\Theta_{j^{\prime}}^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)$ in powers of $\zeta^{\prime}$, we may write the equations of $\mathscr{M}^{\prime}$ under the form $\xi_{j^{\prime}}^{\prime}=\sum_{\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}}\left(\zeta^{\prime}\right)^{\gamma^{\prime}} \Theta_{j^{\prime}, \gamma^{\prime}}^{\prime}\left(t^{\prime}\right)$ for $j^{\prime}=1, \ldots, d^{\prime}$. In terms of such a development, the infinite Segre mapping of $M^{\prime}$ is defined to be the mapping

$$
\mathscr{Q}_{\infty}^{\prime}: \mathbb{C}^{n^{\prime}} \ni t^{\prime} \longmapsto\left(\Theta_{j^{\prime}, \gamma^{\prime}}^{\prime}\left(t^{\prime}\right)\right)_{1 \leqslant j^{\prime} \leqslant d^{\prime}, \gamma^{\prime} \in \mathbb{N}^{m^{\prime}}} \in \mathbb{C}^{\infty} .
$$

Let $k^{\prime} \in \mathbb{N}$. For finiteness reasons, it is convenient to truncate this infinite collection and to define the $k^{\prime}$-th Segre mapping of $M^{\prime}$ by

$$
\mathscr{Q}_{k^{\prime}}^{\prime}: \quad \mathbb{C}^{n^{\prime}} \ni t^{\prime} \longmapsto\left(\Theta_{j^{\prime}, \gamma^{\prime}}^{\prime}\left(t^{\prime}\right)\right)_{1 \leqslant j^{\prime} \leqslant d^{\prime}, \mid \gamma^{\prime} \leqslant k^{\prime}} \in \mathbb{C}^{N_{d^{\prime}, n^{\prime}, k^{\prime}}},
$$

where $N_{d^{\prime}, n^{\prime}, k^{\prime}}=d^{\prime} \frac{\left(n^{\prime}+k^{\prime}\right)!}{n^{\prime}!k^{\prime}!}$. If $k_{2}^{\prime} \geqslant k_{1}^{\prime}$, we have $\pi_{k_{2}^{\prime}, k_{1}^{\prime}}\left[\mathscr{Q}_{k_{2}^{\prime}}^{\prime}\left(t^{\prime}\right)\right]=\mathscr{Q}_{k_{1}^{\prime}}^{\prime}\left(t^{\prime}\right)$. One verifies ([Me2004a]) the following characterizations.
(nd1) $M^{\prime}$ is Levi non-degenerate at the origin if and only if $\mathscr{Q}_{1}^{\prime}$ is of rank $n^{\prime}$ at $t^{\prime}=0^{\prime}$.
(nd2) $M^{\prime}$ is finitely nondegenerate at the origin if and only if there exists an integer $\ell_{0}^{\prime}$ such that $\mathscr{Q}_{k^{\prime}}^{\prime}$ is of rank $n^{\prime}$ at $t^{\prime}=0^{\prime}$, for all $k^{\prime} \geqslant \ell_{0}^{\prime}$.
(nd3) $M^{\prime}$ is essentially finite at the origin if there exists an integer $\ell_{0}^{\prime}$ such that $\mathscr{Q}_{k^{\prime}}^{\prime}$ is a finite holomorphic map at $t^{\prime}=0^{\prime}$, for all $k^{\prime} \geqslant \ell_{0}^{\prime}$.
(nd4) $M^{\prime}$ is Segre nondegenerate at the origin if there exists an integer $\ell_{0}^{\prime}$ such that the restriction of $\mathscr{Q}_{k^{\prime}}^{\prime}$ to the complexified Segre variety $\mathscr{S}_{0^{\prime}}^{\prime}$ (of complex dimension $m^{\prime}$ ) is of generic rank $m^{\prime}$, for all $k^{\prime} \geqslant \ell_{0}^{\prime}$.
(nd5) $M^{\prime}$ is holomorphically nondegenerate if there exists an integer $\ell_{0}^{\prime}$ such that the map $\mathscr{Q}_{k^{\prime}}^{\prime}$ is of maximal possible generic rank, equal to $n^{\prime}$, for all $k^{\prime} \geqslant \ell_{0}^{\prime}$.
3.6. Essential holomorphic dimension and Levi multitype. Assume now that $M^{\prime}$ is not nececessarily local, but connected. Denote by $\ell_{M^{\prime}}^{\prime}$ the smallest integer $k^{\prime}$ such that the generic rank of the jet mappings $\left(t^{\prime}, \tau^{\prime}\right) \mapsto J_{\tau^{\prime}}^{k^{\prime}} \mathscr{S}_{t^{\prime}}^{\prime}$ does not increase after $k^{\prime}$ and denote by $m^{\prime}+n_{M^{\prime}}^{\prime} \leqslant m^{\prime}+n^{\prime}$ the (maximal) generic rank of $\left(t^{\prime}, \tau^{\prime}\right) \mapsto J_{\tau^{\prime}}^{\ell^{\prime}} \underline{\mathscr{S}_{t^{\prime}}^{\prime}}$. Since $w^{\prime} \mapsto \Theta^{\prime}\left(\zeta^{\prime}, z^{\prime}, w^{\prime}\right)$ is of rank $d^{\prime}$ according to Theorem 2.5, the (generic) rank of the zero-th order jet map satisfies

$$
\operatorname{genrk}_{\mathbb{C}}\left(\left(t^{\prime}, \tau^{\prime}\right) \mapsto J_{\tau^{\prime}}^{0} \underline{\mathscr{S}}_{t^{\prime}}^{\prime}=\left(\zeta^{\prime}, \Theta^{\prime}\left(\zeta^{\prime}, z^{\prime}, w^{\prime}\right)\right)\right)=m^{\prime}+d^{\prime}=n^{\prime} .
$$

Thus, $d^{\prime} \leqslant n_{M^{\prime}}^{\prime} \leqslant n^{\prime}$. It is natural to call $n_{M^{\prime}}^{\prime}$ the essential holomorphic dimension of $M^{\prime}$ because of the following.

Proposition 3.7. ([Me2001a, Me2004a]) Locally in a neighborhood of a Zariski-generic point $p^{\prime} \in M^{\prime}$, the generic submanifold $M^{\prime}$ is biholomorphically equivalent to the product $\underline{M}_{p^{\prime}}^{\prime} \times \Delta^{n^{\prime}-n_{M^{\prime}}^{\prime}}$, of a generic submanifold $\underline{M}_{p^{\prime}}^{\prime}$ of codimension $d^{\prime}$ in $\mathbb{C}^{n_{M^{\prime}}^{\prime}}$ by a complex polydisc $\Delta^{n^{\prime}-n_{M^{\prime}}^{\prime}}$.

Generally speaking, we may define $\lambda_{0, M^{\prime}}^{\prime}:=\operatorname{genrk}_{\mathbb{C}}\left(\left(t^{\prime}, \tau^{\prime}\right) \mapsto J_{\tau^{\prime}}^{0} \underline{\mathscr{S}}_{t^{\prime}}^{\prime}\right)-$ $m^{\prime}=d^{\prime}$ and for every $k^{\prime}=1, \ldots, \ell_{M^{\prime}}^{\prime}$,

$$
\lambda_{k^{\prime}, M^{\prime}}^{\prime}:=\operatorname{genrk}_{\mathbb{C}}\left(\left(t^{\prime}, \tau^{\prime}\right) \mapsto J_{\tau^{\prime}}^{k^{\prime}} \underline{\mathscr{S}}_{t^{\prime}}^{\prime}\right)-\operatorname{genrk}_{\mathbb{C}}\left(\left(t^{\prime}, \tau^{\prime}\right) \mapsto J_{\tau^{\prime}}^{k^{\prime}-1} \underline{\mathscr{S}}_{t^{\prime}}^{\prime}\right)
$$

One verifies ([Me2004a]) that $\lambda_{1, M^{\prime}}^{\prime} \geqslant 1, \ldots, \lambda_{\ell_{M^{\prime}}^{\prime}, M^{\prime}}^{\prime} \geqslant 1$. With these definitions, we have the relations

$$
\operatorname{genrk}_{\mathbb{C}}\left(\left(t^{\prime}, \tau^{\prime}\right) \mapsto J_{\tau^{\prime}}^{k^{\prime}} \mathscr{S}_{t^{\prime}}^{\prime}\right)=m^{\prime}+\lambda_{0, M^{\prime}}^{\prime}+\lambda_{1, M^{\prime}}^{\prime}+\cdots+\lambda_{k^{\prime}, M^{\prime}}^{\prime}
$$

for $k^{\prime}=0,1, \ldots, \ell_{M^{\prime}}^{\prime}$ and
$\operatorname{genrk}_{\mathbb{C}}\left(\left(t^{\prime}, \tau^{\prime}\right) \mapsto J_{\tau^{\prime}}^{k^{\prime}} \underline{\mathscr{S}}_{t^{\prime}}^{\prime}\right)=m^{\prime}+d^{\prime}+\lambda_{1, M^{\prime}}^{\prime}+\cdots+\lambda_{\ell_{M^{\prime}}, M^{\prime}}^{\prime}=m^{\prime}+n_{M^{\prime}}^{\prime}$,
for all $k^{\prime} \geqslant \ell_{M^{\prime}}^{\prime}$. It follows that

$$
\ell_{M^{\prime}}^{\prime} \leqslant \lambda_{1, M^{\prime}}^{\prime}+\cdots+\lambda_{\ell_{M^{\prime}}^{\prime}, M^{\prime}}^{\prime}=n_{M^{\prime}}^{\prime}-d^{\prime} \leqslant m^{\prime}
$$

Theorem 3.8. ([Me2004a]) Let $M^{\prime}$ be a connected real algebraic or analytic generic submanifold in $\mathbb{C}^{n^{\prime}}$ of codimension $d^{\prime} \geqslant 1$ and of CR dimension $m^{\prime}=n^{\prime}-d^{\prime} \geqslant 1$. Then there exist well defined integers $n_{M^{\prime}}^{\prime} \geqslant d^{\prime}, \ell_{M^{\prime}}^{\prime} \geqslant 0$, $\lambda_{0, M^{\prime}}^{\prime} \geqslant 1, \lambda_{1, M^{\prime}}^{\prime} \geqslant 1, \ldots, \lambda_{{\ell^{\prime}}_{\prime}^{\prime}, M^{\prime}}^{\prime} \geqslant 1$ and a proper real algebraic or analytic subvariety $E^{\prime}$ of $M^{\prime}$ such that for every point $p^{\prime} \in M^{\prime} \backslash E^{\prime}$ and for every system of coordinates $\left(z^{\prime}, w^{\prime}\right)$ vanishing at $p^{\prime}$ in which $M^{\prime}$ is represented by defining equations $\bar{w}_{j^{\prime}}=\Theta_{j^{\prime}}^{\prime}\left(\bar{z}^{\prime}, t^{\prime}\right), j^{\prime}=1, \ldots, d^{\prime}$, then the following four properties hold:

$$
\text { - } \lambda_{0, M^{\prime}}^{\prime}=d^{\prime}, d^{\prime} \leqslant n_{M^{\prime}}^{\prime} \leqslant n^{\prime} \text { and } \ell_{M^{\prime}}^{\prime} \leqslant n_{M^{\prime}}^{\prime}-d^{\prime}
$$

- For every $k^{\prime}=0,1, \ldots, \ell_{M^{\prime}}^{\prime}$, the mapping of $k^{\prime}$-th order jets of the conjugate complexified Segre varieties $\left(t^{\prime}, \tau^{\prime}\right) \mapsto J_{\tau^{\prime}}^{k^{\prime}} \underline{\mathscr{L}}_{t^{\prime}}^{\prime}$ is of rank equal to $m^{\prime}+\lambda_{0, M^{\prime}}^{\prime}+\cdots+\lambda_{k^{\prime}, M^{\prime}}^{\prime}$ at $\left(t_{p^{\prime}}^{\prime}, t_{p^{\prime}}^{\prime}\right)=\left(0^{\prime}, 0^{\prime}\right)$.
- $n_{M^{\prime}}^{\prime}=d^{\prime}+\lambda_{1, M^{\prime}}^{\prime}+\cdots+\lambda_{\ell_{M^{\prime}}^{\prime}, M^{\prime}}^{\prime}$ and for every $k^{\prime} \geqslant \ell_{M^{\prime}}^{\prime}$, the mapping of $k^{\prime}$-th order jets of the conjugate complexified Segre varieties $\left(t^{\prime}, \tau^{\prime}\right) \mapsto J_{\tau^{\prime}}^{k^{\prime}} \underline{\mathscr{S}}_{t^{\prime}}^{\prime}$ is of rank equal to $n_{M^{\prime}}^{\prime}$ at $\left(0^{\prime}, 0^{\prime}\right)$.
- There exists a local complex algebraic or analytic change of coordinates $t^{\prime \prime}=h^{\prime}\left(t^{\prime}\right)$ fixing $p^{\prime}$ such that the image $M_{p^{\prime}}^{\prime \prime}:=h^{\prime}\left(M^{\prime}\right)$ is locally in a neighborhood of $p^{\prime}$ the product $\underline{M}_{p^{\prime}}^{\prime \prime} \times \Delta^{n^{\prime}-n_{M^{\prime}}^{\prime}}$ of a real algebraic or analytic generic submanifold of codimension $d^{\prime}$ in $\mathbb{C}^{n_{M^{\prime}}^{\prime}}$ by a complex polydisc $\Delta^{n^{\prime}-n_{M^{\prime}}^{\prime}}$. Furthermore, at the central point $\underline{p}^{\prime} \in \underline{M}_{p^{\prime}}^{\prime \prime} \subset \mathbb{C}^{n^{\prime}}{ }_{M^{\prime}}$, the generic submanifold $\underline{M}_{p^{\prime}}^{\prime \prime}$ is $\ell_{M^{\prime}}^{\prime}$-finitely nondegenerate, hence in particular its essential holomorphic dimension $n_{\underline{M}_{p^{\prime}}^{\prime \prime}}^{\prime \prime}$ coincides with $n_{M^{\prime}}^{\prime}$.
In particular, $M^{\prime}$ is holomorphically nondegenerate if and only if $n_{M^{\prime}}^{\prime}=$ $n^{\prime}$ and in this case, $M^{\prime}$ is finitely nondegenerate at every point of the Zariskiopen subset $M^{\prime} \backslash E^{\prime}$.
3.9. CR-horizontal nondegeneracy conditions. As in §3.1, let $h=h(t) \in$ $\mathbb{C} \llbracket t \rrbracket^{n^{\prime}}$ be a formal CR mapping $(M, 0) \rightarrow_{\mathscr{F}}\left(M^{\prime}, 0^{\prime}\right)$. Decompose $h(t)=$ $(f(t), g(t)) \in \mathbb{C} \llbracket t \rrbracket^{m^{\prime}} \times \mathbb{C} \llbracket t \rrbracket^{d^{\prime}}$, as in the splitting $t^{\prime}=\left(z^{\prime}, w^{\prime}\right) \in \mathbb{C}^{m^{\prime}} \times$ $\mathbb{C}^{d^{\prime}}$. Replacing $w$ by $\bar{\Theta}(z, \tau)$ in the fundamental identity $\bar{r}^{\prime}(\bar{h}(\tau), h(t)) \equiv$ $\bar{b}(\tau, t) \bar{r}(\tau, t)$, the right hand side vanishes identically (since $\bar{r}(\tau, t)=w-$ $\bar{\Theta}(z, \tau)$ by definition), and we get a formal identity in $\mathbb{C} \llbracket z, \tau \rrbracket^{d^{\prime}}$ :

$$
g(z, \bar{\Theta}(z, \tau)) \equiv \bar{\Theta}^{\prime}(f(z, \bar{\Theta}(z, \tau)), \bar{h}(\tau)) .
$$

Setting $\tau:=0$, we get $g(z, \bar{\Theta}(z, 0)) \equiv \bar{\Theta}^{\prime}(f(z, \bar{\Theta}(z, 0)), 0)$. In other words, $\left.h\right|_{\mathscr{S}_{0}}$ maps $\mathscr{S}_{0}$ formally to $\mathscr{S}_{0^{\prime}}^{\prime}$. The restriction $\left.h\right|_{\mathscr{O}_{0}}$ coincides with the formal map:

$$
\mathbb{C}^{m} \ni z \longmapsto \mathscr{F}\left(f(z, \bar{\Theta}(z, 0)), \bar{\Theta}^{\prime}(f(z, \bar{\Theta}(z, 0)), 0)\right) \in \mathbb{C}^{m^{\prime}} \times \mathbb{C}^{d^{\prime}}
$$

The rank properties of this formal map are the same as those of its $C R$ horizontal part:

$$
\mathbb{C}^{m} \ni z \longmapsto \mathscr{F} f(z, \bar{\Theta}(z, 0)) \in \mathbb{C}^{m^{\prime}} .
$$

The formal CR mapping $h$ is said ([Me2004a]) to be:
(cr1) $C R$-invertible at the origin if $m^{\prime}=m$ and if its CR-horizontal part is a formal equivalence at $z=0$;
(cr2) $C R$-submersive at the origin if $m^{\prime} \leqslant m$ and if its CR-horizontal part is a formal submersion at $z=0$;
(cr3) CR-finite at the origin if $m^{\prime}=m$ and if its CR-horizontal part is a finite formal map at $z=0$, namely the quotient ring $\mathbb{C} \llbracket z \rrbracket /\left(f_{k^{\prime}}(z, \bar{\Theta}(z, 0))_{1 \leqslant k^{\prime} \leqslant m^{\prime}}\right)$ is finite-dimensional (the requirement $m^{\prime}=m$ is necessary for the reflection principle below);
(cr4) $C R$-dominating at the origin if $m^{\prime} \leqslant m$ and if there exist integers $1 \leqslant k(1)<\cdots<k\left(m^{\prime}\right) \leqslant m$ such that the determinant $\operatorname{det}\left(\left[\partial \phi_{k_{1}^{\prime}} / \partial z_{k\left(k_{2}^{\prime}\right)}\right](z)\right)_{1 \leqslant k_{1}^{\prime}, k_{2}^{\prime} \leqslant m^{\prime}} \not \equiv 0$ does not vanish identically in $\mathbb{C} \llbracket z \rrbracket$, where $\phi_{k^{\prime}}(z):=f_{k^{\prime}}(z, \bar{\Theta}(z, 0))$;
(cr5) CR-transversal at the origin if there does not exist a nonzero formal power series $F^{\prime}\left(f_{1}, \ldots, f_{m^{\prime}}\right) \in \mathbb{C} \llbracket f_{1}, \ldots, f_{m^{\prime}} \rrbracket$ such that $F^{\prime}\left(\phi_{1}(z), \ldots, \phi_{m^{\prime}}(z)\right) \equiv 0$ in $\mathbb{C} \llbracket z \rrbracket$, where $\phi_{k^{\prime}}(z):=f_{k^{\prime}}(z, \bar{\Theta}(z, 0))$.
One verifies ([Me2004a]) biholomorphic invariance and the four implications:

$$
(\operatorname{cr} 1) \Rightarrow(\operatorname{cr} 2) \Rightarrow(\operatorname{cr} 3) \Rightarrow(\mathrm{cr} 4) \Rightarrow(\mathrm{cr} 5),
$$

provided that $m^{\prime}=m$ in the second and in the third. By far, CRtransversality is the most general nondegeneracy condition.
3.10. Nondegeneracy conditions for CR mappings. This subsection explains how to synthetize the combinatorics of various formal reflection principles published in the last decade.

As in §3.1, let $h^{c}:(\mathscr{M}, 0) \rightarrow_{\mathscr{F}}\left(\mathscr{M}^{\prime}, 0\right)$ be a complexified formal CR mapping between two formal, analytic or algebraic complexified generic submanifolds of equations $0=r(t, \tau):=\xi-\Theta(\zeta, t)$ and $0=r^{\prime}\left(t^{\prime}, \tau^{\prime}\right):=$ $\xi^{\prime}-\Theta^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)$. By hypothesis, $r^{\prime}(h(t), \bar{h}(\tau)) \equiv b(t, \tau) r(t, \tau)$. Denoting $h=$ $(f, g) \in \mathbb{C}^{m^{\prime}} \times \mathbb{C}^{d^{\prime}}$, replacing $\xi$ by $\Theta(\zeta, t)$ in $r^{\prime}(h(t), \bar{h}(\tau)) \equiv b(t, \tau) r(t, \tau)$ and developing $\Theta^{\prime}(\bar{f}, h)=\sum_{\gamma^{\prime} \in \mathbb{N}^{\prime}} \bar{f}^{\gamma^{\prime}} \Theta_{\gamma^{\prime}}^{\prime}(h)$, we start with the following fundamental power series identity in $\mathbb{C} \llbracket \zeta, t \rrbracket^{d^{\prime}}$ :

$$
\begin{aligned}
\bar{g}(\zeta, \Theta(\zeta, t)) & \equiv \Theta^{\prime}(\bar{f}(\zeta, \Theta(\zeta, t)), h(t)) \\
& \equiv \sum_{\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}} \bar{f}(\zeta, \Theta(\zeta, t))^{\gamma^{\prime}} \Theta_{\gamma^{\prime}}^{\prime}(h(t)) .
\end{aligned}
$$

Consider the $m$ complex vector fields $\underline{\mathscr{L}}_{1}, \ldots, \underline{\mathscr{L}}_{m}$ tangent to $\mathscr{M}$ that were defined in §2.7. For every $\beta=\left(\beta_{1}, \ldots, \beta_{m}\right) \in \mathbb{N}^{m}$, define the multiple derivation $\mathscr{L}^{\beta}=\mathscr{L}_{1}^{\beta_{1}} \cdots \mathscr{L}_{m}^{\beta_{m}}$. Applying them to the above $d^{\prime}$ scalar equations, observing that they do not differentiate the variables $t=(z, w)$, we get, without writing the arguments:

$$
\underline{\mathscr{L}}^{\beta} \bar{g}_{j^{\prime}}-\sum_{\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}} \underline{\mathscr{L}}^{\beta}\left(\bar{f}^{\gamma^{\prime}}\right) \Theta_{j^{\prime}, \gamma^{\prime}}^{\prime}(h) \equiv 0,
$$

for all $\beta \in \mathbb{N}^{m}$, all $j^{\prime}=1, \ldots, d^{\prime}$ and all $(t, \tau) \in \mathscr{M}$.

Lemma 3.11. ([Me2004a, Me2005]) For every $i^{\prime}=1, \ldots, n^{\prime}$ and every $\beta \in \mathbb{N}^{m}$, there exists a polynomial $P_{i^{\prime}, \beta}$ in the jet $J_{\tau}^{|\beta|} \bar{h}(\tau)$ with coefficients being power series in $(t, \tau)$ which depend only on the defining functions $\xi_{j}-$ $\Theta_{j}(\zeta, t)$ of $\mathscr{M}$ and which can be computed by means of some combinatorial formula, such that

$$
\underline{\mathscr{L}}^{\beta} \bar{h}_{i^{\prime}}(\tau) \equiv P_{i^{\prime}, \beta}\left(t, \tau, J_{\tau}^{|\beta|} \bar{h}(\tau)\right) .
$$

Convention 3.12. Let $k, l \in \mathbb{N}$. On the complexification $\mathscr{M}$, equipped with either the coordinates $(z, \tau)$ or $(\zeta, t)$, which correspond to either replacing $w$ by $\bar{\Theta}(z, \tau)$ or $\xi$ by $\Theta(\zeta, t)$, we shall identify (notationally) a power series written under the complete form

$$
R\left(t, \tau, J^{k} h(t), J^{l} \bar{h}(\tau)\right)
$$

with a power series written under one of the following four forms:

- $R\left(t, \zeta, \Theta(\zeta, t), J^{k} h(t), J^{l} \bar{h}(\zeta, \Theta(\zeta, t))\right)$,
- $R\left(t, \zeta, J^{k} h(t), J^{l} \bar{h}(\zeta, \Theta(\zeta, t))\right)$,
- $R\left(z, \bar{\Theta}(z, \tau), \tau, J^{k} h(z, \bar{\Theta}(z, \tau)), J^{l} \bar{h}(\tau)\right)$,
- $R\left(z, \tau, J^{k} h(z, \bar{\Theta}(z, \tau)), J^{l} \bar{h}(\tau)\right)$.

Thanks to the lemma and to the convention, we may therefore write:

$$
\begin{equation*}
\mathscr{L}^{\beta}\left[\bar{g}_{j^{\prime}}(\tau)-\Theta_{j^{\prime}}^{\prime}(\bar{f}(\tau), h(t))\right]=: R_{j^{\prime}, \beta}^{\prime}\left(t, \tau, J_{\tau}^{|\beta|} \bar{h}(\tau): h(t)\right) \equiv 0, \tag{3.13}
\end{equation*}
$$

for $j^{\prime}=1, \ldots, d^{\prime}$. Remind that $h(t)$ is not differentiated, since the derivations $\underline{\mathscr{L}}^{\beta}$ involve only $\frac{\partial}{\partial \tau_{i}}, i=1, \ldots, n$. This is why we write $h(t)$ after " $:$ ". Furthmerore, the identities " $\equiv 0$ " are understood "on $\mathscr{M}$ ", namely as formal power series identities in $\mathbb{C} \llbracket \zeta, t \rrbracket$ after replacing $\xi$ by $\Theta(\zeta, t)$ or equivalently, as a formal power series identities in $\mathbb{C} \llbracket z, \tau \rrbracket$ after replacing $w$ by $\bar{\Theta}(z, \tau)$.

To understand the reflection principle, it is important to observe immediately that the smoothness of the power series $R_{j^{\prime}, \beta}^{\prime}$ is the minimum of the two smoothnesses of $M$ and of $M^{\prime}$. For instance, the power series $R_{j^{\prime}, \beta}^{\prime}$ are all complex analytic if $M$ is real analytic and if $M^{\prime}$ is real algebraic, even if the power series CR mapping $h(t)$ was assumed to be purely formal and nonconvergent. By a careful inspection of the application of the chain rule in the development of the above equations (3.13) (cf. Lemma 3.11), we even see that each $R_{j^{\prime}, \beta}^{\prime}$ is relatively polynomial with respect to the derivatives of positive order $\left(\partial_{\tau}^{\alpha} \bar{h}(\tau)\right)_{1 \leqslant|\alpha| \leqslant|\beta|}$.
3.14. Nondegeneracy conditions for formal CR mappings. In the equations (3.13), we replace $h(t)$ by a new independent variable $t^{\prime} \in \mathbb{C}^{n^{\prime}}$, we set
$(t, \tau)=(0,0)$, and we define the following collection of power series

$$
\Psi_{j^{\prime}, \beta}^{\prime}\left(t^{\prime}\right):=\left[\underline{\mathscr{L}}^{\beta} \bar{g}_{j^{\prime}}-\sum_{\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}} \mathscr{L}^{\beta}\left(\bar{f}^{\gamma^{\prime}}\right) \Theta_{j^{\prime}, \gamma^{\prime}}^{\prime}\left(t^{\prime}\right)\right]_{t=\tau=0}
$$

for $j^{\prime}=1, \ldots, d^{\prime}$ and $\beta \in \mathbb{N}^{m}$. Here, if $\beta=0$, we mean that $\Psi_{j^{\prime}, 0}^{\prime}\left(t^{\prime}\right)=$ $-\Theta_{j^{\prime}}^{\prime}\left(0, t^{\prime}\right)$. According to (3.13), an equivalent definition is:

$$
\Psi_{j^{\prime}, \beta}^{\prime}\left(t^{\prime}\right):=R_{j^{\prime}, \beta}^{\prime}\left(0,0, J_{\tau}^{|\beta|} \bar{h}(0): t^{\prime}\right) .
$$

Now, just before introducing five new nondegeneracy conditions, we make a crucial heuristic remark. When $n=n^{\prime}, m=m^{\prime}, M=M^{\prime}$ and $h=\mathrm{Id}$, writing $T^{\prime}$ instead of $t^{\prime}$ the special variable above in order to avoid confusion, we get for $j^{\prime}=1, \ldots, d^{\prime}$ and $\beta^{\prime} \in \mathbb{N}^{m^{\prime}}$ :

$$
\begin{aligned}
\Psi_{j^{\prime}, \beta^{\prime}}^{\prime}\left(T^{\prime}\right) & =\left[\underline{\mathscr{L}^{\prime \beta^{\prime}}} \xi_{j^{\prime}}^{\prime}-\sum_{\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}} \underline{\mathscr{L}^{\prime \prime} \beta^{\prime}}\left(\zeta^{\prime}\right)^{\gamma^{\prime}} \Theta_{j^{\prime}, \gamma^{\prime}}^{\prime}\left(T^{\prime}\right)\right]_{t^{\prime}=\tau^{\prime}=0^{\prime}} \\
& =\left[\underline{\mathscr{L}}^{\prime \beta^{\prime}} \Theta_{j^{\prime}}^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)-\beta^{\prime}!\Theta_{j^{\prime}, \beta^{\prime}}^{\prime}\left(T^{\prime}\right)\right]_{t^{\prime}=\tau^{\prime}=0^{\prime}} \\
& =\beta^{\prime}!\left(\Theta_{j^{\prime}, \beta^{\prime}}^{\prime}\left(0^{\prime}\right)-\Theta_{j^{\prime}, \beta^{\prime}}^{\prime}\left(T^{\prime}\right)\right)
\end{aligned}
$$

Consequently, up to a translation by a constant, we recover with $\Psi_{j^{\prime}, \beta^{\prime}}^{\prime}\left(T^{\prime}\right)$ the components of the infinite Segre mapping $\mathscr{Q}_{\infty}^{\prime}$ of $M^{\prime}$. Hence the next definition generalizes the concepts introduced before.

Definition 3.15. The formal CR mapping $h:(M, 0) \rightarrow_{\mathscr{F}}\left(M^{\prime}, 0^{\prime}\right)$ is called
(h1) Levi-nondegenerate at the origin if the mapping

$$
t^{\prime} \mapsto\left(R_{j^{\prime}, \beta}^{\prime}\left(0,0, J_{\tau}^{|\beta|} \bar{h}(0): t^{\prime}\right)\right)_{1 \leqslant j^{\prime} \leqslant d^{\prime},|\beta| \leqslant 1}
$$

is of rank $n^{\prime}$ at $t^{\prime}=0^{\prime}$;
(h2) finitely nondegenerate at the origin if there exists an integer $\ell_{1}$ such that the mapping

$$
t^{\prime} \mapsto\left(R_{j^{\prime}, \beta}^{\prime}\left(0,0, J_{\tau}^{|\beta|} \bar{h}(0): t^{\prime}\right)\right)_{1 \leqslant j^{\prime} \leqslant d^{\prime},|\beta| \leqslant k}
$$

is of rank $n^{\prime}$ at $t^{\prime}=0^{\prime}$, for $k=\ell_{1}$, hence for every $k \geqslant \ell_{1}$;
(h3) essentially finite at the origin if there exists an integer $\ell_{1}$ such that the mapping

$$
t^{\prime} \mapsto\left(R_{j^{\prime}, \beta}^{\prime}\left(0,0, J_{\tau}^{|\beta|} \bar{h}(0): t^{\prime}\right)\right)_{1 \leqslant j^{\prime} \leqslant d^{\prime},|\beta| \leqslant k}
$$

is locally finite at $t^{\prime}=0^{\prime}$, for $k=\ell_{1}$, hence for every $k \geqslant \ell_{1}$;
(h4) Segre nondegenerate at the origin if there exist an integer $\ell_{1}$, integers $j_{*}^{\prime 1}, \ldots, j_{*}^{\prime n^{\prime}}$ with $1 \leqslant j_{*}^{\prime i^{\prime}} \leqslant d^{\prime}$ for $i^{\prime}=1, \ldots, n^{\prime}$ and multiindices $\beta_{*}^{1}, \ldots, \beta_{*}^{n^{\prime}}$ with $\left|\beta_{*}^{i^{\prime}}\right| \leqslant \ell_{1}$ for $i^{\prime}=1, \ldots, n^{\prime}$, such that the determinant
$\operatorname{det}\left(\frac{\partial R_{j_{*}^{\prime}}^{\prime}{ }^{i_{1}^{\prime}, \beta_{*}^{i_{1}^{\prime}}}}{\partial t_{i_{2}^{\prime}}^{\prime}}\left(z, \bar{\Theta}(z, 0), 0,0, J^{\mid \beta_{*}^{i_{1}^{\prime}}} \mid \bar{h}(0): h(z, \bar{\Theta}(z, 0))\right)\right)_{1 \leqslant i_{1}^{\prime}, i_{2}^{\prime} \leqslant n^{\prime}}$
does not vanish identically in $\mathbb{C} \llbracket z \rrbracket$;
(h5) holomorphically nondegenerate at the origin if there exists an integer $\ell_{1}$, integers $j_{*}^{\prime 1}, \ldots, j_{*}^{\prime n^{\prime}}$ with $1 \leqslant j_{*}^{\prime i^{\prime}} \leqslant d^{\prime}$ for $i^{\prime}=1, \ldots, n^{\prime}$ and multiindices $\beta_{*}^{1}, \ldots, \beta_{*}^{n^{\prime}}$ with $\left|\beta_{*}^{i^{\prime}}\right| \leqslant \ell_{1}$ for $i^{\prime}=1, \ldots, n^{\prime}$, such that the determinant

$$
\operatorname{det}\left(\frac{\partial R_{j_{*}^{\prime} i_{1}^{\prime}, \beta_{*}^{i_{1}^{\prime}}}^{\prime}}{\partial t_{i_{2}^{\prime}}^{\prime}}\left(0,0,0,0, J^{\left.\right|_{*} ^{\beta_{1}^{\prime}}} \mid \bar{h}(0): h(t)\right)\right)_{1 \leqslant i_{1}^{\prime}, i_{2}^{\prime} \leqslant n^{\prime}}
$$

does not vanish identically in $\mathbb{C} \llbracket t\rfloor$.
The nondegeneracy of the formal mapping $h$ requires the same nondegeneracy on the target $\left(M^{\prime}, 0^{\prime}\right)$.
Lemma 3.16. ([Me2004a]) Let $h:(M, 0) \rightarrow_{\mathscr{F}}\left(M^{\prime}, 0^{\prime}\right)$ be a formal $C R$ mapping.
(1) If $h$ is Levi-nondegenerate at 0, then $M^{\prime}$ is necessarily Levinondegenerate at $0^{\prime}$.
(2) If $h$ is finitely nondegenerate at 0 , then $M^{\prime}$ is necessarily finitely nondegenerate at $0^{\prime}$.
(3) If $h$ is essentially finite at 0 , then $M^{\prime}$ is necessarily essentially finite at $0^{\prime}$.
(4) If $h$ is Segre nondegenerate at 0 , then $M^{\prime}$ is necessarily Segre nondegenerate at $0^{\prime}$.
(5) If $h$ is holomorphically nondegenerate at 0 , then $M^{\prime}$ is necessarily holomorphically nondegenerate at $0^{\prime}$.

We now show that CR-transversality of the mapping $h$ insures that it enjoys exactly the same nondegeneracy condition as the target $\left(M^{\prime}, 0^{\prime}\right)$.
Theorem 3.17. ([Me2004a]) Assume that the formal CR mapping $h$ : $(M, 0) \rightarrow_{\mathscr{F}}\left(M^{\prime}, 0^{\prime}\right)$ is CR-transversal at 0 . Then the following five implications hold:
(1) If $M^{\prime}$ is Levi nondegenerate at $0^{\prime}$, then $h$ is finitely nondegenerate at 0.
(2) If $M^{\prime}$ is finitely nondegenerate at $0^{\prime}$, then $h$ is finitely nondegenerate at 0 .
(3) If $M^{\prime}$ is essentially finite at $0^{\prime}$, then $h$ is essentially finite at 0 .
(4) If $M^{\prime}$ is Segre nondegenerate at $0^{\prime}$, then $h$ is Segre nondegenerate at 0.
(5) If $M^{\prime}$ is holomorphically nondegenerate, and if moreover $h$ is transversal at 0 , then $h$ is holomorphically nondegenerate at 0 .

The above five implications also hold under the assumption that $h$ is either CR-invertible, or CR-submersive, or CR-finite with $m=m^{\prime}$ or CRdominating: this provides at least 20 more (less refined) versions of the theorem, some of which appear in the literature.

Other relations hold true between the nondegeneracy conditions on $h$ and on the generic submanifolds $(M, 0)$ and $\left(M^{\prime}, 0^{\prime}\right)$. We mention some, concisely. As above, assume that $h:(M, 0) \mapsto_{\mathscr{F}}\left(M^{\prime}, 0\right)$ is a formal CR mapping. Since $d h_{0}\left(T_{0}^{c} M\right) \subset T_{0}^{c} M^{\prime}$, a linear map $d h_{0}^{\text {trv }}: T_{0} M / T_{0}^{c} M \rightarrow$ $T_{0} M^{\prime} / T_{0}^{c} M^{\prime}$ is induced. Assume $d^{\prime}=d$ and $m^{\prime}=m$. The next statement may be interpreted as a kind of Hopf Lemma for CR mappings.

Theorem 3.18. ([BR1990, ER2004]) If $M$ is minimal at 0 and if $h$ is CRdominating at 0 , then $d h_{0}^{\mathrm{trv}}: T_{0} M / T_{0}^{c} M \rightarrow T_{0} M^{\prime} / T_{0}^{c} M^{\prime}$ is an isomorphim.

An open question is to determine whether the condition that the jacobian determinant $\operatorname{det}\left(\frac{\partial h_{i}}{\partial t_{j}}(t)\right)_{1 \leqslant i, j \leqslant n}$ does not vanish identically in $\mathbb{C} \llbracket t \rrbracket$ is sufficient to insure that $d h_{0}^{\text {trv }}: T_{0} M / T_{0}^{c} M \rightarrow T_{0} M^{\prime} / T_{0}^{c} M^{\prime}$ is an isomorphism. A deeper understanding of the constraints between various nondegeneracy conditions on $h, M$ and $M^{\prime}$ would be desirable.
3.19. Classical versions of the reflection principle. Let $h:(M, 0) \rightarrow_{\mathscr{F}}$ $\left(M^{\prime}, 0\right)$ be a formal power series CR mapping between two generic submanifolds. Assume that $M$ is minimal at 0 .

Theorem 3.20. ([BER1999, Me2004a, Me2005]) If $M$ and $M^{\prime}$ are real analytic, if $h$ is either Levi nondegenerate, or finitely nondegenerate, or essentially finite, or Segre nondegenerate at the origin, then $h(t)$ is convergent, namely $h(t) \in \mathbb{C}\{t\}^{n^{\prime}}$. If moreover, $M$ and $M^{\prime}$ are algebraic, then $h$ is algebraic.

If one puts separate nondegeneracy conditions on $h$ and on $M^{\prime}$, as in Theorem 3.17, one obtains a combinatorics of possible statements, some of which appear in the literature.

If $h$ is finitely nondegenerate (level (2)), the (paradigmatic) proof yields more information.

Theorem 3.21. ([BER1999, Me2005]) As above, let $h:(M, 0) \rightarrow\left(M^{\prime}, 0^{\prime}\right)$ be a formal power series CR mapping. Assume that $M$ is minimal at 0 and let $\nu_{0}$ be the integer of Corollary 2.17. Assume also that $h$ is $\ell_{1}$-finitely nondegenerate at 0 . Then there exists a $\mathbb{C}^{n^{\prime}}$-valued power series mapping $H\left(t, J^{2 \nu_{0} \ell_{1}}\right)$ which is constructed algorithmically by means of the defining equations of $(M, 0)$ and of $\left(M^{\prime}, 0^{\prime}\right)$, such that the power series identity

$$
h(t) \equiv H\left(t, J^{2 \nu_{0} \ell_{1}} h(0)\right)
$$

holds in $\mathbb{C} \llbracket t \rrbracket^{n^{\prime}}$. If $M$ and $M^{\prime}$ are real analytic (resp. algebraic), $H$ is holomorphic (resp. complex algebraic) in a neighborhood of $0 \times J^{2 \nu_{0} \ell_{1}} h(0)$.

In [BER1999, GM2004], the above formula $h(t) \equiv H\left(t, J^{2 \nu_{0} \ell_{1}} h(0)\right)$ is studied horoughly in the case where $M^{\prime}=M$ and $h$ is a local holomorphic automorphism of $(M, 0)$ close to the identity.

At level (5), namely with a holomorphically nondegenerate target $\left(M^{\prime}, 0^{\prime}\right)$, the reflection principle is much more delicate. It requires the introduction of a new object, whose regularity properties hold in fact without any nondegeneracy assumption on the target $\left(M^{\prime}, 0^{\prime}\right)$.
3.22. Convergence of the reflection mapping. The reflection mapping associated to $h$ and to the system of coordinates $\left(z^{\prime}, w^{\prime}\right)$ is :

$$
\mathscr{R}_{h}^{\prime}\left(\tau^{\prime}, t\right):=\xi^{\prime}-\Theta^{\prime}\left(\zeta^{\prime}, h(t)\right) \in \mathbb{C} \llbracket \tau^{\prime}, t \rrbracket^{d^{\prime}} .
$$

Since $h$ is formal, it is only a formal power series mapping. As argued in the introduction of [Me2005], it is the most fundamental object in the analytic reflection principle. In the case of CR mappings between essentially finite hypersurfaces, the analytic regularity of the reflection mapping is equivalent to the extension of CR mappings as correspondences, as studied in [DP1995, Sh2000, Sh2003, DP2003]. Without nondegeneracy assumption on $\left(M^{\prime}, 0^{\prime}\right)$, the reflection mapping enjoys regularity properties from which all analytic reflection principles may be deduced. Here is the very main theorem of this Section 3.

Theorem 3.23. ([Me2001b, BMR2002, Me2005]) If $M$ is minimal at the origin and if h is either CR-invertible, or CR-submersive, or CR-finite, or $C R$-dominating, or $C R$-transversal, then for every system of coordinates $\left(z^{\prime}, w^{\prime}\right) \in \mathbb{C}^{m^{\prime}} \times \mathbb{C}^{d^{\prime}}$ in which the extrinsic complexification $\mathscr{M}^{\prime}$ is represented by $\xi^{\prime}=\Theta^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)$, the associated $C R$-reflection mapping is convergent, namely $\mathscr{R}_{h}^{\prime}\left(\tau^{\prime}, t\right) \in \mathbb{C}\left\{\tau^{\prime}, t\right\}^{d^{\prime}}$.

If the convergence property holds in one such system of coordinates, it holds in all systems of coordinates ([Me2005]; Proposition 3.26 below).

Further, if we develope $\Theta^{\prime}\left(\zeta^{\prime}, t^{\prime}\right)=\sum_{\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}}\left(\zeta^{\prime}\right)^{\gamma^{\prime}} \Theta_{\gamma^{\prime}}^{\prime}\left(t^{\prime}\right)$, the convergence of $\mathscr{R}_{h}^{\prime}\left(\tau^{\prime}, t\right)$ has a concrete signification.

Corollary 3.24. All the components $\Theta_{\gamma^{\prime}}^{\prime}(h(t))$ of the reflection mapping are convergent, namely $\Theta_{\gamma^{\prime}}^{\prime}(h(t)) \in \mathbb{C}\{t\}^{d^{\prime}}$ for every $\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}$.

Conversely ([Me2001b, Me2005]), if $\Theta_{\gamma^{\prime}}^{\prime}(h(t)) \in \mathbb{C}\{t\}^{d^{\prime}}$ for every $\gamma^{\prime} \in$ $\mathbb{N}^{m^{\prime}}$, an elementary application of the Artin approximation Theorem 3.28 (below) yields Cauchy estimates: there exist $\rho>0, \sigma>0$ and $C>0$ so that $\left|\Theta_{\gamma^{\prime}}^{\prime}(h(t))\right|<C(\rho)^{-\left|\gamma^{\prime}\right|}$, for every $t \in \mathbb{C}^{n}$ with $|t|<\sigma$. It follows that $\mathscr{R}_{h}^{\prime}\left(\tau^{\prime}, t\right) \in \mathbb{C}\left\{\tau^{\prime}, t\right\}^{d^{\prime}}$.

Taking account of the nondegeneracy conditions (ndi) and (crj), several corollaries may be deduced from the theorem. Most of them are already expressed by Theorem 3.20, except notably the delicate case where $\left(M^{\prime}, 0^{\prime}\right)$ is holomorphically nondegenerate.

Corollary 3.25. ([Me2001b, Me2005]) If $M$ is minimal at the origin, if $\left(M^{\prime}, 0^{\prime}\right)$ is holomorphically nondegenerate and if $h$ is either $C R$-invertible and invertible, or $C R$-submersive and submersive, or $C R$-finite and finite with $m^{\prime}=m$, or $C R$-dominating and dominating, or $C R$-transversal and transversal, then $h(t) \in \mathbb{C}\{t\}^{n^{\prime}}$ is convergent.

It is known ([St1996]) that $\left(M^{\prime}, 0^{\prime}\right)$ is holomorphically degenerate if and only if there exists a nonzero $(1,0)$ vector field $X^{\prime}=\sum_{i^{\prime}=1}^{n^{\prime}} a_{i^{\prime}}^{\prime}\left(t^{\prime}\right) \frac{\partial}{\partial t_{i^{\prime}}^{\prime}}$ having holomorphic coefficients which is tangent to $\left(M^{\prime}, 0^{\prime}\right)$. In the corollary above, holomorphic nondegeneracy is optimal for the convergence of a formal equivalence: if $M^{\prime}$ is holomorphically degenerate, if $\left(s^{\prime}, t^{\prime}\right) \longmapsto$ $\exp \left(s^{\prime} X^{\prime}\right)\left(t^{\prime}\right)$ denotes the local flow of $X^{\prime}$, where $s^{\prime} \in \mathbb{C}, t^{\prime} \in \mathbb{C}^{n^{\prime}}$, there indeed exist ([BER1999, Me2005]) nonconvergent power series $\varpi^{\prime}\left(t^{\prime}\right) \in$ $\mathbb{C} \llbracket t^{\prime} \rrbracket$ such that $t^{\prime} \mapsto \mathscr{F} \exp \left(\varpi^{\prime}\left(t^{\prime}\right) X^{\prime}\right)\left(t^{\prime}\right)$ is a nonconvergent formal equivalence of $M^{\prime}$.

The invariance of the reflection mapping is crucial.
Proposition 3.26. ([Me2002, Me2004a, Me2005]) The convergence of the reflection mapping is a biholomorphically invariant property. More precisely, if $t^{\prime \prime}=\phi^{\prime}\left(t^{\prime}\right)$ is a local biholomorphism fixing $0^{\prime}$ and transforming $\left(M^{\prime}, 0^{\prime}\right)$ into a generic submanifold $\left(M^{\prime \prime}, 0^{\prime}\right)$ of equations $\bar{w}_{j^{\prime}}^{\prime \prime}=\Theta_{j^{\prime}}^{\prime \prime}\left(\bar{z}^{\prime \prime}, t^{\prime \prime}\right)$, $j^{\prime}=1, \ldots, d^{\prime}$, the composed reflection mapping of $\phi^{\prime} \circ h:(M, 0) \rightarrow_{\mathscr{F}}$ ( $M^{\prime \prime}, 0^{\prime}$ ) defined by

$$
\begin{aligned}
\mathscr{R}_{\phi^{\prime} \circ h}^{\prime \prime}\left(\tau^{\prime \prime}, t\right): & =\xi^{\prime \prime}-\Theta^{\prime \prime}\left(\zeta^{\prime \prime}, \phi^{\prime}(h(t))\right) \\
& =\xi^{\prime \prime}-\sum_{\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}}\left(\zeta^{\prime \prime}\right)^{\gamma^{\prime}} \Theta_{\gamma^{\prime}}^{\prime \prime}\left(\phi^{\prime}(h(t))\right)
\end{aligned}
$$

has components $\Theta_{\gamma^{\prime}}^{\prime \prime}\left(\phi^{\prime}(h(t))\right)$ given by formulas

$$
\Theta_{\gamma^{\prime}}^{\prime \prime}\left(\phi^{\prime}(h(t))\right) \equiv S_{\gamma^{\prime}}^{\prime}\left(\left(\Theta_{\gamma_{1}^{\prime}}^{\prime}(h(t))\right)_{\gamma_{1}^{\prime} \in \mathbb{N}^{m^{\prime}}}\right),
$$

where the local holomorphic functions $S_{\gamma^{\prime}}^{\prime}$ depend only on the biholomorphism $t^{\prime \prime}=\phi^{\prime}\left(t^{\prime}\right)$ (they have an infinite number of variables, but the necessary Cauchy estimates insuring convergence are automatically satisfied).

A few words about the proof of the main Theorem 3.23. Although the classical reflection principle deals only with the "reflection identities" (3.13), to get the most adequate version of the reflection principle, it is unavoidable to understand the symmetry between the variables $t$ and the variables $\tau=(\bar{t})^{c}$.

The assumption that $h^{c}$ maps formally $(\mathscr{M}, 0)$ to $\left(\mathscr{M}^{\prime}, 0^{\prime}\right)$ is equivalent to each one of the following two formal identities:

$$
\left\{\begin{array}{l}
\bar{g}(\tau)=\sum_{\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}} \bar{f}(\tau)^{\gamma^{\prime}} \Theta_{\gamma^{\prime}}^{\prime}(h(t)) \\
g(t)=\sum_{\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}} f(t)^{\gamma^{\prime}} \bar{\Theta}_{\gamma^{\prime}}^{\prime}(\bar{h}(\tau))
\end{array}\right.
$$

on $\mathscr{M}$, namely after replacing either $w$ by $\bar{\Theta}(z, \tau)$ or $\xi$ by $\Theta(\zeta, t)$. The symmetry may be pursued by considering the two families of derivations:

$$
\left\{\begin{array}{l}
\mathscr{L}^{\beta}:=\left(\mathscr{L}_{1}\right)^{\beta_{1}}\left(\mathscr{L}_{1}\right)^{\beta_{2}} \cdots\left(\mathscr{L}_{m}\right)^{\beta_{m}} \quad \text { and } \\
\mathscr{L}^{\beta}:=\left(\mathscr{L}_{1}\right)^{\beta_{1}}\left(\mathscr{L}_{1}\right)^{\beta_{2}} \cdots\left(\mathscr{L}_{m}\right)^{\beta_{m}},
\end{array}\right.
$$

where $\beta=\left(\beta_{1}, \beta_{2}, \ldots, \beta_{m}\right) \in \mathbb{N}^{m}$. Applying them to the two formal identities above, if we respect the completeness of the combinatorics, we will get four families of reflection identities. The first pair is obtained by applying $\underline{L}^{\beta}$ to the two formal identities above:

$$
\left\{\begin{aligned}
\mathscr{L}^{\beta} \bar{g}(\tau) & =\sum_{\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}} \mathscr{L}^{\beta}\left[\bar{f}(\tau)^{\gamma^{\prime}}\right] \Theta_{\gamma^{\prime}}^{\prime}(h(t)), \\
0 & =\sum_{\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}} f(t)^{\gamma^{\prime}} \underline{\mathscr{L}^{\beta}}\left[\bar{\Theta}_{\gamma^{\prime}}^{\prime}(\bar{h}(\tau))\right] .
\end{aligned}\right.
$$

The second pair is obtained by applying $\mathscr{L}^{\beta}$, permuting the two lines:

$$
\left\{\begin{aligned}
\mathscr{L}^{\beta} g(t) & =\sum_{\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}} \mathscr{L}^{\beta}\left[f(t)^{\gamma^{\prime}}\right] \bar{\Theta}_{\gamma^{\prime}}^{\prime}(\bar{h}(\tau)), \\
0 & =\sum_{\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}} \bar{f}(\tau)^{\gamma^{\prime}} \mathscr{L}^{\beta}\left[\Theta_{\gamma^{\prime}}^{\prime}(h(t))\right]
\end{aligned}\right.
$$

We immediately see that these two pairs are conjugate line by line. In each pair, we notice a crucial difference between the first and the second line: whereas it is $\bar{g}$ and the power $\bar{f}^{\gamma^{\prime}}$ (or $g$ and $f \gamma^{\prime}$ ) that are differentiated in each first line, in each second line, only the components $\bar{\Theta}_{\gamma^{\prime}}^{\prime}(\bar{h})\left(\right.$ or $\Theta_{\gamma^{\prime}}^{\prime}(h)$ ) of the reflection mapping, which are the right invariant functions, are differentiated. In a certain sense, it is forbidden to differentiate $\bar{g}$ and $\bar{f}^{\gamma^{\prime}}$ (or $g$ and $\left.f \gamma^{\prime}\right)$, because the components $(f, g)$ of $h$ need not enjoy a reflection principle. In fact, in the proof of the main Theorem 3.23, one has to play constantly with the four reflection identities above.

Since we cannot summarize here the long and refined proof, we only formulate the main technical proposition. Denote by $J_{t}^{\ell} \psi$ the $\ell$-th jet of a power series $\psi(t) \in \mathbb{C} \llbracket t \rrbracket^{d^{\prime}}$, for instance $J_{t}^{\ell} \Theta_{\gamma^{\prime}}^{\prime}(h)$ for some $\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}$. Remind that $\underline{\Gamma}_{k}$ and $\Gamma_{k}$ are (conjugate) Segre chains. Let $N_{d^{\prime}, n, \ell}:=d^{\prime} \frac{(n+\ell)!}{n!\ell!}$.

Proposition 3.27. ([Me2005]) For every $k \in \mathbb{N}$ and every $\ell \in \mathbb{N}$, the following two properties hold:

- if $k$ is odd, for every $\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}$ :

$$
\left[J_{t}^{\ell} \Theta_{\gamma^{\prime}}^{\prime}(h)\right]\left(\Gamma_{k}\left(z_{(k)}\right)\right) \in \mathbb{C}\left\{z_{(k)}\right\}^{N_{d^{\prime}, n, \ell}} ;
$$

- if $k$ is even, for every $\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}$ :

$$
\left[J_{\tau}^{\ell} \bar{\Theta}_{\gamma^{\prime}}^{\prime}(\bar{h})\right]\left(\Gamma_{k}\left(z_{(k)}\right)\right) \in \mathbb{C}\left\{z_{(k)}\right\}^{N_{d^{\prime}, n, \ell}} .
$$

With $\ell=0$ and $k=2 \nu_{0}$, thanks to Corollary 2.17, we deduce from this main proposition that $\Theta_{\gamma^{\prime}}^{\prime}(h(t)) \in \mathbb{C}\{t\}^{\gamma^{\prime}}$ for every $\gamma^{\prime} \in \mathbb{N}^{m^{\prime}}$. This yields Theorem 3.23.

The main tool in the proof of this proposition is an approximation theorem saying that a formal power series mapping that is a solution of some analytic equations may be corrected so as to become convergent and still a solution.

Theorem 3.28. (Artin [Ar1968, JoPf2000]) Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, let $n \in \mathbb{N}$ with $n \geqslant 1$, let $\mathrm{x}=\left(\mathrm{x}_{1}, \ldots, \mathrm{x}_{n}\right) \in \mathbb{K}^{n}$, let $m \in \mathbb{N}$, with $m \geqslant 1$, let $\mathrm{y}=\left(\mathrm{y}_{1}, \ldots, \mathrm{y}_{m}\right) \in \mathbb{K}^{n}$, let $d \in \mathbb{N}$ with $d \geqslant 1$ and let $R_{1}(\mathrm{x}, \mathrm{y}), \ldots, R_{d}(\mathrm{x}, \mathrm{y})$ be an arbitrary collection of formal power series in $\mathbb{K}\{\mathrm{x}, \mathrm{y}\}$ that vanish at the origin, namely $R_{j}(0,0)=0, j=1, \ldots, d$. Assume that there exists a formal mapping $h(\mathrm{x})=\left(h_{1}(\mathrm{x}), \ldots, h_{m}(\mathrm{x})\right) \in \mathbb{K} \llbracket \mathrm{x} \rrbracket^{m}$ with $h(0)=0$ such that

$$
R_{j}(\mathrm{x}, h(\mathrm{x})) \equiv 0 \text { in } \mathbb{K} \llbracket \mathrm{x} \rrbracket, \quad \text { for } j=1, \ldots, d
$$

Let $\mathfrak{m}(x):=x_{1} \mathbb{K} \llbracket x \rrbracket+\cdots+x_{n} \mathbb{K} \llbracket x \rrbracket$ be the maximal ideal of $\mathbb{K} \llbracket x \rrbracket$. For every integer $N \geqslant 1$, there exists a convergent power series mapping $h^{N}(x) \in$
$\mathbb{K}\{\mathrm{x}\}^{m}$ such that

$$
R_{j}\left(\mathrm{x}, h^{N}(\mathrm{x})\right) \equiv 0 \text { in } \mathbb{K} \llbracket \mathrm{x} \rrbracket, \quad \text { for } j=1, \ldots, d,
$$

that approximates $h(\mathrm{x})$ to order $N-1$ :

$$
h^{N}(\mathrm{x}) \equiv h(\mathrm{x}) \bmod \left(\mathfrak{m}(x)^{N}\right) .
$$

As an application of the main Theorem 3.23, an approximation property for formal CR mappings holds.

Theorem 3.29. ([Me2005]) Under the assumptions of Theorem 3.23, for every integer $N \geqslant 1$, there exists a convergent power series mapping $\mathrm{H}^{N}(t) \in \mathbb{C}\{t\}^{n^{\prime}}$ with $\mathrm{H}^{N}(t) \equiv h(t) \bmod (\mathfrak{m}(t))^{N}($ whence $H(0)=0)$, that induces a local holomorphic map from $(M, 0)$ to $\left(M^{\prime}, 0^{\prime}\right)$.

Corollary 3.30. ([Me2001b, Me2005]) Assume that $n^{\prime}=n$, that $d^{\prime}=d$, that $M$ is minimal at the origin, and that $h:(M, 0) \rightarrow_{\mathscr{F}}\left(M^{\prime}, 0^{\prime}\right)$ is a formal (invertible) equivalence. Then $M$ and $M^{\prime}$ are biholomorphically equivalent.

It is known ([St1996, BER1999, GM2004]) that a minimal holomorphically nondegenerate real analytic generic submanifold of $\mathbb{C}^{n}$ has finitedimensional local holomorphic automorphism group. Unique determination by a jet of finite order follows from a representation formula, as in Theorem 3.21. More generally:

Corollary 3.31. ([Me2001b, BMR2002, Me2005]) Assume that $m^{\prime}=m$ and $d^{\prime}=d$, that $(M, 0)$ is minimal at the origin and that $\left(M^{\prime}, 0\right)$ is holomorphically nondegenerate. There exists an integer $\kappa=\kappa(m, d)$ such that, if two local biholomorphisms $h^{1}, h^{2}:(M, 0) \rightarrow\left(M^{\prime}, 0\right)$ have the same $\kappa$-th jet at the origin, then $h^{1}=h^{2}$.

From an inspection of the proof, Theorem 3.29 holds without the assumption that $(M, 0)$ is minimal, but with the assumption that its CR orbits have constant dimension in a neighborhood of 0 . However, the case where CR orbits have arbitrary dimension is delicate.
Open question 3.32. Does formal equivalence coincide with biholomorphic equivalence in the category of real analytic generic local submanifolds of $\mathbb{C}^{n}$ whose CR orbits have non-constant dimension?
3.33. Algebraicity of the reflection mapping. We will assume that both $M$ and $M^{\prime}$ are algebraic. Remind that Theorem 3.20 shows the algebraicity of $h$ under some hypotheses. A much finer result is as follows. It synthetizes all existing results ([We1977, SS1996, CMS1999, BER1999, Za1999]) about algebraicity of local holomorphic mappings.
Theorem 3.34. ([Me2001a]) If $h$ is a local holomorphic map $(M, 0) \rightarrow$ $\left(M^{\prime}, 0^{\prime}\right)$, if $M$ and $M^{\prime}$ are algebraic, if $M$ is minimal at the origin and if
$M^{\prime}$ is the smallest (for inclusion) local real algebraic manifold containing $h(M)$, then the reflection mapping $\mathscr{R}_{h}^{\prime}\left(\tau^{\prime}, t\right)$ is algebraic.

Trivial examples ([Me2001a]) show that the algebraicity of $\mathscr{R}_{h}^{\prime}$ need not hold if $M^{\prime}$ is not the smallest one.

In fact, Theorem 3.34 also holds (with the same proof) if one assumes only that the source $M$ is minimal at a Zariski-generic point: it suffices to shrink $M$ and the domain of definition of $h$ around such points, getting local algebraicity of $\mathscr{R}_{h}^{\prime}$ there, and since algebraicity is a global property, $\mathscr{R}_{h}^{\prime}$ is algebraic everywhere.

An equivalent formulation of Theorem 3.34 uses the concept of transcendence degree, studied in [Pu1990, CMS1999, Me2001a]. With $n_{M^{\prime}}^{\prime}$ being the essential holomorphic dimension of $\left(M^{\prime}, 0^{\prime}\right)$ defined in $\S 3.6$, set $\kappa_{M^{\prime}}^{\prime}:=n^{\prime}-n_{M^{\prime}}^{\prime}$. Observe that $\left(M^{\prime}, 0^{\prime}\right)$ is holomorphically nondegenerate precisely when $\kappa_{M^{\prime}}^{\prime}=0$. Denote by $\mathbb{C}[t]$ the ring of complex polynomials of the variable $t \in \mathbb{C}^{n}$ and by $\mathbb{C}(t)$ its quotient field. Let $t^{\prime}=h(t)$ be a local holomorphic mapping as in Theorem 3.34. and let $\mathbb{C}(t)\left(h_{1}(t), \ldots, h_{n^{\prime}}(t)\right)$ be the field generated by the components of $h$.

Theorem 3.35. ([Me2001a]) With the same assumptions as in Theorem 3.34, the transcendence degree of the field extension $\mathbb{C}(t) \rightarrow \mathbb{C}(t)(h(t))$ is less than or equal to $\kappa_{M^{\prime}}^{\prime}$.
Corollary 3.36. ([CMS1999, Za1999, Me2001a]) If $M$ is minimal at a Zariski-generic point and if the real algebraic target $M^{\prime}$ does not contain any complex algebraic curve, then the local holomorphic mapping $h$ is algebraic.

However, in case $h$ is only a formal CR mapping, it is impossible to shift the central point to a nearby minimal point. Putting the simplest rank assumption (invertibility) on $h$, we may thus formulate delicate problems for the future.

Open question 3.37. Let $h$ be a formal equivalence between two real analytic generic submanifolds of $\mathbb{C}^{n}$ which are minimal at a Zariski-generic point.

- Is the reflection mapping convergent?
- Is $h$ uniquely determined by a jet of finite order when the target is holomorphically nondegenerate?
- Is $h$ convergent under the assumption that the real analytic target $M^{\prime}$ does not contain any complex analytic curve ?

For $M^{\prime}$ algebraic containing no complex algebraic curve and $M$ minimal at 0 , the third question has been settled in [MMZ2003b]. However, the assumption of algebraicity of $M^{\prime}$ is strongly used there, because these
authors deal with the transcendence degree of the field extension $\mathbb{C}(t) \rightarrow$ $\mathbb{C}(t)(h(t))$, a concept which is meaningless if $M^{\prime}$ is real analytic. For further (secondary) results and open questions, we refer to [BMR2002, Ro2003]. This closes up our survey of the formal/algebraic/analytic reflection principle.

A generic submanifold $M \subset \mathbb{C}^{n}$ is called locally algebraizable at one of its points $p$ if there exist local holomorphic coordinates centered at $p$ in which it is Nash algebraic. Unlike partial results, the following question remains up to now unsolved.

Open problem 3.38. ([Hu2001, HJY2001, Ji2002, GM2004, Fo2004]) Formulate a necessary and sufficient condition for the local algebraizability of a real analytic hypersurface $M \subset \mathbb{C}^{n}$ in terms of a basis of the (differential) algebra of its Cartan-Hachtroudi-Chern invariants.

To conclude, we would like to mention that the complete theory of CR mappings may be transferred to systems of partial differential equations having finite-dimensional Lie symmetry group. This aspect will be treated in subsequent publications ([Me2006a, Me2006b]).

# III: Systems of vector fields and CR functions 


#### Abstract

Table of contents 1. Sussman's orbit theorem and structural properties of orbits .................. . 49. 2. Finite type systems and their genericity (openess and density) ................. 61 . 3. Locally integrable CR structures . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 72. 4. Smooth generic submanifolds and their CR orbits . . . . . . . . . . . . . . . . . . . . . . . . 83. 5. Approximation and uniqueness principles ......................................... . 102.


Beyond the theorems of Frobenius and of Nagano, Sussmann's theorem provides a means, valid in the smooth category, to construct all the integral manifolds of an arbitrary system of vector fields, as orbits of the pseudo-group actions of global flows. The fundamental properties of such orbits: lower semi-continuity of dimension, local flow box structure, propagation of embeddedness, intersection with a transversal curve in the one-codimensional case, are essentially analogous, but different from the ones known in foliation theory. Orbits possess wide applications in Control Theory, in sub-Riemannian Geometry, in the Analysis of Linear Partial Differential Equations and in Cauchy-Riemann geometry.

Let $L_{j} f=g_{j}, j=1, \ldots, \lambda$, be a linear PDE system with unknown $f$, where $g$ is smooth and where $\left\{L_{k}\right\}_{1 \leqslant k \leqslant r}$ is an involutive (in the sense of Frobenius) system of smooth vector fields on $\mathbb{R}^{n}$ having complex-valued coefficients. Since Lewy's celebrated discovery of an example of a single equation $L f=g$ in $\mathbb{R}^{3}$ without any solution, a major problem in the Analysis of PDE's is to find adequate criterions for the existence of local solutions. Condition (P) of Nirenberg-Treves has appeared to be necessary and sufficient to insure local integrability of a single equation of principal type having simple characteristics. The problem of characterizing systems of several linear first order PDE's having maximal space of solutions is not yet solved in full generality; several fine questions remain open.

Following Treves, to abstract the notion of systems involving several equations, an involutive structure on a smooth $\mu$-dimensional real manifold $M$ is a $\lambda$ dimensional complex subbundle $\mathscr{L}$ of $\mathbb{C} \otimes T M$ satisfying $[\mathscr{L}, \mathscr{L}] \subset \mathscr{L}$. The automatic integrability of smooth almost complex structures (those with $\mathscr{L} \oplus \overline{\mathscr{L}}=$ $\mathbb{C} \otimes T M$ ) and the classical (non)integrability theorems for smooth abstract CR structures (those with $\mathscr{L} \cap \overline{\mathscr{L}}=\{0\}$ ) are inserted in this general framework.

Beyond such problematics, it is of interest to study the analysis and the geometry of subbundles $\mathscr{L}$ whose space of solutions is maximal, viz the preceding question is assumed to be solved, optimally: in a neighborhood of every point of $M$, there exist $(\mu-\lambda)$ local complex valued functions $z_{1}, \ldots, z_{\mu-\lambda}$ having linearly independent differentials which are solutions of $\mathscr{L} z_{k}=0$. Such involutive structures are called locally integrable. Some representative examples are provided by the bundle of anti-holomorphic vector fields tangent to various embedded generic submanifolds
of $\mathbb{C}^{n}$. According to a theorem due to Baouendi-Treves, every local solution of $\mathscr{L} f=0$ may be approximated sharply by polynomials in a set of fundamental solutions $z_{1}, \ldots, z_{\mu-\lambda}$, in the topology of functional spaces as $\mathscr{C}^{\kappa, \alpha}, L_{l o c}^{\mathrm{p}}$, or $\mathscr{D}^{\prime}$.

In a locally integrable structure, the Sussmann orbits of the vector fields $\operatorname{Re} L_{k}, \operatorname{Im} L_{k}$ are then of central importance in analytic and in geometrical questions. They show up propagational aspects, as for instance: the support of a function or distribution solution $f$ of $\mathscr{L} f=0$ is a union of orbits. The approximation theorem also yields an elegant proof of uniqueness in the Cauchy problem. Further propagational aspects will be studied in the next chapters, using the method of analytic discs. Sections 3, 4 and 5 of this chapter and the remainder of the memoir are focused on embedded generic submanifolds.

## §1. SUSSMANN'S THEOREM AND STRUCTURAL PROPERTIES OF ORBITS

1.1. Integral manifolds of a system of vector fields. Ordinary differential equations in the modern sense emerged in the seventieth century, concomitantly with the infinitesimal calculus. Nowadays, in contemporary mathematics, the abstract study of vector fields is inserted in several broad areas of research, among which we perceive the following.

- Control Theory: controllability of vector fields on $\mathscr{C}^{\infty}$ and real analytic manifolds; nonholonomic systems; sub-Riemannian geometry ([GV1987, Bel1996]).
- Dynamical systems: singularities of real or complex vector fields and foliations; normal forms and classification; phase diagrams; Lyapunov theory; Poincaré-Bendixson theory; theory of limit cycles of polynomial and analytic vector fields; small divisors ([Ar1978, Ar1988]).
- Lie-Cartan theory: infinitesimal symmetries of differential equations; classification of local Lie group actions; Lie algebras of vector fields; representations of Lie algebras; exterior differential systems; Cartan-Vessiot-Kähler theorem; Janet-Riquier theory; Cartan's method of equivalence ([Ol1995, Stk2000]).
- Numerical analysis: systems of (non)linear ordinary differential equations; methods of: Euler, Newton-Cotes, Newton-Raphson, Runge-Kutta, Adams-Bashforth, Adams-Moulton ([De1996]).
- PDE theory: Local solvability of linear partial differential equations; uniqueness in the Cauchy problem; propagation of singularities; FBI transform and control of wave front set ([ES1993, Trv1992]).

To motivate the present Part III, let us expose informally two dual questions about systems of vector fields. Consider a set $\mathbb{L}$ of local vector fields defined on a domain of $\mathbb{R}^{n}$. Frobenius' theorem provides local foliations by submanifolds to which every element of $\mathbb{L}$ is tangent, provided $\mathbb{L}$ is closed under Lie brackets. However, for a generic set $\mathbb{L}$, the condition $[\mathbb{L}, \mathbb{L}] \subset \mathbb{L}$ fails and in addition, the tangent spaces spanned by elements of $\mathbb{L}$ are of varying dimension. To surmount these imperfections, two inverse options present themselves:
Sub: find the subsystems $\mathbb{L}^{\prime} \subset \mathbb{L}$ which satisfy Frobenius' condition $\left[\mathbb{L}^{\prime}, \mathbb{L}^{\prime}\right] \subset \mathbb{L}^{\prime}$ and which are maximal in an appropriate sense;
Sup: find the supsystems $\mathbb{L}^{\prime} \supset \mathbb{L}$ which have integral manifolds and which are minimal in an appropriate sense.


The first problem Sub is answered by the Cartan-Vessiot-Kähler theorem, thanks to an algorithm which provides all the minimal Frobenius-integrable subsystems $\mathbb{L}^{\prime}$ of $\mathbb{L}$ (we recommend [Stk2000] for a presentation). Generically, there are infinitely many solutions and their cardinality is described by means of a sequence of integers together with the so-called Cartan character of $\mathbb{L}$. In the course of the proof, the Cauchy-Kowalevskaya integrability theorem, valid only in the analytic category, is heavily used. It was not a serious restriction at the time of É. Cartan, but, in the second half of the twentieth century, the progress of the Analysis of PDE showed deep new phenomena in the differentiable category. Hence, one may raise the:

Open problem 1.2. Find versions of the Cartan-Vessiot-Kähler theorem for systems of vector fields having smooth non-analytic coefficients.

The Cauchy characteristic subsystem of $\mathbb{L}$ ([Stk2000]) is always involutive, hence the smooth Frobenius theorem applies to it ${ }^{4}$. However, for intermediate systems, the question is wide open. Possibly, this question is related to some theorems about local solvability of smooth partial differential equations (cf. Section 3) that were established to understand the Hans Lewy counterexample (§3.1).

[^3]The second problem Sup is already answered by Nagano's theorem (Part II), though only in the analytic category, with a unique integrable minimal supsystem $\mathbb{L}^{\text {lie }} \supset \mathbb{L}$. In the general smooth category, the stronger Chevalley-Lobry-Stefan-Sussmann theorem, dealing with flows of vector fields instead of Lie brackets, shows again that there is a unique integrable sup-system of $\mathbb{L}$ which has integral manifolds. As this theorem will be central in this memoir, it will be exposed thoroughly in the present Section 1.
1.3. Flows of vector fields and their regularity. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Let $D$ be a open connected subset of $\mathbb{K}^{n}$. Let $\mathrm{x}=\left(\mathrm{x}_{1}, \ldots, \mathrm{x}_{n}\right) \in D$. Let $L=$ $\sum_{i=1}^{n} a_{i}(\mathrm{x}) \frac{\partial}{\partial \mathrm{x}_{i}}$ be a vector field defined over $D$. Throughout this section, we shall assume that its coefficients $a_{i}$ are either $\mathbb{K}$-analytic (of class $\mathscr{C}^{\omega}$ ), of class $\mathscr{C}^{\infty}$, or of class $\mathscr{C}^{\kappa, \alpha}$, where $\kappa \geqslant 1$ and $0 \leqslant \alpha \leqslant 1$ (see Section 1(IV) for background about Hölder classes).

By the classical Cauchy-Lipschitz theorem, through each point $x_{0} \in D$, there passes a unique local integral curve of the vector field $L$, namely a local solution $\mathrm{x}(\mathrm{t})=\left(\mathrm{x}_{1}(\mathrm{t}), \ldots, \mathrm{x}_{n}(\mathrm{t})\right)$ of the system of ordinary differential equations:

$$
d \mathrm{x}_{1}(\mathrm{t}) / d \mathrm{t}=a_{1}(\mathrm{x}(\mathrm{t})), \ldots \ldots, d \mathrm{x}_{n}(\mathrm{t}) / d \mathrm{t}=a_{n}(\mathrm{x}(\mathrm{t})),
$$

which satisfies the initial condition $x(0)=x_{0}$. This solution is defined at least for small $\mathrm{t} \in \mathbb{K}$ and is classically denoted by $\mathrm{t} \mapsto \exp (\mathrm{t} L)\left(\mathrm{x}_{0}\right)$, because it has the local pseudogroup property

$$
\exp \left(\mathrm{t}^{\prime} L\right)\left(\exp (\mathrm{t} L)\left(\mathrm{x}_{0}\right)\right)=\exp \left(\left(\mathrm{t}+\mathrm{t}^{\prime}\right) L\right)\left(\mathrm{x}_{0}\right)
$$

whenever the composition is defined. Denote by $\Omega_{x_{0}}$ the largest connected open set containing the origin in $\mathbb{K}$ in which $\exp (\mathrm{t} L)\left(\mathrm{x}_{0}\right)$ is defined. One shows that the union of various $\Omega_{x_{0}}$, for $\mathrm{x}_{0}$ running in $D$, is an open connected set $\Omega_{L}$ of $\mathbb{K} \times \mathbb{K}^{n}$ which contains $\{0\} \times D$. Some regularity with respect to both variables $t$ and $x_{0}$ is got automatically.

Theorem 1.4. ([La1983], [*]) The global flow $\Omega_{L} \ni\left(\mathrm{t}, \mathrm{x}_{0}\right) \mapsto$ $\exp (\mathrm{t} L)\left(\mathrm{x}_{0}\right) \in D$ of a vector field $L=\sum_{i=1}^{n} a_{i}(\mathrm{x}) \partial_{\mathrm{x}_{i}}$ defined in the domain $D$ has exactly the same smoothness as $L$, namely it is $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$.

As a classical corollary, a local straightening property holds : in a neighborhood of a point at which $L$ does not vanish, there exists a $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$ change of coordinates $x^{\prime}=x^{\prime}(x)$ in which the transformed vector field is the unit positive vector field directed by the $x_{1}^{\prime}$ lines, viz $L^{\prime}=\partial / \partial x_{1}^{\prime}$.

Up to the end of this Section 1, we will work with $\mathbb{K}=\mathbb{R}$.
1.5. Searching integral manifolds of a system of vector fields. Let $M$ be a smooth paracompact real manifold, which is $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa+1, \alpha}$, where $\kappa \geqslant 1,0 \leqslant \alpha \leqslant 1$. Let $\mathbb{L}:=\left\{L_{a}\right\}_{a \in A}$ be a collection of vector fields defined on open subsets $D_{a}$ of $M$ and having $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$ coefficients, where $A$ is an arbitrary set. It is no restriction to assume that $\cup_{a \in A} D_{a}=M$, since otherwise, it suffices to shrink $M$. Call $\mathbb{L}$ a system of vector fields on $M$.
Problem 1.6. Find submanifolds $N$ of $M$ such that each element of $\mathbb{L}$ is tangent to $N$.
To analyze this (still imprecise) problem, let $\mathbb{F}_{M}$ denote the collection of all $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$ functions defined on open subsets of $M$, and call the system $\mathbb{L}$ of vector fields $\mathbb{F}_{M}$-linear if every combined vector field $f K+g L$ belongs to $\mathbb{L}$, whenever $f, g \in \mathbb{F}_{M}$ and $K, L \in \mathbb{L}$. Here, $f K+g L$ is defined in the intersection of the domains of definition of $f, g, K$ and $L$. To study the problem, it is obviously no restriction to assume that $\mathbb{L}$ is $\mathbb{F}_{M}$-linear.

For $p \in M$ arbitrary, define

$$
\mathbb{L}(p):=\{L(p): L \in \mathbb{L}\} .
$$

Since $\mathbb{L}$ is $\mathbb{F}_{M}$-linear, this is a linear subspace of $T_{p} M$. So Problem 1.6 is to find submanifolds $N$ satisfying $T_{p} N \supset \mathbb{L}(p)$, for every $p \in N$. Notice that an appropriate answer should enable one to find all such submanifolds. Also, suppose that $N_{1}$ and $N_{2}$ are two solutions with $N_{2} \subset N_{1}$. Then the problem with the pair $(M, N)$ is exactly the same as the problem with the pair $\left(N_{1}, N_{2}\right)$. Hence a better formulation.

Problem 1.6'. Find all the submanifolds $N \subset M$ of smallest dimension that satisfy $T_{p} N \supset \mathbb{L}(p)$, for every $p \in N$.

The classical Frobenius theorem ([Fr1877, Sp1970, BER1999, Bo1991, Ch1991, Stk2000, Trv1992]) provides an answer in the (for us simplest) case where $\mathbb{L}$ is closed under Lie brackets and is of constant dimension: every point $p \in M$ admits an open neighborhood foliated by submanifolds $N$ satisfying $T_{q} N=\mathbb{L}(q)$, for every $q \in N$. The global properties of these submanifolds were not much studied until C. Ehresmann and G. Reeb endeavoured to understand them (birth of foliation theory). A line with irrational slope in the 2 -torus $(\mathbb{R} / \mathbb{Z})^{2}$ shows that it is necessary to admit submanifolds $N$ of $M$ which are not closed. Let $\mathscr{A}_{M}$ denote the manifold structure of $M$.

Definition 1.7. An immersed submanifold of $\left(M, \mathscr{A}_{M}\right)$ is a subset of $N$ of $M$ equipped with its own smooth manifold structure $\mathscr{A}_{N}$, such that the inclusion map $i:\left(N, \mathscr{A}_{N}\right) \rightarrow\left(M, \mathscr{A}_{M}\right)$ is smooth, immersive and injective.

Thus, to keep maximally open Problem 1.6', one should seek immersed submanifolds and make no assumption about closedness under Lie brackets. For later use, recall that an immersed submanifold $N$ of $M$ is embedded
if its own manifold structure coincides with the manifold inherited from the inclusion $N \subset M$. It is well known ([CLN1985]) that an immersed submanifold $N$ is embedded if and only if for every point $p \in N$, there exists a neighborhood $U_{p}$ of $p$ in $M$ such that the pair $\left(U_{p}, N \cap U_{p}\right)$ is diffeomorphic to $\left(\mathbb{R}^{\operatorname{dim} M}, \mathbb{R}^{\operatorname{dim} N}\right)$.
1.8. Maximal strong integral manifolds property. In order to understand Problem 1.6', for heuristic reasons, it will be clever to discuss the differences between the two possibilities $\mathbb{L}(p)=T_{p} N$ and $\mathbb{L}(p) \nsubseteq T_{p} N$. Consider an arbitrary $\mathbb{F}_{M}$-linear system of vector fields $\widehat{\mathbb{L}}$ containing $\mathbb{L}$, for instance $\mathbb{L}$ itself. Let $p \in M$ and define the linear subspace $\widehat{\mathbb{L}}(p):=\{\widehat{L}(p): \widehat{L} \in \widehat{\mathbb{L}}\}$.

Definition 1.9. An immersed submanifold $N$ of $M$ is said to be:

- a strong $\widehat{\mathbb{L}}$-integral manifold if $T_{q} N=\widehat{\mathbb{L}}(q)$, at every point $q \in N$;
- a weak $\mathbb{L}$-integral manifold if $T_{q} N \supset \widehat{\mathbb{L}}(q)$, at every point $q \in N$.

In advance, the answer (Theorem 1.21 below) to Problem 1.6' states that it is possible to construct a unique system of vector fields $\widehat{\mathbb{L}}$ containing $\mathbb{L}$, whose strong integral manifolds coincide with the smallest weak $\mathbb{L}$-integral manifolds $N$. Further definitions are needed.

A system of vector fields $\widehat{\mathbb{L}}$ is said to have the strong integral manifolds property if for every point $p \in M$, there exists a strong $\widehat{\mathbb{L}}$-integral submanifold $N$ passing through $p$. A maximal strong $\widehat{\mathbb{L}}$-integral manifold $N$ is an immersed $\widehat{\mathbb{L}}$-integral manifold with the property that every connected strong $\widehat{\mathbb{L}}$-integral manifold which intersects $N$ is an open submanifold of $N$. Thus, through a point $p \in M$, there passes at most one maximal strong $\widehat{\mathbb{L}}$-integral submanifold. Finally, the system $\widehat{\mathbb{L}}$ has the maximal strong integral manifolds property if, through every point $p \in M$, there passes a maximal strong $\widehat{\mathbb{L}}$-integral manifold. The $\mathbb{F}_{M}$-linear systems $\widehat{\mathbb{L}}$ containing $\mathbb{L}$ are ordered by inclusion. We then admit that Problem 1.6' is essentially reduced to:
Problem 1.6". How to construct the (a posteriori unique) smallest (for inclusion) $\mathbb{F}_{M}$-linear system of vector fields $\widehat{\mathbb{L}}$ containing $\mathbb{L}$ which has the maximal strong integral manifolds property?
1.10. Taking account of the Lie brackets. Here is a basic geometric observation inspired by Frobenius' and Nagano's theorems.
Lemma 1.11. Assume the $\mathbb{F}_{M}$-linear system $\widehat{\mathbb{L}}$ has the strong integral manifolds property. Then for every two vector fields $\widehat{L}, \widehat{L}^{\prime} \in \widehat{\mathbb{L}}$ and for every $p$ in the intersection of their domains, the Lie bracket $\left[\widehat{L}, \widehat{L}^{\prime}\right](p)$ belongs to $\widehat{\mathbb{L}}(p)$.

Proof. Indeed, let $N$ be a strong $\widehat{\mathbb{L}}$-integral manifold, namely satisfying $T N=\left.\widehat{\mathbb{L}}\right|_{N}$. If $\widehat{L}, \widehat{L}^{\prime} \in \widehat{\mathbb{L}}$, the two restrictions $\left.\widehat{L}\right|_{N}$ and $\left.\widehat{L}^{\prime}\right|_{N}$ are tangent to $N$. Hence the restriction to $N$ of the Lie bracket $\left[\widehat{L}, \widehat{L^{\prime}}\right]$ is also tangent to $N$. In conclusion, at every $p \in N$, we have $\left[\widehat{L}, \widehat{L^{\prime}}\right](p) \in T_{p} N=\widehat{\mathbb{L}}(p)$.

So it is a temptation to believe that the smallest system $\mathbb{L}^{\text {lie }}$ of vector fields containing $\mathbb{L}$ which is closed under Lie brackets does enjoy the maximal integral manifolds property. However, just after the statement of Nagano's theorem (Part II), we have already learnt by means of Example 1.6(II) that in the $\mathscr{C}^{\infty}$ and $\mathscr{C}^{\kappa, \alpha}$ categories, the consideration of $\mathbb{L}^{\text {lie }}$ is inappropriate.
1.12. Transport of a vector field by the flow of another vector field. To understand why $\mathbb{L}^{\text {lie }}$ is insufficient, it will be clever to recall one of the classical definitions of the Lie bracket between two vector fields. Let $p \in M$ and let $K$ be a vector field defined in a neighborhood of $p$. Denote by $K(q)$ the value of $K$ at a point $q$ (this is a vector in $T_{q} M$ ), by $g_{*}(K)$ the pushforward of $K$ by a local diffeomorphism $g$, and by $q \mapsto K_{\mathrm{s}}(q)$ [instead of $q \mapsto \exp (\operatorname{s} K)(q)]$ the local diffeomorphism at time s induced by the flow of $K$. If $L$ is a second vector field defined in a neighborhood of $p$, the Lie bracket between $K$ and $L$ at $p$ is defined by:

$$
\begin{equation*}
[K, L](p):=\lim _{\mathrm{s} \rightarrow 0}\left(\frac{L(p)-\left(K_{\mathrm{s}}\right)_{*}\left(L\left(K_{-\mathrm{s}}(p)\right)\right)}{\mathrm{s}}\right) . \tag{1.13}
\end{equation*}
$$

Observe that for every fixed s $\neq 0$, the two vectors $L(p)$ and $\left(K_{\mathrm{s}}\right)_{*}\left(L\left(K_{-\mathrm{s}}(p)\right)\right)$ belong $T_{p} M$.


We explain how to read the right hand side of the diagram. In it: the integral curve of $K$ passing through $p$ is denoted by $\mu$; the integral curve of $L$ passing through the point $K_{-\mathrm{s}}(p)$ for s very small is denoted by $\gamma_{-\mathrm{s}, L}$; its image by the local diffeomorphism $K_{\mathrm{s}}$ is denoted by $K_{\mathrm{s}}\left(\gamma_{-\mathrm{s}, L}\right)$; the vector $L\left(K_{-\mathrm{s}}(p)\right)$ is tangent to $\gamma_{-\mathrm{s}, L}$ at the point $K_{-\mathrm{s}}(p)$; the vector
$\left(K_{\mathrm{s}}\right)_{*}\left(L\left(K_{-\mathrm{s}}(p)\right)\right)$, transported by the differerential of $K_{\mathrm{s}}$, is in general distinct from the vector $L(p)$; in fact, the difference $L(p)-\left(K_{\mathrm{s}}\right)_{*}\left(L\left(K_{-\mathrm{s}}(p)\right)\right)$ divided by s , tends to $[K, L](p)$ as $\mathrm{s} \rightarrow 0$.

Essentially, $\mathbb{L}^{\text {lie }}$ collects all vector fields obtained by taking infinitesimal differences (1.13) between vectors $L(p)$ and transported vectors $\left(K_{\mathrm{s}}\right)_{*}\left(L\left(K_{-\mathrm{s}}(p)\right)\right)$, and then iterating this processus to absorb all multiple Lie brackets.

As suggested in the left hand side of the diagram, instead of taking the infinitesimal differences, it is more general to collect all the vectors of the form $\left(K_{\mathrm{s}}\right)_{*}(L(p))$. This is the clue of Sussmann's theorem. In fact, the system $\widehat{\mathbb{L}}$ which is sought for in Problem 1.6 " should not only contain $\mathbb{L}^{\text {lie }}$, but should also collect all the vector fields of the form $\left(K_{\mathrm{s}}\right)_{*}(L)$, where s is not an infinitesimal.
Lemma 1.14. Let $\widehat{\mathbb{L}}$ be a $\mathbb{F}_{M}$-linear system of vector fields containing $\mathbb{L}$ which has the strong integral manifolds property. Let $p \in M$, let $K, L \in \mathbb{L}$ be two arbitrary vector fields defined in a neighborhood of $p$ and let $q=$ $K_{\mathrm{s}}(p)$ be a point in the integral curve of $K$ issued from $p$, with $\mathrm{s} \in \mathbb{R}$ small. Then the linear subspace $\widehat{\mathbb{L}}(q)$ necessarily contains the transported vector $\left(K_{\mathbf{s}}\right)_{*}(L(p))$.
Proof. Let $N$ be a strong $\widehat{\mathbb{L}}$-integral manifold passing through $p$. As $\widehat{\mathbb{L}}(r)=$ $T_{r} N$ at every point $r \in N$, and as $\mathbb{L}$ is contained in $\widehat{\mathbb{L}}$, it follows that the restricted vector field $\left.K\right|_{N}$ is tangent to $N$. Consequently, the integral curve of $K$ issued from $p$ is locally contained in $N$, hence the point $q=K_{\mathrm{s}}(p)$ belongs to $N$.

Moreover, as $\mathbb{L}$ is contained in $\widehat{\mathbb{L}}$, the vector $L(p)$ is tangent to $N$ at $p$. The differential $\left(K_{\mathrm{s}}\right)_{*}$ being a linear isomorphism between $T_{p} N$ and $T_{q} N$, it follows that the vector $\left(K_{\mathrm{s}}\right)_{*}(L(p))$ belongs to the tangent space $T_{q} N$, which coincides with $\widehat{\mathbb{L}}(q)$ by assumption.
1.15. The smallest $\mathbb{L}$-invariant system of vector fields $\mathbb{L}^{\text {inv }}$. Based on this crucial observation, we may introduce the smallest $\mathbb{F}_{M^{\prime}}$-linear system of vector fields $\mathbb{L}^{\text {inv }}$ ("inv" abbreviates "invariant") containing $\mathbb{L}$ which contains all vectors of the form $\left(K_{\mathrm{s}}\right)_{*}(L)$, whenever $K, L \in \mathbb{L}$ and $\mathrm{s} \in \mathbb{R}$. It follows that $\left(K_{\mathbf{s}}\right)_{*}\left(\mathbb{L}^{\text {inv }}(p)\right)=\mathbb{L}^{\text {inv }}\left(K_{\mathrm{s}}(p)\right)$ : the distribution of linear subspaces $p \mapsto \mathbb{L}^{\text {inv }}(p) \subset T_{p} M$ is invariant under the local flow maps.

In [Su1973], it is shown that $\mathbb{L}^{\text {inv }}$ is concretely and finitely generated as stated in Lemma 1.16 below. At first, some more notation is needed to denote the composition of several local diffeomorphisms of the form $K_{\mathrm{s}}$. Let $\mathbb{X}$ denote the system of all tangent vector fields to $M$, defined on open subsets of $M$. Let $k \in \mathbb{N}$ with $k \geqslant 1$ and let $K=\left(K^{1}, \ldots, K^{k}\right) \in \mathbb{X}^{k}$ be a $k$-tuple of vector fields defined in their domains of definition. If $s=$
$\left(\mathrm{s}_{1}, \ldots, \mathrm{~s}_{k}\right) \in \mathbb{R}^{k}$ is a $k$-tuple of "time" parameters, we will denote by $K_{\mathbf{s}}(p)$ the point

$$
K_{\mathrm{s}_{1}}^{1}\left(\cdots\left(K_{\mathrm{s}_{k}}^{k}(p)\right) \cdots\right):=\exp \left(\mathrm{s}_{1} K^{1}\left(\cdots\left(\exp \left(\mathrm{~s}_{k} K^{k}(p)\right)\right) \cdots\right)\right),
$$

whenever the composition is defined. The $k$-tuple $\left(\mathrm{s}_{1}, \ldots, \mathrm{~s}_{k}\right)$ will also be called a multitime parameter. For s fixed, the map $p \mapsto K_{\mathbf{s}}(p)$ is a local diffeomorphism between two open subsets of $M$. Its local inverse is the map $p \mapsto \widetilde{K}_{-\widetilde{s}}(p)$, where $\widetilde{K}:=\left(K^{k}, \ldots, K^{1}\right) \in \mathbb{L}^{k}$ and $\widetilde{\mathrm{s}}:=\left(\mathrm{s}_{k}, \ldots, \mathrm{~s}_{1}\right)$. Moreover, if we define $\left(\mathrm{s}, \mathrm{s}^{\prime}\right):=\left(\mathrm{s}_{1}, \ldots, \mathrm{~s}_{k}, \mathrm{~s}_{1}^{\prime}, \ldots, \mathrm{s}_{k^{\prime}}^{\prime}\right)$ for general $s=\left(s_{1}, \ldots, s_{k}\right) \in \mathbb{R}^{k}$ and $s^{\prime}=\left(s_{1}^{\prime}, \ldots, s_{k^{\prime}}^{\prime}\right) \in \mathbb{R}^{k^{\prime}}$, we have $K_{\mathrm{s}^{\prime}}^{\prime} \circ K_{\mathrm{s}}=\left(K^{\prime}, K\right)_{\left(\mathrm{s}^{\prime}, \mathrm{s}\right)}$.

After shrinking the domains of definition, the composition of local diffeomorphisms $K_{\mathrm{s}}$ is clearly associative, where it is defined. It follows that the set of local diffeomorphisms $K_{\mathrm{s}}$ constitutes a pseudogroup of local diffeomorphisms. Here, the term "pseudo" stems from the fact that the domains of definitions have to be adjusted; not all compositions are allowed.
Lemma 1.16. ([Su1973]) The system $\mathbb{L}^{\text {inv }}$ is generated by the $\mathbb{F}_{M}$-linear combinations of all vector fields of the form $\left(K_{\mathrm{s}}\right)_{*}(L)$, for all $L \in \mathbb{L}$, all $k$-tuples $K=\left(K^{1}, \ldots, K^{k}\right) \in \mathbb{L}^{k}$ of elements of $\mathbb{L}$ and all multitime parameters $\mathrm{s}=\left(\mathrm{s}_{1}, \ldots, \mathrm{~s}_{k}\right) \in \mathbb{R}^{k}$.

The definitions and the above reasonings show that the $\mathbb{F}_{M}$-linear system $\mathbb{L}^{\text {lie }}$ is a subsystem of the $\mathbb{F}_{M}$-linear system $\mathbb{L}^{\text {inv }}$ (of course, every system is contained in $\mathbb{X}$ ):

$$
\mathbb{L} \subset \mathbb{L}^{\text {lie }} \subset \mathbb{L}^{\text {inv }} \subset \mathbb{X} .
$$

In general, at a fixed point $p \in M$, the inclusions $\mathbb{L}(p) \subset \mathbb{L}^{\text {lie }}(p) \subset$ $\mathbb{L}^{\text {inv }}(p) \subset \mathbb{X}(p)=T_{p} M$ may be all strict.
Example 1.17 . On $\mathbb{R}^{4}$, consider the system $\mathbb{L}$ generated by the three vector fields

$$
\frac{\partial}{\partial \mathrm{x}_{1}}, \quad \mathrm{x}_{1} \frac{\partial}{\partial \mathrm{x}_{2}}, \quad e^{-1 / \mathrm{x}_{1}^{2}} \frac{\partial}{\partial \mathrm{x}_{3}} .
$$

Then it may be checked that

$$
\left\{\begin{aligned}
\mathbb{L}(0) & =\mathbb{R} \partial_{x_{1}}, \\
\mathbb{L}^{\text {lie }}(0) & =\mathbb{R} \partial_{x_{1}} \oplus \mathbb{R} \partial_{x_{2}}, \\
\mathbb{L}^{\text {inv }}(0) & =\mathbb{R} \partial_{x_{1}} \oplus \mathbb{R} \partial_{x_{2}} \oplus \mathbb{R} \partial_{x_{3}}, \\
\mathbb{X}(0) & =\mathbb{R} \partial_{x_{1}} \oplus \mathbb{R} \partial_{x_{2}} \oplus \mathbb{R} \partial_{x_{3}} \oplus \mathbb{R} \partial_{x_{4}}
\end{aligned}\right.
$$

Theorem 1.18. ([Na1966, Su1973]) In the $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ and $\mathscr{C}^{\kappa, \alpha}$ categories, the system $\mathbb{L}^{\text {inv }}$ is the smallest one containing $\mathbb{L}$ that has the maximal strong integral manifolds property. In the $\mathscr{C}^{\omega}$ category, $\mathbb{L}^{\text {inv }}=\mathbb{L}^{\text {lie }}$.

Further structural properties remain to be explained.
1.19. $\mathbb{L}$-orbits. The maximal strong integral manifolds of $\mathbb{L}^{\text {inv }}$ may be defined directly by means of $\mathbb{L}$, without refering to $\mathbb{L}^{\text {inv }}$, as follows. Two points $p, q \in M$ are said to be $\mathbb{L}$-equivalent if there exists a local diffeomorphism of the form $K_{\mathrm{s}}, K=\left(K^{1}, \ldots, K^{k}\right), \mathrm{s}=\left(\mathrm{s}_{1}, \ldots, \mathbf{s}_{k}\right), k \in \mathbb{N}$, with $K_{\mathrm{s}}(p)=q$. This clearly defines an equivalence relation on $M$. The equivalence classes are called the $\mathbb{L}$-orbits of $M$ and will be denoted either by $\mathscr{O}_{\mathbb{L}}(p)$ or shortly by $\mathscr{O}_{\mathbb{L}}$, when the reference to one point of the orbit is superfluous.

Concretely, two points $p, q \in M$ belong to the same $\mathbb{L}$-orbit if and only if there exist a continuous curve $\gamma:[0,1] \rightarrow M$ with $\gamma(0)=p$ and $\gamma(1)=q$ together with a partition of the interval $[0,1]$ by numbers $0=s_{0}<s_{1}<$ $\mathrm{s}_{2}<\cdots<\mathrm{s}_{k}=1$ and vector fields $K^{1}, \ldots, K^{k} \in \mathbb{L}$ such that for each $i=1, \ldots, k$, the restriction of $\gamma$ to the subinterval $\left[\mathrm{s}_{i-1}, \mathrm{~s}_{i}\right]$ is an integral curve of $K^{i}$. Such a curve will be called a piecewise integral curve of $\mathbb{L}$.

Let $p \in M$. Then its $\mathbb{L}$-orbit $\mathscr{O}_{\mathbb{L}}(p)$ may be equipped with the finest topology which makes all the maps $\mathrm{s} \mapsto K_{\mathbf{s}}(p)$ continuous, for all $k \geqslant 1$, all $K=\left(K^{1}, \ldots, K^{k}\right) \in \mathbb{L}^{k}$ and all multitime parameters $\mathrm{s}=\left(\mathrm{s}_{1}, \ldots, \mathrm{~s}_{k}\right)$. This topology is independent of the choice of a central point $p$ inside a given orbit ([Su1973]). Since the maps $\mathbb{R}^{k} \ni \mathrm{~s} \mapsto K_{\mathrm{s}}(p) \in M$ are already continuous, the topology of $\mathscr{O}_{\mathbb{L}}(p)$ is always finer than the topology induced by the inclusion $\mathscr{O}_{\mathbb{L}}(p) \subset M$. It follows that the inclusion map from $\mathscr{O}_{\mathbb{L}}(p)$ into $M$ is continuous. In particular, $\mathscr{O}_{\mathbb{L}}(p)$ is Hausdorff.
1.20. Precise statement of the orbit theorem. We now state in length the fundamental theorem of Sussmann, based on preliminary versions due to Hermann ([He1963]), to Nagano ([Na1966]) and to Lobry ([Lo1970]). It describes $\mathbb{L}$-orbits as immersed submanifolds (1), (2) enjoying the everywhere accessibility conditions (3), (4), together with a local flow-box property (5), useful in applications.

Theorem 1.21. (SuSSmann [Su1973, Trv1992, BM1997, BER1999, BCH2005], [*]) The following five properties hold true.
(1) Every $\mathbb{L}$-orbit $\mathscr{O}_{\mathbb{L}}$, equipped with the finest topology which makes all the maps $\mathrm{s} \mapsto K_{\mathrm{s}}(p)$ continuous, admits $a$ unique differentiable structure with the property that $\mathscr{O}_{\mathbb{L}}$ is an immersed submanifold of $M$, of class $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$.
(2) With this topology, each $\mathbb{L}$-orbit $\mathscr{O}_{\mathbb{L}}$ is simultaneously a (connected) maximal weak integral manifold of $\mathbb{L}$ and a (connected) maximal strong integral manifold of the $\mathbb{L}$-invariant $\mathbb{F}_{M}$-linear system $\mathbb{L}^{\text {inv }}$; thus, for every point $p \in M$, it holds $T_{p} \mathscr{O}_{\mathbb{L}}(p)=\mathbb{L}^{\text {inv }}(p)$, whence in particular $\operatorname{dim} \mathbb{L}^{\operatorname{inv}}(q)=\operatorname{dim} \mathscr{O}_{\mathbb{L}}(p)$ is constant for all $q$ belonging to a given $\mathbb{L}$-orbit $\mathscr{O}_{\mathbb{L}}(p)$.
(3) For every $p \in M, k \geqslant 1, K \in \mathbb{L}^{k}$, $\mathrm{s} \in \mathbb{R}^{k}$ such that $K_{\mathbf{s}}(p)$ is defined, the differential map $\left(K_{\mathrm{s}}\right)_{*}$ makes a linear isomorphism from $T_{p} \mathscr{O}_{\mathbb{L}}(p)=\mathbb{L}^{\text {inv }}(p)$ onto $T_{K_{\mathbf{s}}(p)} \mathscr{O}_{\mathbb{L}}(p)=\mathbb{L}^{\text {inv }}\left(K_{\mathbf{s}}(p)\right)$.
(4) For every $p, q \in M$ belonging to the same $\mathbb{L}$-orbit, there exists an integer $k \geqslant 1$, there exists a $k$-tuple of vector fields $K=$ $\left(K^{1}, \ldots, K^{k}\right) \in \mathbb{L}^{k}$ and there exists a multitime $\mathrm{s}^{*}=\left(\mathrm{s}_{1}^{*}, \ldots, \mathrm{~s}_{k}^{*}\right) \in$ $\mathbb{R}^{k}$ such that $p=K_{\mathrm{s}^{*}}(q)$ and such that the differential at $\mathrm{s}^{*}$ of the map

$$
\begin{equation*}
\mathbb{R}^{k} \ni \mathrm{~s} \mapsto K_{\mathrm{s}}(q) \in \mathscr{O}_{\mathbb{L}}(p) \tag{1.22}
\end{equation*}
$$

is of rank equal to $\operatorname{dim} \mathscr{O}_{\mathbb{L}}(p)$.
(5) For every $p \in M$, there exists an open connected neighborhood $V_{p}$ of $p$ in $M$ and there exists a $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$ diffeomorphism

$$
\begin{equation*}
\square^{e} \times \square^{n-e} \ni(\mathrm{~s}, \mathrm{r}) \longmapsto \varphi(\mathrm{s}, \mathrm{r}) \in V_{p}, \tag{1.23}
\end{equation*}
$$

where $e=\operatorname{dim} \mathscr{O}_{\mathbb{L}}(p)$, where $\square=\{\mathrm{x} \in \mathbb{R}:|\mathrm{x}|<1\}$, such that:

- $\varphi(0,0)=p$;
- the plaque $\varphi\left(\square^{e} \times\{0\}\right)$ is an open piece of the $\mathbb{L}$-orbit of $p$;
- each plaque $\varphi\left(\square^{e} \times\{r\}\right)$ is contained in a single $\mathbb{L}$-orbit; and:
- the set of $r \in \square^{n-e}$ such that $\varphi\left(\square^{e} \times\{r\}\right)$ is contained in the same $\mathbb{L}$-orbit $\mathscr{O}_{\mathbb{L}}(p)$ is either finite or countable.

In general, for $r \neq 0$, the $e$-dimensional plaques $\varphi\left(\square^{e} \times\{r\}\right)$ have positive codimension in the nearby orbits. We draw a diagram, in which $e=\operatorname{dim} \mathscr{O}_{\mathbb{L}}(p)=1$, with the nearby $\mathbb{L}$-orbits $\mathscr{O}_{2}, \mathscr{O}_{2}^{\prime}, \mathscr{O}_{3}$ and $\mathscr{O}_{3}^{\prime}$ having dimensions 2, 2, 3 and 3 .


Local orbit flow box theorem
Property (4) is crucial: the maps (1.22) of rank $\operatorname{dim} \mathscr{O}_{\mathbb{L}}(p)$ are used to define the differentiable structure on $\mathscr{O}_{\mathbb{L}}(p)$; they are also used to obtain the local orbit flow box property (5), as follows.

Let $p \in M$ and choose $q \in \mathscr{O}_{\mathbb{L}}(p)$ with $q \neq p$, to fit with the diagrams ( $q=$ $p$ would also do). Assuming that (4) holds, set $e:=\operatorname{dim} \mathscr{O}_{\mathbb{L}}(p)$, introduce an open subset $T_{e}$ in some $e$-dimensional affine subspace passing through $\mathrm{s}^{*}$ in $\mathbb{R}^{k}$ so that the restriction of the map (1.23) to $T_{e}$ still has rank $e$ at $\mathrm{s}=\mathrm{s}^{*}$. Introduce also an $(n-e)$-dimensional local submanifold $\Lambda_{p}$ passing through $p$ with $T_{p} \Lambda_{p} \oplus T_{p} \mathscr{O}_{\mathbb{L}}(p)=T_{p} M$ and set $\Lambda_{q}:=\widetilde{K}_{-\widetilde{s}}\left(\Lambda_{p}\right)$. Notice that $T_{q} \Lambda_{q} \oplus T_{q} \mathscr{O}_{\mathbb{L}}(p)=T_{q} M$, since the multiple flow map $K_{\mathbf{s}}(\cdot)$ stabilizes $\mathscr{O}_{\mathbb{L}}(p)$. Then, as one of the possible maps $\varphi$ whose existence is claimed in (5), we may choose a suitable restriction of:

$$
T_{e} \times \Lambda_{q} \ni(\mathrm{~s}, \mathrm{r}) \longmapsto K_{\mathrm{s}}(\mathrm{r}) \in M .
$$


1.24. Characterization of embedded $\mathbb{L}$-orbits. A smooth manifold $N$ together with an immersion $i: N \rightarrow M$ is called weakly embedded if for every manifold $P$, every smooth map $\psi: P \rightarrow M$ with $\psi(P) \subset N$, then $\psi: P \rightarrow N$ is in fact smooth ([Sp1970]; the diagram is also borrowed).


An immersion of the real line in $\mathbb{R}^{2}$ that is not weakly embedded
Proposition 1.25. ([Bel1996, BM1997, BCH2005]) Each $\mathbb{L}$-orbit is countable at infinity (second countable) and weakly embedded in $M$.

As the multiple flows are diffeomorphisms, embeddability propagates.
Proposition 1.26. ([Bel1996, BM1997, BCH2005]) Let $\mathscr{O}_{\mathbb{L}}$ be an $\mathbb{L}$-orbit in $M$ and let $e:=\operatorname{dim} \mathscr{O}_{\mathbb{L}}$. The following three conditions are equivalent:

- $\mathscr{O}_{\mathbb{L}}$ is an embedded submanifold of $M$;
- for every point $p \in \mathscr{O}_{\mathbb{L}}$, there exists a straightening map $\varphi$ as in (1.23) with $\mathscr{O}_{\mathbb{L}} \cap \varphi\left(\square^{e} \times \square^{n-e}\right)=\varphi\left(\square^{e} \times\{0\}\right)$;
- there exists at least one point at which the preceding property holds.

Conversely, $\mathscr{O}_{\mathbb{L}}$ is not embedded in $M$ if and only if for every $p \in \mathscr{O}_{\mathbb{L}}$ and for every local straightening map $\varphi$ centered at p as in (1.23), the set of $r \in \square^{n-e}$ such that $\varphi\left(\square^{n-e} \times\{r\}\right)$ is contained in $\mathscr{O}_{\mathbb{L}}=\mathscr{O}_{\mathbb{L}}(p)$ is infinite (nonetheless countable).
1.27. Local $\mathbb{L}$-orbits and their smoothness. For $U$ running in the collection of all nonempty open connected subsets of $M$ containing $p$, consider the localized $\left.\mathbb{L}\right|_{U}$-orbit of $p$ in $U$, denoted by $\mathscr{O}_{\mathbb{L}}(U, p)$. If $p \in U_{2} \subset U_{1}$, then $\mathscr{O}_{\mathbb{L}}\left(U_{2}, p\right) \subset \mathscr{O}_{\mathbb{L}}\left(U_{1}, p\right) \cap U_{2}$, so the dimension of $\mathscr{O}_{\mathbb{L}}(U, p)$ decreases as $U$ shrinks. Consequently, the localized $\mathbb{L}$-orbit $\mathscr{O}_{\mathbb{L}}(U, p)$ stabilizes and defines a unique piece of local ${ }^{5} \mathbb{L}$-integral submanifold through $p_{0}$, called the local $\mathbb{L}$ orbit of $p_{0}$ and denoted by $\mathscr{O}_{\mathbb{L}}^{\text {loc }}(p)$. In the CR context, this concept will be of interest in Parts V and VI. Sometimes, $\mathbb{L}$-orbits (in $M$ ) are called global, to distinguish them and to emphasize their nonlocal, nonpointwise nature.

From the flow regularity Theorem 1.4 and from Theorem 1.21, it follows:
Lemma 1.28. Global and local $\mathbb{L}$-orbits are as smooth as $\mathbb{L}$, i.e. $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$. Furthermore,

$$
T_{p} \mathscr{O}_{\mathbb{L}}^{l o c}(p) \subset T_{p} \mathscr{O}_{\mathbb{L}}(M, p)=\mathbb{L}^{\mathrm{inv}}(p),
$$

for every $p \in M$. This inclusion may be strict in the smooth categories $\mathscr{C}^{\infty}$ and $\mathscr{C}^{\kappa, \alpha}$, whereas, in the $\mathscr{C}^{\omega}$ category, local and global CR orbits have the same dimension.

In the $\mathscr{C}^{\kappa, \alpha}$ category, the maximal integral curve of an arbitrary element of $\mathbb{L}$ is $\mathscr{C}^{\kappa+1, \alpha}$, trivially because the right hand sides of the equations $d \mathrm{x}_{k}(\mathrm{t}) / d \mathrm{t}=a_{k}(\mathrm{x}(\mathrm{t})), k=1, \ldots, n$, are $\mathscr{C}^{\kappa, \alpha}$. May it be induced that general $\mathbb{L}$-orbits are $\mathscr{C}^{\kappa+1, \alpha}$ ? Trivially yes if $\operatorname{dim} \mathbb{L}=1$ at every point.

Another instance is as follows. Let $r \in \mathbb{N}$ with $1 \leqslant r \leqslant n-1$ and let $\mathbb{L}^{0}=\left\{L_{a}\right\}_{1 \leqslant a \leqslant r}$ be a system of $\mathscr{C}^{\kappa, \alpha}$ vector fields defined in a neighborhood of the origin in $\mathbb{R}^{n}$ that are linearly independent there. Consider the system $\mathbb{L}$ generated by linear combinations of elements of $\mathbb{L}^{0}$. Achieving Gaussian elimination and a linear change of coordinates, we may assume that $r$ generators of $\mathbb{L}$, still denoted by $L_{1}, \ldots, L_{r}$, take the form $L_{i}=\frac{\partial}{\partial \mathrm{x}_{i}}+$ $\sum_{j=1}^{n-r} a_{i j}(\mathrm{x}, \mathrm{y}) \frac{\partial}{\partial \mathrm{y}_{j}}, i=1, \ldots, r$, with $(\mathrm{x}, \mathrm{y})=\left(\mathrm{x}_{1}, \ldots, \mathrm{x}_{r}, \mathrm{y}_{1}, \ldots, \mathrm{y}_{n-r}\right)$ and with $a_{i j}(\mathrm{x}, \mathrm{y})$ of class $\mathscr{C}^{\kappa, \alpha}$ in a neighborhood of the origin.

We claim that if $\mathscr{O}_{\mathbb{L}}(0)$ has (minimal possible) dimension $r$, then it is $\mathscr{C}^{\kappa+1, \alpha}$. This happens in particular if $\mathbb{L}$ is Frobenius-integrable.

[^4]Indeed, the local graphed equations of $\mathscr{O}_{\mathbb{L}}(0)$ must then be of the form $\mathrm{y}_{j}=h_{j}(\mathrm{x}), j=1, \ldots, n-r$, with the $h_{j}$ of class at least $\mathscr{C}^{\kappa, \alpha}$, thanks to the lemma above. Observe that the $L_{i}$ are tangent to this submanifold if and only if the $h_{j}$ satisfy the complete system of partial differential equations $\frac{\partial h_{j}}{\partial \mathrm{x}_{i}}(\mathrm{x})=a_{i j}(\mathrm{x}, h(\mathrm{x}))$, for $i=1, \ldots, r, j=1, \ldots, n-r$, implying directly that the $h_{j}$ are $\mathscr{C}^{\kappa+1, \alpha}$. In general, this argument shows that if $\operatorname{dim} \mathscr{O}_{\mathbb{L}}(p)$ coincides with $\operatorname{dim} \mathbb{L}(p)$, the orbit is $\mathscr{C}^{\kappa+1, \alpha}$ at $p$.
Example 1.29. However, this improvement of smoothness is untrue when $\operatorname{dim} \mathbb{L}(p)+1 \leqslant \operatorname{dim} \mathscr{O}_{\mathbb{L}}(p) \leqslant n-1$.

Indeed, pick the function $\chi_{\kappa, \alpha}=\chi_{\kappa, \alpha}(z)$ of $z \in \mathbb{R}$ equal to zero for $z \leqslant 0$ and, for $z \geqslant 0$, defined by:

$$
\chi_{\kappa, \alpha}(z)= \begin{cases}z^{\kappa+\alpha}, & \text { if } 0<\alpha \leqslant 1 \\ z^{\kappa} / \log z, & \text { if } \alpha=0\end{cases}
$$

This function is $\mathscr{C}^{\kappa, \alpha}$ on $\mathbb{R}$, but for $(\lambda, \beta)>(\kappa, \alpha)$, it is not $\mathscr{C}^{\lambda, \beta}$ in any neighborhood of the origin. Then in $\mathbb{R}^{4}$ equipped with coordinates ( $\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{t}$ ), consider the hypersurface $\Sigma$ of equation:

$$
0=\mathrm{t}-\chi_{\kappa+1, \alpha}(\mathrm{y}) \chi_{\kappa, \alpha}(\mathrm{z}),
$$

Then $\Sigma$ is $\mathscr{C}^{\kappa, \alpha}$, not better. The two vector fields $L_{1}:=\frac{\partial}{\partial x}$ and $L_{2}:=$ $\frac{\partial}{\partial y}+\left[\mathrm{x} \chi_{\kappa, \alpha}(-\mathrm{y})\right] \frac{\partial}{\partial z}+\left[\chi_{\kappa+1, \alpha}^{\prime}(\mathrm{y}) \chi_{\kappa, \alpha}(\mathrm{z})\right] \frac{\partial}{\partial \mathrm{t}}$ have $\mathscr{C}^{\kappa, \alpha}$ coefficients and are tangent to $\Sigma$. We claim that $\Sigma$ is the local $\left\{L_{1}, L_{2}\right\}$-orbit of the origin.

Otherwise, there would exist a local two-dimensional submanifold $\{\mathrm{z}=g(\mathrm{x}, \mathrm{y}), \mathrm{t}=h(\mathrm{x}, \mathrm{y})\}$ with $L_{1}$ and $L_{2}$ tangent to it. Then $\left[L_{1}, L_{2}\right]=\chi_{\kappa, \alpha}(-\mathrm{y}) \frac{\partial}{\partial z}$ should also be tangent. However, at points $(0, \mathrm{y}, g(0, \mathrm{y}), h(0, \mathrm{y}))$, with y negative and arbitrarily small, $L_{1}, L_{2}$ and $L_{3}$ are equal to the three linearly independent vectors $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and $\chi_{\kappa, \alpha}(-y) \frac{\partial}{\partial z}$.

## §2. Finite type system and their genericity (OPENESS And DENSITY)

2.1. Systems of vector fields satisfying $\mathbb{L}^{\text {lie }}=\mathbb{L}$. Let $M$ be a $\mathscr{C}^{\kappa}(1 \leqslant \kappa \leqslant$ $\infty$ ) connected manifold of dimension $n \geqslant 1$. By $\mathbb{X}$, denote the system of all vector fields defined on open subsets of $M$ (it is a sheaf). Let

$$
\mathbb{L}^{0}=\left\{L_{a}\right\}_{1 \leqslant a \leqslant r}, \quad r \geqslant 1,
$$

be a finite collection of $\mathscr{C}^{\kappa-1}$ vector fields defined on $M$, namely $L_{a} \in$ $\mathbb{X}(M)$. Unlike in the $\mathscr{C}^{\omega}$ category, in the $\mathscr{C}^{\kappa}$ category, $\mathbb{X}(M)$ is always nonempty and quite large, thanks to partitions of unity. For this reason, we shall not work in the real analytic category, except in some specific local situations. The set of linear combinations of elements of $\mathbb{L}^{0}$ with coefficients
in $\mathscr{C}^{\kappa-1}(M, \mathbb{R})$ will be denoted by $\mathbb{L}\left(\right.$ or $\left.\mathbb{L}^{1}\right)$ and called the $\mathscr{C}^{\kappa-1}(M)$-linear hull of $\mathbb{L}^{0}$.

Definition 2.2. A $\mathscr{C}^{\kappa-1}(M)$-linear system $\mathbb{L} \subset \mathbb{X}$ is said to be of finite type at a point $p \in M$ if $\mathbb{L}^{\text {lie }}(p)=T_{p} M$.

If $\mathbb{L}^{\text {lie }}$ is of finite type at every point, then $\mathbb{L}^{\text {lie }}=\mathbb{L}^{\text {inv }}=\mathbb{X}$ and there is just one maximal $\mathbb{L}$-integral manifold in the sense of Sussmann: $M$ itself.

In 1939, Chow had already shown that the equality $\mathbb{L}^{\text {lie }}=\mathbb{X}$ implies the everywhere accessibility condition: every two points of $M$ may be joined by integral curves of $\mathbb{L}$. In 1967, Hörmander established that every second order partial differential operator $P:=L_{1}^{2}+\cdots+L_{r}^{2}+R_{1}+R_{0}$ on a domain $\Omega \subset \mathbb{R}^{n}$ whose top order part is a sum of squares of $\mathscr{C}{ }^{\infty}$ vector fields $L_{a}, 1 \leqslant$ $a \leqslant r$, such that $\mathbb{L}^{\text {lie }}=\mathbb{X}$ is $\mathscr{C}^{\infty}$-hypoelliptic, namely $\operatorname{Pf} \in \mathscr{C}^{\infty}$ implies $f \in \mathscr{C}^{\infty}$. Vector field systems satisfying $\mathbb{L}^{\text {lie }}=\mathbb{X}$ have been further studied by workers in hypoelliptic partial differential equations and in nilpotent Lie algebras: Métivier, Stein, Mitchell, Stefan, Lobry and others.

In the next Parts V and VI, we will focus on propagational aspects that are enjoyed by the (more general) smooth systems $\mathbb{L}$ that satisfy $\mathbb{L}^{\text {inv }}=\mathbb{X}$, but possibly $\mathbb{L}^{\text {lie }}(p) \neq \mathbb{X}(p)$ at every $p \in M$. Nevertheless, for completeness, we shall survey in the present section some classical geometric properties of finite type systems.
2.3. Lie bracket flags, weights, privilegied coordinates and distance estimate. Define $\mathbb{L}^{1}:=\mathbb{L}$ and by induction, for $s \in \mathbb{N}$ with $2 \leqslant s \leqslant \kappa$, define $\mathbb{L}^{s}$ to be the $\mathscr{C}^{\kappa-s}$-linear hull of $\mathbb{L}^{s-1}+\left[\mathbb{L}^{1}, \mathbb{L}^{s-1}\right]$. Concretely, $\mathbb{L}^{s}$ is generated over $\mathscr{C}^{\kappa-s}$ by iterated Lie brackets of length $\leqslant s$ of the form:

$$
L_{\alpha}=\left[L_{\alpha_{1}},\left[L_{\alpha_{2}}, \ldots,\left[L_{\alpha_{\ell-1}}, L_{\alpha_{\ell}}\right] \ldots\right]\right], \quad 1 \leqslant \ell \leqslant s
$$

Jacobi's identity insures that $\left[\mathbb{L}^{s_{1}}, \mathbb{L}^{s_{2}}\right] \subset \mathbb{L}^{s_{1}+s_{2}}$.
Denote $\mathbb{L}^{s}(p):=\operatorname{Vect}_{\mathbb{R}}\left\{L(p): L \in \mathbb{L}^{s}\right\}$. Clearly, $\mathbb{L}$ is of finite type at $p \in M$ if and only it there exists an integer $d(p) \leqslant \kappa$ with $\mathbb{L}^{d(p)}(p)=T_{p} M$. The smallest $d(p)$ is sometimes called the degree of non-holonomy of $\mathbb{L}$ at $p$. Other authors call it the type of $\mathbb{L}$ at $p$, which we will do. The function $p \mapsto d(p) \in[1, \kappa] \cup\{\infty\}$ is upper-semi-continuous: $d(q) \leqslant d(p)$ for $q$ near p.

Combinatorially, at a finite type point, it is of interest to introduce the Lie bracket flag:

$$
\{0\} \subset \mathbb{L}^{1}(p) \subset \mathbb{L}^{2}(p) \subset \cdots \subset \mathbb{L}^{s}(p) \subset \cdots \subset \mathbb{L}^{d(p)}(p)=T_{p} M
$$

Then a finite type point $p$ is called regular if the integers $n_{s}(q):=\operatorname{dim} \mathbb{L}^{s}(q)$ remain constant in some neighborhood of $p$. It is elementary to verify
([Bel1996]) that, at such a regular point, the dimensions are strictly increasing:

$$
0<n_{1}(p)<n_{2}(p)<\cdots<n_{d(p)}(p)=n .
$$

Fix $p$, not necessarily regular. A local coordinate system $\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ centered at $p$ is linearly adapted at $p$ if:

$$
\left\{\begin{aligned}
& \mathbb{L}^{1}(p)=\operatorname{Vect}_{p}\left(\frac{\partial}{\partial x_{1}}, \ldots, \frac{\partial}{\partial x_{n_{1}(p)}}\right) \\
& \mathbb{L}^{2}(p)=\operatorname{Vect}_{p}\left(\frac{\partial}{\partial x_{1}}, \ldots, \frac{\partial}{\partial x_{n_{1}(p)}}, \ldots, \frac{\partial}{\partial x_{n_{2}(p)}}\right) \\
&\left.\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots, \ldots, \frac{\partial}{\partial x_{n_{2}(p)}}, \ldots \frac{\partial}{\partial x_{n_{d(p)}(p)}}\right) \\
& \mathbb{L}^{d(p)}(p)=\operatorname{Vect}_{p}\left(\frac{\partial}{\partial x_{1}}, \ldots, \frac{\partial}{\partial x_{n_{1}(p)}}, \ldots,\right.
\end{aligned}\right.
$$

Let us assign weights $w_{i}$ to such linearly adapted coordinates $x_{i}$ as follows: the first group $\left(x_{1}, \ldots, x_{n_{1}(p)}\right)$ being linked to $\mathbb{L}^{1}(p)$, their weights will all be equal to one: $w_{1}=\cdots=w_{n_{1}(p)}=1$. The second group $\left(x_{n_{1}(p)+1}, \ldots, x_{n_{2}(p)}\right)$, linked to the quotient $\mathbb{L}^{2}(p) / \mathbb{L}^{1}(p)$, will be assigned uniform weight two: $w_{n_{1}(p)+1}=\cdots=w_{n_{2}(p)}=2$, and so on, until $w_{n_{d(p)-1}(p)+1}=\cdots=w_{n_{d(p)}(p)}=d(p)$.

Provided $\mathbb{L}$ is of finite type at every point, we claim that the original finite collection $\mathbb{L}^{0}$ produces what is called a sub-Riemannian metric; then by means of weights, the topology associated to this metric may be compared to the manifold topology of $M$ in a highly precise way.

Indeed, let us define the (infinitesimal) sub-Riemannian length of a vector $v_{p} \in \mathbb{L}^{1}(p)$ by:

$$
\left\|v_{p}\right\|_{\mathbb{L}^{0}}:=\inf \left\{\left(u_{1}^{2}+\cdots+u_{m}^{2}\right)^{1 / 2}: v_{p}=u_{1} L_{1}(p)+\cdots+u_{r} L_{r}(p)\right\} .
$$

For $v_{p} \notin \mathbb{L}^{1}(p)$, we set $\left\|v_{p}\right\|_{\mathbb{L}^{0}}=\infty$. The length of a piecewise $\mathscr{C}^{1}$ curve $\gamma(t), t \in[0,1]$, will be the integral:

$$
\operatorname{length}_{\mathbb{L}^{0}}(\gamma):=\int_{0}^{1}\|d \gamma(t) / d t\|_{\mathbb{L}^{0}} d t
$$

Finally, the distance associated to the finite collection $\mathbb{L}^{0}$ is:

$$
d_{\mathbb{L}^{0}}(p, q):=\inf _{\gamma: p \rightarrow q} \text { length }_{\mathbb{L}^{0}}(\gamma) .
$$

Assume for instance $d(p)=2$, so that $n_{2}(p)=n$. If the coordinates are linearly adapted, the tangent space $T_{p} M$ then splits in the "horizontal" space, the ( $\left.x_{1}, \ldots, x_{n_{1}(p)}\right)$-plane, together with a (not unique) "vertical" space generated $e . g$. by the remaining coordinates. It is then classical that the distance
from $p$ to a point of coordinates $\left(x_{1}, \ldots, x_{n}\right)$ close to $p$ enjoys the estimate:
$d_{\mathbb{L}^{0}}\left(p,\left(x_{1}, \ldots, x_{n}\right)\right) \asymp\left|x_{1}\right|+\cdots+\left|x_{n_{1}(p)}\right|+\left|x_{n_{1}(p)+1}\right|^{1 / 2}+\cdots+\left|x_{n}\right|^{1 / 2}$.
Here, the abbreviation $\Phi \asymp \Psi$ means that there exists $C>1$ with $C^{-1} \Psi<\Phi<C \Psi$. Notice that the successive exponents coincide with the weights $w_{1}, \ldots, w_{n_{1}(p)}, w_{n_{1}(p)+1}, \ldots, w_{n}$. In particular, to reach a point of coordinates $(0, \ldots, 0, \varepsilon, \ldots, \varepsilon)$, it is necessary to flow along $\mathbb{L}^{0}$ during a time $\sim \operatorname{cst} . \varepsilon^{1 / 2}$. Observe that $\left|x_{1}\right|+\cdots+\left|x_{n}\right|$ is equivalent to the distance from $p$ to $x$ induced by any Riemannian metric. Thus, the modified distance $d_{\mathbb{L}^{0}}$ is just obtained by replacing each $\left|x_{i}\right|$ by $\left|x_{i}\right|^{1 / w_{i}}$, up to a multiplicative constant.

To generalize such a quantitative comparison between the $d_{\mathbb{L}^{0}}$-distance and the underlying topology of $M$, linearly adapted coordinates appear to be insufficient. For $\beta=\left(\beta_{1}, \ldots, \beta_{r}\right) \in \mathbb{N}^{r}$, denote by $L^{\beta}$ the $|\beta|$-th order derivation $L_{1}^{\beta_{1}} L_{2}^{\beta_{2}} \cdots L_{r}^{\beta_{r}}$. Beyond linearly adapted coordinates, one must introduce privileged coordinates, whose existence is assured by the following.

Theorem 2.4. ([Bel1996]) There exist local coordinates $\left(x_{1}, \ldots, x_{n}\right)$ centered at $p$ that are privileged in the sense that each $x_{i}$ is of order exactly equal to $w_{i}$ with respect to $\mathbb{L}^{0}$-derivations, namely, for $i=1, \ldots, n$ :

$$
\begin{array}{cc}
\left.L^{\gamma} x_{i}\right|_{p}=0, & \text { for all } \gamma \text { with }|\gamma| \leqslant w_{i}-1 \\
\left.L^{\beta_{i}^{*}} x_{i}\right|_{p} \neq 0, & \text { for some } \beta_{i}^{*} \text { with }\left|\beta_{i}^{*}\right|=w_{i}
\end{array}
$$

Only if $d(p)=2$, linearly adapted coordinates are automatically privileged ([Bel1996]). As soon as $d(p) \geqslant 3$, privileged systems are unavoidable.

Theorem 2.5. ([Bel1996]) For $x$ in a neighborhood of $p$, the estimate:

$$
d_{\mathbb{L}^{0}}\left(p,\left(x_{1}, \ldots, x_{n}\right)\right) \asymp\left|x_{1}\right|^{1 / w_{1}}+\cdots+\left|x_{n}\right|^{1 / w_{n}}
$$

holds if and only if the coordinates are privileged.
For $\varepsilon>0$ small, define the anisotropic ball $B_{\mathbb{L}^{0}}(p, \varepsilon):=\left\{x: d_{\mathbb{L}^{0}}(p, x)<\right.$ $\varepsilon\}$.

Corollary 2.6. ([Bel1996]) There exist $C>1$ such that

$$
\frac{1}{C} \prod_{i=1}^{n}\left[-\varepsilon^{w_{i}}, \varepsilon^{w_{i}}\right] \subset B_{\mathbb{L}^{0}}(p, \varepsilon) \subset C \prod_{i=1}^{n}\left[-\varepsilon^{w_{i}}, \varepsilon^{w_{i}}\right]
$$

2.7. Local basis. At a non-regular point, the integers $n_{k}(p), k=1, \ldots, d(p)$ are not necessarily strictly increasing. Thus, it is necessary to express the combinatorics of the Lie bracket flag with more precision, in terms of what is sometimes called Hörmander numbers $m_{i}, \ell_{i}$. From now on, we shall assume that the $r$ vector fields $L_{a}, 1 \leqslant a \leqslant r$, are linearly independent at $p$
and have $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\omega}$ coefficients. In both cases, the formal Taylor series of every coefficient exists.

In the flag

$$
\{0\} \subset \mathbb{L}^{1}(p) \subset \mathbb{L}^{2}(p) \subset \cdots \subset \mathbb{L}^{s}(p) \subset \cdots \subset \mathbb{L}^{d(p)}(p)=T_{p} M
$$

let $m_{1}$ denote the smallest $k \geqslant 2$ such that the dimension of $\mathbb{L}^{k}(p)$ is larger than the dimension of $\mathbb{L}^{1}(p)$ (at a regular point, $m_{1}=2$ ) and set $\ell_{1}:=\operatorname{dim} \mathbb{L}^{m_{1}}(p)-r \geqslant 1$. Similarly, let $m_{2}$ denote the smallest $k \geqslant 1+m_{1}$ such that the dimension of $\mathbb{L}^{k}(p)$ is larger than the dimension of $\mathbb{L}^{m_{1}}(p)$ (at a regular point, $m_{2}=3$ ) and set $\ell_{2}:=\operatorname{dim} \mathbb{L}^{m_{2}}(p)-\operatorname{dim} \mathbb{L}^{m_{1}}(p)$. By induction, let $m_{j+1}$ denote the smallest $k \geqslant 1+m_{j}$ such that the dimension of $\mathbb{L}^{k}(p)$ is larger than the dimension of $\mathbb{L}^{m_{j}}(p)$ and set $\ell_{j+1}:=$ $\operatorname{dim} \mathbb{L}^{m_{j+1}}(p)-\operatorname{dim} \mathbb{L}^{m_{j}}(p)$.

Since $p$ is a point of finite type, the process terminates until $m_{h}=d(p)$ reaches the degree of non-holonomy at $p$, for a certain integer $h \geqslant 1$. We thus have extracted the interesting information, namely the strict flag of linear spaces:

$$
\mathbb{L}^{1}(p) \subset \mathbb{L}^{m_{1}}(p) \subset \mathbb{L}^{m_{2}}(p) \subset \cdots \subset \mathbb{L}^{m_{h}}(p)=T_{p} M
$$

with Lie bracket orders $1<m_{1}<m_{2}<\cdots<m_{h}$, whose successive dimensions may be listed parallelly:

$$
r<r+\ell_{1}<r+\ell_{1}+\ell_{2}<\cdots<r+\sum_{1 \leqslant j \leqslant h} \ell_{j}
$$

Next, let $x=\left(x_{1}, \ldots, x_{n}\right)$ be linearly adapted coordinates, vanishing at $p$. We shall denote them by $\left(y, s_{1}, s_{2}, \ldots, s_{h}\right)$, where $y \in \mathbb{R}^{r}$, $s_{1} \in \mathbb{R}^{\ell_{1}}, s_{2} \in \mathbb{R}^{\ell_{2}}, \ldots, s_{h} \in \mathbb{R}^{\ell_{h}}$. As in the preceding paragraph, we assign weight 1 to the $y$-coordinates, weight $m_{1}$ to the $s_{1}$-coordinates, weight $m_{2}$ to the $s_{2}$-coordinates, $\ldots$, weight $m_{h}$ to the $s_{h}$-coordinates. The weight of a monomial $x^{\alpha}=y^{\beta} s_{1}^{\gamma_{1}} s_{2}^{\gamma_{2}} \cdots s_{h}^{\gamma_{h}}$ is obviously defined as $|\beta|+m_{1}\left|\gamma_{1}\right|+m_{2}\left|\gamma_{2}\right|+\cdots+m_{h}\left|\gamma_{h}\right|$. We say that a formal power series $a(x)=a\left(y, s_{1}, \ldots, s_{h}\right)$ is an $\mathrm{O}(\kappa)$ if all its monomials have weight $\geqslant \kappa$. Also, $a(x)$ is called weighted homogeneous of degree $\kappa$ if

$$
a\left(t y, t^{m_{1}} s_{1}, t^{m_{2}} s_{2}, \ldots, t^{m_{h}} s_{h}\right)=t^{\kappa} a\left(y, s_{1}, s_{2}, \ldots, s_{h}\right),
$$

for all $t \in \mathbb{R}$. As in the case of $\mathbb{R} \llbracket z_{1}, \ldots, z_{n} \rrbracket$ with all weights equal to 1 , every formal series $a\left(y, s_{1}, s_{2}, \ldots, s_{h}\right)$ may be decomposed as a countable sum of weighted homogeneous polynomials of increasing degree.

Dually, we also assign weights to all the basic vector fields: $\frac{\partial}{\partial y_{a}}$ will have weight -1 , whereas for $j=1, \ldots, m_{h}$, the $\frac{\partial}{\partial s_{j l}}, l=1, \ldots, \ell_{j}$, will have weight $-m_{j}$. The weight of a monomial vector field $x^{\alpha} \frac{\partial}{\partial x_{i}}$ is defined to be the sum the weights of $x^{\alpha}$ with the weight of $\frac{\partial}{\partial x_{i}}$. Every vector field having
formal power series coefficients may be decomposed as a countable sum of weighted homogeneous vector fields having polynomial coefficients.

Theorem 2.8. ([Bel1996, BER1999]) Assume the local vector fields $L_{a}$, $a=1, \ldots, r$, have $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\omega}$ coefficients and are linearly independend at $p$. If the $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\omega}$ coordinates $x=\left(y, s_{1}, s_{2}, \ldots, s_{h}\right)$ centered at $p$ are priveleged, then each $L_{a}$ may be developed as:

$$
L_{a}=\widehat{L}_{a}+\mathrm{O}(0)
$$

where each vector field:

$$
\widehat{L}_{a}:=\frac{\partial}{\partial y_{a}}+\sum_{1 \leqslant j \leqslant 1} \sum_{1 \leqslant l \leqslant \ell_{j}} p_{a, j, l}\left(y, s_{1}, \ldots, s_{j-1}\right) \frac{\partial}{\partial s_{j, l}},
$$

is homogeneous of degree -1 and has as its coefficients some polynomials $p_{a, j, l}=p_{a, j, l}\left(y, s_{1}, \ldots, s_{j-1}\right)$ that are independent of $s_{j}$ and are homogeneous of degree $m_{j}-1$.

A crucial algebraic information is missing in this statement: what are the nondegeneracy conditions on the $p_{a, j, l}$ that insure that the system is indeed of finite type at $p$ with the combinatorial invariants $m_{j}$ and $\ell_{j}$ ? The real problem is to classify vector field systems that are of finite type, up to local changes of coordinates. At least, the following may be verified.
Theorem 2.9. ([Bel1996, BER 1999]) The vector fields $\widehat{L}_{a}, a=1, \ldots, r$, form a finite type system $\widehat{\mathbb{L}}^{0}$ at $p$ having the same combinatorial invariants $m_{j}$ and $\ell_{j}$ and satisfying the same distance estimate as $d_{\mathbb{L}^{0}}$ in Theorem 2.5. Moreover, the linear hull of $\widehat{\mathbb{L}}^{0}$ generates a Lie algebra $\widehat{\mathbb{L}}^{\text {lie }}$ with the nilpotency property that all Lie brackets of length $\geqslant m_{h}+1$ all vanish.
2.10. Finite-typisation of smooth systems of vector fields. As previously, let $\mathbb{L}^{0}=\left\{L_{a}\right\}_{1 \leqslant a \leqslant r}$ be a finite collection of $\mathscr{C}^{\kappa-1}$ vector fields globally defined on a connected manifold $M$ of class $\mathscr{C}^{\kappa}(1 \leqslant \kappa \leqslant \infty)$ and of dimension $n \geqslant 1$. Let $\mathbb{L}$ be its $\mathscr{C}^{\kappa-1}(M)$-linear hull. If $r=1$, then $\mathbb{L}^{\text {lie }}=\mathbb{L}$, hence $\mathbb{L}$ cannot be of finite type, unless $n=1$. So we assume $n \geqslant 2$ and $r \geqslant 2$. We want to perturb $\mathbb{L}$ slightly to $\mathbb{L}$ so as to get finite-typeness at every point: $\widetilde{\mathbb{L}}^{\text {lie }}(p)=T_{p} M$ at every $p \in M$. Since the composition of Lie brackets of length $\ell$ requires coefficients of vector fields to be at least $\mathscr{C}^{\ell}$, if $\kappa<\infty$, then necessarily $\widetilde{\mathbb{L}}^{\text {lie }}=\widetilde{\mathbb{L}}^{\kappa}$ stops at length $\kappa$.

At a central point, say the origin in $\mathbb{K}^{n}$, and for $\mathbb{K}$-analytic vector field systems, the already presented Theorem 1.11(II) yields small perturbations that are of finite type at 0 . Of course, the same local result holds true for collections of vector fields that are $\mathscr{C}^{\infty}$, or even $\mathscr{C}^{\kappa-1}$ with $\kappa$ large enough. Now, we want a global theorem.

What does it mean for $\widetilde{\mathbb{L}}$ to be close to $\mathbb{L}$ ? A vector field $L \in \mathbb{X}(M)$ may be interpreted as a section of the tangent bundle, in particular a $\mathscr{C}^{\kappa-1}$ map $M \rightarrow T M$. The most useful topology on the set $\mathscr{C}^{\lambda}(M, N)$ of all $\mathscr{C}^{\lambda}$ maps from a manifold $M$ to another manifold $N(e . g . N=T M$ with $\lambda=\kappa-1$ ) is the strong Whitney topology; it controls better than the socalled weak topology the behaviour of maps at infinity in the noncompact case. Essentially, $f, g \in \mathscr{C}^{\lambda}(M, N)$ are (strongly) close to each other if all their partial derivatives of order $\leqslant \lambda$, computed in a countable collection of charts $\varphi_{\nu}: U_{\nu} \rightarrow \mathbb{R}^{n}$ and $\psi_{\nu}: V_{\nu} \rightarrow \mathbb{R}^{m}$ covering $M$ and $N, \nu \in \mathbb{N}$, are $\varepsilon_{\nu}$-close, the smallness of $\varepsilon_{\nu}>0$ depending on the pair of charts $\left(\varphi_{\nu}, \psi_{\nu}\right)$. Precise definitions may be found in the monograph [17]. We then topologize this way the finite product $\mathbb{X}(M)^{r}$.

Already two vector fields may well be of finite type on a manifold of arbitrary dimension, e.g. $\frac{\partial}{\partial x_{1}}$ and $\sum_{i=2}^{n} x_{1}^{i-1} \frac{\partial}{\partial x_{i}}$ on $\mathbb{R}^{n}$.

Theorem 2.11. ([Lo1970]) If the connected manifold $M$ of dimension $n \geqslant 2$ is $\mathscr{C}^{n+n^{2}}$, then the set of pairs of vector fields $\mathbb{L}^{0}:=(K, L) \in \mathbb{X}(M)^{2}$ on $M$ whose $\mathscr{C}^{n^{2}+n-1}$-linear hull $\mathbb{L}$ satisfies $\mathbb{L}^{n^{2}+n}=\mathbb{L}$, is open and dense in the strong Whitney topology.

According to [Su1976], the smoothness $M \in \mathscr{C}^{n+n^{2}}$ in [Lo1970] was improved to $M \in \mathscr{C}^{2 n}$ in Lobry's thesis (unpublished). We will summarize the demonstration in the case $M \in \mathscr{C}^{2 n}$. However, since neither $\mathscr{C}^{2 n}$ nor $\mathscr{C}^{n+n^{2}}$ are optimal, we will improve this result afterwards (Theorem 2.16 below).

Proof. Openness is no mystery. For denseness, we need some preliminary. If $M$ and $N$ are two $\mathscr{C}^{\lambda}$ manifolds, we denote by $J^{\lambda}(M, N)$ the bundle of $\lambda$-th jets of $\mathscr{C}^{\lambda}$ maps from $M$ to $N$. We recall that, to a $\mathscr{C}^{\lambda}$ map $f$ : $M \rightarrow N$ is associated the $\lambda$-th jet map $j^{\lambda} f: M \rightarrow J^{\lambda}(M, N)$, a continuous map that may be considered as a kind of intrinsic collection of all partial derivatives of $f$ up to order $\lambda$. Let $\pi: J^{\lambda}(M, N) \rightarrow M$ be the canonical projection, sending a jet to its base point. For $p \in M$, the fiber $\pi^{-1}(p)$ may be identified with $\mathbb{R}^{N_{m, n, \lambda}}$, where $N_{m, n, \lambda}:=m \frac{(n+\lambda)!}{n!\lambda!}$ counts the number of partial derivatives of order $\leqslant \lambda$ of maps $\mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$.

We will state a lemma which constitutes a special case of the jet transversality theorem. This particular statement (Lemma 2.12 below) generalizes the intuitively obvious statement that any $\mathscr{C}^{0}$ curve graphed over $\mathbb{R} \times\{0\}^{2}$ in $\mathbb{R}^{3}$ may always be slightly perturbed to avoid a given fixed $\mathscr{C}^{1}$ curve $\Sigma$.

Call a subset $\Sigma \subset J^{\lambda}(M, N)$ algebraic in the jet variables if in every pair of local charts, it possesses defining equations that are polynomials in the jet variables $f_{j, \alpha}, 1 \leqslant j \leqslant m, \alpha \in \mathbb{N}^{n},|\alpha| \leqslant \lambda$, whose coefficients are
independent of the coordinates $x \in M$. Of course, after a local diffeomorphism $x \mapsto \bar{x}(x)$ of $M$, a general polynomial in the jet variables which is independent of $x$ almost never remains independent of $\bar{x}$ in the new coordinates. Nevertheless, in the sequel, we shall only encounter special sets $\Sigma \subset J^{\lambda}(M, N)$ which, in any coordinate system, may be defined as zero sets of such special polynomials.

For instance, taking $\left(x_{k}, y_{j}, y_{j, k}\right)$ as coordinates on $J^{1}(M, N)$, where $1 \leqslant$ $k \leqslant n=\operatorname{dim} M$ and $1 \leqslant j \leqslant m=\operatorname{dim} N$, a change of coordinates $x \mapsto$ $\bar{x}(x)$ induces $\left(x_{k}, y_{j}, y_{j, k}\right) \longmapsto\left(\bar{x}_{k}, \bar{y}_{j}, \bar{y}_{j, k}\right)$, where $\bar{y}_{j}=y_{j}$ is unchanged but the new jet variables $y_{j, k}=\sum_{l=1}^{n} \bar{y}_{j, l} \frac{\partial \bar{x}_{l}}{\partial x_{k}}$ involve the variables $x$ (or $\bar{x}$ ). Nevertheless, the equations $\left\{y_{j, k}=0,1 \leqslant j \leqslant m, 1 \leqslant k \leqslant n\right\}$ saying that the first (pure) jet vanishes are equivalent to $\left\{\bar{y}_{j, k}=0,1 \leqslant j \leqslant m, 1 \leqslant\right.$ $k \leqslant n\}$, since the invertible Jacobian matrix $\left(\frac{\partial x_{l}}{\partial x_{k}}\right)$ may be erased: vanishing properties in a jet bundle are intrinsic !

A theorem due to Whitney states that real algebraic sets are stratified, i.e. are finite unions of geometrically smooth real algebraic manifolds. The codimension of $\Sigma$ is thus well-defined.

Lemma 2.12. ([17]) Assume $\Sigma \subset J^{\lambda}(M, N)$ is algebraic in the jet variables and of codimension $\geqslant 1+\operatorname{dim} M$. Then the set of maps $f \in \mathscr{C}^{\lambda}(M, N)$ whose $\lambda$-th prolongation $j^{\lambda} f: M \rightarrow J^{\lambda}(M, N)$ does not meet $\Sigma$ at any point is open and dense in the strong Whitney topology.

Although $j^{\lambda} f$ is only continuous, the fact that the bad set $\Sigma$ is algebraic enables to apply the appropriate version of Sard's theorem that is used in the jet transversality theorem.

We shall apply the lemma by defining a certain bad set $\Sigma$ which, if avoided, means that a pair of vector fields on $M$ is of finite type at every point.

Assume $M \in \mathscr{C}^{2 n}$ and let $(K, L) \in \mathbb{X}(M)^{2}$. Both vector fields have $\mathscr{C}^{2 n-1}$ coefficients. With $\lambda:=2 n-1$, denote by $J^{2 n-1}\left(\mathbb{X}(M)^{2}\right)$ the fiber bundle of the $(2 n-1)$-th jets of these pairs. In some coordinates provided by a local chart $U \ni q \mapsto\left(x^{1}(q), \ldots, x^{n}(q)\right) \in \mathbb{R}^{n}$, with $U \subset M$ open, we may write $K=\sum_{1 \leqslant i \leqslant n} K_{i}(x) \frac{\partial}{\partial x^{i}}$ and $L=\sum_{1 \leqslant i \leqslant n} L_{i}(x) \frac{\partial}{\partial x^{i}}$. In such a chart, the $(2 n-1)$-th jet map $j^{2 n-1}(K, L): U \longrightarrow J^{2 n-1}\left(\left.\mathbb{X}^{2}(M)\right|_{U}\right)$ is concretely given by:

$$
U \ni x \longmapsto\left(\partial_{x}^{\alpha} K_{i}(x), \partial_{x}^{\alpha} L_{i}(x)\right)_{\alpha \in \mathbb{N}^{n},|\alpha| \leqslant 2 n-1,1 \leqslant i \leqslant n} .
$$

We denote by $K_{i, \alpha}$ and $L_{i, \alpha}$ the corresponding jet variables. A $\mathscr{C}^{2 n}$ local diffeomorphism $x \mapsto \bar{x}=\bar{x}(x)$ induces a triangular transformation involving the chain rule between these jets variables, with coefficients depending on
the $2 n$-th jet of $\bar{x}(x)$, some of which are only $\mathscr{C}^{0}$, which might be unpleasant. Fortunately, our bad set $\Sigma$ will be shown to be algebraic with respect to the jet variables $K_{i, \alpha}$ and $L_{i, \alpha}$ in any system of coordinates.

Let $(K, L) \in \mathbb{X}(M)^{2}$. To write shortly iterated Lie brackets, we denote $\operatorname{ad}(K) L:=[K, L]$, so that $\operatorname{ad}(K)^{2} L=[K,[K, L]], \operatorname{ad}(K)^{3} L=$ $[K,[K,[K, L]]]$ and so on. Also, we set $\operatorname{ad}^{0}(K) L:=L$. Define a subset $\Sigma \subset J^{2 n-1}\left(\mathbb{X}^{2}(M)\right)$ as a union $\Sigma=\Sigma^{\prime} \cup \Sigma^{\prime \prime} \cup \Sigma^{\prime \prime \prime}$, where:

- firstly $\Sigma^{\prime}$ is defined by the $2 n$ equations $K_{i, 0}=L_{i, 0}=0$;
- secondly, $\Sigma^{\prime \prime}$ is defined by requiring that all the $n \times n$ minors of the following $n \times(2 n)$ matrix

$$
\left(\operatorname{ad}^{0}(K) L \operatorname{ad}^{1}(K) L \cdots \cdots \operatorname{ad}^{2 n-1}(K) L\right)
$$

vanish;

- thirdly, $\Sigma^{\prime \prime \prime}$ is defined similarly, after exchanging $K$ with $L$.

Lemma 2.13. In the vector space of real $n \times(2 n)$ matrices, isomorphic to $\mathbb{R}^{2 n^{2}}$, the subset of matrices of rank $\leqslant(n-1)$ is a real algebraic subset of codimension equal to $(n+1)$.

Without obtaining a complete explicit expression, it is easily verified that $\operatorname{ad}^{j}(K)(L), 0 \leqslant j \leqslant 2 n-1$, is a universal polynomial in the jet variables $K_{i, \alpha}$ and $L_{i, \alpha}$. Under a local change of coordinates $x \mapsto \bar{x}(x)$, if the two vector fields $K$ and $L$ transform to $\bar{K}$ and to $\bar{L}$ (push-forward), all the multiple Lie brackets $\operatorname{ad}^{j}(K) L$ then transform to $\operatorname{ad}^{j}(\bar{K}) \bar{L}$, thanks to the invariance of Lie brackets. Geometrically, the vanishing of each of the $n \times n$ minors defining $\Sigma^{\prime \prime}$ and $\Sigma^{\prime \prime \prime}$ means the linear dependence of a system of $n$ vectors, thus it is an intrinsic condition. Consequently, although the jet variables $K_{i, \alpha}$ and $L_{i, \alpha}$ are transformed in an unpleasant way through diffeomorphisms, the sets $\Sigma^{\prime}, \Sigma^{\prime \prime}$ and $\Sigma^{\prime \prime \prime}$ may be defined by universal polynomials in the jet variables $K_{i, \alpha}$ and $L_{i, \alpha}$, that are the same in any system of local coordinates.

The lemma above and an inspection of a part of the complete expression of the $\operatorname{ad}^{j}(K)(L), 0 \leqslant j \leqslant 2 n-1$ provides the following information. Details will be skipped.
Lemma 2.14. The two subsets $\Sigma^{\prime \prime}$ and $\Sigma^{\prime \prime \prime}$ of $J^{2 n-1}\left(\mathbb{X}(M)^{2}\right)$ are both algebraic in the jet variables and of codimension $(n+1)$ outside $\Sigma^{\prime}$.

To conclude the proof of the theorem, we have to show that arbitrarily close to $(K, L)$, there are pairs of finite type. Since $\Sigma^{\prime}$ has codimension $2 n>$ $\operatorname{dim} M$, a first application of the avoidance Lemma 2.12 yields a perturbed pair, still denoted by $(K, L)$, with the property that at every point $p \in M$, either $K(p) \neq 0$ or $L(p) \neq 0$. Since $\Sigma^{\prime \prime}$ and $\Sigma^{\prime \prime \prime}$ both have codimension $n+1>\operatorname{dim} M$, a second application of the avoidance Lemma 2.12 yields
a perturbed pair such that the two collections of $2 n$ vector fields $\operatorname{ad}^{j}(K) L$, $0 \leqslant j \leqslant 2 n-1$, and $\operatorname{ad}^{j}(L) K, 0 \leqslant j \leqslant 2 n-1$, generate $T M$ at every point $p \in M$. The proof is complete.

To improve this theorem, let $r \geqslant 2$ and consider the set $\mathbb{X}(M)^{r}$ of collections of $r$ vector fields globally defined on $M$ that are $\mathscr{C}^{\kappa-1}$ for some $\kappa \geqslant 2$ to be chosen later. If $\mathbb{L}^{0}=\left\{L_{1}, L_{2}, \ldots, L_{r}\right\}$, is such a collection, its elements may be expressed in a local chart $\left(x_{1}, \ldots, x_{n}\right)$ as $L_{a}=\sum_{i=1}^{n} \varphi_{a, i}(x) \frac{\partial}{\partial x_{i}}$, for $a=1, \ldots, r$. Since the coefficients are $\mathscr{C}^{\kappa-1}$, it is possible to speak of $\mathbb{L}^{\lambda}$ only for $\lambda \leqslant \kappa$. We want to determine the smallest regularity $\kappa$ such that the set of $r$-tuples $\mathbb{L}^{0} \in \mathbb{X}(M)^{r}$ that are of finite type at every point of $M$ is open and dense in $\mathbb{X}(M)^{r}$ for the strong Whitney topology.

As in $\S 1.8(\mathrm{II})$, let $\mathrm{n}_{\kappa}(r)$ denote the dimension of the subspace $\mathrm{F}_{\kappa}(r)$ of the free Lie algebra $\mathrm{F}(r)$ that is generated as a real vector space by simple words (abstract Lie brackets) of length $\leqslant \kappa$. Then $\mathrm{n}_{\kappa}(r)$ is the maximal possible dimension of $\mathbb{L}^{\kappa}(p)$ at a point $p \in M$. We know that $\mathbb{L}^{\kappa}$ is generated by simple iterated Lie brackets of the form

$$
\left[L_{a_{1}},\left[L_{a_{2}}, \ldots,\left[L_{a_{\kappa-1}}, L_{a_{\lambda}}\right] \ldots\right]\right]
$$

for all $\lambda \leqslant \kappa$ and for certain (not all) $a_{i}$ with $1 \leqslant a_{1}, a_{2}, \ldots, a_{\lambda-1}, a_{\lambda} \leqslant r$ that depend on the choice of a Hall-Witt basis (Definition 1.9(II)) of $F_{\kappa}(r)$.

We choose $\kappa$ minimal so that $\mathrm{n}_{\kappa}(r) \geqslant 2 \operatorname{dim} M=2 n$. This fixes the smoothness of $M$. For $b=r+1, \ldots, \mathrm{n}_{k}(r)$, we order linearly as $L_{b}=\sum_{i=1}^{n} \psi_{b, i}(x) \frac{\partial}{\partial x_{i}}$ the chosen collection of iterated Lie brackets that generate $\mathbb{L}^{\kappa}$. If $\lambda=\lambda(b)$ denotes the length of $L_{b}$, namely $L_{b} \in \mathbb{L}^{\lambda(b)}$ of the form $L_{b}=\left[L_{a_{1}}, \ldots,\left[L_{a_{\lambda(b)-1}}, L_{a_{\lambda(b)}}\right] \ldots\right]$, there are universal differential polynomials $A_{a_{1}, \ldots, a_{\lambda(b)}}^{i}$ in the $(\lambda(b)-1)$-th jet of the coefficients $\varphi_{a, i}$ such that $\psi_{b, i}(x)=A_{a_{1}, \ldots, a_{\lambda(b)}}^{i}\left(J_{x}^{\lambda(b)-1} \varphi(x)\right)$. Also, in a fixed local system of coordinates, we form the $n \times(2 n)$ matrix

$$
\left(\varphi_{1, i} \ldots \varphi_{r, i} \psi_{r+1, i} \ldots \psi_{2 n, i}\right)_{1 \leqslant i \leqslant n}
$$

Similarly as in the proof of the previous theorem, we define a "bad" subset $\Sigma$ of $J^{\kappa-1}\left(\mathbb{X}(M)^{r}\right)$ by requiring that the dimension of $\mathbb{L}^{\kappa}(p)$ is $\leqslant(n-1)$ at every point $p \in M$. This geometric condition is intrinsic and neither depends on the choice of local coordinates nor on the choice of a Hall-Witt basis. Concretely, in a local system of coordinates, $\Sigma$ is described as the zero-set of all $n \times n$ minors of the above matrix. Thanks to Lemma 2.13 and to an inspection of a portion of the explicit expressions of the jet polynomials $A_{a_{1}, \ldots, a_{\lambda(b)}}^{i}\left(J_{x}^{\lambda(b)-1} \varphi(x)\right)$, we may establish the following assertion.

Lemma 2.15. The so defined subset $\Sigma=\left\{\operatorname{dim} \mathbb{L}^{\kappa}(p) \leqslant n-1, \forall p \in M\right\}$ of $J^{\kappa-1}\left(\mathbb{X}(M)^{r}\right)$ is algebraic in the jet variables and of codimension $(n+1)$.

Then an application of the avoidance Lemma 2.12 yields that, after an arbitrarily small perturbation of $\mathbb{L}^{0}$, still denoted by $\mathbb{L}^{0}$, we have $\mathbb{L}^{\kappa}(p)=$ $T_{p} M$ for every $p \in M$. Equivalently, the type $d(p)$ of $p$ is finite at every point and satisfies $d(p) \leqslant \kappa$.

Theorem 2.16. Let $r \geqslant 2$ be an integer and assume that the connected $n$ dimensional abstract manifold $M$ is $\mathscr{C}^{\kappa}$, where $\kappa$ is minimal with the property that the dimension $\mathrm{n}_{\kappa}(r)$ of the vector subspace $\mathrm{F}_{\kappa}(r)$, of the free Lie algebra $\mathrm{F}(r)$ having $r$ generators, that is generated by all brackets of length $\leqslant \kappa$, satisfies

$$
\mathrm{n}_{\kappa}(r) \geqslant 2 \operatorname{dim} M=2 n .
$$

Then the set of collections of $r$ vector fields $\mathbb{L}^{0} \in \mathbb{X}(M)^{r}$ that are of type $\leqslant \kappa$ at every point is open and dense in $\mathbb{X}(M)^{r}$ for the strong Whitney topology.

A more general problem about finite-typisation of vector field structures is concerned with general substructures of a given finite type structure.

Open question 2.17. Given a finite type collection $\mathbb{K}^{0}=\left\{K_{b}\right\}_{1 \leqslant b \leqslant s}, s \geqslant 3$, of $\mathscr{C}^{\kappa-1}$ vector fields on $M$ of class $\mathscr{C}^{\kappa}$ with the property that $\mathbb{K}^{\kappa}(p)=T_{p} M$ at every point and given a $\mathscr{C}^{\kappa-1}$ subsystem $\mathbb{L}^{0}=\left\{L_{a}\right\}_{1 \leqslant a \leqslant r}, 2 \leqslant r \leqslant s-1$, of the form $L_{a}=\sum_{1 \leqslant b \leqslant s} \psi_{a, b} K_{b}$, is it always possible to perturb slightly the functions $\psi_{a, b}: M \rightarrow \mathbb{R}$ so as to render $\mathbb{L}^{0}$ of finite type at every point? If so, what is the smallest regularity $\kappa$, in terms of $r, s$ and the highest type of $\mathbb{K}^{0}$ at points of $M$ ?

Finally, we mention a result similar to Theorem 2.16 that is valid in the $\mathscr{C}^{2}$ category and does not use any Lie bracket. It is based on Sussmann's orbit Theorem 1.21. The reference [Su1976] deals with several other genericity properties, motivated by Control Theory.

Theorem 2.18. ([Su1976]) Assume $r \geqslant 2$ and $\kappa \geqslant 2$. The set of collections $\mathbb{L}^{0}=\left\{L_{a}\right\}_{1 \leqslant a \leqslant r}$ of $r$ vector fields on a connected $\mathscr{C}^{\kappa}$ manifold $M$ so that $M$ consists of a single $\mathbb{L}$-orbit, is open and dense in $\mathbb{X}(M)^{r}$ equipped with the strong Whitney $\mathscr{C}^{\kappa-1}$ topology.
2.19. Transition. The next Section 3 exposes the point of view of Analysis, where vector field systems are considered as partial differential operators, until we come back to the applications of the notion of orbits to CR geometry in Section 4.

## §3. LOCALLY INTEGRABLE CR STRUCTURES

3.1. Local insolvability of partial differential equations. Until the 1950's, among analysts, it was believed and expected that all linear partial differential equations having smooth coefficients had local solutions ([Trv2000]). In fact, elliptic, parabolic, hyperbolic and constant coefficient equations were known to be locally solvable. Although his thesis subject was to confirm this expectation in full generality, in 1957, Hans Lewy ([Lew1957]) exhibited a striking and now classical counterexample of a $\mathscr{C}^{\infty}$ function $g$ in a neighborhood of the origin of $\mathbb{R}^{3}$, such that $\bar{L} f=g$ has no local solution at all. Here, $\bar{L}=\frac{\partial}{\partial \bar{z}}+z \frac{\partial}{\partial v}$ is the generator of the Cauchy-Riemann antiholomorphic bundle tangential to the Heisenberg sphere of equation $v=z \bar{z}$ in $\mathbb{C}^{2}$, equipped with coordinates $(z, w)=(x+i y, u+i v)$.

From the side of Analysis, almost absent in the two grounding works [Po1907] and [Ca1932] of Henri Poincaré and of Élie Cartan, Lewy's discovery constituted the birth of smooth linear PDE theory and of smoooth Cauchy-Riemann geometry. Later, in 1971, the simpler two-variables Mizohata equation $\frac{\partial f}{\partial x}-i x^{k} \frac{\partial f}{\partial y}=g$ was shown by Grushin to be non-solvable, if $k$ is odd, for certain $g$. One may verify that the set of smooth functions $g$ for which Lewy's or Grushin's equation is insolvable, even in the distributional sense, is generic in the sense of Baire. For $k=1$, the Mizohata vector field $\frac{\partial}{\partial x}-i x \frac{\partial}{\partial y}$ intermixes the holomorphic and antiholomorphic structures, depending on the sign of $x$.

In 1973 answering a question of Lewy, Nirenberg ([Ni1973]) exhibited a perturbation $\frac{\partial}{\partial x}-i x(1+\varphi(x, y)) \frac{\partial}{\partial y}$ of the Mizohata vector field, where $\varphi$ is $\mathscr{C}^{\infty}$ and null for $x \leqslant 0$, such that the only local solutions of $L f=0$ are the constants. A year later, in [Ni1974], he exhibited a perturbation of the Lewy vector field having the same property. A refined version is as follows.

Let $\Omega$ be a domain in $\mathbb{R}^{3}$, exhausted by a countable family of compact sets $K_{j}, j=1,2, \ldots$ with $K_{j} \subset \operatorname{Int} K_{j+1}$. If $f \in \mathscr{C}^{\infty}(\Omega, \mathbb{C})$, define the Fréchet semi-norms $\rho_{j}(f):=\max _{x \in K_{j},|\alpha| \leqslant j}\left|\partial_{x}^{\alpha} f(x)\right|$ and topologize $\mathscr{C}^{\infty}(\Omega, \mathbb{C})$ by means of the metric $d(f, g):=\sum_{j=1}^{\infty} \frac{\rho_{j}(f-g)}{1+\rho_{j}(f-g)}$. Consider the set

$$
\widehat{\mathbf{L}}:=\left\{L=\sum_{j=1}^{3} a_{j}(x) \frac{\partial}{\partial x_{j}}: a_{j} \in \mathscr{C}^{\infty}(\Omega, \mathbb{C})\right\},
$$

equipped with this topology for each coefficient $a_{j}$.
Theorem 3.2. ([JT1982, Ja1990]) The set of $L \in \widehat{\mathbf{L}}$ for which the solutions $u \in \mathscr{C}^{1}(\Omega, \mathbb{C})$ of $L u=0$ are the constants only, is dense in $\widehat{\mathbf{L}}$.

These phenomena and others were not suspected at the time of Lie, of Poincaré, of É. Cartan, of Vessiot and of Janet, when PDE theory was focused on the algebraic complexity of systems of differential equations having analytic coefficients. In 1959, Hörmander explained the behavior of the Lewy counter-example, as follows. The references [Trv1970, Trv1986, ES1993, Trv2000] provide further survey informations about operators of principal type, operators with multiple characteristics, pseudodifferential operators, hypoelliptic operators, microlocal analysis, etc.

Let $P=P\left(x, \partial_{x}\right)=\sum_{\alpha \in \mathbb{N}^{n},|\alpha| \leqslant m} a_{\alpha}(x) \partial_{x}^{\alpha}$ be a linear partial differential operator of degree $m$ having $\mathscr{C}^{\infty}$ complex-valued coefficients $a_{\alpha}: \Omega \rightarrow \mathbb{C}$ defined in a domain $\Omega \subset \mathbb{R}^{n}$. Its symbol $P(x, \xi):=\sum_{|\alpha| \leqslant m} a_{\alpha}(x)(i \xi)^{\alpha}$ is a function from the cotangent $T^{*} \Omega \equiv \Omega \times \mathbb{R}^{n}$ to $\mathbb{C}$. Its principal symbol is the homogeneous degree $m$ part $P_{m}(x, \xi):=\sum_{|\alpha|=m} a_{\alpha}(x)(i \xi)^{\alpha}$. The cone of points $(x, \xi) \in \Omega \times\left(\mathbb{R}^{n} \backslash\{0\}\right)$ such that $P_{m}(x, \xi)=0$ is the characteristic set of $P$, the locus of the obstructions to existence as well as to regularity of solutions $f$ of $P\left(x, \partial_{x}\right) f=g$.

The real characteristics of $P$ are called simple if, at every characteristic point $\left(x_{0}, \xi_{0}\right)$ with $\xi_{0} \neq 0$, the differential $d_{\xi} P_{m}=\sum_{k=1}^{n} \frac{\partial P_{m}}{\partial \xi_{k}} d \xi_{k}$ with respect to $\xi$ is nonzero. It follows from homogeneity and from Euler's identity that the zeros of $P$ are simple, so the characteristic set is a regular hypersurface of $\Omega \times\left(\mathbb{R}^{n} \backslash\{0\}\right)$. One can show that this assumption entails that the behaviour of $P$ is the same as that of $P_{m}$ : in a certain rigorous sense, lower order terms may be neglected. In his thesis (1955), Hörmander called such operators of principal type, a label that has stuck ([Trv1970]).

Call $P$ solvable at a point $x_{0} \in \Omega$ if there exists a neighborhood $U$ of $x_{0}$ such that for every $g \in \mathscr{C}^{\infty}(U)$, there exists a distribution $f$ supported in $U$ that satisfies $P f=g$ in $U$. In 1955, Hörmander had shown that a principal type partial differential operator $P$ is locally solvable if all the coefficients $a_{\alpha}(x),|\alpha|=m$, of its principal part $P_{m}$ are real-valued. On the contrary, if they are complex-valued, in 1959, he showed:

Theorem 3.3. ([Нӧ1963]) If the quantity

$$
\sum_{k=1}^{n} \frac{\overline{\partial P_{m}(x, \xi)}}{\partial \xi_{k}} \frac{\partial P_{m}(x, \xi)}{\partial x_{k}}
$$

is nonzero at a characteristic point $\left(x_{0}, \xi_{0}\right) \in T^{*} \Omega$, for some $\xi_{0} \neq 0$, then $P$ is insolvable at $x_{0}$.

With $\bar{P}_{m}\left(x, \partial_{x}\right):=\sum_{|\alpha|=m} \overline{a_{\alpha}(x)} \partial_{x}^{\alpha}$, denote by $C_{2 m-1}(x, \xi)$ the principal symbol of the commutator $\left[P_{m}\left(x, \partial_{x}\right), \bar{P}_{m}\left(x, \partial_{x}\right)\right]$, obviously zero if $P_{m}$ has real coefficients. The above necessary condition for local solvability
may be rephrased as: if $P$ is locally solvable at $x_{0}$, then for all $\xi \in \mathbb{R}^{n} \backslash\{0\}$ :

$$
P_{m}(x, \xi)=0 \Longrightarrow C_{2 m-1}(x, \xi)=0 .
$$

This condition explained the non-solvability of the Lewy operator appropriately.
3.4. Condition ( $\mathbf{P}$ ) of Nirenberg-Treves and local solvability. The geometric content of the above necessary condition was explored and generalized by Nirenberg-Treves ([NT1963, NT1970, Trv1970]). Recall that the Hamiltonian vector field associated to a function $f=f(x, \xi) \in \mathscr{C}^{1}\left(\Omega \times \mathbb{R}^{n}\right)$ is $H_{f}:=\sum_{k=1}^{n}\left(\frac{\partial f}{\partial \xi_{k}} \frac{\partial}{\partial x_{k}}-\frac{\partial f}{\partial x_{k}} \frac{\partial}{\partial \xi_{k}}\right)$. A bicharacteristic of the real part $A(x, \xi)$ of $P_{m}(x, \xi)$ is an integral curve of $H_{A}$, namely:

$$
\frac{d x}{d t}=\operatorname{grad}_{\xi} A(x, \xi), \quad \frac{d \xi}{d t}=-\operatorname{grad}_{x} A(x, \xi)
$$

It follows at once that the function $A(x, \xi)$ must be constant along its bicharacteristics. When the constant is zero, a bicharacteristic is called a null bicharacteristic. In particular, null bicharacteristics are contained in the characteristic set, which explains the terminology.

Then Hörmander's necessary condition may be interpreted as follows. Let $B(x, \xi)$ be the imaginary part of $P_{m}(x, \xi)$. An immediate computation shows that the principal symbol of $\left[A\left(x, \partial_{x}\right), B\left(x, \partial_{x}\right)\right]$ is given by:

$$
C_{1}(x, \xi)=\sum_{k=1}^{n}\left\{\frac{\partial A}{\partial \xi_{k}}(x, \xi) \frac{\partial B}{\partial x_{k}}(x, \xi)-\frac{\partial B}{\partial \xi_{k}}(x, \xi) \frac{\partial A}{\partial x_{k}}(x, \xi)\right\} .
$$

Equivalently,

$$
C_{1}(x, \xi)=(d B / d t)(x, \xi) .
$$

Theorem 3.3 says that the nonvanishing of $C_{1}$ at a characteristic point entails insolvability. In fact, Nirenberg-Treves observed that if the order of vanishing of $B$ along the null characteristic of $A$ is odd then insolvability holds. Beyond finite order of vanishing, what appeared to matter is only the change of sign. Since the equation $P f=g$ has the same solvability properties as $z P f=g$, for all $z \in \mathbb{C} \backslash\{0\}$, this led to the following:

Definition 3.5. ([NT1963, NT1970]) A differential operator $P$ of principal type is said to satisfy condition ( $\mathbf{P}$ ) if, for every $z \in \mathbb{C} \backslash\{0\}$, the function $\operatorname{Im}\left(z P_{m}\right)$ does not change sign along the null bicharacteristic of $\operatorname{Re}\left(z P_{m}\right)$.

The next theorem has been shown for $P$ having $\mathscr{C}^{\omega}$ coefficients and in certain cases for $P$ having $\mathscr{C}^{\infty}$ coefficients by Nirenberg-Treves, and finally, in the general $\mathscr{C}^{\infty}$ category by Beals-Fefferman (sufficiency) and by Moyer (necessity).

Theorem 3.6. ([NT1963, Trv1970, NT1970, BeFe 1973, Trv1986]) Condition ( $\mathbf{P}$ ) is necessary and sufficient for the local solvability in $L^{2}$ of a principal type linear partial differential equation $P f=g$.

Except for complex and strongly pseudoconvex structures, little is known about solvability of $\mathscr{C}^{\infty}$ systems of PDE's, especially overdetermined ones ([Trv2000]). In the sequel, only vector field systems (order $m=1$ ), studied for themselves, will be considered.
3.7. Involutive and CR structures. Following [Trv1981, $\operatorname{Trv} 1992$, BCH2005], let $M$ be a $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}(\kappa \geqslant 2,0<\alpha<1)$ paracompact Hausdorff second countable abstract real manifold of dimension $\mu \geqslant 1$ and let $\mathscr{L}$ be a $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa-1, \alpha}$ complex vector subbundle of $\mathbb{C} T M:=\mathbb{C} \otimes T M$ of rank $\lambda$, with $1 \leqslant \lambda \leqslant \mu$. Denote by $\mathscr{L}_{p}$ its fiber at a point $p \in M$. Denote by $\mathscr{T}$ the orthogonal of $\mathscr{L}$ for the duality between differential forms and vector fields. It is a vector subbundle of $\mathbb{C} T^{*} M$, whose fiber at a point $p \in M$ is $\mathscr{L}_{p}^{\perp}=\left\{\varpi \in \mathbb{C} T_{p}^{*} M: \varpi=0\right.$ on $\left.\mathscr{L}_{p}\right\}$. The characteristic set $\mathscr{C}:=\mathscr{T} \cap T^{*} M\left(\right.$ real $\left.T^{*} M\right)$ is in general not a vector bundle: the dimension of $\mathscr{C}_{p}^{0}$ may vary with $p$, as shown for instance by the bundle generated over $\mathbb{R}^{2}$ by the Mizohata operator $\partial_{x}-i x \partial_{y}$.

From now on, we shall assume that the bundle $\mathscr{L}$ is formally integrable, i.e. that $[\mathscr{L}, \mathscr{L}] \subset \mathscr{L}$. Then $\mathscr{L}$ defines:

- an elliptic structure if $\mathscr{C}_{p}=0$ for all $p \in M$;
- a complex structure of $\mathscr{L}_{p} \oplus \overline{\mathscr{L}_{p}}=\mathbb{C} T_{p} M$ for all $p \in M$;
- a Cauchy-Riemann (CR for short) structure if $\mathscr{L}_{p} \cap \overline{\mathscr{L}_{p}}=\{0\}$ for all $p \in M$;
- an essentially real structure if $\mathscr{L}_{p}=\overline{\mathscr{L}_{p}}$, for all $p \in M$.

In general, $\mathscr{L}$ will be called an involutive structure if $[\mathscr{L}, \mathscr{L}]=\mathscr{L}$. Let us summarize basic linear algebra properties ([Trv1981, Trv1992, BCH2005]). Every essentially real structure is locally generated by real vector fields. Every complex structure is elliptic. If $\mathscr{L}$ is a CR structure (often called abstract), the characteristic set $\mathscr{C}$ is in fact a vector subbundle of $T^{*} M$ of rank $\mu-2 \lambda$; this integer is the codimension of the CR structure. A CR structure is of hypersurface type if its codimension equals 1.
3.8. Local integrability and generic submanifolds of $\mathbb{C}^{n}$. The bundle $\mathscr{L}$ is locally integrable if every $p \in M$ has a neighborhood $U_{p}$ in which there exist $\tau:=\mu-\lambda$ functions $z_{1}, \ldots, z_{\tau}: U_{p} \rightarrow \mathbb{C}$ of class $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$ whose differentials $d z_{1}, \ldots, d z_{\tau}$ are linearly independent and span $\left.\mathscr{T}\right|_{U_{p}}$ (or equivalently, are annihilated by sections of $\mathscr{L}$ ). In other words, the homogeneous PDE system $\mathscr{L} f=0$ has the best possible space of solutions.

Here is a canonical example of locally integrable structure. Consider a generic submanifold $M$ of $\mathbb{C}^{n}$ of class $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}, \kappa \geqslant 1,0 \leqslant \alpha \leqslant 1$, as defined in $\S 2.1$ (II) and in $\S 4.1$ below. Let $d \geqslant 0$ be its codimension and let $m=n-d \geqslant 0$ be its CR dimension. Let $T^{c} M=T M \cap J T M$ (a real vector bundle) and let $\mathbb{C} T M=\mathbb{C} \otimes T M$. Define the two complex subbundles $T^{1,0} M$ and $T^{0,1} M=\overline{T^{1,0} M}$ of $\mathbb{C} T M$ whose fibers at a point $p \in M$ are:
$\left\{\begin{array}{l}T_{p}^{1,0} M=\left\{X_{p}+i J X_{p}: X_{p} \in T_{p}^{c} M\right\}=\left\{Z_{p} \in \mathbb{C} T_{p} M: J Z_{p}=-i Z_{p}\right\}, \\ T_{p}^{0,1} M=\left\{X_{p}-i J X_{p}: X_{p} \in T_{p}^{c} M\right\}=\left\{Z_{p} \in \mathbb{C} T_{p} M: J Z_{p}=i Z_{p}\right\} .\end{array}\right.$
Geometrically, $T^{1,0} M$ and $T^{0,1} M$ are just the traces on $M$ of the holomorphic and anti-holomorphic bundles $T^{1,0} \mathbb{C}^{n}$ and $T^{0,1} \mathbb{C}^{n}$, whose fibers at a point $p$ are $\left.\sum_{k=1}^{n} a_{k} \frac{\partial}{\partial z_{k}}\right|_{p}$ and $\left.\sum_{k=1}^{n} b_{k} \frac{\partial}{\partial \bar{z}_{k}}\right|_{p}$. They satisfy the Frobenius involutivity conditions $\left[T^{1,0} M, T^{1,0} M\right] \subset T^{1,0} M$ and $\left[T^{0,1} M, T^{0,1} M\right] \subset$ $T^{0,1} M$. More detailed background information may be found in [Ch1991, Bo1991, Trv 1992, BER1999].

On such an embedded generic submanifold $M$, choose as structure bundle $\mathscr{L}$ just $T^{0,1} M \subset \mathbb{C} T M$. Then clearly, the $n$ holomorphic coordinate functions $z_{1}, \ldots, z_{n}$ are annihilated by the anti-holomorphic local sections $\sum_{k=1}^{n} b_{k} \frac{\partial}{\partial \bar{z}_{k}}$ of $T^{0,1} M$ and they have linearly independent differential, at every point of $M$. A generic submanifold embedded in $\mathbb{C}^{n}$ carries a locally integrable involutive structure. Conversely:

Lemma 3.9. Every locally integrable CR structure is locally realizable as the anti-holomorphic structure induced on a generic submanifold embedded $\mathbb{C}^{n}$.

Proof. Indeed, if a real $\mu$-dimensional $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$ manifold $M$ bears a locally integrable CR structure, the map $Z=\left(z_{1}, \ldots, z_{\tau}\right)$ produces an embedding of the open set $U_{p}$ as a local generic submanifold $M:=Z\left(U_{p}\right)$ of $\mathbb{C}^{\tau}$, with $Z_{*}(\mathscr{L})=T^{0,1} M$.

A locally integrable CR structure is sometimes called locally realizable or locally embeddable.
3.10. Levi form. Let $\mathscr{L}$ be an involutive structure, not necessarily locally integrable and let $\mathrm{c}_{p} \in \mathscr{C}_{p} \subset T_{p}^{*} M$ be a nonzero characteristic covector at $p$.

Definition 3.11. The Levi form at $p$ in the characteristic codirection $\mathrm{c}_{p} \in$ $\mathscr{C}_{p} \backslash\{0\} \subset T_{p}^{*} M \backslash\{0\}$ is the Hermitian form acting on two vectors $X_{p}, Y_{p} \in$ $\mathscr{L}(p)$ as:

$$
\mathfrak{L}_{p, c_{p}}\left(X_{p}, \bar{Y}_{p}\right):=\frac{1}{2 i} \mathrm{c}_{p}([X, \bar{Y}]),
$$

where $X, Y$ are any two sections of $\mathscr{L}$ defined in a neighborhood of $p$ and satisfying $X(p)=X_{p}, Y(p)=Y_{p}$. The resulting number is independent of the choice of such extensions $X, Y$.

For the study of realizability of CR structures of codimension one, nondegeneracy of the Levi form, especially positivity or negativity, is of crucial importance. An abstract CR structure of hypersurface type whose Levi form has a definite signe is said to be strongly pseudoconvex, since, after a possible rescaling of sign of a nonzero characteristic covector, all the eigenvalues of its Levi form are positive.
3.12. Nonembeddable CR structures. After Lewy's discovery, the first example of a smooth strictly pseudoconvex CR structure in real dimension 3 which is not locally embeddable was produced by Nirenberg in 1973 ([Ni1973]), cf. Theorem 3.3 above. For CR structures of hypersurface type, Nirenberg's work has been generalized in higher dimension under the assumption that the Levi form is neither positive nor negative, in any characteristic codirection. Let $n \geqslant 2$ and let $\varepsilon_{1}=1, \varepsilon_{k}=-2, k=2, \ldots, n$.

Theorem 3.13. ([JT1982, BCH2005]) There exists a $\mathscr{C}^{\infty}$ complex-valued function $g=g(x, y, s)$ defined in a neighborhood of the origin in $\mathbb{C}^{n} \times \mathbb{R}$ and vanishing to infinite order along $\left\{x_{1}=y_{1}=0\right\}$ such that the vector fields:

$$
\widehat{L}_{j}:=\frac{\partial}{\partial \bar{z}_{j}}-i \varepsilon_{j} z_{j}(1+g(x, y, s)) \frac{\partial}{\partial s},
$$

are pairwise commuting and such that every $\mathscr{C}^{1}$ solution $h$ of $\widehat{L}_{j} h=0, j=$ $1,2, \ldots, n$ defined in a neighborhood of the origin must satisfy $\frac{\partial h}{\partial s}(0)=0$.

This entails that the involutive structure spanned by the $\widehat{L}_{j}$ is not locally integrable at 0 . One may establish that the set of such $g$ is generic. Crucially, the Levi-form is of signature $(n-1)$.

Open problem 3.14. Find versions of generic non-embeddability for CR structures of codimension 1 having degenerate Levi-form. Find higher codimensional versions of generic non-embeddability.
3.15. Integrability of complex structures and embeddability of strongly pseudoconvex CR structures. Let us now expose positive results. Every formally integrable essentially real structure $\mathscr{L}=\operatorname{Re} \mathscr{L}$ is locally integrable, thanks to Frobenius' theorem; however the condition $[\mathscr{L}, \mathscr{L}] \subset \mathscr{L}$ entails the involutivity of $\operatorname{Re} \mathscr{L}$ only in this special case. Also, every analytic formally integrable CR structure is locally integrable: it suffices to
complexify the coefficients of a generating set of vector fields and to apply the holomorphic version of Frobenius' theorem. For complex structure, the proof is due to Libermann (1950) and to Eckman-Frölicher (1951), see [AH1972a, Trv1981, Trv 1992].

Theorem 3.16. Smooth complex structures are locally integrable.
This deeper fact has a long history, which we shall review concisely. On real analytic surfaces, isothermal coordinates where discovered by Gauss in 1825-26, before he published his Disquisitiones generales circa superficies curvas. In the 1910's, by a nontrivial advance, Korn and Lichtenstein transferred this theorem to Hölder continuous metrics.

Theorem 3.17. Let $d s^{2}=E d u^{2}+2 F d u d v+G d v^{2}$ be a $\mathscr{C}^{0, \alpha}(0<\alpha<1)$ Gaussian metric defined in some neighborhood of 0 in $\mathbb{R}^{2}$. Then there exists $\underset{\sim}{a} \mathscr{C}^{1, \alpha}$ change of coordinates $(u, v) \mapsto(\tilde{u}, \tilde{v})$ fixing 0 and a $\mathscr{C}^{0, \alpha}$ function $\tilde{\lambda}=\tilde{\lambda}(\tilde{u}, \tilde{v})$ such that:

$$
\tilde{\lambda}\left(d \tilde{u}^{2}+d \tilde{v}^{2}\right)=E d u^{2}+2 F d u d v+G d v^{2} .
$$

A modern proof of this theorem based on the complex notation and on the $\bar{\partial}$ formalism was provided by Bers ([Be1957]) and by Chern ([Ch1955]). In the monograph [Ve1962], deeper weakenings of smoothness assumptions are provided.

As a consequence of this theorem, complex structures of class $\mathscr{C}^{0, \alpha}$ on surfaces may be shown to be locally integrable. Let us explain in length this corollary.

At first, remind that an almost complex structure on $2 n$-dimensional manifold $M$ is a smoothly varying field $J=\left(J_{p}\right)_{p \in M}$ of endomorphisms of $T_{p} M$ satisfying $J_{p} \circ J_{p}=-\mathrm{Id}$. Thanks to $J$, as in the standard complex case, one may define $T_{p}^{0,1} M:=\left\{X_{p}+i J_{p} X_{p}: X_{p} \in T_{p} M\right\}$ and then the bundle $\mathscr{L}:=T^{0,1} M$ is a complex structure in the PDE sense of $\S 3.7$ above. Conversely, given a complex structure $\mathscr{L} \subset \mathbb{C} T M$, then locally in some neighborhood $U_{p}$ of an arbitrary point $p \in M$, there exist local coordinates $\left(x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right)$ vanishing at $p$ so that $n$ complex vector fields of the form:

$$
Z_{j}:=\sum_{k=1}^{n} a_{k, j} \partial_{x_{k}}+i \sum_{k=1}^{n} b_{k, j} \partial_{y_{k}},
$$

with $a_{k, j}(0)=\delta_{k, j}=b_{k, j}(0)$, span $\left.\mathscr{L}\right|_{U_{p}}$. The associated almost complex structure is obtained by declaring that, at a point of coordinates $(x, y)$, one has:

$$
J\left(\sum_{k=1}^{n} a_{k, j} \partial_{x_{k}}\right)=\sum_{k=1}^{n} b_{k, j} \partial_{y_{k}} \text { and } J\left(\sum_{k=1}^{n} b_{k, j} \partial_{y_{k}}\right)=-\sum_{k=1}^{n} a_{k, j} \partial_{x_{k}} .
$$

Lemma 3.18. The bundle $\mathscr{L} \subset \mathbb{C} T M$ satisfies $[\mathscr{L}, \mathscr{L}] \subset \mathscr{L}$ if and only if, for every two vector fields $X$ and $Y$ on $M$, the Nijenhuis expression:

$$
N(X, Y):=[J X, J Y]-J[X, J Y]-J[J X, Y]-[X, Y]
$$

vanishes identically.
The proof is abstract nonsense. Also, one verifies that $N(f X, g Y)=$ $f g N(X, Y)$ for every two smooth local function $f$ and $g$ : the expression $N$ is of tensorial character. In symplectic and in almost complex geometry ([MS1995]), the following is settled.

Definition 3.19. The almost complex structure is called integrable if, in some neighborhood $U_{p}$ of every point $p \in M$ there exist $n$ complex-valued functions $z_{1}, \ldots, z_{n}: U_{p} \rightarrow \mathbb{C}$ of class at least $\mathscr{C}^{1}$ and having linearly independent differentials such that $d z_{k} \circ J=i \circ d z_{k}$, for $k=1, \ldots, n$.

One verifies that it is equivalent to require $\mathscr{L} z_{k}=0, k=1, \ldots, n$ : integrability of an almost complex structure coincides with local integrability of $\mathscr{L}=T^{0,1} M$.

Now, we may come back to the integrability Theorem 3.16. To an arbitrary Gaussian metric $g=d s^{2}$ as in Theorem 3.17, with $E>0, G>0$ and $E G-F^{2}>0$, are associated both a volume form and an almost complex structure:

$$
d \mathrm{vol}_{g}:=\sqrt{E G-F^{2}} d u \wedge d v \text { and } J_{g}:=\frac{1}{\sqrt{E G-F^{2}}}\left(\begin{array}{cc}
-F & -G \\
E & F
\end{array}\right) .
$$

Conversely, given a volume form and an almost complex structure $J$ on a surface, an associated Riemannian metric is provided by:

$$
g(\cdot, \cdot):=d \operatorname{vol}(\cdot, J \cdot)
$$

According to Korn's and Lichtenstein's theorem, there exist coordinates in which the metric is conformally flat, equal to $\lambda\left(d u^{2}+d v^{2}\right)$. In these coordinates, the associated complex structure is obviously the standard one: $J \partial_{u}=\partial_{v}$ and $J \partial_{v}=-\partial_{u}$. In fact, any local change of coordinates $(u, v) \mapsto(\tilde{u}, \tilde{v})$ which respects orthogonality of the curvilinear coordinates, i.e. transforms the Gaussian isothermal metric to a similar one $\tilde{\lambda}\left(d \tilde{u}^{2}+d \tilde{v}^{2}\right)$, commutes with $J$, so that the map $u+i v \mapsto \tilde{u}+i \tilde{v}$ is holomorphic. In conclusion:

Theorem 3.20. $\mathscr{C}^{0, \alpha}(0<\alpha<1)$ complex structures are locally integrable.
The generalization to several variables of the theorem of Korn and Lichtenstein is due to Newlander-Nirenberg, who solved a question raised by Chern. The proof was modified and the smoothness assumption was
perfected by several mathematicians: Nijenhuis-Woolf ([NW1963]), Malgrange, Kohn, Hörmander, Nirenberg, Treves ([Trv 1992]) and finally Webster ([We1989c]) who used the Leray-Koppelman $\bar{\partial}$ homotopy formula together with the Nash-Moser rapidly convergent iteration scheme for solving nonlinear functional equations.
Theorem 3.21. $\left(\mathscr{C}^{2 n, \alpha}, 0<\alpha<1\right.$ : [NN1957]; $\mathscr{C}^{1, \alpha}, 0<\alpha<1$ : [NW1963, We1989c]; $\mathscr{C}^{\infty}$ : [Trv1992]) Suppose that on the real manifold $M$ of dimension $2 n \geqslant 4$, the formally integrable complex structure $\mathscr{L}$ is $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa-1, \alpha}, \kappa \geqslant 2,0<\alpha<1$. Then there exist local complex-valued coordinates $\left(z_{1}, \ldots, z_{n}\right)$ annihilated by $\mathscr{L}$ which are $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$.

Finally, an elementary linear algebra argument ([Trv1981, Trv1992, BCH2005]) enables to deduce local integrability of $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa-1, \alpha}$ elliptic structures from the above theorem. In fact, elliptic structures are shown to be locally isomorphic to $\mathbb{C}^{\tau} \times \mathbb{R}^{\lambda-\tau}$, equipped with $\frac{\partial}{\partial \bar{z}_{i}}, \frac{\partial}{\partial t_{j}}$.
Problem 3.22. Is a formally integrable involutive structure having positivedimensional characteristic set locally integrable?

Again the history is rich. Integrability results are known only for strongly pseudoconvex CR structures of hypersurface type. Solving a question raised by Kohn in 1965, Kuranishi ([Ku1982]) showed in 1982 that $\mathscr{C}{ }^{\infty}$ strongly pseudoconvex abstract CR structures of dimension $\geqslant 9$ are locally realizable. His delicate proof involved a study of the Neumann operator in $L^{2}$ spaces, for solving the tangential Cauchy-Riemann equations, together with the Nash-Moser argument. In 1987, Akahori ([Ak1987]) modified the technique of Kuranishi and included the case of dimension 7.

In 1989, to solve an associated linearized problem, instead of the Neumann operator, Webster used the totally explicit integral operators of Henkin.
Theorem 3.23. ([We1989a, We1989b]) Let $M$ be a strongly pseudoconvex (2n-1)-dimensional CR manifold of class $\mathscr{C}^{\mu}$. Then $M$ admit, locally near each point, a holomorphic embedding of class $\mathscr{C}^{\kappa}$, provided

$$
n \geqslant 4, \quad \kappa \geqslant 21, \quad \mu \geqslant 6 \kappa+5 n-3
$$

The main new ingredient in his proof was Henkin' local homotopy operator $\bar{\partial}_{M}$ on a hypersurface $M \subset \mathbb{C}^{n}$ :

$$
f=\bar{\partial}_{M} P(f)+Q\left(\bar{\partial}_{M} f\right), \quad f \text { a }(0,1)-\text { form },
$$

known to hold for $n \geqslant 4$. For this reason, Webster suspected the existence of refinements based on an insider knowledge of $\bar{\partial}$ techniques. In 1994, using a modified homotopy formula yielding better $\mathscr{C}^{\kappa}$-estimates, MaMichel [MM1994] improved smoothness:

$$
\kappa \geqslant 18, \quad \mu \geqslant \kappa+13 .
$$

Up to now, the five dimensional remains open. In fact, the solvability of $\bar{\partial}_{M} f=g$ for a $(0,1)$-form on a hypersurface of $\mathbb{C}^{3}$ requires a special trick which does not lead to a homotopy formula. Nagel-Rosay [NR1989] showed the nonexistence of a homotopy formula in the 5 -dimensional case, emphasizing an obstacle.
Open problem 3.24. Find generalizations of the Kuranishi-Akahori-Webster-Ma-Michel theorem to higher codimension, using the integral formulas for solving the $\bar{\partial}_{M}$ due to Ayrapetian-Henkin. Replace the assumption of strong pseudoconvexity by finer nondegeneracy conditions, e.g. weak pseudoconvexity and finite type in the sense of Kohn.
3.25. Local generators of locally integrable structures. Abandoning these deep problems of local solvability and of local realizability, let us survey basic properties of locally integrable structures. Thus, let $\mathscr{L}$ be a $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa-1, \alpha}$ locally integrable structure of rank $\lambda$ on a $\mathscr{C}^{\kappa, \alpha}$ or $\mathscr{C}^{\infty}$ manifold $M$ of dimension $\mu$. Denote by $\tau=\mu-\lambda$ the dimension of $\mathscr{T}=\mathscr{L}^{\perp}$. Let $p \in M$ and let $\delta_{p}$ denote the dimension of $\mathscr{C}_{p}=\mathscr{T} \cap T_{p}^{*} M$. Notice that $\left(\tau-\delta_{p}\right)+\left(\tau-\delta_{p}\right)+\delta_{p}+\left(\lambda-\tau+\delta_{p}\right)=\tau+\lambda=\mu$ just below.
Theorem 3.26. ([Trv1981, Trv1992, BCH2005]) There exist real coordinates:

$$
\left(x_{1}, \ldots, x_{\tau-\delta_{p}}, y_{1}, \ldots, y_{\tau-\delta_{p}}, u_{1}, \ldots, u_{\delta_{p}}, s_{1}, \ldots, s_{\lambda-\tau+\delta_{p}}\right),
$$

defined in a neighborhood $U_{p}$ of $p$ and vanishing at $p$, and there exist $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$ functions $\varphi_{j}=\varphi_{j}(x, y, u, s)$ with $\varphi_{j}(0)=0, d \varphi_{j}(0)=0, j=$ $1, \ldots, \delta_{p}$, such that the differentials of the $\tau$ functions:

$$
\left\{\begin{array}{l}
z_{k}:=x_{k}+i y_{k}, \quad k=1, \ldots, \tau-\delta_{p}, \\
w_{j}:=u_{j}+i \varphi_{j}(x, y, u, s), \quad j=1, \ldots, \delta_{p}
\end{array}\right.
$$

$\left.\operatorname{span} \mathscr{T}\right|_{U_{p}}$.
Since $d \varphi_{j}(0)=0$, there exist unique coefficients $b_{l, j}=b_{l, j}(x, y, u, s)$ such that the vector fields:

$$
K_{j}:=\sum_{l=1}^{\delta_{p}} b_{l, j} \frac{\partial}{\partial u_{l}}, \quad k=1, \ldots, \delta_{p}
$$

satisfy $K_{j_{1}}\left(w_{j_{2}}\right)=\delta_{j_{1}, j_{2}}$, for $j_{1}, j_{2}=1, \ldots, \delta_{p}$. Define then the $\lambda$ vector fields:

$$
\left\{\begin{aligned}
\bar{L}_{k}:=\frac{\partial}{\partial \bar{z}_{k}}-i \sum_{l=1}^{\delta_{p}} \frac{\partial \varphi_{l}}{\partial \bar{z}_{k}} K_{l}, & k=1, \ldots, \tau-\delta_{p}, \\
L_{j}^{\prime}:=\frac{\partial}{\partial s_{j}}-i \sum_{l=1}^{\delta_{p}} \frac{\partial \varphi_{l}}{\partial s_{j}} K_{l}, & j=1, \ldots, \lambda-\tau+\delta_{p} .
\end{aligned}\right.
$$

Clearly, $0=\bar{L}_{k_{1}}\left(z_{k_{2}}\right)=\bar{L}_{k}\left(w_{j}\right)=L_{j}^{\prime}\left(z_{k}\right)=L_{j_{1}}^{\prime}\left(w_{j_{2}}\right)$, hence the structure bundle $\left.\mathscr{L}\right|_{U_{p}}$ is spanned by the $L_{k}, L_{j}^{\prime}$. One may verify the commutation relations ([Trv1981, Trv1992, BCH2005]):

$$
\begin{aligned}
& 0=\left[\bar{L}_{k_{1}}, \bar{L}_{k_{2}}\right]=\left[\bar{L}_{k}, L_{j}^{\prime}\right]=\left[L_{j_{1}}^{\prime}, L_{j_{2}}^{\prime}\right], \\
& 0=\left[\bar{L}_{k}, K_{j}\right]=\left[L_{j_{1}}^{\prime}, K_{j_{2}}\right]=\left[K_{j_{1}}, K_{j_{2}}\right] .
\end{aligned}
$$

Remind that if an involutive structure $\mathscr{L}$ is CR , then $\delta_{p}$ is independent of $p$ and equal to the codimension $\mu-2 \lambda=: \delta$. It follows that $\tau-\delta=\tau-\mu+2 \lambda=$ $\lambda$, or $\lambda-\tau+\delta=0$ : this means that the variables $\left(s_{1}, \ldots, s_{\lambda-\tau+\delta_{p}}\right)$ disappear.
Corollary 3.27. In the case of a CR structure of codimension $\delta$, the local integrals are:

$$
\left\{\begin{array}{l}
z_{k}:=x_{k}+i y_{k}, \quad k=1, \ldots, \lambda, \\
w_{j}:=u_{j}+i \varphi_{j}(x, y, u), \quad j=1, \ldots, \delta
\end{array}\right.
$$

and a local basis for the structure bundle $\left.\mathscr{L}\right|_{U_{p}}$ is:

$$
\bar{L}_{k}:=\frac{\partial}{\partial \bar{z}_{k}}-i \sum_{l=1}^{\delta} \frac{\partial \varphi_{l}}{\partial \bar{z}_{k}} K_{l}, \quad k=1, \ldots, \lambda .
$$

We recover a generic submanifold embedded in $\mathbb{C}^{\tau}$ which is graphed by the equations $v_{j}=\varphi_{j}(x, y, u)$, as in Theorem 2.3(II), or as in Theorem 4.2 below.
3.28. Local embedding into a CR structure. But in general, the coordinates $\left(s_{1}, \ldots, s_{\lambda-\tau+\delta_{p}}\right)$ are present. A trick ([Ma1992]) is to introduce extra coordinates $\left(t_{1}, \ldots, t_{\lambda-\tau+\delta_{p}}\right)$ and to define a new structure on the product $U_{p} \times \mathbb{R}^{\lambda-\tau+\delta_{p}}$ generated by the following local integrals:

$$
\begin{cases}\tilde{z}_{k}:=z_{k}, & k=1, \ldots, \tau-\delta_{p}, \\ \tilde{z}_{k}:=s_{k-\tau+\delta_{p}}+i t_{k-\tau+\delta_{p}}, & k=\tau-\delta_{p}+1, \ldots, \lambda, \\ \tilde{w}_{j}:=w_{j}, & j=1, \ldots, \delta_{p} .\end{cases}
$$

The associated structure bundle $\widetilde{\mathscr{L}}$ is spanned by:

$$
\left\{\begin{array}{l}
\widetilde{\widetilde{L}}_{k}:=\bar{L}_{k}, \quad k=1, \ldots, \tau-\delta_{p}, \\
\overline{\widetilde{L}}_{j}^{\prime}:=\frac{1}{2} L_{j}^{\prime}+\frac{i}{2} \frac{\partial}{\partial t_{j}}, \quad j=1, \ldots, \lambda-\tau+\delta_{p}
\end{array}\right.
$$

It is a CR structure of codimension $\delta_{p}$ on $U_{p} \times \mathbb{R}^{\lambda-\tau+\delta_{p}}$. All analyticogeometric objects defined in $U_{p}$ can be lifted to $U_{p} \times \mathbb{R}^{\lambda-\tau+\delta_{p}}$, just by declaring them to be independent of the extra variables $\left(t_{1}, \ldots, t_{\lambda-\tau+\delta_{p}}\right)$.

Such an embedding enables one to transfer elementarily several theorems valid in embedded Cauchy-Riemann Geometry, to the more general setting
of locally integrable structures. For instance, this is true of most of the theorems about holomorphic or CR extension of CR functions presented Part V. In addition, most of the results stated in §3, §4 and §5 below hold in locally integrable structures.
3.29. Transition. However, for reasons of space and because the possible generalizations which we could state by applying this embedding trick would require dry technical details, we will content ourselves to just mention these virtual generalizations, as was done in [MP1999]. For further study of locally integrable structures, we refer mainly to [Trv1992, BCH2005] and to the references therein.

In summary, in this Section 3, we wanted to show how our approach is inserted into a broad architecture of questions about solvability of partial differential equations, about the problem of realizability of abstract CR structures, as well as into hypo-analytic structures. Thus, even if some of the subsequent surveyed results (exempli gratia: the celebrated BaouendiTreves approximation Theorem 5.2) were originally stated in the context of locally integrable structures, even though we could as well state them in this context or at least in the context of locally embeddable CR structures (as was done in [MP1999]), we shall content ourselves to state them in the context of embedded Cauchy-Riemann geometry, just because the very core of the present memoir is concerned by Several Complex Variables topics: analytic discs, envelopes of holomorphy, removable singularities, etc.

## §4. Smooth Generic submanifolds and their CR orbits

4.1. Definitions of CR submanifolds and local graphing equations. We begin by some coordinate-invariant geometric definitions. Some implicit lemmas are involved (the reader is referred to [Ch1989, Ch1991, Bo1991, BER1999]). Let $J$ denote the complex structure of $T \mathbb{C}^{n}$ (see §2.1(II)). A real connected submanifold $M \subset \mathbb{C}^{n}$ of class at least $\mathscr{C}^{1}$ is called:

- Totally real if $T_{p} M \cap J T_{p} M=\{0\}$ for every $p \in M$. Then $M$ has codimension $d \geqslant n$ and is called maximally real if $d=n$. The complex vector subspace $H_{p}:=T_{p} M+J T_{p} M$ of $T_{p} \mathbb{C}^{n}$ has complex dimension $2 n-d$. If $\operatorname{proj}_{H_{p}}(\cdot)$ denotes any $\mathbb{C}$-linear projection of $T_{p} \mathbb{C}^{n}$ onto $H_{p}$ and if $\mathscr{U}_{p}$ is a small neighborhood of $p$ in $\mathbb{C}^{n}$, then $\operatorname{proj}_{H_{p}}\left(M \cap \mathscr{U}_{p}\right)$ is maximally real in $H_{p}$.
- Generic if $T_{p} M+J T_{p} M=T_{p} \mathbb{C}^{n}$ for every $p \in M$. Then $M$ has codimension $d \leqslant n$ and is maximally real only if $d=n$. Then $T_{p} M \cap J T_{p} M$ is the maximal $\mathbb{C}$-linear subspace of $T_{p} M$ and has complex dimension equal to the integer $m:=n-d$, called the $C R$ dimension of $M$. It is obviously constant, as $p$ runs in $M$.
- Cauchy-Riemann ( $C R$ for short) if the maximal $\mathbb{C}$-linear subspace $T_{p} M \cap J T_{p} M$ of $T_{p} M$ has constant dimension $m$ (necessarily $\leqslant n$ ) for $p$ running in $M$. If $M$ has codimension $d$, the integer $c:=d-$ $n+m$ is called the holomorphic codimension of $M$. Then for $p$ running in $M$, the complex vector subspaces $H_{p}:=T_{p} M+J T_{p} M$ of $T_{p} \mathbb{C}^{n}$ have constant complex codimension $c$, which justifies the terminology. If proj$\dot{H}_{p}(\cdot)$ denotes any $\mathbb{C}$-linear projection of $T_{p} \mathbb{C}^{n}$ onto $H_{p}$, and if $\mathscr{U}_{p}$ is a small open neighborhood of $p$ in $\mathbb{C}^{n}$, then

$$
\widetilde{M}_{p}:=\operatorname{proj}_{H_{p}}\left(M \cap \mathscr{U}_{p}\right)
$$

is a generic submanifold of $\mathbb{C}^{n-c}$.
In §2.1(II), we have graphed totally real, generic and, generally, CauchyRiemann local submanifolds $M \subset \mathbb{C}^{n}$, but only in the algebraic and in the analytic category. In the smooth category, the intrinsic complexification $\left\{w_{2}=0\right\}$ disappears, but $H_{p}=T_{p} M+J T_{p} M$ still exists, so that further graphing functions are needed.

Theorem 4.2. ([Ch1989, Ch1991, Bo1991, BER1999, Me2004a]) Let $M \subset$ $\mathbb{C}^{n}$ be a real submanifold of codimension $d$ and let $p \in M$. There exist complex algebraic or analytic coordinates centered at $p$ and $\rho_{1}>0$ such that $M$, supposed to be $\mathscr{C}^{\mathscr{R}}$, where $\mathscr{R}=\infty$ or where $\mathscr{R}=(\kappa, \alpha), \kappa \geqslant 1$, $0 \leqslant \alpha \leqslant 1$, is locally represented as follows:

- If $M$ is totally real, letting $d_{1}=2 n-d \geqslant 0$ and $c=d-n \geqslant 0$, then $d_{1}+c=n$ and

$$
\begin{aligned}
M=\left\{\left(w_{1}, w_{2}\right) \in\right. & \left(\square_{\rho_{1}}^{d_{1}} \times i \mathbb{R}^{d_{1}}\right) \times \mathbb{C}^{c}: \\
& \left.\operatorname{Im} w_{1}=\varphi_{1}\left(\operatorname{Re} w_{1}\right), w_{2}=\psi_{2}\left(\operatorname{Re} w_{1}\right)\right\},
\end{aligned}
$$

for some $\mathbb{R}^{d_{1}}$-valued $\mathscr{C}^{\mathscr{R}}$ map $\varphi_{1}$ and some $\mathbb{C}^{c}$-valued $\mathscr{C}^{\mathscr{R}}$ map $\psi_{2}$ satisfying $\varphi_{1}(0)=0$ and $\psi_{2}(0)=0$.

- If $M$ is generic, letting $m=d-n$, then $m+d=n$ and

$$
M=\left\{(z, w) \in \Delta_{\rho_{1}}^{m} \times\left(\square_{\rho_{1}}^{d} \times i \mathbb{R}^{d}\right): \operatorname{Im} w=\varphi(\operatorname{Re} z, \operatorname{Im} z, \operatorname{Re} w)\right\}
$$

for some $\mathbb{R}^{d}$-valued $\mathscr{C}^{\Omega}$ map $\varphi$ satisfying $\varphi(0)=0$.

- If $M$ is Cauchy-Riemann, letting $m=$ CRdim $M, c=d-n+m \geqslant$ 0 , and $d_{1}=2 n-2 m-d \geqslant 0$, then $m+d_{1}+c=n$ and

$$
\begin{aligned}
M=\{ & \left\{\left(z, w_{1}, w_{2}\right) \in \Delta_{\rho_{1}}^{m} \times\left(\square_{\rho_{1}}^{d_{1}} \times i \mathbb{R}^{d_{1}}\right) \times \mathbb{C}^{c}:\right. \\
& \left.\operatorname{Im} w_{1}=\varphi_{1}\left(\operatorname{Re} z, \operatorname{Im} z, \operatorname{Re} w_{1}\right), w_{2}=\psi_{2}\left(\operatorname{Re} z, \operatorname{Im} z, \operatorname{Re} w_{1}\right)\right\}
\end{aligned}
$$

for some $\mathbb{R}^{d_{1}}$-valued $\mathscr{C}^{\mathscr{R}}$ map $\varphi_{1}$ with $\varphi_{1}(0)=0$ and some $\mathbb{C}^{c}$-valued $\mathscr{C}^{\Re}$ map $\psi_{2}$ with $\psi_{2}(0)=0$ which is $C R$ (definition in $\S 4.25$ below) on the local generic submanifold $M^{1}$ := $\left\{\left(z, w_{1}\right) \in \Delta_{\rho_{1}}^{m} \times\left(\square_{\rho_{1}}^{d_{1}} \times i \mathbb{R}^{d_{1}}\right): \operatorname{Im} w_{1}=\varphi_{1}\left(\operatorname{Re} z, \operatorname{Im} z, \operatorname{Re} w_{1}\right)\right\}$.
An adapted linear change of coordinates insures that the differentials at the origin of the graphing maps all vanish.

A CR manifold $M$ being always locally graphed above a generic submanifold of $\mathbb{C}^{n-c}$, the remainder of this memoir will mostly be devoted to the study of $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$ generic submanifolds of $\mathbb{C}^{n}$. The above local representation of a generic $M$ will be constantly used.
4.3. CR vector fields. Let $M$ be generic, of class at least $\mathscr{C}^{1}$, represented by $v=\varphi(x, y, u)$ as above, in coordinates $(z, w)=(x+i y, u+i v)$. Sometimes, we shall also write $v=\varphi(z, u)$, being it clear that $\varphi$ is not holomorphic with respect to $z$. Here, we provide a description in coordinates of three useful families of vector fields.

There exist $m$ anti-holomorphic vector fields defined in $\Delta_{\rho_{1}}^{m} \times\left(\square_{\rho_{1}}^{d} \times i \mathbb{R}^{d}\right)$ of the form:

$$
\bar{L}_{k}^{\prime}=\frac{\partial}{\partial \bar{z}_{k}}+\sum_{j=1}^{d} a_{j, k}^{\prime}(x, y, u) \frac{\partial}{\partial \bar{w}_{j}},
$$

whose restrictions to $M$ span $T^{0,1} M$. To compute the coefficients $a_{j, k}^{\prime}$, the conditions $0 \equiv \bar{L}_{k}^{\prime}\left(\varphi_{j}(x, y, u)-v_{j}\right)$ yield:

$$
2 \varphi_{j, \bar{z}_{k}}=\sum_{l=1}^{d}\left(i \delta_{j, l}-\varphi_{j, u_{l}}\right) a_{l, k}^{\prime} .
$$

In matrix notation, the solution is: $a^{\prime}=2\left(i I_{d \times d}-\varphi_{u}\right)^{-1} \cdot \varphi_{\bar{z}}$, with $a^{\prime}=\left(a_{j, k}^{\prime}\right)_{1 \leqslant j \leqslant d}^{1 \leqslant k \leqslant m}$ and $\varphi_{\bar{z}}=\left(\varphi_{j, \bar{z}_{k}}\right)_{1 \leqslant j \leqslant d}^{1 \leqslant k \leqslant m}$ both of size $d \times m$. Since $\left[T^{0,1} M, T^{0,1} M\right] \subset T^{0,1} M$, the $\bar{L}_{k}^{\prime}$ commute. They are extrinsic.

Also, there exist $m$ intrinsic sections of $\mathbb{C} T M$ of the form:

$$
\bar{L}_{k}:=\frac{\partial}{\partial \bar{z}_{k}}+\sum_{j=1}^{d} a_{j, k}(x, y, u) \frac{\partial}{\partial u_{j}},
$$

written in the coordinates $(x, y, u)$ of $M$, which span the structure bundle $T^{0,1} M \subset \mathbb{C} T M$. Since $(x, y, u)$ are coordinates on $M$, restricting the $\left.\bar{L}_{k}^{\prime}\right|_{M}$ to $M$ amounts to just drop the terms $\frac{i}{2} \frac{\partial}{\partial v_{j}}$ in each $\frac{\partial}{\partial \bar{w}_{j}}$ appearing in $\bar{L}_{k}^{\prime}$. Hence:

Lemma 4.4. One has $a_{j, k}=\frac{1}{2} a_{j, k}^{\prime}$.

Another argument is to first introduce the $d$ vector fields:

$$
K_{j}=\sum_{l=1}^{d} b_{l, j}(x, y, u) \frac{\partial}{\partial u_{l}},
$$

that are uniquely determined by the conditions

$$
\delta_{j_{1}, j_{2}}=K_{j_{1}}\left(\left.w_{j_{2}}\right|_{M}\right)=K_{j_{1}}\left(u_{j_{2}}+i \varphi_{j_{2}}(x, y, u)\right) .
$$

Equivalently, the coefficients $b_{l, j}$ satisfy:

$$
\delta_{j_{1}, j_{2}}=\sum_{l=1}^{d}\left(\delta_{j_{2}, l}+i \varphi_{j_{2}, u_{l}}\right) b_{l, j_{1}},
$$

whence, in matrix notation: $b=\left(I_{d \times d}+i \varphi_{u}\right)^{-1}$. Here, $b=\left(b_{l, j}\right)_{1 \leqslant l \leqslant d}^{1 \leqslant j \leqslant d}$ and $\varphi_{u}=\left(\varphi_{j, u_{l}}\right)_{1 \leqslant j \leqslant d}^{1 \leqslant l}$ both are $d \times d$ matrices.

Similarly as in Corollary 3.27, the $\bar{L}_{k}$ defined above span the structure bundle $\mathscr{L}=T^{0,1} M$ having the local integrals $z_{1}, \ldots, z_{m},\left.w_{1}\right|_{M}, \ldots,\left.w_{d}\right|_{M}$, if and only if they satisfy $0=\bar{L}_{k}\left(\left.w_{j}\right|_{M}\right)=\bar{L}_{k}\left[u_{j}+i \varphi_{j}(x, y, u)\right]$. Seeking the $\bar{L}_{k}$ under the form $\bar{L}_{k}=\frac{\partial}{\partial \bar{z}_{k}}+\sum_{l=1}^{d} c_{l, k}(x, y, u) K_{l}$, it follows from $\delta_{j_{1}, j_{2}}=K_{j_{1}}\left(u_{j_{2}}+i \varphi_{j_{2}}(x, y, u)\right)$ that $c_{j, k}=-i \varphi_{j, \bar{z}_{k}}$. Reexpressing explicitly the $K_{l}$ in terms of the $\frac{\partial}{\partial u_{j}}$ as achieved above, we finally get in matrix notation $a=\left(i I_{d \times d}-\varphi_{u}\right)^{-1} \cdot \varphi_{\bar{z}}$. This yields a second, more intrinsic computation of the coefficients $a_{j, k}$ and a second proof of $a_{j, k}=\frac{1}{2} a_{j, k}^{\prime}$.

If $\chi$ is a $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$ function on $M$, its differential may be computed as

$$
d \chi=\sum_{k=1}^{m} L_{k}(\chi) d z_{k}+\sum_{k=1}^{m} \bar{L}_{k}(\chi) d \bar{z}_{k}+\left.\sum_{j=1}^{d} K_{j}(\chi) d w_{j}\right|_{M} .
$$

Lemma 4.5. ([Trv1981, BR1987, Trv1992, BCH2005]) The following relations hold:

$$
\left\{\begin{array}{l}
L_{k_{1}}\left(z_{k_{2}}\right)=\delta_{k_{1}, k_{2}}, L_{k}\left(w_{j}\right)=0, K_{j}\left(z_{k}\right)=0, \quad K_{j_{1}}\left(\left.w_{j_{2}}\right|_{M}\right)=\delta_{j_{1}, j_{2}}, \\
{\left[L_{k_{1}}, L_{k_{2}}\right]=\left[L_{k}, K_{j}\right]=\left[K_{j_{1}}, K_{j_{2}}\right]=0 .}
\end{array}\right.
$$

4.6. Vector-valued Levi form. Let $p \in M$ and denote by $\pi_{p}$ the projection $\mathbb{C} T_{p} M \longrightarrow \mathbb{C} T_{p} M /\left(T_{p}^{1,0} M \oplus T_{p}^{0,1} M\right)$.
Definition 4.7. The Levi map at $p$ is the Hermitian $\mathbb{C}^{d}$-valued form acting on two vectors $X_{p}, Y_{p} \in T_{p}^{1,0} M$ as:

$$
\left[\begin{array}{ll}
\mathfrak{L}_{p}: & T_{p}^{1,0} M \times T_{p}^{1,0} M \longrightarrow \mathbb{C} T_{p} M /\left(T_{p}^{1,0} M \oplus T_{p}^{0,1} M\right) \\
& \mathfrak{L}_{p}\left(X_{p}, Y_{p}\right):=\frac{1}{2 i} \pi_{p}([X, \bar{Y}](p))
\end{array}\right.
$$

where $X, Y$ are any two sections of $T^{1,0} M$ defined in a neighborhood of $p$ satisfying $X(p)=X_{p}, Y(p)=Y_{p}$. The resulting number is independent of the choice of such extensions $X, Y$.

As $p$ varies, this yields a smooth bundle map. The Levi map $\mathfrak{L}_{p}$ is nondegenerate at $p$ if its kernel is null: $\mathfrak{L}_{p}\left(X_{p}, Y_{p}\right)=0$ for every $Y_{p}$ implies $X_{p}=0$. On the opposite, $M$ is Levi-flat if the kernel of $\mathfrak{L}_{p}$ equals $T_{p}^{1,0} M$ at every $p$. If $M$ is a hypersurface $(d=1)$, one calls $\mathfrak{L}_{p}$ the Levi form at $p$. Then $M$ is strongly pseudoconvex at $p$ if the Levi form $\mathfrak{L}_{p}$ is definite, positive or negative. These definitions agree with the ones formulated in §3.10 for abstract CR structures.

Theorem 4.8. ([Fr1977, Ch1991]) In a neighborhood $U_{p}$ of a point $p \in M$ in which the kernel of the Levi map is of constant rank and defines a $\mathscr{C}^{\omega}$, $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa-1, \alpha}(\kappa \geqslant 2,0 \leqslant \alpha \leqslant 1)$ distribution of $m_{1}$-dimensional complex planes $P_{q} \subset T_{q} M, \forall q \in U_{p}$, the distribution is Frobenius-integrable, hence $M$ is $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa-1, \alpha}$ foliated by complex manifolds of dimension $m_{1}$.

In particular, a Levi-flat generic submanifold of CR dimension $m$ is foliated by $m$-dimensional complex manifolds. These observations go back to Sommer (1959). In [Ch1991], one founds a systematic study of foliations by complex and by CR manifolds.
4.9. CR orbits. Let $M \subset \mathbb{C}^{n}$ be generic and consider the system $\mathbb{L}$ of sections of $T^{c} M$. To apply the orbit Theorem 1.21 , we need $M$ to be at least $\mathscr{C}^{2}$, in order that the flows are at least $\mathscr{C}^{1}$. By definition, a weak $T^{c} M$ integral submanifold $S \subset M$ satisfies $T_{p} S \supset T_{p}^{c} M$, at every point $p \in S$. Equivalently, $S$ has the same CR dimension as $M$ at every point.

In the theory of holomorphic extension exposed in Part V, local and global CR orbits will appear to be adequate objects of study. They constitute one of the main topics of this memoir.

Proposition 4.10. ([Trp1990, Tu1990, Tu1994a, Me1994, Jö1996, MP1999, MP2002]) Let $M \subset \mathbb{C}^{n}$ be generic of class $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$ with $\kappa \geqslant 2$ and $0 \leqslant \alpha \leqslant 1$.
(a) The (global) $T^{c} M$-orbits are called CR orbits. They are denoted by $\mathscr{O}_{C R}$ or by $\mathscr{O}_{C R}(M, p)$, if the reference to one of point $p \in \mathscr{O}_{C R}$ is needed.
(b) The local CR orbit of a point $p \in M$ is denoted by $\mathscr{O}_{C R}^{l o c}(M, p)$. It is a local submanifold embedded in $M$, closed in a sufficiently small neighborhood of $p$ in $M$.
(c) Local and global CR orbits are $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \beta}$, for every $\beta$ with $0<\beta<\alpha$.
(d) $M$ is partitioned in global CR orbits. Each global CR orbit is injectively immersed and weakly embedded in $M$, is a CR submanifold of $\mathbb{C}^{n}$ contained in $M$ and has the same $C R$ dimension as $M$.
(e) Every (immersed) $C R$ submanifold $S \subset M$ having the same $C R$ dimension as $M$ contains the local CR orbit of each of its points
(f) CR orbits of the smallest possible real dimension $2 m=2$ CRdim $M$ satisfy $T_{p} \mathscr{O}_{C R}=T_{p}^{c} \mathscr{O}_{C R}$ at every point, hence are complex $m$ dimensional submanifolds.

According to Example 1.29, CR orbits should be $\mathscr{C}^{\kappa-1, \alpha}$, not smoother. But in generic submanifolds, they also can be described as boundaries of small attached analytic discs ([Tu1990, Tu1994a, Me1994]) and the $\mathscr{C}^{\kappa, \alpha-0}=\bigcap_{\beta<\alpha} \mathscr{C}^{\kappa, \beta}$ smoothness of the solution in Theorem 3.7(IV) explains (c).

Let us summarize some structural properties of CR orbits, useful in applications. A specialization of Theorem 1.21(4) yields the following.
Proposition 4.11. For every $p \in M$, there exist $k \in \mathbb{N}$, sections $L^{1}, \ldots, L^{k}$ of $T^{c} M$ and parameters $\mathrm{s}_{1}^{*}, \ldots, \mathbf{s}_{k}^{*} \in \mathbb{R}$ such that $L_{\mathrm{s}_{k}^{*}}^{k}\left(\cdots\left(L_{\mathbf{s}_{1}^{*}}^{1}(p)\right) \cdots\right)=p$ and the map $\left(\mathrm{s}_{1}, \ldots, \mathrm{~s}_{k}\right) \longmapsto \quad L_{\mathrm{s}_{k}}^{k}\left(\cdots\left(L_{\mathrm{s}_{1}}^{1}(p)\right) \cdots\right)$ is of rank $\operatorname{dim} \mathscr{O}_{C R}(M, p)$ at $\left(\mathrm{s}_{1}^{*}, \ldots, \mathrm{~s}_{k}^{*}\right)$.

The dimension of any $\mathscr{O}_{C R}$ is equal to $2 m+e$, for some $e \in \mathbb{N}$ with $0 \leqslant e \leqslant d$. Denote:

- $\mathscr{O}_{2 m+e} \subset M$ the union of CR orbits of dimension $=2 m+e$;
- $\mathscr{O}_{2 m+e}^{\geq} \subset M$ the union of CR orbits of dimension $\geqslant 2 m+e$;
- $\mathscr{O}_{2 m+e}^{\leqslant} \subset M$ the union of CR orbits of dimension $\leqslant 2 m+e$.

The function $p \mapsto \operatorname{dim} \mathscr{O}_{C R}(M, p)$ is lower semicontinuous. It follows that $\mathscr{O}_{2 m+e}^{\geq}$is open in $M$ and that $\mathscr{O}_{2 m+e}^{\leq}$is closed in $M$.

Let $p \in M$, let $\mathscr{O}_{p}$ be a small piece of $\mathscr{O}_{C R}(M, p)$ passing through $p$, of dimension $2 m+e_{p}$, for some integer $e_{p}$ with $0 \leqslant e_{p} \leqslant d$, and let $H_{p}$ be a local $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$ submanifold of $M$ passing through $p$ and satisfying $T_{p} H_{p} \oplus T_{p} \mathscr{O}_{p}=T_{p} M$. Call $H_{p}$ a local orbit-transversal. Implicitly, $H_{p}=\emptyset$ if $e_{p}=d$. Then, in a sufficiently small neighborhood of $p$ :

- lower semi-continuity: $H_{p} \cap \mathscr{O}_{2 m+e_{p}+1}^{\geqslant}=\emptyset$;
- equivalently: $H_{p} \cap \mathscr{O}_{2 m+e_{p}}^{\geqslant}=H_{p} \cap \mathscr{O}_{2 m+e_{p}}$;
- $H_{p} \cap \mathscr{O}_{2 m+e_{p}}^{K}$ is closed.

Proposition 4.12. If $M$ is $\mathscr{C}^{\omega}$, then at every point $p \in M$, for every orbittransversal $H_{p}$ passing through $p$, the closed set $H_{p} \cap \mathscr{O}_{2 m+e_{p}}^{\leqslant}$is a local real analytic subset of $H_{p}$ containing $p$.

A CR-invariant subset of $M$ is a union of CR orbits. A closed (for the topology of $M$ ) CR-invariant subset is minimal if it does not contain any proper subset which is also a closed CR-invariant subset.

Problem 4.13. Describe the possible structure of the decomposition of $M$ in CR orbits.

There are differences between embedded and locally embeddable generic submanifolds, which we shall not discuss, assuming that $M$ is embedded in $\mathbb{C}^{n}$ or in $P_{n}(\mathbb{C})$. Also, the $\mathscr{C}^{\omega}$ category enjoys special features.

Indeed, if $M$ is a connected real analytic hypersurface, Proposition 4.12 entails that each minimal closed invariant subset of $M$ is either an embedded complex hypersurface or an open orbit; also if $M$ contains at least one CR orbit of maximal dimension $(2 n-1)$ (hence an open subset of $M$ ), all its CR orbits of codimension one are complex ( $n-1$ )-dimensional embedded submanifold of $M$ (a real analytic subset of codimension one in $\mathbb{R}$ consists of isolated points). In the smooth category things are different.

So, let $M$ be a connected $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}(\kappa \geqslant 2,0 \leqslant \alpha \leqslant 1)$ hypersurface of $\mathbb{C}^{n}$. Its CR-orbits are either $(2 n-1)$-dimensional, i.e. open in $M$, or ( $2 n-2$ )-dimensional and $T^{c} M$-integral, hence complex ( $n-1$ )-dimensional hypersurfaces immersed in $M$.

Proposition 4.14. ([Jö1999a]) In the smooth category, the structure of every minimal closed CR-invariant subset $C$ of $M$ has one of the following types:
(i) $C=M$ consists of a single embedded open CR orbit;
(ii) $C=\bigcup_{a \in A} \mathscr{O}_{C R, a}=M$ is a union of complex hypersurfaces, each being dense in $C$, with $\operatorname{Card} A=\operatorname{Card} \mathbb{R}$;
(iii) $C=\bigcup_{a \in A} \mathscr{O}_{C R, a}$ has empty interior in $M$ and is a union union of complex hypersurfaces, each being dense in $C$, with $\operatorname{Card} A=$ Card $\mathbb{R}$;
(iv) C consists of a single complex hypersurface embedded in $M$.

Furthermore, the closure, with respect to the topology of $M$, of every $C R$ orbit of dimension $(2 n-2)$ is a minimal closed $C R$-invariant subset of $M$.

These four options are known in foliation theory ([HH1983, CLN1985]). One has to remind that each CR orbit contained in $C$ is dense in $C$. In the first two cases, the trace of $C$ on any orbit-transversal is a full open segment; in the third, it is a Cantor set; in the last, it is an isolated point. In the third case, impossible if $M$ is real analytic, $C$ will be called an exceptional minimal $C R$-invariant subset, similarly as in foliation theory. We shall see below that compactness of $M \subset \mathbb{C}^{n}$ imposes a strong restriction on the possible $C$ 's.

We mention that an analog of Proposition 4.14 holds for connected generic submanifolds of codimension $d \geqslant 2$, provided one puts the restrictive assumption that all its CR orbits are of codimension $\leqslant 1$, the only difference being that CR orbits of codimension 1 are not complex manifolds in this case.

The presence of CR orbits of codimension $\geqslant 2$ in $M$ may produce minimal closed CR-invariant subsets with complicated transversal structure, even in the real analytic category ([BM1997]). Also, in a bounded strongly pseudoconvex boundary (see §1.15(V) for background), there may exist a CR orbit of codimension one whose closure constitutes an exceptional minimal CR-invariant subset.

Theorem 4.15. ([Jö1999a]) There exists a bounded strongly pseudoconvex domain $\Omega \subset \mathbb{C}^{3}$ with $\mathscr{C}^{\infty}$ boundary and a compact $\mathscr{C}^{\infty}$ submanifold $M \subset$ $\partial \Omega$ of dimension 4 which is generic in $\mathbb{C}^{3}$ such that:

- $M$ is $\mathscr{C}^{\infty}$ foliated by CR orbits of dimension 3;
- M contains a compact exceptional minimal CR-invariant set, but no compact CR orbit.

Summarized proof. The main idea is to start with an example due to Sacksteder, known in foliation theory, of a compact real analytic 3-dimensional manifold $\mathscr{N}$ equipped with a $\mathscr{C}^{\omega}$ foliation $\mathscr{F}$ of codimension one which carries a compact exceptional minimal set, but no compact leaf. According to [HH1983], there exists such a pair $(\mathscr{N}, \mathscr{F})$, together with a $\mathscr{C}^{\infty}$ diffeomorphism $\varphi_{1}: \mathscr{N} \rightarrow B \times S^{1}$, where $B$ is some compact orientable $\mathscr{C}^{\infty}$ surface of genus 2 embedded in $\mathbb{R}^{3}$, and where $S^{1}$ is the unit circle. Let $B \ni b \mapsto \mathbf{n}(b) \in T_{b} \mathbb{R}^{3}$ denote the $\mathscr{C}^{\infty}$ unit outward normal vector field to such a $B \subset \mathbb{R}^{3}$, and consider $\mathbb{R}^{3}$ to be embedded in $\mathbb{C}^{3}$. For $\varepsilon>0$ small, the map

$$
\varphi_{2}: B \times S^{1} \ni(b, \zeta) \longmapsto b+\mathbf{n}(b) \cdot \varepsilon \zeta \in \mathbb{C}^{3}
$$

may be seen to be a totally real $\mathscr{C}^{\infty}$ embedding. By results of BruhatWhitney and Grauert, one may approximate the $\mathscr{C}^{\infty}$ totally real embedding $\varphi_{2} \circ \varphi_{1}$ by a $\mathscr{C}^{\omega}$ embedding $\varphi: \mathscr{N} \rightarrow \mathbb{C}^{3}$ which is arbitrarily close to $\varphi_{2} \circ \varphi_{1}$ in $\mathscr{C}^{1}$ norm, hence totally real. Denote $N:=\varphi(\mathscr{N})$. The transported foliation $F:=\varphi_{*}(\mathscr{F})$ being real analytic, one may then proceed to an intrinsic complexification of all its totally real 2 -dimensional leaves, getting some 5 -dimensional real analytic hypersurface $N^{i_{c}}$ containing $N$, equipped with a foliation $F^{i_{c}}$ of $N^{i_{c}}$ by 2-dimensional complex manifolds, with $F^{i_{c}} \cap N=F$. This foliation $F^{i_{c}}$ is closed in some tubular neighborhood $\Omega$ of $N$ in $\mathbb{C}^{3}$, say $\Omega:=\left\{z \in \mathbb{C}^{3}: \operatorname{dist}(z, N)<\delta\right\}$, with $\delta>0$ small. Since $N$ is totally real, the boundary $\partial \Omega$ is strongly pseudoconvex (Grauert) and is $\mathscr{C}^{\infty}$. Clearly, the intersection $M:=N^{i_{c}} \cap \partial \Omega$ is a 4-dimensional $\mathscr{C}^{\infty}$
submanifold. The intersections of the 2 -dimensional complex leaves of $F^{i c}$ with $\partial \Omega$ show that $M$ is foliated by strongly pseudoconvex 3 -dimensional boundaries, which obviously consist of a single CR orbit. Thus a CR orbit of $M$ is just the intersection of a global leaf of $F^{i_{c}}$ with $\partial \Omega$. In conclusion, letting $\mathrm{Exc}_{\mathscr{F}}$ be the minimal exceptional set of Sacksteder's example, $M$ contains the exceptional minimal CR-invariant set $\left[\varphi_{*}\left(\operatorname{Exc}_{\mathscr{F}}\right)\right]^{i_{c}} \cap \partial \Omega$ and no compact CR orbit.
4.16. Global minimality and laminations by complex manifolds. The CR orbits being essentially independent bricks, it is natural to define the class of CR manifolds which consist of only one brick.

Definition 4.17. A $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha} \mathrm{CR}$ manifold $M$ is called globally minimal if if consists of a single CR orbit.

Each CR orbit of a CR manifold is a globally minimal immersed CR submanifold of $\mathbb{C}^{n}$. To simplify the overall presentation and not to expose superficial corollaries, almost all the theorems of Parts V and VI in this memoir will be formulated with a global minimality $M$.
Lemma 4.18. ([Gr1968, Jö1995, BCH2005]) A compact connected $\mathscr{C}^{2}$ hypersurface in $\mathbb{C}^{n}$ is necessarily globally minimal.

Proof. Otherwise, the closure of a CR-orbit of codimension one in $M$ would produce a compact CR-invariant subset $C$ which is a union of immersed complex hypersurfaces, each dense in $C$. Looking at a point of $C$ where the pluriharmonic function $r_{i}:=\operatorname{Re} z_{i}$ (or $s_{i}:=\operatorname{Im} z_{i}$ ) is maximal, the maximum principle entails that $r_{i}$ (or $s_{i}$ ) is constant on $C$, for $i=1, \ldots, n$, hence $C=\{p t$. $\}$, contradiction.

More generally, the same simple argument yields:
Corollary 4.19. Any Stein manifold cannot contain a compact set which is laminated by complex manifolds

In the projective space $P_{n}(\mathbb{C})$, one expects compact orientable connected $\mathscr{C}^{2}$ hypersurfaces $M$ to be still globally minimal, but arguments are far to be simple. In fact, there are infinitely many compact projective algebraic complex hypersurfaces $\Sigma$ in $P_{n}(\mathbb{C})$. However, they cannot be contained in such an $M \subset P_{n}(\mathbb{C})$ since, otherwise, their complex normal bundle $\left.T P_{n}(\mathbb{C})\right|_{\Sigma} / T \Sigma$, known to be never trivial, but equal to the complexification of the trvial bundle $\left.T M\right|_{\Sigma} / T \Sigma$, would be trivialized.

Thus, according to Proposition 4.14 above, the very question is about nonexistence of closed CR-invariant sets $C \subset M$ laminated by complex hypersurfaces which either coincide with $M$ or are transversally Cantor sets. If $M \subset P_{n}(\mathbb{C})$ is real analytic, it might only be Levi-flat.

Nonexistence of orientable Levi-flat hypersurfaces in $P_{n}(\mathbb{C})$ was expected, because they would divide the projective space in two smoothly bounded pseudoconvex domains. In the real analytic case, non-existence was verified by Lins-Neto for $n \geqslant 3$ and by Ohsawa for $n=2$; in the smooth (harder) case, see [Si2000].

Open question 4.20 . Is any compact orientable connected $\mathscr{C}^{2}$ hypersurface of $P_{n}(\mathbb{C})$ globally minimal?

So, the expected answer is yes. In fact, the question is a particular case of a deep conjecture stemming from Hilbert's sixteen problem about phase diagrams of vector fields having polynomial coefficients on the twodimensional projective space. This conjecture is inspired by the PoincaréBendixson theory valid over the real numbers, according to which the closure of each leaf of such a foliation over $P_{2}(\mathbb{R})$ contains either a compact leaf or a singularity. In its most general form, it says that $P_{n}(\mathbb{C})$ cannot contain a compact set laminated by $(n-1)$-dimensional complex manifold, unless it is a trivial lamination, viz just a compact complex projective algebraic hypersurface; however, nontrivial laminations by ( $n-2$ )-dimensional complex manifolds may be shown to exist.

A related general open question is to find topologico-geometrical criteria on open subsets of $P_{n}(\mathbb{C})$ insuring the existence of nonconstant holomorphic functions there.
4.21. Finite-typisation of generic submanifolds. Let $M$ be a connected $\mathscr{C}^{\kappa}(2 \leqslant \kappa \leqslant \infty)$ generic submanifold of $\mathbb{C}^{n}$ of codimension $d \geqslant 1$ and of CR dimension $m=n-d \geqslant 1$. The distribution of subspaces $p \mapsto T_{p}^{c} M$ of $T M$ is of constant rank $2 m$. We apply to the complex tangential bundle $T^{c} M$ the concept of finite-type.

Definition 4.22. A point $p \in M$ is said to be of finite type if the system $\mathbb{L}$ of local sections of $T^{c} M$ defined in a neighborhood of $p$ satisfies $\mathbb{L}^{\kappa}(p)=$ $T_{p} M$. The smallest integer $d(p) \leqslant \kappa$ with $\mathbb{L}^{d(p)}(p)=T_{p} M$ is called the type of $M$ at $p$.

We want now to figure out how, in general, a generic submanifold of $\mathbb{C}^{n}$ must be globally minimal and in fact, of finite type at every point. We equip with the strong Whitney topology the set ${ }^{\kappa} \mathscr{G}_{d, m}^{n}$ of $\mathscr{C}^{\kappa}(2 \leqslant \kappa \leqslant \infty)$ connected generic submanifolds $M \subset \mathbb{C}^{n}$ of codimension $d \geqslant 1$ and of CR dimension $m=n-d \geqslant 1$. No restriction is made on the global topology.

As a model case, let $\kappa \geqslant 2$ and consider $M$ to be rigid algebraic represented by

$$
w_{j}=\bar{w}_{j}+i P_{j}(z, \bar{z})=\bar{w}_{j}+i \sum_{|\alpha|+|\beta| \leqslant \kappa} p_{j, \alpha, \beta} z^{\alpha} \bar{z}^{\beta},
$$

where $\alpha, \beta \in \mathbb{N}^{m}$, where the polynomials $P_{j}$ are real, $p_{j, \alpha, \beta}=\bar{p}_{j, \beta, \alpha}$ and have no pluriharmonic term, namely $0 \equiv P_{j}(z, 0) \equiv P_{j}(0, \bar{z})$, for $j=$ $1, \ldots, d$. A basis of $(1,0)$ and of $(0,1)$ vector fields is given by

$$
L_{k}:=\frac{\partial}{\partial z_{k}}+i \sum_{j=1}^{d} P_{j, z_{k}} \frac{\partial}{\partial w_{j}} \quad \text { and } \quad \bar{L}_{k}:=\frac{\partial}{\partial \bar{z}_{k}}-i \sum_{j=1}^{d} \bar{P}_{j, \bar{z}_{k}} \frac{\partial}{\partial \bar{w}_{j}},
$$

for $k=1, \ldots, n$. The Lie algebra $\mathbb{L}^{\kappa}$ generated by all Lie brackets of length $\leqslant \kappa$ of the system $\mathbb{L}:=\left\{L_{k}, \bar{L}_{k}\right\}_{1 \leqslant \kappa \leqslant m}$ contains the subalgebra $\mathbb{L}_{C R, \text { rigid }}^{\kappa}$ generated by the only brackets of the form

$$
\left[L_{\lambda_{1}}, \ldots,\left[L_{\lambda_{a}},\left[\bar{L}_{\mu_{1}}, \ldots,\left[\bar{L}_{\mu_{b}},\left[L_{k_{1}}, \bar{L}_{k_{2}}\right]\right] \ldots\right]\right] \ldots\right]
$$

where $2+a+b \leqslant \kappa$ and where $1 \leqslant \lambda_{1}, \ldots, \lambda_{a}, \mu_{1}, \ldots, \mu_{b}, k_{1}, k_{2} \leqslant m$. One computes

$$
\left[L_{k_{1}}, \bar{L}_{k_{2}}\right]=-i\left(\sum_{j=1}^{d} P_{j, z_{k_{1}} \bar{z}_{k_{2}}} \frac{\partial}{\partial w_{j}}+\sum_{j=1}^{d} \bar{P}_{j, z_{k_{1}} \bar{z}_{k_{2}}} \frac{\partial}{\partial \bar{w}_{j}}\right),
$$

hence the above iterated Lie brackets are equal to

$$
-i\left(\sum_{j=1}^{d} P_{j, z_{\lambda_{1}} \ldots z_{\lambda_{a}} \bar{z}_{\mu_{1}} \ldots \bar{z}_{\mu_{b}} z_{k_{1}} \bar{z}_{k_{2}}} \frac{\partial}{\partial w_{j}}+\sum_{j=1}^{d} \bar{P}_{j, z_{\lambda_{1}} \ldots z_{\lambda_{a}} \bar{z}_{\mu_{1}} \ldots \bar{z}_{\mu_{b}} z_{k_{1}} \bar{z}_{k_{2}}} \frac{\partial}{\partial \bar{w}_{j}}\right) .
$$

This shows that all these brackets are linearly independent as functions of the jets of the $P_{j}$. In fact, the number of such brackets is exactly equal to the dimension of the space of polynomials $P(z, \bar{z})$ of degree $\leqslant \kappa$ having no pluriharmonic term, namely equal to

$$
\frac{(2 m+\kappa)!}{2 m!\kappa!}-2 \frac{(m+\kappa)!}{m!\kappa!}+1 .
$$

For a general $\mathscr{C}^{\kappa}$ submanifold $M$ (not necessarily rigid), one verifies that the same collection of brackets is independent in terms of the jets of the defining equation of $M$. Generalizing slightly Lemma 2.13 , we see that in the vector space of $d \times(d+e)$ (real or complex) matrices, the subset of matrices of rank $\leqslant d-1$ is a real algebraic set of codimension equal to $(e+1)$. If we choose $\kappa$ large enough so that the dimension of $\mathbb{L}_{C R, \text { rigid }}^{\kappa}$ is $\geqslant d+\operatorname{dim} M=2(m+d)=2 n$ (applying the previous assertion with $e:=\operatorname{dim} M)$, if we form the $d \times\left(d+e^{\prime}\right), e^{\prime} \geqslant e$, matrix consisting of the coordinates of the $\frac{\partial}{\partial w}$-part of brackets of length $\leqslant \kappa$ as above, then the set where this matrix is of rank $\leqslant d-1$ is of codimension $\geqslant \operatorname{dim} M+1$ in the space of $\kappa$-th jets of the defining equations of $M$. Consequently, the jet transversality theorem applies.

Theorem 4.23. Let $n \geqslant 1, m \geqslant 1$ and $d \geqslant 1$ be integers satisfying $m+d=$ $n$ and let $\kappa$ be the minimal integer having the property that

$$
\frac{(2 m+\kappa)!}{2 m!\kappa!}-2 \frac{(m+\kappa)!}{m!\kappa!}+1 \geqslant 2(m+d)=2 n .
$$

Then the set of $\mathscr{C}^{\kappa}$ connected generic submanifolds $M \subset \mathbb{C}^{n}$ of codimension $d$ and of $C R$ dimension $m$ that are of finite type $\leqslant \kappa$ at every point is open and dense in the set ${ }^{\kappa \mathscr{C}_{d, m}^{n}}$ of all generic submanifolds.

In particular, a connected $\mathscr{C}^{4}$ (resp. $\mathscr{C}^{3}$, resp. $\mathscr{C}^{2}$ ) hypersurface in $\mathbb{C}^{2}$ (resp. in $\mathbb{C}^{3}$, resp. in $\mathbb{C}^{n}$ for $n \geqslant 4$ ) is of finite type 4 (resp. 3, resp. 2 , or equivalently, is not Levi-flat) at every point after an arbitrarily small perturbation.

Similarly, if instead of the subalgebra $\mathbb{L}_{\mathrm{CR}, \text { rigid }}$, one would have considered the (smaller) subalgebra consisting of only the brackets $\left[L_{\lambda_{1}}, \ldots,\left[L_{\lambda_{a}},\left[L_{k_{1}}, \bar{L}_{k_{2}}\right]\right] \ldots\right]$, where $2+a \leqslant \kappa$ and where $1 \leqslant \lambda_{1}, \ldots, \lambda_{a}, k_{1}, k_{2} \leqslant \kappa$, one would have obtained finite type $\leqslant \kappa$, for $\kappa$ minimal satisfying $2 m \frac{(m+\kappa-1)!}{m!(\kappa-1)!} \geqslant m^{2}+2(2 m+d)$. We also mention that the same technique enables one to prove that, after an arbitrarily small perturbation, $M$ is finitely nondegenerate at every point and of finite nondegeneracy type $\leqslant \ell$, with $\ell$ minimal satisfying $2 d \frac{(\ell+m)!}{\ell!m!} \geqslant 4 n-1$. In particular, $\ell=3$ when $m=d=1$ while $\ell=2$ suffices when $m=1$ for all $d \geqslant 2$. Details are left to the reader.

To conclude, we state the analog of Open question 2.17 for induced CR structures.

Open question 4.24. ([JS2004], [*]) Given a fixed generic submanifold $M$ of class $\mathscr{C}^{\kappa}$ that is of finite type at every point, is it always possible to perturb slightly a $\mathscr{C}^{\kappa}$ submanifold $M_{1}$ of $M$ that is generic in $\mathbb{C}^{n}$, of codimension $d_{1} \geqslant 1$ and of CR dimension $m_{1}=n-d_{1} \geqslant 1$, as a $\mathscr{C}^{\kappa}$ submanifold $\widetilde{M}_{1}$ of $M$ that is of finite type at every point? If so, what is the smallest regularity $\kappa$ in terms of $d, m, d_{1}, m_{1}$ and of the highest type at points of $M$ ?
4.25. Spaces of CR functions and of CR distributions. A $\mathscr{C}^{1}$ function $f: M \rightarrow \mathbb{C}$ is called Cauchy-Riemann ( $C R$ briefly) if it is annihilated by every section of $T^{0,1} M$. Equivalently:

- $d f$ is $\mathbb{C}$-linear on $T^{c} M$;
- df $\left.\wedge d t_{1} \wedge \cdots \wedge d t_{n}\right|_{M}=0$;
- $\int_{M} f \bar{\partial} \omega=0$, for every $\mathscr{C}^{1}$ form $\omega$ of type $(n, m-1)$ in $\mathbb{C}^{n}$ having compact support.
(Remind the local expression of $(r, s)$ forms: $\sum_{I, J} a_{I, J} d t^{I} \wedge d \bar{t}^{J}$, where $I=\left(i_{1}, \ldots, i_{r}\right)$ and $J=\left(j_{1}, \ldots, j_{s}\right)$.) A (only) continuous function
$f: M \rightarrow \mathbb{C}$ is $C R$ if the last condition $\int_{M} f \bar{\partial} \omega=0$ holds. Further, Lebesgue-integrable CR functions, CR measures, CR distributions and CR currents may be defined as follows ([Trv1981, Trv1992, HM1998, Trp1996, Jö1999b]).

Thanks to graphing functions, one may equip locally $M$ with some (in fact many) volume form, or equivalently, some deformation of the canonical $\operatorname{dim} M$-dimensional Legesgue measure defined on tangent spaces. Let p be a real number with $1 \leqslant p \leqslant \infty$. Since two such measures are multiple of each other, it makes sense to speak of $L_{l o c}^{\mathrm{p}}$ functions $M \rightarrow \mathbb{C}$. In this setting, a $L_{l o c}^{\mathrm{p}}(M)$ function $f$ is CR if $\int_{M} f \bar{\partial} \omega=0$, for every $\mathscr{C}^{1}$ form $\omega$ of type $(n, m-1)$ in $\mathbb{C}^{n}$ having compact support.

A distribution T on $M$ is $C R$ if for every section $\bar{L}$ of $T^{0,1} M$ defined in an open subset $U \subset M$ and every $\chi \in \mathscr{C}_{c}^{\infty}(U, \mathbb{C})$, one has $\langle\mathrm{T}, \bar{L}(\chi)\rangle=0$.

A CR distribution of order zero on $M$ is called a $C R$ measure. Equivalently, a CR measure is a continuous linear map $\omega^{\prime} \mapsto \mu\left(\omega^{\prime}\right)$ from compactly supported forms on $M$ of maximal degree $2 m+d$ to $\mathbb{C}$, that is CR in the weak sense, namely $\mu(\bar{\partial} \omega)=0$, for every $\mathscr{C}^{1}$ form $\omega$ of type $(n, m-1)$ having compact support. Once a volume form $\mathrm{dvol}_{M}$ is fixed on $M$, the quantity $\mu \mathrm{dvol}_{M}$ is a CR (Borel) measure on $M$.
4.26. Traces of $\mathbf{C R}$ functions on $\mathbf{C R}$ orbits. A $\mathscr{C}^{1}$ function $f: M \rightarrow \mathbb{C}$ is CR on $M$ if and only if its restriction to every CR orbit of $M$ is CR (obvious). If $f$ is $\mathscr{C}^{0}$ or $L_{l o c}^{\mathrm{p}}$, a similar but nontrivial statement holds. By "almost every CR orbit", we shall mean "except a union of CR orbits whose $\operatorname{dim} M$-dimensional measure vanishes".

Theorem 4.27. ( $d=1$ : [Jö1999b]; $d \geqslant 1$ : [Po1997, MP1999]) Assume that $M$ is at least $\mathscr{C}^{3}$ and let $f$ be a function in $L_{l o c}^{\mathrm{p}}(M)$ with $1 \leqslant \mathrm{p} \leqslant \infty$. Then the restriction $\left.f\right|_{\mathscr{O}_{C R}}$ is in $L_{l o c}^{\mathrm{p}}$ on $\mathscr{O}_{C R}$, for almost every $\mathscr{O}_{C R}$. Furthermore, $f$ is CR if and only if, for almost every CR orbit $\mathscr{O}_{C R}$ of $M$, its restriction $\left.f\right|_{O_{C R}}$ is $C R$.

The theorem also holds for $f$ continuous, with $\left.f\right|_{\mathscr{O}_{C R}}$ being CR for every CR orbit. Here, $\mathscr{C}^{3}$-smoothness is needed. Property (5) of Sussman's orbit Theorem 1.21 together with a topological reasoning yields a covering by orbit-chart which is used in the proof.

Proposition 4.28. ([Jö1999a, Po1997, MP1999]) Assume M is $\mathscr{C}^{\infty}$ or $\mathscr{C}^{\kappa, \alpha}$, with $\kappa \geqslant 2,0 \leqslant \alpha \leqslant 1$ and let $\square:=\{x \in \mathbb{R}:|x|<1\}$. There exists a countable covering $\bigcup_{k \in \mathbb{N}} U_{k}=M$ such that for each $k$, there exist $e_{k} \in \mathbb{N}$ with $0 \leqslant e_{k} \leqslant d$ and $a \mathscr{C}^{\kappa-1, \alpha}$ diffeomorphism:

$$
\varphi_{k}:\left(s_{k}, t_{k}\right) \ni \square^{2 m+e_{k}} \times \square^{d-e_{k}} \longmapsto \varphi_{k}\left(s_{k}, t_{k}\right) \in U_{k},
$$

such that:

- $\varphi_{k}\left(\square^{2 m+e_{k}} \times\left\{t_{k}^{*}\right\}\right)$ is contained in a single CR orbit, for every fixed $t_{k}^{*} \in \square^{d-e_{k}}$;
- for each $p \in M$, there exists $k=k_{p} \in \mathbb{N}$ with $p \in U_{k_{p}}$, viz there exist $s_{k_{p}, p}$ and $t_{k_{p}, p}$ with $\varphi_{k_{p}}\left(s_{k_{p}, p}, t_{k_{p}, p}\right)=p$, such that $\varphi_{k_{p}}\left(\square^{2 m+e_{k_{p}}} \times\left\{t_{k_{p}, p}\right\}\right)$ is an open piece of the CR orbit of $p$, i.e. $\operatorname{dim} \mathscr{O}_{C R}(M, p)=2 m+e_{k_{p}}$.

In the proof of the theorem, $\mathscr{C}^{2}$-smoothness of the maps $\varphi_{k}$ (hence $\mathscr{C}^{3}$ smoothness of $M$ ) is required to insure that the pull-back $\varphi_{k}^{*}\left(\left.T^{c} M\right|_{U_{k}}\right)$ is $\mathscr{C}^{1}$. However, we would like to mention that if $M$ is $\mathscr{C}^{2, \alpha}$ with $0<\alpha<1$ results of [Tu1990, Tu1994a, Tu1996] and Theorem 3.7(IV) insuring the $\mathscr{C}^{2, \beta}$-smoothness of local and global CR orbits, for every $\beta<\alpha$, this would yield orbit-charts $\varphi_{k}$ of class $\mathscr{C}^{2, \beta}$, and then the above theorem holds true with $M$ of class $\mathscr{C}^{2, \alpha}$.

### 4.29. Boundary values of holomorphic functions for functional spaces

 $\mathscr{C}^{\kappa, \alpha}, \mathscr{D}^{\prime}, L_{l o c}^{\mathrm{p}}$. Let $M$ be a generic submanifold of $\mathbb{C}^{n}$ of codimention $d \geqslant 1$ and of nonnegative CR dimension $m \geqslant 0$ (we admit $m=0$ ). Assume $M$ is at least $\mathscr{C}^{1}$. In appropriate coordinates $t=(z, w)=(x+i y, u+i v) \in$ $\mathbb{C}^{n} \times \mathbb{C}^{m}$ centered at one of its points $p$ :$$
M=\left\{(z, w) \in \Delta_{\rho_{1}}^{m} \times\left(\square_{\rho_{1}}^{d} \times i \mathbb{R}^{d}\right): v=\varphi(x, y, u)\right\}
$$

for some $\rho_{1}>0$, with $\varphi(0)=0$ and $d \varphi(0)=0$. Let $\rho$ be a real number with $0 \leqslant \rho \leqslant \rho_{1}$. The height function:

$$
\sigma(\rho):=\max _{|x|,|y|,|u| \leqslant \rho}|\varphi(x, y, u)|
$$

is continuous and tends to 0 , as $\rho$ tends to 0 . For every $\rho \leqslant \rho_{1}$ and every $\sigma>\sigma(\rho)$, the boundary of $M \cap\left[\Delta_{\rho}^{m} \times\left(\square_{\rho}^{d} \times i \square_{\sigma}^{d}\right)\right]$ is contained in the boundary $\partial\left(\Delta_{\rho}^{d} \times \square_{\rho}^{d}\right)$ of the horizontal space, times the vertical space $i \square \square_{\sigma}^{d}$.

Let $C$ be an open convex cone in $\mathbb{R}^{d}$ having vertex 0 . We shall assume it to be salient, namely contained in one side of some hyperplane passing through the origin. Equivalently, its intersection $C \cap S^{d-1}$ with the unit sphere of $\mathbb{R}^{d}$ is open, contained in some open hemisphere and convex in the sense of spherical geometry.

A local wedge of edge $M$ at $p$ directed by $C$ is an open set of the form:

$$
\begin{align*}
\mathscr{W}=\mathscr{W}(\rho, \sigma, C):= & \left\{(x+i y, u+i v) \in \Delta_{\rho}^{m} \times \square_{\rho}^{d} \times i \square_{\sigma}^{d}:\right.  \tag{4.30}\\
& v-\varphi(x, y, u) \in C\},
\end{align*}
$$

for some $\rho, \sigma>0$ satisfying $\rho \leqslant \rho_{1}$ and $\sigma>\sigma(\rho)$. This type of open set is independent of the choice of local coordinates and of local defining functions; in codimension $d \geqslant 2$, it generalizes the notion of local side of a hypersurface. Notice that $\mathscr{W}$ is connected.

If there exists a function $F$ that is holomorphic in $\mathscr{W}$ and that extends continuously up to the edge

$$
M_{\rho}:=M \cap\left[\Delta_{\rho}^{m} \times\left(\square_{\rho}^{d} \times i \mathbb{R}^{d}\right)\right]
$$

of the wedge $\mathscr{W}$, then the limiting values of $F$ define a continuous CR function on $M_{\rho}$.

A more general phenomenon holds. A function $F$, holomorphic in the wedge $\mathscr{W}$, has slow growth up to $M$, if there exist $k \in \mathbb{N}$ and $C>0$ such that

$$
|F(t)| \leqslant C|v-\varphi(x, y, u)|^{-k}, \quad t=(x+i y, u+i v) \in \mathscr{W}
$$

Equivalently, $|F(t)| \leqslant C[\operatorname{dist}(t, M)]^{-k}$, with the same $k$ but a possibly different $C$. As in the cited references, we shall assume $M$ to be $\mathscr{C}{ }^{\infty}$.

Theorem 4.31. ([BCT1983, Hö1985, BR1987, BER1999]) If $F \in \mathscr{O}(\mathscr{W}(\rho, \sigma, C))$ has slow growth up to $M$, it possesses a boundary value $b_{M} F$ which is a CR distribution on the edge $M \cap\left[\Delta_{\rho}^{m} \times\left(\square_{\rho}^{d} \times i \square \square_{\sigma}^{d}\right)\right]$ precisely defined by:

$$
\begin{gathered}
\left\langle\mathrm{b}_{M} F, \chi\right\rangle:=\lim _{C \ni \theta \rightarrow 0} \int_{\Delta_{\rho}^{m} \times \square_{\rho}^{d}} F(x+i y, u+i \varphi(x, y, u)+i \theta) . \\
\cdot \chi(x, y, u) d x d y d u,
\end{gathered}
$$

where $\chi=\chi(x, y, u)$ is a $\mathscr{C}^{\infty}$ function having compact support in $\Delta_{\rho}^{m} \times \square_{\rho}^{d}$.
(i) The limit is independent of the way how $\theta \in C$ approaches $0 \in \mathbb{R}^{d}$.
(ii) If $\mathrm{b}_{M} F$ is $\mathscr{C}^{\lambda, \beta}, \lambda \geqslant 0,0 \leqslant \beta \leqslant 1$, then $F$ extends as a $\mathscr{C}^{\lambda, \beta}$ function on $\mathscr{W}^{\prime} \cup\left(M \cap\left[\Delta_{\rho}^{m} \times\left(\square_{\rho}^{d} \times i \square_{\sigma^{\prime}}^{d}\right)\right]\right)$, for every wedge $\mathscr{W}^{\prime}=$ $\mathscr{W}^{\prime}\left(\rho, \sigma^{\prime}, C^{\prime}\right)$ with $\sigma(\rho)<\sigma^{\prime} \leqslant \sigma$ and with $C^{\prime} \cap S^{d-1} \subset \subset C \cap S^{d-1}$.
(iii) Finally, $F$ vanishes identically in the wege $\mathscr{W}$ if and only if $\mathrm{b}_{M} F$ vanishes on some nonempty open subset of the edge $M_{\rho}$.

The integration is performed on the translation $M_{\rho}^{\theta}:=M_{\rho}+(0, i \theta)$, drawn as follows.


Boundary values of functions holomorphic in a wedge
The proof is standard for $M \equiv \mathbb{R}^{n}$ ([Hö1985]), the main argument going back to Hadamard's finite parts. With technical adaptations in the case of a general generic $M$, several integrations by part are performed on a thin ( $\operatorname{dim} M+1$ )-dimensional cycle delimited by $M_{\rho}^{0}$ and $M_{\rho}^{\theta}$, taking advantage of Cauchy's classical formula, until the rate of explosion of $F$ up to the edge is dominated. The uniqueness property (iii) requires analytic disc methods (Part V).

Boundary values in the $L^{\mathrm{p}}$ sense requires special attention. At first, remind that a function $F$ holomorphic in the unit disc $\Delta$ belongs to the Hardy class $H^{\mathrm{p}}(\Delta)$ if the supremum:

$$
\|F\|_{H^{\mathrm{p}}(\Delta)}:=\sup _{0<r<1}\left(\int_{-\pi}^{\pi}\left|F\left(r e^{i t}\right)\right|^{\mathrm{p}}\right)^{1 / \mathrm{p}}<\infty
$$

is finite. According to Fatou and Privalov, such a function $F$ has radial boundary values $f\left(e^{i t}\right):=\lim _{r \rightarrow 1} F\left(r e^{i t}\right)$, for almost every $t \in[-\pi, \pi]$, so that the boundary value $f$ belongs to $L^{\mathrm{p}}([-\pi, \pi])$. Furthermore, if $1 \leqslant \mathrm{p}<$ $\infty$ :

$$
\lim _{r \rightarrow 1} \int_{-\pi}^{\pi}\left|F\left(r e^{i t}\right)-f\left(e^{i t}\right)\right|^{\mathrm{p}}=0 .
$$

Consider a bounded domain $D \subset \mathbb{C}^{n}$ having boundary of class at least $\mathscr{C}^{2}$, defined by $D=\left\{z \in \mathbb{C}^{n}: \rho(z)<0\right\}$, with $\rho \in \mathscr{C}^{2}$ satisfying $d \rho \neq$ 0 on $\partial D$. For $\varepsilon>0$ small, let $D_{\varepsilon}:=\{z \in D: \rho(z)<-\varepsilon\}$. The induced Euclidean measure on $\partial D_{\varepsilon}$ (resp. $\partial D$ ) is denoted by $d \sigma_{\varepsilon}$ (resp. $d \sigma$ ). Then the Hardy space $H^{\mathrm{p}}(D)$ consists of holomorphic functions $F \in \mathscr{O}(D)$ having the property that the supremum:

$$
\|F\|_{H \mathrm{P}(D)}:=\sup _{\varepsilon>0}\left(\int_{\partial D_{\varepsilon}}|F(z)|^{\mathrm{p}} d \sigma_{\varepsilon}(z)\right)^{1 / \mathrm{p}}<\infty
$$

is finite. The resulting space does not depend on the choice of a defining function $\rho$ ([St1972]). Let $\mathbf{n}_{z}$ be the outward-pointing normal to the boundary at $z \in \partial D$.
Theorem 4.32. ([St1972]) If $F \in H^{\mathrm{P}}(D)$, for almost all $z \in \partial D$, the normal boundary value $f(z):=\lim _{\varepsilon \rightarrow 0} F\left(z-\varepsilon \mathbf{n}_{z}\right)$ exists and defines a function $f$ which belongs to $L^{\mathrm{p}}(\partial D)$. Furthermore, if $1 \leqslant \mathrm{p}<\infty$ :

$$
\lim _{\varepsilon \rightarrow 0} \int_{\partial D}\left|F\left(z-\varepsilon \mathbf{n}_{z}\right)-f(z)\right|^{\mathrm{p}} d \sigma(z)=0 .
$$

In arbitrary codimension, the notion of $L^{\mathrm{p}}$ boundary values may be defined in the local sense as follows. Let $M$ be generic, let $p \in M$ and let $\mathscr{W}=\mathscr{W}(\rho, \sigma, C)$ be a local wedge of edge $M$ at $p$, as defined by (4.30). A holomorphic function $F \in \mathscr{O}(\mathscr{W})$ belongs to the Hardy space $H_{l o c}^{\mathrm{p}}(\mathscr{W})$ if for every cone $C^{\prime} \subset \mathbb{R}^{d}$ with $C^{\prime} \cap S^{d-1} \subset \subset C \cap S^{d-1}$ and every $\rho^{\prime}<\rho$, the supremum:

$$
\sup _{\theta^{\prime} \in C^{\prime}} \int_{\Delta_{\rho^{\prime} \times \square_{\rho^{\prime}}^{m}}^{d}}\left|F\left(x+i y, u+i \varphi(x, y, u)+i \theta^{\prime}\right)\right|^{\mathrm{p}} d x \wedge d y \wedge d u<\infty
$$

is finite. Up to shrinking cubes, polydiscs and cones, the resulting space neither depends on local coordinates nor on the choice of local defining equations.

Theorem 4.33. $(d=1$ : [St1972, Jö1999b]; $d \geqslant 2$ : [Po1997]) If $F \in$ $H_{l o c}^{\mathrm{p}}(\mathscr{W})$, for almost $(x, y, u+i \varphi(x, y, u)) \in M_{\rho}$ and for every cone $C^{\prime}$ with $C^{\prime} \cap S^{d-1} \subset \subset C \cap S^{d-1}$, the boundary value:

$$
f(x, y, u):=\lim _{C^{\prime} \ni \theta^{\prime} \rightarrow 0} F\left(x+i y, u+i \varphi(x, y, y)+i \theta^{\prime}\right)
$$

exists and defines a function $f$ which belongs to $L_{l o c, C R}^{\mathrm{p}}\left(M_{\rho}\right)$. Furthermore, if $1 \leqslant \mathrm{p}<\infty$, for every $\rho^{\prime}<\rho$ :

$$
\begin{aligned}
& \lim _{C^{\prime} \ni \theta^{\prime} \rightarrow 0} \int_{\Delta_{\rho^{\prime}}^{m} \times \square_{\rho^{\prime}}^{d}} \mid F\left(x+i y, u+i \varphi(x, y, y)+i \theta^{\prime}\right)- \\
&-\left.f(x, y, u)\right|^{\mathfrak{p}} d x \wedge d y \wedge d u=0 .
\end{aligned}
$$

4.34. Holomorphic extendability of CR functions in $\mathscr{C}^{\kappa, \alpha}, \mathscr{D}^{\prime}, L_{l o c}^{\mathrm{p}}$. In Part V, we will study sufficient conditions for the existence of wedges to which CR functions and distributions extend holomorphically.
Definition 4.35. A CR function of class $\mathscr{C}^{\kappa, \alpha}$ or $L_{l o c}^{\mathrm{p}}(1 \leqslant \mathrm{p}<\infty)$ or a CR distribution $f$ defined on $M$ is holomorphically extendable if there exists a local wedge $\mathscr{W}=\mathscr{W}(\rho, \sigma, C)$ at $p$ and a holomorphic function $F \in \mathscr{O}(\mathscr{W})$ whose boundary value $\mathrm{b}_{M} F$ equals $f$ on $M_{\rho}$ in the $\mathscr{C}^{\kappa, \alpha}, L^{\mathrm{p}}$ or $\mathscr{D}^{\prime}$ sense.
4.36. Local CR distributions supported by a local CR orbit. Assume now that $M$, of class $\mathscr{C}^{\infty}$ and represented as in $\S 4.3$, is not locally minimal at $p$. Equivalently, $\mathscr{O}_{C R}^{\text {loc }}(M, p)$ is of dimension $2 m+e \leqslant 2 m+d-1$. In a small neighborhood, $S:=\mathscr{O}_{C R}^{l o c}(M, p)$ is a closed connected CR submanifold of $M$ passing through $p$ and having the same CR dimension as $M$. There exist local holomorphic coordinates $(z, w)=\left(z, w_{1}, w_{2}\right) \in \mathbb{C}^{m} \times$ $\mathbb{C}^{e} \times \mathbb{C}^{d-e}$ vanishing at $p$ in which $M$ is represented by $v=\varphi(x, y, u)$ and $S$ is represented by the supplementary (scalar) equation(s) $u_{2}=\lambda_{2}\left(x, y, u_{1}\right)$, with $\varphi$ and $\lambda_{2}$ of class $\mathscr{C}^{\infty}$ satisfying $\varphi(0)=0, d \varphi(0)=0, \lambda_{2}(0)=0$ and $d \lambda_{2}(0)=0$. According to Theorem 4.2, the assumption that $S$ is CR and has the same CR dimension as $M$ may be expressed as follows.

Proposition 4.37. Decomposing $\varphi=\left(\varphi_{1}, \varphi_{2}\right)$ and defining:

$$
\begin{aligned}
& v_{1}=\varphi_{1}\left(x, y, u_{1}, \lambda_{2}\left(x, y, u_{1}\right)\right)=: \mu_{1}\left(x, y, u_{1}\right), \\
& v_{2}=\varphi_{2}\left(x, y, u_{1}, \lambda_{2}\left(x, y, u_{1}\right)\right)=: \mu_{2}\left(x, y, u_{1}\right) .
\end{aligned}
$$

the map:

$$
\psi_{2}\left(x, y, u_{1}\right):=\lambda_{2}\left(x, y, u_{1}\right)+i \mu_{2}\left(x, y, u_{1}\right)
$$

is $C R$ on the generic submanifold $v_{1}=\mu_{1}\left(x, y, u_{1}\right)$ of $\mathbb{C}^{m} \times \mathbb{C}^{e}$.
In a small neighborhood $U$ of $p$, the restrictions

$$
\left.d z_{1}\right|_{S}, \ldots,\left.d z_{m}\right|_{S},\left.d w_{1}\right|_{S}, \ldots,\left.d w_{e}\right|_{S}
$$

span an $(m+e)$-dimensional subbundle of $\mathbb{C} T^{*} S$. Denoting $d z:=d z_{1} \wedge \cdots \wedge$ $d z_{m}, d \bar{z}:=d \bar{z}_{1} \wedge \cdots \wedge d \bar{z}_{m}$ and $d w^{\prime}:=d w_{1} \wedge \cdots \wedge d w_{e}$, for $\chi \in \mathscr{C}_{c}^{\infty}(U, \mathbb{C})$, consider the (localized) distribution defined by:

$$
\langle[\mathrm{S}], \chi\rangle:=\int_{U \cap S} \chi \cdot d z \wedge d w^{\prime} \wedge d \bar{z}
$$

Proposition 4.38. ([Trv1992, HT1993]) Then [S] is a nonzero local CR measure supported by $S \cap U$.
Proof. It is clear that $[\mathrm{S}]$ is supported by $S \cap U$ and is of order zero. Let the $(0,1)$ vector fields $\bar{L}_{k}$ and the complex-transversal ones $K_{j}$ be as in §4.3. Reminding $d \chi=\sum_{k=1}^{m} L_{k}(\chi) d z_{k}+\sum_{k=1}^{m} \bar{L}_{k}(\chi) d \bar{z}_{k}+$ $\left.\sum_{j=1}^{d} K_{j}(\chi) d w_{j}\right|_{M}$, we observe:

$$
\bar{L}_{k}(\chi) d z \wedge d w^{\prime} \wedge d \bar{z}= \pm d\left(\chi \cdot d z \wedge d w^{\prime} \wedge d \bar{z}_{1} \wedge \cdots \wedge \widehat{\bar{z}}_{k} \wedge \cdots \wedge d \bar{z}_{m}\right)
$$

Replacing this volume form in the integrand:

$$
\begin{aligned}
\left\langle\bar{L}_{k}[\mathrm{~S}], \chi\right\rangle & :=-\left\langle[\mathrm{S}], \bar{L}_{k}(\chi)\right\rangle=-\int_{S \cap U} \bar{L}_{k}(\chi) d z \wedge d w^{\prime} \wedge d \bar{z} \\
& = \pm \int_{S \cap U} d\left(\chi \cdot d z \wedge d w^{\prime} \wedge d \bar{z}_{1} \wedge \cdots \wedge{\widehat{d \bar{z}_{k}}} \wedge \cdots \wedge d \bar{z}_{m}\right)
\end{aligned}
$$

and applying Stokes' theorem, we deduce $\left\langle\bar{L}_{k}[\mathrm{~S}], \chi\right\rangle=0$, i.e. $[\mathrm{S}]$ is CR.
The last assertion of Theorem 4.31 and the vanishing of $[S]$ on the dense open set $U \backslash(S \cap U)$ entails the following.
Corollary 4.39. ([Trv1992, HT1993]) The nonzero local CR measure $[\mathrm{S}]$ does not extend holomorphically to any local wedge of edge $M$ at $p$.

By means of this wedge nonextendable CR measure, one may construct non-extendable CR functions of arbitrary smoothness. Indeed, let $M$ be a local generic submanifold with central point $p$, as represented in $\S 4.3$ and let $K_{j}$ be the complex-transversal vector fields satisfying $K_{j_{1}}\left(w_{j_{2}}\right)=\delta_{j_{1}, j_{2}}$ and $\left[K_{j_{1}}, K_{j_{2}}\right]=0$.

Proposition 4.40. ([BT1981, Trv1981, BR1990, Trv1992, HT1996, BER1999]) For every CR distribution $T$ on $M$ and every $\kappa \in \mathbb{N}$, there exist an integer $\mu \in \mathbb{N}$ and a local CR function $f$ of class $\mathscr{C}^{\kappa}$ defined in some neighborhood of $p$ such that:

$$
\mathrm{T}=\left(K_{1}^{2}+\cdots+K_{d}^{2}\right)^{\mu} f
$$

Then with $\mathrm{T}:=[\mathrm{S}]$ and for $\kappa \in \mathbb{N}$, an associated CR function $f$ of class $\mathscr{C}^{\kappa}$ is also shown to be not holomorphically extendable to any local wedge of edge $M$ at $p$. A Baire category argument ([BR1990]) enables to treat the $\mathscr{C}^{\infty}$ case.
Theorem 4.41. ([BR1990, BER1999]) If $M$ is not locally minimal at $p$, then for every $\kappa=0,1,2, \ldots, \infty$, there exists a CR function hof class $\mathscr{C}^{\kappa}$ defined in a neighborhood of $p$ which does not extend holomorphically to any local wedge of edge $M$ at $p$.

Open problem 4.42. Find criteria for the existence of CR distributions or functions supported by a global CR orbit.

In [BM1997], this question is dealt with in the case of CR orbits of hypersurfaces which are immersed or embedded complex manifolds.

To conclude this section, we give the general form of a CR distribution supported by a local CR orbit $S=\mathscr{O}_{C R}^{l o c}(M, p)$. After restriction to $S$, the collection $K_{S}:=\left(K_{e+1}, \ldots, K_{d}\right)$ of vector fields spans the normal bundle to $S$ in $M$, in a neighborhood of $p$. Let T be a local CR distribution defined on $M$ that is supported by $S$.
Theorem 4.43. ([Trv1992, BCH2005]) There exist an integer $\kappa \in \mathbb{N}$, and for all $\beta \in \mathbb{N}^{d-e}$ with $|\beta| \leqslant \kappa$, local CR distributions $\mathrm{T}_{\beta}^{S}$ defined on $S$ such that:

$$
\langle\mathrm{T}, \chi\rangle=\sum_{\beta \in \mathbb{N}^{d-e},|\beta| \leqslant \kappa}\left\langle\mathrm{T}_{\beta}^{S},\left.\left(K_{S}\right)^{\beta}(\chi)\right|_{S}\right\rangle .
$$

## §5. APPROXIMATION AND UNIQUENESS PRINCIPLES

5.1. Approximation of CR functions and of CR distributions. Let $M$ be a generic submanifold of $\mathbb{C}^{n}$. The following approximation theorem has appeared to be a fundamental tool in extending CR functions holomorphically (Part V) and in removing their singularities (Part VI). It is also used naturally in the proof of Theorem 4.43 just above as well as in the Cauchy uniqueness principle Corollary 5.4 below. The statement is valid in the general context of locally integrable structures $\mathscr{L}$, but, as explained in the end of Section 3, we decided to focus our attention on embedded Cauchy-Riemann geometry.

Theorem 5.2. ([BT1981, HM1998, Jö1999b, BCH2005]) For every $p \in M$, there exists a neighborhood $U_{p}$ of $p$ in $M$ such that for every function $f$ or distribution T as defined below, there exists a sequence of holomorphic polynomials $\left(P_{k}(z)\right)_{k \in \mathbb{N}}$ with:

- if $M$ is $\mathscr{C}^{\kappa+2, \alpha}$, with $\kappa \geqslant 0,0 \leqslant \alpha \leqslant 1$, if $f$ is a CR function of class $\mathscr{C}^{\kappa, \alpha}$ on $M$, then $\lim _{k \rightarrow \infty}\left\|P_{k}-f\right\|_{\mathscr{C}^{\kappa, \alpha}\left(U_{p}\right)} \rightarrow 0$; in particular, continuous $C R$ functions on a $\mathscr{C}^{2}$ generic submanifold are approximable sharply by holomorphic polynomials;
- if $M$ is at least $\mathscr{C}^{2}$, if $f$ is a $L_{l o c}^{\mathrm{p}}$ CR function $(1 \leqslant \mathrm{p}<\infty)$, then $\lim _{k \rightarrow \infty}\left\|P_{k}-f\right\|_{L_{\text {loc }}^{\mathrm{p}}\left(U_{p}\right)} \rightarrow 0$;
- if $M$ is $\mathscr{C}^{\kappa+2}$, if T is a CR distribution of order $\leqslant \kappa$ on $M$, then $\lim _{k \rightarrow \infty}\left\langle P_{k}, \chi\right\rangle=\langle\mathrm{T}, \chi\rangle$ for every $\chi \in \mathscr{C}_{c}^{\infty}\left(U_{p}\right)$.

In [HM1998, BCH2005], convergence in Besov-Sobolev spaces $L_{s, l o c}^{\mathrm{p}}$ and in Hardy spaces $h^{\mathrm{p}}$, frequently used as substitutes for the $L^{\mathrm{p}}$ spaces when $0<\mathrm{p}<1$, is also considered, in the context of locally integrable structure.

Proof. Let us describe some ideas of the proof, assuming for simplicity that $M$ is $\mathscr{C}^{2}$ and $f$ is $\mathscr{C}^{1}$. In coordinates $\left(t_{1}, \ldots, t_{n}\right)$ vanishing at $p$, choose a local maximally real $\mathscr{C}^{2}$ submanifold $\Lambda_{0}$ contained in $M$, passing through $p$ and satisfying $T_{p} \Lambda_{0}=\{\operatorname{Re} t=0\}$. Let $V_{p}$ be a small neighborhood of $p$, whose projection to $T_{p} M$ is a $(2 m+d)$-dimensional open ball. We may assume that $\Lambda_{0}$ is contained in $V_{p}$ with boundary $B_{0}:=\overline{\Lambda_{0}} \cap \partial V_{p}$ being diffeomorphic to the $(n-1)$-dimensional sphere. Consider a parameter $u \in \mathbb{R}^{d}$ satisfying $|u|<\delta$, with $\delta>0$ small. We may include $\Lambda_{0}$ in a family $\left(\Lambda_{u}\right)_{|u|<\delta}$ of maximally real $\mathscr{C}^{2}$ submanifolds of $U_{p}$ with $\left.\Lambda_{u}\right|_{u=0}=$ $\Lambda_{0}$, whose boundary is fixed: $\partial \Lambda_{u} \equiv \partial \Lambda_{0}=B_{0}$, such that the $\Lambda_{u}$ foliates a small neighborhood $U_{p}$ of $p$ in $M$. For $t \in U_{p}$, there exists a $u=u(t)$ such that $t$ belongs to $\Lambda_{u(t)}$. We then introduce the entire functions:

$$
F_{k}(t):=\left(\frac{k}{\pi}\right)^{n / 2} \int_{\Lambda_{u(t)}} e^{-k(t-\tau)^{2}} f(\tau) d \tau_{1} \wedge \cdots \wedge d \tau_{n}
$$

where $(t-\tau)^{2}:=\sum_{j=1}^{n}\left(t_{j}-\tau_{j}\right)^{2}$ and where $k \in \mathbb{N}$. Shrinking $V_{p}$ and $U_{p}$ if necessary, we may assume that $|\operatorname{Im}(t-\tau)| \leqslant \frac{1}{2}|\operatorname{Re}(t-\tau)|$ for all $t, \tau \in \Lambda_{u} \cap U_{p}$ and all $|u|<\delta$. Here, the $\mathscr{C}^{2}$-smoothness assumption is used. With this inequality, the above multivariate Gaussian kernel is easily seen to be an approximation of the Dirac distribution at $\tau=t$ on $\Lambda_{u(t)}$. Consequently $F_{k}(t)$ tends to $f(t)$ as $k \rightarrow \infty$. Moreover, the convergence is uniform and holds in $\mathscr{C}^{0}\left(U_{p}\right)$.

We claim that the assumption that $f$ is CR insures that $F_{k}(t)$ has the same value if the integration is performed on $\Lambda_{0}$ :

$$
\begin{equation*}
F_{k}(t)=\left(\frac{k}{\pi}\right)^{n / 2} \int_{\Lambda_{0}} e^{-k(t-\tau)^{2}} f(\tau) d \tau_{1} \wedge \cdots \wedge d \tau_{n} \tag{5.3}
\end{equation*}
$$

Indeed, $\Lambda_{u(t)}$ and $\Lambda_{0}$ bound a $(n+1)$-dimensional submanifold $\Pi_{t}$ contained in $V_{p}$ with $\partial \Pi_{t}=\Lambda_{u(t)}-\Lambda_{0}$. Since $e^{-k(t-\tau)^{2}}$ is holomorphic with respect to $\tau$ and since $\left.d f(\tau) \wedge d \tau_{1} \wedge \cdots \wedge d \tau_{n}\right|_{M}=0$, because $f$ is $\mathscr{C}^{1}$ and CR, the $(n, 0)$ form $\omega:=e^{-k(t-\tau)^{2}} f(\tau) d \tau=0$ is closed: $d \omega=0$. By an application of Stokes' theorem, it follows that $0=\int_{\Pi_{t}} d \omega=\int_{\Lambda_{u(t)}} \omega-\int_{\Lambda_{0}} \omega$, which proves the claim.

Finally, to approximate $f$ by polynomials on $U_{p}$ in the $\mathscr{C}^{0}$ topology, in the above integral (5.3) that is performed on the fixed maximally real submanifold $\Lambda_{0}$, it suffices to develop the exponential in Taylor series and to integrate term by term. In other functional spaces, the arguments have to be adapted.

As a consequence, uniqueness in the Cauchy problem holds. It may be shown ([Trv1981, Trv1992]) that the trace of a CR distribution on a maximally real submanifold always exists, in the distributional sense.

Corollary 5.4. ([Trv1981, Trv1992]) If a CR function or distribution vanishes on a maximally real submanifold $\Lambda$ of $M$, there exists an open neighborhood $U_{\Lambda}$ of $\Lambda$ in $M$ in which it vanishes identically.

Since every submanifold $H$ of $M$ which is generic in $\mathbb{C}^{n}$ contains small maximally real sumanifolds passing through every of its points, the corollary also holds with $\Lambda$ replaced by such a $H$.

Proof. It suffices to localize the above construction in a neighborhood of an arbitrary point $p \in \Lambda$ and to take for $\Lambda_{0}$ a neighborhood of $p$ in $\Lambda$. The integral (5.3) then vanishes identically.

Corollary 5.5. ([Trv1981, Trv1992]) The support of a CR function or distribution on $M$ is a closed CR-invariant subset of $M$.

Proof. By contraposition, if a CR function or distribution vanishes in a neighborhood $U_{p}$ of a point $p$ in $M$, it vanishes identically in the $C R$ invariant hull of $U_{p}$, viz the union of CR orbits of all points $q \in U_{p}$. The CR orbits being covered by concatenations of CR vector fields, neglecting some technicalities, the main step is to establish:

Lemma 5.6. Let $p \in M$, let L be a section of $T^{c} M$ and let $q^{*}=\exp \left(s^{*} L\right)(p)$ for some $\mathrm{s}^{*} \in \mathbb{R}$. If a CR function or distribution vanishes in a neighborhood of $p$, it vanishes also in a neighborhood of $q$.

Indeed, we may construct a one-parameter family $\left(H_{\mathrm{s}}\right)_{0 \leqslant s \leqslant s^{*}}$ of $\mathscr{C}^{2}$ hypersurfaces of $M$ with $q^{*} \in H_{s^{*}}$ and with $H_{0}$ contained in a small neighborhood of $p$ at which the CR function of distribution vanishes already. As illustrated by the following diagram, we can insure that at every point $q_{\mathrm{s}}=\exp (\mathrm{s} L)(p)$, the vector $L\left(q_{\mathrm{s}}\right)$ is nontangent to $H_{\mathrm{s}}$.


It follows that the hypersurfaces $H_{\mathrm{s}}$ are generic in $\mathbb{C}^{n}$, for every s . Then the phrase after Corollary 5.4 applies to each $H_{\mathrm{s}}$ from $H_{0}$ up to $H_{\mathrm{s}^{*}}$, showing the propagation of vanishing.
5.7. Unique continuation principles. At least three unique continuation properties are known to be enjoyed by holomorphic functions $h$ of several complex variables defined in a domain $D \subset \mathbb{C}^{n}$. Indeed, we have $h \equiv 0$ in either of the following three cases:
(ucp1) the restriction of $h$ to some nonempty open subset of $D$ vanishes identically;
(ucp2) the restriction of $h$ to some generic local submanifold $\Lambda$ of $D$ vanishes identically;
(ucp3) there exists a point $p \in D$ at which the infinite jet of $h$ vanishes.
In Complex Analysis and Geometry, the (ucpi) have deep influence on the whole structure of the theory. Finer principles involving tools from Harmonic Analysis appear in [MP2006b].

Problem 5.8. Find generalizations of the (ucpi) to the category of embedded generic submanifolds $M$.

Since a domain $D$ of $\mathbb{C}^{n}$ trivially consists of a single CR orbit, it is natural to assume that the given generic manifold $M$ is globally minimal (although some meaningful questions arise without this assumption, we prefer not to enter such technicalities). In this setting, Corollaries 5.4 and 5.5 provide a complete generalization of (ucp1) and of (ucp2).

A version of (ucp3) with the point $p$ in the boundary $\partial D$ does not hold, even in complex dimension one. Indeed, the function $\exp \left(e^{i 5 \pi / 4} / \sqrt{w}\right)$ is holomorphic in $\mathbb{H}^{+}:=\{w \in \mathbb{C}: \operatorname{Re} w>0\}$, of class $\mathscr{C}^{\infty}$ on $\overline{\mathbb{H}}^{+}$and flat at $w=0$. The restriction of this function to the Heisenberg sphere $\operatorname{Re} w=z \bar{z}$ of $\mathbb{C}^{2}$ provides a CR example.

To generalize rightly (ucp3), let $M$ be a $\mathscr{C}^{1}$ generic submanifold of codimension $d \geqslant 1$ and of CR dimension $m \geqslant 1$ in $\mathbb{C}^{n}$, with $n=m+d$. Let $\Sigma$ be a $\mathscr{C}^{1}$ submanifold of $M$ satisfying:

$$
T_{q}^{c} M \oplus T_{q} \Sigma=T_{q} M, \quad q \in \Sigma
$$

Here, $\Sigma$ plays the rôle of the point $p$ in (ucp3). Denote by $\mathscr{O}_{C R}^{\Sigma}$ the union of CR orbits of points of $\Sigma$, i.e. the $C R$-invariant hull of $\Sigma$. It is an open subset of $M$. We say that a CR function $f: M \rightarrow \mathbb{C}$ of class $\mathscr{C}^{1}$ vanishes to infinite order along $\Sigma$ if for every $p \in \Sigma$, there exists an open neighborhood $U_{p}$ of $p$ in $M$ such that for every $\nu \in \mathbb{N}$, there exists a constant $C>0$ with

$$
|h(t)| \leqslant C[\operatorname{dist}(t, \Sigma)]^{\nu}, \quad t \in U_{p} .
$$

Theorem 5.9. ([Ro1986b, BT1988], [*]) Assume that $\Sigma$ is the intersection with $M$ of some d-dimensional holomorphic submanifold of $\mathbb{C}^{n}$. If a $C R$ function of class $\mathscr{C}^{1}$ vanishes to infinite order along $\Sigma$, then it vanishes identically on the globally minimal generic submanifold $M$.

Assuming that $\Sigma$ is only a conic $d$-codimensional holomorphic submanifold entering a wedge to which all CR functions of $M$ extend holomorphically (Theorem 3.8(V)), the proof of this theorem may be easily generalized.
Open question 5.10. ([Ro1986b, BT1988]) Is the above unique continuation true for $\Sigma$ merely $\mathscr{C}^{1}$ ?

To attack this question, one should start with $M$ being unit sphere $S^{3} \subset$ $\mathbb{C}^{2}$ and $\Sigma \subset S^{3}$ being any $T^{c} S^{3}$-transversal real segment which is nowhere locally the boundary of a complex curve lying inside the ball.

# IV: Hilbert transform and Bishop's equation in Hölder spaces 

Table of contents<br>1. Hölder spaces: basic properties 107.<br>2. Cauchy integral, Sokhotskiï-Plemelj formulas and Hilbert transform ..... 109.<br>3. Solving a local parametrized Bishop equation with optimal loss of smoothness<br>124.<br>4. Appendix: proofs of some lemmas 142.

[1 diagram]

In complex and harmonic analysis, the spaces $\mathscr{C}^{\kappa, \alpha}$ of fractionally differentiable maps, called Hölder spaces, are very flexible to generate inequalities and they yield rather satisfactory norm estimates for almost all the classical singular integral operators, especially when $0<\alpha<1$. For instance, the Cauchy integral of a $\mathscr{C}^{\kappa, \alpha}$ function $f: \Gamma \rightarrow \mathbb{C}$ defined on a $\mathscr{C}^{\kappa+1, \alpha}$ Jordan curve $\Gamma$ of the complex plane produces a sectionally holomorphic function, whose boundary values from one or the either side are $\mathscr{C}^{\kappa, \alpha}$ on the curve. The Sokhotskiĭ-Plemelj formulas show that the arithmetic mean of the two (in general different) boundary values at a point of the curve is given by the principal value of the Cauchy integral at that point.

Harmonic and Fourier analysis on the unit disc $\Delta$ is of particular interest for geometric applications in Cauchy-Riemann geometry. According to a theorem due to Privalov, the Hilbert transform T is a bounded linear endomorphism of $\mathscr{C}^{\kappa, \alpha}(\partial \Delta, \mathbb{R})$ with norm $\|\mathrm{T}\|_{\kappa, \alpha}$ equivalent to $\frac{C}{\alpha(1-\alpha)}$, for some absolute constant $C>0$. This operator produces the harmonic conjugate $\mathrm{T} u$ of any real-valued function $u: \partial \Delta \rightarrow \mathbb{R}$ on the unit circle, so that $u+i \mathrm{~T} u$ always extends holomorphically to $\Delta$. Bishop (1965), Hill-Taiani (1978), Boggess-Pitts (1985) and Tumanov (1990) formulated and solved a functional equation involving T in order to find small analytic discs with boundaries contained in a generic submanifold $M$ of codimension $d$ in $\mathbb{C}^{n}$.

In a general setting, this Bishop-type equation is of the form:

$$
U\left(e^{i \theta}\right)=U_{0}-\mathrm{T}[\Phi(U(\cdot), \cdot, s)]\left(e^{i \theta}\right)
$$

where $U_{0} \in \mathbb{R}^{d}$ is a constant vector, where $\Phi=\Phi\left(u, e^{i \theta}, s\right)$ is an $\mathbb{R}^{d}$-valued $\mathscr{C}^{\kappa, \alpha}$ map, with $\kappa \geqslant 1$ and $0<\alpha<1$, where $u \in \mathbb{R}^{d}$, where $e^{i \theta} \in \partial \Delta$ and where $s \in \mathbb{R}^{b}$ is an additional parameter which is useful in geometric applications. Under some explicit assumptions of smallness of $U_{0}$ and of the first order jet of $\Phi$, the general solution $U=U\left(e^{i \theta}, s, U_{0}\right)$ is of class $\mathscr{C}^{\kappa, \alpha}$ with respect to $e^{i \theta}$ and in addition, for every $\beta$ with $0<\beta<\alpha$, it is of class $\mathscr{C}^{\kappa, \beta}$ with respect to all the variables $\left(e^{i \theta}, s, U_{0}\right)$. These smoothness properties are optimal.

## 1. HÖLDER SPACES: BASIC PROPERTIES

1.1. Background on Hölder spaces. Let $n \in \mathbb{N}$ with $n \geqslant 1$ and let $\mathrm{x}=\left(\mathrm{x}_{1}, \ldots, \mathrm{x}_{n}\right) \in \mathbb{R}^{n}$. On the vector space $\mathbb{R}^{n}$, we choose once for all the maximum norm $|\mathrm{x}|:=\max _{1 \leqslant i \leqslant n}\left|\mathrm{x}_{i}\right|$ and, for any "radius" $\rho$ satisfying $0<\rho \leqslant \infty$, we define the open cube $\square_{\rho}^{n}:=\left\{\mathrm{x} \in \mathbb{R}^{n}:|x|<\rho\right\}$ as a fundamental, concrete open set. For $\rho=\infty$, we identify $\square_{\infty}^{n}$ with $\mathbb{R}^{n}$.

Let $\kappa \in \mathbb{N}$ and let $\alpha \in \mathbb{R}$ with $0 \leqslant \alpha \leqslant 1$. If $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, a scalar function $f: \square_{\rho}^{n} \rightarrow \mathbb{K}$ belongs to the Hölder class $\mathscr{C}^{\kappa, \alpha}\left(\square_{\rho}^{n}, \mathbb{K}\right)$ if, for every multiindex $\delta=\left(\delta_{1}, \ldots, \delta_{n}\right) \in \mathbb{N}^{n}$ of length $|\delta| \leqslant \kappa$, the partial derivative $f_{x^{\delta}}(\mathrm{x}):=\frac{\partial^{|\delta|} f}{\partial x^{\gamma^{1} \ldots}{ }^{\delta} \mathrm{x}^{\delta n}}$ is continuous in $\square_{\rho}^{n}$ and if, moreover, the quantity:

$$
\|f\|_{\kappa, \alpha}:=\sum_{0 \leqslant|\delta| \leqslant \kappa} \sup _{x \in \square_{\rho}^{n}}\left|f_{x^{\delta}}(\mathrm{x})\right|+\sum_{|\delta|=\kappa} \sup _{\mathrm{x}^{\prime \prime} \neq \mathrm{x}^{\prime} \in \square_{\rho}^{n}} \frac{\left|f_{\mathrm{x}^{\delta}}\left(\mathrm{x}^{\prime \prime}\right)-f_{x^{\delta}}\left(\mathrm{x}^{\prime}\right)\right|}{\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right|^{\alpha}}
$$

is finite (if $\alpha=0$, it is understood that the second sum is absent). In case $f=\left(f_{1}, \ldots, f_{m}\right)$ is a $\mathbb{K}^{m}$-valued mapping, with $m \geqslant 1$, we simply define $\|f\|_{\kappa, \alpha}:=\max _{1 \leqslant j \leqslant m}\left\|f_{j}\right\|_{\kappa, \alpha}$. This is coherent with the choice of the maximum norm $|\mathrm{y}|:=\max _{1 \leqslant i \leqslant n}\left|\mathrm{y}_{i}\right|$ on $\mathbb{K}^{m}$. For short, such a map will be said to be $\mathscr{C}^{\kappa, \alpha}$-smooth or of class $\mathscr{C}^{\kappa, \alpha}$ and we write $f \in \mathscr{C}^{\kappa, \alpha}$. One may verify $\left\|f_{1} f_{2}\right\|_{\kappa, \alpha} \leqslant\left\|f_{1}\right\|_{\kappa, \alpha} \cdot\left\|f_{2}\right\|_{\kappa, \alpha}$ and of course $\left\|\lambda_{1} f_{1}+\lambda_{2} f_{2}\right\|_{\kappa, \alpha} \leqslant$ $\left|\lambda_{1}\right|\left\|f_{1}\right\|_{\kappa, \alpha}+\left|\lambda_{2}\right|\left\|f_{2}\right\|_{\kappa, \alpha}$. If $\kappa=0$ and $\alpha=1$, the map $f$ is called Lipschitzian. The condition $\left|f\left(e^{i \theta^{\prime \prime}}\right)-f\left(e^{i \theta^{\prime}}\right)\right| \leqslant C \cdot\left|\theta^{\prime \prime}-\theta^{\prime}\right|$ on the unit circle was first introduced by Lipschitz in 1864 as sufficient for the pointwise convergence of Fourier series.

Thanks to a uniform convergence argument, the space $\mathscr{C}^{\kappa, \alpha}\left(\square_{\rho}^{n}, \mathbb{K}\right)$ is shown to be complete, hence it constitutes a Banach algebra. The space of functions defined on the closure $\overline{\square_{\rho}^{n}}$ also constitutes a Banach algebra. If $\alpha$ is positive, thanks to a prolongation argument, one may verify that $\mathscr{C}^{\kappa, \alpha}\left(\square_{\rho}^{n}, \mathbb{K}\right)$ identifies with the restriction $\left.\mathscr{C}^{\kappa, \alpha}\left(\overline{\square_{\rho}^{n}}, \mathbb{K}\right)\right|_{\square_{\rho}^{n}}$.

Hölder spaces may also be defined on arbitrary convex open subsets. More generally, on an arbitrary subset $\Omega \subset \mathbb{R}^{n}$, it is reasonable to define the Hölder norms $\|\cdot\|_{\kappa, \alpha}, 0<\alpha \leqslant 1$, only if $\operatorname{dist}_{\Omega}\left(\mathrm{x}^{\prime \prime}, \mathrm{x}^{\prime}\right) \leqslant C \cdot\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right|$ for every two points $\mathrm{x}^{\prime \prime}, \mathrm{x}^{\prime} \in \Omega$. This is the case for instance if $\Omega$ is a domain in $\mathbb{R}^{n}$ having piecewise $\mathscr{C}^{1,0}$ boundary.

Introducing the total order $\left(\kappa_{1}, \alpha_{1}\right) \leqslant(\kappa, \alpha)$ defined by: $\kappa_{1}<\kappa$, or: $\kappa_{1}=\kappa$ and $\alpha_{1} \leqslant \alpha$, we verify that $\mathscr{C}^{\kappa, \alpha}$ is contained in $\mathscr{C}^{\kappa_{1}, \alpha_{1}}$ and that:

- $\|f\|_{\kappa, 0} \leqslant\|f\|_{\kappa, \alpha}$ for all $\alpha$ with $0<\alpha \leqslant 1$ and for all $\kappa \in \mathbb{N}$;
- $\|f\|_{\kappa, \alpha_{1}} \leqslant 3\|f\|_{\kappa, \alpha_{2}}$ for all $\alpha_{1}, \alpha_{2}$ with $0<\alpha_{1}<\alpha_{2} \leqslant 1$ and for all $\kappa \in \mathbb{N}$;
- $\|f\|_{\kappa, 1} \leqslant\|f\|_{\kappa+1,0}$, for all $\kappa \in \mathbb{N}$.

The first inequality above is trivial while the third follows from (1.3) below. We explain the factor 3 in the second inequality. Since $\left|x^{\prime \prime}-x^{\prime}\right|^{-\alpha_{1}} \leqslant$ $\left|x^{\prime \prime}-x^{\prime}\right|^{-\alpha_{2}}$ only if $\left|x^{\prime \prime}-x^{\prime}\right| \leqslant 1$, we may estimate:

$$
\sup _{0<\left|x^{\prime \prime}-x^{\prime}\right| \leqslant 1} \frac{\left|f\left(x^{\prime \prime}\right)-f\left(x^{\prime}\right)\right|}{\left|x^{\prime \prime}-x^{\prime}\right|^{\alpha_{1}}} \leqslant \sup _{0<\left|x^{\prime \prime}-x^{\prime}\right| \leqslant 1} \frac{\left|f\left(x^{\prime \prime}\right)-f\left(x^{\prime}\right)\right|}{\left|x^{\prime \prime}-x^{\prime}\right|^{\alpha_{2}}} \leqslant\|f\|_{0, \alpha_{2}} .
$$

On the other hand, if $\left|x^{\prime \prime}-x^{\prime}\right|>1$, we simply apply the (not fine) inequalities:

$$
\frac{\left|f\left(\mathrm{x}^{\prime \prime}\right)-f\left(\mathrm{x}^{\prime}\right)\right|}{\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right|^{\alpha_{1}}} \leqslant\left|f\left(\mathrm{x}^{\prime \prime}\right)-f\left(\mathrm{x}^{\prime}\right)\right| \leqslant 2\|f\|_{0,0} \leqslant 2\|f\|_{0, \alpha_{2}} .
$$

Consequently:

$$
\|f\|_{0, \alpha_{1}}=\|f\|_{0,0}+\sup _{\mathrm{x}^{\prime \prime} \neq \mathrm{x}^{\prime}} \frac{\left|f\left(\mathrm{x}^{\prime \prime}\right)-f\left(\mathrm{x}^{\prime}\right)\right|}{\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right|^{\alpha_{1}}} \leqslant 3\|f\|_{0, \alpha_{2}}
$$

with a factor 3 . For general $\kappa \geqslant 1$, the desired inequality follows:

$$
\begin{aligned}
\|f\|_{\kappa, \alpha_{1}}=\|f\|_{\kappa-1,0}+\sum_{|\delta|=\kappa}\left\|f_{x^{\delta}}\right\|_{0, \alpha_{1}} & \leqslant\|f\|_{\kappa-1,0}+3 \sum_{|\delta|=\kappa}\left\|f_{x^{\delta}}\right\|_{0, \alpha_{2}} \\
& \leqslant 3\|f\|_{\kappa, \alpha_{2}} .
\end{aligned}
$$

In the sequel, sometimes, we might abbreviate $\mathscr{C}^{\kappa, 0}$ by $\mathscr{C}^{\kappa}$, a standard notation. However, we shall never abbreviate $\mathscr{C}^{0, \alpha}$ by $\mathscr{C}^{\alpha}$, in order to avoid the unpleasant ambiguity $\mathscr{C}^{1,0} \equiv \mathscr{C}^{1} \equiv \mathscr{C}^{0,1}$. Without providing proofs, let us state some fundamental structural properties of Hölder spaces. Some of them are in [Kr1983].

- The inclusions $\mathscr{C}^{\lambda, \beta} \subset \mathscr{C}^{\kappa, \alpha}$ for $(\lambda, \beta)>(\kappa, \alpha)$ are all strict. For instance, on $\mathbb{R}$, the function $\chi_{\kappa, \alpha}=\chi_{\kappa, \alpha}(\mathrm{x})$ equal to zero for $\mathrm{x} \leqslant 0$ and, for $x \geqslant 0$ :

$$
\chi_{\kappa, \alpha}(x)= \begin{cases}x^{\kappa+\alpha}, & \text { if } 0<\alpha \leqslant 1, \\ x^{\kappa} / \log x, & \text { if } \alpha=0,\end{cases}
$$

is $\mathscr{C}^{\kappa, \alpha}$ in any neighborhood of the origin, not better.

- If $0<\alpha_{1}<\alpha$, any uniformly bounded set of functions in $\mathscr{C}^{\kappa, \alpha}$ contains a sequence of functions that converges in $\mathscr{C}^{\kappa, \alpha_{1}}$-norm to a function in $\mathscr{C}^{\kappa, \alpha_{1}}$. This is a Hölder-space version of the Arzelà-Ascoli lemma.
- For $0<\alpha \leqslant 1$, define the Hölder semi-norm (notice the wide hat):

$$
\|f\|_{\widehat{0, \alpha}}:=\sup _{x^{\prime \prime} \neq x^{\prime} \in \square_{\rho}^{n}} \frac{\left|f\left(\mathrm{x}^{\prime \prime}\right)-f\left(\mathrm{x}^{\prime}\right)\right|}{\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right|^{\alpha}} .
$$

The constants satisfy $\|c\|_{\widehat{0, \alpha}}=0$ and, of course, we have $\|f\|_{0, \alpha} \equiv\|f\|_{0,0}+$ $\|f\|_{\widehat{0, \alpha}}$. As a function of $\alpha$, the semi-norm is logarithmically convex:

$$
\|f\|_{0, t \alpha_{1}+(1-t) \alpha_{2}}=\left(\|f\|_{\widehat{0, \alpha_{1}}}\right)^{t} \cdot\left(\|f\|_{\widehat{0, \alpha_{2}}}\right)^{1-t}
$$

Here, $0<\alpha_{1}<\alpha_{2} \leqslant 1$ and $0 \leqslant t \leqslant 1$.

- Importantly, if $f$ is $\mathbb{K}^{m}$-valued, if $1 \leqslant l \leqslant m$, from the Taylor integral formula:

$$
\begin{equation*}
f_{l}\left(\mathrm{x}^{\prime \prime}\right)-f_{l}\left(\mathrm{x}^{\prime}\right)=\int_{0}^{1} \sum_{i=1}^{n} \frac{\partial f_{l}}{\partial \mathrm{x}_{i}}\left(\mathrm{x}^{\prime}+\mathrm{s}\left(\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right)\right)\left[\mathrm{x}_{i}^{\prime \prime}-\mathrm{x}_{i}^{\prime}\right] d \mathrm{~s}, \tag{1.2}
\end{equation*}
$$

follows the mean value inequality:

$$
\begin{align*}
\left|f\left(\mathrm{x}^{\prime \prime}\right)-f\left(\mathrm{x}^{\prime}\right)\right| & =\max _{1 \leqslant l \leqslant m}\left|f_{l}\left(\mathrm{x}^{\prime \prime}\right)-f_{l}\left(\mathrm{x}^{\prime}\right)\right|  \tag{1.3}\\
& \leqslant\|f\|_{\widehat{1,0}} \cdot\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right|
\end{align*}
$$

where $\mathrm{x}^{\prime \prime}, \mathrm{x}^{\prime} \in \square_{\rho}^{n}$ are arbitrary, and where

$$
\|f\|_{\widehat{1,0}}:=\max _{1 \leqslant l \leqslant m} \sum_{k=1}^{n} \sup _{|\times|<\rho}\left|f_{l, x_{k}}(\mathrm{x})\right| .
$$

This useful inequality also holds (by definition) if $f$ is merely Lipschitzian, with $\|f\|_{\widehat{1,0}}$ replaced by $\|f\|_{\widehat{0,1}}$.

- If a function $f$ is $\mathscr{C}^{\kappa, 0}$, then for every multiindex $\delta \in \mathbb{N}^{n}$ of length $|\delta| \leqslant \kappa$, the partial derivative $f_{x^{\delta}}$ is $\mathscr{C}^{\kappa-|\delta|, 0}$ and $\left\|f_{x^{\delta}}\right\|_{\kappa-|\delta|, 0} \leqslant\|f\|_{\kappa, 0}$.


## §2. CAUCHY Integral, SokhotskiĬ-Plemelj formulas and Hilbert transform

2.1. Boundary behaviour of the Cauchy integral. Let $\Omega$ be a domain in $\mathbb{C}$, let $z \in \Omega$ and let $\Gamma$ be a $\mathscr{C}^{1}$-smooth simple closed curve surrounding $z$ and oriented counterclockwise. Assume that its interior domain (to which $z$ belongs) is entirely contained in $\Omega$. In case $\Gamma$ is a circle, Cauchy ([Ca1831]) established in 1831 the celebrated representation formula:

$$
f(z)=\frac{1}{2 \pi i} \int_{\Gamma} \frac{f(\zeta) d \zeta}{\zeta-z}
$$

valid for all functions $f \in \mathscr{O}(\Omega)$ holomorphic in $\Omega$. Remarkably, $\Gamma$ may be modified and deformed without altering the value $f(z)$ of the integral.

The best proof of this formula is to derive it from the more general Cauchy-Green-Pompeiu formula, itself being an elementary consequence of the Green-Stokes formula, which is valid for functions $f$ of class only
$\mathscr{C}^{1}$ defined on the closure of a domain $\Omega \subset \mathbb{C}$ having $\mathscr{C}^{1}$-smooth oriented boundary $\partial \Omega$ ([Hö1973]):

$$
f(z)=\frac{1}{2 \pi i} \int_{\partial \Omega} \frac{f(\zeta) d \zeta}{\zeta-z}+\frac{1}{2 \pi i} \iint_{\Omega} \frac{\partial f / \partial \bar{\zeta}}{\zeta-z} d \zeta \wedge d \bar{\zeta}
$$

Indeed, for holomorphic $f$, one clearly sees that the "remainder" double integral disappears.

The holomorphicity of the kernel $\frac{1}{\zeta-z}$ enables then to build concisely the fundamental properties of holomorphic functions from Cauchy's formula: local convergence of Taylor series, residue theorem, Cauchy uniform convergence theorem, maximum principle, etc. ([Hö1973]). Studying the Cauchy integral for itself appeared therefore to be of interest and became a thoroughly investigated subject in the years 1910-1960, under the influence of Privalov.

If $z \in \Omega$ belongs to the exterior of $\Gamma$, i.e. to the unbounded component of $\mathbb{C} \backslash \Gamma$, by a fundamental theorem also due to Cauchy, the integral vanishes: $0=\frac{1}{2 \pi i} \int_{\Gamma} \frac{f(\zeta) d \zeta}{\zeta-z}$. Thus, fixing the countour $\Gamma$, as $z$ moves toward $\Gamma$, the Cauchy integral is constant, either equal to $f(z)$ or to 0 . What happens when $z$ hits the curve $\Gamma$ ?

Denote by $\zeta_{0}$ a point of $\Gamma$ and by $\Delta\left(\zeta_{0}, \varepsilon\right)$ the open disc of radius $\varepsilon>0$ centered at $\zeta_{0}$. If $\Gamma_{\varepsilon}$ denotes the complement $\Gamma \backslash \Delta\left(\zeta_{0}, \varepsilon\right)$, introducing an arc of small circle contained in $\partial \Delta\left(\zeta_{0}, \varepsilon\right)$ to join the two extreme points of $\Gamma_{\varepsilon}$, it may be verified that

$$
\begin{equation*}
\frac{1}{2} f\left(\zeta_{0}\right)=\lim _{\varepsilon \rightarrow 0} \frac{1}{2 \pi i} \int_{\Gamma_{\varepsilon}} \frac{f(\zeta) d \zeta}{\zeta-\zeta_{0}} \tag{2.2}
\end{equation*}
$$

Geometrically speaking, essentially one half of the circle $\partial \Delta\left(\zeta_{0}, \varepsilon\right)$ of radius $\varepsilon$ centered at $\zeta_{0}$ is contained in the domain $\Omega$. Consequently, the "correct value" of the Cauchy integral at a point $\zeta_{0}$ of the curve $\Gamma$ is equal to the arithmetic mean:

$$
\frac{1}{2}\left(\lim _{z \rightarrow \zeta_{0}, z \text { inside }}+\lim _{z \rightarrow \zeta_{0}, z \text { outside }}\right)=\frac{1}{2}\left(f\left(\zeta_{0}\right)+0\right) .
$$

Let us recall briefly why the excision of an $\varepsilon$-neighborhood of $\zeta_{0}$ in the domain of integration is necessary to provide this "correct" average value. Parametrizing $\Gamma$ by a real number, the problem of giving a sense to the singular integral $\int_{\Gamma} \frac{f(\zeta) d \zeta}{\zeta-\zeta_{0}}$ amounts to the following classical definition of the notion of principal value ([Mu1953, Ga1966, EK2000]).
2.3. Principal value integrals. Let $a, b \in \mathbb{R}$ with $a<b$ and let $f$ be a $\mathscr{C}^{1}$ smooth real-valued function defined on the open segment $(a, b)$. Pick $x \in \mathbb{R}$ with $a<\mathrm{x}<b$ and consider the integral $\int_{a}^{b} \frac{d y}{\mathrm{y}-\mathrm{x}}$ whose integrand is singular.

The two integrals avoiding the singularity from the left and from the right, namely:

$$
\begin{aligned}
& \int_{a}^{x-\varepsilon_{1}} \frac{d y}{y-x}=\log \left(\varepsilon_{1}\right)-\log (x-a) \\
& \int_{x+\varepsilon_{2}}^{b} \frac{d y}{y-x}=\log (b-x)-\log \left(\varepsilon_{2}\right)
\end{aligned}
$$

tend to $-\infty$, as $\varepsilon_{1} \rightarrow 0^{+}$, and to $+\infty$ as $\varepsilon_{2} \rightarrow 0^{+}$. Clearly, if $\varepsilon_{2}=\varepsilon_{1}$ (or more generally, if $\varepsilon_{1}$ and $\varepsilon_{2}$ both depend continuously on an auxiliary parameter $\varepsilon>0$ with $\left.1=\lim _{\varepsilon \rightarrow 0^{+}} \frac{\varepsilon_{2}(\varepsilon)}{\varepsilon_{1}(\varepsilon)}\right)$, the positive and the negative parts compensate, so that the principal value:

$$
\text { p.v. } \int_{a}^{b} \frac{d \mathrm{y}}{\mathrm{y}-\mathrm{x}}:=\lim _{\varepsilon \rightarrow 0^{+}}\left(\int_{a}^{\mathrm{x}-\varepsilon}+\int_{\mathrm{x}+\varepsilon}^{b}\right)=\log \frac{b-\mathrm{x}}{\mathrm{x}-a}
$$

exists. Briefly, there is a key cancellation of infinite parts, thanks to the fact that the singular kernel $\frac{1}{\mathrm{y}}$ is odd. This is why in (2.2) above, the integration was performed over the excised curve $\Gamma_{\varepsilon}$.

Generally, if $g:[a, b] \rightarrow \mathbb{R}$ is a real-valued function the principal value integral, defined by:

$$
\text { p.v. } \begin{aligned}
\int_{a}^{b} \frac{g(\mathrm{y}) d \mathrm{y}}{\mathrm{y}-\mathrm{x}} & :=\lim _{\varepsilon \rightarrow 0^{+}}\left(\int_{a}^{\mathrm{x}-\varepsilon}+\int_{\mathrm{x}+\varepsilon}^{b}\right) \\
& =\int_{a}^{b} \frac{g(\mathrm{y})-g(\mathrm{x})}{\mathrm{y}-\mathrm{x}} d \mathrm{y}+g(\mathrm{x}) \text { p.v. } \int_{a}^{b} \frac{d \mathrm{y}}{\mathrm{y}-\mathrm{x}} d \mathrm{y} \\
& =\int_{a}^{b} \frac{g(\mathrm{y})-g(\mathrm{x})}{\mathrm{y}-\mathrm{x}} d \mathrm{y}+g(\mathrm{x}) \log \frac{b-\mathrm{x}}{\mathrm{x}-a}
\end{aligned}
$$

exists whenever the quotient $\frac{g(\mathrm{y})-g(\mathrm{x})}{\mathrm{y}-\mathrm{x}}$ is integrable. This is the case for instance if $g$ is of class $\mathscr{C}^{1,0}$ or of class $\mathscr{C}^{0, \alpha}$, with $\alpha>0$, since $\int_{0}^{1} \mathrm{y}^{\alpha-1} d \mathrm{y}<$ $\infty$. More is true.

Theorem 2.4. ([Mu1953, Ve1962, Dy1991, SME1988, EK2000], [*]) Let $g:[a, b] \rightarrow \mathbb{R}$ be $\mathscr{C}^{\kappa, \alpha}$-smooth, with $\kappa \geqslant 0$ and $0<\alpha<1$. Then for every $x \in(a, b)$, the principal value integral

$$
G(\mathrm{x}):=\text { p.v. } \int_{a}^{b} \frac{g(\mathrm{y}) d \mathrm{y}}{\mathrm{y}-\mathrm{x}}
$$

exists. In every closed segment $\left[a^{\prime}, b^{\prime}\right]$ contained in $(a, b)$, the function $G(x)$ becomes $\mathscr{C}^{\kappa, \alpha}$-smooth and enjoys the norm inequality $\|G\|_{\mathscr{C}^{\kappa, \alpha}\left[a^{\prime}, b^{\prime}\right]} \leqslant$ $\frac{C}{\alpha(1-\alpha)}\|g\|_{\mathscr{C}_{\kappa, \alpha}[a, b]}$, for some constant $C=C\left(\kappa, a, b, a^{\prime}, b^{\prime}\right)$. If $g$ together with its derivatives up to order $\kappa$ vanish at the two extreme points $a$ and $b$,
the function $G(\mathrm{x})$ is $\mathscr{C}^{\kappa, \alpha}$-smooth over $[a, b]$ and enjoys the norm inequality $\|G\|_{\mathscr{C}^{\kappa}, \alpha[a, b]} \leqslant \frac{C}{\alpha(1-\alpha)}\|g\|_{\mathscr{C}^{\kappa}, \alpha[a, b]}$, for some constant $C=C(\kappa, a, b)$.

Notice the presence of the (nonremovable) factor $\frac{1}{\alpha(1-\alpha)}$.
2.5. General Cauchy integral. Beginning with works of Sokhotskiĭ [So1873], of Harnack [Ha1885] and of Morera [Mo1889], the Cauchy integral transform:

$$
F(z):=\frac{1}{2 \pi i} \int_{\Gamma} \frac{f(\zeta) d \zeta}{\zeta-z}
$$

has been studied for itself, in the more general case where $\Gamma$ is an arbitrary closed or non-closed curve in $\mathbb{C}$ and $f$ is an arbitrary smooth complexvalued function defined on $\Gamma$, not necessarily holomorphic in a neighborhood of $\Gamma$ (precise rigorous assumptions will follow; historical account may be found in [Ga1966]). In Sokhotskiǐ's and in Harnack's works, the study of the boundary behaviour of the Cauchy integral was motivated by physical problems; its boundary properties find applications to mechanics, to hydrodynamics and to elasticity theory. Let us restitute briefly the connection to the notion of logarithmic potential ([Mu1953]).

Assuming $\Gamma$ and $f: \Gamma \rightarrow \mathbb{R}$ to be real-valued and of class at least $\mathscr{C}^{1,0}$, parametrize $\Gamma$ by arc-length $\zeta=\zeta(s)$, denote $\mathbf{r}(s):=\zeta(s)-z$ the radial vector from $z$ to $\zeta(s)$, denote $r=r(s)=|\mathbf{r}(s)|$ its euclidean norm, denote $\mathbf{t}(s):=\frac{d \mathbf{r}}{d s}$ the unit tangent vector field to $\Gamma$ and denote $\mathbf{n}(s):=\frac{d \mathbf{r}}{d s} /\left|\frac{d \mathbf{r}}{d s}\right|$ the unit normal vector field to $\Gamma$. Puting $z=x+i y$ and decomposing the Cauchy transform $F(z)=U(x, y)+i V(x, y)$ in real and imaginary parts, the two functions $U$ and $V$ are harmonic in $\mathbb{C} \backslash \Gamma$, since $F$ is clearly holomorphic there. After elementary computations, one shows that $U$ may be expressed under the form:

$$
U(x, y)=\frac{1}{2 \pi} \int_{\Gamma} f \frac{\cos (\mathbf{r}, \mathbf{n})}{r} d s
$$

which, physically, represents the potential of a double layer with momentdensity $\frac{f}{2 \pi}$. Also, $V$ may be expressed under the form:

$$
V(x, y)=\frac{1}{2 \pi} \int_{\Gamma} \frac{d f}{d s} \log r d s
$$

which, in the case where $\Gamma$ consists of a finite number of closed Jordan curves, represents the potential of a single layer with moment-density $-\frac{1}{2 \pi} \frac{d f}{d s}$.
2.6. The Sokhotskiï-Plemelj formulas. Coming back to the mathematical study of the Cauchy integral, we shall assume that the curve $\Gamma$ over which the integration is performed is a connected curve of finite length parametrized by arc length

$$
[a, b] \ni s \longmapsto \zeta(s) \in \Gamma,
$$

where $a<b$, where $\zeta(s)$ is of class $\mathscr{C}^{\kappa+1, \alpha}$ over the closed segment $[a, b]$, and where $\kappa \geqslant 0,0<\alpha<1$. Topologically, we shall assume that $\Gamma=$ $\zeta[a, b]$ is either:

- a Closed Jordan arc, namely $\zeta:[a, b] \rightarrow \mathbb{C}$ is an embedding;
- or a Jordan contour, namely $\zeta:(a, b) \rightarrow \mathbb{C}$ is an embedding, $\zeta(a)=$ $\zeta(b), \zeta$ extends as a $\mathscr{C}^{\kappa+1, \alpha}$-smooth map on the quotient $[a, b] /(a \sim$ $b)$ and $\Gamma=\zeta[a, b]$ is diffeomorphic to a circle.

Various more general assumptions can be made: $\Gamma$ consists of a finite number of connected pieces, $\Gamma$ is piecewise smooth (corners appear), $\Gamma$ possesses certain cusps, $\Gamma$ is only Lipschitz, the length of $\Gamma$ is not finite, $f$ is $L^{\mathrm{p}}$ integrable, $f$ is $L_{\alpha}^{\mathrm{p}}$, i.e. $f \in L^{\mathrm{p}}(\Gamma)$ and $\int_{\Gamma}|f(s+h)-f(s)|^{\mathrm{p}} \leqslant$ Cte $|h|^{\alpha}, f$ belongs to certain Sobolev spaces, $f(\zeta) d \zeta$ is replaced by a measure $d \mu(\zeta)$, etc., but we shall not review the theory (see [Mu1953, Ve1962, Ga1966] and especially [Dy1991]).

The natural orientation of the segment $[a, b]$ induced by the order relation on $\mathbb{R}$ enables to orient the two semi-local sides of $\Gamma$ in $\mathbb{C}$ : the region on the left to $\Gamma$ will be called the positive side ("+"), while the region to the right will be called negative ("-"). In the case where $\Gamma$ is Jordan contour, we assume that $\Gamma$ is oriented counterclockwise, so that the positive region coincides with the bounded component of $\mathbb{C} \backslash \Gamma$.

Theorem 2.7. ([Mu1953, Ve1962, Ga1966, Dy1991, SME1988, EK2000], [*]) Let $\Gamma$ be a $\mathscr{C}^{\kappa+1, \alpha}$-smooth closed Jordan arc or Jordan contour in $\mathbb{C}$ and let $f: \Gamma \rightarrow \mathbb{C}$ be a $\mathscr{C}^{\kappa, \alpha}$-smooth complex-valued function.
(a) If $\Gamma^{\prime}$ is any closed portion of $\Gamma$ having no ends in common with those of $\Gamma$, then for every $\zeta_{1} \in \Gamma^{\prime}$, the Cauchy transform $F(z):=$ $\frac{1}{2 \pi i} \int_{\Gamma} \frac{f(\zeta) d \zeta}{\zeta-z}$ possesses (a priori distinct) limits $F^{+}\left(\zeta_{1}\right)$ and $F^{-}\left(\zeta_{1}\right)$, when $z$ tends to $\zeta_{1}$ from the positive or from the negative side.
(b) These two limits $F^{+}$and $F^{-}$are of class $\mathscr{C}^{\kappa, \alpha}$ on $\Gamma^{\prime}$ with a norm estimate $\left\|F^{ \pm}\right\|_{\mathscr{G} \kappa, \alpha\left(\Gamma^{\prime}\right)} \leqslant \frac{C\left(\kappa, \Gamma^{\prime}, \Gamma\right)}{\alpha(1-\alpha)}\|f\|_{\mathscr{\mathscr { K } , \alpha ( \Gamma )}}$, for some positive constant where $C\left(\kappa, \Gamma^{\prime}, \Gamma\right)$.
(c) Furthermore, if $\omega_{+}^{\prime}$ and $\omega_{-}^{\prime}$ denote an upper and a lower open onesided neighborhood $\Gamma^{\prime}$ in $\mathbb{C}$, the two functions $F^{ \pm}: \omega_{ \pm}^{\prime} \rightarrow \mathbb{C}$ defined by:

$$
\begin{cases}F^{ \pm}(z):=F(z) & \text { if } z \in \omega_{ \pm}^{\prime} \\ F^{ \pm}(z):=F^{ \pm}\left(\zeta_{1}\right) & \text { if } z=\zeta_{1} \in \Gamma^{\prime}\end{cases}
$$

are of class $\mathscr{C}^{\kappa, \alpha}$ in $\omega_{ \pm}^{\prime} \cup \Gamma^{\prime}$, with a similar norm estimate $\left\|F^{ \pm}\right\|_{\left.\mathscr{\mathscr { C } ^ { \kappa , \alpha } ( \omega _ { \pm } ^ { \prime }} \cup \Gamma^{\prime}\right)} \leqslant \frac{C_{1}\left(\kappa \kappa \Gamma^{\prime}, \overline{,}\right)}{\alpha(1-\alpha)}\|f\|_{\mathscr{C}^{\kappa, \alpha}(\Gamma)}$.
(d) Finally, at every point $\zeta_{0}$ of the curve $\Gamma$ not coinciding with its ends, $F^{+}$and $F^{-}$satisfy the two Sokhotskiĭ-Plemelj formulas:

$$
\left\{\begin{aligned}
F^{+}\left(\zeta_{0}\right)-F^{-}\left(\zeta_{0}\right) & =f\left(\zeta_{0}\right) \\
\frac{1}{2}\left[F^{+}\left(\zeta_{0}\right)+F^{-}\left(\zeta_{0}\right)\right] & =\text { p.v. } \frac{1}{2 \pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta-\zeta_{0}} d \zeta
\end{aligned}\right.
$$



Sometimes, $F$ is called sectionnally holomorphic, as it is discontinuous across $\Gamma$. Its jump across $\Gamma$ is provided by the first formula above, while the arithmetic mean $\frac{F^{+}+F^{-}}{2}$ is given by the value of the Cauchy (singular) integral at $\zeta_{0} \in \Gamma$. Morera's classical theorem ([Mo1889]) states that if $F^{+}$ and $F^{-}$match up on the interior of $\Gamma$, then the Cauchy integral is holomorphic in $\mathbb{C}$ minus the endpoints of $\Gamma$. As is known ([Sh1990]), this theorem is also true for an arbitrary holomorphic function $F \in \mathscr{O}(\mathbb{C} \backslash \Gamma)$ which is not necessarily defined by a Cauchy integral.
2.8. Less regular boundaries. The boundary behaviour of the Cauchy transform at the two extreme points $\gamma(a)$ and $\gamma(b)$ of a Jordan arc is studied in [Mu1953]. We refer to [Dy1991] for a survey presentation of the finest condition on $\Gamma$ (namely, it to be a Carleson curve) which insures that the Cauchy integral exists and that the Sokhotskiir-Plemelj formulas hold true, almost everywhere. Let us just mention what happens with the Cauchy inte$\operatorname{gral} F(z)$ in the limit case $\alpha=0$.

If $\Gamma$ is (only) $\mathscr{C}^{1,0}$, if $f$ is (only) $\mathscr{C}^{0,0}$, then for $\zeta_{1}$ in the interior of $\Gamma$, the limit $F^{-}\left(\zeta_{1}\right)$ exists if and only if the limit $F^{+}\left(\zeta_{1}\right)$ exists ([Mu1953]). However, generically, none limit exists.

A more useful statement, valid in the case $\alpha=0$, is as follows. Assume $\Gamma$ to be $\mathscr{C}^{\kappa+1,0}$ with $\kappa \geqslant 0$ and let $\Gamma^{\prime}$ be a closed portion of the interior of $\Gamma$. Parametrize $\Gamma^{\prime}$ by a $\mathscr{C}^{\kappa+1,0}$ map $\zeta^{\prime}:\left[a^{\prime}, b^{\prime}\right] \rightarrow \Gamma^{\prime}$. Extend $\zeta^{\prime}=\zeta^{\prime}(s)$ as a a $\mathscr{C}^{\kappa+1,0}$ embedding $\zeta^{\prime}(s, \varepsilon)$ defined on $\left[a^{\prime}, b^{\prime}\right] \times\left(-\varepsilon_{0}, \varepsilon_{0}\right)$, where $\varepsilon_{0}>0$, with $\zeta^{\prime}(s, 0) \equiv \zeta^{\prime}(s)$ and with $\zeta^{\prime}(s, \varepsilon)$ in the positive side of $\Gamma^{\prime}$ for $\varepsilon>0$. The family of curves $\Gamma_{\varepsilon}^{\prime}:=\zeta^{\prime}\left(\left[a^{\prime}, b^{\prime}\right] \times\{\varepsilon\}\right)$ foliates a strip thickening of $\Gamma^{\prime}$.

Theorem 2.9. ([Mu1953]) For every choice of a $\mathscr{C}^{\kappa+1,0}$ extension $\zeta^{\prime}(s, \varepsilon)$, and every $f \in \mathscr{C}^{\kappa, 0}(\Gamma, \mathbb{C})$, the difference from either side of the Cauchy transform $\left.F\right|_{\Gamma_{\varepsilon}^{\prime}}-\left.F\right|_{\Gamma_{-\varepsilon^{\prime}}^{\prime}}$ tends to $\left.f\right|_{\Gamma^{\prime}}$ in $\mathscr{C}^{\kappa, 0}$ norm as $\varepsilon \rightarrow 0$ :

$$
\lim _{\varepsilon \rightarrow 0} \sup _{s \in\left[a^{\prime}, b^{\prime}\right]}\left\|F\left(\zeta^{\prime}(s, \varepsilon)\right)-F\left(\zeta^{\prime}(s,-\varepsilon)\right)-f\left(\zeta^{\prime}(s)\right)\right\|_{\kappa, 0}=0
$$

To conclude, we state a criterion, due to Hardy-Littlewood, which insures $\mathscr{C}^{\kappa, \alpha}$-smoothness of holomorphic functions up to the boundary.

Theorem 2.10. ([Mu1953, Ga1966], [*]) Let $\Gamma$ be a $\mathscr{C}^{\kappa+1, \alpha}$-smooth Jordan contour, divinding the complex plane in two components $\Omega^{+}$(bounded) and $\Omega^{-}$(unbounded). If $f \in \mathscr{O}\left(\Omega^{ \pm}\right)$satisfies the estimate $\left|\partial_{z}^{\kappa} f(z)\right| \leqslant$ $C(1-|z|)^{1-\alpha}$, for some $\kappa \in \mathbb{N}$, some $\alpha$ with $0<\alpha<1$, and some positive constant $C>0$, then $f$ is of class $\mathscr{C}^{\kappa, \alpha}$ in the closure $\overline{\Omega^{ \pm}}=\Omega^{ \pm} \cup \Gamma$.
2.11. Functions and maps defined on the unit circle. In the sequel, $\Omega$ will be the unit disc $\Delta:=\{\zeta \in \mathbb{C}:|\zeta|<1\}$ having as boundary the unit circle $\partial \Delta:=\{\zeta \in \mathbb{C}:|\zeta|=1\}$. Consider a function $f: \partial \Delta \rightarrow \mathbb{K}$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Parametrizing $\partial \Delta$ by $\zeta=e^{i \theta}$ with $\theta \in \mathbb{R}$, such an $f$ will be considered as the function

$$
\mathbb{R} \ni \theta \longmapsto f\left(e^{i \theta}\right) \in \mathbb{K}
$$

For $j \in \mathbb{N}$, we shall write $f_{\theta^{j}}:=\frac{d^{j} f}{d \theta^{j}}$.
Let $\alpha$ satisfy $0<\alpha \leqslant 1$ and assume that $f \in \mathscr{C}^{0, \alpha}$. We define its $\mathscr{C}^{0, \alpha}$ semi-norm (notice the wide hat) precisely by:

$$
\|f\|_{\widehat{0, \alpha}}:=\sup _{\theta^{\prime \prime} \neq \theta^{\prime}} \frac{\left|f\left(e^{i \theta^{\prime \prime}}\right)-f\left(e^{i \theta^{\prime}}\right)\right|}{\left|\theta^{\prime \prime}-\theta^{\prime}\right|^{\alpha}}
$$

Thanks to $2 \pi$-periodicity, $\sup _{\theta^{\prime \prime} \neq \theta^{\prime}}$ may be replaced by $\sup _{0<\left|\theta^{\prime \prime}-\theta^{\prime}\right| \leqslant \pi}$. According to the definition of $\S 1.1$, the function $f$ is $\mathscr{C}^{\kappa, \alpha}$ if the quantity

$$
\|f\|_{\kappa, \alpha}:=\sum_{0 \leqslant j \leqslant \kappa}\left\|f_{\theta^{j}}\right\|_{0,0}+\left\|f_{\theta^{\kappa}}\right\|_{\widehat{0, \alpha}}<\infty
$$

is finite. Besides Hölder spaces, we shall also consider the Lebesgue spaces $L^{\mathrm{p}}(\partial \Delta)$, with $\mathrm{p} \in \mathbb{R}$ satisfying $1 \leqslant \mathrm{p} \leqslant \infty$. As $\partial \Delta$ is compact, the Hölder inequality entails the (strict) inclusions $L^{\infty}(\partial \Delta) \subset L^{\mathrm{p}^{\prime}}(\partial \Delta) \subset L^{\mathrm{p}}(\partial \Delta) \subset$ $L^{1}(\partial \Delta)$, for $1<\mathrm{p}<\mathrm{p}^{\prime}<\infty$.
2.12. Fourier series of Hölder continuous functions. If $f$ is at least of class $L^{1}$ on $\partial \Delta$, let

$$
\widehat{f_{k}}:=\frac{1}{2 \pi i} \int_{\partial \Delta} \zeta^{-k} f(\zeta) \frac{d \zeta}{\zeta}
$$

denote the $k$-th Fourier coefficient of $f$, where $k \in \mathbb{Z}$. Given $n \in \mathbb{N}$, consider the $n$-th partial sum of the Fourier series of $f$ :

$$
\mathrm{F}_{n} f\left(e^{i \theta}\right):=\sum_{-n \leqslant k \leqslant n} \widehat{f}_{k} e^{i k \theta} .
$$

We remind that Dini's (elementary) criterion:

$$
\int_{0}^{\pi} \frac{\left|f\left(e^{i(\theta+t)}\right)+f\left(e^{i(\theta-t)}\right)-2 f\left(e^{i \theta}\right)\right|}{t} d t<\infty
$$

insures the pointwise convergence $\lim _{n \rightarrow \infty} \mathrm{~F}_{n} f\left(e^{i \theta}\right)=f\left(e^{i \theta}\right)$. If $f$ is $\mathscr{C}^{0, \alpha}$ on $\partial \Delta$, with $0<\alpha \leqslant 1$, the above integral obviously converges at every $e^{i \theta} \in \partial \Delta$, so that we may identify $f$ with its (complete) Fourier series:

$$
f\left(e^{i \theta}\right)=\mathrm{F} f\left(e^{i \theta}\right):=\sum_{k \in \mathbb{Z}} \widehat{f_{k}} e^{i k \theta} .
$$

In fact ([Zy1959]), if $f \in \mathscr{C}^{\kappa, \alpha}$ with $\kappa \in \mathbb{N}$ and $0 \leqslant \alpha \leqslant 1$, then $\left|\widehat{f}_{k}\right| \leqslant \frac{\pi^{1+\alpha}}{|k|^{\kappa+\alpha}}\|f\|_{\widehat{\kappa, \alpha}}$ for all $k \in \mathbb{Z} \backslash\{0\}$. Also, if $f \in \mathscr{C}^{0, \alpha}$, then $\sum_{k \in \mathbb{Z}}\left|\widehat{f}_{k}\right|^{c}$ converges for $c>\frac{2}{2 \alpha+1}$. In 1913, Bersteĭn proved absolute convergence of $\sum_{k \in \mathbb{Z}}\left|\widehat{f}_{k}\right|$ for $\alpha>1 / 2$.
2.13. Three Cauchy transforms in the unit disc. In the case $\Omega=\Delta$, our goal is to formulate Theorem 2.7 with more precision about the constant $C(\kappa, \partial \Omega)$. For $\eta \in \partial \Delta$ in the unit circle and $f \in \mathscr{C}^{\kappa, \alpha}(\partial \Delta, \mathbb{C})$ with $\kappa \geqslant 0$, $0<\alpha<1$, as in §2.6, we define:

$$
\begin{aligned}
\mathrm{C}^{+} f(\eta) & :=\lim _{r \rightarrow 1^{-}} \frac{1}{2 \pi i} \int_{\partial \Delta} \frac{f(\zeta)}{\zeta-r \eta} d \zeta, \\
\mathrm{C}^{0} f(\eta) & :=\text { p.v. } \frac{1}{2 \pi i} \int_{\partial \Delta} \frac{f(\zeta)}{\zeta-\eta} d \zeta, \\
\mathrm{C}^{-} f(\eta) & :=\lim _{r \rightarrow 1^{+}} \frac{1}{2 \pi i} \int_{\partial \Delta} \frac{f(\zeta)}{\zeta-r \eta} d \zeta .
\end{aligned}
$$

The Sokhotskiǐ-Plemelj formulas hold: $f(\eta)=\mathrm{C}^{+} f(\eta)-\mathrm{C}^{-} f(\eta)$ and $\mathrm{C}^{0} f(\eta)=\frac{1}{2}\left[\mathrm{C}^{+} f(\eta)+\mathrm{C}^{-} f(\eta)\right]$. A theorem due to Aleksandrov ${ }^{6}$ enables to obtain a precise estimate of the $\mathscr{C}^{\kappa, \alpha}$ norms of these Cauchy operators. To describe it, define:

$$
\mathscr{M}_{0}^{\alpha}:=\left\{f \in \mathscr{C}^{0, \alpha}(\partial \Delta, \mathbb{C}): \widehat{f_{0}}=0\right\}
$$

Then $\|\cdot\|_{\widehat{0, \alpha}}$ is a norm on $\mathscr{M}_{0}^{\alpha}$, since only the constants $c$ satisfy $\|c\|_{\widehat{0, \alpha}}=0$.
For $p, q \in \mathbb{R}$ with $0<p, q<1$, recall the definition $B(p, q):=\int_{0}^{1} x^{p-1}(1-$ $x)^{q-1} d x$ of the Euler beta function.

[^5]Theorem 2.14. ([Al1975]) The operator $\mathrm{C}^{0} f(\eta):=\mathrm{p} . \mathrm{v} \cdot \frac{1}{2 \pi i} \int_{\partial \Delta} \frac{f(\zeta)}{\zeta-\eta} d \zeta$ is a bounded linear endomorphism of $\mathscr{M}_{0}^{\alpha}$ having norm:

$$
\left\|C^{0}\right\|_{\widehat{0, \alpha}}=\frac{1}{2 \pi} B\left(\frac{\alpha}{2}, \frac{1-\alpha}{2}\right) .
$$

One may easily verify the two equivalences $B\left(\frac{\alpha}{2}, \frac{1-\alpha}{2}\right) \sim \frac{2}{\alpha}$ as $\alpha \rightarrow 0$ and $B\left(\frac{\alpha}{2}, \frac{1-\alpha}{2}\right) \sim \frac{2}{1-\alpha}$ as $\alpha \rightarrow 1$ as well as the two inequalities:

$$
\frac{1}{\alpha(1-\alpha)} \leqslant B\left(\frac{\alpha}{2}, \frac{1-\alpha}{2}\right) \leqslant \frac{4}{\alpha(1-\alpha)} .
$$

Thus, the nonremovable factor $\frac{1}{\alpha(1-\alpha)}$ shows what is the precise rate of explosion of the norm $\left\|C^{0}\right\|_{\widehat{0, \alpha}}$ as $\alpha \rightarrow 0$ or as $\alpha \rightarrow 1$.

Further, if $f \in \mathscr{C}^{0, \alpha}$ does not necessarily belong to $\mathscr{M}_{0}^{\alpha}$, it is elementary to check that $\left\|\mathrm{C}^{0} f\right\|_{0,0} \leqslant \frac{C}{\alpha}\|f\|_{0, \alpha}$, for some absolute constant $C>0$. It follows that the (complete) operator norm $\left\|\mathrm{C}^{0}\right\|_{0, \alpha}$ behaves like $\frac{C}{\alpha(1-\alpha)}$.

In conclusion, thanks to the Sokhotskiǐ-Plemelj formulas $\mathrm{C}^{+} f=\frac{1}{2}\left(\mathrm{C}^{0} f+\right.$ $f)$ and $\mathrm{C}^{-} f=\frac{1}{2}\left(\mathrm{C}^{0} f-f\right)$, we deduce that there exists an absolute constant $C_{1}>1$ such that:

$$
\frac{1 / C_{1}}{\alpha(1-\alpha)} \leqslant\left\|\mathrm{C}^{\mathrm{b}}\right\|_{0, \alpha} \leqslant \frac{C_{1}}{\alpha(1-\alpha)},
$$

where $0<\alpha<1$ and where $\mathrm{b}=-, 0,+$.
Next, what happens with $f \in \mathscr{C}^{\kappa, \alpha}$, for $\kappa \in \mathbb{N}$ arbitrary ? For $\mathbf{b}=-, 0,+$, the $\mathrm{C}^{\mathrm{b}}$ are bounded linear endomorphisms of $\mathscr{C}^{\kappa, \alpha}$ and similarly:

Theorem 2.15. There exists an absolute constant $C_{1}>1$ such that if $\kappa \in \mathbb{N}$ and $0<\alpha<1$, for $\mathrm{b}=-, 0,+$ :

$$
\frac{1 / C_{1}}{\alpha(1-\alpha)} \leqslant\left\|\mathrm{C}^{\mathrm{b}}\right\|_{\kappa, \alpha} \leqslant \frac{C_{1}}{\alpha(1-\alpha)} .
$$

In other words, the constant $C_{1}$ is independent of $\kappa$. To deduce this theorem from the estimates with $\kappa=0$ (with different absolute constant $C_{1}$ ) we proceed as follows, without exposing all the rigorous details.

Inserting the Fourier series $\mathbf{F}\left(f, e^{i \theta}\right)$ in the integrals defining $C^{-}, C^{0}, C^{+}$ and integrating termwise (an operation which may be justified), we get:

$$
\left\{\begin{aligned}
\mathrm{C}^{-} f\left(e^{i \theta}\right) & =-\sum_{k<0} \widehat{f}_{k} e^{i k \theta} \\
\mathrm{C}^{0} f\left(e^{i \theta}\right) & =-\frac{1}{2} \sum_{k<0} \widehat{f_{k}} e^{i k \theta}+\frac{1}{2} \widehat{f_{0}}+\frac{1}{2} \sum_{k>0} \widehat{f}_{k} e^{i k \theta} \\
\mathrm{C}^{+} f\left(e^{i \theta}\right) & =\widehat{f_{0}}+\sum_{k>0} \widehat{f}_{k} e^{i k \theta}
\end{aligned}\right.
$$

If $\kappa \geqslant 1$, by differentiating termwise with respect to $\theta$ these three Fourier representations of the $\mathrm{C}^{\mathrm{b}}$, we see that these operators commute with differentiation.

Lemma 2.16. For every $j \in \mathbb{N}$ with $0 \leqslant j \leqslant \kappa$ and for $\mathrm{b}=-, 0,+$, we have:

$$
\mathrm{C}^{\mathrm{b}}\left(f_{\theta^{j}}\right)=\left(\mathrm{C}^{\mathrm{b}} f\right)_{\theta^{j}}
$$

Dealing directly with the principal value definition of $\mathrm{C}^{0} f$, another proof of this lemma for $\mathrm{C}^{0}$ would consist in integrating by parts, deducing afterwards that $\mathrm{C}^{-}$and $\mathrm{C}^{+}$enjoy the same property, thanks to the SokhotskiìPlemelj formulas.

To establish Theorem 2.15, we introduce another auxiliary $\mathscr{C}^{\kappa, \alpha}$ norm:

$$
\|f\|_{\kappa, \alpha}^{\sim}:=\sum_{0 \leqslant j \leqslant \kappa}\left\|f_{\theta^{j}}\right\|_{0, \alpha}=\|f\|_{\kappa, \alpha}+\sum_{0 \leqslant j \leqslant \kappa-1}\left\|f_{\theta^{j}}\right\|_{\widehat{0_{2}}},
$$

which is equivalent to $\|\cdot\|_{\kappa, \alpha}$, thanks to the elementary inequalities ([ $\left.*\right]$ ):

$$
\|f\|_{\kappa, \alpha} \leqslant\|f\|_{\kappa, \alpha}^{\sim} \leqslant(1+\pi)\|f\|_{\kappa, \alpha} .
$$

Notice that $\|\cdot\|_{0, \alpha}^{\sim}=\|\cdot\|_{0, \alpha}$. The next lemma applies to $\mathrm{L}=\mathrm{C}^{-}, \mathrm{C}^{0}, \mathrm{C}^{+}$and to $\mathrm{L}=\mathrm{T}$, the Hilbert conjugation operator defined below.

Lemma 2.17. ([*]) Let L be a bounded linear endomorphism of all the spaces $\mathscr{C}^{\kappa, \alpha}(\partial \Delta, \mathbb{C})$ with $\kappa \in \mathbb{N}, 0<\alpha<1$, which commutes with differentiations, namely $\mathrm{L}\left(f_{\theta^{j}}\right)=(\mathrm{L} f)_{\theta^{j}}$, for $j \in \mathbb{N}$. Assume that there exist a contant $C_{1}(\alpha)>1$ depending on $\alpha$ such that $C_{1}(\alpha)^{-1} \leqslant\|\mathrm{~L}\|_{0, \alpha} \leqslant C_{1}(\alpha)$. Then for every $\kappa \in \mathbb{N}$ :

Proof. Indeed, if $f \in \mathscr{C}^{\kappa, \alpha}$, we develope a chain of (in)equalities:

$$
\begin{aligned}
\|\mathrm{L} f\|_{\kappa, \alpha} & \leqslant\|\mathrm{L} f\|_{\kappa, \alpha}^{\sim}=\sum_{0 \leqslant j \leqslant \kappa}\left\|(\mathrm{~L} f)_{\theta^{j}}\right\|_{0, \alpha}=\sum_{0 \leqslant j \leqslant \kappa}\left\|\mathrm{~L}\left(f_{\theta^{j}}\right)\right\|_{0, \alpha} \\
& \leqslant C_{1}(\alpha) \sum_{0 \leqslant j \leqslant \kappa}\left\|f_{\theta^{j}}\right\|_{0, \alpha}=C_{1}(\alpha)\|f\|_{\kappa, \alpha}^{\sim} \\
& \leqslant(1+\pi) C_{1}(\alpha)\|f\|_{\kappa, \alpha} .
\end{aligned}
$$

This yields the two majorations. Minorations are obtained similarly.
To conclude this paragraph, we state a Tœplitz type theorem about $\mathrm{C}^{+}$, which will be crucial in solving Bishop's equation with optimal loss of smoothness, as we will see in Section 3. A similar one holds about $\mathrm{C}^{-}$, assuming $\phi \in H^{\infty}(\overline{\mathbb{C}} \backslash \bar{\Delta})$ instead, where $\overline{\mathbb{C}}$ is the Riemann sphere.

Theorem 2.18. ([Tu1994b], [*]) There exists an absolute constant $C_{1}>1$ such that for all $f \in \mathscr{C}^{\kappa, \alpha}, \kappa \in \mathbb{N}, 0<\alpha<1$, and all $\phi \in \mathrm{H}^{\infty}(\Delta):=$ $\mathscr{O}(\Delta) \cap L^{\infty}(\Delta):$

$$
\left\|\mathrm{C}^{+}(f \bar{\phi})\right\|_{\kappa, \alpha} \leqslant \frac{C_{1}}{\alpha(1-\alpha)}\|f\|_{\kappa, \alpha}\|\phi\|_{L^{\infty}} .
$$

Closely related to the Cauchy transform are the Schwarz and the Hilbert transforms.
2.19. Schwarz transform on the unit disc. Let $u \in L^{1}(\partial \Delta, \mathbb{R})$ be realvalued. The Schwarz transform of $u$ is the function of $z \in \Delta$ defined by:

$$
\mathrm{S} u(z):=\frac{1}{2 \pi i} \int_{\partial \Delta} u(\zeta)\left(\frac{\zeta+z}{\zeta-z}\right) \frac{d \zeta}{\zeta} .
$$

Thanks to the holomorphicity of the kernel, $\mathrm{S} u(z)$ is a holomorphic function of $z \in \Delta$. Decomposing it in real and imaginary parts:

$$
\mathrm{S} u(z)=\mathrm{P} u(z, \bar{z})+i \mathrm{~T} u(z, \bar{z}),
$$

we get the Poisson transform of $u$ :

$$
\mathrm{P} u(z, \bar{z}):=\frac{1}{2 \pi i} \int_{\partial \Delta} u(\zeta) \operatorname{Re}\left(\frac{\zeta+z}{\zeta-z}\right) \frac{d \zeta}{\zeta},
$$

together with the Hilbert transform of $u$ :

$$
\mathrm{T} u(z, \bar{z}):=\frac{1}{2 \pi i} \int_{\partial \Delta} u(\zeta) \operatorname{Im}\left(\frac{\zeta+z}{\zeta-z}\right) \frac{d \zeta}{\zeta} .
$$

Thanks to the harmonicity of the two kernels, $\mathrm{P} u$ and $\mathrm{T} u$ are harmonic in $\Delta$. The power series of $\mathrm{C} u$, of $\mathrm{P} u$ and of $\mathrm{T} u$ are given by:

$$
\left\{\begin{aligned}
\mathrm{S} u(z) & =\widehat{u}_{0}+2 \sum_{k>0} \widehat{u}_{k} z^{k}, \\
\mathrm{P} u(z, \bar{z}) & =\sum_{k<0} \widehat{u}_{k} \bar{z}^{k}+\widehat{u}_{0}+\sum_{k>0} \widehat{u}_{k} z^{k}, \\
\mathrm{~T} u(z, \bar{z}) & =\frac{1}{i}\left(-\sum_{k<0} \widehat{u}_{k} \bar{z}^{k}+\sum_{k>0} \widehat{u}_{k} z^{k}\right),
\end{aligned}\right.
$$

where $\widehat{u}_{k}$ is the $k$-th Fourier coefficient of $u$. These three series converge normally on compact subsets of $\Delta$.
2.20. Poisson transform on the unit disc. Let us first summarize the basic properties of the Poisson transform ([Ka1968, DR2002]). Setting $z=r e^{i \theta}$ with $0 \leqslant r<1$ and $\zeta=e^{i t}$, computing $\operatorname{Re}\left(\frac{\zeta+z}{\zeta-z}\right)$ and switching the convolution integral, we obtain:

$$
\mathrm{P} u\left(r e^{i \theta}\right)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} P_{r}(t) u\left(e^{i(\theta-t)}\right) d t=P_{r} * u\left(e^{i \theta}\right),
$$

where

$$
P_{r}(t):=\frac{1-r^{2}}{1-2 r \cos t+r^{2}}
$$

is the Poisson summability kernel. It has three nice properties:

- $P_{r}>0$ on $\partial \Delta$ for $0 \leqslant r<1$,
- $\frac{1}{2 \pi} \int_{-\pi}^{\pi} P_{r}(t) d t=1$ for $0 \leqslant r<1$, and:
- $\lim _{r \rightarrow 1^{-}} P_{r}(t)=0$ for every $t \in[-\pi, \pi] \backslash\{0\}$.

Consequently, $P_{r}$ is an approximation of the Dirac measure $\delta_{1}$ at $1 \in \partial \Delta$. For this reason, the Poisson convolution integral possesses excellent boundary value properties.

Lemma 2.21. ([Ka1968, DR2002]) Convergence in norm holds:
(i) If $u \in L^{\mathrm{p}}$ with $1 \leqslant \mathrm{p}<\infty$ or $p=\infty$ and $u$ is continuous, then $\lim _{r \rightarrow 1^{-}}\left\|P_{r} * u-u\right\|_{L^{\mathrm{p}}}=0$.
(ii) If $u \in \mathscr{C}^{\kappa, \alpha}$ with $\kappa \in \mathbb{N}$ and $0 \leqslant \alpha \leqslant 1$, including $\alpha=0$ and $\alpha=1$, then $\lim _{r \rightarrow 1^{-}}\left\|P_{r} * u-u\right\|_{\kappa, \alpha}=0$.

In $\mathscr{C}^{\kappa, \alpha}$, the pointwise convergence $\lim _{r \rightarrow 1^{-}} P_{r} * u\left(e^{i \theta}\right) \rightarrow u\left(e^{i \theta}\right)$ follows obviously. However, in $L^{\mathrm{p}}$, from convergence in norm one may only deduce pointwise convergence almost everywhere for some sequence $r_{k} \rightarrow 1$ which depends on the function. In $L^{\mathrm{p}}$, almost everywhere pointwise convergence was proved by Fatou in 1906.

Theorem 2.22. ([Fa1906, Ka1968, DR2002]) If $u \in L^{\mathrm{p}}$ with $1 \leqslant \mathrm{p} \leqslant \infty$, then for almost every $e^{i \theta} \in \partial \Delta$, we have:

$$
\lim _{r \rightarrow 1^{-}} P_{r} * u\left(e^{i \theta}\right)=u\left(e^{i \theta}\right) .
$$

In summary, the Poisson transform $\mathrm{P} u$ yields a harmonic extension to $\Delta$ of any function $u \in L^{\mathfrak{p}}(\partial \Delta, \mathbb{R})$ or $u \in \mathscr{C}^{\kappa, \alpha}(\partial \Delta, \mathbb{R})$, with expected boundary value $\mathrm{b}_{\partial \Delta}(\mathrm{P} u)=u$ on $\partial \Delta$.
2.23. Hilbert transform on the unit disc. Next, we survey the fundamental properties of the Hilbert transform. Again, $u$ is real-valued on $\partial \Delta$. Setting $z=r e^{i \theta}$ with $0 \leqslant r<1$ and $\zeta=e^{i t}$, computing $\operatorname{Im}\left(\frac{\zeta+z}{\zeta-z}\right)$ and switching the convolution integral, we obtain:

$$
\mathrm{T} u\left(r e^{i \theta}\right)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} T_{r}(t) u\left(e^{i(\theta-t)}\right) d t
$$

where

$$
T_{r}(t):=\frac{2 r \sin t}{1-2 r \cos t+r^{2}}
$$

is the Hilbert kernel. It is not a summability kernel, being positive and negative with $L^{1}$ norm tending to $\infty$ as $r \rightarrow 1^{-}$; for this reason, the Hilbert transform does not enjoy the same nice boundary value properties as the Poisson transform: Hölder classes are needed.

Setting $r=1$, the Poisson kernel $\mathrm{P}_{1}(t)$ vanishes identically and the Hilbert kernel tends to $\frac{2 \sin t}{2-2 \cos t}=\frac{\cos t / 2}{\sin t / 2}$. Near $t=0$, the function $\cot (t / 2)$ behaves like the function $2 / t$, having infinite $L^{1}$ norm. For $u \in \mathscr{C}^{0, \alpha}(\partial \Delta, \mathbb{R})$, it may be verified that, as $z \rightarrow e^{i \theta} \in \partial \Delta$, the Hilbert transform $\mathrm{T} u(z)$ tends to

$$
\begin{aligned}
\mathrm{T} u\left(e^{i \theta}\right) & :=\text { p.v. } \frac{1}{2 \pi} \int_{-\pi}^{\pi} \frac{u\left(e^{i(\theta-t)}\right)}{\tan (t / 2)} d t \\
& =\text { p.v. } \frac{1}{2 \pi i} \int_{-\pi}^{\pi} u(\zeta) \operatorname{Im}\left(\frac{\zeta+e^{i \theta}}{\zeta-e^{i \theta}}\right) \frac{d \zeta}{\zeta}
\end{aligned}
$$

Since $\operatorname{Re}\left(\frac{\zeta+e^{i \theta}}{\zeta-e^{i \theta}}\right) \equiv 0$ for $\zeta=e^{i t} \in \partial \Delta$, we get $\operatorname{Im}\left(\frac{\zeta+e^{i \theta}}{\zeta-e^{i \theta}}\right)=\frac{1}{i} \frac{\zeta+e^{i \theta}}{\zeta-e^{i \theta}}$ so that we may rewrite

$$
i \mathrm{~T} u\left(e^{i \theta}\right)=\text { p.v. } \frac{1}{2 \pi i} \int_{-\pi}^{\pi} u(\zeta) \frac{\zeta+e^{i \theta}}{\zeta-e^{i \theta}} \frac{d \zeta}{\zeta}
$$

Setting $\mathrm{P}_{0} u:=\frac{1}{2 \pi i} \int_{\partial \Delta} u(\zeta) \frac{d \zeta}{\zeta}=\widehat{u}_{0}$, the algebraic relation $\frac{2}{\zeta-e^{i \theta}}-\frac{1}{\zeta}=$ $\frac{\zeta+e^{i \theta}}{\zeta-e^{i \theta}} \frac{1}{\zeta}$ gives a fundamental relation between $\mathrm{C}^{0}$ and T :

$$
2 \mathrm{C}^{0}-\mathrm{P}_{0}=i \mathrm{~T} .
$$

From Theorem 2.15, we deduce ( $\mathrm{P}_{0}$ is innocuous):
Theorem 2.24. ([Pri1916, HiTa1978, Bo1991, BER1999], [*]) There exist an absolute constant $C_{1}>1$ such that if $\kappa \in \mathbb{N}$ and $0<\alpha<1$ :

$$
\frac{1 / C_{1}}{\alpha(1-\alpha)} \leqslant\|\mathbf{T}\|_{\kappa, \alpha} \leqslant \frac{C_{1}}{\alpha(1-\alpha)} .
$$

It follows that at the level of Fourier series, T transforms $u\left(e^{i \theta}\right)=$ $\mathrm{F} u\left(e^{i \theta}\right)=\sum_{k \in \mathbb{Z}} \widehat{u}_{k} e^{i k \theta}$ to

$$
\mathrm{T} u\left(e^{i \theta}\right):=\frac{1}{i}\left(-\sum_{k<0} \widehat{u}_{k} e^{i k \theta}+\sum_{k>0} \widehat{u}_{k} e^{i k \theta}\right) .
$$

Notice that $(\widehat{\mathrm{T} u})_{0}=0$. In fact, this formula coincides with the series $\frac{1}{i}\left(-\sum_{k<0} \widehat{u}_{k} \bar{z}^{k}+\sum_{k>0} \widehat{u}_{k} z^{k}\right)$, written for $z \rightarrow e^{i \theta}$, the limit existing provided $0<\alpha<1$.

By termwise differentiation of the above formula, $\mathrm{T}\left(u_{\theta^{j}}\right)=(\mathrm{T} u)_{\theta^{j}}$ for $0 \leqslant j \leqslant \kappa$, if $u \in \mathscr{C}^{\kappa, \alpha}$ (some integrations by parts in the singular integral defining $\mathrm{T} u$ would yield a second proof of this property).

The Poisson transform $\mathrm{P} u$ of $u \in \mathscr{C}^{0, \alpha}$ having boundary value $\mathrm{b}_{\partial \Delta}(\mathrm{P} u)=$ $u$ and the Schwarz transform being holomorphic in $\Delta$, we see that the function $u+i \mathrm{~T} u$ on $\partial \Delta$ extends holomorphically to $\Delta$ as $\mathrm{S} u(z)$. So $\mathrm{T} u$ on $\Delta$ is one of the Harmonic conjugates of $u$. In general, these conjugates are defined up to a constant. The property $(\widehat{\mathrm{T} u})_{0}=0$ means that $\mathrm{T} u(0)=0$.

Lemma 2.25. The Hilbert transform $\mathrm{T} u$ on $\partial \Delta$ is the boundary value on $\partial \Delta$ of the unique harmonic conjugate in $\Delta$ of the harmonic Poisson extension $\mathrm{P} u$, that vanishes at $0 \in \Delta$.

$$
\text { For } u \in \mathscr{C}^{\kappa, \alpha}(\partial \Delta, \mathbb{R}), u+i \mathrm{~T} u \text { extends holomorphically to } \Delta \text {. }
$$

Furthermore, $\mathrm{T}(\mathrm{T} u)=-u+\widehat{u}_{0}$.
2.26. Hilbert transform in $L^{p}$ spaces. It is elementary to show that the study of the principal value integral p.v. $\frac{1}{2 \pi} \int_{-\pi}^{\pi} \frac{u\left(e^{i(\theta-t)}\right)}{\tan (t / 2)} d t$ is equivalent to the study of the same singular convolution operator, in which $\cot (t / 2)$ is replaced by $2 / t$. Similarly, one may define the Hilbert transform on the real line:

$$
\mathrm{H} f(\mathrm{x}):=\mathrm{p.v.} \int_{\mathbb{R}} \frac{f(\mathrm{y})}{\mathrm{y}-\mathrm{x}} d \mathrm{y}
$$

If $f$ is $\mathscr{C}^{1,0}$ on $\mathbb{R}$ and has compact support or satisfies $\int_{\mathbb{R}}|f|<\infty$, replacing $f(\mathrm{y})$ in the numerator by $[f(\mathrm{y})-f(\mathrm{x})]+f(\mathrm{x})$ and reasoning as in $\S 2.3$, one straightforwardly shows the existence of the above principal value.

Privalov showed that $\mathrm{H} f(\mathrm{x})$ exists for almost every $\mathrm{x} \in \mathbb{R}$ if $f \in L^{1}(\mathbb{R})$. A theorem due to M. Riesz states that the two Hilbert transforms $H$ on the real line and $T$ on the unit circle are bounded endomorphisms of $L^{\mathrm{p}}$, for $1<\mathrm{p}<\infty$, namely if $f \in L^{\mathrm{p}}(\mathbb{R})$ and $u \in L^{\mathrm{p}}(\partial \Delta)$, then:

$$
\|\mathrm{H} f\|_{L^{\mathrm{p}}(\mathbb{R})} \leqslant C_{\mathrm{p}}\|f\|_{L^{\mathrm{p}}(\mathbb{R})} \quad \text { and } \quad\|\mathrm{T} u\|_{L^{\mathrm{p}}(\partial \Delta)} \leqslant C_{\mathrm{p}}\|u\|_{L^{\mathrm{p}}(\partial \Delta)}
$$

whith the same constant $C_{\mathrm{p}}$ ([Zy 1959], Chapters VII and XVI). In [Pi1972], Zygmund's doctoral student Pichorides obtained the best value of the constant $C_{\mathrm{p}}$ : for $1<\mathrm{p} \leqslant 2, C_{\mathrm{p}}=\tan \frac{\pi}{2 \mathrm{p}}$, while, by a duality argument, $C_{\mathrm{p}}=\cot \frac{\pi}{2 \mathrm{p}}$ for $2 \leqslant \mathrm{p}<\infty$. The two elementary bounds $\tan \frac{\pi}{2 \mathrm{p}} \leqslant \frac{\mathrm{p}}{\mathrm{p}-1}$ for $1<\mathrm{p} \leqslant 2$ and $\cot \frac{\pi}{2 \mathrm{p}} \leqslant \mathrm{p}$ for $2 \leqslant \mathrm{p}<\infty$, yield:
$\|\mathrm{H} f\|_{L^{\mathrm{p}}(\mathbb{R})} \leqslant \frac{\mathrm{p}^{2}}{\mathrm{p}-1}\|f\|_{L^{\mathrm{p}}(\mathbb{R})} \quad$ and $\quad\|\mathrm{T} f\|_{L^{\mathrm{p}}(\partial \Delta)} \leqslant \frac{\mathrm{p}^{2}}{\mathrm{p}-1}\|f\|_{L^{\mathrm{p}}(\partial \Delta)}$,
for $1<\mathrm{p}<\infty$. In $L^{1}$, the Hilbert transform is unbounded but, according to a theorem due to Kolmogorov ([Dy1991, DR2002]), it sastisfies a weak inequality:

$$
\mathfrak{m}\{\mathrm{H} f(\mathrm{x})>a\} \leqslant \frac{C}{a}\|f\|_{L^{1}}
$$

for every $a \in \mathbb{R}$ with $a>0$, where $\mathfrak{m}$ is the Lebesgue measure and where $C>0$ is some absolute constant.
2.28. Pointwise convergence of Fourier series. The boundedness of the Hilbert transform in $L^{\mathrm{p}}$ has a long history, closely related to the problem of pointwise convergence of Fourier series. In 1913, before M. Riesz proved the estimates (2.27), using complex function theory and the Riesz-Fischer theorem, Luzin showed that H is bounded in $L^{2}$ and formulated the celebrated conjecture that Fourier series of $L^{2}$ functions converge pointwise almost everywhere. This "hypothetical theorem" was established by Carleson ([Ca1966]) in 1966 and slightly later by Hunt ([Hu1966]) in $L^{\mathrm{p}}$ for $1<\mathrm{p}<\infty$. A complete self-contained restitution of these results is available in [DR2002]. Let us survey the main theorem.

The $n$-th partial sum of the Fourier series of a function $f$ on $\partial \Delta$ is given by:

$$
\mathrm{F}_{n} f\left(e^{i \theta}\right)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} D_{n}(t) f\left(e^{i(\theta-t)}\right) d t
$$

where

$$
D_{n}(t):=\frac{\sin (n+1 / 2) t}{\sin t / 2}
$$

is the Dirichlet kernel, having unbounded $L^{1}$ norm $\left\|D_{n}\right\|_{L^{1}} \sim \frac{4}{\pi^{2}} \log n$. It is elementary to show that the behaviour of this convolution integral, as $n \rightarrow \infty$, is equivalent to the behaviour of the integral:

$$
\int_{-\pi}^{\pi} \frac{\sin n t}{t} f\left(e^{i(\theta-t)}\right) d t
$$

Without loss of generality, $f$ is assumed to be real-valued, so that the above integral is the imaginary part of the Carleson integral:

$$
\mathrm{C}_{n}\left(f, e^{i \theta}\right):=\text { p.v. } \int_{-\pi}^{\pi} \frac{e^{i n t}}{t} f\left(e^{i(\theta-t)}\right) d t
$$

In, chapters $4,5,6,7,8,9$ and 10 of [DR2002], the main proposition is to prove that the Carleson maximal sublinear operator:

$$
\mathbb{C}^{*} f\left(e^{i \theta}\right):=\sup _{n \in \mathbb{N}}\left|\mathrm{C}_{n}\left(f, e^{i \theta}\right)\right|
$$

is bounded from $L^{\mathrm{p}}$ to $L^{\mathrm{p}}$. The proof involves dyadic partitions, changes of frequency, microscopic Fourier analysis of $f$, choices of allowed pairs and seven exceptional sets. By an elementary argument, one deduces that the maximal Fourier series sublinear operator:

$$
\mathrm{F}^{*} f\left(e^{i \theta}\right):=\sup _{n \in \mathbb{N}}\left|\mathrm{~F}_{n} f\left(e^{i \theta}\right)\right|
$$

is bounded from $L^{\mathrm{p}}$ to $L^{\mathrm{p}}$.
Theorem 2.29. ([Ca1966, Hu1966, DR2002]) If $f \in L^{\mathrm{p}}$ with $1<\mathrm{p}<\infty$, there exists an absolute constant $C>1$ such that:

$$
\left\|\mathrm{F}^{*} f\right\|_{L^{p}} \leqslant C \frac{\mathrm{p}^{4}}{(\mathrm{p}-1)^{3}}\|f\|_{L^{\mathrm{p}}}
$$

Then by a standard argument, $\lim _{n \rightarrow \infty} \mathrm{~F}_{n} f\left(e^{i \theta}\right)=f\left(e^{i \theta}\right)$ almost everywhere.
2.30. Transition. Since the grounding article [HiTa1978], the nice behaviour of the Hilbert transform in the Hölder classes (Theorem 2.24) is the main reason why Bishop analytic discs have been constructed in the category of $\mathscr{C}^{\kappa, \alpha}$ generic submanifolds of $\mathbb{C}^{n}$ ([BPo1982, BPi1985, Tu1990, Trp1990, Bo1991, BRT1994, Tu1994a, Me1994, Trp1996, Jö1996, BER1999]). Perhaps it is also interesting to construct Bishop analytic discs in the Sobolev classes.

## §3. Solving a local parametrized Bishop equation with OPTIMAL LOSS OF SMOOTHNESS

3.1. Analytic dises attached to a generic submanifold of $\mathbb{C}^{n}$. As in Theorem 4.2(III), let $M$ be a $\mathscr{C}^{\kappa, \alpha}$ local graphed generic submanifold of equation $v=\varphi(x, y, u)$, where $\varphi$ is defined for $|x+i y|<\rho_{1},|u|<\rho_{1}$, for some $\rho_{1}>0$ and where $\varphi(0)=0, d \varphi(0)=0$ and $|\varphi|<\rho_{1}$.
Definition 3.2. An analytic disc is a map

$$
\bar{\Delta} \ni \zeta \longmapsto A(\zeta)=(Z(\zeta), W(\zeta)) \in \mathbb{C}^{m} \times \mathbb{C}^{d}
$$

which is holomorphic in the unit disc $\Delta$ and at least $\mathscr{C}^{0,0}$ in $\bar{\Delta}$. It is attached to $M$ if it sends $\partial \Delta$ into $M$.

Thus, suppose that $(Z(\zeta), W(\zeta))$ is attached to $M$ and sufficiently small, namely $\left|[X+i Y]\left(e^{i \theta}\right)\right|<\rho_{1},\left|U\left(e^{i \theta}\right)\right|<\rho_{1}$ and $\left|V\left(e^{i \theta}\right)\right|<\rho_{1}$ on $\partial \Delta$, where $Z(\zeta)=X(\zeta)+i Y(\zeta)$ and $W(\zeta)=U(\zeta)+i V(\zeta)$. Then clearly, the disc sends $\partial \Delta$ to $M$ if and only if

$$
V\left(e^{i \theta}\right)=\varphi\left(X\left(e^{i \theta}\right), Y\left(e^{i \theta}\right), U\left(e^{i \theta}\right)\right)
$$

for every $e^{i \theta} \in \partial \Delta$. Thanks to the Hilbert transform, we claim that we may express analytically the fact that the disc is attached to $M$.

At first, in order to guarantee the applicability of the harmonic conjugation operator T , all our analytic discs will $\mathscr{C}^{\kappa, \alpha}$ on $\bar{\Delta}$, with $\kappa \in \mathbb{N}$ and $0<\alpha<1$. We let T act componentwise on maps $U=\left(U^{1}, \ldots, U^{d}\right) \in$ $\mathscr{C}^{\kappa, \alpha}\left(\partial \Delta, \mathbb{R}^{d}\right)$, namely $\mathrm{T} U:=\left(\mathrm{T} U^{1}, \ldots, \mathrm{~T} U^{d}\right)$. We set $\|\mathrm{T} U\|_{\kappa, \alpha}:=$ $\max _{1 \leqslant j \leqslant d}\left\|\mathrm{~T} U^{j}\right\|_{\kappa, \alpha}$. With a slight change of notation, instead of $\mathrm{P} U(0)$, we denote by $\mathrm{P}_{0} U:=\frac{1}{2 \pi} \int_{-\pi}^{\pi} U\left(e^{i \theta}\right) d \theta$ the value at the origin of the Poisson extension $\mathrm{P} U$. Equivalently, $\mathrm{P}_{0} U=\widehat{U}_{0}$ is the mean value of $U$ on $\partial \Delta$. Here is a summary of the most useful properties of $T$.
Lemma 3.3. The $\mathbb{R}^{d}$-valued Hilbert transform T is a bounded linear endomorphism of $\mathscr{C}^{\kappa, \alpha}\left(\partial \Delta, \mathbb{R}^{d}\right)$ with $\frac{1 / C_{1}}{\alpha(1-\alpha)} \leqslant\|\mathrm{T}\|_{\kappa, \alpha} \leqslant \frac{C_{1}}{\alpha(1-\alpha)}$ satisfying $\mathrm{T}(\mathrm{cst})=0$ and

$$
\mathrm{T}(\mathrm{~T} U)=-U+\mathrm{P}_{0} U
$$

In the sequel, we shall rather use the mild modification $T_{1}$ of $T$ defined by:

$$
\mathrm{T}_{1} U\left(e^{i \theta}\right):=\mathrm{T} U\left(e^{i \theta}\right)-\mathrm{T} U(1) .
$$

In fact, $\mathrm{T}_{1}$ is uniquely determined by the normalizing condition $\mathrm{T}_{1} U(1)=0$. Then $\mathrm{T}_{1}$ is also bounded: $\frac{1 / C_{1}}{\alpha(1-\alpha)} \leqslant\left\|\mathrm{T}_{1}\right\|_{\kappa, \alpha} \leqslant \frac{C_{1}}{\alpha(1-\alpha)}$, also annihilates constants: $\mathrm{T}_{1}($ cst $)=0$ and

$$
\mathrm{T}_{1}\left(\mathrm{~T}_{1} U\right)=-U+U(1)
$$

Furthermore, most importantly:
Lemma 3.4. If $U \in \mathscr{C}^{\kappa, \alpha}\left(\partial \Delta, \mathbb{R}^{d}\right)$, then $U\left(e^{i \theta}\right)+i \mathrm{~T}_{1} U\left(e^{i \theta}\right)$ extends as a holomorphic map $\Delta \rightarrow \mathbb{C}^{d}$ which is $\mathscr{C}^{\kappa, \alpha}$ in the closed disc $\bar{\Delta}$.

To check that the extension is $\mathscr{C}^{\kappa, \alpha}$ in $\bar{\Delta}$, one may introduce the Poisson integral formula and apply Lemma 2.21 (ii).

If $A=(Z, W)$ is an analytic disc attached to $M$, we set $U_{0}:=U(1)$ and $V_{0}:=V(1)$. Since $W$ is holomorphic, necessarily $V\left(e^{i \theta}\right)=\mathrm{T}_{1} U\left(e^{i \theta}\right)+V_{0}$. Applying $\mathrm{T}_{1}$ to both sides, we get $\mathrm{T}_{1} V\left(e^{i \theta}\right)=-U\left(e^{i \theta}\right)+U_{0}$ (the left and the right hand sides vanish at $e^{i \theta}=1$ ). Applying $\mathrm{T}_{1}$ to $V\left(e^{i \theta}\right)=$
$\varphi\left(X\left(e^{i \theta}\right), Y\left(e^{i \theta}\right), U\left(e^{i \theta}\right)\right)$ above and reorganizing, we obtain that $U$ satisfies a functional equation ${ }^{7}$ involving the Hilbert transform:

$$
\begin{equation*}
U\left(e^{i \theta}\right)=-\mathrm{T}_{1}[\varphi(X(\cdot), Y(\cdot), U(\cdot))]\left(e^{i \theta}\right)+U_{0} \tag{3.5}
\end{equation*}
$$

Here, the map $U: \partial \Delta \rightarrow \mathbb{R}^{d}$ is the unknown, whereas the holomorphic map $Z=X+i Y: \partial \Delta \rightarrow \mathbb{C}^{m}$ and the constant vector $U_{0}$ are given data.

Conversely, given $X+i Y$ and $U_{0}$, assume that $U \in \mathscr{C}^{\kappa, \alpha}$ satisfies the above functional equation. Set $V\left(e^{i \theta}\right):=\mathrm{T}_{1} U\left(e^{i \theta}\right)+V_{0}$, where $V_{0}:=$ $\varphi(X(1), Y(1), U(1))$. Then $U\left(e^{i \theta}\right)+i V\left(e^{i \theta}\right)$ extends as a $\mathscr{C}^{\kappa, \alpha}$ map $\bar{\Delta} \ni$ $\zeta \mapsto W(\zeta) \in \mathbb{C}^{d}$ which is holomorphic in $\Delta$. If $\left|[X+i Y]\left(e^{i \theta}\right)\right|<\rho_{1}$, $\left|U\left(e^{i \theta}\right)\right|<\rho_{1}$ and $\left|V\left(e^{i \theta}\right)\right|<\rho_{1}$, the disc $A:=(Z, W)$ is attached to $M$.

Bishop (1965) in the $\mathscr{C}^{\kappa, 0}$ classes and then Hill-Taiani (1978), BoggessPitts (1985) in the Hölder classes $\mathscr{C}^{\kappa, \alpha}$ established existence and uniqueness of the solution $U$ to the fundamental functional equation (3.5).

Theorem 3.6. ([Bi1965, HiTa1978, BPi1985]) If $M$ is at least $\mathscr{C}^{1, \alpha}$, shrinking $\rho_{1}$ if necessary, there exists $\rho_{2}$ with $0<\rho_{2}<\rho_{1}$ such that whenever the data $Z \in \mathscr{C}^{0, \alpha}\left(\bar{\Delta}, \mathbb{C}^{m}\right) \cap \mathscr{O}\left(\Delta, \mathbb{C}^{m}\right)$ and $U_{0} \in \mathbb{R}^{d}$ satisfy $\left|Z\left(e^{i \theta}\right)\right|<\rho_{2}$ on $\partial \Delta$ and $\left|U_{0}\right|<\rho_{2}$, there exists a unique solution $U \in \mathscr{C}^{0, \beta}\left(\partial \Delta, \mathbb{R}^{d}\right)$, $0<\beta<\alpha^{2}$, to the Bishop-type functional equation (3.5) above such that $\left|U\left(e^{i \theta}\right)\right|<\rho_{1}$ on $\partial \Delta$ and such that in addition $\left|V\left(e^{i \theta}\right)\right|<\rho_{1}$ on $\partial \Delta$, where

$$
V\left(e^{i \theta}\right):=\mathrm{T}_{1} U\left(e^{i \theta}\right)+\varphi(X(1), Y(1), U(1))
$$

Consequently, the disc $(Z, U+i V)$ is attached to $M$.
Notice the (substantial) loss of smoothness, occuring also in [BRT1994, BER1999], which is due to an application of a general implicit function theorem in Banach spaces. The main theorem of this chapter ([Tu1990, Tu1996]) refines the preceding result with a negligible loss of smoothness, provided the graphing map $\varphi$ belongs to the Hölder space $\mathscr{C}^{\kappa, \alpha}$. In the geometric applications (Parts V and VI), it is advantageous to be able to solve a Bishop equation like (3.5) which involves supplementary parameters. Thus, instead of $\varphi$, we shall consider an $\mathbb{R}^{d}$-valued $\mathscr{C}^{\kappa, \alpha}$ map $\Phi=\Phi\left(u, e^{i \theta}, s\right)$, where $s$ is a parameter. For fixed $s$, we shall denote by $\left.\Phi\right|_{s}$ the map

[^6]$\square_{\rho_{1}}^{d} \times \partial \Delta \ni\left(u, e^{i \theta}\right) \longmapsto \Phi\left(u, e^{i \theta}, s\right) \in \mathbb{R}^{d}$. In accordance with Section 1, we set:
$$
\left\|\Phi_{u}\right\|_{0,0}:=\max _{1 \leqslant j \leqslant d}\left(\sum_{1 \leqslant l \leqslant d}\left\|\Phi_{u_{l}}^{j}\right\|_{0,0}\right)
$$
and similarly $\left\|\Phi_{\theta}\right\|_{0,0}=\max _{1 \leqslant j \leqslant d}\left\|\Phi_{\theta}^{j}\right\|_{0,0}$.
Theorem 3.7. ([Tu1990, Tu1996], [*]) Let $\Phi=\Phi\left(u, e^{i \theta}, s\right)$ be an $\mathbb{R}^{d}$ valued map of class $\mathscr{C}^{\kappa, \alpha}, \kappa \geqslant 1,0<\alpha<1$, defined for $u \in \mathbb{R}^{d},|u|<\rho_{1}$, $\theta \in \mathbb{R}$ and $s \in \mathbb{R}^{b},|s|<\sigma_{1}$, where $0<\rho_{1}<1$ and $0<\sigma_{1}<1$. Assume that on its domain of definition $\square_{\rho_{1}}^{d} \times \partial \Delta \times \square_{\sigma_{1}}^{b}$, the map $\Phi$ and its derivatives with respect to $u$ and to $\theta$ satisfy the inequalities (nothing is required about $\Phi_{s}$ ):
$$
\|\Phi\|_{0,0} \leqslant \mathrm{c}_{1}, \quad\left\|\Phi_{u}\right\|_{0,0} \leqslant \mathrm{c}_{2}, \quad\left\|\Phi_{\theta}\right\|_{0,0} \leqslant \mathrm{c}_{3},
$$
for some small positive constants $\mathrm{c}_{1}, \mathrm{c}_{2}$ and $\mathrm{c}_{3}$ such that
$$
\text { (3.8) } \mathrm{c}_{1} \leqslant C \alpha \rho_{1}, \quad \mathrm{c}_{2} \leqslant C^{2} \alpha^{2}\left[1+\sup _{|s|<\sigma_{1}}\left\|\left.\Phi\right|_{s}\right\|_{1, \alpha}\right]^{-2}, \quad \mathrm{c}_{3} \leqslant \rho_{1}^{2} \mathrm{c}_{2},
$$
where $0<C<1$ is an absolute constant. Then for every fixed $U_{0}$ satisfying $\left|U_{0}\right|<\rho_{1} / 16$ and every fixed $s \in \square_{\sigma_{1}}^{b}$, the parameterized local Bishop-type functional equation:
$$
U\left(e^{i \theta}\right)=-\mathrm{T}_{1}[\Phi(U(\cdot), \cdot, s)]\left(e^{i \theta}\right)+U_{0}
$$
has a unique solution:
$$
\partial \Delta \ni e^{i \theta} \longmapsto U\left(e^{i \theta}, s, U_{0}\right) \in \mathbb{R}^{d},
$$
with $\|U\|_{0,0} \leqslant \rho_{1} / 4$ which is of class $\mathscr{C}^{\kappa, \alpha}$ on $\partial \Delta$. Furthermore, this solution is of class $\mathscr{C}^{\kappa, \alpha-0}=\bigcap_{\beta<\alpha} \mathscr{C}^{\kappa, \beta}$ with respect to all the variables, including parameters, namely the complete map
$$
\partial \Delta \times \square_{\sigma_{1}}^{b} \times \square_{\rho_{1} / 16}^{d} \ni\left(e^{i \theta}, s, U_{0}\right) \longmapsto U\left(e^{i \theta}, s, U_{0}\right) \in \mathbb{R}^{d}
$$
is $\mathscr{C}^{\kappa, \alpha-0}$.
Since the assumptions involve only the $\mathscr{C}^{1, \alpha}$ norm of $\Phi$, we notice that the theorem is also true with $\Phi \in \mathscr{C}^{\kappa, \alpha-0}$, provided $\kappa \geqslant 2$, except that the solution will only be $\mathscr{C}^{\kappa, \alpha-0}$ with respect to $e^{i \theta}$ : it suffices to apply the theorem by considering that $\Phi \in \mathscr{C}^{\kappa, \beta}$, with $\beta<\alpha$ arbitrary, getting a solution that is $\mathscr{C}^{\kappa, \beta-0}$ with respect to all variables and concluding from $\bigcap_{\beta<\alpha} \mathscr{C}^{\kappa, \beta-0}=\mathscr{C}^{\kappa, \alpha-0}$.

The main purpose of this section is to provide a thorough proof of the theorem. In the sequel, $C, C_{1}, C_{2}, C_{3}$ and $C_{4}$ will denote positive absolute constants. We may assure that they all will be $\geqslant 10^{-5}$ and $\leqslant 10^{5}$.

The smallness of $\|\Phi\|_{0,0}$, of $\left\|\Phi_{u}\right\|_{0,0}$ and of $\left\|\Phi_{\theta}\right\|_{0,0}$ guarantee the smallness of $\|\Phi \mid s\|_{1, \alpha / 2}$ by virtue of an elementary observation.
Lemma 3.9. ([*]) Let $\mathrm{x} \in \square_{\rho}^{n}, n \in \mathbb{N}, n \geqslant 1$, where $0<\rho_{i} \leqslant \infty$, and let $f=f(\mathrm{x})$ be $\mathscr{C}^{0, \alpha}$ function with values in $\mathbb{R}^{d}, d \geqslant 1$. If $\|f\|_{0,0} \leqslant \mathrm{c}$, for some quantity $\mathrm{c}>0$, then:

$$
\|f\|_{\widehat{0, \alpha / 2}} \leqslant \mathrm{c}^{1 / 2}\left[2+\|f\|_{\widehat{0, \alpha}}\right] .
$$

We apply this inequality to $\left.\Phi_{u}\right|_{s}$ and to $\left.\Phi_{\theta}\right|_{s}$, pointing out that for any $\beta$ with $0<\beta \leqslant \alpha$, by definition:

$$
\begin{aligned}
& \left\|\left.\Phi_{u}\right|_{s}\right\|_{\widehat{0, \beta}}=\max _{1 \leqslant j \leqslant d}\left(\sum_{l=1}^{d} \frac{\left|\Phi_{u_{l}}^{j}\left(u^{\prime \prime}, e^{i \theta^{\prime \prime}}, s\right)-\Phi_{u_{l}}^{j}\left(u^{\prime}, e^{i \theta^{\prime}}, s\right)\right|}{\left|\left(u^{\prime \prime}, \theta^{\prime \prime}\right)-\left(u^{\prime}, \theta^{\prime}\right)\right|^{\beta}}\right), \\
& \left\|\left.\Phi_{\theta}\right|_{s}\right\|_{\widehat{0, \beta}}=\max _{1 \leqslant j \leqslant d} \frac{\left|\Phi_{\theta}^{j}\left(u^{\prime \prime}, e^{i \theta^{\prime \prime}}, s\right)-\Phi_{\theta}^{j}\left(u^{\prime}, e^{i \theta^{\prime}}, s\right)\right|}{\left|\left(u^{\prime \prime}, \theta^{\prime \prime}\right)-\left(u^{\prime}, \theta^{\prime}\right)\right|^{\beta}} .
\end{aligned}
$$

Lemma 3.10. ([*]) Independently of s, we have:

$$
\begin{aligned}
\left\|\left.\Phi_{u}\right|_{s}\right\|_{\widehat{0, \alpha / 2}} & \leqslant \mathrm{c}_{2}^{1 / 2}\left[2+\left\|\left.\Phi\right|_{s}\right\|_{1, \alpha}\right] \\
\left\|\left.\Phi_{\theta}\right|_{s}\right\|_{\widehat{0, \alpha / 2}} & \leqslant \mathrm{c}_{3}^{1 / 2}\left[2+\left\|\left.\Phi\right|_{s}\right\|_{1, \alpha}\right] \\
\left\|\left.\Phi\right|_{s}\right\|_{1, \alpha / 2} & \leqslant \mathrm{c}_{1}+\mathrm{c}_{2}+\mathrm{c}_{3}+\left(\mathrm{c}_{2}^{1 / 2}+\mathrm{c}_{3}^{1 / 2}\right)\left[2+\left\|\left.\Phi\right|_{s}\right\|_{1, \alpha}\right] .
\end{aligned}
$$

The presence of the squares in the inequalities of Theorem 3.7 anticipates the roots $\mathrm{c}_{2}^{1 / 2}$ and $\mathrm{c}_{3}^{1 / 2}$ above. These two lemmas and the next involve dry computations with Hölder norms. The detailed proofs are postponed to Section 4.

Lemma 3.11. ([*]) If $U \in \mathscr{C}^{1, \beta}\left(\partial \Delta, \mathbb{R}^{d}\right)$ with $0<\beta \leqslant \alpha$ satisfies $\left|U\left(e^{i \theta}\right)\right|<\rho_{1}$ on $\partial \Delta$, then for every fixed $s \in \square_{\sigma_{1}}^{b}$, we have:

$$
\begin{aligned}
\|\Phi(U(\cdot), \cdot, s)\|_{\mathscr{G} 1, \beta}(\partial \Delta) & \leqslant\|\Phi\|_{0,0}+\left\|\Phi_{\theta}\right\|_{0,0}+\left\|\left.\Phi_{\theta}\right|_{s}\right\|_{\widehat{0, \beta}}\left[1+\left(\|U\|_{\widehat{1,0}}\right)^{\beta}\right]+ \\
& +\left\|\Phi_{u}\right\|_{0,0}\|U\|_{1, \beta}+\left\|\left.\Phi_{u}\right|_{s}\right\|_{\widehat{0, \beta}}\left[\|U\|_{\widehat{1,0}}+\left(\|U\|_{\widehat{1,0}}\right)^{1+\beta}\right] .
\end{aligned}
$$

Remind $\|U\|_{\widehat{1,0}}=\sup _{\theta}\left|U_{\theta}\left(e^{i \theta}\right)\right|$. We then introduce the map:

$$
U \longmapsto \mathfrak{F}(U):=U_{0}-\mathrm{T}_{1}[\Phi(U(\cdot), \cdot, s)]\left(e^{i \theta}\right) .
$$

To construct the solution $U$, we endeavour a Picard iteration process, setting $\left.U_{k}\right|_{k=0}:=U_{0}$ with $\left|U_{0}\right|<\rho_{1} / 16$ and $U_{k+1}:=\mathfrak{F}\left(U_{k}\right)$, for $k \in \mathbb{N}$, whenever $\mathfrak{F}\left(U_{k}\right)$ may be defined, i.e. whenever $\left\|U_{k}\right\|_{0,0}<\rho_{1}$. We shall first work in $\mathscr{C}^{1, \alpha / 2} \subset \mathscr{C}^{\kappa, \alpha}$.

Lemma 3.12. If we choose the absolute constant $C<1$ appearing in the theorem sufficiently small, then independently of $s$, the sequence $U_{k}$ satisfies the uniform boundedness estimate:

$$
\left\|U_{k}\right\|_{1, \alpha / 2} \leqslant \rho_{1} / 4<\rho_{1}
$$

hence it is defined for every $k \in \mathbb{N}$ and each $U_{k}$ belongs to $\mathscr{C}^{1, \alpha / 2}(\partial \Delta)$.
Proof. By Theorem 2.24, there exists an absolute constant $C_{1}>1$ (not exactly the same), such that

$$
\left\|\mathrm{T}_{1}\right\|_{1, \alpha / 2} \leqslant C_{1} / \alpha .
$$

Majorating by means of the $\mathscr{C}^{0, \alpha / 2}$-norm:

$$
\left\|\mathfrak{F}\left(U_{k}\right)\right\|_{1, \alpha / 2} \leqslant\left|U_{0}\right|+\left\|T_{1}\right\|_{1, \alpha / 2}\left\|\Phi\left(U_{k}(\cdot), \cdot, s\right)\right\|_{\mathscr{C}^{1}, \alpha / 2}(\partial \Delta)
$$

Assume by induction that $U_{k}$ is $\mathscr{C}^{1, \alpha / 2}$ and satisfies $\left\|U_{k}\right\|_{1, \alpha / 2} \leqslant \rho_{1} / 4$ (this holds for $k=0$ ). Clearly $U_{k+1}=\mathfrak{F}\left(U_{k}\right)$ is $\mathscr{C}^{1, \alpha / 2}$. Thanks to Lemma 3.11, and to the trivial majoration $\left(\left\|U_{k}\right\|_{\widehat{1,0}}\right)^{\alpha / 2} \leqslant\left(\rho_{1} / 4\right)^{\alpha / 2}<1$ :

$$
\begin{aligned}
& \left\|\Phi\left(U_{k}(\cdot), \cdot, s\right)\right\|_{\mathscr{G}^{1, \alpha / 2}(\partial \Delta)} \leqslant\|\Phi\|_{0,0}+\left\|\Phi_{\theta}\right\|_{0,0}+2\left\|\left.\Phi_{\theta}\right|_{s}\right\|_{\widehat{0, \alpha / 2}}+ \\
& +\left\|\Phi_{u}\right\|_{0,0}\left\|U_{k}\right\|_{1, \alpha / 2}+2\left\|\left.\Phi_{u}\right|_{s}\right\|_{\widehat{0, \alpha / 2}}\left\|U_{k}\right\|_{1, \alpha / 2} .
\end{aligned}
$$

Using then the assumptions (3.8) of the theorem together with Lemma 3.10:

$$
\begin{aligned}
\left\|U_{k+1}\right\|_{1, \alpha / 2} & \leqslant \rho_{1} / 16+C_{1} \alpha^{-1}\left[\mathrm{c}_{1}+\mathrm{c}_{3}+4 \mathrm{c}_{3}^{1 / 2}\left(1+\left\|\left.\Phi\right|_{s}\right\|_{1, \alpha}\right)+\right. \\
& \left.+\left\|U_{k}\right\|_{1, \alpha / 2}\left(\mathrm{c}_{2}+4 \mathrm{c}_{2}^{1 / 2}\left(1+\left\|\left.\Phi\right|_{s}\right\|_{1, \alpha}\right)\right)\right] .
\end{aligned}
$$

Using the two trivial majorations $\mathrm{c}_{3} \leqslant C \alpha \rho_{1}$ and $\mathrm{c}_{2} \leqslant C \alpha$ together with the main assumptions (3.8) to majorate $\mathrm{c}_{2}^{1 / 2}$ and $\mathrm{c}_{3}^{1 / 2}$, we get:

$$
\left\|U_{k+1}\right\|_{1, \alpha / 2} \leqslant \rho_{1} / 16+C_{1} \rho_{1} 6 C+\left\|U_{k}\right\|_{1, \alpha / 2} C_{1} 5 C .
$$

Choosing $C \leqslant \frac{1}{16 C_{1} 6}$ (whence $C \leqslant \frac{1}{2 C_{1} 5}$ ), we finally get:

$$
\left\|U_{k+1}\right\|_{1, \alpha / 2} \leqslant \rho_{1} / 8+(1 / 2)\left\|U_{k}\right\|_{1, \alpha / 2}
$$

By immediate induction, the assumption $\left|U_{0}\right|<\rho_{1} / 16$ and these (strict) inequalities entail that $\left\|U_{k}\right\|_{1, \alpha / 2} \leqslant \rho_{1} / 4$ for every $k \in \mathbb{N}$, as claimed.
Lemma 3.13. ([Tu1990], [*]) For every $\beta$ with $0<\beta \leqslant \alpha$ and every fixed $s \in \square_{\rho_{1},}^{b}$, if two maps $U^{j} \in \mathscr{C}^{1,0}\left(\partial \Delta, \mathbb{R}^{d}\right)$ with $\left\|U^{j}\right\|_{0,0}<\rho_{1} / 3$ for $j=1,2$ are given, the following inequality holds:

$$
\left\|\Phi\left(U^{2}(\cdot), \cdot, s\right)-\Phi\left(U^{1}(\cdot), \cdot, s\right)\right\|_{\mathscr{C}^{0, \beta}(\partial \Delta)} \leqslant \mathrm{C}\left\|U^{2}-U^{1}\right\|_{\mathscr{C}^{0, \beta}(\partial \Delta)}
$$

where

$$
\mathbf{C}=\left\|\Phi_{\mid s}\right\|_{1, \beta} 2\left[1+\left(\left\|U^{1}\right\|_{\widehat{1,0}}\right)^{\beta}+\left(\left\|U^{2}\right\|_{\widehat{1,0}}\right)^{\beta}\right] .
$$

Again, the (latexnically lengthy) proof is postponed to Section 4.
Lemma 3.14. If we choose the absolute constant $C$ of the theorem sufficiently small, then independently of $s$, the map:

$$
U \longmapsto \mathfrak{F}(U):=U_{0}-\mathrm{T}_{1}[\Phi(U(\cdot), \cdot, s)]\left(e^{i \theta}\right),
$$

restricted to the set of those $U \in \mathscr{C}^{1, \alpha / 2}\left(\partial \Delta, \mathbb{R}^{d}\right)$ that satisfy $\|U\|_{1, \alpha / 2} \leqslant$ $\rho_{1} / 4$, is a contraction:

$$
\left\|\mathfrak{F}\left(U^{2}\right)-\mathfrak{F}\left(U^{1}\right)\right\|_{0, \alpha / 2} \leqslant \frac{1}{2}\left\|U^{2}-U^{1}\right\|_{0, \alpha / 2}
$$

Proof. Let $U^{j} \in \mathscr{C}^{1, \alpha / 2}$ with $\left\|U^{j}\right\|_{1, \alpha / 2} \leqslant \rho_{1} / 4$ for $j=1,2$. In particular, $\left\|U^{j}\right\|_{0,0}<\rho_{1} / 3$, so Lemma 3.13 applies. In the majorations below, to pass to the fourth line, we use the assumption $\rho_{1}<1$, which enables us to majorate simply by 3 the three terms in the brackets of the third line:

$$
\begin{aligned}
& \left\|\mathfrak{F}\left(U^{2}\right)-\mathfrak{F}\left(U^{1}\right)\right\|_{0, \alpha / 2}=\left\|\mathrm{T}_{1}\left[\Phi\left(U^{2}(\cdot), \cdot, s\right)-\Phi\left(U^{1}(\cdot), \cdot, s\right)\right]\right\|_{0, \alpha / 2} \\
& \leqslant\left\|\mathbf{T}_{1}\right\|_{0, \alpha / 2}\left\|\Phi\left(U^{2}(\cdot), \cdot, s\right)-\Phi\left(U^{1}(\cdot), \cdot, s\right)\right\|_{0, \alpha / 2} \\
& \leqslant \frac{C_{1}}{\alpha}\left\|\Phi_{\mid s}\right\|_{1, \alpha / 2} 2\left[1+\left(\left\|U^{1}\right\|_{1, \alpha / 2}\right)^{\alpha / 2}+\left(\left\|U^{2}\right\|_{1, \alpha / 2}\right)^{\alpha / 2}\right]\left\|U^{2}-U^{1}\right\|_{0, \alpha / 2} \\
& \leqslant \frac{C_{2}}{\alpha}\left\|\Phi_{\mid s}\right\|_{1, \alpha / 2}\left\|U^{2}-U^{1}\right\|_{0, \alpha / 2}
\end{aligned}
$$

where $C_{2}>1$ is absolute. Then we apply Lemma 3.10 to majorate $\left\|\left.\Phi\right|_{s}\right\|_{1, \alpha / 2}$, we use the three trivial majorations $\mathrm{c}_{1}, \mathrm{c}_{2}, \mathrm{c}_{3} \leqslant C \alpha$ and we majorate $c_{2}^{1 / 2}, c_{3}^{1 / 2}$ by means of (3.8), dropping $\rho_{1}<1$ in $c_{3}^{1 / 2}$, which yields:

$$
\begin{aligned}
\left\|\left.\Phi\right|_{s}\right\|_{1, \alpha / 2} & \leqslant \mathrm{c}_{1}+\mathrm{c}_{2}+\mathrm{c}_{3}+\left(\mathrm{c}_{2}^{1 / 2}+\mathrm{c}_{3}^{1 / 2}\right)\left[2+\left\|\Phi_{\mid s}\right\|_{1, \alpha}\right] \\
& \leqslant 3 C \alpha+2 C \alpha+4 C \alpha=9 C \alpha .
\end{aligned}
$$

Then we conclude that

$$
\left\|\mathfrak{F}\left(U^{2}\right)-\mathfrak{F}\left(U^{1}\right)\right\|_{0, \alpha / 2} \leqslant C C_{3}\left\|U^{2}-U^{1}\right\|_{0, \alpha / 2} .
$$

Choosing the absolute constant $C$ of the theorem $\leqslant \frac{1}{2 C_{3}}$ yields the desired contracting factor $\frac{1}{2}$.

The fixed point theorem then entails that our sequence $U_{k}$ converges in $\mathscr{C}^{0, \alpha / 2}$-norm towards some map $U \in \mathscr{C}^{0, \alpha / 2}\left(\partial \Delta, \mathbb{R}^{d}\right)$. More is true:

Lemma 3.15. For every fixed parameters $\left(s, U^{0}\right)$, this solution $U=$ $U\left(e^{i \theta}, s, U_{0}\right)=\lim _{k \rightarrow \infty} U_{k}$ belongs in fact to $\mathscr{C}^{1, \alpha / 2}\left(\partial \Delta, \mathbb{R}^{d}\right)$ and satisfies $\|U\|_{1, \alpha / 2} \leqslant \rho_{1} / 4$.

Proof. Indeed, since $\left\|U_{k}\right\|_{1, \alpha / 2} \leqslant \rho_{1} / 4$ is bounded, it is possible thanks to the Arzelà-Ascoli lemma to extract some subsequence converging in $\mathscr{C}^{1,0}(\partial \Delta)$ to a map, still denoted by $U=U\left(e^{i \theta}, s, U_{0}\right)$, which is $\mathscr{C}^{1,0}$ on $\partial \Delta$. Still denoting by $U_{k}$ such a subsequence, we observe that the uniform convergence $\left\|U_{k}-U\right\|_{1,0} \rightarrow 0$ plus the boundedness $\left\|U_{k}\right\|_{\widehat{1, \alpha / 2}} \leqslant \rho_{1} / 4$ entail immediately that the following majoration holds:

$$
\frac{\left|U_{\theta}\left(e^{i \theta^{\prime \prime}}\right)-U_{\theta}\left(e^{i \theta^{\prime}}\right)\right|}{\left|\theta^{\prime \prime}-\theta^{\prime}\right|^{\alpha / 2}}=\lim _{k \rightarrow \infty} \frac{\left|U_{k, \theta}\left(e^{i \theta^{\prime \prime}}\right)-U_{k, \theta}\left(e^{i \theta^{\prime}}\right)\right|}{\left|\theta^{\prime \prime}-\theta^{\prime}\right|^{\alpha / 2}} \leqslant \frac{\rho_{1}}{4}
$$

for arbitrary $0<\left|\theta^{\prime \prime}-\theta^{\prime}\right| \leqslant \pi$. Consequently, $U$ belongs to $\mathscr{C}^{1, \alpha / 2}$. Passing to the limit in $\left\|U_{k}\right\|_{1, \alpha / 2} \leqslant \rho_{1} / 4$, we also deduce $\|U\|_{1, \alpha / 2} \leqslant \rho_{1} / 4$.

The next crucial step is to study the regularity of the solution $U=$ $U\left(e^{i \theta}, s, U_{0}\right)$ with respect to $\left(s, U_{0}\right)$.

Lemma 3.16. The solution $U=U\left(e^{i \theta}, s, U_{0}\right)$ satisfies a Lipschitz condition with respect to the parameters $s$ and $U_{0}$.
Proof. Consider two parameters $s^{1}, s^{2} \in \square_{\sigma_{1}}^{b}$ and define $U^{j}:=$ $U\left(e^{i \theta}, s^{j}, U_{0}\right)$ for $j=1,2$. Then substract the two corresponding Bishop equations, insert two innocuous opposite terms and majorate:

$$
\begin{aligned}
\left\|U^{2}-U^{1}\right\|_{0, \alpha / 2} \leqslant\left\|T_{1}\right\|_{0, \alpha / 2} & {\left[\left\|\Phi\left(U^{2}(\cdot), \cdot, s^{2}\right)-\Phi\left(U^{2}(\cdot), \cdot, s^{1}\right)\right\|_{0, \alpha / 2}+\right.} \\
& \left.+\left\|\Phi\left(U^{2}(\cdot), \cdot, s^{1}\right)-\Phi\left(U^{1}(\cdot), \cdot, s^{1}\right)\right\|_{0, \alpha / 2}\right] .
\end{aligned}
$$

To majorate the difference in the second line, we again apply Lemma 3.13. To majorate the difference in the first line, we apply:
Lemma 3.17. ([*]) Let $\beta$ with $0<\beta \leqslant \alpha$, let $U \in \mathscr{C}^{1,0}\left(\partial \Delta, \mathbb{R}^{d}\right)$ with $\|U\|_{0,0}<\rho_{1}$ and let two parameters $s^{1}, s^{2} \in \square_{\sigma_{1}}^{b}$. Then

$$
\left\|\Phi\left(U(\cdot), \cdot, s^{2}\right)-\Phi\left(U(\cdot), \cdot, s^{1}\right)\right\|_{0, \beta} \leqslant\left|s^{2}-s^{1}\right|\left(\|\Phi\|_{1,0}+\|\Phi\|_{1, \beta}\left[1+\left(\|U\|_{\widehat{1,0}}\right)^{\beta}\right]\right) .
$$

With $\beta:=\frac{\alpha}{2}$, we thus obtain:

$$
\begin{aligned}
\left\|U^{2}-U^{1}\right\|_{0, \alpha / 2} & \leqslant \frac{C_{1}}{\alpha}\left[\left|s^{2}-s^{1}\right|\left(\|\Phi\|_{1,0}+\|\Phi\|_{1, \alpha / 2}\left[1+\left(\left\|U^{2}\right\|_{1,0}\right)^{\frac{\alpha}{2}}\right]\right)+\right. \\
& \left.+\sup _{|s|<\sigma_{1}}\left\|\Phi_{\mid s}\right\|_{1, \alpha / 2} 2\left[1+\left(\left\|U^{1}\right\|_{\widehat{1,0}}\right)^{\frac{\alpha}{2}}+\left(\left\|U^{2}\right\|_{\widehat{1,0}}\right)^{\frac{\alpha}{2}}\right]\right]\left\|U^{2}-U^{1}\right\|_{0, \alpha / 2}
\end{aligned}
$$

Then we apply the majoration of Lemma 3.10 to $\left\|\Phi_{\mid s}\right\|_{1, \alpha / 2}$ and we use $\rho_{1}<$
1 to majorate by 1 the terms $\left\|U^{j}\right\|_{\widehat{1,0}} \leqslant \rho_{1} / 4$ :

$$
\left\|U^{2}-U^{1}\right\|_{0, \alpha / 2} \leqslant C_{1} \alpha^{-1}\left|s^{2}-s^{1}\right| 3\|\Phi\|_{1, \alpha / 2}+C C_{2}\left\|U^{2}-U^{1}\right\|_{0, \alpha / 2}
$$

Setting K $:=C_{1} \alpha^{-1} 3\|\Phi\|_{1, \alpha / 2}$, requiring $C \leqslant \frac{1}{2 C_{2}}$ and reorganizing we obtain that $U\left(e^{i \theta}, s, U_{0}\right)$ is Lipschitzian with respect to $s$ :

$$
\left\|U^{2}-U^{1}\right\|_{0,0} \leqslant\left\|U^{2}-U^{1}\right\|_{0, \alpha / 2} \leqslant 2 \mathrm{~K}\left|s^{2}-s^{1}\right| .
$$

The proof that $U\left(e^{i \theta}, s, U_{0}\right)$ is Lipschitzian with respect to $U_{0}$ is similar and even simpler.

In summary, the solution $U=U\left(e^{i \theta}, s, U_{0}\right)$ is $\mathscr{C}^{1, \alpha / 2}$ with respect to $e^{i \theta}$ and Lipschitzian with respect to all the variables $\left(e^{i \theta}, s, U_{0}\right)$.

Consequently, according to a theorem due to Rademacher ([Ra1919, Fe1969]), the partial derivatives $U_{s_{k}}, k=1, \ldots, b$ and $U_{U_{0}^{m}}, m=1, \ldots, d$ exist in $L^{\infty}$. We then have to differentiate the Bishop-type equation of Theorem 3.7 with respect to $\theta$, to $s_{k}$ and to $U_{0}^{m}$. However, the linear operator $\mathrm{T}_{1}$ is not bounded in $L^{\infty}$; in fact, according to (2.27), $\|T\|_{L^{p}} \sim \mathrm{p}$ as $\mathrm{p} \rightarrow \infty$. So we need more information.

Lemma 3.18. There exists a null-measure subset $\mathfrak{N} \subset \square_{\sigma_{1}}^{b} \times \square_{\rho_{1} / 16}^{d}$ and there exists a quantity $\mathrm{K}>0$ such that at every $\left(s, U_{0}\right) \notin \mathfrak{N}$, for every $k=1, \ldots, b$ and for every $m=1, \ldots, d$ :
(i) the partial derivatives $U_{s_{k}}\left(e^{i \theta}, s, U_{0}\right)$ and $U_{U_{0}^{m}}\left(e^{i \theta}, s, U_{0}\right)$ exist for every $e^{i \theta} \in \partial \Delta$;
(ii) the maps $e^{i \theta} \mapsto U_{s_{k}}\left(e^{i \theta}, s, U_{0}\right)$ and $e^{i \theta} \mapsto U_{U_{0}^{m}}\left(e^{i \theta}, s, U_{0}\right)$ are $\mathscr{C}^{0, \alpha / 2}$ on $\partial \Delta$ and satisfy the uniform inequality

$$
\left\|U_{s_{k}}\left(\cdot, s, U_{0}\right)\right\|_{\mathscr{C}^{0, \alpha / 2}(\partial \Delta)} \leqslant \mathrm{K} \quad \text { and } \quad\left\|U_{U_{0}^{m}}\left(\cdot, s, U_{0}\right)\right\|_{\mathscr{C}^{0, \alpha / 2}(\partial \Delta)} \leqslant \mathrm{K}
$$

Proof. Since $U$ is almost everywhere differentiable with respect to all its arguments, there exist a subset $\mathfrak{F} \subset \square_{\sigma_{1}}^{b} \times \square_{\rho_{1} / 16}^{d} \times \partial \Delta$ having full measure, namely its complement has null measure, such that for every $\left(e^{i \theta}, s, U_{0}\right) \in$ $\mathfrak{F}$, all partial derivatives $U_{\theta}, U_{s_{k}}, U_{U_{0}^{m}}$ exist at $\left(e^{i \theta}, s, U_{0}\right)$. Since $\mathfrak{F}$ has full measure, there exists a null measure subset $\mathfrak{N} \subset \square_{\sigma_{1}}^{b} \times \square_{\rho_{1} / 16}^{d}$ such that for every $\left(s, U_{0}\right) \notin \mathfrak{N}$, the slice

$$
\mathfrak{F}_{s, U_{0}}:=\left(\partial \Delta \times\{s\} \times\left\{U_{0}\right\}\right) \cap \mathfrak{F}
$$

is a subset of $\partial \Delta$ having full measure, so that $U_{\theta}, U_{s_{k}}, U_{U_{0}^{m}}$ exist at $\left(e^{i \theta}, s, U_{0}\right)$ with $e^{i \theta} \in \mathfrak{F}_{s, U_{0}}$.

Fix $\left(s, U_{0}\right) \notin \mathfrak{N}$. We will treat only the partial derivatives with respect to the $s_{k}$, arguments being similar for the $U_{U_{0}^{m}}$. In the end of the proof of Lemma 3.17, we have shown:

$$
\left\|U^{2}-U^{1}\right\|_{0, \alpha / 2} \leqslant \mathrm{~K}\left|s^{2}-s^{1}\right|
$$

for some (not the same) quantity $\mathrm{K}>0$. Fix $k \in\{1,2, \ldots, b\}$, take $s^{2}$ and $s^{1}$ with $s_{k}^{2} \neq s_{k}^{1}$ but $s_{l}^{2}=s_{l}^{1}$ for $l \neq k$. The inequality above says that for every $e^{i \theta}$, $e^{i \theta^{\prime}}, e^{i \theta^{\prime \prime}} \in \partial \Delta$ with $0<\left|\theta^{\prime \prime}-\theta^{\prime}\right| \leqslant \pi$, we have

$$
\begin{aligned}
& \left|\frac{U\left(e^{i \theta}, s^{2}, U_{0}\right)-U\left(e^{i \theta}, s^{1}, U_{0}\right)}{s_{k}^{2}-s_{k}^{1}}\right|+ \\
& \quad+\left\lvert\, \frac{U\left(e^{i \theta^{\prime \prime}}, s^{2}, U_{0}\right)-U\left(e^{i \theta^{\prime \prime}}, s^{1}, U_{0}\right)}{s_{k}^{2}-s_{k}^{1}}-\right. \\
& \quad-\frac{U\left(e^{i \theta^{\prime}}, s^{2}, U_{0}\right)-U\left(e^{i \theta^{\prime}}, s^{1}, U_{0}\right)}{s_{k}^{2}-s_{k}^{1}}\left|/\left|\theta^{\prime \prime}-\theta^{\prime}\right|^{\alpha / 2} \leqslant \mathrm{~K} .\right.
\end{aligned}
$$

Assume $e^{i \theta}, e^{i \theta^{\prime}}, e^{i \theta^{\prime \prime}} \in \mathfrak{F}_{s^{1}, U_{0}}$, let $s_{k}^{2} \rightarrow s_{k}^{1}$ (the limits of the quotients above exist) and rename $s^{1}$ by $s$ :

$$
\left|U_{s_{k}}\left(e^{i \theta}, s, U_{0}\right)\right|+\frac{\left|U_{s_{k}}\left(e^{i \theta^{\prime \prime}}, s, U_{0}\right)-U_{s_{k}}\left(e^{i \theta^{\prime}}, s, U_{0}\right)\right|}{\left|\theta^{\prime \prime}-\theta^{\prime}\right| \alpha / 2} \leqslant \mathrm{~K} .
$$

This inequality says that $U_{s_{k}}\left(\cdot, s, U_{0}\right)$ is $\mathscr{C}^{0, \alpha / 2}$ almost everywhere on $\partial \Delta$. The next extension lemma concludes the proof.

Lemma 3.19. Let $n \geqslant 1$, let $\mathrm{x} \in \mathbb{R}^{n}$, let $m \geqslant 1$, let $\mathrm{y} \in \mathbb{R}^{m}$, let $\rho>0$, let $\sigma>0$, and let $f=f(\mathrm{x}, \mathrm{y})$ be a function defined (only) in a full-measure subset $\mathfrak{F} \subset \square_{\rho}^{n} \times \square_{\sigma}^{m}$, so that there exists a null-measure subset $\mathfrak{N} \subset \square_{\sigma}^{m}$ with the property that for every y $\notin \mathfrak{N}$, the slice $\mathfrak{F}_{\mathrm{y}}:=\left(\square_{\rho}^{n} \times\{\mathrm{y}\}\right) \times \mathfrak{F}$ has full measure in $\square_{\rho}^{n}$. Assume that for every $\mathrm{y} \notin \mathfrak{N}$, every $\mathrm{x}, \mathrm{x}^{\prime}, \mathrm{x}^{\prime \prime} \in \mathfrak{F}_{\mathrm{y}}$, we have

$$
|f(\mathrm{x}, \mathrm{y})|+\frac{\left|f\left(\mathrm{x}^{\prime \prime}, \mathrm{y}\right)-f\left(\mathrm{x}^{\prime}, \mathrm{y}\right)\right|}{\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right|^{\beta}} \leqslant \mathrm{K}
$$

for some $\beta$ with $0<\beta \leq 1$ and some quantity $\mathrm{K}>0$. Then for every y $\notin \mathfrak{N}$, the function $\mathrm{x} \mapsto f(\mathrm{x}, \mathrm{y})$ admits a unique continuous prolongation to $\square_{\rho}^{n}$, still denoted by $f(\cdot, \mathrm{y})$, that is $\mathscr{C}^{0, \beta}$ in $\square_{\rho}^{n}$ with

$$
\|f(\cdot, \mathrm{y})\|_{\mathscr{C}^{0, \beta}\left(\square_{\rho}^{n}\right)} \leqslant \mathrm{K} .
$$

Thus, for every $\left(s, U_{0}\right) \notin \mathfrak{N}$, the partial derivatives $U_{s_{k}}, U_{U_{0}^{m}}$ belong to $\mathscr{C}^{0, \alpha / 2}\left(\partial \Delta, \mathbb{R}^{d}\right)$. Since the operator $\mathrm{T}_{1}$ is linear and bounded in $\mathscr{C}^{0, \alpha / 2}$, we may differentiate the $d$ scalar Bishop-type equations of Theorem 3.7 with respect to $\theta$, to $s_{k}, k=1, \ldots, b$ and to $U_{0}^{m}, m=1, \ldots, d$, which yields, for
$j=1, \ldots, d$ :
(3.20)

$$
\left\{\begin{aligned}
U_{\theta}^{j}\left(e^{i \theta}\right) & =-\mathrm{T}_{1}\left[\sum_{1 \leqslant l \leqslant d} \Phi_{u_{l}}^{j}(U(\cdot), \cdot, s) U_{\theta}^{l}(\cdot)+\Phi_{\theta}^{j}(U(\cdot), \cdot, s)\right]\left(e^{i \theta}\right), \\
U_{s_{k}}^{j}\left(e^{i \theta}\right) & =-\mathrm{T}_{1}\left[\sum_{1 \leqslant l \leqslant d} \Phi_{u_{l}}^{j}(U(\cdot), \cdot, s) U_{s_{k}}^{l}(\cdot)+\Phi_{s_{k}}^{j}(U(\cdot), \cdot, s)\right]\left(e^{i \theta}\right), \\
U_{U_{0}^{m}}^{j}\left(e^{i \theta}\right) & =\delta_{m}^{j}-\mathrm{T}_{1}\left[\sum_{1 \leqslant l \leqslant d} \Phi_{u_{l}}^{j}(U(\cdot), \cdot, s) U_{U_{0}^{m}}^{l}(\cdot)\right]\left(e^{i \theta}\right),
\end{aligned}\right.
$$

for every $e^{i \theta} \in \partial \Delta$, provided $\left(s, U_{0}\right) \notin \mathfrak{N}$. In the first line, we noticed that $\left(\mathrm{T}_{1} V\right)_{\theta}=\mathrm{T}\left(V_{\theta}\right)$. We observe that as $U$ is Lipschitzian, as $\Phi \in \mathscr{C}^{\kappa, \alpha}$ and as $\kappa \geqslant 1$, the composite functions $\Phi_{u_{l}}^{j}, \Phi_{\theta}^{j}, \Phi_{s_{i}}^{j}\left(U\left(e^{i \theta}, s, U_{0}\right), e^{i \theta}, s\right)$ are of class $\mathscr{C}^{0, \alpha}$ with respect to all variables.

We notice that in each of the three linear systems of Bishop-type equations (3.20) above, $t:=\left(s, U_{0}\right)$ is a parameter and there appears the same matrix coefficients:

$$
p_{l}^{j}\left(e^{i \theta}, t\right):=\Phi_{u_{l}}^{j}\left(U\left(e^{i \theta}, s, U_{0}\right), e^{i \theta}, s\right)
$$

for $j, l=1, \ldots, d$. For any $\beta$ with $0<\beta \leqslant \alpha$, in order to be coherent with the definition of $\left\|\left.\Phi_{u}\right|_{s}\right\|_{\widehat{0, \beta}}$ given after Lemma 3.9, we set:

$$
\left\|\left.p\right|_{t}\right\|_{\widehat{0, \beta}}:=\max _{1 \leqslant j \leqslant d}\left(\sum_{1 \leqslant l \leqslant d}\left\|\left.p_{l}^{j}\right|_{t}\right\|_{\widehat{0, \beta}}\right) .
$$

We also set:

$$
\|p\|_{0,0}:=\max _{1 \leqslant j \leqslant d}\left(\sum_{1 \leqslant l \leqslant d}\left\|p_{l}^{j}\right\|_{0,0}\right)=\left\|\Phi_{u}\right\|_{0,0} .
$$

With these definitions, it is easy to check the inequality:

$$
\left\|\left.p\right|_{t}\right\|_{0, \beta} \leqslant\left\|\left.\Phi_{u}\right|_{s}\right\|_{0, \beta}\left[1+\left(\|U\|_{1,0}\right)^{\beta}\right] .
$$

As $\|U\|_{1,0} \leqslant \rho_{1} / 4<1$, with $\beta:=\alpha$, we deduce:

$$
\left\|\left.p\right|_{t}\right\|_{0, \alpha} \leqslant 2\left\|\left.\Phi_{u}\right|_{s}\right\|_{0, \alpha} \leqslant 2\left\|\left.\Phi\right|_{s}\right\|_{1, \alpha} .
$$

Taking $\sup _{s}$ and then $\sup _{U_{0}}$, adding 1 , squaring and inverting:

$$
\left[1+\sup _{s}\left\|\left.\Phi\right|_{s}\right\|_{1, \alpha}\right]^{-2} \leqslant 4\left[1+\sup _{s, U_{0}}\left\|\left.p\right|_{t}\right\|_{0, \alpha}\right]^{-2}
$$

Consequently, the main assumption of the next Proposition 3.21, according to which:

$$
\|p\|_{0,0} \leqslant C^{2} \alpha^{2}\left[1+\sup _{t}\left\|\left.p\right|_{t}\right\|_{0, \alpha}\right]^{-2}
$$

for some positive absolute constant $C<1$, is superseded by one of the main assumptions, made in the theorem, according to which:

$$
\left\|\Phi_{u}\right\|_{0,0} \leqslant C^{2} \alpha^{2}\left[1+\sup _{s}\left\|\left.\Phi\right|_{s}\right\|_{0, \alpha}\right]^{-2}
$$

for some (a priori distinct) positive absolute constant $C<1$.
The following proposition applies to the three systems (3.20) and suffices to conclude the proof of Theorem 3.7 in the case $\kappa=1$. The case $\kappa \geqslant 2$ shall be discussed afterwards.

Proposition 3.21. ([Tu1996], [*]) Let $c \in \mathbb{N}$ with $c \geqslant 1$, let $\tau_{1}=$ $\left(\tau_{1,1}, \ldots, \tau_{1, c}\right) \in \mathbb{R}^{c}$ with $0<\tau_{1, i} \leqslant \infty, i=1, \ldots, c$, and denote by $\square_{\tau_{1}}^{c}$ the polycube $\left\{t \in \mathbb{R}^{c}:\left|t_{i}\right|<\tau_{1, i}\right\}$. Consider vector-valued and matrix-valued Hölder data:

$$
\begin{aligned}
& q=\left(q^{j}\left(e^{i \theta}, t\right)\right)^{1 \leqslant j \leqslant d} \in \mathscr{C}^{0, \alpha}\left(\partial \Delta \times \square_{\tau_{1}}^{c}, \mathbb{R}^{d}\right), \\
& p=\left(p_{l}^{j}\left(e^{i \theta}, t\right)\right)_{1 \leqslant l \leqslant d}^{1 \leqslant j \leqslant d} \in \mathscr{C}^{0, \alpha}\left(\partial \Delta \times \square_{\tau_{1}}^{c}, \mathbb{R}^{d \times d}\right),
\end{aligned}
$$

with $0<\alpha<1$. Suppose that a bounded measurable map:

$$
u=\left(u^{j}\left(e^{i \theta}, t\right)\right)^{1 \leqslant j \leqslant d} \in L^{\infty}\left(\partial \Delta \times \square_{\tau_{1}}^{c}, \mathbb{R}^{d}\right)
$$

is $\mathscr{C}^{0, \alpha / 2}$ on $\partial \Delta$ for every fixed $t$ not belonging to some null-measure subset $\mathfrak{N}$ of $\square \square_{\tau_{1}}^{c}$ and suppose that it satisfies the system of linear Bishop-type equations in $\mathscr{C}^{0, \alpha / 2}\left(\partial \Delta, \mathbb{R}^{d}\right)$ :

$$
\begin{equation*}
u^{j}=\mathrm{T}_{*}\left(\sum_{1 \leqslant l \leqslant d} p_{l}^{j} u^{l}\right)+q^{j}, \tag{3.22}
\end{equation*}
$$

for $j=1, \ldots, d$, where $\mathrm{T}_{*}=\mathrm{T}$ or $\mathrm{T}_{*}=\mathrm{T}_{1}$. Assume that the norm of the matrix p satisfies:

$$
\|p\|_{0,0}=\max _{1 \leqslant j \leqslant d}\left(\sum_{1 \leqslant l \leqslant d}\left\|p_{l}^{j}\right\|_{0,0}\right) \leqslant c_{4},
$$

for some small positive constant $\mathrm{c}_{4}<1 / 16$ such that

$$
\begin{equation*}
\mathrm{c}_{4} \leqslant C^{2} \alpha^{2}\left[1+\sup _{|t|<\tau_{1}}\left\|\left.p\right|_{t}\right\|_{0, \alpha}\right]^{-2} \tag{3.23}
\end{equation*}
$$

where $C<1$ is a positive absolute constant. Then, after a correction of $u$ on $\mathfrak{N}$ :
(i) on its full domain of definition $\partial \Delta \times \square_{\tau_{1}}^{c}$, the corrected map $u\left(e^{i \theta}, t\right)$ is $\mathscr{C}^{0, \alpha-0}=\bigcap_{0<\beta<\alpha} \mathscr{C}^{0, \beta}$ and furthermore:
(ii) for every fixed $t$, the map $e^{i \theta} \mapsto u\left(e^{i \theta}, t\right)$ is $\mathscr{C}^{0, \alpha}$ on $\partial \Delta$.

In general, the Hilbert transform T does not preserve $\mathscr{C}^{0, \alpha}$ smoothness with respect to parameters, so that the above solution $u=u\left(e^{i \theta}, t\right)$ is not better than $\mathscr{C}^{0, \alpha-0}$.

Example 3.24. ([Tu 1996]) If $s \in \mathbb{R}$ with $|s|<1$ is a parameter, the function:

$$
\begin{aligned}
u\left(e^{i \theta}, s\right) & :=|s|^{\alpha} \text { if }-\pi \leqslant \theta \leqslant-|s|^{1 / 2}, \\
& :=\theta^{2 \alpha} \quad \text { if }-|s|^{1 / 2} \leqslant \theta \leqslant 0, \\
& :=\theta^{\alpha} \quad \text { if } 0 \leqslant \theta \leqslant|s|, \\
& :=|s|^{\alpha} \text { if }|s| \leqslant \theta \leqslant \pi,
\end{aligned}
$$

is $2 \pi$-periodic with respect to $\theta$ and $\mathscr{C}^{0, \alpha}$ with respect to $\left(e^{i \theta}, s\right)$. As the function $\cot (t / 2)-2 / t$ is $\mathscr{C}^{0,0}$ on $[-\pi, \pi]$, the regularity properties of the singular integral $\mathrm{T} u\left(e^{i \theta}\right)=$ p.v. $\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{u\left(e^{i(\theta-t)}\right)}{\tan (t / 2)} d t$ are the same as those of:

$$
\widetilde{\mathrm{T}} u\left(e^{i \theta}\right):=\text { p.v. } \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{u\left(e^{i(\theta-t)}\right)}{t} d t .
$$

However $\widetilde{\mathrm{T}} u(1)$ involves the term $|s|^{\alpha} \log |s|$ which is $\mathscr{C}^{0, \alpha-0}$ but not $\mathscr{C}^{0, \alpha}$ :

$$
\begin{aligned}
\widetilde{\mathrm{T}} u(1) & =\frac{1}{\pi}\left(\int_{-|s|^{1 / 2}}^{-|s|} \frac{|s|^{\alpha}}{t} d t+\int_{-|s|}^{0} \frac{(-t)^{\alpha}}{t} d t+\int_{0}^{|s|^{1 / 2}} \frac{t^{2 \alpha}}{t} d t\right) \\
& =\frac{1}{2 \pi}\left(|s|^{\alpha} \log |s|-\frac{|s|^{\alpha}}{\alpha}\right) .
\end{aligned}
$$

Proof of the proposition. We shall drop the indices, writing $u, p$ and $q$, without arguments. Assume that $t \notin \mathfrak{N}$. For future majorations, it is necessary to have $\mathrm{P}_{0} u=0$. If this is not the case, we set $u^{\prime}:=u-\mathrm{P}_{0} u$ in order that $\mathrm{P}_{0} u^{\prime}=0$. Since $u$ satisfies either $u=\mathrm{T}(p u)+q$ or $u=\mathrm{T}(p u)-\mathrm{T}[p u](1)+q$, it follows that $u^{\prime}$ satisfies similar equations: either $u^{\prime}=\mathrm{T}\left(p u^{\prime}\right)+q^{\prime}$, with $q^{\prime}:=q-\mathrm{P}_{0} u$ or $u^{\prime}=\mathrm{T}\left(p u^{\prime}\right)+q^{\prime}$, with $q^{\prime}:=q-\mathrm{P}_{0} u-\mathrm{T}(p u)(1)$. Notice that $p$ is unchanged. It then suffices to establish the improvements of smoothness (i) and (ii) for $u^{\prime}$. Equivalently, we may assume that $\widehat{u}_{0}=\mathrm{P}_{0} u=0$ in the proposition.

For $t \notin \mathfrak{N}$, the function $\phi$ is $\mathscr{C}^{0, \alpha / 2}$ on $\partial \Delta$. Applying T either to the equation $u=\mathrm{T}(p u)+q$ or to the equation $u=\mathrm{T}(p u)-\mathrm{T}[p u](1)+q$, we get the same equation for both:

$$
\mathrm{T} u=-p u+\mathrm{P}_{0}(p u)+\mathrm{T} q .
$$

As $\widehat{u}_{0}=0$, we may write $u\left(e^{i \theta}\right)=\sum_{k<0} \widehat{u}_{k} e^{i k \theta}+\sum_{k>0} \widehat{u}_{k} e^{i k \theta}=: \bar{\phi}+\phi$, where $\phi$ extends holomorphically to $\Delta$. In fact, $\phi$ is determined up to a imaginary constant $i A$ and we choose $A:=-\mathrm{P}_{0}(p u) / 2$, so that:

$$
\begin{equation*}
\phi=\frac{u+i \mathrm{~T} u-i \mathrm{P}_{0}(p u)}{2} \tag{3.25}
\end{equation*}
$$

Equivalently:

$$
\left\{\begin{aligned}
u & =\phi+\bar{\phi} \\
\mathrm{T} u & =\mathrm{P}_{0}(p u)-i(\phi-\bar{\phi}) .
\end{aligned}\right.
$$

Substituting, we rewrite (3.25) as:

$$
-i(\phi-\bar{\phi})=-p(\phi+\bar{\phi})+\mathrm{T} q
$$

or under the more convenient form:

$$
\phi=\bar{\phi}+P \bar{\phi}+Q,
$$

where the $d \times d$-matrix $P:=-2 i p(I+i p)^{-1}$ and the $d$-vector $Q:=$ $i(I+i p)^{-1} \mathrm{~T} q$ both belong to $\mathscr{C}^{0, \alpha}$.

First of all, we establish (ii) before any correction of $u$.
Lemma 3.26. For $t \notin \mathfrak{N}$, the map $e^{i \theta} \mapsto u\left(e^{i \theta}, t\right)$ is $\mathscr{C}^{0, \alpha}$ on $\partial \Delta$.
Proof. By assumption, the map $e^{i \theta} \mapsto \phi\left(e^{i \theta}, t\right)$ is $\mathscr{C}^{0, \alpha / 2}$ on $\partial \Delta$. Since $\mathrm{C}^{+}$is bounded in $\mathscr{C}^{0, \alpha / 2}$, we may apply $\mathrm{C}^{+}$to the vectorial equation $\phi=$ $\bar{\phi}+P \bar{\phi}+Q \bar{\phi}$, noticing that $\mathrm{C}^{+}(\phi)=\phi$ and that $\mathrm{C}^{+}(\bar{\phi})=\mathrm{P}_{0}(\bar{\phi})$, since, by construction, $\phi$ extends holomorphically to $\Delta$. We thus get:

$$
\begin{equation*}
\phi=\mathrm{P}_{0} \bar{\phi}+\mathrm{C}^{+}[P \bar{\phi}+Q] . \tag{3.27}
\end{equation*}
$$

Remind that $\mathrm{P}_{0} \psi=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \psi\left(e^{i \theta}\right) d \theta$, so that $\left\|\mathrm{P}_{0} \psi\right\|_{0,0} \leqslant\|\psi\|_{0,0}$. Notice ${ }^{8}$ that $\|\phi(\cdot, t)\|_{0,0}<\infty$ for $t \notin \mathfrak{N}$. Taking the $\mathscr{C}^{0, \alpha}$ norm to (3.27) and applying crucially Theorem 2.18 in its $\mathbb{R}^{d}$-valued version, as in [Tu1996], we get:

$$
\begin{aligned}
\|\phi\|_{0, \alpha} & \leqslant\left\|\mathrm{P}_{0} \bar{\phi}\right\|_{0,0}+\left\|\mathrm{C}^{+}[P \bar{\phi}+Q]\right\|_{0, \alpha} \\
& \leqslant\|\phi\|_{0,0}+\frac{C_{1}}{\alpha(1-\alpha)}\left(\|P\|_{0, \alpha}\|\phi\|_{0,0}+\|Q\|_{0, \alpha}\right) \\
& <\infty
\end{aligned}
$$

whence $\phi=\phi(\cdot, t)$ is $\mathscr{C}^{0, \alpha}$ on $\partial \Delta$, as claimed.

[^7]Next, we shall establish (i) before any correction of $u$. To this aim, let $t^{1}, t^{2} \notin \mathfrak{N}$ and simply denote by $u^{1}, u^{2}$, by $\phi^{1}, \phi^{2}$, by $P^{1}, P^{2}$ and by $Q^{1}, Q^{2}$ the functions on $\partial \Delta$ with these two values of $t$. In the theorem, to establish that $u$ is $\mathscr{C}^{0, \alpha-0}$, it is natural to show that $u$ is $\mathscr{C}^{0, \beta}$ for every $\beta<\alpha$ arbitrarily close to $\alpha$.

Lemma 3.28. For every $\beta$ with $\frac{\alpha}{2}<\beta<\alpha$, every two $t^{1}, t^{2} \notin \mathfrak{N}$, we have $\left\|u^{2}-u^{1}\right\|_{0,0} \leqslant 4 \mathrm{~K}_{\alpha, \beta}\left|t^{2}-t^{1}\right|^{\beta}$, for some positive quantity $\mathrm{K}_{\alpha, \beta}<\infty$.

We shall obtain $\mathrm{K}_{\alpha, \beta}$ involving a nonremovable factor $1 /(\alpha-\beta)$. This will confirm the optimality of $\mathscr{C}^{0, \alpha-0}$-smoothness of $u$ with respect to the parameter $t$.

Proof. Since $\mathrm{P}_{0} u=\mathrm{P}_{0}(\mathrm{~T} u)=0$, applying $\mathrm{P}_{0}$ to the conjugate of (3.25), we get $\mathrm{P}_{0} \bar{\phi}=\frac{i}{2} \mathrm{P}_{0}(p u)$, so that we may rewrite (3.27) under the form:

$$
\phi=\frac{i}{2} \mathrm{P}_{0}(p u)+\mathrm{C}^{+}[P \bar{\phi}+Q] .
$$

We may then organize the difference $\phi^{2}-\phi^{1}$ as follows:

$$
\begin{aligned}
\phi^{2}-\phi^{1}= & \frac{i}{2} \mathrm{P}_{0}\left(p^{2}\left(u^{2}-u^{1}\right)\right)+\frac{i}{2} \mathrm{P}_{0}\left(\left(p^{2}-p^{1}\right) u^{1}\right)+ \\
& +\mathrm{C}^{+}\left(\left(P^{2}-P^{1}\right) \bar{\phi}^{2}+Q^{2}-Q^{1}\right)+\mathrm{C}^{+}\left(P^{1}\left(\bar{\phi}^{2}-\bar{\phi}^{1}\right)\right) \\
= & E_{1}+E_{2}+E_{3}+E_{4}
\end{aligned}
$$

To majorate these four $E_{i}$ 's, we shall need various preliminaries.
Firstly, to majorate $E_{1}$, we first observe the elementary inequality:

$$
\begin{equation*}
\left\|u^{2}-u^{1}\right\|_{0,0} \leqslant 2\left\|\phi^{2}-\phi^{1}\right\|_{0,0} . \tag{3.29}
\end{equation*}
$$

Also, we notice that:

$$
\left\|p^{2}\right\|_{0,0}=\left\|p\left(\cdot, t^{2}\right)\right\|_{\mathscr{C ^ { 0 , 0 } ( \partial \Delta )}} \leqslant\|p\|_{0,0} \leqslant \mathrm{c}_{4} .
$$

The majoration of $E_{1}$ is then easy:

$$
\left\|E_{1}\right\|_{0,0} \leqslant \frac{1}{2}\left\|p^{2}\right\|_{0,0}\left\|u^{2}-u^{1}\right\|_{0,0} \leqslant 4 \mathbf{c}_{4}\left\|\phi^{2}-\phi^{1}\right\|_{0,0}
$$

Secondly, to majorate $E_{2}$, let again $\beta$ with $\frac{\alpha}{2}<\beta<\alpha$. Using the inequality $\|p\|_{0, \beta} \leqslant 3\|p\|_{0, \alpha}$ proved in the beginning of Section 1, we may majorate $E_{2}$ :

$$
\left\|E_{2}\right\|_{0,0} \leqslant \frac{1}{2}\left\|u^{1}\right\|_{0,0}\left\|p^{2}-p^{1}\right\|_{0,0} \leqslant \frac{3}{2}\left\|u^{1}\right\|_{0,0}\|p\|_{0, \alpha}\left|t^{2}-t^{1}\right|^{\beta}
$$

Thirdly, to majorate $E_{3}$, we need:

Lemma 3.30. Let $f=f(\mathrm{x}, \mathrm{y})$ be a $\mathscr{C}^{0, \alpha}$ function, defined in the product open cube $\square_{\rho}^{n} \times \square_{\rho}^{m}$, where $0<\alpha<1$ and $\rho>0$. Let $\beta$ with $0<\beta<\alpha$. For $\mathrm{x}^{\prime}, \mathrm{x}^{\prime \prime} \in \square_{\rho}^{n}$ arbitrary, the $\mathscr{C}^{0, \alpha-\beta}$-norm of the function $\square_{\rho}^{m} \ni \mathrm{y} \longmapsto$ $f\left(\mathrm{x}^{\prime \prime}, \mathrm{y}\right)-f\left(\mathrm{x}^{\prime}, \mathrm{y}\right) \in \mathbb{R}$ satisfies:

$$
\left\|f\left(x^{\prime \prime}, \cdot\right)-f\left(x^{\prime}, \cdot\right)\right\|_{0, \alpha-\beta} \leqslant 4\|f\|_{0, \alpha}\left|x^{\prime \prime}-x^{\prime}\right|^{\beta} .
$$

Again, assume $\frac{\alpha}{2}<\beta<\alpha$. Thanks to Theorem 2.18 and to the lemma above, we may majorate $E_{3}$ :

$$
\begin{aligned}
\left\|E_{3}\right\|_{0,0} \leqslant\left\|E_{3}\right\|_{0, \alpha-\beta} & \leqslant \frac{C_{1}}{\alpha-\beta}\left(\left\|P^{2}-P^{1}\right\|_{0, \alpha-\beta}\left\|\phi^{2}\right\|_{0,0}+\left\|Q^{2}-Q^{1}\right\|_{0, \alpha-\beta}\right) \\
& \leqslant \frac{C_{2}}{\alpha-\beta}\left(\|P\|_{0, \alpha}\left\|\phi^{2}\right\|_{0,0}+\|Q\|_{0, \alpha}\right)\left|t^{2}-t^{1}\right|^{\beta}
\end{aligned}
$$

Fourthly, to majorate $E_{4}$, we need a statement whose proof is similar to that of Lemma 3.10.

Lemma 3.31. As $\|p\|_{0,0} \leqslant \mathrm{c}_{4}$, then independently of $t \notin \mathfrak{N}$, we have:

$$
\left\|p_{\mid t}\right\|_{0, \alpha / 2} \leqslant \mathrm{c}_{4}+\mathrm{c}_{4}^{1 / 2}\left[2+\sup _{t}\left\|\left.p\right|_{t}\right\|_{0, \alpha}\right]
$$

Reminding the main assumption $\mathrm{c}_{4} \leqslant C^{2} \alpha^{2}\left[1+\sup _{t}\left\|\left.p\right|_{t}\right\|_{0, \alpha}\right]^{-2}$, whence $\mathrm{c}_{4} \leqslant C \alpha$, we deduce:

$$
\left\|p_{\mid t}\right\|_{0, \alpha / 2} \leqslant 3 C \alpha
$$

Choosing then $C<1 / 6$, we may insure that $\left\|p_{\mid t}\right\|_{0, \alpha / 2} \leqslant 1 / 2$ for every fixed $t$. It follows in particular that we may develope $P=-2 i p \sum_{k \in \mathbb{N}}(-i p)^{k}$ and deduce the norm inequality:

$$
\left\|P_{\mid t}\right\|_{0, \alpha / 2} \leqslant \frac{2\left\|p_{\mid t}\right\|_{0, \alpha / 2}}{1-\left\|p_{\mid t}\right\|_{0, \alpha / 2}} \leqslant 4\left\|p_{\mid t}\right\|_{0, \alpha / 2}
$$

valid for every fixed $t$. Then again thanks to Theorem 2.18 and thanks to the previous inequalities:

$$
\begin{aligned}
\left\|E_{4}\right\|_{0,0} \leqslant\left\|E_{4}\right\|_{0, \alpha / 2} & \leqslant C_{1} \alpha^{-1}\left\|P^{1}\right\|_{0, \alpha / 2}\left\|\phi^{2}-\phi^{1}\right\|_{0,0} \\
& \leqslant C_{2} \alpha^{-1}\left\|\left.p\right|_{t^{1}}\right\|_{0, \alpha / 2}\left\|\phi^{2}-\phi^{1}\right\|_{0,0} \\
& \leqslant C_{3} C\left\|\phi^{2}-\phi^{1}\right\|_{0,0} \\
& \leqslant 4^{-1}\left\|\phi^{2}-\phi^{1}\right\|_{0,0},
\end{aligned}
$$

provided $C<\frac{1}{4 C_{3}}$. We then insert these four majorations in the inequality

$$
\left\|\phi^{2}-\phi^{1}\right\|_{0,0} \leqslant\left\|E_{1}\right\|_{0,0}+\left\|E_{2}\right\|_{0,0}+\left\|E_{3}\right\|_{0,0}+\left\|E_{4}\right\|_{0,0},
$$

and we get after reorganization:

$$
\left\|\phi^{2}-\phi^{1}\right\|_{0,0}\left(1-4 \mathrm{c}_{4}-4^{-1}\right) \leqslant \mathrm{K}_{\alpha, \beta}\left|t^{2}-t^{1}\right|^{\beta}
$$

for some quantity $\mathrm{K}_{\alpha, \beta}<\infty$ whose precise expression is:

$$
K_{\alpha, \beta}:=\frac{C_{2}}{\alpha-\beta}\left(\|P\|_{0, \alpha}\left\|\phi^{2}\right\|_{0,0}+\|Q\|_{0, \alpha}\right)+\frac{3}{2}\left\|u^{1}\right\|_{0,0}\|p\|_{0, \alpha} .
$$

It suffices then to remind that $\mathrm{c}_{4}<1 / 16$ in the assumptions of the theorem to insure the uniform Hölder condition:

$$
\left\|\phi^{2}-\phi^{1}\right\|_{0,0} \leqslant 2 \mathrm{~K}_{\alpha, \beta}\left|t^{2}-t^{1}\right|^{\beta}
$$

valid for $t^{1}, t^{2} \notin \mathfrak{N}$. From (3.29), we conclude that $\left\|u^{2}-u^{1}\right\|_{0,0} \leqslant$ $4 \mathrm{~K}_{\alpha, \beta}\left|t^{2}-t^{1}\right|^{\beta}$. The proof of Lemma 3.28 is complete.

Then the correction of $u$ is provided by the following statement.
Lemma 3.32. ([*]) Let $f=f(\mathrm{x}, \mathrm{y}): \square_{\rho}^{n} \times\left(\square_{\rho}^{m} \backslash \mathfrak{N}\right) \rightarrow \mathbb{R}$ be a measurable $L^{\infty}$ map defined only for y not belonging to some null-measure subset $\mathfrak{N} \subset$ $\square_{\rho}^{m}$ and let $\beta$ with $0<\beta<\alpha$. If the map $\mathrm{x} \mapsto f(\mathrm{x}, \mathrm{y})$ is $\mathscr{C}^{0, \beta}$ for every $\mathrm{y} \notin \mathfrak{N}$ and if there exists $\mathrm{K}<\infty$ such that:

$$
\sup _{\mathrm{x}}\left|f\left(\mathrm{x}, \mathrm{y}^{2}\right)-f\left(\mathrm{x}, \mathrm{y}^{1}\right)\right| \leqslant \mathrm{K}\left|\mathrm{y}^{2}-\mathrm{y}^{1}\right|^{\beta}
$$

for every two $\mathrm{y}^{1}, \mathrm{y}^{2} \notin \mathfrak{N}$, then $f$ may be extended as a $\mathscr{C}^{0, \beta}$ map in the full domain $\square_{\rho}^{n} \times \square_{\rho}^{m}$.

In sum, still denoting by $u$ the $\mathscr{C}^{0, \alpha-0}$ extension of $u$ through $\mathfrak{N}$, property (i) of the proposition is proved. To obtain that $u$ is $\mathscr{C}^{0, \alpha}$ with respect to $e^{i \theta}$, we re-apply the reasoning of Lemma 3.26 to this extension.

The proof of Proposition 3.21 is complete.
In conclusion, the functions $U_{\theta}^{j}, U_{s_{k}}^{j}$ and $U_{U_{0}^{m}}^{j}$ are $\mathscr{C}^{0, \alpha-0}$ with respect to $\left(e^{i \theta}, s, U_{0}\right)$ and $\mathscr{C}^{\alpha, 0}$ with respect to $e^{i \theta}$. Thus, the theorem is achieved if $\kappa=1$.

If $\kappa=2$, the composite functions $\Phi_{u_{l}}^{j}, \Phi_{\theta}^{j}, \Phi_{s_{i}}^{j}\left(U\left(e^{i \theta}, s, U_{0}\right), e^{i \theta}, s\right)$ are then of class $\mathscr{C}^{1, \alpha-0}$ with respect to $\left(e^{i \theta}, s, U_{0}\right)$ and of class $\mathscr{C}^{1, \alpha}$ with respect to $e^{i \theta}$. We then apply the next lemma to the three families of Bishop-type vector equations (3.20).
Lemma 3.33. Let $t \in \square_{\tau_{1}}^{c}$ be a parameter with $c \in \mathbb{N}, 0<\tau_{1, i} \leqslant \infty$, $i=1, \ldots, c$, and consider vector-valued and matrix-valued Hölder data $q=\left(q^{j}\left(e^{i \theta}, t\right)\right)^{1 \leqslant j \leqslant d}$ and $p=\left(p_{l}^{j}\left(e^{i \theta}, t\right)\right)_{1 \leqslant l \leqslant d}^{1 \leqslant l}$ that are $\mathscr{C}^{1, \alpha-0}$ with respect to $\left(e^{i \theta}, t\right)$ and $\mathscr{C}^{1, \alpha}$ with respect to $e^{i \theta}$. Suppose that a given map $u=\left(u^{j}\left(e^{i \theta}, t\right)\right)^{1 \leqslant j \leqslant d}$ which is $\mathscr{C}^{0, \alpha-0}$ with respect to $\left(e^{i \theta}, t\right)$ and $\mathscr{C}^{0, \alpha}$ with
respect to $e^{i \theta}$ satisfies the linear Bishop-type equation $u=\mathrm{T}_{*}(p u)+q$, where $\mathrm{T}_{*}=\mathrm{T}$ or $\mathrm{T}_{*}=\mathrm{T}_{1}$. Assume that the norm of the matrix $p$ satisfies the same inequality as in the proposition: $\|p\|_{0,0} \leqslant \mathrm{c}_{4}$, for some small positive constant $\mathrm{c}_{4} \leqslant C^{2} \alpha^{2}\left[1+\sup _{t}\left\|\left.p\right|_{t}\right\|_{0, \alpha}\right]^{-2}$, where $0<C<1$ is some absolute constant. Then $u$ is $\mathscr{C}^{1, \alpha}$ with respect to $e^{i \theta}$ and satisfies the Lipschitz condition

$$
\left\|u\left(\cdot, t^{2}\right)-u\left(\cdot, t^{1}\right)\right\|_{0, \alpha / 2} \leqslant \mathrm{~K}\left|t^{2}-t^{1}\right|,
$$

for some quantity $\mathrm{K}<\infty$. Furthermore, there exists a null-measure subset $\mathfrak{N} \subset \square_{\tau_{1}}^{c} \times \square_{\rho_{1} / 16}^{d}$ such that at every $t \notin \mathfrak{N}$, for every $l=1, \ldots, c$ :
(i) the partial derivative $u_{t_{l}}\left(e^{i \theta}, t\right)$ exists for every $e^{i \theta} \in \partial \Delta$;
(ii) the map $e^{i \theta} \mapsto u_{t_{l}}\left(e^{i \theta}, t\right)$ is $\mathscr{C}^{0, \alpha / 2}$ on $\partial \Delta$.

Proof. The fact that $u$ is $\mathscr{C}^{1, \alpha}$ with respect to $e^{i \theta}$ is proved as in Lemma 3.26. For the Lipschitz condition, the reasoning is simpler than Lemma 3.17, due to the linearity of $u=\mathrm{T}_{*}(p u)+q$. Indeed, for two parameters $t^{1}, t^{2} \in \square_{\tau_{1}}^{c}$, if we take the $\mathscr{C}^{0, \alpha / 2}$-norm of the difference:

$$
u^{2}-u^{1}=\mathbf{T}_{*}\left(p^{2}\left(u^{2}-u^{1}\right)\right)+\mathbf{T}_{*}\left(\left(p^{2}-p^{1}\right) u^{1}\right)+q^{2}-q^{1},
$$

we get:

$$
\left\|u^{2}-u^{1}\right\|_{0, \alpha / 2} \leqslant C_{1} \alpha^{-1}\left\|p^{2}\right\|_{0, \alpha / 2}\left\|u^{2}-u^{1}\right\|_{0, \alpha / 2}+\mathrm{K}\left|t^{2}-t^{1}\right|,
$$

and after substraction, taking account of Lemma 3.31:

$$
\left\|u^{2}-u^{1}\right\|_{0, \alpha / 2} \leqslant 2 \mathrm{~K}\left|t^{2}-t^{1}\right| .
$$

Then the sequel of the reasoning is already known.
So for $l=1, \ldots, c$, the partial derivatives $u_{t_{l}}$ exist almost everywhere and they satisfy:

$$
u_{t_{l}}=\mathrm{T}_{*}\left(p u_{t_{l}}\right)+q_{l},
$$

with the same matrix $p$, where $q_{l}:=\mathrm{T}_{*}\left(p_{t_{l}} u\right)+q_{t_{l}}$ is $\mathscr{C}^{0, \alpha-0}$ with respect to $\left(e^{i \theta}, t\right)$ and $\mathscr{C}^{0, \alpha}$ with respect to $e^{i \theta}$.

Lemma 3.34. Proposition 3.21 holds true if more generally, $q$ is only assumed to be $\mathscr{C}^{0, \alpha-0}$ with respect to $\left(e^{i \theta}, t\right)$ and $\mathscr{C}^{0, \alpha}$ with respect to $e^{i \theta}$, with the same conclusion.
(It suffices only to inspect the majoration of $E_{3}$.) Consequently, with $u=U_{\theta}, U_{s_{k}}, U_{U_{0}^{m}}$ in the three equations (3.20), we have verified that the partial derivatives $u_{\theta}, u_{s_{k}}$ and $u_{U_{0}^{m}}$ exist everywhere, are $\mathscr{C}^{0, \alpha-0}$ with respect to $\left(e^{i \theta}, s, U_{0}\right)$ and are $\mathscr{C}^{0, \alpha}$ with respect to $e^{i \theta}$. In summary, the theorem is achieved if $\kappa=2$.

Needless to say, we have clarified how to cover the case of general $\kappa \geqslant$ 2 by pure logical induction. In conclusion, the proof of Theorem 3.7 is complete.

Open problem 3.35. Solve parametrized Bishop-type equations in Sobolev spaces.

## §4. Appendix: proofs of some lemmas

4.1. Proofs of Lemmas 3.9 and 3.10. Let $x \in \mathbb{R}^{n}, n \geqslant 1$, with $|x|<\rho$, where $0<\rho \leqslant \infty$. Assuming $\|f\|_{0,0} \leqslant \mathrm{c}$, we estimate:

$$
\|f\|_{\widehat{0, \alpha / 2}} \leqslant \sup _{\mathrm{x}^{\prime \prime} \neq \mathrm{x}^{\prime}} \frac{\left|f\left(\mathrm{x}^{\prime \prime}\right)-f\left(\mathrm{x}^{\prime}\right)\right|}{\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right|^{\alpha / 2}}=\max (\mathrm{A}, \mathrm{~B}) \leqslant \mathrm{A}+\mathrm{B}
$$

where $A:=\sup _{0<\left|x^{\prime \prime}-x^{\prime}\right|<c^{1 / \alpha}}$ and $B:=\sup _{\left|x^{\prime \prime}-x^{\prime}\right| \geqslant c^{1 / \alpha}}$ satisfy:

$$
\begin{aligned}
& \mathrm{A}=\sup _{0<\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right|<\mathrm{c}^{1 / \alpha}}\left(\frac{\left|f\left(\mathrm{x}^{\prime \prime}\right)-f\left(\mathrm{x}^{\prime}\right)\right|}{\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right|^{\alpha}}\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right|^{\alpha / 2}\right) \leqslant\|f\|_{\widehat{0, \alpha}} \mathrm{c}^{1 / 2}, \\
& \mathrm{~B}=\sup _{\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right| \geqslant \mathrm{c}^{1 / \alpha}} \frac{\left|f\left(\mathrm{x}^{\prime \prime}\right)-f\left(\mathrm{x}^{\prime}\right)\right|}{\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right|^{\alpha / 2}} \leqslant \frac{2\|f\|_{0,0}}{\mathrm{c}^{1 / 2}} \leqslant 2 \mathrm{c}^{1 / 2} .
\end{aligned}
$$

Lemma 3.9 is proved.
Applying this to $x=(u, \theta)$, we deduce:

$$
\begin{aligned}
& \left\|\left.\Phi_{u}\right|_{s}\right\|_{\widehat{0, \alpha / 2}} \leqslant c_{2}^{1 / 2}\left[2+\left\|\left.\Phi_{u}\right|_{s}\right\|_{\widehat{0, \alpha}}\right] \leqslant c_{2}^{1 / 2}\left[2+\left\|\left.\Phi\right|_{s}\right\|_{1, \alpha}\right], \\
& \left\|\left.\Phi_{\theta}\right|_{s}\right\|_{\widehat{0, \alpha / 2}} \leqslant c_{3}^{1 / 2}\left[2+\left\|\left.\Phi_{\theta}\right|_{s}\right\|_{\widehat{0, \alpha}}\right] \leqslant c_{3}^{1 / 2}\left[2+\left\|\left.\Phi\right|_{s}\right\|_{1, \alpha}\right] .
\end{aligned}
$$

Consequently:

$$
\begin{aligned}
\left\|\left.\Phi\right|_{s}\right\|_{1, \alpha / 2} & =\left\|\left.\Phi\right|_{s}\right\|_{0,0}+\left\|\left.\Phi_{u}\right|_{s}\right\|_{0,0}+\left\|\left.\Phi_{\theta}\right|_{s}\right\|_{0,0}+\left\|\left.\Phi_{u}\right|_{s}\right\|_{\widehat{0, \alpha / 2}}+\left\|\left.\Phi_{\theta}\right|_{s}\right\|_{\widehat{0, \alpha / 2}} \\
& \leqslant \mathrm{c}_{1}+\mathrm{c}_{2}+\mathrm{c}_{3}+\left(\mathrm{c}_{2}^{1 / 2}+\mathrm{c}_{3}^{1 / 2}\right)\left[2+\|\Phi\|_{1, \alpha}\right] .
\end{aligned}
$$

Lemma 3.10 is proved.
4.2. Proof of Lemma 3.11. We shall abbreviate $\sup _{0<\left|\theta^{\prime \prime}-\theta^{\prime}\right| \leqslant \pi}$ by $\sup _{\theta^{\prime \prime} \neq \theta^{\prime}}$. By definition:

$$
\begin{aligned}
& \|\Phi(U(\cdot), \cdot, s)\|_{\mathscr{C}^{1, \beta}(\partial \Delta)}= \\
& =\sup _{\theta}\left|\Phi\left(U\left(e^{i \theta}\right), e^{i \theta}, s\right)\right|+ \\
& +\sup _{\theta}\left|\sum_{1 \leqslant l \leqslant d} \Phi_{u_{l}}\left(U\left(e^{i \theta}\right), e^{i \theta}, s\right) U_{\theta}^{l}\left(e^{i \theta}\right)+\Phi_{\theta}\left(U\left(e^{i \theta}\right), e^{i \theta}, s\right)\right|+ \\
& +\sup _{\theta^{\prime \prime} \neq \theta^{\prime}} \mid \sum_{1 \leqslant l \leqslant d} \Phi_{u_{l}}\left(U\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime \prime}}, s\right) U_{\theta}^{l}\left(e^{i \theta^{\prime \prime}}\right)+\Phi_{\theta}\left(U\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime \prime}}, s\right)- \\
& \quad-\sum_{1 \leqslant l \leqslant d} \Phi_{u_{l}}\left(U\left(e^{i \theta^{\prime}}\right), e^{i \theta^{\prime}}, s\right) U_{\theta}^{l}\left(e^{i \theta^{\prime}}\right)-\Phi_{\theta}\left(U\left(e^{i \theta^{\prime}}\right), e^{i \theta^{\prime}}, s\right)| | \theta^{\prime \prime}-\left.\theta^{\prime}\right|^{-\beta}
\end{aligned}
$$

Majorating and inserting some appropriate new terms whose sum is zero:

$$
\begin{aligned}
\leqslant & \|\Phi\|_{0,0}+\left\|\Phi_{u}\right\|_{0,0}\|U\|_{\widehat{1,0}}+\left\|\Phi_{\theta}\right\|_{0,0}+ \\
& +\sup _{\theta^{\prime \prime} \neq \theta^{\prime}}\left|\sum_{1 \leqslant l \leqslant d} \Phi_{u_{l}}\left(U\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime \prime}}, s\right)\left[U_{\theta}^{l}\left(e^{i \theta^{\prime \prime}}\right)-U_{\theta}^{l}\left(e^{i \theta^{\prime}}\right)\right]\right|\left|\theta^{\prime \prime}-\theta^{\prime}\right|^{-\beta}+ \\
& +\sup _{\theta^{\prime \prime} \neq \theta^{\prime}}\left|\sum_{1 \leqslant l \leqslant d}\left[\Phi_{u_{l}}\left(U\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime \prime}}, s\right)-\Phi_{u_{l}}\left(U\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime}}, s\right)\right] U_{\theta}^{l}\left(e^{i \theta^{\prime}}\right)\right|\left|\theta^{\prime \prime}-\theta^{\prime}\right|^{-\beta}+ \\
& +\sup _{\theta^{\prime \prime} \neq \theta^{\prime}}\left|\sum_{1 \leqslant l \leqslant d}\left[\Phi_{u_{l}}\left(U\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime}}, s\right)-\Phi_{u_{l}}\left(U\left(e^{i \theta^{\prime}}\right), e^{i \theta^{\prime}}, s\right)\right] U_{\theta}^{l}\left(e^{i \theta^{\prime}}\right)\right|\left|\theta^{\prime \prime}-\theta^{\prime}\right|^{-\beta}+ \\
& +\sup _{\theta^{\prime \prime} \neq \theta^{\prime}}\left|\Phi_{\theta}\left(U\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime \prime}}, s\right)-\Phi_{\theta}\left(U\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime}}, s\right)\right|\left|\theta^{\prime \prime}-\theta^{\prime}\right|^{-\beta}+ \\
& +\sup _{\theta^{\prime \prime} \neq \theta^{\prime}}\left|\Phi_{\theta}\left(U\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime}}, s\right)-\Phi_{\theta}\left(U\left(e^{i \theta^{\prime}}\right), e^{i \theta^{\prime}}, s\right)\right|\left|\theta^{\prime \prime}-\theta^{\prime}\right|^{-\beta}
\end{aligned}
$$

Majorating:

$$
\begin{aligned}
\leqslant & \|\Phi\|_{0,0}+\left\|\Phi_{u}\right\|_{0,0}\|U\|_{\widehat{1,0}}+\left\|\Phi_{\theta}\right\|_{0,0}+ \\
& +\left\|\Phi_{u}\right\|_{0,0}\|U\|_{\widehat{1, \beta}}+\left\|\left.\Phi_{u}\right|_{s}\right\|_{\widehat{0, \beta}}\|U\|_{\widehat{1,0}}+\left\|\left.\Phi_{u}\right|_{s}\right\|_{\widehat{0, \beta}}\left(\|U\|_{\widehat{1,0}}\right)^{\beta}\|U\|_{\widehat{1,0}}+ \\
& +\left\|\left.\Phi_{\theta}\right|_{s}\right\|_{\widehat{0, \beta}}+\left\|\left.\Phi_{\theta}\right|_{s}\right\|_{\widehat{0, \beta}}\left(\|U\|_{\widehat{1,0}}\right)^{\beta},
\end{aligned}
$$

which yields the lemma, noticing that $\left\|\Phi_{u}\right\|_{0,0}\left(\|U\|_{\widehat{1,0}}+\|U\|_{\widehat{1, \beta}}\right) \leqslant$ $\left\|\Phi_{u}\right\|_{0,0}\|U\|_{1, \beta}$.

### 4.3. Proof of Lemma 3.13. We need two preparatory lemmas.

Lemma 4.4. Let $n \geqslant 1$, let $\mathrm{x} \in \mathbb{R}^{n}$, let $m \geqslant 1$, let $\mathrm{y} \in \mathbb{R}^{m}$, let $\rho>0$ and let $f=f(\mathrm{x}, \mathrm{y})$ be $a \in \mathscr{C}^{1, \alpha}$ map, with $0<\alpha \leqslant 1$, defined in the strip $\left\{(\mathrm{x}, \mathrm{y}) \in \mathbb{R}^{m} \times \mathbb{R}^{n}:|\mathrm{x}+\mathrm{y}|<\rho\right\}$ and valued in $\mathbb{R}^{d}, d \geqslant 1$. If four vertices $\left(x^{\prime}, y^{\prime}\right),\left(x^{\prime \prime}, y^{\prime}\right),\left(x^{\prime}, y^{\prime \prime}\right)$ and $\left(x^{\prime \prime}, y^{\prime \prime}\right)$ of a rectangle belong to the strip, then:

$$
\left|f\left(\mathrm{x}^{\prime \prime}, \mathrm{y}^{\prime \prime}\right)-f\left(\mathrm{x}^{\prime}, \mathrm{y}^{\prime \prime}\right)-f\left(\mathrm{x}^{\prime \prime}, \mathrm{y}^{\prime}\right)+f\left(\mathrm{x}^{\prime}, \mathrm{y}^{\prime}\right)\right| \leqslant\|f\|_{\widehat{1, \alpha}}\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right|\left|\mathrm{y}^{\prime \prime}-\mathrm{y}^{\prime}\right|^{\alpha} .
$$

A similar inequality holds by reversing the rôles of the variables x and y .
Proof. We apply twice the Taylor integral formula (1.2) with respect to the variable $x$ and we majorate:

$$
\begin{aligned}
& \left|\left(f\left(\mathrm{x}^{\prime \prime}, \mathrm{y}^{\prime \prime}\right)-f\left(\mathrm{x}^{\prime}, \mathrm{y}^{\prime \prime}\right)\right)-\left(f\left(\mathrm{x}^{\prime \prime}, \mathrm{y}^{\prime}\right)-f\left(\mathrm{x}^{\prime}, \mathrm{y}^{\prime}\right)\right)\right| \leqslant \\
& \quad \leqslant \int_{0}^{1} \sum_{1 \leqslant i \leqslant n}\left|\frac{\partial f}{\partial \mathrm{x}_{i}}\left(\mathrm{x}^{\prime}+\mathrm{s}\left(\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right), \mathrm{y}^{\prime \prime}\right)-\frac{\partial f}{\partial \mathrm{x}_{i}}\left(\mathrm{x}^{\prime}+\mathrm{s}\left(\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right), \mathrm{y}^{\prime}\right)\right|\left|\mathrm{x}_{i}^{\prime \prime}-\mathrm{x}_{i}^{\prime}\right| d \mathrm{~s} \\
& \quad \leqslant\|f\|_{1, \alpha}\left|\mathrm{y}^{\prime \prime}-\mathrm{y}^{\prime}\right|^{\alpha}\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right| .
\end{aligned}
$$

Lemma 4.5. Let $n \geqslant 1$, let $\mathrm{x} \in \mathbb{R}^{n}$, let $\rho>0$ and let $H=H(\mathrm{t})$ be a $\mathscr{C}^{1, \alpha}$ map, with $0<\alpha \leqslant 1$, defined in the open cube $\square_{\rho}^{n}=\{|t|<\rho\}$ and valued in $\mathbb{R}^{d}$. Let $\mathrm{x}, \mathrm{y}, \mathrm{z}$ with $|\mathrm{x}|,|\mathrm{y}|,|\mathrm{z}| \leqslant \rho / 3$, so that the four points $\mathrm{x}, \mathrm{y}, \mathrm{z}$ and $\mathrm{x}+\mathrm{y}-\mathrm{z}$ constitute the vertices of a parallelogram which is contained in $\square{ }_{\rho}^{n}$. Then:

$$
|H(\mathrm{x}+\mathrm{y}-\mathrm{z})-H(\mathrm{x})-H(\mathrm{y})+H(\mathrm{z})| \leqslant 4\|H\|_{\widehat{1, \alpha}}|\mathrm{y}-\mathrm{z}||\mathrm{x}-\mathrm{z}|^{\alpha}
$$

A similar inequality holds after exchanging x and y .
Proof. To estimate the second difference of $H$, we introduce a new map

$$
f(\mathrm{x}, \mathrm{y}):=H(\mathrm{x}+\mathrm{y})
$$

of $(\mathrm{x}, \mathrm{y}) \in \mathbb{R}^{n} \times \mathbb{R}^{n}$, whose domain is the strip $\{|\mathrm{x}+\mathrm{y}|<\rho\}$. Let $\mathrm{x}, \mathrm{y}, \mathrm{z}$ with $|x|,|y|,|z| \leqslant \rho / 3$. Fixing $x^{\prime} \in \mathbb{R}^{n}$ arbitrary, there exist unique $y^{\prime}, x^{\prime \prime}$ and $y^{\prime \prime}$ solving the linear system:

$$
\left\{\begin{aligned}
& x^{\prime}+y^{\prime}=z, \\
& x^{\prime \prime}+y^{\prime \prime}=x, \\
& x^{\prime \prime}=x+y-y^{\prime}=y
\end{aligned}\right.
$$

In fact, $y^{\prime}=z-x^{\prime}, x^{\prime \prime}=y-z+x^{\prime}$ and $y^{\prime \prime}=x-x^{\prime \prime}$. Taking the norms $|\cdot|$ of the four equations above, we see that the rectangle $\left(x^{\prime}, y^{\prime}\right),\left(x^{\prime \prime}, y^{\prime}\right),\left(x^{\prime}, y^{\prime \prime}\right)$, $\left(x^{\prime \prime}, y^{\prime \prime}\right)$ is contained in the strip $\{|x+y|<\rho\}$. Applying then Lemma 4.4 (with $m=n$ ), we get:

$$
\begin{aligned}
& |H(\mathrm{x}+\mathrm{y}-\mathrm{z})-H(\mathrm{x})-H(\mathrm{y})+H(\mathrm{z})|= \\
& \quad=\left|f\left(\mathrm{x}^{\prime \prime}, \mathrm{y}^{\prime \prime}\right)-f\left(\mathrm{x}^{\prime}, \mathrm{y}^{\prime \prime}\right)-f\left(\mathrm{x}^{\prime \prime}, \mathrm{y}^{\prime}\right)+f\left(\mathrm{x}^{\prime}, \mathrm{y}^{\prime}\right)\right| \\
& \quad \leqslant\|f\|_{\widehat{1, \alpha}}\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right|\left|\mathrm{y}^{\prime \prime}-\mathrm{y}^{\prime}\right|^{\alpha} \\
& \quad=\|f\|_{\widehat{1, \alpha}}|\mathrm{y}-\mathrm{z}||\mathrm{x}-\mathrm{z}|^{\alpha} .
\end{aligned}
$$

We claim that $\|f\|_{\widehat{1, \alpha}} \leqslant 4\|H\|_{\widehat{1, \alpha}}$, which will conclude. Carefulness and rigor are required. In fact, to estimate:

$$
\|f\|_{\widehat{1, \alpha}}=\sum_{i=1}^{n} \sup _{\left(x^{\prime \prime}, y^{\prime \prime}\right) \neq\left(x^{\prime}, y^{\prime}\right)} \frac{\left|f_{x_{i}}\left(x^{\prime \prime}, \mathrm{y}^{\prime \prime}\right)-f_{x_{i}}\left(x^{\prime}, \mathrm{y}^{\prime}\right)\right|+\left|f_{y_{i}}\left(\mathrm{x}^{\prime \prime}, \mathrm{y}^{\prime \prime}\right)-f_{\mathrm{y}_{i}}\left(\mathrm{x}^{\prime}, \mathrm{y}^{\prime}\right)\right|}{\left|\left(\mathrm{x}^{\prime \prime}, \mathrm{y}^{\prime \prime}\right)-\left(\mathrm{x}^{\prime}, \mathrm{y}^{\prime}\right)\right|^{\alpha}},
$$

we shall first transform the denominator. By definition:

$$
\left|\left(x^{\prime \prime}, y^{\prime \prime}\right)-\left(x^{\prime}, y^{\prime}\right)\right|=\max \left(\left|x^{\prime \prime}-x^{\prime}\right|,\left|y^{\prime \prime}-y^{\prime}\right|\right) .
$$

If we set $a:=\left|\mathrm{x}^{\prime \prime}-\mathrm{x}^{\prime}\right|$ and $b:=\left|\mathrm{y}^{\prime \prime}-\mathrm{y}^{\prime}\right|$ and if we invert the inequality $(a+b)^{\alpha} \leqslant 2 \max \left(a^{\alpha}, b^{\alpha}\right)$, noticing $2^{\alpha} \leqslant 2$, we obtain:

$$
\begin{aligned}
\frac{1}{\left|\left(x^{\prime \prime}, y^{\prime \prime}\right)-\left(x^{\prime}, y^{\prime}\right)\right|^{\alpha}}=\frac{1}{\max \left(a^{\alpha}, b^{\alpha}\right)} \leqslant \frac{2}{(a+b)^{\alpha}} & =\frac{2}{\left(\left|x^{\prime \prime}-x^{\prime}\right|+\left|y^{\prime \prime}-y^{\prime}\right|\right)^{\alpha}} \\
& \leqslant \frac{2}{\left|x^{\prime \prime}+y^{\prime \prime}-x^{\prime}-y^{\prime}\right|^{\alpha}}
\end{aligned}
$$

Replacing the denominator above, we then transform and majorate the numerator:

$$
\begin{aligned}
\|f\|_{\widehat{1, \alpha}} & \leqslant 2 \sum_{i=1}^{n} \sup _{\left(\mathrm{x}^{\prime \prime}, \mathrm{y}^{\prime \prime}\right) \neq\left(\mathrm{x}^{\prime}, \mathrm{y}^{\prime}\right)}\left(\frac{\left|H_{\mathrm{t}_{i}}\left(\mathrm{x}^{\prime \prime}+\mathrm{y}^{\prime \prime}\right)-H_{\mathrm{t}_{i}}\left(\mathrm{x}^{\prime}+\mathrm{y}^{\prime}\right)\right|+\left|H_{\mathrm{t}_{i}}\left(\mathrm{x}^{\prime \prime}+\mathrm{y}^{\prime \prime}\right)-H_{\mathrm{t}_{i}}\left(\mathrm{x}^{\prime}+\mathrm{y}^{\prime}\right)\right|}{\left|\mathrm{x}^{\prime \prime}+\mathrm{y}^{\prime \prime}-\mathrm{x}^{\prime}-\mathrm{y}^{\prime}\right|^{\alpha}}\right) \\
& \leqslant 4 \sum_{i=1}^{n} \sup _{\mathrm{t}^{\prime \prime} \neq \mathrm{t}^{\prime}}\left(\frac{\left|H_{\mathrm{t}_{i}}\left(\mathrm{t}^{\prime \prime}\right)-H_{\mathrm{t}_{i}}\left(\mathrm{t}^{\prime}\right)\right|}{\left|\mathrm{t}^{\prime \prime}-\mathrm{t}^{\prime}\right|^{2}}\right) \\
& =4\|H\|_{\widehat{1, \alpha}} .
\end{aligned}
$$

This completes the proof of Lemma 4.5.
We can now state a slightly simplified version of Lemma 3.13.
Lemma 4.6. ([Tu1990], [*]) Let $u \in \mathbb{R}^{d}, d \geqslant 1$, let $\rho_{1}>0$ and let $\Psi=$ $\Psi(u)$ be a $\mathscr{C}^{1, \alpha}$ map, with $0<\alpha \leqslant 1, u \in \mathbb{R}^{d}$, defined in the cube $\left\{|u|<\rho_{1}\right\}$ and valued in $\mathbb{R}^{d}$. Let $U^{1}, U^{2} \in \mathscr{C}^{1,0}\left(\partial \Delta, \mathbb{R}^{d}\right)$ with $\left|U^{j}\left(e^{i \theta}\right)\right|<\rho_{1} / 3$ on $\partial \Delta$, for $j=1,2$. For every $\beta$ with $0<\beta \leqslant \alpha$ the following inequality holds:

$$
\left\|\Psi\left(U^{2}(\cdot)\right)-\Psi\left(U^{1}(\cdot)\right)\right\|_{\mathscr{C} 0, \beta}(\partial \Delta) \leqslant \mathrm{D}\left\|U^{2}-U^{1}\right\|_{0, \beta}
$$

with

$$
\mathrm{D}=\|\Psi\|_{1, \beta}\left[1+2\left(\left\|U^{1}\right\|_{\widehat{1,0}}\right)^{\beta}+2\left(\left\|U^{2}\right\|_{\widehat{1,0}}\right)^{\beta}\right]
$$

Proof. Firstly and obviously:

$$
\left\|\Psi\left(U^{2}\right)-\Psi\left(U^{1}\right)\right\|_{0,0} \leqslant\|\Psi\|_{\widehat{1,0}}\left\|U^{2}-U^{1}\right\|_{0,0}
$$

Secondly, we have $\left\|\Psi\left(U^{2}\right)-\Psi\left(U^{1}\right)\right\|_{\widehat{0, \beta}}=\sup _{0<\left|\theta^{\prime \prime}-\theta^{\prime}\right| \leqslant \pi}\left(\mathrm{Q} /\left|\theta^{\prime \prime}-\theta^{\prime}\right|^{\beta}\right)$, where:

$$
\mathbf{Q}:=\left|\Psi\left(U^{2}\left(e^{i \theta^{\prime \prime}}\right)\right)-\Psi\left(U^{1}\left(e^{i \theta^{\prime \prime}}\right)\right)-\Psi\left(U^{2}\left(e^{i \theta^{\prime}}\right)\right)+\Psi\left(U^{1}\left(e^{i \theta^{\prime}}\right)\right)\right| .
$$

To majorate $\mathbf{Q}$, we start by inserting the term $\Psi\left[U^{1}\left(e^{i \theta^{\prime \prime}}\right)+U^{2}\left(e^{i \theta^{\prime}}\right)-\right.$ $\left.U^{1}\left(e^{i \theta^{\prime}}\right)\right]$, well-defined, thanks to the assumption $\left\|U^{j}\right\|_{0,0}<\rho_{1} / 3$ :

$$
\begin{aligned}
& \mathbf{Q} \leqslant\left|\Psi\left(U^{2}\left(e^{i \theta^{\prime \prime}}\right)\right)-\Psi\left[U^{1}\left(e^{i \theta^{\prime \prime}}\right)+U^{2}\left(e^{i \theta^{\prime}}\right)-U^{1}\left(e^{i \theta^{\prime}}\right)\right]\right|+ \\
&+\mid \Psi\left[U^{1}\left(e^{i \theta^{\prime \prime}}\right)+U^{2}\left(e^{i \theta^{\prime}}\right)-U^{1}\left(e^{i \theta^{\prime}}\right)\right]-\Psi\left(U^{1}\left(e^{i \theta^{\prime \prime}}\right)\right)- \\
&-\Psi\left(U^{2}\left(e^{i \theta^{\prime}}\right)\right)+\Psi\left(U^{1}\left(e^{i \theta^{\prime}}\right)\right) \mid .
\end{aligned}
$$

To estimate the second absolute value, we apply Lemma 4.5 with $\mathrm{x}=$ $U^{1}\left(e^{i \theta^{\prime \prime}}\right)$, with $\mathbf{y}=U^{2}\left(e^{i \theta^{\prime}}\right)$ and with $\mathbf{z}=U^{1}\left(e^{i \theta^{\prime}}\right)$ :

$$
\begin{aligned}
Q \leqslant & \|\Psi\|_{\widehat{1,0}}\left|\left[U^{2}-U^{1}\right]\left(e^{i \theta^{\prime \prime}}\right)-\left[U^{2}-U^{1}\right]\left(e^{i \theta^{\prime}}\right)\right|+ \\
& +4\|\Psi\|_{\widehat{1, \beta}}\left|U^{2}\left(e^{i \theta^{\prime}}\right)-U^{1}\left(e^{i \theta^{\prime}}\right)\right|\left|U^{1}\left(e^{i \theta^{\prime \prime}}\right)-U^{1}\left(e^{i \theta^{\prime}}\right)\right|^{\beta} .
\end{aligned}
$$

We then achieve the remaining majorations:

$$
\begin{aligned}
Q \leqslant & \|\Psi\|_{\widehat{1,0}}\left\|U^{2}-U^{1}\right\|_{\widehat{0, \beta}}\left|\theta^{\prime \prime}-\theta^{\prime}\right|^{\beta}+ \\
& +4\|\Psi\|_{\widehat{1, \beta}}\left\|U^{2}-U^{1}\right\|_{0,0}\left(\left\|U^{1}\right\|_{\widehat{1,0}}\right)^{\beta}\left|\theta^{\prime \prime}-\theta^{\prime}\right|^{\beta}
\end{aligned}
$$

Reminding that $\left\|U^{2}-U^{1}\right\|_{0,0}+\left\|U^{2}-U^{1}\right\|_{\widehat{0, \beta}}=\left\|U^{2}-U^{1}\right\|_{0, \beta}$ and summing, we obtain:

$$
\left\|\Psi\left(U^{2}\right)-\Psi\left(U^{1}\right)\right\|_{0, \beta} \leqslant\|\Psi\|_{1, \beta}\left[1+4\left(\left\|U^{1}\right\|_{\widehat{1,0}}\right)^{\beta}\right]\left\|U^{2}-U^{1}\right\|_{0, \beta} .
$$

A similar inequality holds with $\left(\left\|U^{2}\right\|_{\widehat{1,0}}\right)^{\beta}$ instead of $\left(\left\|U^{1}\right\|_{\widehat{1,0}}\right)^{\beta}$. Taking the arithmetic mean, we find the symmetric quantity $D$ of the lemma. The proof is complete.

## Proof of Lemma 3.13. By definition:

$$
\begin{aligned}
\mathrm{R}:= & \left\|\Phi\left(U^{2}(\cdot), \cdot, s\right)-\Phi\left(U^{1}(\cdot), \cdot, s\right)\right\|_{\mathscr{C} 0, \beta}(\partial \Delta) \\
= & \sup _{\theta}\left|\Phi\left(U^{2}\left(e^{i \theta}\right), e^{i \theta}, s\right)-\Phi\left(U^{1}\left(e^{i \theta}\right), e^{i \theta}, s\right)\right|+ \\
& +\sup _{\theta^{\prime \prime} \neq \theta^{\prime}} \mid \Phi\left(U^{2}\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime \prime}}, s\right)-\Phi\left(U^{1}\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime \prime}}, s\right)- \\
& \quad-\Phi\left(U^{2}\left(e^{i \theta^{\prime}}\right), e^{i \theta^{\prime}}, s\right)+\Phi\left(U^{1}\left(e^{i \theta^{\prime}}\right), e^{i \theta^{\prime}}, s\right)| | \theta^{\prime \prime}-\left.\theta^{\prime}\right|^{-\beta} .
\end{aligned}
$$

In the numerator, we insert $-\Phi\left(U^{2}\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime}}, s\right)+\Phi\left(U^{1}\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime}}, s\right)$ plus its opposite:

$$
\begin{aligned}
\mathrm{R} \leqslant & \|\Phi\|_{\widehat{1,0}}\left\|U^{2}-U^{1}\right\|_{0,0}+ \\
+\sup _{\theta^{\prime \prime} \neq \theta^{\prime}} & \Phi\left(U^{2}\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime \prime}}, s\right)-\Phi\left(U^{1}\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime \prime}}, s\right)- \\
& -\Phi\left(U^{2}\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime}}, s\right)+\Phi\left(U^{1}\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime}}, s\right)| | \theta^{\prime \prime}-\left.\theta^{\prime}\right|^{-\beta}+ \\
+\sup _{\theta^{\prime \prime} \neq \theta^{\prime}} & \mid \Phi\left(U^{2}\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime}}, s\right)-\Phi\left(U^{1}\left(e^{i \theta^{\prime \prime}}\right), e^{i \theta^{\prime}}, s\right)- \\
& \quad-\Phi\left(U^{2}\left(e^{i \theta^{\prime}}\right), e^{i \theta^{\prime}}, s\right)+\Phi\left(U^{1}\left(e^{i \theta^{\prime}}\right), e^{i \theta^{\prime}}, s\right)| | \theta^{\prime \prime}-\left.\theta^{\prime}\right|^{-\beta}
\end{aligned}
$$

To majorate the first $\sup _{\theta^{\prime \prime} \neq \theta^{\prime}}$, we apply Lemma 4.4 with $\mathrm{x}^{\prime}=U^{1}\left(e^{i \theta^{\prime \prime}}\right)$, $\mathrm{y}^{\prime}=e^{i \theta^{\prime}}, \mathrm{x}^{\prime \prime}=U^{2}\left(e^{i \theta^{\prime \prime}}\right)$ and $\mathrm{y}^{\prime \prime}=e^{i \theta^{\prime}}$, where $s$ is considered as a dumb parameter. To majorate the second $\sup _{\theta^{\prime \prime} \neq \theta^{\prime}}$, we apply Lemma 4.6 to $\Psi(u):=\Phi\left(u, e^{i \theta}, s\right)$, where $\left(e^{i \theta}, s\right)$ are considered as dumb parameters. We get:

$$
\begin{aligned}
\mathrm{R} \leqslant & \|\Phi\|_{\widehat{1,0}}\left\|U^{2}-U^{1}\right\|_{0,0}+ \\
& +\|\Phi\|_{\widehat{1, \beta}} \sup _{\theta}\left|U^{2}\left(e^{i \theta}\right)-U^{1}\left(e^{i \theta}\right)\right|+ \\
& +\|\Phi\|_{1, \beta}\left[1+2\left(\left\|U^{1}\right\|_{\widehat{1,0}}\right)^{\beta}+2\left(\left\|U^{2}\right\|_{\widehat{1,0}}\right)^{\beta}\right]\left\|U^{2}-U^{1}\right\|_{0, \beta} .
\end{aligned}
$$

To conclude, we use $\|\Phi\|_{\widehat{1,0}}+\|\Phi\|_{\widehat{1, \beta}} \leqslant\|\Phi\|_{1, \beta}$ and we get the term $C$ of Lemma 3.13.

With these techniques, the proofs of Lemmas 3.17, 3.19, 3.30, 3.31 and 3.32 are easily guessed and even simpler.

# V: Holomorphic extension of CR functions 


#### Abstract

Table of contents 1. Hartogs theorem, jump formula and domains having the holomorphic extension property 149. 2. Trépreau's theorem, deformations of Bishop dises and propagation on hypersurfaces 162. 3. Tumanov's theorem, deformations of Bishop discs and propagation on generic submanifolds 174. 4. Holomorphic extension on globally minimal generic submanifolds ........ 189.


[19 diagrams]

The method of analytic discs is rooted in the very birth of the theory of functions of several complex variables. The discovery by Hurwitz and Hartogs of the compulsory extension of holomorphic functions relied upon an application of Cauchy's integral formula along a family of analytic discs surrounding an illusory singularity. Since H. Cartan, Thullen, Behnke and Sommer, various versions of this argument were coined "Kontinuitättsatz" or "Continuity principle".

The removal of compact singularities culminated in the so-called HartogsBochner theorem, usually proved by means of integral formulas or thanks to the resolution of a $\bar{\partial}$ problem with compact support. Contradicting all expectations, a subtle example due to Fornaess (1998), shows that on a non-pseudoconvex domain, the disc method may fail to fill in the domain, if the discs are required to stay inside the domain.

Nevertheless, it is of the highest prize to build constructive methods in order to describe significant parts of the envelope of holomorphy of a domain, of a CR manifold, as well as the polynomial hull of certain compact sets. In such problems, analytic discs with boundary in the domain, the CR manifold or the compact set remain the most adequate tools.

The precise existence Theorem 3.7(IV) for the solutions of Bishop's equation that was established in the previous Part IV may now be applied systematically to a variety of geometric situations. In this respect, we just followed Bishop's genuine philosophy that required to ensure an explicit control of the size of solutions in terms of the size of data, instead of appealing to some general, imprecise version of the implicit function theorem.

Thanks to the jump theorem, holomorphic extension of CR functions defined on a hypersurface $M$ is equivalent to the extension of the functions that are holomorphic in one of the two sides to the other side. Trépreau's original theorem (1986) states that such an extension holds at a point $p$ if and only if there does not exist a local complex hypersurface $\Sigma$ of $\mathbb{C}^{n}$ with $p \in \Sigma \subset M$. A deeper phenomenon of propagation (Trépreau, 1990) holds: if CR functions extend holomorphically to one side at a point $p$, a similar extension holds at every point of the CR orbit of $p$ in $M$. By means of deformations of attached Bishop discs, there is an elementary
(and folklore) proof that contains both the local and the global extension theorems on hypersurfaces, yielding a satisfactory understanding of the phenomenon.

On a generic submanifold $M$ of $\mathbb{C}^{n}$ of higher codimension, the celebrated Tu manov extension theorem (1988) states that CR functions defined on $M$ extend holomorphically to a local wedge of edge $M$ at a point $p$ if the local CR orbit of $p$ contains a neighborhood of $p$ in $M$. A globalization of this statement, obtained independently by Jöricke and the first author in 1994, states that the same extension phenomenon holds if $M$ consists of a single CR orbit, i.e. is globally minimal. Both proofs heavily relied on the local Tumanov theorem and on a precise control of the propagation of directions of extension.

A clever proof that treats both locally minimal and globally minimal generic submanifolds on the same footing constitutes the main Theorem 4.12 of the present Part V: if $M$ is a globally minimal $\mathscr{C}^{2, \alpha}(0<\alpha<1)$ generic submanifold of $\mathbb{C}^{n}$ of codimension $\geqslant 1$ and of $C R$ dimension $\geqslant 1$, there exists a wedgelike domain $\mathscr{W}$ attached to $M$ such that every continuous $C R$ function $f \in \mathscr{C}_{C R}^{0}(M)$ possesses a holomorphic extension $F \in \mathscr{O}(\mathscr{W}) \cap \mathscr{C}^{0}(M \cup \mathscr{W})$ with $\left.F\right|_{M}=f$. This basic statement as well as the techniques underlying its proof will be the very starting point of the study of removable singularities in Parts VI and in [26].

## §1. HARTOGS THEOREM, JUMP FORMULA AND DOMAINS HAVING THE EXTENSION PROPERTY

1.1. Hartogs extension theorem: brief history. ${ }^{9}$ In 1897, Hurwitz showed that a function holomorphic in $\mathbb{C}^{2} \backslash\{0\}$ extends holomorphically through the origin. In his thesis (1906), Hartogs generalized this discovery, emphasizing the compulsory holomorphic extendability of functions that are defined on the nowadays celebrated Hartogs skeleton (diagram below). The main argument is to apply Cauchy's integral formula along families of analytic discs having their boundary inside the domain and whose interior goes outside the domain. In fact, the thinness of an embedded circle in $\mathbb{C}^{n}(n \geqslant 2)$ offers much freedom to include illusory singularities inside a disc.

In 1924, Osgood stated the ultimate generalization of the discovery of Hurwitz and Hartogs: if $\Omega \subset \mathbb{C}^{n}(n \geqslant 2)$ is a domain and if $K \subset \Omega$ is any compact such that $\Omega \backslash K$ connected, then $\mathscr{O}(\Omega \backslash K)=\left.\mathscr{O}(\Omega)\right|_{\Omega \backslash K}$. This statement is nowadays called the Hartogs-Bochner theorem. Although the proof of Osgood was correct for geometrically simple complements $\Omega \backslash K$, as for instance spherical shells, it was incomplete for general $\Omega \backslash K$. In fact, unpleasant topological and monodromy obstructions occur for general $\Omega \backslash K$ when pushing analytic discs. In 1998, Fornaess exhibited certain domains in which discs are forced to first leave some intermediate domain $\Omega \backslash K_{1}$, $K_{1} \subset K$, before $\Omega$ may be filled in.

[^8]In the late 1930's, a rigorous proof of Osgood's general statement was obtained by Fueter, by means of a generalization of the classical Cauchy-Green-Pompeiu integral formula to several variables, in the context of complex and quaternionic functions (Moisil 1931, Fueter 1935). In 1943, Martinelli simplified the formal treatment of Fueter. Then Bochner observed that the same result holds more generally if one assumes given on $\partial \Omega$ just a CR function.

1.2. Hartogs domain. Consider the $\varepsilon$-Hartogs skeleton (pot-looking) domain:
$\mathscr{H}_{\varepsilon}:=\left\{\left(z_{1}, z_{2}\right) \in \mathbb{C}^{2}:\left|z_{1}\right|<1,\left|z_{2}\right|<\varepsilon\right\} \bigcup\left\{1-\varepsilon<\left|z_{1}\right|<1,\left|z_{2}\right|<1\right\}$.
We draw two diagrams: in $\left(\left|z_{1}\right|,\left|z_{2}\right|\right)$ and in $\left(x_{1}, y_{1},\left|z_{2}\right|\right)$ coordinates.
Lemma 1.3. Every holomorphic function $f \in \mathscr{O}\left(\mathscr{H}_{\varepsilon}\right)$ extends holomorphically to the bidisc $\Delta \times \Delta$, the convex hull of $\mathscr{H}_{\varepsilon}$.

Proof. Letting $\delta$ with $0<\delta<\varepsilon$, for every $z_{2} \in \mathbb{C}$ with $\left|z_{2}\right|<1$, the analytic disc

$$
\Delta \ni \zeta \longmapsto A_{z_{2}}(\zeta):=\left([1-\delta] \zeta, z_{2}\right) \in \mathbb{C}^{2}
$$

has its boundary $A_{z_{2}}(\partial \Delta)$ contained in $\mathscr{H}_{\varepsilon}$, the domain where the function $f$ is defined. Thus, we may compute the Cauchy integral

$$
F\left(z_{1}, z_{2}\right):=\frac{1}{2 \pi i} \int_{\partial \Delta} \frac{f\left(A_{z_{2}}(\zeta)\right)}{\zeta-z_{1}} d \zeta .
$$

Differentiating under the sum, this extension $F$ is seen to be holomorphic. In addition, for $\left|z_{2}\right|<\varepsilon$, it coincides with $f$. Obviously, the discs $A_{z_{2}}(\Delta)$ fill in the hole of the domain $\mathscr{H}_{\varepsilon}$.
1.4. Bounded domains in $\mathbb{C}^{n}$ and Hartogs-Bochner extension phenomenon. Let $\Omega$ be a connected open subset of $\mathbb{C}^{n}$, a domain. We assume it to be bounded, i.e. $\bar{\Omega}$ is compact and that its boundary $\partial \Omega:=\bar{\Omega} \backslash \Omega$ is a hypersurface of $\mathbb{C}^{n}$ of class at least $\mathscr{C}^{1}$. By means of a partition of unity, one can construct a real-valued function $r$ defined on $\mathbb{C}^{n}$ such that $\Omega=\{z: r(z)<0\}$ and $\partial \Omega=\{z: r(z)=0\}$, with $d r(z) \neq 0$ for every $z \in \partial \Omega$. Then $\partial \Omega$ is orientable.

Extensions of the above disc argument led to the most general ${ }^{10}$ form of the Hartogs theorem: if $\Omega$ is a bounded domain in $\mathbb{C}^{n}(n \geqslant 2)$ having connected boundary $\partial \Omega$, then every function holomorphic in a neighborhood of $\partial \Omega$ uniquely extends as a function holomorphic in $\Omega$. There are three classical methods of proof:

- using the Bochner-Martinelli kernel;
- using solutions of $\bar{\partial} u=v$ having compact support;
- pushing analytic discs, in successive Hartogs skeletons.

The first two are rigorously established and we shall review the first in a while. For almost one hundred years, it has been a folklore belief that the third method could be accomplished somehow. Let us be precise.
1.5. Fornaess' counterexample and a disc theorem. Thus, let $\Omega$ be a bounded domain of $\mathbb{C}^{2}$ having connected $\mathscr{C}^{1}$ boundary. For $\delta>0$ small, consider the one-sided neighborhood of $\partial \Omega$ defined by:

$$
\widetilde{\Omega}_{\delta}:=\{z \in \Omega: \operatorname{dist}(z, \partial \Omega)<\delta\}
$$

The complement $\Omega \backslash \widetilde{\Omega}_{\delta}$ is a compact hole. Remind that the bidisc $\Delta^{2}$ is the convex hull of the Hartogs skeleton $\mathscr{H}_{\varepsilon}$. Following [F1998], we say that $\Omega$ can be filled in by analytic discs if for every $\delta>0$, there exist a finite sequence of subdomains of $\Omega$ having $\mathscr{C}^{1}$ boundary, $\widetilde{\Omega}_{\delta}=\Omega_{1} \subset \Omega_{2} \subset$ $\cdots \subset \Omega_{k}=\Omega$ and for each $j=1, \ldots, k-1$, an $\varepsilon_{j}>0$ and a univalent holomorphic map $\Phi_{j}$ defined in a neighborhood of $\bar{\Delta}^{2}$ such that:
(1) $\Omega_{j+1} \subset \Omega_{j} \cup \Phi_{j}\left(\Delta^{2}\right) \subset \Omega$;
(2) $\Phi_{j}\left(\mathscr{H}_{\varepsilon}\right) \subset \Omega_{j}$;
(3) $\Omega_{j} \cap \Phi_{j}\left(\Delta^{2}\right)$ is connected;
(4) $\Omega_{j+1} \cap \Phi_{j}\left(\Delta^{2}\right)$ is connected.

For such domains, by pushing analytic discs in the embedded Hartogs figure, taking account of connectedness, we have $\left.\mathscr{O}\left(\Omega_{j+1}\right)\right|_{\Omega_{j}}=\mathscr{O}\left(\Omega_{j}\right)$. Then by induction, uniquely determined holomorphic extension holds from

[^9]$\Omega_{1}$ up to $\Omega$. Importantly, the intermediate domains are required to be all contained in $\Omega$.

In 1998, Fornaess [F1998] constructed a topologically strange domain $\Omega \subset \mathbb{C}^{2}$ that cannot be filled in this way. This example shows that the requirement that $\Omega_{j} \subset \Omega_{j+1} \subset \Omega$ is too stringent.

Nevertheless, taking care of monodromy and working in the envelope of holomorphy of $\Omega$, one may push analytic discs by allowing them to wander in the outside, in order to get the general Hartogs theorem stated above. As a preliminary, one perturbs and smoothes out the boundary. Denote by $\|z\|:=\left(\left|z_{1}\right|^{2}+\cdots+\left|z_{n}\right|^{2}\right)^{1 / 2}$ the Euclidean norm of $z=\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{C}^{n}$ and by $\mathbb{B}^{n}(p, \delta):=\{\|z-p\|<\delta\}$ the open ball of radius $\delta>0$ centered at a point $p$.
Theorem 1.6. ([MP2006c]) Let $M \Subset \mathbb{C}^{n}(n \geqslant 2)$ be a connected $\mathscr{C}^{\infty}$ hypersurface bounding a domain $\Omega_{M} \Subset \mathbb{C}^{n}$. Suppose to fix ideas that $2 \leqslant$ dist $\left(0, \bar{\Omega}_{M}\right) \leqslant 5$ and assume that the restriction $r_{M}:=\left.r\right|_{M}$ of the distance function $r(z)=\|z\|$ to $M$ is a Morse function having only a finite number $\kappa$ of critical points $\widehat{p}_{\lambda} \in M, 1 \leqslant \lambda \leqslant \kappa$, located on different sphere levels:

$$
2 \leqslant \widehat{r}_{1}:=r\left(\widehat{p}_{1}\right)<\cdots<\widehat{r}_{\kappa}:=r\left(\widehat{p}_{\kappa}\right) \leqslant 5+\operatorname{diam}\left(\bar{\Omega}_{M}\right) .
$$

Then there exists $\delta_{1}>0$ such that for every $\delta$ with $0<\delta<\delta_{1}$, the (tubular) neighborhood

$$
\mathscr{V}_{\delta}(M):=\cup_{p \in M} \mathbb{B}^{n}(p, \delta)
$$

enjoys the global Hartogs extension property into $\Omega_{M}$ :

$$
\mathscr{O}\left(\mathscr{V}_{\delta}(M)\right)=\left.\mathscr{O}\left(\Omega_{M} \cup \mathscr{V}_{\delta}(M)\right)\right|_{\mathscr{V}_{\delta}(M)},
$$

by "pushing" analytic discs inside a finite number of Hartogs figures, without using neither the Bochner-Martinelli kernel, nor solutions of some auxiliary $\bar{\partial}$ problem.
1.7. Hartogs-Bochner theorem via the Bochner-Martinelli kernel. By $\mathscr{O}(C)$, where $C \subset \mathbb{C}^{n}$ is closed, we mean $\mathscr{O}(\mathscr{V}(C))$ for some open neighborhood $\mathscr{V}(C)$ of $C$. Here is the general statement.
Theorem 1.8. ([HeLe1984, 15, 29]) Let $\Omega$ be a bounded domain in $\mathbb{C}^{n}$ having connected boundary. Then for every neighborhood $U$ of $\partial \Omega$ in $\mathbb{C}^{n}$ and every holomorphic function $f \in \mathscr{O}(U)$, there exists a function $F \in \mathscr{O}(\bar{\Omega})$ with $\left.F\right|_{\partial \Omega}=\left.f\right|_{\partial \Omega}$.

In the thin neighborhood $U$ of the not necessarily smooth boundary $\partial \Omega$, by means of a partion of unity, one may construct a connected boundary $\partial \Omega_{1} \subset U$ close to $\partial \Omega$ which is $\mathscr{C}^{1}$, or $\mathscr{C}^{\infty}$, or even $\mathscr{C}^{\omega}$, using Whitney approximation ([17]; in addition, one may assure that $\left.r(z)\right|_{\partial \Omega_{1}}$ is as in Theorem 1.6, whence both statements are equivalent). Then the restriction $\left.F\right|_{\partial \Omega_{1}}$ is CR on $\partial \Omega_{1}$ and the previous theorem is a consequence of the next.

Theorem 1.9. ( $[29,15])$ Let $\Omega$ be a bounded domain in $\mathbb{C}^{n}(n \geqslant 2)$ having connected $\mathscr{C}^{\kappa, \alpha}$ boundary, with $1 \leqslant \kappa \leqslant \infty, 0 \leqslant \alpha \leqslant 1$. Then for every CR function $f: \partial \Omega \rightarrow \mathbb{C}$ of class $\mathscr{C}^{\kappa, \alpha}$, there exists a function $F \in \mathscr{O}(\Omega) \cap$ $\mathscr{C}^{\kappa, \alpha}(\bar{\Omega})$ with $\left.F\right|_{\partial \Omega}=f$.

Some words about the proof. With $\zeta, z \in \mathbb{C}^{n}$, consider the BochnerMartinelli kernel:

$$
\mathrm{BM}(\zeta, z):=\frac{(n-1)!}{(2 \pi i)^{n}}|\zeta-z|^{-2 n} \sum_{j=1}^{n}\left(\bar{\zeta}_{j}-\bar{z}_{j}\right) d \zeta_{j} \wedge_{k \neq j} d \bar{\zeta}_{k} \wedge d \zeta_{k}
$$

This is a $(n, n-1)$-form which is $\mathscr{C}^{\omega}$ off the diagonal $\{\zeta=z\}$. For $n=2$, it coincides with the Cauchy kernel $\frac{1}{2 \pi i} \frac{1}{\zeta-z}$. If $f$ and $\partial \Omega$ are $\mathscr{C}^{1}$, the integral formula:

$$
F(z):=\int_{\partial \Omega} f(\zeta) \mathrm{BM}(\zeta, z)
$$

provides the holomorphic extension $F$.
1.10. Hypersurfaces of $\mathbb{C}^{n}$ and jump theorem for CR functions. Let $M$ be a real hypersurface of $\mathbb{C}^{n}$ without boundary. In the sequel, we shall mainly deal with three geometric situations.

- Local: $M$ is defined in a small open polydisc centered at one point $p \in M$.
- Global: $M$ is a connected orientable embedded submanifold of $\mathbb{C}^{n}$.
- Boundary: $\mathbb{C}^{n} \backslash M$ consists of two open sets $\Omega^{+}$, bounded and $\Omega^{-}$, unbounded.
Then there exists some appropriate neighborhood $\mathscr{M}$ of $M$ in $\mathbb{C}^{n}$ in which $M$ is relatively closed, in the sense that $\bar{M} \cap \mathscr{M}=M$.

More generally, let $\mathscr{M}$ be an arbitrary complex manifold of dimension $n \geqslant 1$ and let $M \subset \mathscr{M}$ be a hypersurface of class at least $\mathscr{C}^{1}$ which is relatively closed in $\mathscr{M}$ and oriented. The complement $\mathscr{M} \backslash M$ then consists of two connected components $\Omega^{+}$and $\Omega^{-}$, where $\Omega^{+}$is located on the positive side to $M$. Also, let $f: M \rightarrow \mathbb{C}$ be a CR function of class at least $\mathscr{C}^{0}$. By definition, $f$ is CR if the current of integration on $M$ of bidegree $(0,1)$ defined by ${ }^{11}$ :

$$
f_{M}(\omega):=\int_{M} f \omega, \quad \omega \in \mathscr{D}^{n, n-1}
$$

satisfies $\int_{M} f \bar{\partial} \varpi=0$ for every $\varpi \in \mathscr{D}^{n, n-2}$. Equivalently, $\bar{\partial} f_{M}=0$ in the sense of currents, where $f_{M}$ is interpreted as a $(0,1)$-form having measure coefficients.

[^10]To formulate the jump theorem in arbitrary complex manifolds, we shall mainly assume that the Dolbeault $\bar{\partial}$-complex on $\mathscr{M}$ is exact in bidegree $(0,1)$, namely $H_{\bar{\partial}}^{0,1}(\mathscr{M})=0$. This assumption holds for instance when $\mathscr{M}=\Delta^{n}, \mathbb{C}^{n}$ or $P_{n}(\mathbb{C})$. It means that the equation $\bar{\partial} u=v$, where $v$ is a $\bar{\partial}$-closed $(0,1)$-form on $\mathscr{M}$ having $\mathscr{C}^{\infty}, L^{2}$ or distributional coefficients has a $\mathscr{C}^{\infty}, L^{2}$ or distributional solution $u$ on $\mathscr{M}$.

Consequently, there exists a distribution $F$ on $\mathscr{M}$ with $\bar{\partial} F=f_{M}$. As $\operatorname{supp} f_{M} \subset M$, such a function $F$ is holomorphic in $\mathscr{M} \backslash M$. The difference $F_{2}-F_{1}$ of two solutions to $\bar{\partial} F=f_{M}$ is holomorphic in $\mathscr{M}$. In the case where $\mathscr{M}=\mathbb{C}^{n}$, a solution to $\bar{\partial} F=f_{M}$ may be represented ([Ch1975, 29]) by means of the Bochner-Martinelli kernel as $F(z):=\int_{M} f(\zeta) \mathrm{BM}(\zeta, z)$. In complex dimension $n=1$, such a solution coincides with the classical Cauchy transform.

In 1975, after previous work of Andreotti-Hill ([AH1972b]), Chirka obtained a several complex variables version of the Sokhotskiǐ-Plemelj Theorem 2.7(IV).
Theorem 1.11. ([Ch1975]) Assume that $H_{\bar{\partial}}^{0,1}(\mathscr{M})=0$ and that the hypersurface $M \subset \mathscr{M}$ is orientable and relatively closed, i.e. $\bar{M} \cap \mathscr{M}=M$. Assume $\operatorname{dim} \mathscr{M}=n \geqslant 1$ and let $(\kappa, \alpha)$ with $0 \leqslant \kappa \leqslant \infty, 0<\alpha<1$. If $M$ is $\mathscr{C}^{\kappa+1, \alpha}$ and if the current $f_{M}$ associated to a $\mathscr{C}^{\kappa, \alpha}$ function $f: M \rightarrow \mathbb{C}$ is $C R$, then every distributional solution $F \in \mathscr{O}(\mathscr{M} \backslash M)$ to $\bar{\partial} F=f_{M}$ extends to be $\mathscr{C}^{\kappa, \alpha}$ in the two closures $\overline{\Omega^{ \pm}}=\Omega^{ \pm} \cup M$, yielding two functions $F^{ \pm} \in \mathscr{O}\left(\Omega^{ \pm}\right) \cap \mathscr{C}^{\kappa, \alpha}\left(\Omega^{ \pm} \cup M\right)$ whose jump across $M$ equals $f$ :

$$
F^{+}(z)-F^{-}(z)=f(z), \quad \forall z \in M
$$

A similar jump formula holds for $f \in L_{\text {loc,CR }}^{\mathrm{p}}(M)$, with $M$ at least $\mathscr{C}^{1}$ (or a Lipschitz graph) and for $f \in \mathscr{D}_{C R}^{\prime}(M)$, with $M \in \mathscr{C}^{\infty}$.

When $\mathscr{M}=\mathbb{C}$, the conditions that $f$ is CR and that $H_{\bar{\partial}}^{0,1}(\mathscr{M})=0$ are automatically satisfied and we recover the Sokhotskiir-Plemelj jump formula. However, we mention that in several complex variables $(n \geqslant 2)$, there is no analog of the second formula $\frac{1}{2}\left[F^{+}\left(\zeta_{0}\right)+F^{-}\left(\zeta_{0}\right)\right]=$ p.v. $\frac{1}{2 \pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta-\zeta_{0}} d \zeta$. The reason is the inexistence of a universal integral formula solving $\bar{\partial} F=$ $f_{M}$. Nevertheless, there should exist generalized principal value integrals which depend on the kernel.

If $M$ is only $\mathscr{C}^{1}$ and $f$ is only $\mathscr{C}^{0}$, it is in general untrue that $F^{-}$and $F^{+}$extend continuously to $M$. Fortunately, there is a useful substitute result, analog to Theorem 2.9 (IV). Consider a open subset $M^{\prime} \subset M$ having compact closure $\overline{M^{\prime}}$ not meeting $\partial M=\bar{M} \backslash M$. We may embedd $M^{\prime}$ in a one-parameter family $\left(M_{\varepsilon}^{\prime}\right)_{|\varepsilon|<\varepsilon_{0}}, \varepsilon_{0}>0$, of hypersurfaces that foliates a strip thickening of $M^{\prime}$.

Theorem 1.12. ([Ch1975]) If $f$ is $C R$ and $\mathscr{C}^{\kappa}$ on a $\mathscr{C}^{\kappa+1}$ hypersurface $M$, then

$$
\lim _{\varepsilon \rightarrow 0}\left\|\left.F\right|_{M_{\varepsilon}^{\prime}}-\left.F\right|_{M_{-\varepsilon}^{\prime}}-f\right\|_{\kappa}=0
$$

1.13. CR extension in the projective space. Unlike in $\mathbb{C}^{n}$, there is no privileged "interior" side of an orientable connected hypersurface in the projective space $P_{n}(\mathbb{C}), n \geqslant 2$. Nevertheless, a version of the Hartogs-Bochner theorem holds. The proof is an illustration of the use of the jump theorem.
Theorem 1.14. ( $n \geqslant 3$ : [HL1975]; $n=2$ : [Sa1999, DM2002]) Let $M$ be a compact orientable connected $\mathscr{C}^{2}$ real hypersurface of $P_{n}(\mathbb{C})$ that divides the projective space into two domains $\Omega^{-}$and $\Omega^{+}$. Then:
(i) there exists a side, $\Omega^{-}$or $\Omega^{+}$, to which every function holomorphic in some neighborhood of $M$ extends holomorphically ${ }^{12}$;
(ii) every function that is holomorphic in the union of the other side of $M$ together with a neighborhood of $M$ must be constant.

Let us summarize the proof. Let $f$ be holomorphic in some neighborhood $\mathscr{V}(M)$ of $M$ in $P_{n}(\mathbb{C})$. As the Dolbeault cohomology group $H^{0,1}\left(P_{n}(\mathbb{C})\right)$ vanishes for $n \geqslant 2$ ([HeLe1984, 15]), thanks to Theorem 1.11 above, the CR function $\left.f\right|_{M}$ on $M$ can be decomposed as the jump $f=F^{+}-F^{-}$between two functions $F^{ \pm}$holomorphic in $\Omega^{ \pm}$which are (at least) continuous up to $M$. It suffices then to show that either $F^{+}$or $F^{-}$is constant, since clearly, if $F^{+}$(resp. $F^{-}$) is constant equal to $c^{+}$(resp. $c^{-}$), then $f$ extends holomorphically to $\Omega^{-}$(resp. to $\Omega^{+}$) as $c^{+}-F^{-}$(resp. as $F^{+}-c^{-}$).

By contradiction, assume that both $F^{+}$and $F^{-}$are nonconstant. We choose two domains $U^{+}$and $U^{-}$with $\mathscr{V}(M) \cup \Omega^{ \pm} \supset U^{ \pm} \supset M \cup \Omega^{ \pm}$. By a preliminary (technical) deformation argument, we may assume that $F^{ \pm}$is holomorphic in $U^{ \pm}$. According to a theorem due to Takeuchi [Ta1964], holomorphic functions in an arbitrary domain of $P_{n}(\mathbb{C})(n \geqslant 2)$, either are constant or separate points. Since $F^{-}$is nonconstant, $\mathscr{O}\left(U^{-}\right)$separates points. Conjugating with elements of the group $\operatorname{PGL}(n, \mathbb{C})$ of projective automorphisms of $P_{n}(\mathbb{C})$, shrinking $\mathscr{V}(M)$ and $U^{-}$slightly if necessary, we may verify ([DM2002]) that $\mathscr{O}\left(U^{-}\right)$separates points and provides local system of holomorphic coordinates at every point. By standard techniques of Stein theory ([Hö1973]), it follows that $U^{-}$is embeddable in some $\mathbb{C}^{N}$, with $N$ large. The image of $M$ under such an embedding $\Phi$ is a compact CR submanifold of $\mathbb{C}^{N}$ that is filled by the relatively compact complex manifold $\Sigma^{-}=\Phi\left(U^{-}\right)$with boundary $\Phi(M)$. Two applications of the maximum principle to the nonconstant holomorphic function $F^{+} \circ \Phi^{-1}$ say that it must

[^11]decrease inside $\Sigma^{-}$, since $\Sigma^{-}$is interior to $\Phi(M)$ in $\mathbb{C}^{N}$, and that it must increase, since the one-sided neighborhood $U^{-} \cap \mathscr{V}(M)$ is exterior to $U^{+}$. This is the desired contradiction.
1.15. Levi extension theorem. A $\mathscr{C}^{2}$ hypersurface $M \subset \mathbb{C}^{n}$ may always be represented as $M=\{z \in U: r(z)=0\}$, where $U$ is some open neighborhood of $M$ in $\mathbb{C}^{n}$, and where $r: U \rightarrow \mathbb{R}$ is a $\mathscr{C}^{2}$ implicit defining function that satisfies $d r(q) \neq 0$ at every point $q$ of $M$. Two defining functions $r^{1}, r^{2}$ are nonzero multiple of each other in some neighborhood $V \subset U$ of $M$ : there exists $\lambda: V \rightarrow \mathbb{R}$ nowhere vanishing with $r_{2}=\lambda r_{1}$.

At a point $p \in M$, the Levi form of $r$ :

$$
\mathfrak{L}_{p} r\left(L_{p}, \bar{L}_{p}\right):=\sum_{1 \leqslant j, k \leqslant n} \frac{\partial^{2} r}{\partial z_{j} \partial \bar{z}_{k}}(p) L_{p}^{j} \bar{L}_{p}^{k}, \quad L_{p} \in T_{p}^{1,0} M
$$

is a Hermitian form that may be diagonalized. Its signature at $p$ :

$$
\left(a_{p}, b_{p}\right):=(\# \text { positive eigenvalues, \# negative eigenvalues })
$$

is the same for $r_{1}$ and $r_{2}$ if they are positive multiples of each other. It is also invariant through local biholomorphic changes of coordinates $z \mapsto \widetilde{z}(z)$ that do not reverse the orientation of $M$. Reversing the orientation or taking a negative factor $\lambda$ corresponds to the transposition $\left(a_{p}, b_{p}\right) \mapsto\left(b_{p}, a_{p}\right)$.

The Levi form may be read off a graphed equation $v=\varphi(x, y, u)$ for $M$.
Lemma 1.16. There exist local holomorphic coordinates $(z, u+i v) \in$ $\mathbb{C}^{n-1} \times \mathbb{C}$ centered at $p$ in which $M$ is represented as a graph of the form:

$$
v=\varphi(z, u)=\sum_{1 \leqslant k \leqslant a_{p}+b_{p}} \varepsilon_{k} z_{k} \bar{z}_{k}+\mathrm{o}\left(|z|^{2}\right)+\mathrm{O}(|z||u|)+\mathrm{O}\left(|u|^{2}\right),
$$

where $\varepsilon_{k}=+1$ for $1 \leqslant k \leqslant a_{p}, \varepsilon_{k}=-1$ for $a_{p}+1 \leqslant k \leqslant a_{p}+b_{p}$. If $a_{p}=n-1$, the open set $\{v>\varphi\}$ is strongly convex (in the real sense) in a neighborhood of $p$.

Assuming that $M$ is orientable, it is surrounded by two open sides. By an open side of $M$, we mean a connected component of $\mathscr{V} \backslash M$ for a (thin) neighborhood $\mathscr{V}$ of $M$ which is divided by $M$ in two components. As germs of open sets along $M$, there exist two open sides (if $M$ were not orientable, there would exist only one).

Assuming that $M$ is represented either by an implicit equation $r=0$ or as a graph $-v+\varphi(x, y, u)=0$, we adopt the convention of denoting:

$$
\begin{array}{lll}
\Omega^{+}:=\{r<0\} & \text { or } & \Omega^{+}:=\{v>\varphi(x, y, u)\} \\
\Omega^{-}:=\{r>0\} & \text { or } & \Omega^{-}:=\{v<\varphi(x, y, u)\}
\end{array}
$$

Once a local side $\Omega$ of $M$ has been fixed, $M$ is oriented and the indetermination $r \leftrightarrow-r$ disappears. By convention, we will always represent
$\Omega=\{r<0\}$. Then the number of positive and of negative eigenvalues of the Levi-form of $r$ at a point $p \in \partial \Omega$ is an invariant. By common abuse of language, we speak of the Levi form of $\partial \Omega$.
At one of its points $p$, a boundary $\partial \Omega$ is called strongly pseudoconvex (resp. strongly pseudoconcave) if its Levi form has all its eigenvalues $>0$ (resp. $<0$ ) at $p$. It is called weakly pseudoconvex (resp. weakly pseudoconcave) at $p$ if all eigenvalues are $\geqslant 0$ (resp. $\leqslant 0$ ). Often, the term "weakly" is dropped in common use.

Definition 1.17. If $\Omega$ is an open side of $M$, we say that $\Omega$ is holomorphically extendable at $p$ if for every open ${ }^{13}$ polydisc $U_{p}$ centered at $p$, there exists an open polydisc $V_{p}$ centered at $p$ such that for every $f \in \mathscr{O}\left(\Omega \cap U_{p}\right)$, there exists $F \in \mathscr{O}\left(V_{p}\right)$ with $\left.F\right|_{\Omega \cap V_{p}}=\left.f\right|_{\Omega \cap V_{p}}$.

In 1910, Levi localized the Hartogs extension phenomenon.
Theorem 1.18. ([Bo1991, Trp1996, Tu1998, BER1999]) If the Levi form of $M \subset \partial \Omega$ has one negative eigenvalue at a point $p$, then $\Omega$ is holomorphically extendable at $p$.

Proof. As $\Omega$ is given by $\left\{v>-z_{1} \bar{z}_{1}+\cdots\right\}$, for $\varepsilon>0$ small, the disc $A_{\varepsilon}(\zeta):=(\varepsilon \zeta, 0, \ldots, 0)$ has its boundary $A_{\varepsilon}(\partial \Delta)$ contained in $\Omega$ near $p$.

Lemma 1.19. Assume $M$ is $\mathscr{C}^{1}$, let $p \in M$ and $\Omega$ be an open side of $M$ at p. Suppose that for every open polydisc $U_{p}$ centered at $p$, there exists an analytic disc $A: \Delta \rightarrow U_{p}$ continuous in $\bar{\Delta}$ with $A(0)=p$ and $A(\partial \Delta) \subset \Omega$. Then $\Omega$ is holomorphically extendable at $p$.

To draw $A(\Delta)$, decreasing by 1 its dimension, we represent it as a curve. The cusp illustrates the fact that $A(\Delta)$ is not assumed to be embedded.


[^12]To prove the lemma, we may assume that $p=0$. Since $A(\partial \Delta)$ is contained in $\Omega$, for $z \in \mathbb{C}^{n}$ very small, say $|z|<\delta$, the translates $z+A(\partial \Delta)$ of the disc boundary are also contained in $\Omega$.

Consequently, the Cauchy integral:

$$
F(z):=\frac{1}{2 \pi i} \int_{\partial \Delta} f(z+A(\zeta)) \frac{d \zeta}{\zeta}
$$

is meaningful and it defines a holomorphic function of $z$ in the polydisc $V_{p}:=\{|z|<\delta\}$.

Does it coincide with $f$ in $V_{p} \cap \Omega$ ? The assumption that $M$ is $\mathscr{C}^{1}$ yields a local real segment $\ell_{p}$ transversal to $M$ at $p$. If $U_{p}$ is sufficiently small and if $z \in \ell_{p} \cap \Omega$ goes sufficiently deep in $\Omega$, the disc $z+A(\bar{\Delta})$ is contained in $\Omega$, so that the Cauchy integral $F(z)$ coincides with $f(z)$ for those $z$.
1.20. Contact of weakly pseudoconvex domains with complex hypersurfaces. The domain $\Omega$ is said to admit a support complex hypersurface at $p \in \partial \Omega$ if there exists a local (possibly singular) complex hypersurface $\Sigma$ passing through $p$ that does not intersect $\Omega$. In this situation, if $\Sigma=\{h(z)=0\}$ with $h$ holomorphic, the function $1 / h$ does not extend holomorphically at $p$, being unbounded. With $\alpha>0$ not integer, one may define branches $h^{\alpha}$ which are uniform in $\Omega$ and continuous up to $\partial \Omega$, but whose extension through $p$ would be ramified around $\Sigma$. Consequently, the existence of a support complex hypersurface prevents $\mathscr{O}(\Omega)$ to be holomorphically extendable at $p$. Is it the right obstruction? For instance, at a strongly pseudoconvex boundary point, the complex tangent plane is support.

Nevertheless, in 1973, Kohn-Nirenberg constructed a special pseudoconvex domain $\Omega_{\mathrm{K} N}^{+}$in $\mathbb{C}^{2}$ showing that:

- not every weakly pseudoconvex smoothly bounded domain is locally biholomorphically equivalent to a domain which is weakly convex in the real sense;
- the holomorphic non-extendability of $\mathscr{O}(\Omega)$ at $p$ is totally independent from the existence of a local supporting complex hypersurface at $p$.
The boundary of this domain

$$
M_{\mathrm{KN}}:=\left\{(z, w) \in \mathbb{C}^{2}: \operatorname{Im} w=|z w|^{2}+z^{4} \bar{z}^{4}+15\left(z^{7} \bar{z}+\bar{z}^{7} z\right) / 14\right\}
$$

may be checked to be strongly pseudoconvex at every point except the origin, where it is weakly pseudoconvex. Hence $\mathscr{O}\left(\Omega_{K N}^{+}\right)$is not holomorphically extendable at the origin. However, $M_{\mathrm{KN}}$ has the striking property that every local (possibly singular) complex hypersurface $\Sigma$ passing through the origin meets both sides of $M_{\mathrm{KN}}$ in every neighborhood of the origin. By
means of a Puiseux parametrization ([22]), such a complex curve $\Sigma$ is always the image of a certain holomorphic disc $\lambda: \Delta \rightarrow \mathbb{C}^{2}$ with $\lambda(0)=0$.

Theorem 1.21. ([KN1973]) Whatever the holomorphic disc $\lambda$, for every $\varepsilon>0$, there are $\zeta^{-}$and $\zeta^{+}$in $\Delta$ with $\left|\zeta^{ \pm}\right|<\varepsilon$ such that $\lambda\left(\zeta^{ \pm}\right) \in \Omega_{\mathrm{KN}}^{ \pm}$.

Clearly, $\Omega_{\mathrm{KN}}^{+}$is not locally convexifiable at the origin (otherwise, the biholomorphic image of the complex tangent line would be support).

Often for technical reasons, certain results in several complex variables require boundaries to be convex in the real sense. Although this condition is not biholomorphically invariant, it is certainly meaningful to characterize the class of convexifiable domains, at least locally: does there exist an analytico-geometric criterion enabling to recognize local convexifiability by reading a defining equation ?
1.22. Holomorphic extendability across finite type hypersurfaces. Let $M$ be a $\mathscr{C}^{\omega}$ hypersurface and let $p \in M$. Then $M$ is of type $m$ at $p$, in the sense of Definition 4.22(III), if and only if there exists a local graphed equation centered at $p$ of the form:

$$
v=\varphi_{m}(z, \bar{z})+\mathrm{O}\left(|z|^{m+1}\right)+\mathrm{O}(|z||u|)+\mathrm{O}\left(|u|^{2}\right),
$$

where $\varphi_{m} \in \mathbb{C}[z, \bar{z}]$ is a nonzero homogeneous real-valued polynomial of degree $m$ having no pluriharmonic term, namely $0 \equiv \varphi_{m}(0, \bar{z}) \equiv \varphi_{m}(z, 0)$. The restriction $\varphi_{m}(\ell(\zeta), \overline{\ell(\zeta)})$ of $\varphi_{m}$ to a complex line $\mathbb{C} \ni \zeta \mapsto \ell(\zeta) \in$ $\mathbb{C}^{n-1}, \ell(0)=0$, is a polynomial in $\mathbb{C}[\zeta, \bar{\zeta}]$. For almost every choice of $\ell$, this polynomial is nonzero, homogeneous of the same degree $m$ and contains no harmonic term. After a rotation, such a line is the complex $z_{1}$-axis. Denoting $z^{\prime}=\left(z_{2}, \ldots, z_{n-1}\right)$, we obtain:

$$
\begin{equation*}
v=\varphi_{m}\left(z_{1}, \bar{z}_{1}\right)+\mathrm{O}\left(\left|z_{1}\right|^{m+1}\right)+O\left(\left|z^{\prime}\right|\right)+\mathrm{O}(|z||u|)+\mathrm{O}\left(|u|^{2}\right) \tag{1.23}
\end{equation*}
$$

Theorem 1.24. ([BeFo1978, R1983, BT1984]) If $m$ is even, at least one side $\Omega^{+}$or $\Omega^{-}$is holomorphically extendable at $p$. If $m$ is odd, both sides have this property.

Proof. Let $\varepsilon>0$ arbitrarily small, let $a \in \mathbb{C}$ with $|a|<1$, let $\zeta$ be in the closed unit disc $\bar{\Delta}$, and introduce a $\mathbb{C}^{n}$-valued analytic disc:

$$
A_{\varepsilon}(\zeta):=\left(\varepsilon(a+\zeta), 0, \ldots, 0, \varepsilon^{m} W(\zeta)\right)
$$

having zero $z^{\prime}$-component and $z_{1}$-component being a disc of radius $\varepsilon$ centered at $-a$. We assume its $w$-component $W(\zeta)$ be defined by requiring that the $\mathbb{C}^{2}$-valued disc

$$
B_{\varepsilon}(\zeta):=\left(\varepsilon(a+\zeta), \varepsilon^{m} W(\zeta)\right)
$$

has its boundary $B_{\varepsilon}(\partial \Delta)$ attached to $v=\varphi_{m}\left(z_{1}, \bar{z}_{1}\right)$. By homogeneity, $\varepsilon^{m}$ drops and it is equivalent to require that $B_{1}$ is attached to $v=\varphi_{m}\left(z_{1}, \bar{z}_{1}\right)$. Equivalently, the imaginary part $V(\zeta)$ of $W=U+i V$ should satisfy:

$$
V\left(e^{i \theta}\right)=\varphi_{m}\left(a+e^{i \theta}, \bar{a}+e^{-i \theta}\right),
$$

for all $e^{i \theta} \in \partial \Delta$. To obtain a harmonic extension to $\Delta$ of the function $V$ thus defined on $\partial \Delta$, no Bishop equation is needed. It suffices to take the harmonic prolongation by means of Poisson's formula, as in §2.20(IV):

$$
V(\eta)=\mathrm{P} V(\eta)=\frac{1}{2 \pi i} \int_{\partial \Delta} \varphi_{m}(a+\zeta, \bar{a}+\bar{\zeta}) \frac{1-|\eta|^{2}}{|\zeta-\eta|^{2}} \frac{d \zeta}{\zeta} .
$$

Since $\varphi_{m}$ has no harmonic term, it may be factored as $\varphi_{m}=z_{1} \bar{z}_{1} \psi_{1}\left(z_{1}, \bar{z}_{1}\right)$, with $\psi_{1} \in \mathbb{C}\left[z_{1}, \bar{z}_{1}\right]$ homogeneous of degree $(m-2)$ and nonzero. In the integral above, we put $\eta:=-a$ and we replace $\varphi_{m}=z_{1} \bar{z}_{1} \psi_{1}$ to get the value of $V$ at $-a$ :

$$
\begin{aligned}
V(-a) & =\frac{1}{2 \pi i} \int_{\partial \Delta} \varphi_{m}(a+\zeta, \bar{a}+\bar{\zeta}) \frac{1-|a|^{2}}{|\zeta+a|^{2}} \frac{d \zeta}{\zeta} \\
& =\frac{1-|a|^{2}}{2 \pi} \int_{-\pi}^{\pi} \psi_{1}\left(a+e^{i \theta}, \bar{a}+e^{-i \theta}\right) d \theta .
\end{aligned}
$$

As a function of $a \in \Delta$, the last integral is identically zero if and only if the polynomial $\psi_{1}$ is zero. Thus, there exists $a$ such that $V(-a) \neq 0$. (However, we have no information about the possible signs of $V(-a)$ in terms of $\varphi_{m}$.) Then we define $U(\zeta)$ to be the harmonic prolongation of $-\mathrm{T} V$ that vanishes at $-a$ and $B_{\varepsilon}(\zeta):=\left(\varepsilon(a+\zeta), \varepsilon^{m} W(\zeta)\right)$.

The positivity (resp. negativity) of the sign of $V(-a)$ means that $B_{\varepsilon}(-a)=(0, i V(-a))$ is in $\Omega^{+}$(resp. $\left.\Omega^{-}\right)$. Then after translating slightly $B_{\varepsilon}$ in the right direction along the $v$-axis, Lemma 1.19 applies to deduce that $\Omega_{1}^{-} \subset \mathbb{C}^{2}\left(\right.$ resp. $\left.\Omega_{1}^{+} \subset \mathbb{C}^{2}\right)$ is holomorphically extendable at the origin. Since $\varphi_{m}\left(-z_{1},-\bar{z}_{1}\right)=(-1)^{m} \varphi_{m}\left(z_{1}, \bar{z}_{1}\right)$, in the case where $m$ is odd, the disc $-B_{\varepsilon}$ will also be attached to $M_{1}$ and will provide extendability of the other side.

Thanks to basic majorations of the " O " remainders in the equation (1.23) of $M$, if $\varepsilon>0$ is sufficiently small, then $\Omega^{-} \subset \mathbb{C}^{n}$ (resp. $\Omega^{+} \subset \mathbb{C}^{n}$ ) has the same extendability property.

If $M$ is a real analytic hypersurface, it is easily seen, by inspecting the Taylor series of its graphing function, that $M$ is not of finite type at a point $p$ if and only if it may be represented by $v=u \widetilde{\varphi}(z, u)$, with $\widetilde{\varphi} \in \mathscr{C}^{\omega}$. Then the local complex hypersurface $\{v=u=0\}$ is contained in $M$.

Corollary 1.25. ([BeFo1978, R1983, BT1984]) If $M$ is $\mathscr{C}^{\omega}$ and if $p \in M$, the following properties are equivalent:

- $M$ has finite type at $p$;
- M does not contain any local complex analytic hypersurface passing through $p$;
- $\Omega^{+}$or $\Omega^{-}$is holomorphically extendable at $p$.
1.26. Which side is holomorphically extendable? We claim that it suffices to study osculating domains in $\mathbb{C}^{2}$ of the form:

$$
\Omega_{\varphi_{m}}^{+}:=\left\{-v+\varphi_{m}(z, \bar{z})<0\right\}, \quad z \in \mathbb{C}, w=u+i v \in \mathbb{C},
$$

where $\varphi_{m} \neq 0$ is real, homogeneous of degree $m \geqslant 2$ and has no harmonic term. Indeed, extendability properties of such domains transfer to perturbations (1.23). Also, extendability properties of $\Omega_{\varphi_{m}}^{-}$are just the same, via $\varphi_{m} \leftrightarrow-\varphi_{m}$. For this reason, if $m$ is odd, both $\Omega_{\varphi_{m}}^{+}$and $\Omega_{\varphi_{m}}^{-}$are holomorphically extendable at $p$.

The local complex line $\{(z, 0)\}$ intersects the closure $\overline{\Omega_{\varphi_{m}}^{+}}$in regions that are closed angular sectors (cones), due to homogeneity. We call these regions interior. The complement $\mathbb{C}^{2} \backslash \Omega_{\varphi_{m}}^{+}$intersects $\{(z, 0)\}$ in open, exterior sectors.

Theorem 1.27. ([R1983, BT1984, FR1985]) If there exists an interior sector of angular width $>\frac{\pi}{m}$, then $\Omega_{\varphi_{m}}^{+}$is holomorphically extendable at $p$.

The proof consists in choosing an appropriate truncated angular sector as the $z$-component of a disc attached to $\partial \Omega_{\varphi_{m}}^{+}$, instead of the round disc $\zeta \mapsto \varepsilon(a+\zeta)$.

Example 1.28. Every homogeneous quartic $v=\varphi_{4}(z, \bar{z})$ in $\mathbb{C}^{2}$ is biholomorphically equivalent to a model

$$
0=r_{a}:=-v+z^{2} \bar{z}^{2}+a z \bar{z}\left(z^{2}+\bar{z}^{2}\right)
$$

for some $a \in \mathbb{R}$. Such a hypersurface bounds two open sides $\Omega_{a}^{+}=\left\{r_{a}<\right.$ $0\}$ and $\Omega_{a}^{-}=\left\{r_{a}>0\right\}$ which enjoy the following properties ([R1983, BT1984]):

- $\Omega_{a}^{-}$is holomorphically extendable at $p$, for every $a$;
- $|a|<2 / 3$ if and only if $\Omega_{a}^{+}$is everywhere strongly pseudoconvex;
- $|a| \leqslant 1 / \sqrt{2}$ if and only if $\Omega_{a}^{+}$is not holomorphically extendable at $p$;
- $|a|>1 / \sqrt{2}$ if and only if $\Omega_{a}^{+}$is holomorphically extendable at $p$;
- the above extendability property holds true for any perturbation of $\partial \Omega$ by higher order terms.

Finer results, strictly more general than the above theorem that apply to sixtics, were obtained in [FR1985]. If we remove all exterior sectors of angular width $\geqslant \frac{\pi}{m}$, the rest of the complex line $\{(z, 0)\}$ is formed by disjoint closed sectors, which are called supersectors of order $m$ of $\Omega_{\varphi_{m}}^{+}$at $p$. A supersector is proper if it contains points of $\Omega_{\varphi_{m}}^{+}$.
Theorem 1.29. ([FR1985])
(i) If $\Omega_{\varphi_{m}}^{+}$has a proper supersector of angular width $>\frac{\pi}{m}$, then $\Omega_{\varphi_{m}}^{+}$is holomorphically extendable at $p$.
(ii) If all supersectors of $\Omega_{\varphi_{m}}^{+}$have angular width $<\frac{\pi}{m}$, then there exists $f \in \mathscr{O}\left(\Omega_{\varphi_{m}}^{+}\right) \cap \mathscr{C}^{0}\left(\overline{\Omega_{\varphi_{m}}^{+}}\right)$that does not extend holomorphically at $p$.

Even in the case $m=6$, some cases in this theorem are left open. Examples may be found in [FR1985].

Open problem 1.30. In the case where $m$ is even, find a necessary and sufficient condition for $\Omega_{\varphi_{m}}^{+}=\left\{v>\varphi_{m}(z, \bar{z})\right\}$ to be holomorphically extendable at $p$, or show that the problem is undecidable.

One could generalize this (already wide open) question to a not necessarily finite type boundary, $\mathscr{C}^{\omega}, \mathscr{C}^{\infty}, \mathscr{C}^{2}$ or even $\mathscr{C}^{0}$ graph.

## §2. TRÉPREAU'S THEOREM, DEFORMATIONS OF BISHOP DISCS AND PROPAGATION ON HYPERSURFACES

2.1. Holomorphic extension of $\mathbf{C R}$ functions via jump. Let $M$ be a hypersurface in $\mathbb{C}^{n}$ of class at least $\mathscr{C}^{1, \alpha}$ with $0<\alpha<1$ and let $f$ be a continuous CR function on $M$. At each point $p$ of $M$, we may restrict $f$ to a small open ball (or polydisc) $\Omega_{p}$ centered at $p$. Applying the jump Theorem 1.11, we may represent $f=F^{+}-F^{-}$, with $F^{ \pm} \in \mathscr{O}\left(\Omega_{p}^{ \pm}\right) \cap \mathscr{C}^{0}\left(\overline{\Omega_{p}^{ \pm}}\right)$. If $\Omega_{p}^{+}$(resp. $\Omega_{p}^{-}$) is holomorphically extendable at $p$, then $F^{+}$(resp. $F^{-}$) extends to a neighborhood $\omega_{p}$ of $p$ in $\mathbb{C}^{n}$ as $G \in \mathscr{O}\left(\omega_{p}\right)$ (resp. $H \in \mathscr{O}\left(\omega_{p}\right)$ ). Then $f$ extends holomorphically to the small one-sided neighborhood $\omega_{p}^{-}$(resp. to $\left.\omega_{p}^{+}\right)$as $G-F^{-}\left(\right.$resp. as $\left.F^{+}-H\right)$.

Lemma 2.2. On hypersurfaces, at a given point, local holomorphic extendability of CR functions to one side is equivalent to holomorphic extendability to the same side of the holomorphic functions defined in the opposite side.

Consequently, the theorems of $\S 1.22$ yield gratuitously extension results about CR functions. For instance:

Corollary 2.3. ([BeFo1978, R1983, BT1984]) On a real analytic hypersurface $M$, at a given point p, continuous CR functions extend holomorphically to one side if and only if $M$ does not contain any local complex hypersurface passing through $p$.

The assumption of real analyticity, or the assumption of finite typeness in case $M$ is $\mathscr{C}^{\infty}$, both consume much smoothness. The removal of these assumptions was accomplished by Trépreau in 1986.
Theorem 2.4. ([Trp1986]) Let $M$ be a $\mathscr{C}^{2}$ hypersurface of $\mathbb{C}^{n}, n \geqslant 2$ and let $p \in M$. The following two conditions are equivalent:

- $M$ does not contain any local complex hypersurface passing through p.
- for every open subset $U_{p} \subset M$ containing p, there exists a onesided neighborhood $\omega_{p}^{ \pm}$of $M$ at $p$ with $\overline{\omega_{p}^{ \pm}} \cap M \Subset U_{p}$ such that for every $f \in \mathscr{C}_{C R}^{0}\left(U_{p}\right)$, there exists $F \in \mathscr{O}\left(\omega_{p}^{ \pm}\right) \cap \mathscr{C}^{0}\left(\omega^{ \pm} \cup U_{p}\right)$ with $\left.F\right|_{U_{p}}=f$.

We have seen that characterizing the side of extension is an open question, even for rigid polynomial hypersurfaces $v=\varphi_{m}(z, \bar{z})$ and even for $m=$ 6. Although the above theorem constitutes a neat answer for holomorphic extension to some imprecise side, it does not provide any control of the side of extension.

Let $M$ be a $\mathscr{C}^{2}$ orientable connected hypersurface and let $\Omega_{M}^{+}$be an open side of $M$. One could hope to characterize holomorphic extension to the other side at every point of $M$, since weak pseudoconvexity characterizes holomorphic non-extendability at every point of $M$, by Oka's theorem.
Example 2.5. ([Trp1992]) In $\mathbb{C}^{3}$, let $\Omega_{M}^{+}$be $\left\{v>\varphi_{m}\left(z_{1}, \bar{z}_{1}\right)-\left|z_{2}\right|^{2}\left|z_{1}\right|^{2 N}\right\}$ where $\varphi_{m} \not \equiv 0$, of degree $m$ with $3 \leqslant m<N$ is as in Open problem 1.30. One verifies that holomorphic extension at every point of $M$ entails a characterization of holomorphic extension at the origin for the domain $\left\{v>\varphi_{m}(z, \bar{z})-\varepsilon|z|^{2 N}\right\}$.

In the sequel, we shall abandon definitely the difficult, still open question of how to control sides of holomorphic extension.

Although Theorem 2.4 is well known in Several Complex Variables, there is a more general formulation with a simpler proof than the original one. The remainder of this section will expose such a proof.

By a global one-sided neighborhood of a connected (not necessarily orientable) hypersurface $M \subset \mathbb{C}^{n}$, we mean a domain $\Omega_{M}$ with $\bar{\Omega}_{M} \supset M$ such that for every point $q \in M$, at least one open side $\omega_{q}^{ \pm}$of $M$ at $q$ is contained in $\Omega_{M}$. In fact, to insure connectedness, $\Omega_{M}$ is the interior of the closure of the union $\cup_{q \in M} \omega_{q}^{ \pm}$of all (possibly shrunk) one-sided neighborhoods.


Then $\Omega_{M}$ contains a neighborhood in $\mathbb{C}^{n}$ of every point $r \in M$ which belongs to at least two one-sided neighborhoods that are opposite. The classical Morera theorem insures holomorphicity in a neighborhood of such points $r$.

Remind that $M$ is called globally minimal if it consists of a single CR orbit. The assumption that $M$ does not contain any complex hypersurface at any point means that for every $p \in M$, every open $U_{p} \ni p$, the CR orbit $\mathscr{O}_{C R}\left(U_{p}, p\right)$ contains a neighborhood of $p$ in $M$. This implies that $M$ is globally minimal and hence, Theorem 2.4 is less general than the following.

Theorem 2.6. ([Trp1990, Tu1994a]) Let M be ${ }^{14}$ a connected $\mathscr{C}^{2, \alpha}(0<\alpha<$ 1) hypersurface of $\mathbb{C}^{n}(n \geqslant 2)$. If $M$ is globally minimal, then there exists a global one-sided neighborhood $\Omega_{M}$ of $M$ such that for every continuous CR function $f \in \mathscr{C}_{C R}^{0}(M)$, there exists $F \in \mathscr{O}\left(\Omega_{M}\right) \cap \mathscr{C}^{0}\left(\Omega_{M} \cup M\right)$ with $\left.F\right|_{M}=f$.

It will appear that $\Omega_{M}$ is contructed by gluing discs to $M$ and to subsequent open sets $\Omega^{\prime} \subset \Omega_{M}$ which are all contained in the polynomial hull of M:

$$
\widehat{M}:=\left\{z \in \mathbb{C}^{n}:|P(z)| \leqslant \sup _{w \in M}|P(w)|, \forall P \in \mathbb{C}[z]\right\}
$$

Let us summarize the proof. Although the assumption of global minimality is so weak that $M$ may incorporate large open Levi-flat regions, there exists at least one point $p \in M$ in a neighborhood of which

$$
T_{q} M=T_{q}^{c} M+\left[T_{q}^{c} M, T_{q}^{c} M\right], \quad q \in U_{p} .
$$

Otherwise, the distribution $p \mapsto T_{p}^{c} M$ would be Frobenius-integrable and all CR orbits would be complex hypersurfaces ! At such a point $p$, the classical Lewy extension theorem ( $\$ 2.10$ below) insures that $\mathscr{C}_{C R}^{0}(M)$ extends holomorphically to (at least) one side at $p$.

Theorem 2.7. ([Trp1990, Tu1994a]) Let $M$ be a connected $\mathscr{C}^{2, \alpha}$ hypersurface, not necessarily globally minimal. If $\mathscr{C}_{C R}^{0}(M)$ extends holomorphically

[^13]to a one-sided neighborhood at some point $p \in M$, then holomorphic extension to one side $\omega_{q}^{ \pm}$holds at every point $q \in \mathscr{O}_{C R}(M, p)$.

When $\mathscr{O}_{C R}(M, p)=M$ as in Theorem 2.6, the global one-sided neighborhood $\Omega_{M}$ will be the interior of the closure of the union $\cup_{q \in M} \omega_{q}^{ \pm}$of all (possibly shrunk) one-sided neighborhoods.

The next paragraphs are devoted to expose a detailed proof of both the Lewy theorem and of the above propagation theorem.
2.8. Approximation theorem and maximum principle. According to the approximation Theorem 5.2(III), for every $p \in M$, there exist a neighborhood $U_{p}$ of $p$ in $M$ and a sequence $\left(P_{\nu}(z)\right)_{\nu \in \mathbb{N}}$ of holomorphic polynomials with $\lim _{\nu \rightarrow \infty}\left\|P_{\nu}-f\right\|_{\mathscr{C}^{0}\left(U_{p}\right)}=0$.

Lemma 2.9. For every analytic disc $A \in \mathscr{O}(\Delta) \cap \mathscr{C}^{0}(\bar{\Delta})$ with $A(\partial \Delta) \subset U_{p}$, the sequence $P_{\nu}$ also converges uniformly on the closed disc $A(\bar{\Delta})$, even if $A(\bar{\Delta})$ goes outside $U_{p}$.
Proof. By assumption, $\lim _{\nu, \mu \rightarrow \infty}\left\|P_{\nu}-P_{\mu}\right\|_{\mathscr{C O}_{0}\left(U_{p}\right)}=0$. Let $\eta \in \bar{\Delta}$ arbitrary. Thanks to the maximum principle and to $A(\partial \Delta) \subset U_{p}$ :

$$
\begin{aligned}
\left\|P_{\nu}(A(\eta))-P_{\mu}(A(\eta))\right\| & \leqslant \max _{\zeta \in \partial \Delta}\left\|P_{\nu}(A(\zeta))-P_{\mu}(A(\zeta))\right\| \\
& \leqslant \sup _{z \in U_{p}}\left\|P_{\nu}(z)-P_{\mu}(z)\right\| \longrightarrow 0 .
\end{aligned}
$$

The same argument shows that $P_{\nu}$ converges uniformly in the polynomial hull of $U_{p}$ (we shall not need this).

Next, suppose that we have some family of analytic discs $A_{s}$, with $s$ a small parameter, such that $\cup_{s} A_{s}(\Delta)$ contains an open set in $\mathbb{C}^{n}$, for instance a one-sided neighborhood at $p \in M$. Then $\left(P_{\nu}\right)_{\nu \in \mathbb{N}}$ converges uniformly on $\cup_{s} A_{s}(\Delta)$ and a theorem due to Cauchy assures that the limit is holomorphic in the interior of $\cup_{s} A_{s}(\Delta)$. It then follows that $f$ extends holomorphically to the interior of $\cup_{s} A_{s}(\Delta)$.

Remarkably, this short argument based on an application of the approximation Theorem 5.2(III) shows that ${ }^{15}$ :

In order to establish local holomorphic extension of CR functions, it suffices to glue appropriate families of analytic discs to CR manifolds.
In the sequel, the geometry of glued discs will be studied for itself; thus, it will be understood that statements about holomorphic or CR extension follow immediately; elementary details about continuity of the obtained extensions will be skipped.

[^14]2.10. Lewy extension. Since $M$ is globally minimal, there exists a point $p$ at which $T_{p} M=T_{p}^{c} M+\left[T_{p}^{c} M, T_{p}^{c} M\right]$. Granted Lemma 2.2, holomorphic extension to one side at such a point $p$ has already been proved in Theorem 1.18. Nevertheless, we want to present a geometrically different proof that will produce preliminaries and motivations for the sequel.

Since $T^{c} M=\operatorname{Re} T^{1,0} M=\operatorname{Re} T^{0,1} M$, we have equivalently $\left[T^{1,0} M, T^{0,1} M\right](p) \not \subset \mathbb{C} \otimes T_{p}^{c} M$, namely the intrinsic Levi form of $M$ at $p$ is nonzero. In other words, there exists a local section $L$ of $T^{1,0} M$ with $L(p) \neq 0$ and $[L, \bar{L}](p) \notin \mathbb{C} \otimes T_{p}^{c} M$. After a complex linear transformation of $T_{p}^{c} M$, we may assume that $L(p)=\frac{\partial}{\partial z_{1}}$. After removing the second order pluriharmonic terms, there exist local coordinates $\left(z_{1}, z^{\prime}, w\right)$ vanishing at $p$ such that $M$ is represented by

$$
v=-z \bar{z}_{1}+\mathrm{O}\left(\left|z_{1}\right|^{2+\alpha}\right)+\mathrm{O}\left(\left|z^{\prime}\right|\right)+\mathrm{O}(|z||u|)+\mathrm{O}\left(|u|^{2}\right) .
$$

The minus sign is set for clarity in the diagram of $\S 2.12$ below. We denote by $\varphi\left(z_{1}, z^{\prime}, u\right)$ the right hand side. Let $\varepsilon_{1}>0$ be small. For $\varepsilon$ satisfying $0<\varepsilon \leqslant \varepsilon_{1}$, we introduce the analytic disc

$$
A_{\varepsilon}(\zeta):=\left(\varepsilon(1-\zeta), 0^{\prime}, U_{\varepsilon}(\zeta)+i V_{\varepsilon}(\zeta)\right)
$$

with zero $z^{\prime}$-component, with $z_{1}$-component equal to a (reverse) round disc of radius $\varepsilon$ centered at $1 \in \mathbb{C}$ and with $u$-component $U_{\varepsilon}$ satisfying the Bishop-type equation:

$$
U_{\varepsilon}\left(e^{i \theta}\right)=-\mathrm{T}_{1}\left[\varphi\left(\varepsilon(1-\cdot), 0^{\prime}, U_{\varepsilon}(\cdot)\right)\right]\left(e^{i \theta}\right) .
$$

Acoording to Theorem 3.7(IV), a unique solution $U_{\varepsilon}\left(e^{i \theta}\right)$ exists and is $\mathscr{C}^{2, \alpha-0}$ with respect to $\left(e^{i \theta}, \varepsilon\right)$. Since $\mathrm{T}_{1}(\psi)(1)=0$ by definition, we have $U_{\varepsilon}(1)=0$ and then $V_{\varepsilon}:=\mathrm{T}_{1}\left(U_{\varepsilon}\right)$ also satisfies $V_{\varepsilon}(1)=0$. Consequently, $A_{\varepsilon}(1)=0$. By applying $\mathrm{T}_{1}$ to both sides of the above equation, we see that the disc is attached to $M$ :

$$
V_{\varepsilon}\left(e^{i \theta}\right)=\varphi\left(\varepsilon\left(1-e^{i \theta}\right), 0^{\prime}, U_{\varepsilon}\left(e^{i \theta}\right)\right) .
$$

We shall prove that for $\varepsilon_{1}$ sufficiently small, every disc $A_{\varepsilon}(\Delta)$ with $0<\varepsilon \leqslant$ $\varepsilon_{1}$ is not tangent to $M$ at $p$. We draw two diagrams: a 2 -dimensional and a 3 -dimensional view. In both, the $v$-axis is vertical, oriented down.


Just now, we need a geometrical remark. Let $A \in \mathscr{O}(\Delta) \cap \mathscr{C}^{1}(\bar{\Delta})$ be an arbitrary but small analytic disc attached to $M$ with $A(1)=0$. We use polar coordinates to denote $\zeta=r e^{i \theta}$.


The holomorphicity of $A$ yields the following identities between vectors in $T_{p} \mathbb{C}^{n}$ :

$$
\left.i \frac{\partial A}{\partial \theta}\left(e^{i \theta}\right)\right|_{\theta=0}=-\left.\frac{\partial A}{\partial r}(r)\right|_{r=1}=-\left.\frac{\partial A}{\partial \zeta}(\zeta)\right|_{\zeta=1}
$$

The multiplication by $i$ (or equivalently the complex structure $J$ ) provides an isomorphism $T_{p} \mathbb{C}^{n} / T_{p} M \rightarrow T_{p} M / T_{p}^{c} M$; in coordinates, $T_{p} \mathbb{C}^{n} / T_{p} M \simeq \mathbb{R}_{v}$, $T_{p} M / T_{p}^{c} M \simeq \mathbb{R}_{u}$ and $J$ sends $\mathbb{R}_{u}$ to $\mathbb{R}_{v}$. It follows that $\frac{\partial A}{\partial r}(1)$ is not tangent to $M$ at $p$ if and only if $\frac{\partial A}{\partial \theta}(1)$ is not complex tangent to $M$ at $p$.

Coming back to $A_{\varepsilon}$, we call the vector

$$
-\frac{\partial A_{\varepsilon}}{\partial r}(1) \bmod T_{p} M=-\frac{\partial W_{\varepsilon}}{\partial r}(1) \bmod T_{p} M
$$

the exit vector of $A_{\varepsilon}$. By differentiating $V_{\varepsilon}=\varphi$ at $\theta=0$, taking account of $d \varphi(0)=0$, we get $\frac{\partial V_{\varepsilon}}{\partial \theta}(1)=0$. So only the real part $\frac{\partial U_{\varepsilon}}{\partial \theta}(1)$ of $\frac{\partial W_{\varepsilon}}{\partial \theta}(1)$ may be nonzero.

Lemma 2.11. Shrinking $\varepsilon_{1}$ if necessary, the exit vector of every disc $A_{\varepsilon}$ with $0<\varepsilon \leqslant \varepsilon_{1}$ is nonzero:

$$
-\frac{\partial W_{\varepsilon}}{\partial r}(1)=i \frac{\partial W_{\varepsilon}}{\partial \theta}(1)=i \frac{\partial U_{\varepsilon}}{\partial \theta}(1) \neq 0 .
$$

Proof. The principal term of $\varphi$ is $-z_{1} \bar{z}_{1}$. We compute first:

$$
\begin{aligned}
\mathrm{T}_{1}\left[-Z_{1}(\zeta) \bar{Z}_{1}(\zeta)\right] & =\mathrm{T}_{1}\left[\varepsilon^{2}\left(e^{-i \theta}-2+e^{i \theta}\right)\right] \\
& =\frac{1}{i} \varepsilon^{2}\left(-e^{-i \theta}+e^{i \theta}\right) .
\end{aligned}
$$

Proceeding as carefully as in Section 3(IV), we may verify that

$$
\begin{aligned}
U_{\varepsilon}\left(e^{i \theta}\right) & =-\mathbf{T}_{1}\left[-Z_{1}(\zeta) \bar{Z}_{1}(\zeta)+\text { Remainder }\right]\left(e^{i \theta}\right) \\
& =-2 \varepsilon^{2} \sin \theta+\widetilde{U}_{\varepsilon}\left(e^{i \theta}\right),
\end{aligned}
$$

with a $\mathscr{C}^{2, \alpha-0}$ remainder satisfying $\left\|\widetilde{U}_{\varepsilon}\right\|_{1,0} \leqslant \mathrm{~K} \varepsilon^{2+\alpha}$, for some quantity $\mathrm{K}>0$. So $\frac{\partial U_{\varepsilon}}{\partial \theta}(1)=-2 \varepsilon^{2}+\mathrm{O}\left(\varepsilon^{2+\alpha}\right) \neq 0$.
2.12. Translations of a nontangent analytic disc. We now fix $\varepsilon$ with $0<$ $\varepsilon \leqslant \varepsilon_{1}$ and we denote simply by $A$ the disc $A_{\varepsilon}$. So the vector

$$
\frac{\partial A}{\partial \theta}(1)=\left(-i \varepsilon, 0^{\prime},-2 \varepsilon^{2}+\mathrm{O}\left(\varepsilon^{2+\alpha}\right)\right)
$$

is not tangent to $T_{p}^{c} M=\{v=u=0\}$ at the origin. Furthermore, it is not tangent to the $(2 n-2)$-dimensional sub-plane $\left\{y_{1}=v=0\right\}$ of $T_{p} M=\{v=0\}$.

We now introduce parameters of translation $x_{1}^{0} \in \mathbb{R}, z_{0}^{\prime} \in \mathbb{C}^{n-2}$ and $u_{0} \in \mathbb{R}$ with $\left|x_{1}^{0}\right|,\left|z_{0}^{\prime}\right|,\left|u_{0}\right|<\delta_{1}$, where $0<\delta_{1} \ll \varepsilon$. The points in $M$ of coordinates

$$
\left(x_{1}^{0}, z_{0}^{\prime}, u_{0}+i \varphi\left(x_{1}^{0}, z_{0}^{\prime}, u_{0}\right)\right)
$$

cover a small $(2 n-2)$-dimensional submanifold $K_{p}$ with $T_{p} K_{p}=\left\{y_{1}=\right.$ $v=0\}$ transverse to the disc boundary $A_{\varepsilon}(\partial \Delta)$ at $p$ that we draw below.


To conclude the proof of one-sided holomorphic extension at the Levi nondegenerate point $p$, it suffices to deform the disc $A_{x_{1}^{0}, z_{0}^{\prime}, u_{0}}$ so that its distinguished point $A_{x_{1}^{0}, z_{0}^{\prime}, u_{0}}(1)$ covers the submanifold $K_{p}$, namely

$$
\begin{equation*}
A_{x_{1}^{0}, z_{0}^{\prime}, u_{0}}(1)=\left(x_{1}^{0}, z_{0}^{\prime}, u_{0}+i \varphi\left(x_{1}^{0}, z_{0}^{\prime}, u_{0}\right)\right) . \tag{2.13}
\end{equation*}
$$

This may be achieved easily by defining

$$
\left(Z_{1, x_{1}^{0}}(\zeta), Z_{z_{0}^{\prime}}^{\prime}(\zeta)\right):=\left(\varepsilon_{1}(1-\zeta)+x_{1}^{0}, z_{0}^{\prime}\right)
$$

and by solving the Bishop-type equation:

$$
\begin{equation*}
U_{x_{1}^{0}, z_{0}^{\prime}, u_{0}}\left(e^{i \theta}\right)=u_{0}-\mathrm{T}_{1}\left[\varphi\left(Z_{1 ; x_{1}^{0}}(\cdot), Z_{z_{0}^{\prime}}^{\prime}(\cdot), U_{x_{1}^{0}, z_{0}^{\prime}, u_{0}}(\cdot)\right)\right]\left(e^{i \theta}\right) \tag{2.14}
\end{equation*}
$$

for the $u$-component of the sought disc $A_{x_{1}^{0}, z_{0}^{\prime}, u_{0}}$. Thanks to Theorem $3.7(\mathrm{IV})$, the solution exists and is $\mathscr{C}^{2, \alpha-0}$ with respect to all the
variables. We finally define the $v$-component of $A_{x_{1}^{0}, z_{0}^{\prime}, u_{0}}$ :

$$
\begin{equation*}
V_{x_{1}^{0}, z_{0}^{\prime}, u_{0}}\left(e^{i \theta}\right):=\mathbf{T}_{1}\left[U_{x_{1}^{0}, z_{0}^{\prime}, u_{0}}(\cdot)\right]\left(e^{i \theta}\right)+\varphi\left(x_{1}^{0}, z_{0}^{\prime}, u_{0}\right) . \tag{2.15}
\end{equation*}
$$

Applying $\mathrm{T}_{1}$ to (2.14), we see that this disc is attached to $M$; also, putting $e^{i \theta}:=1$ in (2.14) and in (2.15), we see that (2.13) holds. Geometrically, the $(2 n-2)$ added parameters $\left(x_{1}^{0}, z_{0}^{\prime}, u_{0}\right)$ correspond to translations in $M$ of the original disc $A_{\varepsilon_{1}}$.


Define the open circular region $\Delta_{1}:=\left\{\zeta \in \Delta:|\zeta-1|<\delta_{1}\right\}$ around 1 in the unit disc. Then we claim that shrinking $\delta_{1}>0$ if necessary, the set

$$
\left\{A_{x_{1}^{0}, z_{0}^{\prime}, u_{0}}(\zeta): \zeta \in \Delta_{1},\left|x_{1}^{0}\right|<\delta_{1},\left|z_{0}^{\prime}\right|<\delta_{1},\left|u_{0}\right|<\delta_{1}\right\}
$$

contains a one-sided neighborhood of $M$ at $p=A_{0,0,0}(1)$. Indeed, by computation, one may check that the $2 n$ vectors of $T_{p} \mathbb{C}^{n}$
$\frac{\partial A_{0,0,0}}{\partial x_{1}}(1), \frac{\partial A_{0,0,0}}{\partial \theta}(1), \frac{\partial A_{0,0,0}}{\partial x_{k}^{\prime}}(1), \frac{\partial A_{0,0,0}}{\partial y_{k}^{\prime}}(1), \frac{\partial A_{0,0,0}}{\partial u}(1),-\frac{\partial A_{0,0,0}}{\partial r}(1)$,
are linearly independent; geometrically and by construction, the first ( $2 n-1$ ) of these vectors span $T_{p} M$ and the last one is linearly independent, since by construction the exit vector of $A_{\varepsilon_{1}}$ is nontangent to $M$ at $p$.

Incidentally, we have proved an elementary but crucial statement: by "translating" (through a suitable Bishop-type equation) any small attached disc whose exit vector is nonzero, we may always cover a one-sided neighborhood.

Lemma 2.16. If a small disc $A$ attached to a hypersurface $M$ satisfies $\frac{\partial A}{\partial \theta}(1) \notin T_{A(1)}^{c} M$, or equivalently $-\frac{\partial A}{\partial r}(1) \notin T_{A(1)} M$, then continuous $C R$ functions on $M$ extend holomorphically at $A(1)$ to the side in which points the nonzero exit vector $i \frac{\partial A}{\partial \theta}(1)=-\frac{\partial A}{\partial r}(1)$.

Of course, the choice of the point $1 \in \partial \Delta$ is no restriction at all, since after a Möbius reparametrization, any given point $\zeta_{0} \in \partial \Delta$ becomes $1 \in \partial \Delta$.
2.17. Propagation of holomorphic extension. The Levi form assumption $T_{p} M=T_{p}^{c} M+\left[T_{p}^{c} M, T_{p}^{c} M\right]$ was strongly used to insure the existence of a disc having a nonzero exit vector at $p$. But if a disc $A$ is attached to a highly degenerate part of $M$, for instance to a region where the Levi form is nearly flat, the disc $A$ might well satisfy $\frac{\partial A}{\partial \theta}\left(\zeta_{0}\right) \in T_{A\left(\zeta_{0}\right)}^{c} M$ (or equivalently, $\left.-\frac{\partial A}{\partial r}\left(\zeta_{0}\right) \in T_{A\left(\zeta_{0}\right)} M\right)$, for every $\zeta_{0} \in \partial \Delta$. Then we are stuck.

To go through, two strategies are known in the literature.

- Devise refined pointwise "finite type" assumptions insuring the existence of small discs having nonzero exit vector at a given central point.
- Devise deformation arguments that propagate holomorphic extension from Levi nondegenerate regions up to highly degenerate regions.

Unfortunately, the first, more developed strategy is unable to provide any proof of Theorem 2.6. Indeed, a smooth globally minimal hypersurface may well contain large Levi-flat regions, as for instance $\left\{(z, w) \in \mathbb{C}^{2}\right.$ : $v=\varpi(x)\}$ with a $\mathscr{C}^{\infty}$ function $\varpi$ satisfying $\varpi(x) \equiv 0$ for $x \leqslant 0$ and $\varpi(x)>0$ for $x>0$ (to check global minimality, proceed as in Example 3.10); Theorem 4.8(III) shows that a Levi-flat portion $M_{\text {LF }}$ of a hypersurface $M$ is locally foliated by complex $(n-1)$-dimensional submanifolds; the uniqueness in Bishop's equation ${ }^{16}$ then entails that every small analytic disc $A \in \mathscr{O}(\Delta) \cap \mathscr{C}^{1}(\bar{\Delta})$ with $A(\partial \Delta) \subset M_{\text {LF }}$ must satisfy $A(\partial \Delta) \subset \Sigma$, where $\Sigma \subset M_{\mathrm{LF}}$ is the unique local complex connected $(n-1)$-dimensional submanifold of the foliation that contains $A(1)$; then the uniqueness principle for holomorphic maps between complex manifolds yields $A(\bar{\Delta}) \subset \Sigma$; finally, $-\frac{\partial A}{\partial r}\left(\zeta_{0}\right) \in T_{A\left(\zeta_{0}\right)} \Sigma=T_{A\left(\zeta_{0}\right)}^{c} M$ has exit vector tangential to $M$ at every $\zeta_{0} \in \partial \Delta$.

For this reason, we will focus our attention only on the second, most powerful strategy, starting with a review.

After works of Sjöstrand ([HS1982, Sj1982a, Sj1982b]) on propagation of singularities for certain classes of partial differential operators, of Baouendi-Chang-Treves [BCT1983], and of Hanges-Treves [HT1983], Trépreau [Trp1990] showed that the hypoanalytic wave-front set of a CR function or distribution propagates along complex-tangential curves. The microlocal technique involves deforming $T^{*} M$ inside conic sets and controlling a certain oscillatory integral called Fourier-Bros-Iagolnitzer (FBI) transform. In 1994, Baouendi-Rothschild-Trépreau [BRT1994] showed how to

[^15]deform analytic discs attached to a hypersurface in order to get some propagation results (however, Theorem 2.7 which appears in [Trp1990] is not formulated in [BRT1994]). Then Tumanov [Tu1994a] showed how to deformation discs attached to generic submanifolds of arbitrary codimension and provided extension results that cannot be obtained by means of the usual microlocal analysis.

Until the end of Section 4, our goal will be to describe and to exploit this technique of propagation. The geometric idea is as follows.

As in Theorem 2.7, assume that holomorphic extension is already known to hold in a one-sided neighborhood $\omega_{q}^{ \pm}$at some point $q \in M$. Referring to the diagram after the main Proposition 2.21 below, we may pick a disc $A$ with $A(-1)=q$. Then a small part of its boundary, namely for $e^{i \theta}$ near -1 , lies in $\overline{\omega_{q}^{ \pm}}$. If the vector $\frac{\partial A}{\partial \theta}(1)$ is not complex tangential at the opposite point $p=A(1)$, it suffices to apply Lemma 2.16 just above to get holomorphic extension at $p$, almost gratuitously. On the contrary, if $\frac{\partial A}{\partial \theta}(1)$ is complex tangential at $p$, we may well hope that by slightly deforming $M$ as a hypersurface $M^{\mathrm{d}}$ which goes inside $\omega_{q}^{ \pm}$a bit, there exists a deformed disc $A^{\mathrm{d}}$ attached to $M^{\mathrm{d}}$ with again $A^{\mathrm{d}}(1)=p$ that will be not tangential: $-\frac{\partial A^{\mathrm{d}}}{\partial r}(1) \notin T_{A^{\mathrm{d}}(1)} M$. Then a translation of the disc $A^{\mathrm{d}}$ as in Lemma 2.16 will provide holomorphic extension at $p$.
2.18. Approximation theorem and chains of analytic discs. To prove Theorem 2.7, we first formulate a version of the approximation theorem which is apppropriate for our purposes.

Lemma 2.19. ([Tu1994a]) For every $p \in M$, there exists a neighborhood $U_{p}$ of $p$ in $M$ such that for every $q \in U_{p}$, for every one-sided neighborhood $\Omega_{q}^{ \pm}$of $U_{p}$ at $q$, there exists a smaller one-sided neighborhood $\omega_{q}^{ \pm} \subset \Omega_{q}^{ \pm}$of $U_{p}$ at $q$ such that the following approximation property holds:

- for every $F \in \mathscr{C}^{0}\left(M \cup \Omega_{q}^{ \pm}\right)$which is $C R$ on $M$ and holomorphic in $\Omega_{q}^{ \pm}$, there exists a sequence of holomorphic polynomials $\left(P_{\nu}(z)\right)_{\nu \in \mathbb{N}}$ such that $0=\lim _{\nu \rightarrow \infty}\left\|P_{\nu}-f\right\|_{\mathscr{C}^{0}\left(U_{p} \cup \omega_{q}^{ \pm}\right)}$.

The proof is an adaptation of Theorem 5.2(III). It suffices to allow the maximally real submanifolds $\Lambda_{u} \subset M$ be slightly deformed in $\Omega_{q}^{ \pm}$. With a control of the smallness of their $\mathscr{C}^{1}$ norm, one may insure that they cover not only $U_{p}$ but also $\omega_{q}^{ \pm}$. Further details will not be provided.

To establish local holomorphic extension of CR functions, it is allowed to glue discs not only to $M$ but also to previously constructed one-sided neighborhoods.

Pursuing, we formulate a lemma and a main proposition.

Lemma 2.20. ([Tu1994a]) Let $p \in M$ and let $U_{p}$ be a neighborhood of $p$ in $M$, arbitrarily small. For every $q \in \mathscr{O}_{C R}(M, p)$ and every small $\varepsilon>0$, there exist $\ell \in \mathbb{N}$ with $\ell=\mathrm{O}(1 / \varepsilon)$ and a chain of $\mathscr{C}^{2, \alpha-0}$ analytic discs $A^{1}, A^{2}, \ldots, A^{\ell-1}, A^{\ell}$ attached to $M$ with the properties:

- $A^{1}(-1) \in U_{p}$, i.e. this point is arbitrarily close to $p$;
- $A^{1}(1)=A^{2}(-1), A^{2}(1)=A^{3}(-1), \ldots, A^{\ell-1}(1)=A^{\ell}(-1)$;
- $A^{\ell}(1)=q$;
- $\left\|A^{k}\right\|_{\mathscr{C}^{1,0}(\bar{\Delta})} \leqslant \varepsilon$, for $k=1,2, \ldots, \ell$;
- each $A^{k}$ is an embedding $\bar{\Delta} \rightarrow \mathbb{C}^{n}$.


Such a chain of analytic discs will be constructed by approximating a complex-tangential curve that goes from $q$ to $p$, using families of discs $B_{q, v_{q}, t}(\zeta)$ to be introduced in a while. The above lemma is essentially obvious, whereas the next proposition constitutes the very core of the argument.

Proposition 2.21. ([BRT1994, Tu1994a]) (Propagation along a disc) Let A be a small $\mathscr{C}^{2, \alpha-0}$ analytic disc attached to $M$ which is an embedding $\bar{\Delta} \rightarrow \mathbb{C}^{n}$. If $\mathscr{C}_{C R}^{0}(M)$ extends holomorphically to a one-sided neighborhood $\omega_{A(-1)}^{ \pm}$at the point $A(-1)$, then it also extends holomorphically to $a$ onesided neighborhood at $A(1)$. With more precisions:

- if the exit vector $-\frac{\partial A}{\partial r}(1)$ is not tangent to $M$ at $A(1)$, extension holds to the side in which points $-\frac{\partial A}{\partial r}(1)$ : this is already known, by Lemma 2.16;
- if the exit vector $-\frac{\partial A}{\partial r}(1)$ is tangent to $M$ at $A(1)$, there exists an arbitrarily small deformation $A^{\mathrm{d}}$ of $A$ with $A^{\mathrm{d}}(1)=A(1)$ having boundary $A^{\mathrm{d}}(\partial \Delta)$ contained in $M \cup \omega_{A(-1)}^{ \pm}$such that the new exit vector $-\frac{\partial A^{\mathrm{d}}}{\partial r}(1)$ is not tangent to $M$ at $A^{\mathrm{d}}(1)$; then by translating $A^{\mathrm{d}}$ as in Lemma 2.16, holomorphic extension holds at $A(1)$.


Indeed, thanks to the flexibility of the solutions to the parametrized Bishop equation provided by Theorem 3.7(IV), we can easily, as in Lemma 2.16, add translation parameters $\left(x_{1}^{0}, z_{0}^{\prime}, u_{0}\right)$ to a slightly deformed disc $A^{\mathrm{d}}$ attached to $M \cup \omega_{A(-1)}^{ \pm}$and then $A_{x_{1}^{0}, z_{0}^{\prime}, u_{0}}^{\mathrm{d}}\left(\Delta_{1}\right)$ covers a small one-sided neighborhood of $M$ at $A(1)=A^{\mathrm{d}}(1)$, thanks to the crucial condition $-\frac{\partial A^{\mathrm{d}}}{\partial r}(1) \neq 0$. We shall not copy the details.

We claim that the proposition ends the proof of Theorem 2.7. By assumption, $\mathscr{C}_{C R}^{0}(M)$ extends holomorphically to a one-sided neighborhood $\omega_{p}^{ \pm}$at $p$. The closure $\overline{\omega_{p}^{ \pm}}$contains an open neighborhood $U_{p}$ of $p$. Let $q \in \mathscr{O}_{C R}(M, p)$ and construct a chain of analytic discs from $q$ up to a point $p^{\prime} \in U_{p}$. The endpoint $p^{\prime}=A^{1}(-1)$ of the chain of analytic discs being arbitrarily close to $p$, hence in $U_{p}$, holomorphic extension holds at $A^{1}(-1)$. We then apply the proposition successively to the discs $A^{1}, A^{2}, \ldots, A^{\ell}$ and deduce holomorphic extension at $q$.

We now explain Lemma 2.20. To approximate a complex-tangential curve, it suffices to construct families of analytic discs that are essentially directed along given vectors $v_{q} \in T_{q}^{c} M$.
Lemma 2.22. For every point $q \in M$ and every nonzero complex tangent vector $v_{q} \in T_{q}^{c} M \backslash\{0\}$, there exists a family of $\mathscr{C}^{2, \alpha-0}$ analytic discs $B_{q, v_{q}, t}(\zeta)$ parametrized by $t \in \mathbb{R}$ with $|t|<t_{1}$, for some $t_{1}>0$, that satisfies:

- $B_{q, v_{q}, t}(\partial \Delta) \subset M$;
- $q=B_{q, v_{q}, t}(1)$;
- $v_{q}=\frac{\partial B_{q, v, 0}}{\partial t}(-1)$;
- $\left\|B_{q, v_{q}, t}\right\|_{\mathscr{C}^{1,0}(\bar{\Delta})} \leqslant \mathrm{K} t$, for some $\mathrm{K}>0$.

Proof. In coordinates centered at $q$, represent $M$ by $v=\varphi(z, u)$ with $\varphi(0)=$ 0 and $d \varphi(0)=0$. The vector $v_{q} \in T_{p}^{c} M=\{w=0\}$ has coordinates $\left(\dot{z}_{q}, 0\right)$ for some nonzero $\dot{z}_{q} \in \mathbb{C}^{n-1}$. Introduce the family of analytic discs

$$
B_{q, v_{q}, t}(\zeta):=\left(t \dot{z}_{q}(1-\zeta) / 2, W_{t}(\zeta)\right),
$$

where the real part $U_{t}$ of $W_{t}$ is the unique $\mathscr{C}^{2, \alpha-0}$ solution of the Bishop-type equation:

$$
U_{t}\left(e^{i \theta}\right)=-\mathrm{T}_{1}\left[\varphi\left(t \dot{z}_{q}(1-\cdot) / 2, U_{t}(\cdot)\right)\right]\left(e^{i \theta}\right)
$$

Proceeding as carefully as in Section 3(IV), we may verify that the assumption $d \varphi(0)=0$ implies that $\left\|W_{t}\right\|_{1,0}=\mathrm{O}\left(|t|^{2}\right)$. Then it is obvious that $v_{q}=\left(\dot{z}_{q}, 0\right)=\frac{\partial B_{q, v_{q}, 0}}{\partial t}(-1)$.

We now complete the proof of Lemma 2.20. Any point $q \in \mathscr{O}_{C R}(M, p)$ is the endpoint of a finite concatenation of integral curves of sections $L$ of $T^{c} M$. It suffices to construct the chain of discs for a single such curve $\exp (t L)(p)$. After multiplying $L$ by a suitable function, we may assume that $q$ is the time-one endpoint $q=\exp (L)(p)$.

Moving backwards, we start from $q_{\ell}:=q$, we define $A^{\ell}(\zeta):=$ $B_{q_{\ell},-L\left(q_{\ell}\right), 1 / \ell}(\zeta)$ and we set $q_{\ell-1}:=B_{q_{\ell},-L\left(q_{\ell}\right), 1 / \ell}(-1)$. Clearly, $q_{\ell-1}=$ $q_{\ell}-\frac{1}{\ell} L\left(q_{\ell}\right)+\mathrm{O}\left(\frac{1}{\ell^{2}}\right)$. Starting again from $q_{\ell-1}$, we again move backwards and so on, i.e. we define by descending induction:

- $A^{k}(\zeta):=B_{q_{k},-L\left(q_{k}\right), 1 / \ell}(\zeta)$;
- $q_{k-1}:=B_{q_{k},-L\left(q_{k}\right), 1 / \ell}(-1)$,
until $k=1$. Since $q_{k-1}=q_{k}-\frac{1}{\ell} L\left(q_{k}\right)+\mathrm{O}\left(\frac{1}{\ell^{2}}\right)$ for $k=1, \ldots, \ell$, the sequence of points $q_{k}$ is a discrete approximation of the integral curve of $L$, hence the endpoint $q_{0}=A^{1}(-1)$ is arbitrarily close to $p$, provided $\ell$ is large enough. Finally, by construction $\left\|A^{k}\right\|_{1,0}=\mathrm{O}\left(\frac{1}{\ell}\right)$.

The proof of the main Proposition 2.21 does not use special features of hypersurfaces. For this reason, we will directly deal with generic submanifolds of arbitrary codimension, passing to a new section.

## §3. Tumanov's theorem, deformations of Bishop discs AND PROPAGATION ON GENERIC MANIFOLDS

3.1. Wedges and CR-wedges. Assume now that $M$ is a connected generic submanifold in $\mathbb{C}^{n}$ of codimension $d \geqslant 1$ and of CR dimension $m=d-n \geqslant$ 1. The case $d=1$ corresponds to a hypersurface. The notion of local wedge at a point $p$ generalizes to codimension $d \geqslant 2$ the notion of one-sided neighborhood at a point of a hypersurface.

More briefly that was has been done in Section 4(III), a wedge may be defined as follows. Choose a $d$-dimensional real subspace $H_{p}$ of $T_{p} \mathbb{C}^{n}$ satisfying $T_{p} \mathbb{C}^{n}=T_{p} H_{p} \oplus T_{p} M$ and a small convex open salient truncated cone $C_{p} \subset H_{p}$ with vertex $p$. Then a local wedge of edge $M$ at $p$ is:

$$
\mathscr{W}\left(U_{p}, C_{p}\right):=\left\{q+\mathrm{c}: q \in U_{p}, \mathrm{c} \in C_{p}\right\} .
$$

This is not yet the most effective definition. Up to shrinking open sets and parameter spaces, all definitions of local wedges will coincide. Concretely,
the wedges we shall construct will always been obtained as unions of small pieces of families of analytic discs partly attached to $M$. So we formulate all the technical conditions that will insure that such pieces of discs cover a wedge.

Definition 3.2. A local wedge of edge $M$ at $p$ is a set of the form:

$$
\mathscr{W}_{p}:=\left\{A_{t, s}\left(r e^{i \theta}\right):|t|<t_{1},|s|<s_{1},|\theta|<\theta_{1}, r_{1}<r<1\right\},
$$

where, $t \in \mathbb{R}^{d-1}$ is a rotation parameter, $t_{1}>0$ is small, $s \in \mathbb{R}^{2 m+d-1}$ is a translation parameter, $s_{1}>0$ is small, $\theta_{1}>0$ is small, $r_{1}<1$ is close to 1 and $A_{t, s}(\zeta)$, with $\zeta \in \bar{\Delta}$, is a parametrized family of $\mathscr{C}^{2, \alpha-0}$ analytic discs satisfying:

- $A_{t, 0}(1)=p$ for every $t$;
- the boundaries $A_{t, s}(\partial \Delta)$ are partly (sometimes completely) attached to $M$, namely $A_{t, s}\left(e^{i \theta}\right) \in M$, at least for $|\theta| \leqslant \frac{3 \pi}{2}$;
- for every fixed $t$, the mapping $\left(s, e^{i \theta}\right) \mapsto A_{t, s}\left(e^{i \theta}\right)$ is a diffeomorphism from $\left\{|s|<s_{1}\right\} \times\left\{|\theta|<\theta_{1}\right\}$ onto a neighborhood of $p$ in M;
- the exit vector $-\frac{\partial A_{0,0}}{\partial r}(1)$ is not tangent to $M$ at $p$, namely it has nonzero projection $\operatorname{proj}_{T_{p} \mathbb{C}^{n} / T_{p} M}\left(-\partial A_{t, 0} / \partial r(1)\right)$ onto the normal space $T_{p} \mathbb{C}^{n} / T_{p} M$ to $M$ at $p$;
- choose any linear subspace $H_{p}$ of $T_{p} \mathbb{C}^{n}$ satisfying $T_{p} H_{p} \oplus T_{p} M=$ $T_{p} \mathbb{C}^{n}$, so that $H_{p} \simeq T_{p} \mathbb{C}^{n} / T_{p} M$, denote by $\operatorname{proj}_{H_{p}}: T_{p} \mathbb{C}^{n} \rightarrow H_{p}$ the projection onto $H_{p}$ parallel to $T_{p} M$, define the associated exit vector

$$
\operatorname{ex}\left(A_{t, 0}\right):=\operatorname{proj}_{H_{p}}\left(-\frac{\partial A_{t, 0}}{\partial r}(1)\right) \in H_{p}
$$

and the associated normalized exit vector $\mathrm{n}-\operatorname{ex}\left(A_{t, 0}\right) \quad:=$ $\operatorname{ex}\left(A_{t, 0}\right) /\left|\operatorname{ex}\left(A_{t, 0}\right)\right|$; then the rank at $t=0$ of the mapping

$$
\mathbb{R}^{d-1} \ni t \longmapsto \mathrm{n}-\mathrm{ex}\left(A_{t, 0}\right) \in S^{d-1} \subset \mathbb{R}^{d}
$$

should be maximal equal to $d-1$.


The last, most significant condition means that $\mathrm{n}-\mathrm{ex}\left(A_{t, 0}\right)$ describes an open neighborhood of n -ex $\left(A_{0,0}\right)$ in the unit sphere $S^{d-1} \subset \mathbb{R}^{d}$. This is of course independent of the choice of $H_{p}$. Then, fixing $s=0$ and $\theta=0$, as the rotation parameter $t \in \mathbb{R}^{d-1}$ varies with $|t|<t_{1}$, and as the radius $r$ with $r_{1}<r<1$ varies, the curves $A_{t, 0}(r)$ generate an open truncated (curved) cone in some $d$-dimensional local submanifold transverse to $M$ at $p$. Finally, as the translation parameter $s$ varies, the points $A_{t, s}\left(r e^{i \theta}\right)$ describe a (curved) local wedge of edge $M$ at $p$.

Lemma 3.3. Shrinking $t_{1}>0, s_{1}>0, \theta_{1}>0$ and $1-r_{1}>0$ if necessary, the points of $\mathscr{W}_{p}$ are covered injectively: $A_{t, s}\left(r e^{i \theta}\right)=A_{t^{\prime}, s^{\prime}}\left(r^{\prime} e^{i \theta^{\prime}}\right)$ if and only if $t=t^{\prime}, s=s^{\prime}, r=r^{\prime}$ and $\theta=\theta^{\prime}$.

This property follows directly from all the rank conditions. It will be useful to insure uniqueness of holomorphic extension (monodromy).

Definition 3.4. ([Tu1990, Trp1990]) A local CR-wedge of edge $M$ at $p$ of dimension $2 m+d+e$, with $1 \leqslant e \leqslant d$, is a set $\mathscr{W}_{p}^{C R, e}$ defined similarly as a local wedge, but assuming that the rotation parameter $t$ belongs to $\mathbb{R}^{e-1}$ and that the rank of the normalized exit vector mapping

$$
\mathbb{R}^{e-1} \ni t \longmapsto \mathrm{n}-\mathrm{ex}\left(A_{t, 0}\right) \in S^{d-1} \subset \mathbb{R}^{d}
$$

is equal to $e-1$.
Then, fixing $s=0$ and $\theta=0$, as the rotation parameter $t \in \mathbb{R}^{e-1}$ with $|t|<t_{1}$ varies, and as the radius $r$ with $r_{1}<r<1$ varies, the curves $A_{t, 0}(r)$ describe an open truncated (curved) cone in some $e$-dimensional local submanifold transverse to $M$ at $p$. These intermediate wedges of smaller dimension will play a crucial technical rôle in the sequel.

The case $e=1$ deserves special attention. A CR-wedge is then just a manifold with boundary $M_{p}^{1}$ with $\operatorname{dim} M_{p}^{1}=1+\operatorname{dim} M$ that is attached to $M$ at $p$, namely there exists an open neighborhood $U_{p}$ of $p$ in $M$ with $U_{p} \subset$ $\partial M_{p}^{1}$. If in addition $M$ has codimension $d=1$, we recover the notion of onesided neighborhood. It is clear that after a possible shrinking, every $\mathscr{C}^{2, \alpha-0}$
manifold with boundary $M_{p}^{1}$ attached to $M$ at $p$ may be prolonged as a local $\mathscr{C}^{2, \alpha-0}$ generic submanifold $\mathscr{M}_{p}^{1} \equiv \mathscr{W}_{p}^{C R, 1}$ containing a neighborhood of $p$ in $M$ (as shown in the right diagram).


By elementary differential geometry, for $e \geqslant 2$, it may be verified that a local CR-wedge $\mathscr{W}_{p}^{C R, e}$ of edge $M$ at $p$ defined by means of a $\mathscr{C}^{2, \alpha-0}$ family of discs, namely

$$
\mathscr{W}_{p}^{C R, e}:=\left\{A_{t, s}\left(r e^{i \theta}\right):|t|<t_{1},|s|<s_{1},|\theta|<\theta_{1}, r_{1}<r<1\right\}
$$

may also be prolonged as a local generic submanifold $\mathscr{M}_{p}^{e}$ of dimension $2 m+d+e$ containing a neighborhood of $p$ in $M$. The left diagram is an illustration; in it, $e=d=2$, so that $M$ of codimension 2 is (unfortunately for intuition) collapsed to $p$.

However, the smoothness of $\mathscr{M}_{p}^{e}$ can decrease to $\mathscr{C}^{1, \alpha-0}$, because as in a standard local blowing down $\left(z_{1}, z_{2}\right) \mapsto\left(z_{1}, z_{1} z_{2}\right)$, the rank of the map $(r, \theta, s, t) \longmapsto A_{t, s}\left(r e^{i \theta}\right)$ degenerates when $r=1$, since the discs (partial) boundaries $\left\{A_{t, s}\left(e^{i \theta}\right):|\theta| \leqslant \frac{3 \pi}{2}\right\}$ are constrained to stay in $M$. For technical reasons, we will need in the sequel the existence of a prolongation $\mathscr{M}_{p}^{e}$ that is $\mathscr{C}^{2, \alpha-0}$ also when $e \geqslant 2$. The following modification of the definition of $\mathscr{W}_{p}^{C R, e}$ insures the existence of a $\mathscr{C}^{2, \alpha-0}$ prolongation $\mathscr{M}_{p}^{e}$. It will be applied implicitly in the sequel without further mention.

So, assume $e \geqslant 2$, let $A_{t, s}$ be a family of discs as in Definition 3.4 with $\operatorname{ex}\left(A_{0,0}\right) \neq 0$ in $T_{p} \mathbb{C}^{n} / T_{p} M$ and $t \mapsto \mathrm{n}-\mathrm{ex}\left(A_{t, 0}\right)$ of rank $e-1$ at $t=0$. Fix $t:=0$ and define firstly

$$
\mathscr{W}_{p}^{C R, 1}:=\left\{A_{0, s}\left(r e^{i \theta}\right):|s|<s_{1},|\theta|<\theta_{1}, r_{1}<r<1\right\} .
$$

This is a manifold with boundary attached to $M$ at $p$. So there is a small $\mathscr{C}^{2, \alpha-0}$ prolongation $\mathscr{M}_{p}^{1} \supset \mathscr{W}_{p}^{C R, 1}$.

Choose $t \neq 0$ small with $A_{t, 0}$ having exit vector nontangent to $\mathscr{M}_{p}^{1}$ at $p$. Introduce a one-parameter family $M_{\sigma}, \sigma \in \mathbb{R},|\sigma|<\sigma_{1}, \sigma_{1}>0$, of generic submanifolds obtained by deforming slightly $M$ inside $\mathscr{M}_{p}^{1}$ near $p$, with $M_{\sigma} \subset M \cup \mathscr{W}_{p}^{C R, 1}$ for $\sigma \geqslant 0$. The $M_{\sigma}$ are "translates" of $M$ in $\mathscr{M}_{p}^{1}$ near $p$. To understand the process, we draw two diagrams in different dimensions.


Thanks to the flexibility of Bishop's equation (Theorem 3.7(IV)), the $A_{t, s}$ may be deformed as a $\mathscr{C}^{2, \alpha-0}$ family $A_{t, s, \sigma}$ and we define secondly

$$
\mathscr{W}_{p}^{C R, 2}:=\left\{A_{t, s, \sigma}\left(r e^{i \theta}\right):|s|<s_{1}, 0<\sigma<\sigma_{1},|\theta|<\theta_{1}, r_{1}<r<1\right\} .
$$

Then this set constitutes a local CR-wedge of dimension $2 m+d+2$ with edge $M$ at $p$. Letting $\sigma$ run in $\left(-\sigma_{1}, \sigma_{1}\right)$ above, we get instead a certain manifold with boundary attached to $\mathscr{M}_{p}^{1}$ that may be extended as a $\mathscr{C}^{2, \alpha-0}$ generic submanifold $\mathscr{M}_{p}^{2}$ of dimension $2 m+d+2$. Then $\mathscr{W}_{p}^{C R, 2}$ is essentially one quarter of $\mathscr{M}_{p}^{2}$. We neither draw $\mathscr{W}_{p}^{C R, 2}$ nor $\mathscr{W}_{p}^{2}$ in the right diagram above, but the reader sees them. By induction, using that $t \mapsto \mathrm{n}-\mathrm{ex}\left(A_{t, 0}\right)$ has rank $e-1$ at $t=0$, we get the following.

Lemma 3.5. After a possible shrinking, a suitably constructed local $\mathscr{C}^{2, \alpha-0}$ $C R$-wedge $\mathscr{W}_{p}^{C R, e}$ of edge $M$ at $p$ may be prolonged as a local $\mathscr{C}^{2, \alpha-0}$ generic submanifold $\mathscr{M}_{p}^{e}$ of dimension $2 m+d+e$ containing a neighborhood of $p$ in $M$.

In the sequel, similar technical constructions will be applied to insure the existence of $\mathscr{C}^{2, \alpha-0}$ prolongations $\mathscr{M}_{p}^{e} \supset \mathscr{W}_{p}^{C R, e}$ without further mention.
3.6. Holomorphic extension of $\mathbf{C R}$ functions in higher codimension. In 1988, Tumanov [Tu1988] established a theorem that is nowadays celebrated in Several Complex Variables. Recall that by definition, $M$ is locally minimal at $p$ if the local CR orbit $\mathscr{O}_{C R}^{\text {loc }}(M, p)$ contains a neighborhood of $p$ in $M$. Equivalently, $M$ does not contain any local submanifold $N$ passing through $p$ with CRdim $N=\mathrm{CR} \operatorname{dim} M$ and $\operatorname{dim} N<\operatorname{dim} M$.

Theorem 3.7. ([Tu1988, BRT1994, Trp1996, Tu1998, BER1999]) Let M be a local $\mathscr{C}^{2, \alpha}$ generic submanifold of $\mathbb{C}^{n}$ and let $p \in M$. If $M$ is locally minimal at $p$, then there exists a local wedge $\mathscr{W}_{p}$ of edge $M$ at $p$ such that every $f \in \mathscr{C}_{C R}^{0}(M)$ possesses a holomorphic extension $F \in \mathscr{O}\left(\mathscr{W}_{p}\right) \cap \mathscr{C}^{0}\left(M \cup \mathscr{W}_{p}\right)$ with $\left.F\right|_{M}=f$.

Conversely, recall that according to Theorem 4.41(III), if $M$ is not locally minimal at $p$, there exists a local continuous CR function that is not holomorphically extendable to any local wedge at $p$.

Since the literature already contains abundant restitutions ${ }^{17}$, we will focus instead on propagation phenomena that are less known.

In 1994, as an answer to a conjecture formulated by Trépreau in [Trp1990], it was shown simultaneously by Jöricke and by the first author that Tumanov's theorem generalizes to globally minimal $M$. The preceding statement is a direct corollary of the next. Its proof given in [Me1994, Jö1996] used techniques and ideas of Tumanov [Tu1988, Tu1994a] and of Trépreau [Trp1990].

Theorem 3.8. ([Me1994, Jö1996]) Let M be a connected $\mathscr{C}^{2, \alpha}$ generic submanifold of $\mathbb{C}^{n}$. If $M$ is globally minimal then at every point $p \in M$, there exists a local wedge $\mathscr{W}_{p}$ of edge $M$ at $p$ such that every continuous CR function $f \in \mathscr{C}_{C R}^{0}(M)$ possesses a holomorphic extension $F \in$ $\mathscr{O}\left(\mathscr{W}_{p}\right) \cap \mathscr{C}^{0}\left(M \cup \mathscr{W}_{p}\right)$ with $\left.F\right|_{M}=f$.

With this statement, the extension theorem for CR function has reached a final, most general form. Philosophically, the main reason why it is true lies in the propagation of holomorphic extendability along complex-tangential curves. This was developed by Trépreau in 1990, using microlocal analysis.

Theorem 3.9. ([Trp1990]) Let $M$ be a connected $\mathscr{C}^{\infty}$ generic submanifold of $\mathbb{C}^{n}$. If $\mathscr{C}_{C R}^{0}(M)$ extends holomorphically to a local wedge at some point $p \in M$, then at every point $q \in \mathscr{O}_{C R}(M, p)$, there exists a local wedge $\mathscr{W}_{q}$ of edge $M$ at $q$ such that every $f \in \mathscr{C}_{C R}^{0}(M)$ possesses a holomorphic extension $F \in \mathscr{O}\left(\mathscr{W}_{q}\right) \cap \mathscr{C}\left(M \cup \mathscr{W}_{q}\right)$ with $\left.F\right|_{M}=f$.

Before surveying the original proof ([Me1994, Jö1996]) of this theorem in Section 5, we shall expose in length a substantially simpler proof of Theorem 3.8 that was devised by the second author in [Po2004]. This neat proof treats locally and globally minimal generic submanifolds on the same footing. It relies partly upon a natural deformation proposition due to Tumanov in [Tu1994a], but without any notion of defect of an analytic disc, without any needs to control the variation of the direction of CR-extendability, and without any partial connection, as in [Trp1990, Tu1994a, Me1994]. The

[^16]next paragraphs and Section 4 are devoted to the proof of this most general Theorem 3.8.

Example 3.10. A globally minimal manifold may well be not locally minimal at any point.

Indeed, let $\chi: \mathbb{R} \rightarrow \mathbb{R}^{+}$be $\mathscr{C}^{\infty}$ with $\chi=0$ on $(-\infty, 1]$, with $\chi>0$ on $(1,+\infty)$ and with second derivative $\chi_{x x}>0$ on $(1,+\infty)$. Consider the generic manifold $M$ of $\mathbb{C}^{3}$ defined by the two equations

$$
v_{1}=\chi(x), \quad v_{2}=\chi(-x),
$$

in coordinates $\left(x+i y, u_{1}+i v_{1}, u_{2}+i v_{2}\right)$. Then $T^{1,0} M$ is generated by

$$
L=\frac{\partial}{\partial z}+i \chi_{x}(x) \frac{\partial}{\partial w_{1}}-i \chi_{x}(-x) \frac{\partial}{\partial w_{2}} .
$$

In terms of the four coordinates $\left(x, y, u_{1}, u_{2}\right)$ on $M$, the two vector fields generating $T^{c} M$ are

$$
\begin{aligned}
& L^{1}:=2 \operatorname{Re} L=\frac{\partial}{\partial x}, \\
& L^{2}:=2 \operatorname{Im} L=\frac{\partial}{\partial y}+\chi_{x}(x) \frac{\partial}{\partial u_{1}}-\chi_{x}(-x) \frac{\partial}{\partial u_{2}}
\end{aligned}
$$

(we have dropped $\chi_{x}(x) \frac{\partial}{\partial v_{1}}-\chi_{x}(-x) \frac{\partial}{\partial v_{2}}$ in $2 \operatorname{Re} L$ ). Denote by $\mathbb{L}^{0}$ the system of these two vector fields $\left\{L^{1}, L^{2}\right\}$ on $\mathbb{R}^{4} \simeq M$ and by $\mathbb{L}$ the $\mathscr{C}^{\infty}\left(\mathbb{R}^{4}\right)$ hull of $\mathbb{L}^{0}$. Observe that the Lie bracket

$$
\left[L^{1}, L^{2}\right]=\chi_{x x}(x) \frac{\partial}{\partial u_{1}}+\chi_{x x}(-x) \frac{\partial}{\partial u_{2}}
$$

is zero at points $p=\left(x_{p}, y_{p}, u_{1}^{p}, u_{2}^{p}\right)$ with $-1<x_{p}<1$, has non-zero $\frac{\partial}{\partial u_{2}}$-component at points $p$ with $x_{p}<-1$ and has non-zero $\frac{\partial}{\partial u_{1}}$-component at points $p$ with $x_{p}>1$. It follows that the local $\mathbb{L}$-orbit of a point $p$ with $x_{p}<-1$ is $\left\{u_{1}=u_{1}^{p}\right\}$, of a point $p$ with $-1<x_{p}<1$ is $\left\{u_{1}=u_{1}^{p}, u_{2}=u_{2}^{p}\right\}$ and of a point $x_{p}$ with $x_{p}>1$ is $\left\{u_{2}=u_{2}^{p}\right\}$. Also, observe that since the vector field $L^{1}=\frac{\partial}{\partial x}$ belongs to $\mathbb{L}$, the local $\mathbb{L}$-orbit of any point $p=\left(x_{p}, y_{p}, u_{1}^{p}, u_{2}^{p}\right)$ contains points of coordinates $\left(x_{p}+t, y_{p}, u_{1}^{p}, u_{2}^{p}\right)$, with $t$ small. We deduce that the local $\mathbb{L}$-orbit of points $p$ with $x_{p}=-1$ or $x_{p}=1$ are three-dimensional, hence in conclusion:

$$
\mathscr{O}_{\mathbb{L}}^{\text {loc }}\left(\mathbb{R}^{4}, p\right)= \begin{cases}U_{p} \cap\left\{u_{1}=u_{1}^{p}\right\} & \text { if } x_{p} \leqslant-1, \\ U_{p} \cap\left\{u_{1}=u_{1}^{p}, u_{2}=u_{2}^{p}\right\} & \text { if }-1<x_{p}<1, \\ U_{p} \cap\left\{u_{2}=u_{2}^{p}\right\} & \text { if } x_{p} \geqslant 1,\end{cases}
$$

where $U_{p}$ is a neighborhood of $p$ in $M$. So $\mathbb{L}$ is nowhere locally minimal.
Lemma 3.11. The system $\mathbb{L}$ is globally minimal.

Proof. We check that any two points $p, q \in \mathbb{R}^{4}$ are in the same $\mathbb{L}$-orbit. Using the flow of $L^{1}=\frac{\partial}{\partial x}$ and then the flow of $L^{2}$ on $\{x=0\}$, the original two points $p$ and $q$ may be joined to points, still denoted by $p=\left(0,0, u_{1}^{p}, u_{2}^{p}\right)$ and $q=\left(0,0, u_{1}^{q}, u_{2}^{q}\right)$, having zero $x$-component and zero $y$-component.

We claim that the global $\mathbb{L}$-orbit $\mathscr{O}_{\mathbb{L}}\left(\mathbb{R}^{4}, p\right)$ of every point $p=\left(0,0, u_{1}^{p}, u_{2}^{p}\right)$ contains a neighborhood of $p$ in $\mathbb{R}^{4}$. Since the twodimensional plane $\{x=y=0\}$ is connected, this will assure that any two points $p=\left(0,0, u_{1}^{p}, u_{2}^{p}\right)$ and $q=\left(0,0, u_{1}^{q}, u_{2}^{q}\right)$ are in the same $\mathbb{L}$-orbit.

Indeed, by means of $\frac{\partial}{\partial x}$, every point $p=\left(0,0, u_{1}^{p}, u_{2}^{p}\right)$ is joined to the two points $p^{-}:=\left(-1,0, u_{1}^{p}, u_{2}^{p}\right)$ and $p^{+}:=\left(1,0, u_{1}^{p}, u_{2}^{p}\right)$. Let $U_{p^{-}}$and $U_{p^{+}}$be small neighborhoods of $p^{-}$and of $p^{+}$. Denote by $H^{-}:=\left\{u_{1}=u_{1}^{p}\right\} \cap U_{p^{-}}$ and by $H^{+}:=\left\{u_{2}=u_{2}^{p}\right\} \cap U_{p^{+}}$small pieces of the three-dimensional local $\mathbb{L}$-orbits of $p^{-}$and of $p^{+}$.


The flow of $L^{1}=\frac{\partial}{\partial x}$ being a translation, we deduce:

$$
\begin{aligned}
\exp \left(L^{1}\right)\left(H^{-}\right) & =\left\{u_{1}=u_{1}^{p}\right\} \cap U_{p}, \\
\exp \left(-L^{1}\right)\left(H^{+}\right) & =\left\{u_{2}=u_{2}^{p}\right\} \cap U_{p},
\end{aligned}
$$

where $U_{p}$ is a small neighborhood of $p$ in $M \simeq \mathbb{R}^{4}$. Observe that the two 3-dimensional planes are transversal in $T_{p} \mathbb{R}^{4}$. Lemma 1.28(III) yields:

$$
\begin{aligned}
& \mathbb{L}^{\text {inv }}\left(p^{-}\right) \supset T_{p^{-}} \mathscr{O}_{\mathbb{L}}^{\text {loc }}\left(p^{-}\right)=\left\{u_{1}=u_{1}^{p}\right\}, \\
& \mathbb{L}^{\text {inv }}\left(p^{+}\right) \supset T_{p^{+}} \mathscr{O}_{\mathbb{L}}^{\text {loc }}\left(p^{+}\right)=\left\{u_{2}=u_{2}^{p}\right\} .
\end{aligned}
$$

By the very definition of $\mathbb{L}^{\text {inv }}$, we necessarily have:

$$
\begin{aligned}
\mathbb{L}^{\text {inv }}(p) & \supset \exp \left(L^{1}\right)_{*}\left(\mathbb{L}^{\text {inv }}\left(p^{-}\right)\right)+\exp \left(-L^{1}\right)_{*}\left(\mathbb{L}^{\text {inv }}\left(p^{+}\right)\right) \\
& =\left\{u_{1}=u_{1}^{p}\right\}+\left\{u_{2}=u_{2}\right\} \\
& =T_{p} \mathbb{R}^{4},
\end{aligned}
$$

so $\mathbb{L}^{\text {inv }}(p)=T_{p} \mathbb{R}^{4}$. Consequently, $\mathscr{O}_{\mathbb{L}}\left(\mathbb{R}^{4}, p\right)$ contains a neighborhood of $\left(0,0, u_{1}^{p}, u_{2}^{p}\right)$ in $\mathbb{R}^{4}$.
3.12. Setup for propagation. Let $M$ be connected, generic and $\mathscr{C}^{2, \alpha}$, let $q \in M$ and let $\mathscr{W}_{q}^{C R, e}$ be a CR-wedge of dimension $2 m+d+e$ at $q$, with $1 \leqslant e \leqslant d$. For short, we will say that $\mathscr{C}_{C R}^{0}(M)$ extends to be $C R$ on $\mathscr{W}_{q}^{C R, e}$ if for every $f \in \mathscr{C}_{C R}^{0}$, there exists $F \in \mathscr{C}_{C R}^{0}\left(M \cup \mathscr{W}_{q}^{C R, e}\right)$ with $\left.F\right|_{M}=f$.

Theorem 3.13. Let $e \in \mathbb{N}$ with $1 \leqslant e \leqslant d$. Assume that $\mathscr{C}_{C R}^{0}(M)$ extends to be CR on a CR-wedge $\mathscr{W}_{p}^{C R, e}$ of dimension $2 m+d+e$ at some point $p \in M$. Then for every $q \in \mathscr{O}_{C R}(M, p)$, there exists a CR-wedge $\mathscr{W}_{q}^{C R, e}$ at $q$ of the same dimension $2 m+d+e$ to which $\mathscr{C}_{C R}^{0}(M)$ extends to be $C R$.

In the case $e=d$, we recover ${ }^{18}$ Trépreau's Theorem 3.9, since continuous CR functions on an open set of $\mathbb{C}^{n}$ (here a usual wedge) are just holomorphic. If $M$ is globally minimal, then extension holds at every $q \in M$. Notice that this statement covers the propagation Theorem 2.7, stated previously in the hypersurface case $d=e=1$.

Let us start the proof. Through a chain of small analytic discs, every $q \in \mathscr{O}_{C R}(M, p)$ is joined to a point $p^{\prime}$ arbitrarily close to $p$ : indeed, Lemma 2.20 and its proof remain the same in arbitrary codimension $d \geqslant 1$. At $p^{\prime}$, CR extension holds, because the edge of $\mathscr{W}_{p}^{C R, e}$ contains a small open neighborhood $U_{p}$ of $p$ in $M$. To deduce CR extension at $q$, it suffices therefore to propagate CR extension along a single disc, as stated in the next main proposition.


Proposition 3.14. (Propagation along a disc) ([Tu1994a, MP1999], [*]) Let $A$ be a small $\mathscr{C}^{2, \alpha-0}$ analytic disc attached to $M$ which is an embedding $\bar{\Delta} \rightarrow \mathbb{C}^{n}$. Let $e \in \mathbb{N}$ with $1 \leqslant e \leqslant d$. Assume that there exists a $\mathscr{C}^{2, \alpha-0}$ CR-wedge $\mathscr{W}_{A(-1)}^{C R, e}$ at $A(-1)$ of dimension $2 m+d+e$ to which $\mathscr{C}_{C R}^{0}(M)$

[^17]extends to be $C R$. Then there exists a $\mathscr{C}^{2, \alpha-0} C R$-wedge $\mathscr{W}_{A(1)}^{C R, e}$ at $A(1)$ of the same dimension $2 m+d+e$ to which $\mathscr{C}_{C R}^{0}(M)$ extends to be $C R$.

With more precisions, the CR-wedge $\mathscr{W}_{A(1),}^{C R, e}$ is constructed by translating a certain family of analytic discs $A_{t^{\prime}}$ having the following properties. Setting $p:=A(1)$, there exists a $\mathscr{C}^{2, \alpha-0}$ family $A_{t^{\prime}}$ of analytic discs, $t^{\prime} \in \mathbb{R}^{e},\left|t^{\prime}\right|<$ $t_{1}^{\prime}, t_{1}^{\prime}>0$, with $\left.A_{t^{\prime}}\right|_{t^{\prime}=0}=A$, with $A_{t^{\prime}}(1)=p$, satisfying $A_{t^{\prime}}\left(e^{i \theta}\right) \in M$ for $|\theta| \leqslant \frac{3 \pi}{2}$ and having their boundaries $A_{t^{\prime}}(\partial \Delta) \subset M \cup \mathscr{W}_{A(-1)}^{C R, e}$ for $t^{\prime}$ belonging to some open truncated cone $\mathrm{C}^{\prime} \subset \mathbb{R}^{e}$, such that the exit vector mapping:

$$
\mathbb{R}^{e} \ni t^{\prime} \longmapsto \operatorname{ex}\left(A_{t^{\prime}}\right)=\operatorname{proj}_{H_{p}}\left(-\frac{\partial A_{t^{\prime}}}{\partial r}(1)\right) \in \mathbb{R}^{d}
$$

is of maximal rank equal to e at $t^{\prime}=0$, where $H_{p} \simeq \mathbb{R}^{d}$ is any linear subspace of $T_{p} \mathbb{C}^{n}$ such that $H_{p} \oplus T_{p} M=T_{p} \mathbb{C}^{n}$, and where $\operatorname{proj}_{H_{p}}$ is the linear projection parallel to $T_{p} M$.

Geometrically, as $t^{\prime}$ varies, the exit vectors ex $\left(A_{t^{\prime}}\right)$ describe an open cone $C_{p} \subset H_{p}$, drawn in the diagram.

We claim that this statement covers the second, delicate case of Proposition 2.21. Indeed assuming that $e=d=1$ and that the exit vector $-\frac{\partial A}{\partial r}(1)$ is tangent to $M$ at $A(1)$, the above proposition includes $A$ in a one-parameter family $A_{t^{\prime}}$ whose direction of exit in the normal bundle has nonzero derivative with respect to $t^{\prime}$. Hence for every nonzero $t^{\prime}$, the direction of exit of $A_{t^{\prime}}$ is not tangent to $M$ at $p$. Thus, a non-tangential deformed disc $A^{\mathrm{d}}$ as in Proposition 2.21 may be chosen to be any $A_{t^{\prime}}$, with $t^{\prime} \neq 0$.

Proof of Proposition 3.14. We first explain how to get CR extension at $p$ from the family $A_{t^{\prime}}$, taking for granted its existence.
(I) Suppose firstly that the exit vector of $A=A_{0}$ is non-tangential to $M$ at $p$. We have to restrict the parameter space $t^{\prime} \in \mathbb{R}^{e}$ to some parameter space $t \in \mathbb{R}^{e-1}$ so as to reach Definition 3.4.

Let us take for granted the fact that the exit vector mapping has rank $e$ at $t^{\prime}=0$. Then the normalized exit vector mapping

$$
\mathbb{R}^{e} \ni t^{\prime} \longmapsto \mathrm{n}-\operatorname{ex}\left(A_{t^{\prime}}\right)=\operatorname{ex}\left(A_{t^{\prime}}\right) /\left|\operatorname{ex}\left(A_{t^{\prime}}\right)\right| \in S^{d-1}
$$

has rank $\geqslant e-1$ at $t^{\prime}=0$. So there exists a small piece of an $(e-1)$ dimensional linear subspace $\Lambda_{0}$ of $\mathbb{R}^{d}$, parameterized as $t^{\prime}=\phi(t)$ for some linear map $\phi$, with $t \in \mathbb{R}^{e-1}$ small, namely $|t|<t_{1}$, for some $t_{1}>0$, such that $t \mapsto \mathrm{n}-\mathrm{ex}\left(A_{\phi(t)}\right)$ has rank $(e-1)$ at $t=0$.

Setting $A_{t}:=A_{\phi(t)}$, we thus reach Definition 3.4, without the translation parameter $s$.

But proceeding exactly as in the hypersurface case, it is easy to include some translation parameter getting a family $\left(A_{t^{\prime}}\right)_{s}=A_{t^{\prime}, s}$. The proof is
postponed to the end $\S 3.24$. Then the desired family $A_{t, s}$ of the proposition is just $A_{\phi(t), s}$, shrinking $t_{1}>0$ and $s_{1}>0$ if necessary.

Lemma 3.15. There exists a deformation $A_{t^{\prime}, s}$ of $A_{t^{\prime}}$, with $s \in \mathbb{R}^{2 m+d-1}$, $|s|<s_{1}$, such that:

- the boundaries $A_{t^{\prime}, s}(\partial \Delta)$ are contained in $M \cup \mathscr{W}_{A(-1)}^{C R, e}$ and $A_{t^{\prime}}\left(e^{i \theta}\right) \in M$ for $|\theta| \leqslant \frac{3 \pi}{2}$;
- for every fixed $t^{\prime}$, the mapping $\left(s, e^{i \theta}\right) \longmapsto A_{t^{\prime}, s}\left(e^{i \theta}\right)$ is a diffeomorphism from $\left\{|s|<s_{1}\right\} \times\left\{|\theta|<\theta_{1}\right\}$ onto a neighborhood of $p$ in $M$.

Therefore, the final family $A_{t, s}$ yields a CR-wedge $\mathscr{W}_{p}^{C R, e}$ at $p=A(1)$, as in Definition 3.4. The mild generalization of the approximation Theorem 5.2 (III) stated as Lemma 2.19 above in the case $d=1$ holds in the general case $d \geqslant 1$ without modification. Consequently, $\mathscr{C}_{C R}^{0}(M)$ extends to be CR on $\mathscr{W}_{p}^{C R, e}$.
(II) Suppose secondly that the exit vector of $A=A_{0}$ is tangential to $M$ at $p$. Thanks to the fact that the exit vector mapping has rank $e$ at $t^{\prime}=0$, for every nonzero $t_{0}^{\prime}$, the disc $A_{t_{0}^{\prime}}$ is nontangential to $M$ at $p$. In this case, we fix a small $t_{0}^{\prime} \neq 0$ and we proceed with $A_{t^{\prime}+t_{0}^{\prime}}$ just as above.

In summary, it remains only to construct the family $A_{t^{\prime}}$ having the crucial property that the exit vector mapping has rank $e$ at $t^{\prime}=0$.
3.16. Normal deformations of analytic discs. Thus, we now expose how to construct $A_{t^{\prime}}$. We shall introduce a parameterized family $M_{t^{\prime}}$ of $\mathscr{C}^{2, \alpha-0}$ generic submanifolds by pushing $M$ near $A(-1)$ inside $\mathscr{W}_{A(-1)}^{C R, e}$ in $e$ independent normal directions, $e$ being the number of degrees of freedom offered by $\mathscr{W}_{A(-1)}^{C R, e}$. Outside a neighborhood of $A(-1)$, each $M_{t^{\prime}}$ shall coincides with $M$ and also $\left.M_{t^{\prime}}\right|_{t^{\prime}=0}=M$.
We may assume that the point $p:=A(1)$ is the origin in coordinates $(z, u+i v) \in \mathbb{C}^{m} \times \mathbb{C}^{d}$ in which $M$ is represented by $v=\varphi(z, u)$, where $\varphi$ satisfies $\varphi(0)=0$ and $d \varphi(0)=0$. Let $t^{\prime} \in \mathbb{R}^{e}$ be small, namely $\left|t^{\prime}\right|<t_{1}^{\prime}$, with $t_{1}^{\prime}>0$.

In terms of graphing equations, the deformation $M_{t^{\prime}}$ may be represented by

$$
v=\Phi\left(z, u, t^{\prime}\right),
$$

with $\Phi \in \mathscr{C}^{2, \alpha-0}$ defined for $\left|t^{\prime}\right|<t_{1}^{\prime}$ satisfying $\Phi(z, u, 0) \equiv \varphi(z, u)$. The point $A(-1)$ has small coordinates $\left(z_{-1}, u_{-1}+i \varphi\left(z_{-1}, u_{-1}\right)\right)$. We require that the $e$ vectors

$$
\Phi_{t_{k}^{\prime}}\left(z_{-1}, u_{-1}, 0\right), \quad k=1, \ldots, e
$$

are linearly independent. There exists a truncated open cone $\mathrm{C}^{\prime} \subset \mathbb{R}^{e}$ with the property that

$$
M_{t^{\prime}} \subset M \cup \mathscr{W}_{A(-1)}^{C R, e},
$$

whenever $t^{\prime} \in \mathrm{C}^{\prime}$. In fact, we implicitly assume in Proposition 3.14 that the CR-wedge based at $A(-1)$ may be extended as a $\mathscr{C}^{2, \alpha-0}$ generic submanifold $\mathscr{M}_{A(-1)}^{e}$ of dimension $2 m+d+e$ passing through $A(-1)$ so that $M_{t^{\prime}}$ is contained in $M \cup \mathscr{M}_{A(-1)}^{e}$, for every $\left|t^{\prime}\right|<t_{1}^{\prime}$. The original CR-wedge $\mathscr{W}_{A(-1)}^{C R, e}$ may then be viewed as a curved real wedge of edge $M$ which is contained inside $\mathscr{M}_{A(-1)}^{C R, e}$.

The starting $\mathscr{C}^{2, \alpha} \operatorname{disc} A(\zeta)=(Z(\zeta), W(\zeta))$ with $W(\zeta)=(U(\zeta)+$ $i V(\zeta))$ is attached to $M$ with $A(1)=0$. Equivalently:

$$
\left\{\begin{array}{l}
V\left(e^{i \theta}\right)=\varphi\left(Z\left(e^{i \theta}\right), U\left(e^{i \theta}\right)\right), \\
U\left(e^{i \theta}\right)=-\mathrm{T}_{1}[\varphi(Z(\cdot), U(\cdot))]\left(e^{i \theta}\right),
\end{array}\right.
$$

for every $e^{i \theta} \in \partial \Delta$. Thanks to the existence Theorem 3.7(IV), there exists a $\mathscr{C}^{2, \alpha-0}$ deformation $A_{t^{\prime}}$ of $A$, where each $A_{t^{\prime}}(\zeta):=\left(Z(\zeta), W\left(\zeta, t^{\prime}\right)\right)$ with $A_{t^{\prime}}(1)=p$ has the same $z$-component ${ }^{19}$ as $A$ and is attached to $M_{t^{\prime}}$, namely:

$$
\left\{\begin{array}{l}
V\left(e^{i \theta}, t^{\prime}\right)=\Phi\left(Z\left(e^{i \theta}\right), U\left(e^{i \theta}, t^{\prime}\right), t^{\prime}\right),  \tag{3.17}\\
U\left(e^{i \theta}, t^{\prime}\right)=-\mathrm{T}_{1}\left[\Phi\left(Z(\cdot), U\left(\cdot, t^{\prime}\right), t^{\prime}\right)\right]\left(e^{i \theta}\right),
\end{array}\right.
$$

for every $e^{i \theta} \in \partial \Delta$. Observe that $W\left(e^{i \theta}, 0\right) \equiv W\left(e^{i \theta}\right)$. We then differentiate the first line above with respect to $t_{k}^{\prime}$ at $t^{\prime}=0$, for $k=1, \ldots, e$, which yields in matrix notation:

$$
\begin{equation*}
V_{t_{k}^{\prime}}\left(e^{i \theta}, 0\right)=\Phi_{u}\left(Z\left(e^{i \theta}\right), U\left(e^{i \theta}\right), 0\right) U_{t_{k}^{\prime}}\left(e^{i \theta}, 0\right)+\Phi_{t_{k}^{\prime}}\left(Z\left(e^{i \theta}\right), U\left(e^{i \theta}\right), 0\right) . \tag{3.18}
\end{equation*}
$$

Also, the $\mathscr{C}^{1, \alpha-0} \operatorname{discs} A_{t_{k}^{\prime}}(\zeta, 0)$ satisfy the linear Bishop-type equation

$$
U_{t_{k}^{\prime}}\left(e^{i \theta}, 0\right)=-\mathrm{T}_{1}\left[\Phi_{u}(Z(\cdot), U(\cdot), 0) U_{t_{k}^{\prime}}(\cdot, 0)+\Phi_{t_{k}^{\prime}}(Z(\cdot), U(\cdot), 0)\right]\left(e^{i \theta}\right)
$$

As a supplementary space to $T_{p} M$ in $T_{p} \mathbb{C}^{n}$, we choose $H_{p}:=\{0\} \times i \mathbb{R}^{d}=$ $\{w=0, u=0\}$. Then $\operatorname{proj}_{H_{p}}\left(-\partial A_{t^{\prime}}(1) / \partial r\right)=-\partial V\left(1, t^{\prime}\right) / \partial r$, which yields after differentiating with respect to $t_{k}^{\prime}$ at $t^{\prime}=0$ :

$$
\begin{equation*}
\left.\frac{\partial}{\partial t_{k}^{\prime}}\right|_{t^{\prime}=0} \operatorname{proj}_{H_{p}}\left(-\frac{\partial A_{t^{\prime}}}{\partial r}(1)\right)=-\frac{\partial V_{t_{k}^{\prime}}}{\partial r}(1,0), \tag{3.19}
\end{equation*}
$$

[^18]for $k=1, \ldots, e$. We will establish that if the local deformations $M_{t^{\prime}}$ of $M$ inside the CR-wedge $\mathscr{W}_{A(-1)}^{C R, e}$ are concentrated in a sufficiently thin neighborhood of $A(-1)$, then the above $e$ vectors $-\partial V_{t_{k}^{\prime}} / \partial r(1,0), k=1, \ldots, e$, are linearly independent. This will complete the proof of the proposition.

There is a singular integral operator $\mathscr{J}$ which yields the interior normal derivative at $1 \in \partial \Delta$ of any $\mathscr{C}^{1, \alpha-0}$ mapping $v=\bar{\Delta} \rightarrow \mathbb{R}^{d}$ which is harmonic in $\Delta$ and vanishes at $1 \in \partial \Delta$ :

$$
\begin{equation*}
\mathscr{J}(v):=\text { p.v. } \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{v\left(e^{i \theta}\right)}{\left|e^{i \theta}-1\right|^{2}} d \theta=-\frac{\partial v}{\partial r}(1) \tag{3.20}
\end{equation*}
$$

The proof is postponed to Lemma 3.25 below. If $h: \bar{\Delta} \rightarrow \mathbb{C}^{d}$ is $\mathscr{C}^{1, \alpha-0}$ and holomorphic in $\Delta$, we have in addition

$$
\mathscr{J}(h)=-\frac{\partial h}{\partial r}(1)=i \frac{\partial h}{\partial \theta}(1) .
$$

With the singular integral $\mathscr{J}$, we may thus reformulate (3.19):

$$
\operatorname{proj}_{H_{p}}\left(-\frac{\partial^{2} A_{0}}{\partial t_{k}^{\prime} \partial r}(1)\right)=\mathscr{J}\left(V_{t_{k}^{\prime}}\right) \text {. }
$$

Lemma 3.21. Let $u, v \in \mathscr{C}^{1, \alpha-0}\left(\bar{\Delta}, \mathbb{R}^{d}\right)$ be harmonic in $\Delta$ and vanish at $1 \in \partial \Delta$. Then:

$$
0=\mathscr{J}\left(u v-\mathrm{T}_{1} u \mathrm{~T}_{1} v\right) .
$$

In addition, $u$ (and also $v$ ) satisfies the two equations:

$$
\mathscr{J}(u)=-\frac{\partial\left(\mathrm{T}_{1} u\right)}{\partial \theta}(1) \quad \text { and } \quad \mathscr{J}\left(\mathrm{T}_{1} u\right)=\frac{\partial u}{\partial \theta}(1) .
$$

Proof. The holomorphic product $w:=\left(u+i \mathbf{T}_{1} u\right)\left(v+i \mathbf{T}_{1} v\right)$ vanishes to second order at $1 \in \partial \Delta$, so $\mathscr{J}(w)=0$, hence

$$
0=\operatorname{Re} \mathscr{J}(w)=\mathscr{J}\left(u v-\mathrm{T}_{1} u \mathrm{~T}_{1} v\right) .
$$

The pair of equations satisfied by $u$ is obtained by identifying the real and imaginary parts of $\mathscr{J}(h)=i \frac{\partial h}{\partial \theta}(1)$, where $h:=u+i \mathrm{~T}_{1} u$.

Following [Tu1994a], we now introduce a $d \times d$ matrix $G$ of $\mathscr{C}^{1, \alpha}$ functions on $\partial \Delta$ defined by the functional equation

$$
G\left(e^{i \theta}\right)=I+\mathrm{T}_{1}\left[G(\cdot) \Phi_{u}(Z(\cdot), U(\cdot), 0)\right]\left(e^{i \theta}\right)
$$

Here $\Phi_{u}=\left(\Phi_{u_{l}}^{j}\right)_{1 \leqslant j \leqslant d}^{1 \leqslant l}$ is a $d \times d$ matrix. Since $\Phi_{u}(z, u, 0) \equiv \varphi_{u}(z, u)$ is small, the solution $G$ exists and is unique, by an application of Proposition 3.21(IV). Notice that $G(1)=I$. Applying $\mathrm{T}_{1}$ to both sides, we get $\mathrm{T}_{1} G=-G \Phi_{u}+$ cst., without writing the arguments. In fact, the constant vanishes, since $\Phi_{u}(0,0,0)=\varphi_{u}(0,0)=0$. So we get:

$$
\mathrm{T}_{1} G=-G \Phi_{u}
$$

We also notice that $V_{t_{k}^{\prime}}=\mathrm{T}_{1} U_{t_{k}^{\prime}}$ and $U_{t_{k}^{\prime}}=-\mathrm{T}_{1} V_{t_{k}^{\prime}}$.
Next, we rewrite (3.18) without arguments: $\Phi_{t_{k}^{\prime}}=V_{t_{k}^{\prime}}-\Phi_{u} U_{t_{k}^{\prime}}, k=$ $1, \ldots, e$, we apply the matrix $G$ to both sides, we replace $G \Phi_{u}$ by $-\mathrm{T}_{1} G$ as well as $U_{t_{k}^{\prime}}$ by $-\mathrm{T}_{1} V_{t_{k}^{\prime}}$ and we let appear a term $u v-\mathrm{T}_{1} u \mathrm{~T}_{1} v$ :

$$
\begin{aligned}
G \Phi_{t_{k}^{\prime}} & =G V_{t_{k}^{\prime}}-G \Phi_{u} U_{t_{k}^{\prime}} \\
& =G V_{t_{k}^{\prime}}-\left(\mathrm{T}_{1} G\right)\left(\mathrm{T}_{1} V_{t_{k}^{\prime}}\right) \\
& =V_{t_{k}^{\prime}}+(G-I) V_{t_{k}^{\prime}}-\mathrm{T}_{1}(G-I) \mathrm{T}_{1} V_{t_{k}^{\prime}} .
\end{aligned}
$$

Finally ${ }^{20}$, applying the singular operator $\mathscr{J}$ and remembering Lemma 3.21, we obtain:

$$
\begin{equation*}
\mathscr{J}\left(G \Phi_{t_{k}^{\prime}}\right)=\mathscr{J}\left(V_{t_{k}^{\prime}}\right) . \tag{3.22}
\end{equation*}
$$

We claim that if the support of the deformation $M_{t^{\prime}}$ is sufficiently concentrated near $A(-1)$, the $e$ vectors $\mathscr{J}\left(V_{t_{k}^{\prime}}\right)=\mathscr{J}\left(G \Phi_{t_{k}^{\prime}}\right) \in \mathbb{R}^{d}$ are linearly independent.

Indeed, since the deformations $M_{t^{\prime}}$ are localized near $A(-1)$, we have $\Phi_{t_{k}^{\prime}}\left(Z\left(e^{i \theta}\right), U\left(e^{i \theta}\right), 0\right) \equiv 0$, except for $|\theta+\pi|<\theta_{2}$, with $\theta_{2}>0$ small. We deduce:

$$
\begin{align*}
\mathscr{J}\left(G \Phi_{t_{k}^{\prime}}\right) & =\frac{1}{\pi} \int_{|\theta+\pi|<\theta_{2}} \frac{G\left(e^{i \theta}\right) \Phi_{t_{k}^{\prime}}\left(Z\left(e^{i \theta}\right), U\left(e^{i \theta}\right), 0\right)}{\left|e^{i \theta}-1\right|^{2}} d \theta  \tag{3.23}\\
& \approx \frac{1}{\pi} \frac{G(-1)}{4} \int_{|\theta+\pi|<\theta_{2}} \Phi_{t_{k}^{\prime}}\left(Z\left(e^{i \theta}\right), U\left(e^{i \theta}\right), 0\right) d \theta .
\end{align*}
$$

Since, by assumption, the $e$ vectors $\Phi_{t_{k}^{\prime}}\left(z_{-1}, u_{-1}, 0\right)$ are linearly independent, the linear independence of the above (concentrated) vector-valued integrals follows.

The proofs of Proposition 3.14 and of Theorem 3.13 are complete.
3.24. Proofs of two lemmas. Firstly, we check formula (3.20).

Lemma 3.25. Let $u \in \mathscr{C}^{1, \beta}(\bar{\Delta})(0<\beta<1)$ be harmonic in $\Delta$, real-valued and satisfying $u(1)=0$. Then the interior normal derivative of $u$ at $1 \in \partial \Delta$ is given by:

$$
-\frac{\partial u}{\partial r}(1)=\text { p.v. } \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{u\left(e^{i \theta}\right)}{\left|e^{i \theta}-1\right|^{2}} d \theta=\text { p.v. } \frac{i}{\pi} \int_{\partial \Delta} \frac{u(\zeta)}{(\zeta-1)^{2}} d \zeta .
$$

Proof. The function $h:=u+i T u$ is holomorphic in $\Delta$ and $\mathscr{C}^{1, \beta}$ in $\bar{\Delta}$. Since $\mathrm{T} u$ is also harmonic in $\Delta$, since $\frac{\partial h}{\partial r}(1)=\frac{\partial u}{\partial r}(1)+i \frac{\partial \mathrm{~T} u}{\partial r}(1)$, and since

[^19]the kernel $\left|e^{i \theta}-1\right|^{-2}$ is real, we may prove the lemma with $u$ replaced by $h \in \mathscr{O}(\Delta) \cap \mathscr{C}^{1, \beta}(\bar{\Delta})$.

Let $\zeta=r e^{i \theta}$ and denote $h_{1}:=\frac{\partial h}{\partial \zeta}(1)=\frac{\partial h}{\partial r}(1)$, so that $h(\zeta)=$ $(\zeta-1) h_{1}+\mathrm{O}\left(|\zeta-1|^{1+\beta}\right)$. We remind that, for any $\zeta_{0} \in \partial \Delta$, by an elementary modification of Cauchy's formula, we have p.v. $\frac{1}{2 \pi i} \int_{\partial \Delta} \frac{d \zeta}{\zeta-\zeta_{0}}=\frac{1}{2}$. We deduce that the linear term $(\zeta-1) h_{1}$ provides the main contribution:

$$
\text { p.v. } \frac{i}{\pi} \int_{\partial \Delta} \frac{(\zeta-1) h_{1}}{(\zeta-1)^{2}} d \zeta=-2 h_{1} \text { p.v. } \frac{1}{2 \pi i} \int_{\partial \Delta} \frac{d \zeta}{\zeta-1}=-h_{1} .
$$

Thus, we have to prove that the remainder $r(\zeta):=h(\zeta)-(\zeta-1) h_{1}$, which belongs to $\mathscr{O}(\Delta) \cap \mathscr{C}^{1, \beta}(\bar{\Delta})$, gives no contribution, namely satisfies $\int_{\partial \Delta} \frac{r(\zeta)}{(\zeta-1)^{2}} d \zeta=0$.

Set $s(\zeta):=\frac{r(\zeta)}{(\zeta-1)^{2}}$. Then $s \in \mathscr{O}(\Delta)$ is continuous on $\bar{\Delta} \backslash\{1\}$ and satisfies $|s(\zeta)| \leqslant \mathrm{K}|\zeta-1|^{\beta-1}$ for some $\mathrm{K}>0$. We claim that by an application of Cauchy's theorem, the integral $\int_{\partial \Delta} s(\zeta) d \zeta$, which exists without principal value, vanishes.

Indeed, let $\varepsilon$ with $0<\varepsilon \ll 1$ and consider the open disc $\Delta(1, \varepsilon)$ of radius $\varepsilon$ centered at 1 . The drawing of this disc delineates three $\operatorname{arcs}$ of $\bar{\Delta}$ :
(i) the open $\operatorname{arc} \partial \Delta \backslash \overline{\Delta(1, \varepsilon)}$, of length $\approx 2 \pi-2 \varepsilon$; ;
(ii) the closed $\operatorname{arc} \partial \Delta \cap \overline{\Delta(1, \varepsilon)}$, of length $\approx 2 \varepsilon$;
(iii) the closed arc $\partial \Delta(1, \varepsilon) \cap \bar{\Delta}$, of length is $\approx \pi \varepsilon$.


We then decompose the integral of $s$ on $\partial \Delta$ as integrals on the first two arcs:

$$
\int_{\partial \Delta} s(\zeta) d \zeta=\int_{\partial \Delta \backslash \overline{\Delta(1, \varepsilon)}} s(\zeta) d \zeta+\int_{\partial \Delta \cap \overline{\Delta(1, \varepsilon)}} s(\zeta) d \zeta
$$

The estimate $|s(\zeta)| \leqslant \mathrm{K}|\zeta-1|^{\beta-1}$ insures the smallness of the second integral:

$$
\left|\int_{\partial \Delta \cap \overline{\Delta(1, \varepsilon)}} s(\zeta) d \zeta\right| \leqslant C_{1} \varepsilon^{\beta} .
$$

To transform the first integral, we observe that Cauchy's theorem entails that integration of $s(\zeta) d \zeta$ on the closed contour $[\partial \Delta \backslash \overline{\Delta(1, \varepsilon)}] \cup[\partial \Delta(1, \varepsilon) \cap \bar{\Delta}]$ vanishes:

$$
0=\int_{\partial \Delta \backslash \overline{\Delta(1, \varepsilon)}} s(\zeta) d \zeta+\int_{\partial \Delta(1, \varepsilon) \cap \bar{\Delta}} s(\zeta) d \zeta .
$$

Hence the first integral $\int_{\partial \Delta \backslash \overline{\Delta(1, \varepsilon)}}$ may be replaced by the integral $-\int_{\partial \Delta(1, \varepsilon) \cap \bar{\Delta}}$ on the third arc. The estimate $|s(\zeta)| \leqslant \mathrm{K}|\zeta-1|^{\beta-1}$ again insures that this second integral is bounded by $C_{2} \varepsilon^{\beta}$. In conclusion $\left|\int_{\partial \Delta} s(\zeta) d \zeta\right| \leqslant\left(C_{1}+C_{2}\right) \varepsilon^{\beta}$.

Proof of Lemma 3.15. Secondly, we provide the details for the translation of the family $A_{t^{\prime}}$. Let $v=\varphi(z, u)$ represent $M$ in a neighborhood of $p$. By assumption, $A_{t^{\prime}}(\zeta)=\left(Z(\zeta), W\left(\zeta, t^{\prime}\right)\right)$ is attached to $M_{t^{\prime}}$, with $A_{t^{\prime}}(1)=$ $p$. Equivalently, the two equations (3.17) hold. Since $A=\left.A_{t^{\prime}}\right|_{t^{\prime}=0}$ is an embedding, the vector $v_{p}:=\frac{\partial A}{\partial \theta}(1) \in T_{p} M$ is nonzero. As in $\S 2.12$, we choose a small $(2 m+d-1)$-dimensional submanifold $K_{p}$ passing through $p$ with $\mathbb{R} v_{p} \oplus T_{p} K_{p}=T_{p} M$ and we parametrize it by $s \mapsto(z(s), u(s)+$ $i \varphi(z(s), u(s)))$, where $s \in \mathbb{R}^{2 m+d-1}$ is small, $|s|<s_{1}, s_{1}>0$. Then the translation

$$
A_{t^{\prime}, s}(\zeta)=\left(Z(\zeta)+z(s), W\left(\zeta, t^{\prime}, s\right)\right)
$$

is constructed by perturbing the two equations (3.17), requiring only that

$$
A_{t^{\prime}, s}(1)=(z(s), u(s)+i \varphi(z(s), u(s))) .
$$

This is easily done:

$$
\left\{\begin{array}{l}
V\left(e^{i \theta}, t^{\prime}, s\right)=\Phi\left(Z\left(e^{i \theta}\right)+z(s), U\left(e^{i \theta}, t^{\prime}, s\right), t^{\prime}\right) \\
U\left(e^{i \theta}, t^{\prime}, s\right)=u(s)-\mathrm{T}_{1}\left[\Phi\left(Z(\cdot)+z(s), U\left(\cdot, t^{\prime}, s\right), t^{\prime}\right)\right]\left(e^{i \theta}\right)
\end{array}\right.
$$

The non-tangency of $v_{p}$ with $K_{p}$ at $p$ then insures that for every small fixed $t^{\prime}$, the mapping $(\theta, s) \mapsto A_{t^{\prime}, s}\left(e^{i \theta}\right)$ is a diffeomorphism onto a neighborhood of $p$ in $M$.

## §4. Holomorphic extension ON GLOBALLY MINIMAL GENERIC SUBMANIFOLDS

4.1. Structure of the proof of Theorem 3.8. Let $M$ be a $\mathscr{C}^{2, \alpha}$ globally minimal generic submanifold of $\mathbb{C}^{n}$. For clarity, we begin by a summary of the main steps of the proof of Theorem 3.8.
(a) Since $M$ is globally minimal, the distribution $q \mapsto T_{q}^{c} M$ must be somewhere not involutive, namely there must exist a point $p \in M$ and a section $L$ of $T^{1,0} M$ defined in an open neighborhood $U_{p}$ of $p$ in $M$ with $L(p) \neq 0$ such that $[L, \bar{L}](p) \notin T_{p}^{1,0} M \oplus T_{p}^{0,1} M$.
(b) Thanks to an easy generalization of the Lewy extension theorem (§2.10), there exists a manifold $M_{p}^{1}$ attached to $M$ at $p$ with $\operatorname{dim} M_{p}^{1}=1+\operatorname{dim} M$ to which $\mathscr{C}_{C R}^{0}(M)$ extends to be CR.
(c) Thanks to the main propagation Proposition 3.14, CR extension to a similar manifold $M_{q}^{1}$ attached to $M$ holds at every point $q \in M=$ $\mathscr{O}_{C R}(M, p)$.
(d) Since there are as many manifolds with boundary as points in $M$, it may well happen that at some point $p \in M$ which belongs to the edge of two different manifolds $M_{p^{\prime}}^{1}$ and $M_{p^{\prime \prime}}^{1}$, the tangent spaces
$T_{p} M_{p^{\prime}}^{1}$ and $T_{p} M_{p^{\prime \prime}}^{1}$ are distinct. Refering to the diagram of $\S 4.5$ below, we may then immediately profit of such a situation, if it occurs.
(e) Indeed, in this case, an appropriate version of the edge-of-the-wedge theorem guarantees that $\mathscr{C}_{C R}^{0}(M)$ extends to be CR on a $\mathscr{C}^{2, \alpha-0} \mathrm{CR}$ wedge $\mathscr{W}_{p}^{C R, e}$ at $p$ whose dimension $e$ is $\geqslant 1+1=2$.
(f) To reason abstractly, let $e_{\max }$ be the maximal integer $e$ with $1 \leqslant$ $e \leqslant d$ such that there exists a point $p \in M$ and a $\mathscr{C}^{2, \alpha-0}$ CR-wedge $\mathscr{W}_{p}^{C R, e}$ at $p$ of dimension $2 m+d+e$ to which $\mathscr{C}_{C R}^{0}(M)$ extends to be CR. Thanks to the main propagation Proposition 3.14, CR extension to a $\mathscr{C}^{2, \alpha-0}$ CR-wedge $\mathscr{W}_{q}^{C R, e_{\max }}$ holds at every point $q \in M=$ $\mathscr{O}_{C R}(M, p)$.
(g) If $e_{\max }=d$, we are done, Theorem 3.8 is proved. Assuming $e_{\max } \leqslant$ $d-1$, we must construct a contradiction in order to complete the proof.
(h) Since $e_{\max }$ is maximal, again because of the edge-of-the-wedge theorem, the transversal situation (d) cannot occur; in other words, every point $p \in M$ that belongs to the edges of two different CRwedges $\mathscr{W}_{p^{\prime}}^{C R, e_{\max }}$ and $\mathscr{W}_{p^{\prime \prime}}^{C R, e_{\max }}$ has the property that $T_{p} \mathscr{W}_{p^{\prime}}^{C R, e_{\max }}=$ $T_{p} \mathscr{W}_{p^{\prime \prime}}^{C R, e_{\text {max }}}$.
(i) It follows that, as $p$ runs in $M$, the $(2 m+d+e)$-dimensional tangent planes $T_{p} \mathscr{W}_{p}^{C R, e_{\max }} \cap T_{p} M$ glue together and they define a $\mathscr{C}^{1, \alpha-0}$ sub-distribution $K M$ of the tangent bundle $T M$, of dimension $2 m+$ $e_{\text {max }}$, which contains $T^{c} M$.
(j) Since $M$ is globally minimal, such a distribution $p \mapsto K M(p)$ must be somewhere not involutive, namely there must exist a point $p \in M$ such that $[K M, K M](p) \not \subset K M(p)$.
(k) The $\mathscr{C}^{2, \alpha-0}$ CR-wedge $\mathscr{W}_{p}^{C R, e_{\text {max }}}$ may be included in some $\mathscr{C}^{2, \alpha-0}$ local generic submanifold $\mathscr{M}_{p}^{e_{\max }}$ passing through $p$ and containing $M$ in a neighborhood of $p$.
(l) Multiplication by $i$ gives $T_{p}^{c} \mathscr{M}_{p}^{e_{\text {max }}}=K M(p)+i K M(p)$ and the nondegeneracy $[K M, K M](p) \not \subset K M(p)$ implies that the Levi-form of $\mathscr{M}_{p}^{e_{\max }}$ is not identically zero at $p$, namely $\left[T_{p}^{c} \mathscr{M}_{p}^{e_{\max }}, T_{p}^{c} \mathscr{M}_{p}^{e_{\text {max }}}\right](p) \not \subset T_{p}^{c} \mathscr{M}_{p}^{e_{\max }}$.
(m) Then a version of the Lewy-extension theorem on conic generic manifolds having a generic edge guarantees that $\mathscr{C}_{C R}^{0}(M)$ extends to be CR on a CR-wedge $\widetilde{\mathscr{W}}_{p}^{C R, 1+e_{\text {max }}}$ of dimension $2 m+d+1+e_{\text {max }}$ at $p$. This new CR-wedge is constructed by means of discs attached to $M \cup \mathscr{W}_{p}^{C R, e_{\max }}$, exploiting the nondegeneracy of the Levi form of $\mathscr{M}_{p}^{e_{\text {max }}}$. This contradicts the assumption that $e_{\text {max }} \leqslant d-1$ was maximal, hence completes the proof of Theorem 3.8.

The remainder of Section 4 is devoted to provide all the details of the proof.
4.2. Lewy extension in arbitrary codimension. As observed in (a) above, there exists a point $p \in M$ and a local section $L$ of $T^{1,0} M$ with $L(p) \neq 0$ such that $[L, \bar{L}](p) \not \subset \mathbb{C} \otimes T_{p}^{c} M$.
Lemma 4.3. ([We1982, BPo1982]) There exists a manifold with boundary $M_{p}^{1}$ attached to a neighborhood of $p$ in $M$ with dim $M_{p}^{1}=1+\operatorname{dim} M$ to which $\mathscr{C}_{C R}^{0}(M)$ extends to be $C R$.

We shall content ourselves with only one direction of extension, since this will be sufficient for the sequel. Nevertheless, we mention that finer results expressed in terms of the Levi-cone of $M$ at $p$ may be found in [BPo1982, Bo1991]. Anyway, all the extension results that are based on pointwise nondegeneracy conditions as the openness of Levi-cone or the finite typeness of $M$ at a point are by far less general than Theorem 3.8, in which propagational aspects are involved.

Proof. The arguments are an almost straightforward generalization of the proof of the Lewy extension theorem (hypersurface case), already exposed in $\S 2.10$ above. Here is a summary.

By linear algebra reasonings, we may find local coordinates $(z, w) \in$ $\mathbb{C}^{m} \times \mathbb{C}^{d}$ vanishing at $p$ with $L(p)=\left.\frac{\partial}{\partial z_{1}}\right|_{p}$, with $M$ given by $v=\varphi(z, u)$, where $\varphi(0)=0, d \varphi(0)=0$, and with first equation given by

$$
v_{1}=\varphi_{1}=z_{1} \bar{z}_{1}+\mathrm{O}\left(\left|z_{1}\right|^{2+\alpha}\right)+\mathrm{O}(|\widetilde{z}|)+\mathrm{O}(|z||u|)+\mathrm{O}\left(|u|^{2}\right),
$$

where we have split further the coordinates as $\left(z_{1}, \widetilde{z}, w_{1}, \widetilde{w}\right)$, with $\widetilde{z} \in \mathbb{C}^{m-1}$ and $\widetilde{w} \in \mathbb{C}^{d-1}$. For $\varepsilon>0$ small, we introduce the disc defined by

$$
A_{\varepsilon}(\zeta):=\left(\varepsilon(1-\zeta), \widetilde{0}, W_{\varepsilon}^{1}(\zeta), \widetilde{W}_{\varepsilon}(\zeta)\right)
$$

where $W_{\varepsilon}(\zeta)=U_{\varepsilon}(\zeta)+i V_{\varepsilon}(\zeta)$ is uniquely defined by requiring that $A_{\varepsilon}$ is attached to $M$ and satisfies $A_{\varepsilon}(1)=p$. As in $\S 2.10$, one verifies that

$$
-\frac{\partial V_{\varepsilon}^{1}}{\partial r}(1)=2 \varepsilon^{2}+\mathrm{O}\left(\varepsilon^{2+\alpha}\right)
$$

Hence the exit vector of $A_{\varepsilon}$ at $1 \in \partial \Delta$ is nontangential to $M$ at $p$, provided $\varepsilon>0$ is small enough and fixed. By translating $A_{\varepsilon}$, we construct the desired manifold with boundary $M_{p}^{1}$.
4.4. Maximal dimension for $\mathbf{C R}$ extension. As in $\S 4.1(\mathbf{f})$, let $e_{\max }$ be the maximal integer $e \leqslant d$ such that there exists a point $p \in M$ and a $\mathscr{C}^{2, \alpha-0}$ CR-wedge $\mathscr{W}_{p}^{C R, e}$ at $p$ of dimension $2 m+d+e$ to which $\mathscr{C}_{C R}^{0}(M)$ extends to be CR. By the above Lewy extension, we have $e_{\max } \geqslant 1$. Thanks to the main propagation Proposition 3.14, it immediately follows that CR extension to a
$\mathscr{C}^{2, \alpha-0}$ CR-wedge $\mathscr{W}_{q}^{C R, e_{\max }}$ holds at every point $q \in M=\mathscr{O}_{C R}(M, p)$. If $e_{\max }=d$, Theorem 3.8 is proved, gratuitously.

Assuming that $1 \leqslant e_{\max } \leqslant d-1$, in order to establish Theorem 3.8, we must construct a contradiction. In the sequel, we shall simply denote $e_{\max }$ by $e$.

To proceed further, we must reformulate with high precision how were constructed all the CR-wedges obtained by the propagation Proposition 3.14.

For every point $p \in M$, there exists a local CR-wedge $\mathscr{W}_{p}^{C R, e}$ attached to a neighborhood of $p$ in $M$ which is described by means of a family of analytic discs $A_{p, t, s}(\zeta)$, where $t$ and $s$ are parameters. Here, the subscript $p$ is not a parameter, it indicates only that $p$ is the base point of $A_{p, t, s}$, namely $A_{p, t, 0}(1)=p$. The family $A_{p, t, s}$ enjoys properties that are listed below. In this list, the conditions are more uniform than those formulated in Definition 3.4, but one immediately verifies that both formulations are equivalent, up to a shrinking of $t_{1}(p)>0$, of $s_{1}(p)>0$, of $\theta_{1}(p)>0$ and of $1-r_{1}(p)>0$.

- The rotation parameter $t \in \mathbb{R}^{e-1}$ runs in $\left\{|t|<t_{1}(p)\right\}$, for some small $t_{1}(p)>0$.
- The translation parameter $s \in \mathbb{R}^{2 m+d-1}$ runs in $\left\{|s|<s_{1}(p)\right\}$, for some small $s_{1}(p)>0$.
- The point $q(p):=A_{p, 0,0}(-1) \in M$ is close to $p$.
- At $q(p)$, there is a CR-wedge $\mathscr{W}_{q(p)}^{C R, e}$.
- The family $A_{p, t, s}$ satisfies $A_{p, t, s}(\partial \Delta) \subset M \cup \mathscr{W}_{q(p)}^{C R, e}$.
- A small angle $\theta_{1}(p)>0$ and a radius $r_{1}(p)>0$ close to 1 are chosen.
- A family $H_{p^{\prime}}$ of linear subspaces of $T_{p^{\prime}} \mathbb{C}^{n}$ satisfying $T_{p^{\prime}} H_{p^{\prime}} \oplus$ $T_{p^{\prime}} M=T_{p^{\prime}} \mathbb{C}^{n}$ for all $p^{\prime} \in M$ in a neighborhood of $p$ is chosen.
- For every $t$ with $|t|<t_{1}(p)$, every $s$ with $|s|<s_{1}(p)$ and every $\theta$ with $|\theta|<\theta_{1}(p)$, the exit vector of $A_{p, t, s}\left(e^{i \theta}\right)$ at $e^{i \theta}$ is not tangent to M:

$$
\operatorname{ex}\left(A_{p, t, s}\right)\left(e^{i \theta}\right):=\operatorname{proj}_{H_{A_{p, t, s}\left(e^{i \theta}\right)}}\left(i \frac{\partial A_{p, t, s}}{\partial \theta}\left(e^{i \theta}\right)\right) \neq 0
$$

- For every fixed $s$ with $|s|<s_{1}(p)$ and every fixed $\theta$ with $|\theta|<\theta_{1}(p)$, the normalized exit vector mapping

$$
\mathbb{R}^{e-1} \ni t \longmapsto \mathrm{n}-\operatorname{ex}\left(A_{p, t, s}\right)\left(e^{i \theta}\right) \in S^{d-1}
$$

is of rank $(e-1)$ at every $t \in\left\{|t|<t_{1}(p)\right\}$.

- For some $t_{2}(p), s_{2}(p), \theta_{2}(p)$ and $r_{2}(p)$ satisfying $0<t_{2}(p)<t_{1}(p)$, $0<s_{2}(p)<s_{1}(p), 0<\theta_{2}(p)<\theta_{1}(p)$ and $0<1-r_{2}(p)<$ $1-r_{1}(p)<1$, the CR-wedge is precisely defined as:

$$
\mathscr{W}_{p}^{C R, e}:=\left\{A_{p, t, s}\left(r e^{i \theta}\right):|t|<t_{2}(p),|s|<s_{2}(p),|\theta|<\theta_{2}(p), r_{2}(p)<r<1\right\} .
$$

- Finally, the CR-wedge $\mathscr{W}_{p}^{C R, e}$ is contained in a $\mathscr{C}^{2, \alpha-0}$ local generic submanifold $\mathscr{M}_{p}^{e}$ of the same dimension $2 m+d+e$ that contains a neighborhood of $p$ in $M$. At a point $p^{\prime}=A_{p, t^{\prime}, s^{\prime}}\left(e^{i \theta^{\prime}}\right) \in M$ of the edge of $\mathscr{W}_{p}^{C R, e}$, the tangent space of $\mathscr{M}_{p}^{e}$ is:

$$
T_{p^{\prime}} \mathscr{M}_{p}^{e}=T_{p} M \oplus \mathbb{R}\left(i \frac{\partial A_{p, t^{\prime}, s^{\prime}}}{\partial \theta}\left(e^{i \theta^{\prime}}\right)\right) \bigoplus_{1 \leqslant k \leqslant e-1} \mathbb{R}\left(i \frac{\partial^{2} A_{p, t^{\prime}, s^{\prime}}}{\partial \theta \partial t_{k}}\left(e^{i \theta^{\prime}}\right)\right) .
$$

4.5. An edge-of-the-wedge theorem. There are as many generic submanifolds $\mathscr{M}_{p^{\prime}}^{C R, e}$ of codimension $d-e$ as points $p^{\prime} \in M$. At a point $p=$ $A_{p^{\prime}, t^{\prime}, s^{\prime}}\left(e^{i \theta^{\prime}}\right)$ that belongs to the edge of such an $\mathscr{M}_{p^{\prime}}^{C R, e}$, we may define a linear subspace of $T_{p} M$ by

$$
K M_{p^{\prime}}(p):=T_{p}^{c} \mathscr{M}_{p^{\prime}}^{e} \cap T_{p} M
$$

Since $\mathscr{M}_{p^{\prime}}^{e}$ is generic and contains $M$ in a neighborhood of $p$, this space $K M_{p^{\prime}}(p)$ contains $T_{p}^{c} M$ and is $(2 m+e)$-dimensional. Also, multiplication by $i$ induces an isomorphism $K M_{p^{\prime}}(p) / T_{p}^{c} M \simeq T_{p} \mathscr{M}_{p^{\prime}}^{e} / T_{p} M$.

In general, two different $K M_{p^{\prime}}(p)$ and $K M_{p^{\prime \prime}}(p)$ need not coincide, or equivalently, two different tangent spaces $T_{p} \mathscr{M}_{p^{\prime}}^{e}$ and $T_{p} \mathscr{M}_{p^{\prime \prime}}^{e}$ are unequal.


More precisely, there is a dichotomy.
(I) Either for every two points $p^{\prime}, p^{\prime \prime} \in M$ such that there exists a point $p$ belonging to the intersection of the edges of the two CR-wedges $\mathscr{W}_{p^{\prime}}^{C R, e}$ and $\mathscr{W}_{p^{\prime \prime}}^{C R, e}$, namely of the form:

$$
p=A_{p^{\prime}, t^{\prime}, s^{\prime}}\left(e^{i \theta^{\prime}}\right)=A_{p^{\prime \prime}, t^{\prime \prime}, s^{\prime \prime}}\left(e^{i \theta^{\prime \prime}}\right),
$$

for some values

$$
\begin{array}{ll}
\left|t^{\prime}\right|<t_{2}\left(p^{\prime}\right), & \left|s^{\prime}\right|<s_{2}\left(p^{\prime}\right), \\
\left|t^{\prime \prime}\right|<\theta_{2}\left(p^{\prime \prime}\right), & \left|s^{\prime \prime}\right|<\theta_{2}\left(p^{\prime}\left(p^{\prime \prime}\right),\right. \\
\left|\theta^{\prime \prime}\right|<\theta_{2}\left(p^{\prime \prime}\right),
\end{array}
$$

the two spaces $T_{p} \mathscr{M}_{p^{\prime}}^{e}$ and $T_{p} \mathscr{M}_{p^{\prime \prime}}^{e}$ coincide. Equivalently, $K M_{p^{\prime}}(p)=K M_{p^{\prime \prime}}(p)$.
(II) Or there exist two points $p^{\prime}, p^{\prime \prime} \in M$ and a point $p=A_{p^{\prime}, t^{\prime}, s^{\prime}}\left(e^{i \theta^{\prime}}\right)=$ $A_{p^{\prime \prime}, t^{\prime \prime}, s^{\prime \prime}}\left(e^{i \theta^{\prime \prime}}\right)$ in the intersection of the edges of the two CR-wedges $\mathscr{W}_{p^{\prime}}^{C R, e}$ and $\mathscr{W}_{p^{\prime \prime}}^{C R, e}$ such that

$$
T_{p} \mathscr{M}_{p^{\prime}}^{e} \neq T_{p} \mathscr{M}_{p^{\prime \prime}}^{e} .
$$

Lemma 4.6. The case $T_{p} \mathscr{M}_{p^{\prime}}^{e} \neq T_{p} \mathscr{M}_{p^{\prime \prime}}^{e}$ implies that $\mathscr{C}_{C R}^{0}(M)$ extends to be CR on a CR-wedge $\widetilde{\mathscr{W}}_{p}^{C R, 1+e}$ at $p$ whose dimension equals $2 m+d+1+e$, contradicting the maximality of $e=e_{\max }$.

Of course, this lemma follows by a known CR version of the edge-of-thewedge theorem ([Ai1989]), but for completeness, we summarize a shorter proof that exploits the existence of the discs $A_{p^{\prime}, t, s}$, as in [Po2004].
Proof. By construction, the family $A_{p^{\prime}, t, s}(\zeta)$ covers the CR-wedge $\mathscr{W}_{p^{\prime}}^{C R, e}$. The point $p$ belongs to the edge of $\mathscr{W}_{p^{\prime}}^{C R, e}$.

Since $T_{p} \mathscr{M}_{p^{\prime}}^{e} \neq T_{p} \mathscr{M}_{p^{\prime \prime}}^{e}$, there exists a manifold $M_{p}^{1} \subset \mathscr{W}_{p^{\prime \prime}}^{C R, e}$ attached to $M$ at $p$ with $\operatorname{dim} M_{p}^{1}=1+\operatorname{dim} M$ such that

$$
1+e=\operatorname{dim}\left(\left[T_{p} M_{p}^{1}+T_{p} \mathscr{W}_{p^{\prime}}^{C R, e}\right] / T_{p} M\right) .
$$



We may deform the family $A_{p^{\prime}, t, s}$ by translating it along $M_{p}^{1}$, as in the diagram. So we introduce a supplementary parameter $\sigma>0$ and we require that the point $A_{p^{\prime}, t, s, \sigma}(1)$ should cover a one-sided neighborhood of $p$ in $M_{p}^{1}$ as $\sigma$ runs in $\left(0, \sigma_{1}\right)$, for some small $\sigma_{1}>0$, and as the previous translation parameter $s \in \mathbb{R}^{2 m+d-1}$ runs in $\left\{|s|<s_{2}\left(p^{\prime}\right)\right\}$. Thanks to Theorem 3.7(IV), the corresponding Bishop-type equation has $\mathscr{C}^{2, \alpha-0}$ solutions.

If we choose $t_{3}>0$ with $\left|t^{\prime}\right|+t_{3}<t_{2}\left(p^{\prime}\right), s_{3}>0$ with $\left|s^{\prime}\right|+s_{3}<s_{2}\left(p^{\prime}\right)$, $\theta_{3}>0$ with $\left|\theta^{\prime}\right|+\theta_{3}<\theta_{2}\left(p^{\prime}\right), \sigma_{3}>0$ with $\sigma_{3}<\sigma_{1}$ and $r_{3}<1$ with $r_{2}\left(p^{\prime}\right)<r_{3}<1$, the set:

$$
\begin{aligned}
\widetilde{W}_{p}^{C R, 1+e}:=\left\{A_{p^{\prime}, t, s, \sigma}\left(r e^{i \theta}\right):\right. & \left|t-t^{\prime}\right|<t_{3},\left|s-s^{\prime}\right|<s_{3}, \\
& \left.\left|\theta-\theta^{\prime}\right|<\theta_{3}, r_{3}<r<1,0<\sigma<\sigma_{3}\right\}
\end{aligned}
$$

will constitute a CR-wedge of dimension $2 m+d+1+e$ at $p$. By a technical adaptation of the approximation Theorem 5.2(III) (cf. Lemma 2.19), $\mathscr{C}_{C R}^{0}(M)$ extends to be CR on $\widetilde{W}_{p}^{C R, 1+e}$.
4.7. Definition of the (non-integrable) subbundle $K M \subset T M$. Consequently, case (II) cannot occur, because of the definition of $e=e_{\max }$. Thus, case (I) holds. In other words, as $p^{\prime}$ runs in $M$, the $\mathscr{C}^{1, \alpha-0}$ distributions $p \mapsto K M_{p^{\prime}}(p)$ defined for $p$ in the edge of $\mathscr{W}_{p^{\prime}}^{C R, e}$ (a neighborhood of $p^{\prime}$ in $M$ ) glue together in a well-defined $\mathscr{C}^{1, \alpha-0}$ vector subbundle of $T M$. Observe that $T^{c} M$ is a subbundle of $K M$ of codimension $e$. For every point $p \in M$, we have:

$$
T_{p}^{c} M \subset K M(p)=T_{p}^{c} \mathscr{M}_{p}^{e} \cap T_{p} M
$$

As in $\S 4.1(\mathbf{j})$, since $M$ is globally minimal and since $K M$ is of codimension $d-e \geqslant 1$ in $T M$, there must exist a point $p \in M$ such that $[K M, K M](p) \not \subset K M(p)$.

Lemma 4.8. At such a point p, the Levi form of $\mathscr{M}_{p}^{e}$ does not vanish identically:

$$
\left[T^{c} \mathscr{M}_{p}^{e}, T^{c} \mathscr{M}_{p}^{e}\right](p) \not \subset T_{p}^{c} \mathscr{M}_{p}^{e}
$$

Proof. We reason by contradiction, assuming that $\left[T^{c} \mathscr{M}_{p}^{e}, T^{c} \mathscr{M}_{p}^{e}\right](p) \subset$ $T_{p}^{c} \mathscr{M}_{p}^{e}$. Let $K^{1}$ and $K^{2}$ be two arbitrary $\mathscr{C}^{1, \alpha-0}$ sections of $K M$ defined in a small neighborhood $U_{p}$ of $p$ in $M$. Since $\left.K M\right|_{U_{p}}$ is a subbundle of $\left.T M\right|_{U_{p}}$, we have

$$
\left[K^{1}, K^{2}\right](p) \in T_{p} M
$$

We may extend $K^{1}$ and $K^{2}$ to a neighborhood $\mathscr{U}_{p}$ of $p$ in $\mathscr{M}_{p}^{e}$ that contains $U_{p}$ as sections $\mathscr{K}^{1}$ and $\mathscr{K}^{2}$ of $\left.T^{c} \mathscr{M}_{p}^{e}\right|_{\mathscr{U}_{p}}$. Since $K^{1}$ and $K^{2}$ are tangent to $M \cap U_{p}$, one verifies that, independently of the extension:

$$
\left[K^{1}, K^{2}\right](p)=\left[\mathscr{K}^{1}, \mathscr{K}^{2}\right](p) \in T_{p}^{c} \mathscr{M}_{p}^{e},
$$

where the second Lie bracket belongs to $T_{p}^{c} \mathscr{M}_{p}^{e}$, because we assumed that the Levi form of $\mathscr{M}_{p}^{e}$ vanishes at $p$. We deduce

$$
\left[K^{1}, K^{2}\right](p) \in T_{p}^{c} \mathscr{M}_{p}^{e} \cap T_{p} M=K M(p) .
$$

This contradicts $[K M, K M](p) \not \subset K M(p)$.
4.9. Lewy extension on CR-wedges. To contradict the maximality of $e=$ $e_{\max }$ at a point $p$ at which $[K M, K M](p) \not \subset K M(p)$, we formulate a Lewy extension theorem on the conic manifold with edge $\mathscr{W}_{p}^{C R, e}$.

Proposition 4.10. Let $p \in M$ and assume that $\left[T_{p}^{c} \mathscr{M}_{p}^{e}, T_{p}^{c} \mathscr{M}_{p}^{e}\right](p) \not \subset$ $T_{p}^{c} \mathscr{M}_{p}^{e}$. Then there exists a $(2 m+d+1+e)$-dimensional local CR-wedge $\widetilde{W}_{p}^{C R, 1+e}$ of edge $M$ at p to which $\mathscr{C}_{C R}^{0}\left(M \cup \mathscr{W}_{p}^{C R, e}\right)$ extends to be $C R$.

Thus, this proposition concludes the proof of Theorem 3.8.
Proof. There exists a local section $L$ of $T^{1,0} \mathscr{M}_{p}^{e}$ with $L(p) \neq 0$ such that $[L, \bar{L}](p) \notin T_{p}^{1,0} \mathscr{M}_{p}^{e} \oplus T_{p}^{0,1} \mathscr{M}_{p}^{e}$. It is appropriate to distinguish two cases.
Firstly, assume that $L(p) \in T^{1,0} M$. Then as in $\S 4.2$, we may construct a small analytic disc $A_{\varepsilon}$ attached to $M$ in a neighborhood of $p$ having exit vector $-\frac{\partial A_{\varepsilon}}{\partial r}(1)$ approximately directed by $[L, \bar{L}](p) \notin \mathbb{C} \otimes T_{p} \mathscr{M}_{p}^{e}$. So this disc has exit vector nontangential to $\mathscr{M}_{p}^{e}$ at $p$. By translating it along $M$ and along the $e$ supplementary directions offered by $\mathscr{W}_{p}^{C R, e}$, we deduce CR extension to a $(2 m+d+1+e)$-dimensional CR wedge $\widetilde{W}_{p}^{C R, 1+e}$.


Secondly, assume that $L(p) \notin T_{p}^{1,0} M$ for every local section $L$ of $T^{1,0} M$ such that $[L, \bar{L}](p) \notin T_{p}^{1,0} \mathscr{M}_{p}^{e} \oplus T_{p}^{0,1} \mathscr{M}_{p}^{e}$.

We explain the case $d=2, e=1$ first, since this case is easier to understand. Under this assumption, $\mathscr{M}_{p}^{1}$ is a hypersurface of $\mathbb{C}^{n}$ divided in two parts by $M$, one part being $\mathscr{W}_{p}^{C R, 1}$. We draw a diagram.


DISC ATTACHED TO A HALF HYPERSURFACE AND HAVING NONTANGENT EXIT VECTOR

There exist coordinates $\left(z, w^{\prime}, w^{\prime \prime}\right) \in \mathbb{C}^{n-2} \times \mathbb{C} \times \mathbb{C}$ centered at $p$ in which $\mathscr{M}_{p}^{1}$ is given by $v^{\prime \prime}=\psi\left(z, w^{\prime}, u^{\prime \prime}\right)$, with $\psi(0)=0$ and $d \psi(0)=0$ and in which $M$ is given by a supplementary equation $v^{\prime}=\varphi^{\prime}\left(z, u^{\prime}, u^{\prime \prime}\right)$ with $\varphi^{\prime}(0)=0$ and $d \varphi^{\prime}(0)=0$. Changing the orientation of the $v^{\prime}$-axis if necessary, it follows $\mathscr{W}_{p}^{C R, 1}$ is given by the equation $v^{\prime \prime}=\psi\left(z, w^{\prime}, u^{\prime \prime}\right)$ and the inequation $v^{\prime}>\varphi^{\prime}\left(z, u^{\prime}, u^{\prime \prime}\right)$, with $\varphi^{\prime}(0)=0$ and $d \varphi^{\prime}(0)=0$. In the diagram, $T_{p}^{c} \mathscr{M}_{p}^{1}$ is the direct sum of the $z$-coordinate space with the $u^{\prime}+i v^{\prime}$-coordinate axis.

The Levi form of $\mathscr{M}_{p}^{1}$ is represented by a scalar Hermitian form $H\left(z, w^{\prime}, \bar{z}, \bar{w}^{\prime}\right)$. By assumption, its restriction to $T_{p}^{c} M$ vanishes (otherwise, the first case holds), so $H(z, 0, \bar{z}, 0) \equiv 0$. The assumption that the Levi form of $\mathscr{M}_{p}^{1}$ does not vanish identically insures that $H$ is nonzero. To proceed further, we need $H\left(0, w^{\prime}, 0, \bar{w}^{\prime}\right) \not \equiv 0$. If $H\left(0, w^{\prime}, 0, \bar{w}^{\prime}\right) \equiv 0$, since $H$ is nonzero, by a linear coordinate change of the form $\widetilde{w}^{\prime}=w^{\prime}$, $\widetilde{z}_{k}=z_{k}+a_{k} w^{\prime}, k=1, \ldots, n-2$, $\widetilde{w}^{\prime \prime}=w^{\prime \prime}$, we may insure that $H\left(0, w^{\prime}, 0, \bar{w}^{\prime}\right) \not \equiv 0$. Observe that such a change of coordinates stabilizes both $T_{p} M$ and $T_{p} \mathscr{M}_{p}^{1}$. After a real dilation, we can assume that the equation of $\mathscr{M}_{p}^{1}$ is of the form:
$v^{\prime \prime}=w^{\prime} \bar{w}^{\prime}+\mathrm{O}\left(\left|w^{\prime}\right|^{2+\alpha-0}\right)+\mathrm{O}\left(|z|\left|\left(z, w^{\prime}\right)\right|\right)+\mathrm{O}\left(\left|u^{\prime \prime}\right|\left|\left(z, w^{\prime}\right)\right|\right)+\mathrm{O}\left(\left|u^{\prime \prime}\right|^{2}\right)$.
To the hypersurface $\mathscr{M}_{p}^{1}$, we attach a disc $A_{\varepsilon}(\zeta)$ with zero $z$-component, with $w^{\prime}$-component equal to $i \varepsilon(1-\zeta)$ and with $w^{\prime \prime}$-component $\left(U_{\varepsilon}^{\prime \prime}(\zeta)+\right.$ $\left.i V_{\varepsilon}^{\prime \prime}(\zeta)\right)$ of class $\mathscr{C}^{2, \alpha-0}$ satisfying the corresponding Bishop-type equation. Exactly as in the Lewy extension theorem (§2.10), for $\varepsilon>0$ small enough and fixed, the exit vector of $A_{\varepsilon}$ at $p$ is nontangent to $\mathscr{M}_{p}^{1}$ (this is uneasy to draw in the diagram above, but imagine that the disc drawn in $\$ 2.10$ is attached to a half-paraboloid). Furthermore, using the inequality $v^{\prime}\left(e^{i \theta}\right)=$ $\varepsilon(1-\cos \theta) \geqslant \varepsilon \frac{\theta^{2}}{\pi}$ for $|\theta| \leqslant \pi$ together with the property $d \varphi^{\prime}(0)=0$, it is elementary to verify that $A_{\varepsilon}(\partial \Delta \backslash\{1\})$ is contained in the open halfhypersurface $\left\{v^{\prime}>\varphi^{\prime}\right\}$, as shown in the diagram.

Since the exit vector of $A_{\varepsilon}$ is nontangent to $\mathscr{M}_{p}^{1}$, in order to get holomorphic extension to a wedge at $p$, it suffices to translate the disc $A_{\varepsilon}$ in the half-hypersurface $\mathscr{W}_{p}^{C R, 1}$.

However, if we translate $A_{\varepsilon}$ as usual by requiring that the base point $A_{\varepsilon, s}(1)=p_{s}$, with $s \in \mathbb{R}^{2 n-2}$ small, covers a neighborhood of $p$ in $M$, it may well happen that, due to the curvature of $M$ in a neighborhood of $p$, the boundary of the translated disc enters slightly the other side of $\mathscr{M}_{p}^{1}$, which is forbidden.

To remedy this imperfection, two equally good options present themselves. The first option would be to rotate slightly the translated disc $A_{\varepsilon, s}$ in order that it becomes tangent to $M$ at the point $p_{s}=A_{\varepsilon, s}(1)$. Then adding
a small parameter $\sigma>0$, we would translate it slightly in the positive direction of $\mathscr{W}_{p}^{C R, 1}$, essentially along the positive $v^{\prime}$-direction.

The second option is to introduce a family of complex affine biholomorphisms $\Psi_{s}$ that transfer $p_{s} \in M$ to the origin and transfer the tangent spaces at $p_{s}$ of $\mathscr{M}_{p}^{1}$ and of $M$ to $\left\{v^{\prime \prime}=0\right\}$ and to $\left\{v^{\prime \prime}=v^{\prime}=0\right\}$. So $\Psi_{s}\left(\mathscr{M}_{p}^{1}\right)$ is given by $v^{\prime \prime}=\psi^{\prime \prime}\left(z, w^{\prime}, u^{\prime \prime}: s\right)$ with $\psi$ of class $\mathscr{C}^{2, \alpha-0}$ with respect to all variables and with the map $\left(z, w^{\prime}, u^{\prime \prime}\right) \mapsto \psi^{\prime \prime}\left(z, w^{\prime}, u^{\prime \prime}: s\right)$ vanishing to second order at the origin for every $s \in \mathbb{R}^{2 n-2}$ small. Also, $\Psi_{s}\left(\mathscr{W}_{p}^{C R, 1}\right)$ is given by a supplementary inequation $v^{\prime}>\varphi^{\prime}\left(z, u^{\prime}, u^{\prime \prime}: s\right)$, with $\varphi^{\prime}$ of class $\mathscr{C}^{2, \alpha}$ (the smoothness of $M$ ) with respect to all variables and with $\left(z, u^{\prime}, u^{\prime \prime}\right) \mapsto \varphi\left(z, u^{\prime}, u^{\prime \prime}: s\right)$ vanishing to second order at the origin.

To the hypersurface $\Psi_{s}\left(\mathscr{M}_{p}^{1}\right)$, we attach the family of discs

$$
\widetilde{A}_{\varepsilon, s, \sigma}(\zeta)=\left(0, i \sigma+i \varepsilon(1-\zeta), \widetilde{U}_{\varepsilon, s, \sigma}^{\prime \prime}(\zeta)+i \widetilde{V}_{\varepsilon, s, \sigma}^{\prime \prime}(\zeta)\right)
$$

having zero $z$-component and $w^{\prime}$-component equal to $i \sigma+i \varepsilon(1-\zeta)$, where $\sigma \in \mathbb{R}$ with $|\sigma|<\sigma_{1}, \sigma_{1}>0$, is a small parameter of translation along the $v^{\prime}$-axis. Of course:

$$
\left\{\begin{array}{l}
\widetilde{U}_{\varepsilon, s, \sigma}^{\prime \prime}\left(e^{i \theta}\right)=-\mathbf{T}_{1}\left[\psi\left(0, i \sigma+i \varepsilon(1-\cdot), \widetilde{U}_{\varepsilon, s, \sigma}^{\prime \prime}(\cdot): s\right)\right]\left(e^{i \theta}\right), \\
\widetilde{V}_{\varepsilon, s, \sigma}^{\prime \prime}\left(e^{i \theta}\right)=\mathbf{T}_{1}\left[\widetilde{U}_{\varepsilon, s, \sigma}^{\prime \prime}\right]\left(e^{i \theta}\right) .
\end{array}\right.
$$

By means of elementary computations involving Taylor's formula, we verify two facts.

- If $\varepsilon>0$ is sufficiently small and fixed, $\widetilde{A}_{\varepsilon, s, 0}(\partial \Delta \backslash\{1\})$ is contained in the open half-hypersurface $\left\{v^{\prime}>\varphi^{\prime}\left(z, u^{\prime}, u^{\prime \prime}: s\right)\right\}$, for all $s \in$ $\mathbb{R}^{2 n-2}$ with $|s|<s_{1}, s_{1}>0$ small.
- Furthermore, for all $\sigma$ with $0<\sigma \leqslant \sigma_{1}$, and all $s$ with $|s|<s_{1}$, the disc boundary $\widetilde{A}_{\varepsilon, s, \sigma}(\partial \Delta)$ is contained in the open half-hypersurface $\left\{v^{\prime}>\varphi^{\prime}\left(z, u^{\prime}, u^{\prime \prime}: s\right)\right\}$.

Coming back to the old system of coordinates, it follows that the family of $\operatorname{discs} A_{\varepsilon, s, \sigma}:=\Psi_{s}^{-1} \circ \widetilde{A}_{\varepsilon, s, \sigma}$ has base point $A_{\varepsilon, s, \sigma}(1)$ covering a neighborhood of $p$ in the half-hypersurface $\mathscr{W}_{p}^{C R, 1}$, as $s$ and $\sigma$ vary. Since the exit vector of $A_{\varepsilon}$ is not tangent to $\mathscr{M}_{p}^{1}$ at $p$, this family of discs covers a $2 n$-dimensional wedge $\widetilde{W}_{p}^{C R, 2 n}$ of edge $M$ at $p$. This completes the proof of the second case of the proposition when $e=1$ and $d=2$.

Based on these explanations, we may now summarize the general case. There exist coordinates $\left(z, w^{\prime}, w^{\prime \prime}\right) \in \mathbb{C}^{m} \times \mathbb{C}^{e} \times \mathbb{C}^{d-e}$ vanishing at $p$ in which the $\mathscr{C}^{2, \alpha-0}$ generic submanifold $\mathscr{M}_{p}^{e}$ is represented by $v^{\prime \prime}=\psi\left(z, w^{\prime}, u^{\prime \prime}\right)$, with $\psi(0)=0$ and $d \psi(0)=0$. After killing the second order pluriharmonic quadratic terms in every right hand side $\psi_{j^{\prime \prime}}\left(z, w^{\prime}, 0\right), j^{\prime \prime}=1, \ldots, d-e$, we may assume that the quadratic terms are Hermitian forms $H_{j^{\prime \prime}}\left(z, w^{\prime}, \bar{z}, \bar{w}^{\prime}\right)$.

After a linear change of coordinates in the $w^{\prime}$-space, $T_{p} M=\left\{v^{\prime}=v^{\prime \prime}=\right.$ $0\}$, the $\mathscr{C}^{2, \alpha}$ generic edge $M$ is defined by $v=\varphi(z, u)$ with $\varphi(0)=0$, $d \varphi(0)=0$ and the conic open submanifold $\mathscr{W}_{p}^{C R, e}$ of $\mathscr{M}_{p}^{e}$ is defined by $v^{\prime \prime}=\psi\left(z, w^{\prime}, u^{\prime \prime}\right)$ together with the inequations

$$
v_{j^{\prime}}^{\prime}>\varphi_{j^{\prime}}^{\prime}\left(z, u^{\prime}, u^{\prime \prime}\right), \quad j^{\prime}=1, \ldots, e
$$

where $\varphi=\left(\varphi^{\prime}, \varphi^{\prime \prime}\right)$. In fact, we may assume that the cone defining the CRwedge on the tangent space is slightly larger than the salient cone $v_{j^{\prime}}^{\prime}>0$, $j^{\prime}=1, \ldots, e$.

The nonvanishing of the Levi form of $\mathscr{M}_{p}^{e}$ at $p$ entails that at least one Hermitian form $H_{j^{\prime \prime}}\left(z, w^{\prime}, \bar{z}, \bar{w}^{\prime}\right)$ is nonzero. After renumbering, $H_{1}$ is nonzero. Also, since $T_{p}^{c} M$ is the $z$-coordinate space, we have $H_{1}(z, 0, \bar{z}, 0) \equiv 0$ (otherwise, the first case holds). After a complex linear coordinate change of the form $\widetilde{w}^{\prime}=w^{\prime}, \widetilde{z}_{k}=z_{k}+\sum_{j^{\prime}=1}^{e} a_{k}^{j^{\prime}} w_{j^{\prime}}^{\prime}, \widetilde{w}^{\prime \prime}=w^{\prime \prime}$, we may insure that $H_{1}\left(0, w^{\prime}, 0, \bar{w}^{\prime}\right) \not \equiv 0$. Then the set of vectors $\left(0, w^{\prime}\right)$ on which $H_{1}$ vanishes is a proper real quadratic cone of $\mathbb{C}^{e}$. Consequently, for almost every real vector $\left(0, i v^{\prime}\right)$, the quadratic form $H_{1}$ is nonzero on the complex line $\mathbb{C}\left(0, i v^{\prime}\right)$. Since the cone defining $\mathscr{W}_{p}^{C R, e}$ is open and may be slightly shrunk, we can assume that $H_{1}$ does not vanish on $\mathbb{C}\left(0, i v_{1}^{\prime}\right)$, with $v_{1}^{\prime}=(1, \ldots, 1) \in \mathbb{R}^{e}$. It follows that the disc $A_{\varepsilon}$ attached to $\mathscr{M}_{p}^{e}$ having zero $z$-component and $w^{\prime}$-component equal to $(i \varepsilon(1-\zeta), \ldots, i \varepsilon(1-\zeta))$ is nontangent to $\mathscr{M}_{p}^{e}$ at $p$.

Furthermore, letting a point $p_{s} \in M$ of coordinates $s:=(z, u)$ vary in a small neighborhood of $p$ in $M$, we may construct a family of biholomorphisms $\Psi_{s}$ sending $p_{s}$ to the origin and normalizing the equations of $M$, of $\mathscr{M}_{p}^{e}$ and of $\mathscr{W}_{p}^{C R, e}$ under the form $v=\varphi(z, u: s), v^{\prime \prime}=\psi\left(z, w^{\prime}, u^{\prime \prime}: s\right)$ and $v_{j^{\prime}}^{\prime}>\varphi_{j^{\prime}}^{\prime}\left(z, u^{\prime}, u^{\prime \prime}: s\right)$, with $\varphi$ being $\mathscr{C}^{2, \alpha}$ and with $\psi$ being $\mathscr{C}^{2, \alpha-0}$ with respect to all variables and both vanishing to second order at the origin.

Let $\sigma \in \mathbb{R}^{e},|\sigma|<\sigma_{1}$, be a small parameter of translation along the $v^{\prime}$ coordinate space. To the generic submanifold $\Psi_{s}\left(\mathscr{M}_{p}^{e}\right)$, we attach the family of discs

$$
\widetilde{A}_{\varepsilon, s, \sigma}(\zeta)=\left(0, W_{\varepsilon, \sigma}^{\prime}(\zeta), U_{\varepsilon, s, \sigma}^{\prime \prime}(\zeta)+i V_{\varepsilon, s, \sigma}^{\prime \prime}(\zeta)\right),
$$

where

$$
W_{\varepsilon, \sigma}^{\prime}(\zeta)=\left(i \sigma_{1}+i \varepsilon(1-\zeta), \ldots, i \sigma_{e}+i \varepsilon(1-\zeta)\right)
$$

and where

$$
\left\{\begin{array}{l}
\widetilde{U}_{\varepsilon, s, \sigma}^{\prime \prime}\left(e^{i \theta}\right)=-\mathbf{T}_{1}\left[\psi\left(0, i \sigma+i \varepsilon(1-\cdot), \widetilde{U}_{\varepsilon, s, \sigma}^{\prime \prime}(\cdot): s\right)\right]\left(e^{i \theta}\right), \\
\widetilde{V}_{\varepsilon, s, \sigma}^{\prime \prime}\left(e^{i \theta}\right)=\mathrm{T}_{1}\left[\widetilde{U}_{\varepsilon, s, \sigma}^{\prime \prime}\right]\left(e^{i \theta}\right) .
\end{array}\right.
$$

By means of elementary computations involving Taylor's formula, we may verify that for all $\sigma \in \mathbb{R}^{e}$ with $0<\sigma_{j^{\prime}} \leqslant \sigma_{1}, j^{\prime}=1, \ldots, e$, and
all $s \in \mathbb{R}^{2 m+d},|s|<s_{1}$, the disc boundary $\widetilde{A}_{\varepsilon, s, \sigma}(\partial \Delta)$ is contained in $\left\{v_{j^{\prime}}^{\prime}>\varphi_{j^{\prime}}^{\prime}\left(z, u^{\prime}, u^{\prime \prime}: s\right), j^{\prime}=1, \ldots, e\right\}$.

Coming back to the old system of coordinates, it follows that the family of $\operatorname{discs} A_{\varepsilon, s, \sigma}:=\Psi_{s}^{-1} \circ \widetilde{A}_{\varepsilon, s, \sigma}$ has base point $A_{\varepsilon, s, \sigma}(1)$ covering a neighborhood of $p$ in the CR-wedge $\mathscr{W}_{p}^{C R, e}$, as $s$ and $\sigma$ vary. Since its exit vector is not tangent to $\mathscr{M}_{p}^{1}$ at $p$, this family of discs covers a $(2 m+d+1+e)$-dimensional CR-wedge $\widetilde{W}_{p}^{C R, 1+e}$ of edge $M$ at $p$.

The proofs of the proposition and of Theorem 3.8 are complete.
4.11. Wedgelike domains. On a globally minimal $M$, at every point $p \in$ $M$, we have constructed a local wedge $\mathscr{W}_{p}$ by gluing deformations of discs. It may well happen that at a point $p$ that belongs to the edges of two different wedges $\mathscr{W}_{q^{\prime}}$ and $\mathscr{W}_{q^{\prime \prime}}$, the wedges have empty intersection in $\mathbb{C}^{n}$ (imagine two thin opposite cones having vertex at $0 \in \mathbb{R}^{2}$ ). Fortunately, by means of the translation trick presented in $\S 4.5$ ( $c f$. the diagram), we can fill in the space in between. Achieving this systematically, by a sort of gluingshrinking processus, we obtain some connected open set $\mathscr{W}$ attached to $M$ containing possibly smaller wedges $\mathscr{W}_{p}^{\prime} \subset \mathscr{W}_{p}$ at every point.

To set-up a useful definition, by a wedgelike domain $\mathscr{W}$ attached to $M$ we mean a connected open set that contains a local wedge of edge $M$ at every point. Geometrically speaking, the requirement of connectedness prevents jumps of the directions of local wedges in the normal bundle $\left.T \mathbb{C}^{n}\right|_{M} / T M$.

We may finally conclude this section by the formulation of a statement that is the very starting point of the study of removable singularities for CR functions ([MP1998, MP1999, MP2000, MP2002, 26]).

Theorem 4.12. ([Me1997, MP1999]) If $M$ is a globally minimal $\mathscr{C}^{2, \alpha}$ generic submanifold of $\mathbb{C}^{n}$, there exists a wedgelike domain $\mathscr{W}$ attached to $M$ such that every continuous $C R$ function $f \in \mathscr{C}_{C R}^{0}(M)$ possesses a holomorphic extension $F \in \mathscr{O}(\mathscr{W}) \cap \mathscr{C}^{0}(M \cup \mathscr{W})$ with $\left.F\right|_{M}=f$.

Its $L^{\text {p }}$ version deserves special attention. Let $\mathscr{W}$ be a wedgelike domain attached to $M$. A holomorphic function $F \in \mathscr{O}(\mathscr{W})$ is said to belong to the Hardy space $H_{l o c}^{\mathrm{p}}(\mathscr{W})$ if, for every $p \in M$, for every local coordinate system centered at $p$ in which $M$ is given by $v=\varphi(x, y, u)$, for every local wedge of edge $M$ at $p$ contained in $\mathscr{W}$ of the form

$$
\begin{aligned}
\mathscr{W}=\mathscr{W}(\rho, \sigma, C):=\{ & (x+i y, u+i v) \in \Delta_{\rho}^{m} \times \square_{\rho}^{d} \times i \square_{\sigma}^{d}: \\
& v-\varphi(x, y, u) \in C\},
\end{aligned}
$$

as defined in $\S 4.29$ (III), for every cone $C^{\prime} \subset \mathbb{R}^{d}$ with $C^{\prime} \cap S^{d-1} \subset \subset C \cap S^{d-1}$ and for every $\rho^{\prime}<\rho$, the supremum:

$$
\sup _{\theta^{\prime} \in C^{\prime}} \int_{\Delta_{\rho^{\prime} \times \square_{\rho^{\prime}}^{d}}^{d}}\left|F\left(x+i y, u+i \varphi(x, y, u)+i \theta^{\prime}\right)\right|^{\mathrm{p}} d x \wedge d y \wedge d u<\infty
$$

is finite. An adaptation of the proof of the preceding theorem yields its $L^{p}$ version.

Theorem 4.13. ([Po1997, Po2000]) If $M$ is a globally minimal $\mathscr{C}^{2, \alpha}$ generic submanifold of $\mathbb{C}^{n}$, there exists a wedgelike domain $\mathscr{W}$ attached to $M$ such that every function $f \in L_{l o c, C R}^{\mathrm{p}}, 1 \leqslant \mathrm{p} \leqslant \infty$, possesses a Hardy space holomorphic extension $F \in \mathrm{H}_{l o c}^{\mathrm{p}}(\mathscr{W})$.

To conclude, we would like to mention that arguments similar to those of Theorem 4.12 yield a mild generalization, worth to be mentioned: the CR extension theory is valid for $\mathscr{C}^{2, \alpha} \mathrm{CR}$ manifolds that are only locally embeddable.

However, for concreteness reasons, we preferred to set up the theory in a globally embedded context. In the remainder of the memoir, not to enter superficial corollaries, we will formulate all our results under the paradigmatic assumption of global minimality. Thus, Theorems 4.12 and 4.13 will be our basic main starting point.

The two monographs [Trv1992, BCH2005] deal not only with embedded structures but also with locally integrable structures. Nevertheless, most topics exposed here are not yet embraced in a comprehensive theory ( $c f$. $\S 3.29($ III) ). So it is an open direction of research to transfer the theory of holomorphic extension of CR functions (including removable singularities) to locally integrable structures.

## VI: Removable singularities

Table of contents

1. Removable singularities for linear partial differential operators ..... 203.
2. Removable singularities for holomorphic functions of one or several complexvariables207.
3. Hulls and removable singularities at the boundary ..... 225.
4. Smooth and metrically thin removable singularities for $\mathbf{C R}$ functions ..... 240.
5. Removable singularities in CR dimension 1 ..... 251.

Removable singularities for general linear partial differential operators $P=$ $\sum_{\beta \in \mathbb{N}^{m}} a_{\beta}(x) \partial_{x}^{\beta}$ on domains $\Omega \subset \mathbb{R}^{n}$ having order $m \geqslant 1$ and $\mathscr{C}^{\infty}$ coefficients have been studied by Harvey and Polking (1970) in a general setting. Assumptions of metrical thinness of singularities, in the sense of Minkowski content or of Hausdorff measure, insure automatic removability. For instance, relatively closed sets $C \subset \Omega$ whose $(n-m)$-dimensional Hausdorff measure is null are $\left(P, L_{\text {loc }}^{\infty}\right)$-removable. For structural reasons, these general results (valid whatever the structure of the operator) necessitate a control of growth when dealing with $L_{\text {loc }}^{1}$-removability. In addition, when $P$ is an embedded complex-tangential operator, this approach does not convey to the adequate results, because removable singularities for holomorphic or CR functions must take advantage of automatic extension to larger sets.

Since almost two decades, thanks to the impulse of Stout, removable singularities have attracted much attention in several complex variables. A natural question is whether the Hartogs-Bochner extension Theorem 1.9(V) holds when considering CR functions that are defined only in the complement $\partial \Omega \backslash K$ of some compact set $K \subset \partial \Omega$ of a connected smooth boundary $\partial \Omega \subset \mathbb{C}^{n}$. In complex dimension $n=2$, Stout showed that the answer is positive if and only if $K$ is convex with respect to the space of functions that are holomorphic in a neighborhood of $\bar{\Omega}$. In complex dimension $n \geqslant 3$, a complete cohomological characterization of different nature was obtained by Lupacciolu (1994).

In another direction, by means of the above-cited global continuity principle, Jöricke (1995) generalized Stout's theorem to weakly pseudoconvex domains. Recently, Jöricke and the second author were able to remove the pseudoconvexity assumption by applying purely geometrical constructions without integral formulas, controlling uniqueness of the extension (monodromy) by fine arguments.

Within the general framework of CR extension theory (exposed in Part V), the study of removable singularities has been endeavoured by Jöricke in the hypersurface case since 1988, and after by the two authors in arbitrary codimension since 1995. The notions of CR-, $\mathscr{W}$ - and $L^{\text {p }}$-removability, although different, may be shown to be essentially equivalent, thanks to technical deformation arguments. All the surveyed results hold in $L_{l o c}^{\mathrm{p}}$ with $1 \leqslant \mathrm{p} \leqslant \infty$, including $\mathrm{p}=1$ and without
any growth assumption near the singularity. On a generic globally minimal $\mathscr{C}^{2, \alpha}$ generic submanifold $M$ of $\mathbb{C}^{n}$, closed sets $C \subset M$ having vanishing ( $\operatorname{dim} M-2$ )dimensional Hausdorff measure are CR-, $\mathscr{W}$ - and $L^{\mathrm{p}}$-removable. As an application, CR meromorphic functions defined on an everywhere locally minimal $M$ do extend meromorphically to a wedgelike domain attached to $M$.

In conjunction with the Harvey-Lawson complex Plateau theorem, singularities $C$ that are a priori contained in a 2 -codimensional $\mathscr{C}^{2, \alpha}$ submanifold $N$ of a strongly pseudoconvex $\mathscr{C}^{2, \alpha}$ boundary $\partial \Omega \subset \mathbb{C}^{n}(n \geqslant 3)$ are shown by Jöricke to be not removable if and only if $N$ is a maximally complex cycle. The condition that $N$ be somewhere generic was shown by the two authors to be sufficient for its removability in arbitrary codimension.

Concerning more massive singularities, a compact subset $K$ of a onecodimensional submanifold $M^{1} \subset \partial \Omega \subset \mathbb{C}^{n}$ is CR-, $\mathscr{W}$ - and $L^{\mathrm{p}}$-removable provided the CR dimension of $\partial \Omega$ is $\geqslant 2$ (viz. $n \geqslant 3$ ) and provided $K$ does not contain any CR orbit of $M^{1}$ (Jöricke, 1999). The second author generalized this theorem to higher codimension, assuming that $M$ is globally minimal of CR dimension $m \geqslant 2$. The main geometric argument (called sweeping out by wedges) being available only in CR dimension $m \geqslant 2$, the more delicate case of CR dimension $m=1$ is studied extensively in the research article [26], placed in direct continuation to this survey.

## §1. REMOVABLE SINGULARITIES FOR LINEAR PARTIAL DIFFERENTIAL OPERATORS

1.1. Hausdorff measure. Let $M$ be a $\mathscr{C}^{1}$ abstract manifold of dimension $n \geqslant 1$ equipped with some Riemannian metric. For $\ell \in \mathbb{R}$ with $0 \leqslant \ell \leqslant n$, we remind ([Ch1989]) the definition of the notion of $\ell$-dimensional Hausdorff measure $\mathrm{H}^{\ell}$ on $M$, that generalizes the notion of integer dimension of submanifolds.

If $C \subset M$ is an arbitrary subset and if $\delta>0$ is small, we define
$\mathrm{H}_{\delta}^{\ell}(C)=\inf \left\{\sum_{j=1}^{\infty} r_{j}^{\ell}: C\right.$ is covered by geodesic balls $B_{j}$ of radius $\left.r_{j} \leqslant \delta\right\}$. Clearly, $\mathrm{H}_{\delta}^{\ell}(C) \leqslant \mathrm{H}_{\delta^{\prime}}^{\ell}(C)$, for $\delta^{\prime} \leqslant \delta$, so the limit $\mathrm{H}^{\ell}(C)=\lim _{\delta \rightarrow 0^{+}} \mathrm{H}_{\delta}^{\ell}(C)$ exists in $[0, \infty]$. This limit is called the $\ell$-dimensional Hausdorff measure of $C$. The value of $\mathrm{H}^{\ell}(C)$ depends on the choice of a metric, but the two properties $\mathrm{H}^{\ell}(C)=0$ and $\mathrm{H}^{\ell}(C)=\infty$ are independent. The most significant property is that there exists a critical exponent $\ell_{C} \geqslant 0$, called the Hausdorff dimension of $C$, such that $\mathrm{H}^{\ell}(C)=\infty$ for all $\ell<\ell_{C}$ and such that $\mathrm{H}^{\ell}(C)=0$ for all $\ell>\ell_{C}$. Then the value $\mathrm{H}^{\ell_{C}}(C)$ may be arbitrary in $[0, \infty]$.

Proposition 1.2. ([Fe1969, Ch1989]) The following properties hold true:
(1) $\mathrm{H}^{0}(C)=\operatorname{Card}(C)$;
(2) $\mathrm{H}^{n}(C)$ coincides with the outer Lebesgue measure of $C \subset M$;
(3) $a \mathscr{C}^{1}$ submanifold $N \subset M$ has Hausdorff dimension $\ell_{N}=\operatorname{dim} N$;
(4) if $\mathrm{H}^{n-1}(C)=0$, then $M \backslash C$ is locally connected;
(5) if $f: M \rightarrow N$ is a $\mathscr{C}^{1}$ map and if $C \subset M$ satisfies $\mathrm{H}^{\ell}(C)=0$ for some $\ell \geqslant \operatorname{dim} N$, then for almost every $q \in N$, it holds that $\mathrm{H}^{\ell-\operatorname{dim}^{N}}\left(C \cap f^{-1}(q)\right)=0$.
(6) $\mathrm{H}^{\ell}(C)=0$ if and only if $\mathrm{H}^{\ell}(K)=0$ for each compact set $K \subset C$.
1.3. Metrically thin singularities of linear partial differential operators. Let $\Omega$ be a domain in $\mathbb{R}^{n}$, where $n \geqslant 1$. We shall denote the Lebesgue measure by $\mathrm{H}^{n}$. Consider a class of $\mathscr{F}(\Omega)$ of distributions defined on $\Omega$, for instance $L_{\mathrm{loc}}^{\mathrm{p}}(\Omega), \mathscr{C}^{\kappa, \alpha}(\Omega)(\kappa \in \mathbb{N}, 0 \leqslant \alpha \leqslant 1)$ or $\mathscr{C}^{\infty}(\Omega)$. Consider a linear partial differential operator

$$
P=P\left(x, \partial_{x}\right)=\sum_{\beta \in \mathbb{N}^{n},|\beta| \leqslant m} a_{\beta}(x) \partial_{x}^{\beta}
$$

of order $m \geqslant 1$, defined in $\Omega$ and having $\mathscr{C}^{\infty}$ coefficients $a_{\beta}(x)$.
Definition 1.4. A relatively closed subset $C$ of $\Omega$ is called ( $P, \mathscr{F}$ )removable if every $f \in \mathscr{F}(\Omega)$ satisfying $P f=0$ in $\Omega \backslash C$ does satisfy $P f=0$ in all of $\Omega$, in the sense of distributions.

For instance, according to the classical Riemann removability theorem, discrete subsets $\left\{p_{k}\right\}_{k \in \mathbb{N}}$ of a domain $\Omega$ in $\mathbb{C}$ are $\left(\bar{\partial}, L^{\infty}\right)$-removable. In fact, since every distribution solution of $\bar{\partial}$ is holomorphic (hypoellipticity), functions extend to be true holomorphic functions in a neighborhood of each $p_{k}$. The Riemann removability theorem also holds under the weaker assumption that $f \in \mathscr{O}\left(\Omega \backslash\left\{p_{k}\right\}_{k \in \mathbb{N}}\right)$ satisfies $f\left(z-p_{k}\right)=\mathrm{o}\left(\left|z-p_{k}\right|^{-1}\right)$ as $z$ approaches $p_{k}$.

In several complex variables, the classical Riemann removability theorem may be stated as follows.

Theorem 1.5. ([Ch1989]) Let $\Sigma$ be a complex analytic subset of $\Omega$. Holomorphic functions in $\Omega \backslash \Sigma$ extend uniquely through $\Sigma$ either if $\operatorname{dim}_{\mathbb{C}} \Sigma \leqslant$ $n-2$ or if $\operatorname{dim}_{\mathbb{C}} \Sigma=n-1$ and they belong to $L_{\text {loc }}^{\infty}(\Omega)$.

The second case also holds true for functions that belong to $L_{l o c}^{2}(\Omega)$. The proofs are elementary and short: in one or several complex variables, everything comes down to observing that $\frac{1}{z}$ is a true $\mathrm{O}\left(\frac{1}{|z|}\right)$ near $z=0$ and does not belong to $L_{l o c}^{2}$.

These preliminary statements are superseded by more general removability theorems, exposed in [HP1970], that we shall now restitute. Some of the (elementary) proofs will be surveyed to give the flavour of the arguments.

In 1956, S. Bochner ([Bo1956, HP1970]) established remarkable removability theorems, valid for general linear differential operators $P$, in which the metrical conditions on the size of the singularity $C$ depend only on the order $m$ of $P$. Some preliminary material is needed.
Lemma 1.6. Let $K \subset \mathbb{R}^{n}$ be a compact set. For every $\varepsilon>0$, there exists a function $\varphi_{\varepsilon} \in \mathscr{C}_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ with $\varphi_{\varepsilon} \equiv 1$ in a neighborhood of $K$ and with $\operatorname{supp} \varphi_{\varepsilon} \subset K_{\varepsilon}$ such that $\left|\partial_{x}^{\beta} \varphi_{\varepsilon}(x)\right| \leqslant C_{\beta} \varepsilon^{-|\beta|}$ for all $x \in \mathbb{R}^{n}$ and all $\beta \in$ $\mathbb{N}^{n}$.

Proof. Denote by $\mathbf{1}_{B}(\cdot)$ the characteristic function of a set $B \subset \mathbb{R}^{n}$. It suffices to define the (rescaled) convolution integral $\varphi_{\varepsilon}(x):=\varepsilon^{-n} \int_{\mathbb{R}^{n}} \mathbf{1}_{K_{\varepsilon / 2}}(y) \psi((x-y) / \varepsilon) d y$, where $\psi \in \mathscr{C}_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ has support contained in $\{|x| \leqslant 1 / 3\}$ and satisfies $\int \psi(y) d y=1$.

It may happen that $C$ is not $(P, \mathscr{F})$-removable, whereas $C$ is removable for some individual function $f \in \mathscr{F}(\Omega)$ satisfying certain supplementary conditions. In this case, we shall say that $C$ is an illusory singularity of $f$.
Theorem 1.7. ([Bo1956, HP1970]) Let $f \in L_{l o c}^{1}(\Omega)$. If, for each compact set $K \subset C$, we have

$$
\liminf _{\varepsilon \rightarrow 0^{+}}\left[\varepsilon^{-m}\left\|f \mathbf{1}_{K_{\varepsilon}}\right\|_{L^{1}}\right]=0
$$

then $C$ is an illusory singularity of $f$.
Whenever the integral is meaningful, for instance if $f \in L_{l o c}^{1}(\Omega)$ and $\varphi \in \mathscr{C}_{c}^{\infty}(\Omega)$, we define $(f, \varphi):=\int_{\Omega} f \varphi$, where the integral is computed with respect to the Lebesgue measure. The formal adjoint of $P$, denoted by ${ }^{t} P$, satisfies the relations $(P \varphi, \psi)=\left(\varphi,{ }^{t} P \psi\right)$ for all $\varphi, \psi \in \mathscr{C}_{c}^{\infty}(\Omega)$, and these relations define it uniquely as

$$
{ }^{t} P(\varphi):=\sum_{|\beta| \leqslant m}(-1)^{|\beta|} \partial_{x}^{\beta}\left(a_{\beta} \varphi\right) .
$$

Proof of Theorem 1.7. Let $K:=(\operatorname{supp} \varphi) \cap C$ and let $\varphi_{\varepsilon}$ be the family of functions constructed in Lemma 1.6. Since $\operatorname{supp} P f \subset C$, we have $(P f, \varphi)=\left(P f, \varphi_{\varepsilon} \varphi\right)=\left(f,{ }^{t} P\left(\varphi_{\varepsilon} \varphi\right)\right)$. Lemma 1.6 entails that $\left\|^{t} P\left(\varphi_{\varepsilon} \varphi\right)\right\|_{L^{\infty}} \leqslant C \varepsilon^{-m}$, for some quantity $C>0$ that is independent of $\varepsilon$. We deduce that $|(P f, \varphi)| \leqslant C \varepsilon^{-m}\left\|f \mathbf{1}_{K_{\varepsilon}}\right\|_{L^{1}}$ for all $\varepsilon>0$. Thanks to the main assumption, this implies that $(P f, \varphi)=0$.

If $p \in \mathbb{R}$ with $1 \leqslant \mathrm{p} \leqslant \infty$ is the exponent of an $L^{\mathrm{p}}$-space, we denote by $\mathrm{p}^{\prime}:=\frac{\mathrm{p}}{\mathrm{p}-1} \in[1, \infty]$ the conjugate exponent, also defined by the relation $1=\frac{1}{\mathrm{p}}+\frac{1}{\mathrm{p}^{\prime}}$. By Hölder's inequality, we have $\left\|f \mathbf{1}_{K_{\varepsilon}}\right\|_{L^{1}} \leqslant$ $\left(\mathrm{H}^{n}\left(K_{\varepsilon}\right)\right)^{1 / \mathrm{p}^{\prime}}\left\|f \mathbf{1}_{K_{\varepsilon}}\right\|_{L^{\mathrm{p}}}$.

Corollary 1.8. Let $f \in L_{l o c}^{\mathrm{p}}(\Omega)$, where $1 \leqslant \mathrm{p} \leqslant \infty$. If, for each compact set $K \subset C$,

$$
\liminf _{\varepsilon \rightarrow 0^{+}}\left[\left(\varepsilon^{-m \mathrm{p}^{\prime}} \mathbf{H}^{n}\left(K_{\varepsilon}\right)\right)^{1 / \mathrm{p}^{\prime}}\left\|f \mathbf{1}_{K_{\varepsilon}}\right\|_{L^{\mathrm{p}}}\right]=0
$$

then $C$ is an illusory singularity of $f$.
The next theorem translates Corollary 1.9 in terms of Hausdorff measures, a finer concept than the Minkowski content.

Theorem 1.9. ([HP1970]) (i) Let $1<\mathrm{p}<\infty$ and assume that $n-m \mathrm{p}^{\prime} \geqslant 0$. If $\mathrm{H}^{n-m \mathrm{p}^{\prime}}(K)<\infty$ for every compact set $K \subset C$, then $C$ is $\left(P, L_{l o c}^{\mathrm{p}}\right)$ removable.
(ii) Let $\mathrm{p}=\infty$ and assume that $n-m \geqslant 0$. If $\mathrm{H}^{n-m}(C)=0$, then $C$ is ( $P, L_{\text {loc }}^{\infty}$ )-removable.
(iii) Let $\mathrm{p}=\infty$ and assume that $n-m \geqslant 0$. If, $\mathrm{H}^{n-m}(K)<\infty$ for each compact set $K \subset C$, then Pf is a measure supported on $C$, for every $f \in L_{\text {loc }}^{\infty}$ satisfying $P f=0$ on $\Omega \backslash C$.

An application of (ii) to $P=\bar{\partial}$ in one or several complex variables yields the Riemann removability Theorem 2.31 below.

Proof. We survey only the proof of (i). Let $\varphi \in \mathscr{C}_{c}^{\infty}(\Omega)$ and set $K:=$ $C \cap \operatorname{supp} \varphi$.

Lemma 1.10. ([HP1970]) Let $K \subset \mathbb{R}^{n}$ be a compact set. Let $\mathrm{p}^{\prime}$ with $1 \leqslant$ $\mathrm{p}^{\prime}<\infty$ and assume $n-m \mathrm{p}^{\prime} \geqslant 0$. For every $\varepsilon>0$, there exists $\varphi_{\varepsilon} \in$ $\mathscr{C}_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ with $\varphi_{\varepsilon} \equiv 1$ in a neighborhood of $K$ and with $\operatorname{supp} \varphi_{\varepsilon} \subset K_{\varepsilon}$ such that for all $\beta \in \mathbb{N}^{n}$ with $|\beta| \leqslant m$, we have

$$
\left\|\partial_{x}^{\beta} \varphi_{\varepsilon}\right\|_{L^{p^{\prime}}} \leqslant C \varepsilon^{m-|\beta|}\left(\mathrm{H}^{n-m \mathrm{p}^{\prime}}(K)+\varepsilon\right)^{1 / \mathrm{p}^{\prime}}
$$

where $C>0$ is independent of $\varepsilon$.
With such cut-off functions $\varphi_{\varepsilon}$, since supp $P f \subset C$, we have $(P f, \varphi)=$ $\left(P f, \varphi_{\varepsilon} \varphi\right)=\left(f,{ }^{t} P\left(\varphi_{\varepsilon} \varphi\right)\right)$. By Hölder's inequality and the preceding lemma:
$|(P f, \varphi)| \leqslant\left\|f \mathbf{1}_{K_{\varepsilon}}\right\|_{L^{p}}\left\|^{t} P\left(\varphi_{\varepsilon} \varphi\right)\right\|_{L^{p^{\prime}}} \leqslant C\left\|f \mathbf{1}_{K_{\varepsilon}}\right\|_{L^{p}}\left(\mathrm{H}^{n-m p^{\prime}}(K)+\varepsilon\right)^{1 / \mathrm{p}^{\prime}}$.
The theorem follows from

$$
\lim _{\varepsilon \rightarrow 0^{+}}\left\|f \mathbf{1}_{K_{\varepsilon}}\right\|_{L^{\mathrm{p}}}=0
$$

since $\mathrm{H}^{n}\left(K_{\varepsilon}\right) \rightarrow 0\left(\right.$ remind $\left.\mathrm{H}^{n-m \mathrm{p}^{\prime}}(K)<\infty\right)$.
It seems impossible to get $L^{1}$ removability without an assumption of growth. At the opposite, in a CR context, the techniques introduced
in [Jö1999b, MP1999] that are developed in Section 5 and in [26] will exhibit $L^{1}$-removability of certain closed subsets of generic submanifolds with only metrico-geometric assumptions.

## §2. REMOVABLE SINGULARITIES FOR HOLOMORPHIC FUNCTIONS OF ONE OR SEVERAL COMPLEX VARIABLES

2.1. Painlevé problem, zero length and analytic capacity. The classical Painlevé problem ([Pa1888, Ah1947]) is to find metric or geometric characterizations of compact sets $K \subset \mathbb{C}$ that are $\left(\bar{\partial}, L^{\infty}\right)$-removable, i.e. such that every $f \in \mathscr{O}(\mathbb{C} \backslash K) \cap L^{\infty}(\mathbb{C} \backslash K)$ extends holomorphically through $K$.

Theorem 1.9(ii) says that $\mathrm{H}^{1}(K)=0$ suffices. It is also known ([Ma1984, Pa2005]) that if $\mathrm{H}^{1+\varepsilon}(K)>0$ for some $\varepsilon>0$, then $K$ has positive analytic capacity (definition below) and is never ( $\bar{\partial}, L^{\infty}$ )-removable. Furthermore, Garnett ([Gar1970]) constructed a self-similar Cantor compact set $K \subset \mathbb{C}$ with $0<\mathrm{H}^{1}(K)<+\infty$ which is $\left(\bar{\partial}, L^{\infty}\right)$-removable. Consequently, Hausdorff measure is not fine enough.

Under a geometric tameness assumption a converse to the sufficiency of $H^{1}(K)=0$ holds and is usually called the solution to Denjoy's conjecture.

Theorem 2.2. ([Cal1977, CMM1982]) A compact set $K \subset \mathbb{C}$ that is a priori contained in a Lipschitz curve is $\left(\bar{\partial}, L^{\infty}\right)$-removable if and only if it has zero 1-dimensional Hausdorff measure.

Classically, this statement is an application of the celebrated result of Calderón, Coifman, McIntosh and Meyer about the $L^{2}$-boundedness of the Cauchy integral on Lipschitz curves. Let us survey one of the simplified proofs ([MV1995]) which involves Menger curvature, a concept useful in a recent answer to Painlevé's problem obtained in [To2003].

Let $\Gamma:=\left\{(x, y) \in \mathbb{R}^{2}: y=\varphi(x)\right\}$ be a (global) Lipschitz graph; here $\varphi \in \mathscr{C}^{0,1}$ is locally absolutely continuous and $\varphi^{\prime}$ exists almost everywhere (a.e.) with $\left\|\varphi^{\prime}\right\|_{L^{\infty}}<+\infty$.

Theorem 2.3. ([Cal1977, CMM1982, MV1995]) If $f \in L^{2}(\Gamma)$, the Cauchy principal value integral

$$
\mathrm{C}^{0} f(z):=\lim _{\varepsilon \rightarrow 0} \frac{1}{2 \pi i} \int_{|\zeta-z|>\varepsilon} \frac{f(\zeta)}{\zeta-z} d \zeta
$$

exists for almost every $z \in \Gamma$ and defines a function $\mathrm{C}^{0} f(z)$ on $\Gamma$, the Cauchy transform of $f$, which belongs to $L^{2}(\Gamma)$ and satisfies in addition

$$
\left\|\mathrm{C}^{0} f\right\|_{L^{2}(\Gamma)} \leqslant C_{1}\|f\|_{L^{2}(\Gamma)}
$$

for some positive constant $C_{1}=C_{1}\left(\left\|\varphi^{\prime}\right\|_{L^{\infty}}\right)$.

Parametrizing $\Gamma$ by $\zeta(t)=t+i \varphi(t)$, dropping the innocuous factor $1+$ $i \varphi^{\prime}(t)$ and setting $z:=x+i \varphi(x)$, one has to estimate the $L^{2}$-norm of the truncated integral

$$
\mathrm{C}_{\varepsilon}^{\prime}(f)(x):=\int_{|t-x|>\varepsilon} \frac{f(t)}{\zeta(t)-\zeta(x)} d t,
$$

with a constant independent of $\varepsilon$. Even more, interpolation arguments reduce the task to a single estimate of the form

$$
\int_{\mathbb{R}}\left|C_{\varepsilon}^{\prime}\left(\chi_{I}\right)\right|^{2} \leqslant C_{1}|I|,
$$

where $C_{1}=C_{1}\left(\left\|\varphi^{\prime}\right\|_{L^{\infty}}\right)$ and where $\chi_{I}$ is the characteristic function of an interval $I \subset \mathbb{R}$ of length $|I|$. Following [MV1995], a symmetrization of the (implicitely triple) integral $\int_{I} \mathrm{C}_{\varepsilon}^{\prime}\left(\chi_{I}\right) \overline{\bar{C}_{\varepsilon}^{\prime}\left(\chi_{I}\right)}$ provides

$$
6 \int_{I}\left|\mathrm{C}_{\varepsilon}^{\prime}\left(\chi_{I}\right)\right|^{2}=\iiint_{S_{\varepsilon}}\left(\sum_{\sigma \in \mathfrak{G}_{3}} \frac{1}{\zeta\left(x_{\sigma(2)}\right)-\zeta\left(x_{\sigma(1)}\right)} \frac{1}{\overline{\zeta\left(x_{\sigma(3)}\right)-\zeta\left(x_{\sigma(1)}\right)}}\right)
$$

where $S_{\varepsilon}:=\left\{(x, y, t) \in I^{3}:|y-x|>\varepsilon,|t-x|>\varepsilon,|t-y|>\varepsilon\right\}$ and where $\mathfrak{S}_{3}$ is the permutation group of $\{1,2,3\}$.

Then a "magic" ([Po2005]) formula enters the scene:

$$
\left(\frac{4 S\left(z_{1}, z_{2}, z_{3}\right)}{\left|z_{1}-z_{2}\right|\left|z_{1}-z_{3}\right|\left|z_{2}-z_{3}\right|}\right)^{2}=\sum_{\sigma \in \mathfrak{S}_{3}} \frac{1}{\zeta\left(x_{\sigma(2)}\right)-\zeta\left(x_{\sigma(1)}\right)} \frac{1}{\zeta\left(x_{\sigma(3)}\right)-\zeta\left(x_{\sigma(1)}\right)},
$$

where $S\left(z_{1}, z_{2}, z_{3}\right)$ denotes the enclosed area; the left hand side measures the "flatness" of the triangle. This crucial formula enables one to link rectifiability properties to the Cauchy kernel.
Definition 2.4. The Menger curvature of the triple $\left\{z_{1}, z_{2}, z_{3}\right\}$ is the square root of the above

$$
c\left(z_{1}, z_{2}, z_{3}\right):=\frac{4 S\left(z_{1}, z_{2}, z_{3}\right)}{\left|z_{1}-z_{2}\right|\left|z_{1}-z_{3}\right|\left|z_{2}-z_{3}\right|}
$$

one sets $c:=0$ if the points are aligned. One also verifies that $c\left(z_{1}, z_{2}, z_{3}\right)=$ $1 / R\left(z_{1}, z_{2}, z_{3}\right)$, where $R$ is the radius of the circumbscribed circle.

Thanks to the nice formula and to the basic inequality

$$
c(\zeta(x), \zeta(y), \zeta(t)) \leqslant 2\left|\frac{\frac{\varphi(y)-\varphi(x)}{y-x}-\frac{\varphi(t)-\varphi(x)}{t-x}}{|t-y|}\right|
$$

the previous symmetric Cauchy triple integral is transformed to an integral involving geometric Lipschitz properties of $\Gamma$. After some computations ([MV1995]), one gets $\int_{I}\left|\mathrm{C}_{\varepsilon}^{\prime}\left(\chi_{I}\right)\right|^{2} \leqslant C_{1}\left(\left\|\varphi^{\prime}\right\|_{L^{\infty}}\right) \cdot|I|$.

Menger curvature also appears in a recent result, considered to be an answer to Painlevé's problem.

Theorem 2.5. ([To2003, Pa2005]) A compact set $K \subset \mathbb{C}$ is not removable for $\mathscr{O}(\mathbb{C} \backslash K) \cap L^{\infty}(\mathbb{C} \backslash K)$ if and only if there exists a nonzero positive Radon measure $\mu$ with $\operatorname{supp} \mu \subset K$ such that

- there exists $C_{1}>0$ with $\mu(\Delta(z, \rho)) \leqslant C_{1} \rho$ for every $z \in \mathbb{C}$ and $\rho>0$;
- $\iiint[c(x, y, z)]^{2} d \mu(x) d \mu(y) d \mu(z)<+\infty$.

The first condition concerns the size of $K$; the second one is of quantitative-geometric nature.

We conclude by mentioning a classical functional characterization due to Ahlfors, usually considered to be only a reformulation of Painlevé's problem, but which has already found generalizations in locally integrable structures ( $\$ 2.16$ below). The analytic capacity of a compact set $K \subset \mathbb{C}$ is ${ }^{21}$

$$
\operatorname{an}-\operatorname{cap}(K):=\sup \left\{\left|f^{\prime}(\infty)\right|: f \in H^{\infty}(\mathbb{C} \backslash K),\|f\|_{L^{\infty}} \leqslant 1\right\}
$$

where $H^{\infty}(\mathbb{C} \backslash K)$ denotes the space of bounded holomorphic functions defined in $\mathbb{C} \backslash K$ (or defined in the complement of $K$ in the Riemann sphere $\mathbb{C} \cup\{\infty\}$, because $\{\infty\}$ is removable).
Theorem 2.6. ([Ah1947, Ma1984, HT1997]) A compact set $K \subset \mathbb{C}$ is removable for $\mathscr{O}(\mathbb{C} \backslash K) \cap L^{\infty}(\mathbb{C} \backslash K)$ if and only if an-cap $(K)=0$.
2.7. Radó-type theorems. A classical theorem due to Radó ([Ra1924, Stu1968, RS1989, Ch1994]) asserts that a continuous function $f$ defined in a domain $\Omega \subset \mathbb{C}$ that is holomorphic outside its zero-set $f^{-1}(0)$ is in fact holomorphic everywhere. By a separate holomorphicity argument, this statement extends directly to several complex variables. In [Stu 1993], it is shown that $f^{-1}(0)$ may be replaced by $f^{-1}(E)$, where $E \subset \mathbb{C}$ is compact and has null analytic capacity. In [RS1989], it is shown that a continuous function defined in a strongly pseudoconvex $\mathscr{C}^{2}$ hypersurface $M \subset \mathbb{C}^{n}$ $(n \geqslant 2)$ that is CR outside its zero-set is CR everywhere; a thin subset of weakly pseudoconvex points is allowed, but the case of general hypersurfaces is not covered. In [Al1993], it is shown that closed sets $f^{-1}(E)$ are removable in the same situation, wehere $E \subset \mathbb{C}$ is a closed polar set, viz.

[^20]$E \subset\{u=-\infty\}$ for some subharmonic function $u \not \equiv-\infty$. Chirka strengthens these results in the following theorem, where no assumption is made on the geometry of the hypersurface.

Remind ([Ch1989, 7]) that $E \subset \mathbb{C}^{m}$ is called complete pluripolar if $E=$ $\{\varphi=-\infty\}$ for some plurisubharmonic function $\varphi \not \equiv-\infty$ on $\mathbb{C}^{m}$.
Theorem 2.8. ([Ch1994]) Let $M \subset \mathbb{C}^{n}(n \geqslant 2)$ be hypersurface that is a local Lipschitz $\left(\mathscr{C}^{0,1}\right)$ graph at every point, let $C$ be a closed subset of $M$ and let $f: M \backslash C \rightarrow \mathbb{C}^{m} \backslash E$ be a continuous mapping satisfying $\|f\|_{\mathscr{C}^{0}(M \backslash C)}<$ $\infty$ such that the set of limit values of $f$ from $M \backslash C$ up to $C$ is contained in a closed complete pluripolar set $E \subset \mathbb{C}^{m}(m \geqslant 1)$. Then the trivial extension $\tilde{f}$ of $f$ to $C$ defined by $\widetilde{f}:=0$ on $C$ is a CR mapping of class $L^{\infty}$ on the whole of $M$.

In higher codimension, nothing is known.
Open question 2.9. Let $M \subset \mathbb{C}^{n}(n \geqslant 3)$ be a generic submanifold of codimension $d \geqslant 2$ and of CR dimension $m \geqslant 1$ that is at least $\mathscr{C}^{1}$. Let $f \in \mathscr{C}^{0}(M)$ that is CR outside its zero-set $f^{-1}(0)$. Is $f \mathrm{CR}$ everywhere?

Remind that condition (P) (Definition 3.5(III)) for a linear partial differential operator $P$ of principal type assures local solvability of the equations $P f=g$. Remind also that nowhere vanishing vector fields are of principal type.
Theorem 2.10. ([HT1993]) Let $\Omega \subset \mathbb{R}^{n}(n \geqslant 2)$ be a domain and let $L$ be a nowhere vanishing vector field on $\Omega$ having $\mathscr{C}^{\infty}$ complex-valued coefficients and satisfying condition $(\mathbf{P})$ of Nirenberg-Treves. If $f \in \mathscr{C}^{0}(\Omega)$ satisfies $L f=0$ in $\Omega \backslash f^{-1}(0)$ in the sense of distributions, then $f$ is a weak solution of $L f=0$ all over $\Omega$.

### 2.11. Capacity and partial differential operators having constant coef-

 ficients. The preceding results admit partial generalizations to vector field systems. Let $\Omega \subset \mathbb{R}^{n}$ be an open set and let $P=P\left(\partial_{x}\right)=\sum_{\beta \in \mathbb{N}^{n}} a_{\beta} \partial_{x}^{\beta}$ be a linear partial differential operator having constant coefficients $a_{\beta} \in \mathbb{C}$. By a theorem due to Malgrange, Ehrenpreis and Palamodov ([Hö1963]), such a $P$ always admits a fundamental solution, namely there exists a distribution $E \in \mathscr{D}^{\prime}\left(\mathbb{R}^{n}\right)$ such that $P\left(\partial_{x}\right) E=\delta_{0}$ is the Dirac measure at the origin.Let $\mathscr{F} \subset \mathscr{D}^{\prime}(\Omega)$ be a Banach space, e.g. $\mathscr{F}=L^{\mathrm{p}}(\Omega)$ with $1 \leqslant \mathrm{p} \leqslant \infty$, or $\mathscr{F}=L^{\infty}(\Omega) \cap \mathscr{C}^{0}(\Omega)$, or $\mathscr{F}=\mathscr{C}^{0, \alpha}(\Omega)$ with $0<\alpha \leqslant 1$.

Definition 2.12. For each relatively closed set $C \subset \Omega$, the $\mathscr{F}$-capacity of $C$ with respect to $P$ is

$$
\mathscr{F}-\operatorname{cap}_{P}(C, \partial \Omega):=\sup \left\{\left|\left(P f, \mathbf{1}_{\Omega}\right)\right|: f \in \mathscr{F},\|f\|_{\mathscr{F}} \leqslant 1, \operatorname{supp}(P f) \Subset C\right\} .
$$

If a closed set $C \subset \Omega$ is $(P, \mathscr{F})$-removable, by definition $P f=0$ everywhere, hence $\mathscr{F}-\operatorname{Cap}_{P}(C, \Omega)=0$. The following theorem establishes the converse for a wide class of differential operators having constant coefficients. For $\beta \in \mathbb{N}^{n}$, denote by $Q^{(\beta)}(x):=\partial_{x}^{\beta} Q(x)$ the $\beta$-th partial derivative of a polynomial $Q(x) \in \mathbb{R}[x]$.
Theorem 2.13. ([HP1972]) Assume that P possesses a fundamental solution $E \in \mathscr{D}^{\prime}\left(\mathbb{R}^{n}\right)$ such that $P^{(\beta)}\left(\partial_{x}\right) E$ is a regular Borel measure on $\mathbb{R}^{n}$ for every $\beta \in \mathbb{N}^{n}$. Let $\Omega \Subset \mathbb{R}^{n}$ be a bounded domain and let $C \subset \Omega$ be a relatively closed subset. Then

- $C$ is $\left(P, L_{l o c}^{\mathrm{p}}\right)$-removable, $1<\mathrm{p} \leqslant \infty$, if and only if $L^{\mathrm{p}}$ $\operatorname{cap}_{P}(C, \Omega)=0$;
- $C$ is $\left(P, L^{\infty} \mathscr{C}^{0}\right)$-removable if and only if $L^{\infty} \mathscr{C}^{0}-\operatorname{cap}_{P}(C, \Omega)=0$;
- $C$ is $\left(P, \mathscr{C}^{0, \alpha}\right)$-removable, $0<\alpha \leqslant 1$, if and only if $\mathscr{C}^{0, \alpha_{-}}$ $\operatorname{cap}_{P}(C, \Omega)=0$.

This hypothesis about $P$ is satisfied by elliptic, semi-elliptic, and parabolic operators and also by the wave operator in $\mathbb{R}^{2}$ ([HP1972]). The theorem (whose proof is rather short) also holds true if $P=P\left(x, \partial_{x}\right)$ has real analytic coefficients and admits a fundamental solution $E$ such that $P^{(\beta)} E$ is a regular Borel measure for every $\beta \in \mathbb{N}^{n}$. But it is void in $L^{1}$.

Theorem 2.14. ([HP1972]) There is a unique function, called a capacitary extremal, $f^{\text {cap }} \in L^{\mathrm{p}}(\Omega)$ with $\left\|f^{\text {cap }}\right\|_{L^{\mathrm{p}}} \leqslant 1$ and $P f^{\text {cap }}=0$ in $\Omega \backslash K$ such that $\left(P f^{\text {cap }}, \mathbf{1}_{\Omega}\right)=L^{\mathrm{p}}-\operatorname{cap}_{P}(K, \Omega)$.

We observe that the definition of $L^{\mathrm{p}}-\operatorname{cap}_{P}(K, \Omega)$ is inspired from Ahlfors' notion of analytic capacity and we mention that the capacitary extremal $f^{\text {cap }}$ is linked to the Riemann uniformization theorem.

Example 2.15. In fact, with $\Omega=\mathbb{C}$ and $P=\partial / \partial \bar{z}=: \bar{\partial}$, the $L^{\infty}$-capacity of a compact set $K \subset \mathbb{C}$ with respect to $\bar{\partial}$ may be shown to be equal, up to the constant $\pi$, to the analytic capacity of $K$, namely

$$
L^{\infty}-\operatorname{cap}_{\bar{\partial}}(K, \mathbb{C})=\pi \operatorname{an}-\operatorname{cap}(K)
$$

Indeed, letting $f \in L^{\infty}(\mathbb{C})$, assuming that $\bar{\partial} f$ is supported by $K$, choosing a big open disc $D \Subset \mathbb{C}$ containing $K$, integrating by parts (Riemann-Green) and performing the change of variables $w:=1 / z$, we may compute
$\left(\bar{\partial} f, \mathbf{1}_{\mathbb{C}}\right)=\left(\bar{\partial} f, \mathbf{1}_{D}\right)=\frac{1}{2 i} \iint_{D} \frac{\partial f}{\partial \bar{z}} d \bar{z} \wedge d z=\frac{1}{2 i} \int_{\partial D} f(z) d z=\pi f^{\prime}(\infty)$.
Remind (footnote) that in the definition of an-cap $(K)$ given in $\S 2.1$, one may assume that $f(\infty)=0$. If in addition the complement of $K$ in the Riemann sphere $\mathbb{C} \cup\{\infty\}$ is simply connected, the unique solution $f^{\text {cap }}$ of
$\left(\bar{\partial} f^{\text {cap }}, \mathbf{1}_{\mathbb{C}}\right)=L^{\infty}-\operatorname{cap}_{\bar{\partial}}(K, \mathbb{C})$ asserted by Theorem 2.14 , viz. the unique solution of the extremal problem
$\sup \left\{\left|f^{\prime}(\infty)\right|: f \in L^{\infty}(\mathbb{C}), \partial f / \partial \bar{z}=0\right.$ in $\mathbb{C} \backslash K, f(\infty)=0$ and $\left.\|f\|_{L^{\infty}} \leqslant 1\right\}$
is the (unique) Riemann uniformization map $f^{\text {cap }}:(\mathbb{C} \cup\{\infty\}) \backslash K \rightarrow \Delta$ satisfying $f^{\text {cap }}(\infty)=0$ and $\partial_{z} f^{\text {cap }}(\infty)>0$.
2.16. Removable singularities of locally solvable vector fields. Let

$$
\mathscr{S}\left(\mathbb{R}^{n}\right):=\left\{f \in \mathscr{C}^{\infty}\left(\mathbb{R}^{n}\right): \lim _{|x| \rightarrow \infty}\left|x^{\alpha} \partial_{x}^{\beta} f(x)\right|=0, \forall \alpha, \beta \in \mathbb{N}^{n}\right\}
$$

be the space of $\mathscr{C}^{\infty}$ functions defined in $\mathbb{R}^{n}$ and having tempered growth. As is known ([Hö1963]), the Fourier transform

$$
\mathrm{F} f(\xi):=\int_{\mathbb{R}^{n}} e^{-2 \pi i\langle x, \xi\rangle} f(x) d x, \quad f \in \mathscr{S}\left(\mathbb{R}^{n}\right)
$$

$\langle x, \xi\rangle:=\sum_{k=1}^{n} x_{k} \xi_{k}$, is an automorphism of $\mathscr{S}\left(\mathbb{R}^{n}\right)$ having as inverse

$$
\mathrm{F}^{-1} f(\xi):=\int_{\mathbb{R}^{n}} e^{2 \pi i\langle x, \xi\rangle} f(x) d x=\mathrm{F} f(-\xi)
$$

Equipping $\mathscr{S}\left(\mathbb{R}^{n}\right)$ with the countable family of semi-norms $p_{\alpha, \beta}(f):=$ $\sup _{x \in \mathbb{R}^{n}}\left|x^{\alpha} \partial_{x}^{\beta} f(x)\right|$, the space $\mathscr{S}^{\prime}\left(\mathbb{R}^{n}\right)$ of tempered distributions consists of linear functionals T on $\mathscr{S}\left(\mathbb{R}^{n}\right)$ that are continuous, viz. there exists $C>0$ and $\alpha, \beta \in \mathbb{N}^{n}$ such that $|\langle\mathrm{T}, f\rangle| \leqslant C p_{\alpha, \beta}(f)$ for every $f \in \mathscr{S}\left(\mathbb{R}^{n}\right)$.

For $\mathrm{p} \in \mathbb{R}$ with $1 \leqslant \mathrm{p} \leqslant \infty$ and for $\sigma \in \mathbb{R}$, we remind the definition of the Sobolev space

$$
L_{\sigma}^{\mathrm{p}}\left(\mathbb{R}^{n}\right):=\left\{\mathrm{T} \in \mathscr{S}^{\prime}\left(\mathbb{R}^{n}\right):\|\mathrm{T}\|_{L_{\sigma}^{\mathrm{p}}}:=\left\|\Lambda^{-\sigma} \mathrm{T}\right\|_{L^{\mathrm{p}}}<\infty\right\}
$$

where $\Lambda^{-\sigma} \mathrm{T}(x):=\mathrm{F}^{-1}\left[\left(1+|\xi|^{2}\right)^{-\sigma / 2} \mathrm{~F} \mathrm{~T}(\xi)\right](x)$. For $\sigma=\kappa \in \mathbb{N}$ and p in the range $1<\mathrm{p}<\infty$, the space $L_{\kappa}^{\mathrm{p}}\left(\mathbb{R}^{n}\right)$ is exactly the subspace of functions $u \in L^{\mathrm{p}}\left(\mathbb{R}^{n}\right)$ whose partial derivatives of order $\leqslant \kappa$ (in the distributional sense) belong to $L^{\mathrm{p}}\left(\mathbb{R}^{n}\right)$. This space is equivalently normed by $\|u\|_{L_{k}^{\mathrm{p}}}:=$ $\sum_{|\beta| \leqslant \kappa}\left\|\partial_{x}^{\beta} u\right\|_{L^{p}}$.

Let $\Omega \subset \mathbb{R}^{n}$ be a domain and let $P=P\left(x, \partial_{x}\right)=\sum_{|\beta| \leqslant m} a_{\beta}(x) \partial_{x}^{\beta}$ be a linear partial differential operator of order $m \geqslant 1$ defined in $\Omega$ and having $\mathscr{C}^{\infty}$ coefficients $a_{\beta}(x)$.
Definition 2.17. We say that $P$ is locally solvable in $L^{\mathrm{p}}$ with one loss of derivative if every point $p \in \Omega$ has an open neighborhood $U_{p} \subset \Omega$ such that for every compactly supported $\mathrm{T} \in L_{\sigma}^{\mathrm{p}}\left(U_{p}\right)$, the equation $\mathrm{PS}=\mathrm{T}$ has a solution $\mathrm{S} \in L_{\sigma+m-1}^{\mathrm{p}}\left(U_{p}\right)$.

Theorem 2.18. ([BeFe1973, HP1996]) Let L be a nowhere vanishing vector field having $\mathscr{C}^{\infty}$ coefficients in a domain $\Omega \subset \mathbb{R}^{n}(n \geqslant 2)$ and assume that $L$ satisfies condition ( $\mathbf{P}$ ) of Nirenberg-Treves. Then for every $\mathrm{p} \in \mathbb{R}$ with $1<\mathrm{p}<\infty$, the operator $L$ is locally solvable in $L^{\mathrm{p}}$ with loss of one derivative.

Example 2.19. As discovered in [HT1996], local solvability fails to hold in $L^{\infty}$ for the (locally solvable) vector field $\frac{\partial}{\partial x}-\frac{i}{x^{2}} e^{-1 /|x|} \frac{\partial}{\partial y}$ satisfying $(\mathbf{P})$ on $\mathbb{R}^{2}$.

Removable singularities for vector fields in $L^{\mathrm{p}}$ have been studied in [HT1996, HT1997]. Because of the example, results in $L^{\mathrm{p}}$ with $1<\mathrm{p}<\infty$ differ from results in $L^{\infty}$.

Definition 2.20. A relatively closed set $C \subset \Omega$ of an open set $\Omega \subset \mathbb{R}^{n}$ is everywhere ( $P, L^{\mathrm{p}}$ )-removable if for every open subset $U \subset \Omega$ and for every $f \in L^{\mathrm{p}}(U)$ satisfying $P f=0$ in $U \backslash C$, then $f$ also satisfies $P f=0$ in all of $U$.

Theorem 2.21. ([HT1996, HT1997]) Let L be a nowhere vanishing vector field having $\mathscr{C}^{\infty}$ coefficients in an open subset $\Omega \subset \mathbb{R}^{n}(n \geqslant 2)$ and assume that $L$ satisfies condition $(\mathbf{P})$. Let $\mathrm{p} \in \mathbb{R}$ with $1<\mathrm{p}<\infty$. Then a relatively closed set $C \subset \Omega$ is everywhere $\left(L, L^{\mathrm{p}}\right)$-removable if and only if there is an open covering of $C$ by open sets $\Omega_{j}, j \in J$, such that

$$
L^{\mathrm{p}-\operatorname{cap}_{L}\left(C \cap \Omega_{j}, \Omega_{j}\right)=0, ~}
$$

for every $j \in J$.
In $L^{\infty}$, when trying to perform the proof of this theorem, local solvability of positive multiples of $L$ is technically needed. Observing that $P f=0$ is equivalent to $e^{\psi} P f=0$, the following notion appeared to be appropriate to deal with $\left(L, L^{\infty}\right)$-removability.
Definition 2.22. ([HT1996, HT1997]) The full $L^{\infty}$-capacity of a relatively closed set $C \subset \Omega$ with respect to $L$ is

$$
\text { full }-L^{\infty}-\operatorname{cap}_{L}(C, \Omega):=\sup _{\widetilde{L}}\left\{L^{\infty}-\operatorname{cap}_{\tilde{L}}(C, \Omega)\right\},
$$

where the supremum is taken over all vector fields $\widetilde{L}=e^{\psi} L$ with $\psi \in$ $\mathscr{C}^{\infty}(\Omega)$ satisfying $\sup _{\Omega}\left|\partial_{x}^{\alpha} \psi(x)\right| \leqslant 1$ for $|\alpha| \leqslant 1$.

By a fine analysis of the degeneracies of $L$ and of the structure of the Sussmann orbits of $\{\operatorname{Re} L, \operatorname{Im} L\}$, Hounie-Tavares were able to substantially generalize Ahlfors' characterization.

Theorem 2.23. ([HT1996, HT1997]) A relatively closed set $C \subset \Omega$ is everywhere $\left(L, L^{\infty}\right)$-removable if and only if there is an open covering of $C$ by open sets $\Omega_{j}, j \in J$, such that

$$
\text { full- } L^{\infty}-\operatorname{cap}_{L}\left(C \cap \Omega_{j}, \Omega_{j}\right)=0,
$$

for every $j \in J$.
On orbits of dimension one, $L$ behaves as a multiple of a real vector field (one-dimensional behavior); on orbits of dimension two, $L$ has the behavior of $\bar{\partial}$ on a Riemann surface $\Sigma \subset \mathscr{O}$, but on the complement $\mathscr{O} \backslash \Sigma$ which is a union of curves with different endpoints along which $\operatorname{Re} L$ and $\operatorname{Im} L$ are both tangent (degeneracy), $L$ behaves again as a multiple of a real vector field (one-dimensional behavior). As shown in [HT1996] (main Theorem 7.3 there) a relatively closed set $C \subset \Omega$ is everywhere removable if and only if $C$ does not disconnect almost every curve on which $L$ has one-dimensional behavior and furthermore, the intersection of $C$ with almost every (reduced) orbit of dimension two has zero analytic capacity for its natural holomorphic structure.

Open problem 2.24. Study removability of a $\mathscr{C}^{\infty}$ locally integrable involutive structure of rank $\lambda \geqslant 2$ in terms of analytic capacity.
2.25. Cartan-Thullen argument and a local continuity principle. The Behnke-Sommer Kontinuitätssatz, alias Continuity Principle, states informally as follows ([Sh1990]). Let $\left(\Sigma_{\nu}\right)_{\nu \in \mathbb{N}}$ be a sequence of complex manifolds with boundary $\partial \Sigma_{\nu}$ contained in a domain $\Omega$ of $\mathbb{C}^{n}$. If $\Sigma_{\nu}$ converges to a set $\Sigma_{\infty} \subset \bar{\Omega}$ and if $\partial \Sigma_{\infty}$ is contained in $\Omega$, then every holomorphic function $f \in \mathscr{O}(\Omega)$ extends holomorphically to a neighborhood of the set $\Sigma_{\infty}$ in $\mathbb{C}^{n}$. The geometries of the $\Sigma_{\nu}$ and of $\Sigma_{\infty}$ have to satisfy certain assumptions in order that the statement be correct; in addition, monodromy questions have to be considered carefully. For applications to removable singularities in [26], the rigorous Theorem 2.27 below is formulated, with the $\Sigma_{\nu}$ being embedded analytic discs.

We denote by $z=\left(z_{1}, \ldots, z_{n}\right)$ the complex coordinates on $\mathbb{C}^{n}$ and by $|z|=\max _{1 \leqslant i \leqslant n}\left|z_{i}\right|$ the polydisc norm. If $E \subset \mathbb{C}^{n}$ is an arbitrary subset, for $\rho>0$, we denote by

$$
\mathscr{V}_{\rho}(E):=\bigcup_{p \in E}\left\{z \in \mathbb{C}^{n}:|z-p|<\rho\right\}
$$

the union of all open polydiscs of radius $\rho$ centered at points of $E$.
Lemma 2.26. ([Me1997]) Let $\Omega$ be a nonempty domain of $\mathbb{C}^{n}$ and let $A$ : $\bar{\Delta} \rightarrow \Omega, A \in \mathscr{O}(\Delta) \cap \mathscr{C}^{1}(\bar{\Delta})$, be an analytic disc contained in $\Omega$ having the property that there exist two constants $c$ and $C$ with $0<c<C$ such that

$$
c\left|\zeta_{2}-\zeta_{1}\right|<\left|A\left(\zeta_{2}\right)-A\left(\zeta_{1}\right)\right|<C\left|\zeta_{2}-\zeta_{1}\right|,
$$

for all distinct points $\zeta_{1}, \zeta_{2} \in \bar{\Delta}$. Set

$$
\rho:=\inf \{|z-A(\zeta)|: z \in \partial \Omega, \zeta \in \partial \Delta\},
$$

namely $\rho$ is the polydisc distance between $A(\partial \Delta)$ and $\partial \Omega$, and set $\sigma:=$ $\rho c / 2 C$. Then for every holomorphic function $f \in \mathscr{O}(\Omega)$, there exists a holomorphic function $F \in \mathscr{O}\left(\mathscr{V}_{\sigma}(A(\bar{\Delta}))\right)$ such that $F=f$ on $\mathscr{V}_{\sigma}(A(\partial \Delta))$.


The inequalities involving $c$ and $C$ are satisfied for instance when $A$ is $\mathscr{C}^{1}$ embedding of $\bar{\Delta}$ into $\Omega$. Whereas $A(\bar{\Delta})$ is contained in $\Omega$, the neighborhood $\mathscr{V}_{\sigma}(A(\bar{\Delta}))$ is allowed to go beyond. We do not claim that the two functions $f \in \mathscr{O}(\Omega)$ and $F \in \mathscr{O}\left(\mathscr{V}_{\sigma}(A(\bar{\Delta}))\right)$ stick together as a holomorphic function globally defined in $\Omega \cup \mathscr{V}_{\sigma}(A(\bar{\Delta}))$. In fact, $\Omega \cap \mathscr{V}_{\sigma}(A(\bar{\Delta}))$ may have several connected components.


In the geometric situations we encounter in [Me1997, MP1999, MP2002, 26], after shrinking $\Omega$ somehow slightly to some subdomain $\Omega^{\prime}$, we shall be able to insure that the intersection $\Omega^{\prime} \cap \mathscr{V}_{\sigma}(A(\bar{\Delta}))$ is connected and that the union $\Omega^{\prime} \cup \mathscr{V}_{\sigma}(A(\bar{\Delta}))$ is still significantly "bigger" than $\Omega$.

Proof of Lemma 2.26. Let $f \in \mathscr{O}(\Omega)$. For $\zeta \in \bar{\Delta}$ arbitrary, we consider the locally converging Taylor series $\sum_{\alpha \in \mathbb{N}^{n}} f_{\alpha}(z-A(\zeta))^{\alpha}$ of $f$ at $A(\zeta)$. For $\rho^{\prime}$ with $0<\rho^{\prime}<\rho$ arbitrarily close to $\rho$, since $\mathscr{V}_{\rho^{\prime}}(A(\partial \Delta)) \Subset \Omega$, the quantity

$$
M_{\rho^{\prime}}(f):=\sup \left\{|f(z)|: z \in \mathscr{V}_{\rho^{\prime}}(A(\partial \Delta))\right\}<\infty,
$$

is finite (it may explode as $\rho^{\prime} \rightarrow \rho$ ). Thus, Cauchy's inequality on a polydisc of radius $\rho^{\prime}$ centered at an arbitrary point $A\left(e^{i \theta}\right)$ of $\partial \Delta$ yields

$$
\frac{1}{\alpha!}\left|\frac{\partial^{|\alpha|} f}{\partial z^{\alpha}}\left(A\left(e^{i \theta}\right)\right)\right| \leqslant \frac{M_{\rho^{\prime}}(f)}{\rho^{\prime \alpha}}
$$

uniformly for all $e^{i \theta} \in \partial \Delta$. Then the maximum principle applied to the function $\zeta \mapsto \frac{\partial^{|\alpha|} f}{\partial z^{\alpha}}(A(\zeta))$ holomorphic in $\Delta$ provides the crucial inequalities (Cartan-Thullen argument):

$$
\begin{aligned}
\left|f_{\alpha}\right|=\frac{1}{\alpha!}\left|\frac{\partial^{|\alpha|} f}{\partial z^{\alpha}}(A(\zeta))\right| & \leqslant \frac{1}{\alpha!} \sup _{e^{i \theta} \in \partial \Delta}\left|\frac{\partial^{|\alpha|} f}{\partial z^{\alpha}}\left(A\left(e^{i \theta}\right)\right)\right| \\
& \leqslant \frac{M_{\rho^{\prime}}(f)}{\rho^{\prime \alpha}}
\end{aligned}
$$

Consequently, the Taylor series of $f$ converges normally in the polydisc $\Delta_{\rho}^{n}(A(\zeta))$ of center $A(\zeta)$ and of radius $\rho$, this being true for every $A(\zeta) \in A(\bar{\Delta})$. These series define a collection of holomorphic functions $F_{A(\zeta), \rho} \in \mathscr{O}\left(\Delta_{\rho}^{n}(A(\zeta))\right)$ parametrized by $\zeta \in \bar{\Delta}$. We claim that the restrictions of all these functions to the smaller polydiscs $\Delta_{\sigma}^{n}(A(\zeta))$ stick together in a well defined holomorphic function $F \in \mathscr{O}\left(\mathscr{V}_{\sigma}(A(\bar{\Delta}))\right)$.

Indeed, assume that two distinct points $\zeta_{1}, \zeta_{2} \in \bar{\Delta}$ are such that the intersection of the two small polydiscs $\Delta_{\sigma}^{n}\left(A\left(\zeta_{1}\right)\right) \cap \Delta_{\sigma}^{n}\left(A\left(\zeta_{2}\right)\right)$ is nonempty, so $\left|A\left(\zeta_{2}\right)-A\left(\zeta_{1}\right)\right|<2 \sigma$. It follows that for every $\zeta$ belonging to the segment $\left[\zeta_{1}, \zeta_{2}\right]$, we have:

$$
\left|\zeta-\zeta_{1}\right| \leqslant\left|\zeta_{2}-\zeta_{1}\right|<\left|A\left(\zeta_{2}\right)-A\left(\zeta_{1}\right)\right| / c<2 \sigma / c
$$

whence

$$
\left|A(\zeta)-A\left(\zeta_{1}\right)\right|<C\left|\zeta-\zeta_{1}\right|<2 C \sigma / c=\rho .
$$

This means that the curved segment $A\left(\left[\zeta_{1}, \zeta_{2}\right]\right)$ is contained in the connected intersection of the two large polydiscs $\Delta_{\rho}^{n}\left(A\left(\zeta_{1}\right)\right) \cap \Delta_{\rho}^{n}\left(A\left(\zeta_{2}\right)\right)$. In a small neighborhood of $A\left(\zeta_{1}\right)$ and of $A\left(\zeta_{2}\right)$, the two holomorphic functions $F_{A\left(\zeta_{1}\right), \rho}$ and $F_{A\left(\zeta_{2}\right), \rho}$ coincide with $f$ by construction. Thanks to the principle of analytic continuation, it follows that they even coincide with $f$ in a thin connected neighborhood of the segment $A\left(\left[\zeta_{1}, \zeta_{2}\right]\right)$. Again thanks to the principle of analytic continuation, $F_{A\left(\zeta_{1}\right), \rho}$ and $F_{A\left(\zeta_{2}\right), \rho}$ coincide in the connected intersection $\Delta_{\rho}^{n}\left(A\left(\zeta_{1}\right)\right) \cap \Delta_{\rho}^{n}\left(A\left(\zeta_{2}\right)\right)$. It follows that they stick together to provide a well defined function $F_{A\left(\zeta_{1}\right), A\left(\zeta_{2}\right), \rho}$ that is holomorphic in $\Delta_{\rho}^{n}\left(A\left(\zeta_{1}\right)\right) \cup \Delta_{\rho}^{n}\left(A\left(\zeta_{2}\right)\right)$. In conclusion, the restriction $\left.F_{A\left(\zeta_{1}\right), A\left(\zeta_{2}\right), \rho}\right|_{\Delta_{\sigma}\left(A\left(\zeta_{1}\right)\right) \cup \Delta_{\sigma}\left(A\left(\zeta_{2}\right)\right)}$ is holomorphic in the union of the two small polydiscs $\Delta_{\sigma}\left(A\left(\zeta_{1}\right)\right) \cup \Delta_{\sigma}\left(A\left(\zeta_{2}\right)\right)$, whenever the intersection
$\Delta_{\sigma}\left(A\left(\zeta_{1}\right)\right) \cap \Delta_{\sigma}\left(A\left(\zeta_{2}\right)\right)$ is nonempty. This proves that all the restrictions $\left.F_{A(\zeta), \rho}\right|_{\Delta_{\sigma}^{n}(A(\zeta))}$ stick together in a well defined holomorphic function $F \in \mathscr{O}\left(\mathscr{V}_{\sigma}(A(\bar{\Delta}))\right)$.

In the next theorem (a local continuity principle often used in [Me1997, MP1999, MP2002, 26]), $A_{1}(\bar{\Delta}) \subset \Omega$, but contrary to Lemma 2.26, $A_{s}(\bar{\Delta})$ may well be not contained in $\Omega$ for $s<1$; nevertheless, the boundaries $A_{s}(\partial \Delta)$ must always stay in $\Omega$.

Theorem 2.27. ([Me 1997]) Let $\Omega$ be a nonempty domain in $\mathbb{C}^{n}$ and let $A_{s}$ : $\bar{\Delta} \rightarrow \mathbb{C}^{n}, A_{s} \in \mathscr{O}(\Delta) \cap \mathscr{C}^{1}(\bar{\Delta})$, be a one-parameter family of analytic discs, where $s \in[0,1]$. Assume that there exist two constants $c_{s}$ and $C_{s}$ with $0<c_{s}<C_{s}$ such that

$$
c_{s}\left|\zeta_{2}-\zeta_{1}\right|<\left|A\left(\zeta_{2}\right)-A\left(\zeta_{1}\right)\right|<C_{s}\left|\zeta_{2}-\zeta_{1}\right|,
$$

for all distinct points $\zeta_{1}, \zeta_{2} \in \bar{\Delta}$. Assume that $A_{1}(\bar{\Delta}) \subset \Omega$, set

$$
\rho_{s}:=\inf \left\{\left|z-A_{s}(\zeta)\right|: z \in \partial \Omega, \zeta \in \partial \Delta\right\},
$$

namely $\rho_{s}$ is the polydisc distance between $A_{s}(\partial \Delta)$ and $\partial \Omega$, and set $\sigma_{s}:=$ $\rho_{s} c_{s} / 2 C_{s}$. Then for every holomorphic functions $f \in \mathscr{O}(\Omega)$, and for all $s \in[0,1]$, there exist holomorphic functions $F_{s} \in \mathscr{O}\left(\mathscr{V}_{\sigma_{s}}\left(A_{s}(\bar{\Delta})\right)\right)$ such that $F_{s}=f$ in $\mathscr{V}_{\sigma_{s}}\left(A_{s}(\partial \Delta)\right)$.

Proof. Let $\mathscr{I} \subset[0,1]$ be the connected set of real $s_{0}$ such that the statement is true for all $s$ with $s_{0} \leqslant s \leqslant 1$. By Lemma 2.26, we already know that $1 \in \mathscr{I}$. We want to prove that $\mathscr{I}=[0,1]$. It suffices to prove that $\mathscr{I}$ is both open and closed.

The fact that $\mathscr{I}$ is closed follows by "abstract nonsense". We claim that $\mathscr{I}$ is also open. Indeed, let $s_{0} \in \mathscr{I}$ and let $s_{1}<s_{0}$ be such that $A_{s_{1}}(\bar{\Delta})$ is contained in $\mathscr{V}_{\sigma_{0_{0}}}\left(A_{s_{0}}(\bar{\Delta})\right)$. Since $F_{s_{0}}=f$ in $\mathscr{V}_{s_{s_{0}}}\left(A_{s_{0}}(\partial \Delta)\right)$ and since the polydisc distance between $A_{s_{1}}(\partial \Delta)$ and $\partial \Omega$ is equal to $\rho_{s_{1}}$, it follows as in the first part of the proof of Lemma 2.26, that the Taylor series of $F_{s_{0}}$ at arbitrary points of the form $A_{s_{1}}(\zeta), \zeta \in \bar{\Delta}$, converges in the polydisc $\Delta_{\rho_{s_{1}}}^{n}\left(A_{s_{1}}(\zeta)\right)$. This gives holomorphic functions $F_{A(\zeta), \rho_{s_{1}}} \in$ $\mathscr{O}\left(\Delta_{\rho_{s_{1}}}^{n}\left(A_{s_{1}}(\zeta)\right)\right)$, for every $\zeta \in \bar{\Delta}$. Reasoning as in the second part of the proof of Lemma 2.26, we obtain a function $F_{s_{1}} \in \mathscr{O}\left(\mathscr{V}_{\sigma_{s_{1}}}\left(A_{s_{1}}(\bar{\Delta})\right)\right)$ with $F_{s_{1}}=f$ in $\mathscr{V}_{\sigma_{s_{1}}}\left(A_{s_{1}}(\partial \Delta)\right)$. This shows that $\mathscr{I}$ is open, as claimed and completes the proof.
2.28. Singularities as complex hypersurfaces. Let $\Omega \subset \mathbb{C}^{n}(n \geqslant 2)$ be a domain. A typical elementary singularity in $\Omega$ is just the zero set $Z_{f}:=\{f=0\}$ of a holomorphic function $f \in \mathscr{O}(\Omega)$ since for instance, the functions $1 / f^{k}, k \geqslant 1$, and $e^{1 / f}$ are holomorphic in $\Omega \backslash Z_{f}$ and singular along $Z_{f}$. Because $\mathbb{C}$ is algebraically closed, the closure in $\bar{\Omega}$ of such
$Z_{f}$ 's necessarily intersects $\partial \Omega$. Early in the twentieth century, the italian mathematicians Levi, Severi and B. Segre ([Se1932]) interpreted Hartogs' extension theorem as saying that compact sets $K \subset \Omega$ are removable, confirming the observation $\overline{Z_{f}} \cap \partial \Omega \neq \emptyset$.
Definition 2.29. (i) A relatively closed subset $C$ of a domain $\Omega \subset \mathbb{C}^{n}$ is called removable if the restriction map $\mathscr{O}(\Omega) \rightarrow \mathscr{O}(\Omega \backslash C)$ is surjective.
(ii) Such a set $C$ is called locally removable if for every $p \in C$, there exists an open neighborhood $U_{p}$ of $p$ in $\Omega$ such that the restriction map $\mathscr{O}\left(\left(U_{p} \cup\right.\right.$ $\Omega) \backslash C) \rightarrow \mathscr{O}(\Omega \backslash C)$ is surjective.

Under the assumption that $C$ is contained in a real submanifold of $\Omega$, the general philosophy of removable singularities is that a set too small to be a $Z_{f}$ (viz. a complex ( $n-1$ )-dimensional variety) is removable. The following theorem collects five statements saying that $C$ is removable provided it cannot contain any complex hypersurface of $\Omega$. Importantly, our submanifolds $N$ of $\Omega$ will always be assumed to be embedded, namely for every $p \in N$, there exist an open neighborhood $U_{p}$ of $p$ in $\Omega$ and a diffeomorphism $\psi_{p}: U_{p} \rightarrow \mathbb{R}^{2 n}$ such that $\psi_{p}\left(N \cap U_{p}\right)=\mathbb{R}^{\operatorname{dim} N} \times\{0\}$.
Theorem 2.30. Let $\Omega$ be a domain of $\mathbb{C}^{n}(n \geqslant 2)$ and let $C \subset \Omega$ be a relatively closed subset. The restriction map $\mathscr{O}(\Omega) \rightarrow \mathscr{O}(\Omega \backslash C)$ is surjective, namely $C$ is removable, under each one of the following five circumstances.
(rm1) $C$ is contained in a connected submanifold $N \subset \Omega$ of codimension $\geqslant 3$.
$(\mathbf{r m 2}) \mathrm{H}^{2 n-2}(C)=0$.
(rm3) $C$ is a relatively closed proper subset of a connected $\mathscr{C}^{2}$ submanifold $N \subset \Omega$ of codimension 2.
(rm4) $C=N$ is a connected $\mathscr{C}^{2}$ submanifold $N \subset \Omega$ that is not a complex hypersurface of $\Omega$.
(rm5) $C$ is a closed subset of a connected $\mathscr{C}^{2}$ real hypersurface $M^{1} \subset \Omega$ that does not contain any CR orbit of $M^{1}$.

In (rm1) and in (rm2), $C$ is in fact locally removable. In (rm5), two kinds of CR orbits coexist: those of real dimension $(2 n-2)$, that are necessarily complex hypersurfaces, and those of real dimension $(2 n-1)$, that are open subsets of $M$. It is necessary to exclude them also. Indeed, if for instance $\Omega$ is divided in two connected components $\Omega^{ \pm}$by a globally minimal $M^{1}$, taking $C=M^{1}$, any locally constant function on $\Omega \backslash M^{1}$ equal to two distinct constants $c^{ \pm}$in $\Omega^{ \pm}$does not extend holomorphically through $M^{1}$. The proof of the theorem is elementary ${ }^{22}$ and we will present it in $\S 2.34$ below,

[^21]as a relevant preliminary to Theorems 4.9, 4.10, 4.31 and 4.32, and to the main Theorem 1.7 of [26]).

We mention that under the assumption of local boundedness, more massive singularities may be removed. An application of Theorem 1.9(ii) to several complex variables deserves to be emphasized.
Theorem 2.31. ([HP1970]) If $\Omega \subset \mathbb{C}^{n}$ is a domain and if $C \subset \Omega$ is a relatively closed subset satisfying $\mathrm{H}^{2 n-1}(C)=0$, then $C$ is removable for functions holomorphic in $\Omega \backslash C$ that are locally bounded in $\Omega$.

Following [Stu1989] and [Lu1990], we now provide variations on (rm2). Any global complex variety of codimension one in $\mathbb{C}^{n}$ is certainly of infinite ( $2 n-2$ )-dimensional area.

Theorem 2.32. ([Stu1989]) Every closed set $C \subset \mathbb{C}^{n}$ satisfying $\mathrm{H}^{2 n-2}(C)<\infty$ is removable for $\mathscr{O}\left(\mathbb{C}^{n}\right)$.
The result also holds true in the unit ball $\mathbb{B}_{n}$ of $\mathbb{C}^{n}$, provided one computes the $(2 n-2)$-dimensional Hausdorff measure with respect to the distance function derived from the Bergman metric ([Stu1989]). Also, if $\Sigma$ is an arbitrary complex $k$-dimensional closed submanifold of $\mathbb{C}^{n}$, every closed subset $C \subset \Sigma$ with $\mathrm{H}^{2 k-2}(C)<\infty$ is removable for $\mathscr{O}(\Sigma \backslash C)$ ([Stu1989]).

A finer variation on the theme requires that $\mathrm{H}^{2 n-2}\left(C \cap R \mathbb{B}_{n}\right)$ does not grow too rapidly as a function of the radius $R \rightarrow \infty$. For instance ([Stu1989]), a closed subset $C \subset \mathbb{C}^{2}$ that satisfies $\mathrm{H}^{2}\left(C \cap R \mathbb{B}_{n}\right)<c R^{2}$ for all large $R$ is removable, provided $c<\frac{\pi^{2}}{4 \sqrt{2}}$. It is expected that $c<\pi$ is optimal, since the line $L:=\left\{\left(z_{1}, 0\right)\right\}$ satisfies $\mathrm{H}^{2}\left(L \cap R B_{2}\right)=\pi R^{2}$.

Yet another variation, raised in [Stu1989], is as follows. Consider a closed set $C$ in the complex projective space $P_{n}(\mathbb{C})(n \geqslant 2)$ such that the Hausdorff ( $2 n-2$ )-dimensional measure (with respect to the Fubini-Study metric) of $C$ is strictly less than that of any complex algebraic hypersurface of $P_{n}(\mathbb{C})$. Is it true that $C$ is a removable singularity for meromorphic functions, in the sense that every meromorphic function on $P_{n}(\mathbb{C}) \backslash C$ extends meromorphically through $C$ ? This question was answered by Lupacciolu.

Let $d_{\mathrm{FS}}(z, w)$ denote the geodesic distance between two points $z, w \in$ $P_{n}(\mathbb{C})$ relative to the Fubini-Study metric and let $\mathrm{H}_{\mathrm{FS}}^{\ell}$ denote the $\ell$ dimensional Hausdorff measure in $P_{n}(\mathbb{C})$ computed with $d_{\mathrm{FS}}$. Given a nonempty closed subset $C \subset P_{n}(\mathbb{C})$, define:

$$
\rho(C):=\frac{\max _{z \in P_{n}(\mathbb{C})} d_{\mathrm{FS}}(z, C)}{\max _{z, w \in P_{n}(\mathbb{C})} d_{\mathrm{FS}}(z, w)}=\frac{\max _{z \in P_{n}(\mathbb{C})} d_{\mathrm{FS}}(z, C)}{\operatorname{diam}_{\mathrm{FS}}\left(P_{n}(\mathbb{C})\right)} \leqslant 1 .
$$

If the Fubini-Study metric is normalized so that $\operatorname{vol} P_{n}(\mathbb{C})=1$, the $(2 n-2)$ dimensional volume of an irreducible complex algebraic hypersurface $\Sigma \subset$ $P_{n}(\mathbb{C})$ is equal to $\operatorname{deg} V$ and $\mathrm{H}_{\mathrm{FS}}^{2 n-2}(C)=(4 / \pi)^{n-1}(n-1)!\operatorname{deg} V$. It follows that the minimum value of $\mathrm{H}^{2 n-2}(\Sigma)$ is equal to $(4 / \pi)^{n-1}(n-1)$ ! and
is attained for $V$ equal to any hyperplane of $P_{n}(\mathbb{C})$. Let $\mathscr{M}$ denote the sheaf of meromorphic functions on $P_{n}(\mathbb{C})$.

Theorem 2.33. ([Lu 1990]) Let $C \subset P_{n}(\mathbb{C})$ be a closed subset such that

$$
\mathrm{H}_{\mathrm{FS}}^{2 n-2}(C)<[\rho(C)]^{4 n-4}(4 / \pi)^{n-1}(n-1)!
$$

Then the restriction map $\mathscr{M}\left(P_{n}(\mathbb{C})\right) \longrightarrow \mathscr{M}\left(P_{n}(\mathbb{C}) \backslash C\right)$ is onto.
2.34. Proof of Theorem 2.30. We claim that we may focus our attention only on (rm2) and on (rm5). Indeed, since a submanifold $N \subset \Omega$ of codimension $\geqslant 3$ satisfies $\mathrm{H}^{2 n-2}(N)=0,(\mathbf{r m 1})$ is a corollary of (rm2).

In both (rm3) and (rm4), we may include $N$ in some $\mathscr{C}^{2}$ hypersurface $M^{1}$ of $\mathbb{C}^{n}$, looking like a thin strip elongated along $N$. We claim that $C$ then does not contain any CR orbit of any such $M^{1}$, so that (rm5) applies. Indeed, CR orbits of $M^{1}$ being of dimension $(2 n-2)$ or $(2 n-1)$ and $C$ being already contained in the $(2 n-2)$-dimensional $N \subset M^{1}$, it could only happen that $C=N=\Sigma$ identifies as a whole to a connected (CR orbit) complex hypersurface $\Sigma \subset M^{1}$. But this is excluded by the assumption that $C \neq N$ in (rm3) and by the existence of generic points in (rm4).

Firstly, we prove (rm2). We show that $C$ is locally removable. Let $p \in C$ and let $B_{p} \subset \Omega$ be a small open ball centered at $p$. By a relevant application of Proposition 1.2(5), one may verify that for almost every complex line $\ell$ passing through $p$, the intersection $\ell \cap B_{p} \cap C$ is reduced to $\{p\}$. Choose such a line $\ell_{1}$. Centering coordinates at $p$ and rotating them if necessary, we may assume that $\ell_{1}=\left\{\left(z_{1}, 0, \ldots, 0\right)\right\}$, whence for $\varepsilon>0$ small and fixed, the disc $A_{\varepsilon}(\zeta):=(\varepsilon \zeta, 0, \ldots, 0)$ satisfies $A_{\varepsilon}(\partial \Delta) \cap C=\emptyset$.

Fix such a small $\varepsilon_{0}>0$ and set $\rho_{0}:=\operatorname{dist}\left(A_{\varepsilon_{0}}(\partial \Delta), C\right)>0$. For $\tau=$ $\left(\tau_{2}, \ldots, \tau_{n}\right) \in \mathbb{C}^{n-1}$ satisfying $|\tau|<\frac{1}{2} \rho_{0}$, set $A_{\varepsilon_{0}, \tau}(\zeta):=\left(\varepsilon_{0} \zeta, \tau_{2}, \ldots, \tau_{n}\right)$. Letting $s \in[0,1]$, we interpolate between $A_{\varepsilon_{0}, 0}$ and $A_{\varepsilon_{0}, \tau}$ by defining

$$
A_{\varepsilon_{0}, \tau, s}(\zeta):=\left(\varepsilon_{0} \zeta, s \tau_{2}, \ldots, s \tau_{n}\right)
$$

Since $|\tau|<\frac{1}{2} \rho_{0}$, these discs all satisfy dist $\left(A_{\varepsilon_{0}, \tau, s}(\partial \Delta), C\right) \geqslant \frac{1}{2} \rho_{0}$. Since the embedded disc $A_{\varepsilon_{0}, \tau, 1}(\bar{\Delta})$ is 2-dimensional and since $\mathrm{H}^{2 n-2}(C)=0$, Proposition 1.2(5) assures that for almost every $\tau$ with $|\tau|<\frac{1}{2} \rho_{0}$, its intersection with $C$ is empty. Thus, we may apply the continuity principle Theorem 2.27, setting $c_{s}=\frac{1}{2} \varepsilon_{0}, C_{s}=2 \varepsilon_{0}$ and $\rho_{s}:=\frac{1}{2} \rho_{0}$ uniformly for every $s \in[0,1]$, whence $\sigma_{s}=\frac{1}{8} \rho_{0}$ independently of the smallness of $\tau$ : for every $f \in \mathscr{O}(\Omega \backslash C)$, there exists $F_{0} \in \mathscr{O}\left(\mathscr{V}_{\frac{\rho_{0}}{8}}\left(A_{\varepsilon_{0}, \tau, 0}(\bar{\Delta})\right)\right)$ with $F_{0}=f$ in $\mathscr{V}_{\frac{\rho_{0}}{8}}\left(A_{\varepsilon_{0}, \tau, 0}(\partial \Delta)\right)$. But since $H^{2 n-2}(C)=0$, for every connected
open set $\mathscr{V} \subset \Omega$, the intersection $\mathscr{V} \cap(\Omega \backslash C)$ is also connected (Proposition 1.2(4)), so $F_{0}$ and $f$ stick together as a well defined function holomorphic in $\Omega \cup \mathscr{V}_{\frac{\rho_{0}}{8}}\left(A_{\varepsilon_{0}, \tau, 0}(\bar{\Delta})\right)$. If $\tau$ was chosen sufficiently small, it is clear that $p=0 \in \mathbb{C}^{n}$ is absorbed in $\mathscr{V}_{\frac{\rho_{0}}{8}}\left(A_{\varepsilon_{0}, \tau, 0}(\bar{\Delta})\right)$, hence removable.

Secondly, we prove (rm5). Let $C \subset M^{1}$ containing no CR orbit and define
$\mathscr{C}^{\prime}:=\left\{C^{\prime} \subset C\right.$ closed, $\forall f \in \mathscr{O}(\Omega \backslash C), \exists f^{\prime} \in \mathscr{O}\left(\Omega \backslash C^{\prime}\right)$ with $\left.\left.f^{\prime}\right|_{\Omega \backslash C}=f\right\}$.
Lemma 2.35. If $C_{1}^{\prime}, C_{2}^{\prime} \in \mathscr{C}^{\prime}$, then $C_{1}^{\prime} \cap C_{2}^{\prime} \in \mathscr{C}^{\prime}$.
Proof. Let $f_{j}^{\prime} \in \mathscr{O}\left(\Omega \backslash C_{j}^{\prime}\right), j=1,2$, with $\left.f_{j}^{\prime}\right|_{\Omega \backslash C}=f$. We claim that $f_{1}^{\prime}$ and $f_{2}^{\prime}$ match up on $C \backslash\left(C_{1}^{\prime} \cup C_{2}^{\prime}\right)$, hence define together a holomorphic function $f_{12}^{\prime} \in \mathscr{O}\left(\Omega \backslash\left(C_{1}^{\prime} \cap C_{2}^{\prime}\right)\right)$ with $\left.f_{12}^{\prime}\right|_{\Omega \backslash C}=f$. Indeed, choose an arbitrary point $p \in C \backslash\left(C_{1}^{\prime} \cup C_{2}^{\prime}\right)$. There exists a small open ball $B_{p}$ centered at $p$ with $B_{p} \cap\left(C_{1}^{\prime} \cup C_{2}^{\prime}\right)=\emptyset$. Since $f_{1}^{\prime}=f_{2}^{\prime}=f$ at least in the dense subset $B_{p} \backslash M^{1}$ of $M^{1} \cap B_{p}$, by continuity of $f_{1}^{\prime}$ and of $f_{2}^{\prime}$ in $B_{p}$, necessarily $f_{1}^{\prime}(p)=f_{2}^{\prime}(p)$.

Next, we define

$$
\widetilde{C}:=\bigcap_{C^{\prime} \in \mathscr{C}^{\prime}} C^{\prime}
$$

Intuitively, $\widetilde{C}$ is the "nonremovable core" of $C$. By the lemma, for every $f \in \mathscr{O}(\Omega \backslash C)$, there exists $\widetilde{f} \in \mathscr{O}(\Omega \backslash \widetilde{C})$ with $\left.\widetilde{f}\right|_{\Omega \backslash C}=f$. To prove (rm5), we must establish that $\widetilde{C}=\emptyset$. Reasoning by contradiction, we assume that $\widetilde{C} \neq \emptyset$ and we apply to $C:=\widetilde{C}$ the lemma below, which is in fact a corollary of Trépreau's Theorem $2.4(\mathrm{~V})$. Of course, $\widetilde{C}$ cannot contain any CR orbit of $M^{1}$. Remind ( $\S 1.27($ III ), $\S 4.9$ (III)) that for us, local CR orbits are not germs, but local CR submanifolds of a certain small size.
Lemma 2.36. Let $M^{1} \subset \mathbb{C}^{n}(n \geqslant 2)$ be a $\mathscr{C}^{2}$ hypersurface and let $C \subset M^{1}$ be a closed subset containing no CR orbit of $M^{1}$. Then there exists at least one point $p \in C$ such that for every neighborhood $V_{p}^{1}$ of $p$ in $M^{1}$, we have:

$$
V_{p}^{1} \cap \mathscr{O}_{C R}^{l o c}\left(M^{1}, p\right) \not \subset C,
$$

and in addition, all such points $p$ are locally removable.
Then $\tilde{f}$ extends holomorphically to a neighborhood of all such points $p \in$ $\widetilde{C}$, contradicting the definition of $\widetilde{C}$, hence completing the proof of ( $\mathbf{r m 5 )}$.
Proof. If $\mathscr{O}_{C R}^{l o c}\left(M^{1}, q\right) \subset C$ for every $q \in C$, then small complex-tangential curves issued from $q$ necessarily remain in $\mathscr{O}_{C R}^{l o c}\left(M^{1}, q\right)$, hence in $C$, and pursuing from point to point, global complex-tangential curves issued from $q$ remain in $C$, whence $\mathscr{O}_{C R}\left(M^{1}, q\right) \subset C$, contrary to the assumption.

So, let $p \in C$ with $\mathscr{O}_{C R}^{\text {loc }}\left(M^{1}, p\right) \not \subset C$. To pursue, we need that $M^{1}$ is minimal at $p$. Since $M^{1}$ is a hypersurface, it might only happen that $\mathscr{O}_{C R}^{\text {loc }}\left(M^{1}, p\right)$ is a local complex hypersurface, a bad situation that has to be changed in advance.

Fortunately, without altering the conclusion of the lemma (and of (rm5)), we have the freedom of perturbing the auxiliary hypersurface $M^{1}$, leaving $C$ fixed of course. Thus, assuming that $\mathscr{O}_{C R}\left(M^{1}, p\right)$ is a complex hypersurface, we claim that there exists a small (in $\mathscr{C}^{2}$ norm) deformation $M_{d}^{1}$ of $M^{1}$ supported in a neighborhood of $p$ with $M_{d}^{1} \supset C$ such that $M_{d}^{1}$ is minimal at $p$.

Indeed, let $\left(q_{k}\right)_{k \in \mathbb{N}}$ be a sequence of points tending to $p$ in $\mathscr{O}_{C R}^{\text {loc }}\left(M^{1}, p\right)$ with $q_{k} \notin C$. To destroy the local complex hypersurface $\mathscr{O}_{C R}^{l o c}\left(M^{1}, p\right)$, it suffices to achieve, by means of cut-off functions, small bump-deformations of $M$ centered at all the $q_{k}$; it is easy to write the technical details in terms of a local graphed representation $v=\varphi^{1}(z, u)$ for $M^{1}$. Outside a small neighborhood of the union of the $q_{k}, M_{d}^{1}$ coincides with $M^{1}$. Then the resulting $M_{d}^{1}$ is necessarily minimal at $p$, since if it where not, the uniqueness principle ${ }^{23}$ for complex manifolds would force $\mathscr{O}_{C R}^{l o c}\left(M^{1}, p\right)=\mathscr{O}_{C R}^{l o c}\left(M_{d}^{1}, p\right)$, but $M_{d}^{1}$ does not contains the $q_{k}$.

So we can assume that $M^{1}$ is minimal at every point $p \in C$ at which $\mathscr{O}_{C R}^{\text {loc }}\left(M^{1}, p\right) \not \subset C$. Let $B_{p} \subset \mathbb{C}^{n}$ be a small open ball centered at $p$ with $B_{p} \cap$ $M^{1} \subset \mathscr{O}_{C R}^{\text {loc }}\left(M^{1}, p\right)$. We will show that $\mathscr{O}(\Omega \backslash C)$ extends holomorphically to $B_{p}$. By assumption, $B_{p} \cap M^{1} \not \subset C$, hence $B_{p} \backslash C$ is connected, a fact that will insure monodromy.

Fixing $v_{p} \in T_{p} \mathbb{C}^{n} \backslash\{0\}$ with $v_{p} \notin T_{p} M^{1}$, we consider the global translations

$$
M_{s}^{1}:=M^{1}+s v_{p}, \quad s \in \mathbb{R},
$$

of $M^{1}$. Let $f \in \mathscr{O}(\Omega \backslash C)$ be arbitrary. For small $s \neq 0, M_{s}^{1} \cap B_{p}$ does not intersect $M^{1}$, hence the restriction $\left.f\right|_{M_{s}^{1} \cap B_{p}}$ is a $\mathscr{C}^{2} \mathrm{CR}$ function on $M_{s}^{1} \cap B_{p}$ (but $\left.f\right|_{M_{0}^{1} \cap B_{p}}$ has possible singularities at points of $C \cap B_{p}$ ).

With $U_{p}:=M^{1} \cap B_{p}$, Theorem $2.4(\mathrm{~V})$ says that $\mathscr{C}_{C R}^{0}\left(U_{p}\right)$ extends holomorphically to some one-sided neighborhood $\omega_{p}^{ \pm}$of $M^{1}$ at $p$. Reorienting if necessary, we may assume that the extension side is $\omega_{p}^{-}$and that $p+s v_{p} \in B_{p}^{+}$for $s>0$ small. The statement and the proof of Theorem $2.4(\mathrm{~V})$ are of course invariant by translation. Hence $\mathscr{C}_{C R}^{0}\left(U_{p}+s v_{p}\right)$ extends holomorphically to $\omega_{p}^{-}+s v_{p}$, for every $s>0$. It is geometrically clear that for $s>0$ small enough, $\omega_{p}^{-}+s v_{p}$ contains $p$. Thus $\left.f\right|_{M_{s}^{1} \cap B_{p}}$ extends holomorphically to a neighborhood of $p$ for such $s$. Monodromy of

[^22]the extension follows from the fact that $B_{p}^{\prime} \backslash C$ is connected for every open ball $B_{p}^{\prime}$ centered at $p$. This completes the proof of the lemma.
2.37. Removability and extension of complex hypersurfaces. Let $\Omega \subset \mathbb{C}^{n}$ be a domain. Theorem 2.30 (rm4) shows that a connected 2-codimensional submanifold $N \subset \Omega$ is removable provided it is not a complex hypersurface, or equivalently, is generic somewhere. Conversely, assume that $\Omega$ is pseudoconvex and let $H \subset \Omega$ be a (not necessarily connected) closed complex hypersurface. Then $\Omega \backslash H$ is (obviously) locally pseudoconvex at every point, hence the characterization of domains of holomorphy yields a function $f \in \mathscr{O}(\Omega \backslash H)$ whose domain of existence is exactly $\Omega \backslash H$. Thus, $H$ is nonremovable. But in a nonpseudoconvex domain, closed complex hypersurfaces may be removable.

Example 2.38. For $\varepsilon>0$ small, consider the following nonpseudoconvex subdomain of $\mathbb{B}_{2}$, defined as the union of a spherical shell together with a thin $\operatorname{rod}$ of radius $\varepsilon$ directed by the $y_{2}$-axis:

$$
\Omega_{\varepsilon}:=\left\{1 / 2<\left|z_{1}\right|^{2}+\left|z_{2}\right|^{2}<1\right\} \bigcup\left(\mathbb{B}_{2} \cap\left\{x_{1}^{2}+y_{1}^{2}+x_{2}^{2}<\varepsilon^{2}\right\}\right) .
$$

Then the intersection of the $z_{1}$-axis with the spherical shell, namely

$$
H:=\left\{\left(z_{1}, 0\right): 1 / 2<\left|z_{1}\right|<1\right\},
$$

is a relatively closed complex hypersurface of $\Omega_{\varepsilon}$ homeomorphic to an open annulus. We claim that $H$ is removable.

Indeed, applying the continuity principle along discs parallel to the $z_{1}$ axis, $\mathscr{O}\left(\Omega_{\varepsilon}\right)$ extends holomorphically to $\mathbb{B}_{2} \backslash\left\{z_{2}=0\right\}$. Since the open small disc $\left\{\left(z_{1}, 0\right):\left|z_{1}\right|<\varepsilon\right\}$, considered as a subset of the closed complex hypersurface $\widetilde{H}$ of $\mathbb{B}_{2}$ defined by

$$
\widetilde{H}:=\left\{\left(z_{1}, 0\right):\left|z_{1}\right|<1\right\}
$$

is contained in the thin rod, hence in $\Omega_{\varepsilon}$, Theorem $2.30(\mathbf{r m 3})$ finishes to show that

$$
\mathrm{E}\left(\Omega_{\varepsilon} \backslash H\right)=\mathrm{E}\left(\Omega_{\varepsilon}\right)=\mathbb{B}_{2}
$$

In such an example, we point out that the closed complex hypersurface $H \subset$ $\Omega_{\varepsilon}$ extends as the closed complex hypersurface $\widetilde{H} \subset \mathrm{E}\left(\Omega_{\varepsilon} \backslash H\right)$ but that the intersection

$$
\widetilde{H} \cap \Omega_{\varepsilon}=\left\{\left(z_{1}, 0\right):\left|z_{1}\right|<\varepsilon\right\} \bigcup\left\{\left(z_{1}, 0\right): 1 / 2<\left|z_{1}\right|<1\right\}
$$

is strictly bigger than $H$.
Problem 2.39. Understand which relatively closed complex hypersurfaces of a general domain $\Omega \subset \mathbb{C}^{n}$ are removable.

We thus consider a (possibly singular and reducible) closed complex hypersurface $H$ of $\Omega$. Basic properties of complex analytic sets ([Ch1991]) insure that $H=\bigcup_{j \in J} H_{j}$ decomposes into at most countably many closed complex hypersurfaces $H_{j} \subset \Omega$ that are irreducible.

Definition 2.40. We say that $H_{j}$ allows an $H$-compatible extension to $\mathrm{E}(\Omega)$ if there exists an irreducible closed complex hypersurface $\widetilde{H}_{j}$ of $\mathrm{E}(\Omega)$ extending $H_{j}$ in the sense that $H_{j} \subset \widetilde{H}_{j} \cap \Omega$ whose intersection with $\Omega$ remains contained in $H$ :

$$
\widetilde{H}_{j} \cap \Omega \subset \bigcup_{j^{\prime} \in J} H_{j^{\prime}} .
$$

The principle of analytic continuation for irreducible complex analytic sets ([Ch1991]) assures that $H_{j}$ has at most one $H$-compatible extension. On the other hand, $\widetilde{H}_{j}$ may be an $H$-compatible extension of several $H_{j^{\prime}}$. In the above example, the removable annulus $H$ had no $H$-compatible extension to $\mathrm{E}(\Omega)$.

Theorem 2.41. ([Dl1977, 22]) Let $\Omega \subset \mathbb{C}^{n}(n \geqslant 2)$ be a domain and let $H=\bigcup_{j \in J} H_{j}$ be a closed complex hypersurface of $\Omega$, decomposed into irreducible components $H_{j}$. Set

$$
J_{\text {comp }}:=\left\{j \in J: H_{j} \text { allows an } H \text {-compatible extension } \widetilde{H}_{j} \text { to } \mathrm{E}(\Omega)\right\} .
$$

Then

$$
\mathrm{E}(\Omega \backslash H)=\mathrm{E}(\Omega) \backslash \bigcup_{j \in J_{\text {comp }}} \widetilde{H}_{j} .
$$

In particular, $H$ is removable (resp. nonremovable) if and only if $J_{\text {comp }}=\emptyset$ (resp. $J_{\text {comp }} \neq \emptyset$ ).

This statement was obtained after a chain of generalizations originating from the classical results of Hartogs [Ha1909] and of Oka [Ok1934]. In [GR1956] it was proved for the case that $H$ is of the form $\Omega \cap \widetilde{H}, \widetilde{H} \subset \mathrm{E}(\Omega)$, and in [Nis1962] under the additional assumption that $\mathrm{E}(\Omega \backslash H)$ is a subset of $\mathrm{E}(\Omega)$ (a priori, it is only a set over $\mathrm{E}(\Omega)$ ). Actually Theorem 2.41 was stated in [D11977] even for Riemann domains $\Omega$. But it was remarked in [22] (p. 306) that the proof in [D11977] is complete only if the functions of $\mathscr{O}(\Omega)$ separate the points of $\Omega$, i.e. if $\Omega$ can be regarded as a subdomain of $\mathrm{E}(\Omega)$. Actually the proof in [D11977] starts from the special case where $\Omega$ is a Hartogs figure, which can be treated by a subtle geometric examination. Then a localization argument shows that extension of hypersurfaces which are singularity loci of holomorphic functions cannot stop when passing from $\Omega$ to $\mathrm{E}(\Omega)$. But in the nonseparated case, the global effect of identifying points of $\Omega$ interferes nastily, and it is unclear how to justify the localization
argument. The final step for general Riemann domains was achieved by the second author by completely different methods.
Theorem 2.42. ([Po2002]) Let $\pi: X \rightarrow \mathbb{C}^{n}$ be an arbitrary Riemann domain, and let $H \subset X$ be a closed complex hypersurface. Denote by $\alpha: X \rightarrow \mathrm{E}(X)$ the canonical immersion of $X$ into $\mathrm{E}(X)$. Then there is a closed complex hypersurface $\widetilde{H}$ of $\mathrm{E}(X)$ with $\alpha^{-1}(\widetilde{H}) \subset H$ such that

$$
\mathrm{E}(X \backslash H)=\mathrm{E}(X) \backslash \widetilde{H}
$$

Let us briefly sketch the main idea of the proof ([Po2002]). The essence of the argument is to reduce extension of hypersurfaces to that of meromorphic functions. For every pseudoconvex Riemann domain $\pi: X \rightarrow \mathbb{C}^{n}$, there exists $f \in \mathscr{O}(X) \cap L^{2}(X)$ having $X$ as domain of existence whose growth is controlled by some power of the polydisc distance to the abstract boundary $\breve{\partial} X$. At boundary points where $\breve{\partial} X$ can be locally identified with a complex hypersurface, $f$ has just a pole of positive order. One can deduce that those hypersurfaces $H$ of $X$ along which some holomorphic function on $X \backslash H$ becomes singular can be represented as the polar locus of some meromorphic function $g$ defined in $X$. But $g$ extends meromorphically to $\mathrm{E}(X)$, and the polar locus of the extension yields the desired extension of $H$.

## §3. Hulls and removable singularities at the boundary

3.1. Motivations for removable singularities at the boundary. As already observed in Section 1, beyond the harmonious realm of pseudoconvexity, the general problem of understanding compulsory holomorphic (or CR) extension is intrinsically rich and open. Some elementary Baire category arguments show that most domains are not pseudoconvex, most CR manifolds have nontrivial disc-envelope, and most compact sets have nonempty essential polynomial hull. Hence, the Grail for the theory of holomorphic extension would comprise:

- a geometric and constructive view of the envelope of holomorphy of most domains, following the Behnke-Sommer Kontinuitätssatz and Bishop's philosophy;
- a clear correspondence between function-theoretic techniques, for instance those involving $\bar{\partial}$ arguments, and geometric techniques, for instance those involving families of complex analytic varieties.
Several applications of the study of envelopes of holomorphy appear, for instance in the study of boundary regularity of solutions of the $\bar{\partial}$-complex, in the complex Plateau problem, in the study of CR mappings, in the computation of polynomial hulls and in removable singularities, the topics of this Part VI and of [26].

In the 1980's, rapid progress in the understanding of the boundary behavior of holomorphic functions led many authors to study the structure of singularities up to the boundary. In $\S 2.28$, we discussed removability of relatively closed subsets $C$ of domains $\Omega \subset \mathbb{C}^{n}$, i.e. the problem whether $\mathscr{O}(\Omega) \rightarrow \mathscr{O}(\Omega \backslash C)$ is surjective. Typically $C$ was supposed to be lowerdimensional and its geometry near $\partial \Omega$ was irrelevant. Now we assume $\Omega$ to be bounded in $\mathbb{C}^{n}(n \geqslant 2)$ and we consider compact subsets $K$ of $\bar{\Omega}$, possibly meeting $\partial \Omega$.

Problem 3.2. Find criteria of geometric, or of function-theoretic nature, assuring that the restriction map $\mathscr{O}(\Omega) \rightarrow \mathscr{O}(\Omega \backslash K)$ is surjective.

If $K \subset \bar{\Omega} \cap \partial \Omega=\emptyset$ and $\Omega \backslash K$ is connected, surjectivity follows from the Hartogs-Bochner extension Theorem 1.9(V). Since this theorem even gives extension of CR functions on $\partial \Omega$, it seems reasonable to ask for holomorphic extension of CR functions on $\partial \Omega \backslash K$, and then it is natural to assume that $K$ is contained in $\partial \Omega$. Hence the formulation of a second trend of questions ${ }^{24}$.

Problem 3.3. Let $K$ is a compact subset of $\partial \Omega$ such that $\partial \Omega \backslash K$ is a hypersurface of class at least $\mathscr{C}^{1}$. Understand under which circumstances CR functions of class $\mathscr{C}^{0}$ or $L_{l o c}^{\mathrm{p}}$ on $\partial \Omega \backslash K$ extend holomorphically to $\Omega$.

A variant of these two problems consists in assuming that functions are holomorphic in some thin (one-sided) neighborhood of $\partial \Omega \backslash K$. In all the theorems that will be surveyed below, it appears that the thinness of the (one-sided) neighborhood of $\partial \Omega \backslash K$ has no influence on extension, as in the original Hartogs theorem. In this respect, it is of interest to immediately indicate the connection of these two problems with the problem of determining certain envelopes of holomorphy.

In the second problem, the hypersurface $\partial \Omega \backslash K$ is often globally minimal, a fact that has to be verified or might be one of the assumptions of a theorem. For instance, several contributions deal with the paradigmatic case where $\partial \Omega$ is at least $\mathscr{C}^{2}$ and strongly pseudoconvex (hence obviously globally mini$\mathrm{mal})$. Then thanks to the elementary Levi-Lewy extension theorem (Theorem $1.18(\mathrm{~V})$, lemma $2.2(\mathrm{~V})$ and $\S 2.10(\mathrm{~V})$ ), there exists a one-sided neighborhood $\mathscr{V}(\partial \Omega \backslash K)$ of $\partial \Omega \backslash K$ contained in $\Omega$ to which both $\mathscr{C}_{C R}^{0}(\partial \Omega \backslash K)$ and $L_{\text {loc, } C R}^{\mathrm{p}}(\partial \Omega \backslash K)$ extend holomorphically. The size of $\mathscr{V}(\partial \Omega \backslash K)$ depends only on the local geometry of $\partial \Omega$, because $\mathscr{V}(\partial \Omega \backslash K)$ is obtained by gluing small discs (Part V). In fact, an inspection of the proof of the Levi-Lewy extension theorem together with an application of the continuity principle shows also that the envelope of holomorphy of any thin one-sided

[^23]neighborhood $\mathscr{V}^{\prime}(\partial \Omega \backslash K)$ (not necessarily contained in $\Omega$ !) contains a onesided neighborhood $\mathscr{V}(\partial \Omega \backslash K)$ of $\partial \Omega \backslash K$ contained in the pseudoconvex domain $\Omega$ that has a fixed, incompressible size ${ }^{25}$.

As they are formulated, the above two problems turn out to be slightly too restrictive. In fact, the final goal is to understand the envelope $\mathrm{E}(\mathscr{V}(\partial \Omega \backslash K))$, or at least to describe some significant part of $\mathrm{E}(\mathscr{V}(\partial \Omega \backslash K))$ lying above $\Omega$. Of course, the question to which extent is the geometry of $\mathrm{E}(\mathscr{V}(\partial \Omega \backslash K))$ accessible (constructively speaking) depends sensitively on the shape of $\Omega$. Surely, the strictly pseudoconvex case is the easiest and the best understood up to now. In what follows we will encounter situations where $\mathrm{E}(\mathscr{V}(\partial \Omega \backslash K))$ contains $\Omega \backslash \widehat{K}$, for some subset $\widehat{K} \subset \bar{\Omega}$ defined in function-theoretic terms and depending on $K \subset \partial \bar{\Omega}$. We will also encounter situations where $\mathrm{E}(\mathscr{V}(\Omega \backslash K))$ is necessarily multisheeted over $\mathbb{C}^{n}$. In this concern, we will see a very striking difference between the complex dimensions $n=2$ and $n \geqslant 3$.

In the last two decades, a considerable interest has been devoted to a subproblem of these two problems, especially with the objective of characterizing the singularities at the boundary that are removable.

Definition 3.4. In the second Problem 3.3, the compact subset $K \subset \partial \Omega$ is called $C R$-removable if for every CR function $f \in \mathscr{C}_{C R}^{0}(\partial \Omega \backslash K)$ (resp. $f \in L_{l o c, C R}^{\mathrm{p}}(\partial \Omega \backslash K)$ ), there exists $F \in \mathscr{O}(\Omega) \cap \mathscr{C}^{0}(\bar{\Omega} \backslash K)$ (resp. $F \in$ $\mathscr{O}(\Omega) \cap H_{l o c}^{\mathrm{p}}(\bar{\Omega} \backslash K)$ ) with $\left.F\right|_{\partial \Omega \backslash K}=f$ (resp. locally at every point $p \in$ $\partial \Omega \backslash K$, the $L_{\text {loc, } C R}^{\mathrm{p}}$ boundary value of $F$ equals $f$ ).

Before exposing and surveying some major results we would like to mention that a complement of information and different approaches may be found in the two surveys [Stu1993, 6], in the two monographs [Ky1995, Lt1997] and in the articles [Stu1981, LT1984, Lu1986, Lu1987, Jö1988, Lt1988, Stu1989, Ky1990, Ky1991, Stu1991, FS1991, Jö1992, KN1993, Du1993, LS1993, Lu1994, AC1994, Jö1995, KR1995, Jö1999a, Jö1999b, JS2000, 21, JS2004].
3.5. Characterization of removable sets contained in strongly pseudoconvex boundaries. Taking inspiration from the pivotal Oka theorem, one of the goals of the study of removable singularities ([Stu1993]) is to characterize removability in function-theoretically significant terms, especially in terms of convexity with respect to certain spaces of functions. In the very beginnings of Several Complex Variables, polynomial convexity appeared

[^24]in connexion with holomorphic approximation. According to the Oka-Weil theorem ([AW1998]), functions that are holomorphic in some neighborhood of a polynomially convex compact set $K \subset \mathbb{C}^{n}$ may be approximated uniformly by polynomials. Later on, holomorphic convexity appeared to be central in Stein theory ([Hö1973]), one of the seminal frequently used idea being to encircle convex compact sets by convenient analytic polyhedra.

The notion of convexity adapted to our pruposes is the following. By $\mathscr{O}(\bar{\Omega})$, we denote the ring of functions that are holomorphic in some neighborhood of the closure $\bar{\Omega}$ of a domain $\Omega \subset \mathbb{C}^{n}$. As in the concept of germs, the neighborhood may depend on the function.

Definition 3.6. Let $\Omega \Subset \mathbb{C}^{n}$ be a bounded domain and let $K \subset \bar{\Omega}$ be a compact set. The $\mathscr{O}(\bar{\Omega})$-convex hull of $K$ is

$$
\widehat{K}_{\mathscr{O}(\bar{\Omega})}:=\left\{z \in \bar{\Omega}:|g(z)| \leqslant \max _{w \in K}|g(w)| \text { for all } g \in \mathscr{O}(\bar{\Omega})\right\} .
$$

If $K=\widehat{K}_{\mathscr{O}(\bar{\Omega})}$, then $K$ is called $\mathscr{O}(\bar{\Omega})$-convex.
If $\Omega$ is strongly pseudoconvex, a generalization of the Oka-Weil theorem shows that every function which is holomorphic in a neighborhood of some $\mathscr{O}(\bar{\Omega})$-convex compact set $K \subset \bar{\Omega}$ may be approximated uniformly on $K$ by functions of $\mathscr{O}(\bar{\Omega})$ (nevertheless, for nonpseudoconvex domains, this approximation property fails ${ }^{26}$ ).

We may now begin with the formulation of a seminal theorem due to Stout that inspired several authors. We state the CR version, due to Lupacciolu ${ }^{27}$.

Theorem 3.7. ([Stu1981, Lu 1986, Stu 1993]) In complex dimension $n=2$, a compact subset $K$ of a $\mathscr{C}^{2}$ strongly pseudoconvex boundary $\partial \Omega \Subset \mathbb{C}^{2}$ is CR-removable if and only if it is $\mathscr{O}(\bar{\Omega})$-convex.

The "only if" part is the easiest, relies on a lemma due to Słodkowski ([RS1989, Stu1993]) and will be presented after Lemma 3.11. Let us sketch the beautiful key idea of the "if" part ([Stu1981, Lu1986, Stu1993, Po1997]).

[^25]From §1.7(V), remind the expression of the Bochner-Martinelli kernel:

$$
\operatorname{BM}(\zeta, z)=\frac{1}{(2 \pi i)^{2}|\zeta-z|^{4}}\left[\overline{\left(\zeta_{2}-z_{2}\right)} d \bar{\zeta}_{1}-\overline{\left(\zeta_{1}-z_{1}\right)} d \bar{\zeta}_{2}\right] \wedge d \zeta_{1} \wedge d \zeta_{2}
$$

Let $\mathscr{M} \subset \mathbb{C}^{2}$ be a thin strongly pseudoconvex neighborhood of $\bar{\Omega}$. By means of a fixed function $g \in \mathscr{O}(\mathscr{M})$, it is possible to construct some explicit primitive of BM as follows. This idea goes back to Martinelli and has been exploited by Stout, Lupacciolu, Leiterer, Laurent-Thiébaut, Kytmanov and others. By a classical result ([HeLe1984]), $g$ admits a Hefer decomposition

$$
g(\zeta)-g(z)=g_{1}(\zeta, z)\left[\zeta_{1}-z_{1}\right]+g_{2}(\zeta, z)\left[\zeta_{2}-z_{2}\right],
$$

with $g_{1}, g_{2} \in \mathscr{O}(\mathscr{M} \times \mathscr{M})$. Then a direct calculation shows that for $z \in \mathscr{M}$ fixed, the ( 0,2 )-form

$$
\Theta_{g, z}(\zeta)=\frac{g_{2}(\zeta, z) \overline{\left(\zeta_{1}-z_{1}\right)}-g_{1}(\zeta, z) \overline{\left(\zeta_{2}-z_{2}\right)}}{(2 \pi i)^{2}|\zeta-z|^{2}[g(\zeta)-g(z)]} d \zeta_{1} \wedge d \zeta_{2}
$$

satisfies

$$
\bar{\partial}_{\zeta} \Theta_{g, z}(\zeta)=d_{\zeta} \Theta_{g, z}(\zeta)=\mathrm{BM}(\zeta, z)
$$

on $\{\zeta \in \mathscr{M}: g(\zeta) \neq g(z)\}$, i.e. provides a primitive of BM outside some thin set. In $\mathbb{C}^{n}$ for $n \geqslant 3$, there is also a similar explicit primitive.

Let $K \subset \partial \Omega$ be as in Theorem 3.7 and fix $z \in \bar{\Omega} \backslash K$. By $\mathscr{O}(\bar{\Omega})$-convexity of $K$, there exists $g \in \mathscr{O}(\bar{\Omega})$ with $g(z)=1$ and $\max _{w \in K}|g(w)|<1$. After a slight elementary modification of $g$ ([Jö1995, Po1997]), one can insure that the set $\{w \in \mathscr{M}:|g(w)|=1\}$ is a geometrically smooth $\mathscr{C}^{\omega}$ Levi-flat hypersurface of $\mathscr{M}$ transverse to $\partial \Omega$. Then the region $\Omega_{g}:=\Omega \cap\{|g|>1\}$ has piecewise smooth connected boundary

$$
\partial \Omega_{g}=(\partial \Omega \cap\{|g|>1\}) \bigcup(\Omega \cap\{|g|=1\})
$$

and its closure $\bar{\Omega}_{g}$ in $\mathbb{C}^{2}$ does not intersect $K$.
Let $f$ be an arbitrary continuous CR function on $\partial \Omega \backslash K$. Supposing for a while that $f$ already enjoys a holomorphic extension $F \in \mathscr{O}(\Omega) \cap \mathscr{C}^{0}(\bar{\Omega} \backslash K)$, the Bochner-Martinelli representation formula then provides for every $z \in$ $\Omega_{g}$ the value

$$
F(z)=\int_{\partial \Omega_{g}} f(\zeta) \mathrm{BM}(\zeta, z)
$$

Decomposing $\partial \Omega_{g}$ as above and using the primitive $\Theta_{g, z}$, we may write

$$
F(z)=\int_{\partial \Omega \cap\{|g|>1\}} f(\zeta) \mathrm{BM}(\zeta, z)+\int_{\Omega \cap\{|g|=1\}} f(\zeta) d_{\zeta} \Theta_{g, z}(\zeta) .
$$

Supposing $f \in \mathscr{C}^{1}$ and applying Stokes' theorem ${ }^{28}$ to the (Levi-flat) hypersurface $\Omega \cap\{|g|=1\}$ with boundary equal to $\partial \Omega \cap\{|g|=1\}$, we get

$$
F(z)=\int_{\partial \Omega \cap\{|g|>1\}} f(\zeta) \mathrm{BM}(\zeta, z)+\int_{\partial \Omega \cap\{|g|=1\}} f(\zeta) \Theta_{g, z}(\zeta) .
$$

But the holomorphic extension $F$ of an arbitrary $f \in \mathscr{C}_{C R}^{0}(\partial \Omega \backslash K)$ is still unknown and in fact has to be constructed! Since the two integrations in the above formula are performed on parts of $\partial \Omega \backslash K$ where $f$ is defined, we are led to set:

$$
F_{g}(z)=\int_{\partial \Omega \cap\{|g|>1\}} f(\zeta) \mathrm{BM}(\zeta, z)+\int_{\partial \Omega \cap\{|g|=1\}} f(\zeta) \Theta_{g, z}(\zeta),
$$

as a candidate extension of $f$ at every $z \in \Omega_{g}$. Since $K$ is $\mathscr{O}(\bar{\Omega})$-convex, $\Omega$ is the union of the regions $\Omega_{g}$ for $g$ running in $\mathscr{O}(\bar{\Omega})$, but these extensions $F_{g}(z)$ do depend on $g$, because $\Theta_{g, z}$ does. The remainder of the proof ([Stu1993, Po1997]) then consists in:
(a) verifying that $F_{g}$ is holomorphic (the kernels are not holomorphic with respect to $z$ );
(a) showing that two differente candidates $F_{g_{1}}$ and $F_{g_{2}}$ coincide in fact on $\Omega_{g_{1}} \cap \Omega_{g_{2}}$;
(b) verifying that at least one candidate $F_{g}$ has boundary value equal to $f$ on some controlled piece of $\partial \Omega \backslash K$.
The reader is referred to [Stu1981, Lu1986, Stu1993] for complete arguments.

In the above construction, the strict pseudoconvexity of $\Omega$ insured the existence of a Stein (i.e. pseudoconvex) neighborhood basis $\left(\mathscr{M}_{j}\right)_{j \in J}$ of $\bar{\Omega}$ which guaranteed in turn the existence of a Hefer decomposition. It was pointed out by Ortega that Hefer decomposition (called Gleason decomposition in [Or1987]) holds on $\mathscr{C}^{\infty}$ pseudoconvex boundaries $\partial \Omega \Subset \mathbb{C}^{n}$, but may fail in the nonpseudoconvex realm. So, let $\Omega \Subset \mathbb{C}^{n}$ be a bounded domain having $\mathscr{C}^{\infty}$ boundary. Denote by $A^{\infty}(\Omega):=\mathscr{O}(\Omega) \cap \mathscr{C}^{\infty}(\bar{\Omega})$ the ring of holomorphic functions in $\Omega$ that are $\mathscr{C}^{\infty}$ up to the boundary.

Theorem 3.8. ([Or1987]) If $\partial \Omega \Subset \mathbb{C}^{n}(n \geqslant 1)$ is $\mathscr{C}^{\infty}$ and pseudoconvex, then every $g \in A^{\infty}(\Omega)$ has a decomposition

$$
g(z)-g(w)=\sum_{k=1}^{n} g_{k}(z, w)\left[z_{k}-w_{k}\right],
$$

with the $g_{k} \in A^{\infty}(\Omega \times \Omega)$.

[^26]This decomposition formula also holds under the assumption that $\Omega$ is a domain of holomorphy (having possibly nonsmooth boundary), but provided that $\bar{\Omega}$ has a basis of neighborhoods consisting of Stein domains. However, not every $\mathscr{C}^{\infty}$ weakly pseudoconvex boundary admits a Stein neighborhood basis, as is shown by the so-called worm domains ([DF1977, FS1987]).

Example 3.9. Furthermore, the above decomposition theorem fails to hold on general domains. Following [Or1987], consider the union $\Omega_{1} \cup \Omega_{2}$ in $\mathbb{C}^{2}$ of the two sets

$$
\begin{aligned}
& \Omega_{1}:=\left\{-4<x_{1}<0,\left|z_{2}\right|<e^{x_{1}}\right\} \quad \text { and } \\
& \Omega_{2}:=\left\{0 \leqslant x_{1}<4, e^{-1 / x_{1}}<\left|z_{2}\right|<1\right\} .
\end{aligned}
$$

The continuity principle along families of analytic discs parallel to the $z_{2}-$ axis shows that the envelope of holomorphy of $\Omega_{1} \cup \Omega_{2}$ contains $\Omega_{1} \cup \Omega_{3}$, where $\Omega_{3}:=\left\{0 \leqslant x_{1}<4,\left|z_{2}\right|<1\right\}$.

The holomorphic mapping $R\left(z_{1}, z_{2}\right):=\left(e^{i z_{1}}, z_{2}\right)$ is one-to-one from $\Omega_{1} \cup$ $\Omega_{2}$ onto its image $R\left(\Omega_{1} \cup \Omega_{2}\right)$. However, the extension of $R$ to $\Omega_{1} \cup \Omega_{3}$ is not injective, because $R$ takes the same value at the two points $\left( \pm \pi, e^{-2 \pi}\right) \in$ $\Omega_{1} \cup \Omega_{3}$. If Theorem 3.8 were true on the domain $R\left(\Omega_{1} \cup \Omega_{2}\right)$, pulling the decomposition formula back to $\Omega_{1} \cup \Omega_{2}$, it would follow that every $g \in$ $\mathscr{O}\left(\Omega_{1} \cup \Omega_{2}\right)$ has a decomposition

$$
\begin{aligned}
g(z)-g(w) & =\widetilde{g}_{1}\left(e^{i z}, w\right)\left[e^{i z_{1}}-e^{i z_{2}}\right]+\widetilde{g}_{2}\left(e^{i z}, w\right)\left[z_{2}-w_{2}\right] \\
& =g_{1}(z, w)\left[e^{i z_{1}}-e^{i z_{2}}\right]+g_{2}(z, w)\left[z_{2}-w_{2}\right] .
\end{aligned}
$$

Then the same decomposition would hold for every $g \in \mathscr{O}\left(\Omega_{1} \cup \Omega_{3}\right)$, by automatic holomorphic extension of $g, g_{1}, g_{2}$. Choosing $z=\left(-\pi, e^{-2 \pi}\right)$, $w=\left(\pi, e^{-2 \pi}\right)$ and $g$ such that $g(z) \neq g(w)\left(g:=z_{1}\right.$ will do $\left.!\right)$, we reach a contradiction.

Corollary 3.10. ([Or 1987, LP2003]) Every function holomorphic in a domain $\Omega \subset \mathbb{C}^{n}$ enjoys the Hefer division property precisely when the envelope of holomorphy of $\Omega$ is schlicht.

The above results mean that a direct application of the integral formula approach sketched above becomes impossible for domains having nonschlicht envelope. Nevertheless, in [Lt1988], using more general divison methods ([HeLe1984]), a Bochner-Martinelli kernel on an arbitrary Stein manifold was constructed that enabled to obtain Theorem 3.28 below, valid for nonpseudoconvex domains.

We conclude our presentation of Theorem 3.7 by exposing the "only if" of Theorem 3.7.

Lemma 3.11. ([RS1989, Stu1993]) Let $\partial \Omega \Subset \mathbb{C}^{2}$ be a $\mathscr{C}^{2}$ strongly pseudoconvex boundary and let $K \subset \partial \Omega$ be a compact set. Then $\Omega \backslash \widehat{K}_{\mathcal{O}(\bar{\Omega})}$ is pseudoconvex.

Taking for granted the lemma, by contraposition, suppose that $K \subset \partial \Omega$ is not $\mathscr{O}(\bar{\Omega})$-convex, viz. $K \varsubsetneqq \widehat{K}_{\mathscr{O}(\bar{\Omega})}$ and show that $K$ is not removable. It follows from strict pseudoconvexity of $\partial \Omega$ that $\Omega \cap \widehat{K}_{\mathscr{O}(\bar{\Omega})}$ is nonempty ([Stu1993]). Leaving $K$ fixed, by deforming $\partial \Omega$ away from $\Omega$, we may enlarge slightly $\Omega$ as a domain $\Omega^{\prime} \supset \Omega$ with $\partial \Omega^{\prime} \supset K$ and $\Omega^{\prime} \supset \partial \Omega \backslash K$, having $\mathscr{C}^{2}$ boundary $\partial \Omega^{\prime}$ close to $\partial \Omega$, as illustrated. Since by the lemma, $\Omega \backslash \widehat{K}_{\mathscr{O}(\bar{\Omega})}$ is pseudoconvex, it follows easily ([Stu1993]) that $\Omega^{\prime} \backslash \widehat{K}_{\mathscr{O}(\bar{\Omega})}$ is also pseudoconvex. Consequently ([Hö1973]), there exists a holomorphic function $F^{\prime} \in \mathscr{O}\left(\Omega^{\prime} \backslash \widehat{K}_{\mathscr{O}(\bar{\Omega})}\right)$ that does not extend holomorphically at any point of the boundary of $\Omega^{\prime} \backslash \widehat{K}_{\mathscr{O}(\bar{\Omega})}$. The restriction of $F^{\prime}$ to $\partial \Omega \backslash K$ is a CR function on $\partial \Omega \backslash K$ for which $K$ is not removable, since $\Omega \cap \widehat{K}_{\mathscr{O}(\bar{\Omega})} \neq \emptyset$.
3.12. Removability, polynomial hulls and Cantor sets. A generalization of Theorem 3.7, essentially with the same proof (excepting notational complications) holds in arbitrary complex dimension $n \geqslant 2$.
Theorem 3.13. ([Lu1986, Stu1993]) Let $\Omega \Subset \mathbb{C}^{n}, n \geqslant 2$, be a bounded pseudoconvex domain such that $\bar{\Omega}$ has a Stein neighborhood basis. If $K \subset$ $\partial \Omega$ is compact and $\mathscr{O}(\bar{\Omega})$-convex, and if $\partial \Omega=K \cup M$, where $M$ is a connected $\mathscr{C}^{1}$ hypersurface of $\mathbb{C}^{n} \backslash K$, then $K$ is $C R$-removable.
Example 3.14. ([6, Jö1999a]) Let $M$ be a connected compact orientable ( $2 n-3$ )-dimensional maximally complex (Definition 4.7 below) CR manifold of class $\mathscr{C}^{1}$ contained in the unit sphere $\partial \mathbb{B}_{n}(n \geqslant 2)$ with empty boundary in the sense of currents. Such an $M$ is called a maximally complex cycle. By a theorem due to Harvey-Lawson (reviewed as Theorem 4.16 below), if $M$ satisfies the moments' condition, then $M$ is the boundary of a unique complex $(n-1)$-dimensional complex subvariety $\Sigma \subset \mathbb{B}_{n}$. Since the cohomology group $H^{2}\left(\mathbb{B}_{n}, \mathbb{Z}\right)$ vanishes, by a standard Cousin problem, $\Sigma$ may be defined as the zero-set of some global holomorphic function $f \in \mathscr{O}\left(\mathbb{B}_{n}\right) \cap \mathscr{C}^{0}\left(\overline{\mathbb{B}}_{n}\right)$. The maximum principle yields that the compact set $K:=\Sigma \cup M=\bar{\Sigma}$ is $\mathscr{O}\left(\overline{\mathbb{B}}_{n}\right)$-convex. Consequently, the envelope of holomorphy of an arbitrarily thin one-sided neighborhood of $\partial \mathbb{B}_{n} \backslash M$ is equal to the pseudoconvex domain $\mathbb{B}_{n} \backslash(M \cup \Sigma)$.

If in addition $\Omega$ is Runge ([Hö1973]) or if $\bar{\Omega}$ is polynomially convex, then every $f \in \mathscr{O}(\bar{\Omega})$ may be approximated uniformly by polynomials on some sufficiently small neighborhood of $\bar{\Omega}$ (whose size depends on $f$ ). It then follows that polynomial convexity and $\mathscr{O}(\bar{\Omega})$-convexity are equivalent. As a paradigmatic example, this holds when $\Omega=\mathbb{B}_{n}$ is the unit ball.

Corollary 3.15. ([Stu1993]) Let $\Omega$ and $K \subset \Omega$ be as in Theorem 3.13 and assume that $\Omega$ is Runge in $\mathbb{C}^{n}$, for instance $\Omega=\mathbb{B}_{n}$. If $K$ is polynomially convex, then $K$ is $C R$-removable. If $n=2$, the $C R$-removability of $K$ is equivalent to its polynomial convexity.

Although the last necessary and sufficient condition seems to be satisfactory, we must point out that concrete geometric characterizations of polynomial convexity usually are hard to provide. In $\S 5.14$ below, we shall describe a class of removable compact sets whose polynomial convexity may be established directly.

A compact subset $K$ of $\mathbb{R}^{n}(n \geqslant 1)$ is a Cantor set if it is perfect, viz. coincides with its first derived set $K^{\prime}$. It is called tame if there is a homeomorphism of $\mathbb{R}^{n}$ onto itself that carries $K$ onto the standard middle-third Cantor set contained in the coordinate line $\mathbb{R}_{x_{1}}$.

Tame Cantor sets $K$ in a $\mathscr{C}^{2}$ strongly pseudoconvex boundary $\partial \Omega \Subset \mathbb{C}^{2}$ were shown to be CR-removable in [FS1991], provided there exists a Stein neighborhood $\mathscr{D}$ of $K$ in $\mathbb{C}^{2}$ such that $K$ is $\mathscr{O}(\bar{D})$-convex. By further analysis, this last assumption was shown later to be redundant and in general, tame Cantor sets are CR-removable. It was then suggested in [Stu1993] that all Cantor subsets of $\partial \mathbb{B}_{n}(n \geqslant 2)$ are removable, or equivalently polynomially convex. Nevertheless, Rudin and then Vitushkin, Henkin and others had constructed Cantor sets $K \subset \mathbb{C}^{2}$ having large polynomial hull $\widehat{K}$, e.g. so that $\widehat{K}$ contains a complex curve, or even contains interior points. Recently, in a beautiful paper, Jöricke showed how to put such sets in the 3 -sphere $\partial \mathbb{B}_{2}$, thus solving the question in the negative.

Theorem 3.16. ([Jo2005]) For every positive number $r<1$, there exists a Cantor set $K \subset \partial \mathbb{B}_{2}$ whose polynomial hull $\widehat{K}$ contains the closed ball $r \overline{\mathbb{B}}_{2}$.
3.17. $L^{\mathrm{p}}$-removability and further results. In the definition of CRremovability, nothing is assumed about the behavior from $\bar{\Omega} \backslash K$ up to $K$ : the rate of growth may be arbitrarily high. If, differently, functions are assumed to be tame on $\partial \Omega$ (including $K$ ), better removability assertions hold.

Definition 3.18. A compact subset $K$ of a $\mathscr{C}^{1}$ boundary $\partial \Omega \Subset \mathbb{C}^{n}(n \geqslant 2)$ is called $L^{\mathrm{p}}$-removable $(1 \leqslant \mathrm{p} \leqslant \infty)$ if every function $f \in L^{\mathrm{p}}(\partial \Omega)$ which is CR on $\partial \Omega \backslash K$ is in fact CR on the whole boundary $\partial \Omega$.

Then by the Hartogs-Bochner theorem, $f$ admits a holomorphic extension to $\Omega$ that may be checked to belong to $H^{\mathrm{P}}(\Omega)$.

Theorem 3.19. ([AC1994]) Let $\Omega \Subset \mathbb{C}^{n}(n \geqslant 2)$ be a bounded domain having $\mathscr{C}^{2}$ boundary $\partial \Omega$ and let $M$ be a $\mathscr{C}^{2}$ totally real embedded submanifold of $\partial \Omega$. If $K \subset M$ is a polynomially convex compact subset, then $K$ is $L^{\mathrm{p}}$-removable.

In complex dimension $n \geqslant 3$, the two extension Theorems 3.13 and 3.19 are not optimal. In general, additional extension phenomena occur, which are principally overlooked by assumptions on the hull of the singularity. A more geometric point of view ( $\$ 3.23$ below) shows that these theorems may be established by means of holomorphic extension along one-parameter families of complex analytic hypersurfaces, whereas the (finer) Kontinuitätssatz holds along families of analytic discs, whose thinness offers more freedom to fill in maximal domains of extension.

Example 3.20. Let $\Omega:=\mathbb{B}_{3}$ be the unit ball in $\mathbb{C}^{3}$, and let

$$
K=\left\{\left(z_{1}, z_{2}, 0\right) \in \partial \mathbb{B}_{3}:\left|z_{1}\right| \geqslant 1 / 2\right\}
$$

be a 3 -dimensional ring in the intersection of $\partial \mathbb{B}_{3}$ with the $\left(z_{1}, z_{2}\right)$-plane. The maximum principle along discs parallel to the $z_{2}$-axis yields:

$$
\widehat{K}_{O\left(\overline{\mathbb{B}}_{3}\right)}=\left\{\left(z_{1}, z_{2}, 0\right) \in \overline{\mathbb{B}}_{3}:\left|z_{1}\right| \geqslant 1 / 2\right\} \neq K,
$$

so $K$ is not $\mathscr{O}\left(\overline{\mathbb{B}}_{3}\right)$-convex. Nevertheless, this $K$ is removable. Indeed, applying the continuity principle, we may first fill in $\mathbb{B}_{3} \backslash \widehat{K}$ by means of discs parallel to the $z_{2}$-axis and then fill in the complete ball $\mathbb{B}_{3}$, by means of discs parallel to the $z_{3}$-axis.

In higher dimensions $n \geqslant 3$, the relevant characterizations of CRremovable compact sets contained in strongly pseudoconvex frontiers are of cohomological nature ( $\$ 3.33$ below). In another vein, the assumption that $\bar{\Omega}$ possesses a Stein neighborhood basis in Theorem 3.13 above inspired some authors to generalize Stout's theorem as follows.
Definition 3.21. Let $\Omega$ be a relatively compact domain of a Stein manifold $\mathscr{M}$ and let $K \subset \bar{\Omega}$ be a compact set. The $\mathscr{O}(\mathscr{M})$-convex hull of $K$ is

$$
\widehat{K}_{\mathscr{O}(\mathscr{M})}:=\left\{z \in \mathscr{M}:|g(z)| \leqslant \max _{w \in K}|g(w)| \text { for all } g \in \mathscr{O}(\mathscr{M})\right\} .
$$

If $K=\widehat{K}_{\mathscr{O}(\mathscr{M})}$, then $K$ is called $\mathscr{O}(\mathscr{M})$-convex.
In $\mathbb{C}^{n}$, the $\mathscr{O}(\mathscr{M})$-convex hull coincides with the polynomial hull. Notice that the next theorem is valid without pseudoconvexity assumption on $\Omega$.
Theorem 3.22. ([Stu1981, Lt1988, Ky1991, Stu1993, Jö1995]) Let $\mathscr{M}$ be a Stein manifold of dimension $n \geqslant 2$, let $\Omega \Subset \mathscr{M}$ be a relatively compact domain such that $\mathscr{M} \backslash \bar{\Omega}$ is connected and let $K \subset \bar{\Omega}$ be a compact set with
$K=\widehat{K}_{O(M)} \cap \partial \Omega$. Then every CR function $f$ defined on $\partial \Omega \backslash K$ extends holomorphically to $\Omega \backslash \widehat{K}_{\mathscr{O}(\mathbb{M})}$, i.e.:

- if $\partial \Omega \backslash K$ is a $\mathscr{C}^{\kappa, \alpha}$ hypersurface, with $\kappa \geqslant 1$ and $0 \leqslant \alpha \leqslant 1$, and if $f \in \mathscr{C}_{C R}^{\kappa, \alpha}(\partial \Omega \backslash K)$, then the holomorphic extension $F \in \mathscr{O}(\Omega \backslash K)$ belongs to the class $\mathscr{C}^{\kappa, \alpha}\left(\bar{\Omega} \backslash \widehat{K}_{\mathscr{O}(\mathcal{M})}\right)$;
- if $\partial \Omega \backslash K$ is a $\mathscr{C}^{1}$ hypersurface and if $f \in L_{l o c}^{\mathrm{p}}(\partial \Omega \backslash K)$ with $1 \leqslant$ $p \leqslant \infty$, then at every point $p \in \partial \Omega \backslash K$, the holomorphic extension $F \in \mathscr{O}\left(\bar{\Omega} \backslash \widehat{K}_{\mathscr{O}(\mathscr{M})}\right)$ belongs to the Hardy space $H_{l o c}^{\mathrm{p}}\left(U_{p} \cap \Omega\right)$, for some small neighborhood $U_{p}$ of $p$ in $\mathscr{M}$.
3.23. $A(\Omega)$-hull and removal of singularities on pseudoconvex boundaries. Following [Jö1995, Po1997, 21], we now expose a geometric aspect of some of the preceding removability theorems. Let $\Omega \Subset \mathbb{C}^{n}$ with $n \geqslant 2$ be a bounded domain having frontier of class at least $\mathscr{C}^{1}$. By $A(\Omega)=\mathscr{O}(\Omega) \cap \mathscr{C}^{0}(\bar{\Omega})$, we denote the ring of holomorphic functions in $\Omega$ that are continuous up to the boundary. Let $K \subset \bar{\Omega}$ be a compact set.

Definition 3.24. The $A(\Omega)$-hull of $K$ is

$$
\widehat{K}_{A(\Omega)}:=\left\{z \in \bar{\Omega}:|g(z)| \leqslant \max _{w \in \bar{\Omega}}|g(w)| \text { for all } g \in \mathscr{A}(\bar{\Omega})\right\} .
$$

If $K=\widehat{K}_{A(\Omega)}$, then $K$ is called $A(\Omega)$-convex. If $K=\partial \Omega \cap \widehat{K}_{A(\Omega)}$, then $K$ is called $C R$-convex.

The next theorem is stronger than Theorem 3.13 in two aspects:

- the inclusion $\widehat{K}_{A(\Omega)} \subset \widehat{K}_{\mathcal{O}(\bar{\Omega})}$ holds in general and may be strict;
- it is not assumed that the pseudoconvex domain $\Omega$ has a Stein neighborhood basis.

Theorem 3.25. ([Jö1995]) Let $\Omega$ be a bounded weakly pseudoconvex domain in $\mathbb{C}^{2}$ having frontier of class $\mathscr{C}^{2}$ and let $K$ be a compact subset of $\partial \Omega$ with $K \neq \partial \Omega$ such that $K$ is CR-convex, namely $K=\partial \Omega \cap \widehat{K}_{A(\Omega)}$. Then the following are true.

1) Let $\mathscr{V}(\partial \Omega \backslash K)$ be an interior one-sided neighborhood of $\partial \Omega \backslash K$ with the property that each connected component of $\mathscr{V}(\partial \Omega \backslash K)$ contains in its boundary exactly one component of $\partial \Omega \backslash K$ and no other point of $\partial \Omega \backslash K$. Then for every holomorphic function $f \in \mathscr{O}(\mathscr{V}(\partial \Omega \backslash K))$, there exists a holomorphic function $F \in \mathscr{O}\left(\Omega \backslash \widehat{K}_{A(\Omega)}\right)$ with $F=f$ in $\mathscr{V}(\partial \Omega \backslash K)$.
2) ([AS1990]) There is a one-to-one correspondence between connected components of $\partial \Omega \backslash K$ and connected components of $\Omega \backslash \widehat{K}_{A(\Omega)}$, namely the boundary of each component of $\Omega \backslash \widehat{K}_{A(\Omega)}$ contains exactly one connected component of $\partial \Omega \backslash K$ and does not intersect any other component.
3) If the boundary $\partial \Omega$ is of class $\mathscr{C}^{\infty}$, then $\Omega \backslash \widehat{K}_{A(\Omega)}$ is pseudoconvex, hence it is the envelope of holomorphy of $\mathscr{V}(\partial \Omega \backslash K)$.

If $K$ is not CR-convex, the one-to-one correspondence between the connected components of $\partial \Omega \backslash K$ and those of $\partial \Omega \backslash \widehat{K}_{A(\Omega)}$ may fail.
Example 3.26. Indeed, let $\Omega:=\mathbb{B}_{2} \cap\left\{x_{1}<\frac{1}{2}\right\}$ be a truncation of the unit ball and let $K:=\partial \mathbb{B}_{2} \cap\left\{x_{1}=\frac{1}{2}\right\}$ be the intersection of the three-sphere $\partial \mathbb{B}_{2}$ with the real hyperplane $\left\{x_{1}=\frac{1}{2}\right\}$ (see only the left hand side of the diagram).


The Levi-flat 3-ball $\mathbb{B}_{2} \cap\left\{x_{1}=\frac{1}{2}\right\}$ being foliated by complex discs, the maximum principle entails that $\widehat{K}_{A(\Omega)}=\overline{\mathbb{B}}_{2} \cap\left\{x_{1}=\frac{1}{2}\right\}=\widehat{K}_{A(\Omega)} \cap \partial \Omega \neq K$, hence $K$ is not CR-convex. Also, $\partial \Omega \backslash K$ has two connected components $\partial \mathbb{B}_{2} \cap\left\{x_{1}<\frac{1}{2}\right\}$ and $\mathbb{B}_{2} \cap\left\{x_{1}=\frac{1}{2}\right\}$, whereas $\partial \Omega \backslash \widehat{K}_{A(\Omega)}=\partial \mathbb{B}_{2} \cap\left\{x_{1}<\frac{1}{2}\right\}$ is connected. Any function on $\partial \Omega \backslash K$ equal to two distinct constants on the two connected components of $\partial \Omega \backslash K$ is CR and not holomorphically extendable to $\Omega=\Omega \backslash \widehat{K}_{A(\Omega)}$. Finally, by smoothing out $\partial \Omega$ near the twosphere $\partial \mathbb{B}_{2} \cap\left\{x_{1}=\frac{1}{2}\right\}$, we obtain an example with $\mathscr{C}^{\infty}$ boundary.
3.27. Hulls and holomorphic extension from nonpseudoconvex boundaries. Since the work [Lu1986] of Lupacciolu, the extension of Theorem 3.7 to nonpseudoconvex boundaries was a daring open problem ([Stu1993]).

Theorem 3.28. ([Po1997, 21, LP2003]) Let $\Omega$ be a not necessarily pseudoconvex bounded domain in $\mathbb{C}^{n}(n \geqslant 2)$ having connected $\mathscr{C}^{2}$ frontier and let
$K \subset \partial \Omega$ be a compact set with $\partial \Omega \backslash K$ connected such that $K=\partial \Omega \cap \widehat{K}_{A(\Omega)}$. Then for every continuous $C R$ function $f \in \mathscr{C}_{C R}^{0}(\partial \Omega \backslash K)$, there exists a holomorphic function $F \in \mathscr{O}\left(\Omega \backslash \widehat{K}_{A(\Omega)}\right) \cap \mathscr{C}^{0}\left(\left[\Omega \backslash \widehat{K}_{A(\Omega)}\right] \cup[\partial \Omega \backslash K]\right)$ such that $\left.F\right|_{\partial \Omega \backslash K}=f$.

A purely geometrical proof applying a global continuity principle together with a fine control of monodromy may be found in [Po1997, 21]; $c f$. also [27]. By a topological device, a second proof ([LP2003]) derives the theorem from the following statement, established by means of $\bar{\partial}$ techniques.
Theorem 3.29. ([Lt1988]) Let $\mathscr{M}$ be a Stein manifold of complex dimension $n \geqslant 2$, let $K \subset \mathscr{M}$ be a compact set that is $\mathscr{O}(\mathscr{M})$-convex and let $\Omega \subset \mathscr{M}$ be a relatively compact not necessarily pseudoconvex domain such that $\partial \Omega \backslash K$ is a connected $\mathscr{C}^{1}$ hypersurface of $\mathscr{M} \backslash K$. Then for every continuous CR function $f$ on $\partial \Omega \backslash K$, there exists a holomorphic function $F \in \mathscr{O}(\Omega \backslash K) \cap \mathscr{C}^{0}(\bar{\Omega} \backslash K)$ with $\left.F\right|_{\partial \Omega \backslash K}=f$.

Contrary to the case where $\partial \Omega$ is pseudoconvex (as in Theorem 3.25), even if $K$ is CR-convex, the one-to-one correspondence between the connected components of $\partial \Omega \backslash K$ and those of $\Omega \backslash \widehat{K}_{A(\Omega)}$ may fail to hold. For this reason, $\partial \Omega \backslash K$ is assumed to be connected in Theorem 3.29.
Example 3.30. ([LP2003]) We modify Example 3.26 so as to get a nonpseudoconvex boundary as follows (see the right hand side of the diagram above). Let $\Omega^{\prime}$ be the unit ball $\mathbb{B}_{2}$ from which we substract the closed ball $\overline{B(q, 1)}$ of radius 1 centered at the point $q$ of coordinates ( 1,0 ). A computation with defining (in)equations shows that $\Omega^{\prime}$ is contained in $\left\{x_{1}<\frac{1}{2}\right\}$. Notice that $\Omega^{\prime}$ is not pseudoconvex and in fact, its envelope of holomorphy is single-sheeted and equal to the domain $\Omega=\mathbb{B}_{2} \cap\left\{x_{1}<\frac{1}{2}\right\}$ drawn in the left hand side. Let $K^{\prime}:=\mathbb{B}_{2} \cap\left\{x_{1}=\frac{1}{2}\right\} \subset \partial \Omega^{\prime}$ (this set is the same 2 -sphere as the set $K$ of the preceding example). Then $K^{\prime}$ is CR-convex, since the candidate for its $A\left(\Omega^{\prime}\right)$-hull is the three-sphere $\overline{\mathbb{B}_{2}} \cap\left\{x_{1}=\frac{1}{2}\right\}$ that lies outside $\Omega^{\prime}$. However, $\partial \Omega^{\prime} \backslash K^{\prime}$ has two connected components, namely $\partial \mathbb{B}_{2} \cap\left\{x_{1}<\frac{1}{2}\right\}$ and $\partial B(q, 1) \cap\left\{x_{1}<\frac{1}{2}\right\}$, whereas $\Omega^{\prime} \backslash \widehat{K}_{A(\Omega)}^{\prime}=\Omega^{\prime} \backslash K^{\prime}=\Omega^{\prime}$ is connected. Hence any CR function equal to two distinct constants on these two components fails to extend holomorphically to $\Omega^{\prime}$. Finally, by smoothing out $\partial \Omega$ near the two-sphere $\partial \mathbb{B}_{2} \cap\left\{x_{1}=\frac{1}{2}\right\}$, we obtain an example with $\mathscr{C}^{\infty}$ boundary.

If we drop CR-convexity of $K$, viz. if $K \neq \widehat{K}_{A(\Omega)} \cap \partial \Omega$, then monodromy problems come on scene: the natural embedding of $\Omega \backslash \widehat{K}_{A(\Omega)}$ into the envelope of holomorphy of a one-sided neighborhood of $\partial \Omega \backslash K$ may fail to be one-to-one.

Example 3.31. ([LP2003]) Consider the real four-dimensional open cube $C:=(-1,1) \times i(-1,1) \times(-1,1) \times i(-1,1)$ in $\mathbb{C}^{2} \simeq \mathbb{R}^{4}$.


Choose $\varepsilon>0$ small and remove from this cube $C$ firstly the narrow tunnel $\mathscr{T}_{1}:=\left\{\left|z_{2}\right| \leqslant \varepsilon,\left|x_{1}-1 / 2\right| \leqslant \varepsilon\right\}$ having an entrance and an exit and secondly the (incomplete) narrow tunnel $\mathscr{T}_{2}:=\left\{\left|z_{2}\right| \leqslant \varepsilon,\left|x_{1}+1 / 2\right| \leqslant\right.$ $\left.\varepsilon,-1<y_{1} \leqslant 1 / 2\right\}$ having only an entrance, and call $\Omega$ the obtained domain. Let $K:=\partial C \cap\left\{y_{1}=0\right\}$. The complete tunnel insures that $\partial \Omega \backslash K$ is connected. Moreover, the maximum principle along families of analytic discs parallel to the complex $z_{2}$-axis enables to verify that
$\widehat{K}_{A(\Omega)}=\left(\Omega \cap\left\{y_{1}=0\right\}\right) \bigcup K \bigcup\left(\partial \mathscr{T}_{1} \cap\left\{y_{1}=0\right\}\right) \bigcup\left(\partial \mathscr{T}_{2} \cap\left\{y_{1}=0\right\}\right)$.
It follows that $\partial \Omega \backslash \widehat{K}_{A(\Omega)}$ has three connected components, firstly the part $T_{1}$ of $\partial \Omega$ that lies in the half-space $\left\{y_{1}<0\right\}$; secondly the dead-lock part $T_{2}$ of the second tunnel that lies in $\left\{y_{1}>0\right\}$; and thirdly, the remainder $T_{3}$ of the boundary, that lies in $\left\{y_{1}>0\right\}$.

The branch of $\log z_{1}$ satisfying $\log 1=0$ is uniquely defined in $\mathbb{C}^{2} \backslash\left\{\left(x_{1}, z_{2}\right): x_{1} \leqslant 0\right\}$, hence $\log z_{1}$ is holomorphic in a neighborhood of $\partial \Omega \backslash \bar{T}_{2}$, where $\bar{T}_{2}:=\partial \mathscr{T}_{2} \cap\left\{y_{1} \geqslant 0\right\}$. In addition, $\log z_{1}$ extends from points near $\mathscr{T}_{2}$ in $\left\{y_{1}<0\right\}$ to a neighborhood of $\bar{T}_{2}$. In sum, it defines a single-valued function that is holomorphic in a neighborhood of $\partial \Omega$.

Observe that $\left(-\frac{1}{2}+\frac{i}{2}, 0\right) \in T_{2} \subset \partial \Omega$. The value of $\log z_{1}$ thus defined at this point is $\log \left(\frac{1}{\sqrt{2}} e^{-i 5 \pi / 4}\right)=\log \frac{1}{\sqrt{2}}-i \frac{5 \pi}{4}$. On the other hand, $\log z_{1}$ restricted to a neighborhood of $\partial C \cap\left\{y_{1}>0\right\} \subset \partial \Omega$ extends holomorphically to $C \cap\left\{y_{1}>0\right\}$ (by means of unit discs parallel to the $z_{2}$-axis) as $\log z_{1}$ itself! But the value of this extension at $\left(-\frac{1}{2}+\frac{i}{2}, 0\right)$ is different: $\log \left(\frac{1}{\sqrt{2}} e^{i 3 \pi / 4}\right)=\log \frac{1}{\sqrt{2}}+i \frac{3 \pi}{4}$.

To conclude this paragraph, before surveying the cohomological characterizations of removable singularities in dimension $n \geqslant 3$, we reformulate the obtained characterization in complex dimension $n=2$. It is known that a compact set $K \subset \mathbb{C}^{n}$ is polynomially convex if and only if the $\bar{\partial}$ cohomology group $H_{\bar{\partial}}^{0,1}(K)$ is trivial and holomorphic functions in a neighborhood of $K$ can be approximated by polynomials uniformly on $K$. Thus, we can state a complete formulation of Theorem 3.7, with the supplementary assumption that $\mathscr{O}(\bar{\Omega})$ may be approximated uniformly by polynomials. This insures that polynomial convexity coincides with $\mathscr{O}(\bar{\Omega})$-convexity. As a major example, the theorem holds for $\Omega$ equal to the unit ball $\mathbb{B}_{2}$ (Corollary 3.15).

Theorem 3.32. ([Stu1989, Stu1993, Lu1994, 6]) The following four conditions for a compact subset $K$ of a $\mathscr{C}^{2}$ strongly pseudoconvex compact boundary $\partial \Omega \Subset \mathbb{C}^{2}$ with $\Omega$ Runge or $\bar{\Omega}$ polynomially convex are equivalent:

- $K$ is $\mathscr{O}(\bar{\Omega})$-convex.
- $K$ is polynomially convex.
- $H_{\bar{\partial}}^{0,1}(K)=0$ and holomorphic functions in a neighborhood of $K$ can be approximated by polynomials uniformly on $K$
- $K$ is removable.

Thus, in this situation, removability amounts to polynomial convexity. Nevertheless, the problem of characterizing geometrically the polynomial convexity of compact sets hides several fine questions. We shall come back to this topic in Section 5.
3.33. Luppaciolu's characterizations. An outstanding theorem due to Lupacciolu provides complete cohomological characterizations of removable sets that are contained in strongly pseudoconvex boundaries, for general $n \geqslant 2$.

Let $\mathscr{M}$ be a Stein manifold of dimension $n \geqslant 2$ and let $\Omega \Subset \mathscr{M}$ be a relatively compact strongly pseudoconvex domain having $\mathscr{C}^{2}$ boundary.

Let $H_{\bar{\partial}}^{p, q}:=\mathscr{Z}_{\bar{\partial}}^{p, q} / \bar{\partial} \mathscr{E}^{\mathscr{P}, q-1}$ denote the usual $(p, q)$-th Dolbeault cohomology group ${ }^{29}$. We endow the space $\mathscr{Z}_{\bar{\partial}}^{n, n-2}(K)$ of $\bar{\partial}$-closed $(n, n-2)$-forms defined in a neighborhood of a compact set $K \subset \mathscr{M}$ with the standard locally convex inductive limit topology derived from the inductive system of the Fréchet-Schwartz spaces $\mathscr{Z}_{\bar{\partial}}^{n, n-2}(U)$, as $U$ ranges through a fundamental system of open neighborhoods of $K$ in $\mathscr{M}$.

[^27]Theorem 3.34. ([Lu1994, 6]) Assume that $\bar{\Omega}$ is $\mathscr{O}(\mathscr{M})$-convex. A proper closed subset $K$ of $\partial \Omega$ is removable if and only if $H_{\bar{\partial}}^{n, n-1}(K)=0$ and the restriction map $\mathscr{Z}_{\bar{\partial}}^{n, n-2}(\mathscr{M}) \rightarrow \mathscr{Z}_{\bar{\partial}}^{n, n-2}(K)$ has dense image.

For $n=2$, the two conditions of the theorem reduce to the $\mathscr{O}(\mathscr{M})$ convexity of $K$ ([Lu1994, 6]). For $n \geqslant 3$, the following improvement is valid. By ${ }^{\sigma} E$, we denote the separated space associated to a given topological vector space $E$, namely the quotient $E / \overline{0}$ of $E$ by the closure of 0 .

Theorem 3.35. ([Lu 1994, 6]) Assume that $n \geqslant 3$. Without the assumption that $\bar{\Omega}$ is $\mathscr{O}(\mathscr{M})$-convex, the compact set $K \subset \partial \Omega$ is removable if and only if $H_{\bar{\partial}}^{n, n-1}(K)=0$ and ${ }^{\sigma} H_{\bar{\partial}}^{n, n-2}(K)=0$.

Lupacciolu also obtains an extrinsic characterization as follows. Let $\Phi$ be the paracompactifying family of all closed subsets of $\mathscr{M} \backslash K$ that have compact closure in $\mathscr{M}$. Let $H_{\Phi}^{p, q}$ the Dolbeault cohomology groups with support in $\Phi$.

Theorem 3.36. ([Lu1994, 6]) For $n \geqslant 3$, a compact subset $K$ of the boundary $\partial \Omega$ of a $\mathscr{C}^{2}$-bounded strongly pseudoconvex domain $\Omega \Subset \mathscr{M}$ is removable if and only if $H_{\Phi}^{0,1}(\mathscr{M} \backslash K)=0$.

Notice that, for $n \geqslant 3$, this theorem has the striking consequence that the condition that $K$ be removable in a strongly pseudoconvex boundary does not depend on the domain in question, but rather on the situation of $K$ itself in the ambient manifold. Also, Lupacciolu provides analogous characterizations for weak removability ([Lu1994, 6]).

## §4. Smooth and metrically thin removable singularities for CR functions

4.1. Three notions of removability. We formulate the concerned notions of removability directly in arbitrary codimension. Let $M \subset \mathbb{C}^{n}$ be a $\mathscr{C}^{2, \alpha}$ generic submanifold of positive codimension $d \geqslant 1$ and of positive CR dimension $m \geqslant 1$. Such $M$ will always be supposed connected. In the sequel, not to mention superficial corollaries, we will systematically assume that $M$ is globally minimal.

Definition 4.2. ([Me 1997, MP1998, Jö1999a, Jö1999b, MP1999, MP2002]) A closed subset $C$ of $M$ is said to be:

- CR-removable if there exists a wedgelike domain $\mathscr{W}$ attached to $M$ to which every continuous CR function $f \in \mathscr{C}_{C R}^{0}(M \backslash C)$ extends holomorphically;
- $\mathscr{W}$-removable if for every wedgelike domain $\mathscr{W}_{1}$ attached to $M \backslash C$, there is a wedgelike domain $\mathscr{W}_{2}$ attached to $M$ and a wedgelike domain $\mathscr{W}_{3} \subset \mathscr{W}_{1} \cap \mathscr{W}_{2}$ attached to $M \backslash C$ such that for every holomorphic function $f \in \mathscr{O}\left(\mathscr{W}_{1}\right)$, there exists a holomorphic function $F \in \mathscr{O}\left(\mathscr{W}_{2}\right)$ which coincides with $f$ in $\mathscr{W}_{3}$;
- $L^{\mathrm{p}}$-removable, where $1 \leqslant \mathrm{p} \leqslant \infty$, if every locally integrable function $f \in L_{l o c}^{\mathrm{p}}(M)$ which is CR in the distributional sense on $M \backslash C$ is in fact CR on all of $M$.

A few comments are welcome. CR-removability requires at least $M \backslash C$ to be globally minimal, in order that the main Theorem $4.12(\mathrm{~V})$ applies, yielding a wedgelike domain $\mathscr{W}_{1}$ attached to $M \backslash C$. Then $\mathscr{W}$-removability of $C$ implies its CR-removability. In both CR- and $\mathscr{W}$-removabililty, after the removal of $C$, nothing is demanded about the growth of the holomorphic extension to a global wedgelike domain $\mathscr{W}_{2}$ attached to $M$. Such extensions might well have essential singularities at some points of $C$, although they are holomorphic in $\mathscr{W}_{2}$. On the contrary, for $L^{\text {p }}$-removability of $C$, CR functions on $M \backslash C$ should really extend to be CR through $C$.

Notwithstanding this difference, the sequel will reveal that $L^{\mathrm{p}}$ removability is also a consequence of $\mathscr{W}$-removability, thanks to some Hardy-space control of the holomorphic extension $F \in \mathscr{O}\left(\mathscr{W}_{2}\right)$. In fact, functions are assumed to be $L_{l o c}^{\mathrm{p}}$ (a variant is to assume continuity on $M$ instead of integrability) even near points of $C$. This strong assumption enables to get a control of the growth of the wedge extension. Before providing more explanations, we assert in advance that $\mathscr{W}$-removability is the most general notion of removability, focusing the question on envelopes of holomorphy.

In codimension $d=1$, wedgelike domains identify to one-sided neighborhoods. Then $\mathscr{W}$-removability of $C$ means that the envelope of holomorphy of every (arbitrarily thin) one-sided neighborhood of $M \backslash C$ contains a complete one-sided neighborhood of the hypersurface $M$ in $\mathbb{C}^{n}$. If $M=\partial \Omega$ is the boundary of a bounded domain $\Omega \subset \mathbb{C}^{n}$ (having connected boundary), then $\mathscr{W}$-removability of a compact set $K \subset \partial \Omega$ entails its removability in the sense of Problem 3.2, thanks to Hartogs Theorem 1.8(V).

As in [Jö1999b, MP1999], we would like to emphasize that all the general theorems presented in Sections 3 and 4 are void for $L_{l o c}^{1}$ functions, or require a strong assumption of growth. On the contrary, the results that will be presented below hold in all spaces $L_{l o c}^{\mathrm{p}}$ with $1 \leqslant \mathrm{p} \leqslant \infty$, without any assumption of growth. The concept of $\mathscr{W}$-removability, interpreted as a result about envelopes of holomorphy, yields a (crucial) external drawing near the illusory singularity, an opportunity that is intrinsically attached to locally
embeddable Cauchy-Riemann structures, but is of course absent for general linear partial differential operators.
4.3. Removable singularities on hypersurfaces. In [LS1993], it is shown that if $\Omega \subset \mathbb{C}^{n}$ is a pseudoconvex bounded domain having $\mathscr{C}^{2}$ boundary, then every compact subset $K \subset \partial \Omega$ with $\mathrm{H}^{2 n-3}(K)=0$ is removable in the sense of Definition 3.4. In fact, Lemma 4.18(III) shows that $\partial \Omega$ is globally minimal and the next lemma shows that in codimension $d=1$, metrically thin singularities do not perturb global minimality.
Lemma 4.4. ([MP2002]) If $M \subset \mathbb{C}^{n}$ is a globally minimal $\mathscr{C}^{2}$ hypersurface, then for every closed set $C \subset M$ with $\mathrm{H}^{2 n-3}(C)=0$, the complement $M \backslash C$ is also globally minimal.
Example 4.5. However, this is untrue if $\mathrm{H}^{2 n-3}(C)>0$. Let $n \geqslant 2$ and $\varphi(z, u)$ be $\mathscr{C}^{2}$ defined for $|z|,|u|<1$ and satisfying $\varphi(z, 0) \equiv 0$ for $\operatorname{Re} z_{1} \leqslant 0$. Let $M \subset \mathbb{C}^{n}$ be the graph $v=\varphi(z, u)$ and define $C:=\left\{\left(i y_{1}, z_{2}, \ldots, z_{n-1}, 0\right)\right\}$. Clearly $\operatorname{dim} C=2 n-3, \mathrm{H}^{2 n-3}(C)>0$ and $\left\{(z, 0): \operatorname{Re} z_{1}<0\right\}$ is a single CR orbit $\mathscr{O}_{-}$of $M \backslash C$. Also, the function $\varphi$ may be chosen so that $M$ is of finite type at every point of $M \backslash \mathscr{O}_{-}$, whence $M \backslash C$ consists of exactly two CR orbits, namely $\mathscr{O}_{-}$and $M \backslash\left(\mathscr{O}_{-} \cup C\right)$. It follows that $M$ is globally minimal.

Theorem 4.6. ([LS1993, 6, MP1998, MP2002]) If $M \subset \mathbb{C}^{n}$ is a globally minimal $\mathscr{C}^{2, \alpha}(0<\alpha<1)$ hypersurface, then every closed set $C \subset M$ with $\mathrm{H}^{2 n-3}(C)=\mathrm{H}^{\operatorname{dim} M^{M-2}}(C)=0$ is locally $C R$-, $\mathscr{W}$ - and $L^{\mathrm{p}}$-removable.

Sometimes, we shall say that $C$ is of codimension $2^{+0}$ in $M$. This is a version of (rm1) and of (rm2) of Theorem 2.30 for CR functions on general hypersurfaces. Except for $L^{\text {p }}$-removability, refinements about smoothness assumptions may be found in [6].

The smallest (Hausdorff) dimension of $C \subset M \subset \mathbb{C}^{n}$ for which its removability may fail is equal to $2 n-3$. Indeed, if $C=M \cap \Sigma$ is equal to the intersection of $M$ with some local complex hypersurface $\Sigma=\{f=0\}$, the functions $1 / f^{k}, k \geqslant 1$ and $e^{1 / f}$ restrict to be CR on $M \backslash C$, but not holomorphically extendable to a one-sided neighborhood at points of $C$, since $\Sigma$ visits both sides of $M$. In such a situation, the real hypersurface $M \cap \Sigma$ of the complex hypersurface $\Sigma$ has dimension $(2 n-3)$ and CR dimension ( $n-2$ ).
Definition 4.7. A CR submanifold $N \subset \mathbb{C}^{n}$ is called maximally complex if it is of odd dimension satisfying $\operatorname{dim} N=1+2$ CRdim $N$.

Every real hypersurface of a complex manifold is maximally complex. The next step in to study singularities $C$ contained in $(2 n-3)$-dimensional submanifolds $N \subset M$.

Example 4.8. We show the necessity of assuming that $M \backslash C$ is also globally minimal ([MP1999]). Take the complex hypersurface $\mathscr{O}_{-}$of the preceding example having boundary $\partial \mathscr{O}_{-}=C=N$. Applying Proposition 4.38(III) to $S:=\mathscr{O}_{-}$, we may construct a measure on $M \backslash C$ supported by $\mathscr{O}_{-}$that is CR on $M \backslash C$ but does not extend holomorphically to a wedge at any point of $\overline{\mathscr{O}}_{-}=\mathscr{O}_{-} \cup C$, for the same reason as in Corollary 4.39(III).

Because of this example, we shall systematically assume that $M \backslash C$ is also globally minimal, if this is not a consequence of other hypotheses. Here is a CR version of (rm3) and of (rm4) of Theorem 2.30. It says that true singularities should be maximally complex. Before stating it, we point out that all submanifolds of given manifolds will constantly be assumed to be embedded submanifolds. Also, all subsets $C$ of a submanifold $N$ of manifold $M$ that are called closed are assumed to be closed both in $M$ and in $N$.

Theorem 4.9. ([Jö1992, Me1997, Jö1999a, Jö1999b]) Let $M \subset \mathbb{C}^{n}$ be a $\mathscr{C}^{2, \alpha}(0<\alpha<1)$ globally minimal hypersurface and let $N \subset M$ be a connected $\mathscr{C}^{2, \alpha}$ embedded submanifold of dimension ( $2 n-3$ ), viz. of codimension 2 in $M$. A closed set $C \subset N$ such that $M \backslash C$ is also globally minimal is $C R$-, $\mathscr{W}$ - and $L^{\mathrm{p}}$-removable under each one of the following two circumstances:
(i) $n \geqslant 2$ and $C \neq N$;
(ii) $n \geqslant 3$ and $C=N$ is not maximally complex, viz. there exists at least one point $p \in N$ at which $N$ is generic.

One may verify ([Jö1999a, MP1999]) that generic points of $N$ are locally removable and then after erasing them by deforming slightly $M$ inside the extensional wedge existing above, (ii) is seen to be a consequence of (i). For various smoothness refinements, the reader is referred to [Jö1992, 6, MP1998, Jö1999a, Jö1999b, MP1999]. One may also combine Theorem 4.6 and 4.9 , assuming that the submanifold $N$ is smooth, except perhaps at all points of some metrically thin closed subset. The proof will not be restituted.

The study of more massive singularities contained in ( $2 n-2$ )-dimensional submanifolds has been initiated by Jöricke ([Jö1988]), having in mind some generalization of Denjoy's approach to Painlevé's problem.
Theorem 4.10. ([Jö1999a, Jö1999b]) Let $M \subset \mathbb{C}^{n}$ be a $\mathscr{C}^{2, \alpha}(0<\alpha<$ 1) globally minimal hypersurface and let $M^{1} \subset M$ be a connected $\mathscr{C}^{2, \alpha}$ embedded submanifold ${ }^{30}$ of dimension $(2 n-2)$, viz. of codimension 1 in

[^28]$M$, that is generic in $\mathbb{C}^{n}$. If $n \geqslant 3$, a closed set $C \subset M^{1}$ is $C R$-, $\mathscr{W}$ - and $L^{\mathrm{p}}$-removable provided it does not contain any $C R$ orbit of $M^{1}$.

It may be established (see e.g. Lemma 3.3 in [26]) that $M^{1} \backslash C^{\prime}$ is also globally minimal for every closed $C^{\prime} \subset M^{1}$ containing no CR orbit of $M^{1}$.

We would like to mention that the removability of two-codimensional singularities (Theorem 4.9) is not a consequence of the removability of the bigger one-codimensional singularities (Theorem 4.10). Indeed, it may happen that $T_{p} N$ contains $T_{p}^{c} M$ at several points $p \in N$ in Theorem 4.9, preventing the existence of a generic $M^{1} \subset M$ containing $N$. In addition, even if $T_{p} N \not \supset T_{p}^{c} M$ for every $p \in N$, Theorem 4.9 is not anymore a corollary of Theorem 4.10. Indeed, with $m=2$ and $d=1$, choosing a local hypersurface $M \subset \mathbb{C}^{3}$ containing a complex curve $\Sigma$, choosing $N \subset M$ of dimension 3 containing $\Sigma$ and being maximally real outside $\Sigma$, and choosing an arbitrary generic $M^{1} \subset M$ containing $N$ (some explicit local defining equations may easily be written), then $\Sigma$ is a CR orbit of $M^{1}$, so $N \supset \Sigma$ is not considered to be removable by Theorem 4.10, whereas Theorem 4.9(ii) asserts that $N$ is removable.

Although singularities are more massive in Theorem 4.10, the assumption $n \geqslant 3$ in it entails that the CR dimension $(n-1)$ of $M$ is $\geqslant 2$, whence $M^{1}$ has positive CR dimension $\geqslant 1$. This insures the existence of small analytic discs with boundary in $M^{1}$. Section 5 below and [26] as a whole are devoted to the more delicate case where $M^{1}$ has null CR dimension.

Example 4.11. ([Jö1999a]) In $\mathbb{C}^{3}$, let $M=\partial \mathbb{B}_{3}$ and let $M^{1}:=\left\{\left(z_{1}, z_{2}, z_{3}\right)\right.$ : $\left.0<x_{1}<1 / 2, y_{1}=0\right\}$. Clearly, $M^{1}$ is foliated by the 3 -spheres

$$
S_{x_{1}^{*}}^{3}:=\left\{z_{1}=x_{1}^{*},\left|z_{2}\right|^{2}+\left|z_{3}\right|^{2}=1-\left|x_{1}^{*}\right|^{2}\right\},
$$

$x_{1}^{*} \in(0,1 / 2)$, that are globally minimal compact 3 -dimensional strongly pseudoconvex maximally complex CR submanifolds of CR dimension 1 bounding the 2 -dimensional complex balls

$$
\mathbb{B}_{2, x_{1}^{*}}:=\left\{z_{1}=x_{1}^{*},\left|z_{2}\right|^{2}+\left|z_{3}\right|^{2}<1-\left|x_{1}^{*}\right|^{2}\right\} .
$$

Theorem 4.10 asserts that a compact set $K \subset M^{1}$ is removable if and only if it does not contain a whole sphere $S_{x_{1}^{*}}$, for some $x_{1}^{*} \in(0,1 / 2)$. If $K$ contains such a sphere $S_{x_{1}^{*}}^{3}$, the complex 2-ball $\mathbb{B}_{2, x_{1}^{*}}$ coincides with the $A(\Omega)-$ hull of $S_{x_{1}^{*}}$ and is nonremovable. More generally, an application of both Theorems 4.10 and 3.25 yields the following.
Corollary 4.12. Let $K$ be a compact subset of $M^{1}$. For every (interior) onesided neighborhood $\mathscr{V}^{-}\left(\partial \mathbb{B}_{3} \backslash K\right)$ that is contained in $\mathbb{B}_{3}$ and every function

[^29]$f$ holomorphic in $\mathscr{V}^{-}\left(\partial \mathbb{B}_{3} \backslash K\right)$, there exists a function $F$ holomorphic in $\mathbb{B}_{3} \backslash \bigcup_{x_{1}^{*}: S_{x_{1}^{*}}^{3} \subset K} \mathbb{B}_{2, x_{1}^{*}}$ with $F=f$ in $\mathscr{V}^{-}\left(\partial \mathbb{B}_{3} \backslash K\right)$.

By means of the complex Plateau problem, the next paragraph discusses the necessity for $N$ not to be maximally complex in Theorem 4.9 and for $M^{1}$ not to contain any CR orbit in Theorem 4.10, in a more general context than $M=\partial \mathbb{B}_{n}$.
4.13. Complex Plateau problem and nonremovable singularities contained in strongly pseudoconvex boundaries. Let $\mathscr{M}$ be a complex manifold of dimension $n \geqslant 2$. If $\Sigma \subset \mathscr{M}$ is a closed pure $k$-dimensional complex subvariety, we denote by $[\Sigma]$ the current of integration on $\Sigma$, whose existence was established by Lelong in 1957 ([Ch1989, 7]).

Definition 4.14. ([HL1975, Ha1977]) A current T on $\mathscr{M}$ is called a holomorphic $k$-chain if it is of the form

$$
\mathrm{T}=\sum_{\text {finite }} n_{j}\left[\Sigma_{j}\right],
$$

where the $\Sigma_{j}$ denote the irreducible components of a pure $k$-dimensional complex subvariety $\Sigma$ of $\mathscr{M}$ and where the multiplicity $n_{j}$ of each $\Sigma_{j}$ is an integer.

The complex Plateau problem consists in filling boundaries $N$ by complex subvarieties $\Sigma$, or more generally by holomorphic chains T. Maximal complexity of the boundary $N$ is naturally required and since $N$ might encounter singular points of $\Sigma$, it should be allowed in advance to be "scarred" somehow. Also, the boundary $N$ inherits an orientation from $\Sigma$ and as the boundary of $\Sigma$, it should have empty boundary.

Definition 4.15. A scarred $\mathscr{C}^{\kappa}(1 \leqslant \kappa \leqslant \infty)$ maximally complex cycle of dimension $(2 m+1), m \geqslant 0$, is a compact subset $N \subset \mathscr{M}$ together with a thin compact scar set $\mathrm{sc}_{N} \subset N$ such that

- $\mathrm{H}^{2 m+1}\left(\mathbf{s c}_{N}\right)=0$;
- $N \backslash \mathbf{~ s c}_{N}$ is an oriented $(2 m+1)$-dimensional embedded maximally complex $\mathscr{C}^{\kappa}$ submanifold of $\mathscr{M} \backslash \mathbf{S C}_{N}$ having finite $(2 m+1)$ dimensional Hausdorff measure;
- the current of integration over $N \backslash \mathbf{s c}_{N}$, denoted by $[N]$, has no boundary: $d[N]=0$.

This definition was essentially devised by Harvey-Lawson and appears to be adequately large, but sufficiently stringent to maintain the possibility of filling a maximally complex cycle by a complex analytic set.

Theorem 4.16. ([HL1975, Ha1977]) Suppose $N$ is a scarred $\mathscr{C}^{\kappa}(1 \leqslant \kappa \leqslant$ $\infty$ ) maximally complex cycle of dimension $(2 m+1), m \geqslant 0$, in a Stein manifold $\mathscr{M}$.

- If $m=0$, assume that $N$ satisfies the moment condition, viz. $\int_{N} \omega=0$ for every holomorphic 1 -form $\omega=\sum_{k=1}^{n} \omega_{k}(z) d z_{k}$ having entire coefficients $\omega_{k} \in \mathscr{O}\left(\mathbb{C}^{n}\right)$.
- If $m \geqslant 1$, assume nothing, since the corresponding appropriate moment condition follows automatically from the assumption of maximal complexity ([HL1975]).
Then there exists a unique holomorphic ( $m+1$ )-chain T in $\mathscr{M} \backslash N$ having compact support and finite mass in $\mathscr{M}$ such that

$$
d \mathrm{~T}=[N]
$$

in the sense of currents in $\mathscr{M}$. Furthermore, there is a compact subset $K$ of $N$ with $\mathrm{H}^{2 m+1}(K)=0$ such that every point of $N \backslash\left(K \cup \mathbf{s c}_{N}\right)$ possesses a neighborhood in which $(\operatorname{supp} \mathrm{T}) \cup N$ is a regular $\mathscr{C}^{\kappa}$ complex manifold with boundary.

A paradigmatic example, much considered since Milnor studied it, consists in intersecting a complex algebraic subvariety of $\mathbb{C}^{n}$ passing through the origin with a spere centered at 0 ; topologists usually require that 0 is an isolated singularity and that the sphere is small or that the defining polynomial is homogeneous.

We apply this filling theorem in a specific situation. Let $\partial \Omega \Subset \mathbb{C}^{n}(n \geqslant 3)$ be a strongly pseudoconvex $\mathscr{C}^{2}$ boundary and let $M^{1} \subset \partial \Omega$ be an embedded $\mathscr{C}^{2}$ one-codimensional submanifold that is generic in $\mathbb{C}^{n}$. We assume that $M^{1}$ has no boundary and is closed, viz. is a compact submanifold. Since $M^{1}$ has CR dimension $(n-2)$, its CR orbits have dimension equal to either $(2 n-4)$, or to $(2 n-3)$ or to $(2 n-2)$. Because of Corollary 4.19(III), no CR orbit of $M^{1}$ can be an immersed complex 2-codimensional submanifold, of real dimension $(2 n-4)$, since its closure in $M^{1}$ would be a compact set laminated by complex manifolds. Nevertheless, there may exist $(2 n-3)$ dimensional CR orbits.

Proposition 4.17. ([Jö1999a]) Every CR orbit $\mathscr{O}_{C R}^{1}$ of a connected $\mathscr{C}^{2}$ hypersurface $M^{1} \subset \partial \Omega$ of a $\mathscr{C}^{2}$ strongly pseudoconvex boundary $\partial \Omega \Subset \mathbb{C}^{n}$ is of the following types:
(i) $\mathscr{O}_{C R}^{1}$ is an open subset of $M$;
(ii) $\mathscr{O}_{C R}^{1}$ is a closed maximally complex $\mathscr{C}^{1}$ cycle embedded in $M^{1}$;
(iii) $\mathscr{O}_{C R}^{1}$ is a maximally complex $\mathscr{C}^{1}$ submanifold injectively immersed in $M^{1}$ whose closure $C$ consists of an uncountable union of similar CR orbits.

In the last situation, $C$ will be called a maximally complex exceptional minimal compact $C R$-invariant set. The intersection of $C$ with a local curve transversal to a piece CR orbit in $M^{1}$ may consist of either an open segment or of a Cantor (perfect) subset.

Here is the desired converse to both Theorems 4.9 and 4.10 in a situation where the Plateau complex filling works.
Corollary 4.18. ([Jö1999a]) Suppose that $\partial \Omega \in \mathscr{C}^{2, \alpha}$ contains a compact embedded ( $2 n-3$ )-dimensional maximally complex submanifold $N$ (without boundary). Then $N$ is not removable.

Proof. Indeed, the scar set of $N$ is empty and the filling of $N$ by a holomorphic chain consists of an irreducible complex subvariety $\Sigma$ that is necessarily contained in $\Omega$, since $\partial \Omega$ is strongly pseudoconvex. Then the domain $\Omega \backslash \Sigma$ is seen to be pseudoconvex and $\widehat{N}_{A(\Omega)}=N \cup \Sigma$. Theorem 3.25 entails that CR functions on $\partial \Omega \backslash N$ extend holomorphicaly to $\Omega \backslash \Sigma$.

A very natural problem, raised in [Jö1999a] and inspired by a perturbation of Example 4.11, is to determine for which compact CR-invariant subsets $K$ of a strongly pseudoconvex boundary $\partial \Omega \subset \mathbb{C}^{n}$ the envelope of holomorphy of $\partial \Omega \backslash K$ is multi-sheeted.

Theorem 4.19. ([JS2004]) Let $M^{1} \subset \partial \mathbb{B}_{n}$ be an orientable ( $2 n-2$ )dimensional generic $\mathscr{C}^{2, \alpha}$ submanifold of $\partial \mathbb{B}_{n}(n \geqslant 3)$ and let $K \subset M^{1}$ be a compact $C R$-invariant subset of $M^{1}$ such that

- the boundary of $K$ in $M^{1}$ is the disjoint union of finitely many connected compact maximally complex $C R$ manifolds $N_{1}, \ldots, N_{\ell}$ of dimension $(2 n-3)$ that are $\mathscr{C}^{2, \alpha-0} C R$ orbits of $M^{1}$;
- the interior of $K$ with respect to $M^{1}$ is globally minimal.

Then the envelope of holomorphy $\mathrm{E}\left(\mathscr{V}\left(\partial \mathbb{B}_{n} \backslash K\right)\right)$ is multi-sheeted in every neighborhood $\bar{U}_{p} \subset \overline{\mathbb{B}_{n}}$ of every point $p \in \operatorname{Int} K$.

We conclude these considerations by formulating a deeply open problem raised by Jöricke. The complex Plateau problem for laminated boundaries is a virgin mathematical landscape.

Open question 4.20. ([Jö1999a]) Let $\partial \Omega \Subset \mathbb{C}^{n}, n \geqslant 3$, be a strongly pseudoconvex boundary of class at least $\mathscr{C}^{2}$. Suppose that $\partial \Omega$ contains a maximally complex exceptional minimal compact CR-invariant set $C$. Does $C$ bound a relatively compact subset $\Sigma \subset \Omega$ laminated by complex manifolds ?

As observed in [DH1997, MP1998, Sa1999, DS2001], removable singularities have an unexpected interesting application to wedge extension of CR-meromorphic functions.
4.21. CR-meromorphic functions and metrically thin singularities. For $n \geqslant 2$, a local meromorphic map $f$ from a domain $\Omega \subset \mathbb{C}^{n}$ to the Riemann sphere $P_{1}(\mathbb{C})$ has an exceptional locus $I_{f} \subset \Omega$, at every point $p$ of which the value $f(p)$ is undefined. For instance the origin $(0,0) \in \mathbb{C}^{2}$ with $f=$ $\frac{z_{1}}{z_{2}}$ (notice that every complex number in $\mathbb{C} \cup\{\infty\}$ is a limit of $\frac{z_{1}}{z_{2}}$ ). This exceptional set $I_{f}$ is a complex analytic subset of $\Omega$ having codimension $\geqslant 2$ ([7]). It is called the indeterminacy set of $f$.

A meromorphic function may be more conveniently defined as a $n$ dimensional irreducibe complex analytic subset $\Gamma_{f}$ of $\Omega \times P_{1}(\mathbb{C})$ having surjective projection onto $\Omega$, viz. $\pi_{\Omega}\left(\Gamma_{f}\right)=\Omega$. Here, $\Omega$ might be any complex manifold. Indeterminacy points correspond precisely to points $p \in \Omega$ satisfying $\pi_{\Omega}^{-1}(p) \cap \Gamma_{f}=\{p\} \times P_{1}(\mathbb{C})$. So, the generalization of meromorphy to the CR category incorporates indeterminacy points.
Definition 4.22. ([HL1975, DH1997, MP1998, Sa1999]) Let $M \subset \mathbb{C}^{n}$ be a scarred $\mathscr{C}^{1}$ generic submanifold of codimension $d \geqslant 1$ and of CR dimension $m=n-d \geqslant 1$. Then a $C R$ meromorphic function on $M$ with values in $P_{1}(\mathbb{C})$ consists of a triple $\left(f, \mathscr{D}_{f}, \Gamma_{f}\right)$ such that:

1) $\mathscr{D}_{f} \subset M$ is a dense open subset of $M$ and $f: \mathscr{D}_{f} \rightarrow P_{1}(\mathbb{C})$ is a $\mathscr{C}^{1}$ map;
2) the closure $\Gamma_{f}$ in $\mathbb{C}^{n} \times P_{1}(\mathbb{C})$ of the graph $\left\{(p, f(p)): p \in \mathscr{D}_{f}\right\}$ defines an oriented scarred $\mathscr{C}^{1}$ CR submanifold of $\mathbb{C}^{n} \times P_{1}(\mathbb{C})$ of the same CR dimension as $M$ having empty boundary in the sense of currents.
The indeterminacy locus of $f$ is denoted by

$$
I_{f}:=\left\{p \in M:\{p\} \times P_{1}(\mathbb{C}) \subset \Gamma_{f}\right\} .
$$

In the CR category, $I_{f}$ is not as thin as in the holomorphic category (where it has real codimension $\geqslant 4$ ), but it is nevertheless thin enough for future purposes, as we shall see. A standard argument from geometric measure theory yields almost everywhere smoothness of almost every level set.
Lemma 4.23. ([Fe1969, HL1975, Ha1977]) Let $M \subset \mathbb{C}^{n}$ be a scarred $\mathscr{C}^{1}$ generic submanifold. Let $\left(f, \mathscr{D}_{f}, \Gamma_{f}\right)$ be a CR meromorphic function on $M$. Then for almost every $w \in P_{1}(\mathbb{C})$, the level set

$$
N_{f}(w):=\left\{p \in M:(p, w) \in \Gamma_{f}\right\}
$$

is a scarred 2-codimensional $\mathscr{C}^{1}$ submanifold of $M$.
Let $p \in I_{f}$. Since $(p, w) \in \Gamma_{f}$ for every $w \in P_{1}(\mathbb{C})$, it follows that $I_{f} \subset N_{f}(w)$ for every $w$. Fixing such a $w \in P_{1}(\mathbb{C})$, we simply denote $N_{f}:=$ $N_{f}(w)$. In particular, the scar set $\mathbf{S C}_{N_{f}}$ of $N_{f}$ is always of codimension $2^{+0}$ in $M$, namely $\mathrm{H}^{\operatorname{dim} M-2}\left(\mathrm{sc}_{N_{f}}\right)=0$.

So $I_{f} \subset N_{f}$ and by definition $I_{f} \times P_{1}(\mathbb{C}) \subset \Gamma_{f}$. We claim that, in addition, $I_{f}$ has empty interior in $N_{f} \backslash \mathbf{S C}_{N_{f}}$. Otherwise, there exist a point $p \in N_{f} \backslash \mathbf{S C}_{N_{f}}$ and a neighborhood $U_{p}$ of $p$ in $M$ with $U_{p} \cap \mathbf{s C}_{N_{f}}=\emptyset$ such that $I_{f}$ contains $U_{p} \cap N_{f}$, whence

$$
\left(U_{p} \cap N_{f}\right) \times P_{1}(\mathbb{C}) \subset \Gamma_{f}
$$

Since $\left(U_{p} \cap N_{f}\right) \times P_{1}(\mathbb{C})$ has dimension equal to $\operatorname{dim} M=\operatorname{dim} \Gamma_{f}$, it follows that

$$
\Gamma_{f} \cap\left(U_{p} \times P_{1}(\mathbb{C})\right) \equiv\left(U_{p} \cap N_{f}\right) \times P_{1}(\mathbb{C})
$$

But $U_{p} \cap N_{f}$ having codimension two in $U_{p}$, this contradicts the assumption that $\Gamma_{f}$ is a (nonempty!) graph above the dense open subset $U_{p} \cap \mathscr{D}_{f}$ of $U_{p}$.
Lemma 4.24. ([MP1998, Sa1999]) The indeterminacy set $I_{f}$ of $f$ is a closed set of empty interior contained in some 2-codimensional scarred $\mathscr{C}^{1}$ submanifold $N_{f}$ of $M$. Moreover, the scar set $\mathbf{S C}_{N_{f}}$ of $N_{f}$ is always of codimension $2^{+0}$ in $M$, viz. $\mathrm{H}^{2 m+d-2}\left(\mathbf{S C}_{N_{f}}\right)=0$.

The statement below and its proof are clear if $\mathscr{D}_{f}=M$; in it, the condition $d\left[\Gamma_{f}\right]=0$ helps in an essential way to keep it true when the closure of $\Gamma_{f}$ possesses a nonempty scar set.

Proposition 4.25. ([MP1998, Sa1999]) There exists a unique CR measure $\mathrm{T}_{f}$ on $M \backslash I_{f}$ with $\left.\mathrm{T}_{f}\right|_{\mathscr{D}_{f}}$ coinciding with the $\mathscr{C}^{1} C R$ function $f: \mathscr{D}_{f} \rightarrow$ $P_{1}(\mathbb{C})$.

It is defined locally as follows. Let $p \in M \backslash I_{f}$ and let $U_{p}$ be an open neighborhood of $p$ in $M$. Since $p \notin I_{f}$, there exists $w_{p} \in P_{1}(\mathbb{C})$ with $\left(p, w_{p}\right) \notin \Gamma_{f}$. Composing with an automorphism of $P_{1}(\mathbb{C})$ and shrinking $U_{p}$, we may assume that $w_{p}=\infty$ and that $\left(U_{p} \times\{\infty\}\right) \cap \Gamma_{f}=\emptyset$. Letting $d \mathrm{Vol}_{U_{p}}$ be some $(2 m+d)$-dimensional volume form on $U_{p}$, letting $\pi_{\Gamma_{f}}: \Gamma_{f} \rightarrow M$ denote the natural projection, the CR measure $\left.\mathrm{T}_{f}\right|_{U_{p}}$ is defined by

$$
\left\langle\mathbf{T}_{f}, \varphi\right\rangle:=\int_{\Gamma_{f}} w \cdot \pi_{\Gamma_{f}}^{*}\left(\varphi d \operatorname{Vol}_{U_{p}}\right)
$$

for every $\varphi \in \mathscr{C}_{c}^{1}\left(U_{p}\right)$.
Thus, on $M \backslash I_{f}$, the CR-meromorphic function $\left(f, \mathscr{D}_{f}, \Gamma_{f}\right)$ behaves like an order zero CR distribution. With $\mathscr{C}_{C R}^{0}, L_{C R, l o c}^{\mathrm{p}}$, it therefore enjoys the extendability properties of Part V on $M \backslash I_{f}$, provided that $M$ is $\mathscr{C}^{2, \alpha}$. The next theorem should be applied to $C:=I_{f}$. Its final proof ([MP2002]) under the most general assumptions combines both the CR extension theory and the application of the Riemann-Hilbert problem to global discs attached to maximally real submanifolds ([Gl1994, Gl1996]). We cannot restitute the proof here.

Theorem 4.26. ([MP1998, DS2001, MP2002]) Suppose $M \subset \mathbb{C}^{n}$ is $\mathscr{C}^{2, \alpha}$ $(0<\alpha<1)$ of codimension $d \geqslant 1$ and of $C R$ dimension $m \geqslant 1$. Then every closed subset $C$ of $M$ such that $M$ and $M \backslash C$ are globally minimal and such that $\mathrm{H}^{2 m+d-2}(C)=0$ is $C R$-, $\mathscr{W}$ - and $L^{\mathrm{p}}$-removable.

However, if $f$ is a CR-meromorphic function defined on such a $M$, with $\mathscr{C}^{1}$ replaced by $\mathscr{C}^{2, \alpha}$ in Definition 4.22, the complement $M \backslash I_{f}$ need not be globally minimal if $M$ is, and it is easy to construct manifolds $M$ and closed sets $C \subset M$ with $\mathrm{H}^{2 m-1}(C)<\infty$ which perturb global minimality, cf. Example 4.8. It is therefore natural to make the additional assumption that $M$ is locally minimal at every point. This assumption is the weakest one that insures that $M \backslash C$ is globally minimal, for arbitrary closed sets $C \subset M$.
Corollary 4.27. Assume that $M \in \mathscr{C}^{2, \alpha}$ is locally minimal at every point and let $f$ be a $C R$-meromorphic function. Then $I_{f}$ is $C R$-, $\mathscr{W}$ - and $L^{\mathrm{p}}$ removable.
Proof. Lemma 4.24 holds with $\mathscr{C}^{1}$ replaced by $\mathscr{C}^{2, \alpha}$. It says that $I_{f}$ is a closed subset with empty interior of some scarred $\mathscr{C}^{2, \alpha}$ submanifold $N_{f}$ of $M$. The removability of the portion of $I_{f}$ that is contained in the regular part of $N_{f}$ follows from Theorem 4.9(i). The removability of the remaining scar set $\mathrm{SC}_{N_{f}}$ follows from Theorem 4.26 above.

Thus the CR measure $\mathrm{T}_{f}$ on $M \backslash I_{f}$ (Proposition 4.25) extends holomorphically to some wedgelike domain $\mathscr{W}_{1}$ attached to $M \backslash I_{f}$. The $\mathscr{W}$ removability of $I_{f}$ entails that the envelope of holomorphy of $\mathscr{W}_{1}$ contains a wedgelike domain $\mathscr{W}_{2}$ attached to $M$. Performing supplementary gluing of discs, the CR extension theory (Part V) insures that such a $\mathscr{W}_{2}$ depends only on $M$, not on $f$. As envelopes of meromorphy and envelopes of holomorphy of domains in $\mathbb{C}^{n}$ coincide by a theorem going back to Levi ([KS1967, Iv1992]), we may conclude.
Theorem 4.28. ([MP2002]) Suppose $M \subset \mathbb{C}^{n}$ is $\mathscr{C}^{2, \alpha}$ and locally minimal at every point. Then there exists a wedgelike domain $\mathscr{W}$ attached to $M$ to which every CR-meromorphic function on $M$ extends meromorphically.
4.29. Peak and smooth removable singularities in arbitrary codimen-
sion. A closed set $C \subset M$ is called a $\mathscr{C}^{0, \beta}$ peak set, $0<\beta<1$, if there exists a nonconstant function $\varpi \in \mathscr{C}_{C R}^{0, \beta}(M)$ such that $C=\{\varpi=1\}$ and $\max _{p \in M}|\varpi(p)| \leqslant 1$.
Theorem 4.30. ([KR1995, MP1999]) Let $M$ be $\mathscr{C}^{2, \alpha}(0<\alpha<1)$ globally minimal. Then every $\mathscr{C}^{0, \beta}$ peak set $C$ satisfies $\mathrm{H}^{\operatorname{dim} M}(C)=0$ and is $L^{\mathrm{p}}$ removable.

To conclude, we mention two precise generalizations of Theorems 4.9 and 4.10 to higher codimension. If $\Sigma=\{z: g(z)=0\}$ is a local complex
hypersurface passing through a point $p$ of a generic submanifold $M \subset \mathbb{C}^{n}$ that is transverse to $M$ at $p$, viz. $T_{p} \Sigma+T_{p} M=T_{p} \mathbb{C}^{n}$, the intersection $\Sigma \cap M$ is a two-codimensional submanifold of $M$ that is nowhere generic in a neighborhood of $p$ and certainly not (locally) removable, since the CR function $\left.\frac{1}{g(z)}\right|_{M \backslash(\Sigma \cap M)}$ is not extendable to any local wedge at $p$.

Theorem 4.31. ([Me1997, MP1999]) Let $M \subset \mathbb{C}^{n}$ be a $\mathscr{C}^{2, \alpha}(0<\alpha<1)$ globally minimal generic submanifold of positive codimension $d \geqslant 1$ and of positive $C R$ dimension $m=n-d \geqslant 1$. Let $N \subset M$ be a connected twocodimensional $\mathscr{C}^{2, \alpha}$ submanifold and assume that $M \backslash N$ is also globally minimal. A closed set $C \subset N$ is $C R$-, $\mathscr{W}$ - and $L^{\mathrm{p}}$-removable under each one of the following two circumstances:
(i) $m \geqslant 1$ and $C \neq N$;
(ii) $m \geqslant 2$ and there exists at least one point $p \in N$ at which $N$ is generic.

In (ii), the assumption that $m \geqslant 2$ is essential. Generally, if $m=1$, whence $d=n-1$ and $\operatorname{dim} M=n+1$, a local transverse intersection $C=\Sigma \cap M$ has dimension $n-1$, hence cannot be generic, and is not (locally) removable by construction. In the next statement, the similar assumption that $m \geqslant 2$ is strongly used in the proof: the one-codimensional submanifold $M^{1} \subset M$ has then CR dimension $m-1 \geqslant 1$, hence there exist small Bishop discs attached to $M^{1}$.

Theorem 4.32. ([Po1997, Me1997, Po2000]) Let $M \subset \mathbb{C}^{n}$ be a $\mathscr{C}^{2, \alpha}(0<$ $\alpha<1$ ) globally minimal generic submanifold of positive codimension $d \geqslant$ 1. Assume that the CR dimension $m=n-d$ of $M$ satisfies $m \geqslant 2$. Let $M^{1} \subset$ $M$ be a connected $\mathscr{C}^{2, \alpha}$ one-codimensional submanifold that is generic in $\mathbb{C}^{n}$. A closed set $C \subset M^{1}$ is $C R$-, $\mathscr{W}$ - and $L^{\mathrm{p}}$-removable provided it does not contain any $C R$ orbit of $M^{1}$.

Three geometrically different proofs of this theorem will be restituted in Section 10 of [26]. The next Section 5 and [26] are devoted to the study of the more delicate case where $m=1$ and where $C$ is contained in some one-codimensional submanifold $M^{1} \subset M$.

## §5. Removable singularities in CR Dimension 1

5.1. Removability of totally real discs in strongly pseudoconvex boundaries. In 1988, applying a global version of the Kontinuitätssatz, Jöricke [Jö1988] established a remarkable theorem, opening the way to a purely geometric study of removable singularities.

Theorem 5.2. ([Jö1988]) Let $\partial \Omega \Subset \mathbb{C}^{2}$ be a strongly pseudoconvex $\mathscr{C}^{2}$ boundary and let $D \subset \partial \Omega$ be a $\mathscr{C}^{2}$ one-codimensional submanifold that is diffeomorphic to the unit open 2 -disc of $\mathbb{R}^{2}$ and maximally real at every point. Then every compact subset $K$ of $D$ is $C R$-, $L^{\infty}$ and $\mathscr{W}$-removable.

By maximal reality of $D$, the line distribution $D \ni p \mapsto \ell_{p}:=T_{p} D \cap T_{p}^{c} M$ is nowhere vanishing and may be integrated. This yields the characteristic foliation $\mathscr{F}_{D}^{c}$ on $D$. The compact set $K$ is contained in a slightly smaller disc $D^{\prime} \Subset D$ having $\mathscr{C}^{2}$ boundary $\partial D^{\prime}$. Poincaré-Bendixson's theorem on such a disc $D^{\prime}$ together with the inexistence of singularities of $\mathscr{F}_{D}^{c}$ entail that every characteristic curve that enters into $D^{\prime}$ must exit from $D^{\prime}$. Orienting then the real 2 -disc $D$ and its characteristic foliation, we have the following topological observation (at the very core of the theorem) saying that there always exists a characteristic leaf that is not crossed by the removable compact set.

$\mathscr{F}_{D}^{c}\{K\}:$ For every compact subset $K^{\prime} \subset K$, there exists a Jordan curve $\gamma:[-1,1] \rightarrow D$, whose range is contained in a single leaf of the characteristic foliation $\mathscr{F}_{D}^{c}$, with $\gamma(-1) \notin K^{\prime}, \gamma(0) \in K^{\prime}$ and $\gamma(1) \notin K^{\prime}$, such that $K^{\prime}$ lies completely in one closed side of $\gamma[-1,1]$ with respect to the topology of $D$ in a neighborhood of $\gamma[-1,1]$.
In the more general context of [26], we will argue that $\mathscr{F}_{D}^{c}\{K\}$ is the very reason why $K$ is removable. We will then remove locally a well chosen special point $p_{\mathrm{sp}}^{\prime} \in K^{\prime} \cap \gamma[-1,1]$. In fact, we shall establish removability of compact subsets $K$ of general surfaces $S$ that are not necessarily diffeomorphic to the unit 2-disc, provided that an analogous topological condition holds. Also, getting rid of strong pseudoconvexity, we shall work with a globally minimal $\mathscr{C}^{2, \alpha}$ hypersurface of $\mathbb{C}^{2}$. Finally, we shall relax slightly the assumption of total reality, admitting some complex tangencies.

Example 5.3. Let $\Omega=\mathbb{B}_{2}$ and let $P(z) \in \mathbb{C}[z]$ be a homogeneous polynomial of degree $\geqslant 2$ having 0 has its only singularity. The intersection $K:=\partial \mathbb{B}_{2} \cap\{P=0\}$ is a finite union of closed real algebraic curves $\simeq S^{1}$ that are everywhere transverse to $T^{c} \partial \mathbb{B}_{2}$. We may enlarge each curve of $K$ as a thin $\mathscr{C}^{\omega}$ annulus. There is much freedom, but every such annulus is necessarily totally real. Denote by $S$ the union of all annuli, a surface in $\partial \mathbb{B}_{2}$. Clearly, no component of $K$ is removable. But the theorem does not apply: on each annulus, the characteristic foliation $\mathscr{F}_{S}^{c}$ is radial and $K$ crosses each characteristic leaf.

Example 5.4. The theorem may fail with the disc $D$ replaced by a surface $S$ having nontrivial fundamental group, even with $S$ compact without boundary. For instance, in $\partial \mathbb{B}_{2}=\left\{\left|z_{1}\right|^{2}+\left|z_{2}\right|^{2}=1\right\}$, the two-dimensional torus $T_{2}:=\left\{\left(\frac{1}{\sqrt{2}} e^{i \theta_{1}}, \frac{1}{\sqrt{2}} e^{i \theta_{2}}\right): \theta_{1}, \theta_{2} \in \mathbb{R}\right\}$ is compact and $K:=T_{2}$ is not removable, since $\partial \mathbb{B}_{2} \backslash T_{2}$ has exactly two connected components.

Example 5.5. ([Jö1988]) In the same torus $T_{2}$, consider instead the proper compact subset $K:=\left\{\left(\frac{1}{\sqrt{2}} e^{i \theta_{1}}, \frac{1}{\sqrt{2}} e^{i \theta_{2}}\right):\left|\theta_{1}\right| \leqslant \frac{3 \pi}{2}, \theta_{2} \in \mathbb{R}\right\}$, diffeomorphic to a closed annulus. It is a set fibered by circles (contained in $\mathbb{C}_{z_{2}}$ ) over the curve $\widehat{\gamma}:=\left\{\frac{1}{\sqrt{2}} e^{i \theta_{1}}:\left|\theta_{1}\right| \leqslant \frac{3 \pi}{2}\right\}$ that is contained in $\mathbb{C}_{z_{1}}$. One may verify that the condition $\mathscr{F}_{T_{2}}^{c}\{K\}$ insuring removability does not hold. In fact, applying Theorem 2.2 (in the much simpler version due to Denjoy where the curve is real analytic), the curve $\widehat{\gamma}$ is not $\left(\bar{\partial}, L^{\infty}\right)$-removable in $\mathbb{C}_{z_{1}}$. So we may pick a holomorphic function $\widehat{f}\left(z_{1}\right) \in \mathscr{O}(\mathbb{C} \backslash \widehat{\gamma})$ that is bounded in $\mathbb{C} \cup\{\infty\}$ but does not extend holomorphically through $\widehat{\gamma}$. The restriction $\left.\widehat{f}\right|_{\partial \mathbb{B}_{2} \backslash K}$ belongs to $L^{\infty}\left(\partial \mathbb{B}_{2}\right)$, is CR on $\partial \mathbb{B}_{2} \backslash K$ but does not extend holomorphically to $\mathbb{B}_{2}$.

Before pursuing, we compare Theorem 5.2 and Theorem 4.10.
In codimension $\geqslant 2\left(e . g\right.$. for curves in $\left.\mathbb{R}^{3}\right)$, no satisfactory generalization of the Poincaré-Bendixson theory is known and perhaps is out of reach. This gap is caused by the complexity of the topology of phase diagrams, by the freedom that curves have to wind wildly around limit cycles, and by the intricate structure of singular points.

Nevertheless, in higher complex dimension $n \geqslant 3$, CR orbits are thicker than curves and often of codimension $\leqslant 1$. For triples $\left(M, M^{1}, C\right)$ as in Theorem 4.10 with $M=\partial \Omega$ being strongly pseudoconvex, one could expect that a statement analogous to Theorem 5.2 holds true, in which the assumption that $M^{1}$ has simple topology would imply automatic removability of every compact subset $K \subset M^{1}$.

To be precise, let $\partial \Omega \Subset \mathbb{C}^{n}(n \geqslant 3)$ be a $\mathscr{C}^{2, \alpha}$ strongly pseudoconvex boundary and let $M^{1} \subset \partial \Omega$ be a $\mathscr{C}^{2, \alpha}$ one-codimensional submanifold that is generic in $\mathbb{C}^{n}$. Strong pseudoconvexity of $\partial \Omega$ entails that CR orbits of $M^{1}$
are necessarily of codimension $\leqslant 1$ in $M^{1}$. Remind that Theorem 4.10 says that a compact subset $K$ of $M^{1}$ is removable provided it does not contain any CR orbit of $M^{1}$. Conversely, in the case where $M^{1}$ has no exceptional CR orbit, if $K$ contains a (then necessarily compact and maximally complex) CR orbit $N$ of $M^{1}$, then $K$ is not removable, since $N$ is fillable by some ( $n-1$ )-dimensional complex subvariety $\Sigma \subset \Omega$ with $\partial \Sigma=N$. Thus, while comparing the two Theorems 4.10 and 5.2 , the true question is whether the assumption that $M^{1} \subset \partial \Omega=M$ be diffeomorphic to the real $(2 n-2)$ dimensional real ball $B^{2 n-2} \subset \mathbb{R}^{2 n-2}$ prevents the existence of compact ( $2 n-3$ )-dimensional CR orbits of $M^{1}$. This would yield a neat statement, valid in arbitrary complex dimension.

For instance, let $N:=\partial \mathbb{B}_{n} \cap H$ be the intersection of the sphere $\partial \mathbb{B}_{n} \simeq$ $S^{2 n-1}$ with a complex linear hyperplane $H \subset \mathbb{C}^{n}$. With such a simple $N$ homeomorphic to a $(2 n-3)$-dimensional sphere, one may verify that every $\mathscr{C}^{\infty}$ submanifold $M^{1} \subset \partial \Omega$ containing $N$ which is diffeomorphic to $B^{2 n-2}$ must contain at least one nongeneric point. Nevertheless, admitting that $N$ has slightly more complicated topology, the expected generalization of Theorem 5.2 appears to fail, according to a discovery of Jöricke-Shcherbina. This confirms the strong differences between CR dimension $m=1$ and $\mathbf{C R}$ dimension $m \geqslant 2$.

Theorem 5.6. ([JS2000]) For $\varepsilon \in \mathbb{R}$ with $0<\varepsilon<1$ close to 1 , consider the intersection

$$
N_{\varepsilon}:=\left\{z_{1} z_{2} z_{3}=\varepsilon\right\} \cap \sqrt{3} \partial \mathbb{B}_{3}
$$

of the complex cubic $\left\{z_{1} z_{2} z_{3}=\varepsilon\right\}$ with the sphere $\sqrt{3} \partial \mathbb{B}_{3}=\left\{\left|z_{1}\right|^{2}+\right.$ $\left.\left|z_{2}\right|^{2}+\left|z_{3}\right|^{2}=3\right\}$. Then $N_{\varepsilon}$ is a maximally complex cycle diffeomorphic to $S^{1} \times S^{1} \times S^{1}$ bounding the (nonempty) complex surface $\Sigma_{\varepsilon}:=\left\{z_{1} z_{2} z_{3}=\right.$ $\varepsilon\} \cap \mathbb{B}_{3}$. Furthermore, there exists a suitably constructed $\mathscr{C}^{\infty}$ generic onecodimensional submanifold $M^{1} \subset \partial \mathbb{B}_{3}$ diffeomorphic to the real $(2 n-2)$ dimensional unit ball $B^{2 n-2}$ containing $N_{\varepsilon}$. Finally, since $N_{\varepsilon}$ bounds $\Sigma_{\varepsilon}$, every compact subset $K \subset M^{1}$ containing $N_{\varepsilon}$ is nonremovable.
5.7. Elliptic isolated complex tangencies and Bishop discs. Coming back to complex dimension $n=2$, we survey known properties of isolated CR singularities of surfaces. So, let $S$ be a two-dimensional surface $S$ in $\mathbb{C}^{2}$ of class at least $\mathscr{C}^{2}$. At a point $p \in S$, the complex tangent plane $T_{p} S$ is either totally (and in fact maximally) real, viz. $T_{p} S \cap J T_{p} S=\{0\}$ or it is a complex line, viz. $T_{p} S=J T_{p} S=T_{p}^{c} S$. An appropriate application of the jet transversality theorem shows that after an arbitrarily small perturbation, the number of complex tangencies of $S$ is locally finite.

If $S$ has an isolated complex tangency at one of its points $p$, Bishop ([Bi1965]) showed that there exist local coordinates $(z, w)$ centered at $p$ in which $S$ may be represented by $w=z \bar{z}+\lambda\left(z^{2}+\bar{z}^{2}\right)+\mathrm{o}\left(|z|^{2}\right)$, where
the real parameter $\lambda \in[0, \infty]$ is a biholomorphic invariant of $S$. The point $p$ is said to be elliptic if $\lambda \in\left[0, \frac{1}{2}\right)$, parabolic if $\lambda=\frac{1}{2}$ and hyperbolic if $\lambda \in\left(\frac{1}{2}, \infty\right]$. The case $\lambda=\infty$ should be understood as the surface $w=z^{2}+\bar{z}^{2}+\mathrm{o}\left(|z|^{2}\right)$. The shape of the projection of such a surface onto the real hyperplane $\{\operatorname{Im} w=0\} \simeq \mathbb{R}^{3}$ is essentially ellipsoid-like for $0<\lambda<1 / 2$ and essentially saddle-like for $\lambda>1 / 2$.

In the seminal article [Bi1965], Bishop introduced this terminology and showed that at an elliptic point, $S$ has a nontrivial polynomial hull $\widehat{S}$, foliated by a continuous one-parameter family of analytic discs attached to $M$. The geometric structure of this family has been explored further by Kenig and Webster.

Theorem 5.8. ([KW1982, BG1983, KW1984, Hu1998]) Let $S \subset \mathbb{C}^{2}$ be a $\mathscr{C}^{\kappa}(\kappa \geqslant 7)$ surface having an elliptic complex tangency at one of its points p. Then there exists a $\mathscr{C}^{(\kappa-7) / 3}$ one-parameter family of disjoint regularly embdedded analytic discs attached to $S$ and converging to $p$. If $S$ is $\mathscr{C}^{5}$, then $\widehat{S}$ is $\mathscr{C}^{0,1}$. Furthermore, every small analytic disc attached to $M$ near $p$ is a reparametrization of one of the discs of the family.

For $\kappa=\infty$, the union of these discs form a $\mathscr{C}^{\infty}$ hypersurface $\widehat{S}$ with boundary $\partial \widehat{S}=S$ in a neighborhood of $p$. Furthermore, $\widehat{S}$ is the local hull of holomorphy of $S$ at $p$.

In the case where $S$ is real analytic, local normal forms may be found that provide a classification up to biholomorphic changes of coordinates.
Theorem 5.9. Let $S: w=z \bar{z}+\lambda\left(z^{2}+\bar{z}^{2}\right)+\mathrm{O}\left(|z|^{3}\right)$ be a local real analytic surface in $\mathbb{C}^{2}$ passing through the origin and having an elliptic complex tangency there.

- ([MW1983]) For every $\lambda$ satisfying $0<\lambda<1 / 2$, either $S$ is locally biholomorphic to the quadric $w=z \bar{z}+\lambda\left(z^{2}+\bar{z}^{2}\right)$ or there exists an integer $s \in \mathbb{N}, s \geqslant 1$, such that $S$ is locally biholomorphic to $w=z \bar{z}+\left[\lambda+\delta u^{s}\right]\left(z^{2}+\bar{z}^{2}\right)$, where $u=\operatorname{Re} w$ and $\delta= \pm 1$.
- ([Mo1985]) For $\lambda=0$, either $S$ is locally biholomorphic to $w=$ $z \bar{z}+z^{s}+\bar{z}^{s}+\mathrm{O}\left(|z|^{s+1}\right)$ for some integer $s \geqslant 3$ or $S$ is locally biholomorphic to $w=z \bar{z}$.
- ([HuKr1995]) For $\lambda=0$ and $s<\infty$, the surface $S$ is locally biholomorphic to the surface $w=z \bar{z}+z^{s}+\bar{z}^{s}+\sum_{j+k>s} a_{j k} z^{j} \bar{z}^{k}$, with $a_{j k}=\bar{a}_{k j}$.
In all cases, after the straightening, $S$ is contained in the real hyperplane $\{\operatorname{Im} w=0\}$.
In the third case $\lambda=0, s<\infty$, it is still unknown how many biholomorphic invariants $S$ can have.
5.10. Hyperbolic isolated complex tangencies. The existence of small Bishop discs attached to $S$ and growing at an elliptic complex tangency impedes local polynomial convexity. At the opposite, if $S$ is hyperbolic, Bishop's construction fails, discs are inexistent, and in fact $S$ is locally polynomially convex.

Theorem 5.11. ([FS1991]) Let $S \subset \mathbb{C}^{2}$ be a $\mathscr{C}^{2}$ surface represented by $w=z \bar{z}+\lambda\left(z^{2}+\bar{z}^{2}\right)+r(z, \bar{z})$, with a $\mathscr{C}^{2}$ remainder $r=\mathrm{o}\left(|z|^{2}\right)$. If $\lambda>1 / 2$, viz. if $S$ is hyperbolic at the origin, then for every $\rho_{1}>0$ sufficiently small, $S \cap\left(\rho_{1} \overline{\mathbb{B}_{2}}\right)$ is polynomially convex.

The Oka-Weil approximation theorem then assures that continuous functions in $S \cap\left(\rho_{1} \overline{\mathbb{B}}_{2}\right)$ are uniformly approximable by polynomials.

A local Bishop surface $S$ is called quadratic if it is locally biholomorphic to the quadric $w=z \bar{z}+\lambda\left(z^{2}+\bar{z}^{2}\right)$. An isolated complex point $p$ of $S$ is called holomorphically flat if there exist local coordinates centered at $p$ in which $S$ is locally contained in $\{\operatorname{Im} w=0\}$. Unlike elliptic points of $\mathscr{C}^{\omega}$ surfaces that are always flat, hyperbolic complex points of $\mathscr{C}^{\omega}$ surfaces may fail to be flat.

Example 5.12. ([MW1983]) The algebraic hyperbolic surface $(\lambda>1 / 2)$

$$
w=z \bar{z}+\lambda\left(z^{2}+\bar{z}^{2}\right)+\lambda z^{3} \bar{z}
$$

cannot be biholomorphically transformed into a real hyperplane.
Theorem 5.11 establishes local polynomial pseudoconvexity of surfaces at hyperbolic complex tangencies. By patching together local plurisubharmonic defining functions, one may easily construct a Stein neighborhood basis of every surface having only finitely many hyperbolic complex tangencies. Unfortunately, in this way one does not control well the topology of such neighborhoods. A finer result answering a question of Forstneric is as follows.

Theorem 5.13. ([S12004]) Let $S$ be a compact real $\mathscr{C}^{\infty}$ surface embedded in a complex surface $\mathscr{X}$ having only finitely many complex points that are all hyperbolic and holomorphically flat. Then $S$ possesses a basis of open neighborhoods $\left(\mathscr{V}_{\varepsilon}\right)_{0<\varepsilon<\varepsilon_{1}}, \varepsilon_{1}>0$, such that:

- $S=\bigcap_{\varepsilon>0} \mathscr{V}_{\varepsilon}$;
- $\mathscr{V}_{\varepsilon}=\bigcup_{\varepsilon^{\prime}<\varepsilon} \mathscr{V}_{\varepsilon^{\prime}}$;
- $\overline{\mathscr{V}}_{\varepsilon}=\bigcap_{\varepsilon^{\prime}>\varepsilon} \mathscr{V}_{\varepsilon^{\prime}} ;$
- each $\mathscr{V}_{\varepsilon}$ has a $\mathscr{C}^{\infty}$ strongly pseudoconvex boundary $\partial \mathscr{V}_{\varepsilon}$;
- for every $\varepsilon$ with $0<\varepsilon<\varepsilon_{1}$, the surface $S$ is a strong deformation retract of $\mathscr{V}_{\varepsilon}$.

It is expected that the same statement remains true without the flatness assumption.
5.14. Real surfaces in strongly pseudoconvex boundaries. Coming back to removable singularities, let $\partial \Omega \Subset \mathbb{C}^{2}$ be a $\mathscr{C}^{2}$ strongly pseudoconvex boundary and let $S \subset \partial \Omega$ be a compact surface, with or without boundary. It will be no restriction to assume that $S$ is connected. Suppose that $S$ has a finite (possibly null) number of complex tangencies. These points then constitute the only singular points of the characteristic foliation of $S$. At an elliptic (resp. hyperbolic) complex tangency, the phase diagram simply looks like a focus (resp. saddle).
Theorem 5.15. ([FS1991]) Let $\mathscr{M}$ be a two-dimensional Stein manifold, let $\partial \Omega \Subset \mathscr{M}$ be a strongly pseudoconvex $\mathscr{C}^{2}$ boundary and let $D$ be a $\mathscr{C}^{2}$ one-codimensional submanifold that is diffeomorphic to the unit open 2disc of $\mathbb{R}^{2}$ and is maximally complex, except at a finite number of hyperbolic complex tangencies. Then every compact subset $K$ of $D$ is $C R$ - and $\mathscr{W}$ removable.

Indirectly, the characterizing Theorem 3.7 of Stout yields the following.
Corollary 5.16. Every compact subset $K \subset D \subset \partial \Omega$ is $\mathscr{O}(\bar{\Omega})$-convex. In particular, such a $K$ is polynomially convex if $\mathscr{M}=\mathbb{C}^{2}$ and if $\bar{\Omega}$ is Runge or polynomially convex, e.g. if $\Omega=\mathbb{B}_{2}$.

The (short) proof mainly relies upon the (very recent in 1991 and since then famous) works [BK1991] and [Kr1991] by Bedford-Klingenberg and by Kruzhilin about the hulls of two-dimensional spheres contained in such strictly pseudoconvex boundaries $\Omega \subset \mathscr{M}$, which may be filled by Levi-flat three-dimensional spheres after an arbitrarily small perturbation.

Theorem 5.17. ([BK1991, Kr1991]) Let $\Omega \Subset \mathbb{C}^{2}$ be a $\mathscr{C}^{6}$ strongly pseudoconvex domain and let $S \subset \partial \Omega$ be a two-dimensional sphere of class $\mathscr{C}^{6}$ embdedded into $\partial \Omega$ that is totally real outside a finite subset consisting of $k$ hyperbolic and $k+2$ elliptic points. Then there exist:

1) a smooth domain $B \subset \mathbb{R}^{3}\left(x_{1}, x_{2}, x_{3}\right)$ with boundary $\partial B$ diffeomorphic to $S$ such that $x_{3}: B \rightarrow \mathbb{R}$ is a Morse function on $\partial B$ having $k+2$ extreme points and $k$ saddle points, whose level sets $\left\{x_{3}=c s t.\right\} \cap B$ are unions of finite numbers of topological discs; and:
2) a continuous injective map $\Phi: B \rightarrow \Omega$ sending $\partial B$ to $S$, the extreme and saddle points of $x_{3}$ on $\partial B$ to the elliptic and hyperbolic points of $S$ and the connected components of $\left\{x_{3}=\mathrm{cst}.\right\} \cap B$ to geometrically smooth holomorphic discs.

The set $\Phi(B)$ is the envelope of holomorphy of $S$ as well as its $\mathscr{O}(\bar{\Omega})$-hull, i.e. its polynomial hull in case $\bar{\Omega}$ is polynomially convex.

In [Du1993], motivated by the problem of understanding polynomial convexity in geometric terms, the question of $\mathscr{O}(\bar{\Omega})$-convexity (instead of removability) of compact subsets of arbitrary surfaces $S \subset \partial \Omega$ (not necessarily diffeomorphic to a 2 -disc) is dealt with directly. If $K$ is a compact subset of a totally real surface $S \subset \partial \Omega$, denote by $\widehat{K}_{\text {ess }}:=\widehat{K}_{\mathscr{O}(\bar{\Omega})} \backslash K$ the essential $\mathscr{O}(\bar{\Omega})$-hull of $K$. An application of Hopf's lemma shows that if $K=A(\partial \Delta)$ is the boundary of a $\mathscr{C}^{1}$ analytic disc $A \in \mathscr{O}(\Delta) \cap \mathscr{C}^{1}(\bar{\Delta})$ attached to the surface $S$, necessarily $K=\widehat{K}_{\text {ess }}$ is an immersed $\mathscr{C}^{1}$ curve that is everywhere transversal to the characteristic foliation of $S$. If $S$ has a hyperbolic complex tangency at one of its points $p$ and if $A(1)=p$, then $A(\partial \Delta)$ must cross at least one separatrix in every neighborhood of $p$. When $\widehat{K}_{\text {ess }}$ contains no analytic disc, similar transversality properties hold.
Theorem 5.18. ([Du1993]) Let $K \Subset S \subset \partial \Omega \Subset \mathbb{C}^{2}$ be as above, with $\partial \Omega \in$ $\mathscr{C}^{2}$ strongly pseudoconvex and $S \in \mathscr{C}^{2}$ having finitely many hyperbolic complex tangencies. In the totally real part of $S$, the essential $\mathscr{O}(\bar{\Omega})$-hull $\widehat{K}_{\text {ess }}$ of $K$ crosses every characteristic curve that it meets. If $\widehat{K}_{\text {ess }}$ meets $a$ hyperbolic complex tangency, then it meets at least two hyperbolic sectors in every neighborhood of $p$.

As a consequence ([Du1993]), every compact subset $K$ of a twodimensional disc $D \subset \partial \Omega$ that has only finitely many hyperbolic complex tangencies is $\mathscr{O}(\bar{\Omega})$-convex.
5.19. Totally real dises in nonpseudoconvex boundaries. All the above results heavily relied on strong pseudoconvexity, in contrast to the removability theorems presented in Section 6, where the adequate statements, based on general CR extension theory, are formulated in terms of CR orbits rather than in terms of Levi curvature. The first theorem for the non-pseudoconvex situation was established by the second author.
Theorem 5.20. ([Po2003]) Let $M$ be a $\mathscr{C}^{\infty}$ globally minimal hypersurface of $\mathbb{C}^{2}$ and let $D \subset M$ be a $\mathscr{C}^{\infty}$ one-codimensional submanifold that is diffeomorphic to the unit open 2-disc of $\mathbb{R}^{2}$ and maximally real at every point. Then every compact subset $K$ of $D$ is $C R$-, $L^{\mathrm{p}}$ - and $\mathscr{W}$-removable.

We would like to point out that, seeking theorems without any assumption of pseudoconvexity leads to substantial open problems, because one loses almost all of the strong interweavings between function-theoretic tools and geometric arguments which are valid in the pseudoconvex realm, for instance: Hopf Lemma, plurisubharmonic exhaustions, envelopes of function spaces, local maximum modulus principle, Stein neighborhood basis, etc.

We sketch the proof of the theorem. We first claim that $M \backslash K$ is (also) globally minimal. Indeed, if there were a lower-dimensional orbit $\mathscr{O}$ of $M \backslash K$, we would obtain a lower-dimensional orbit of $M$ by adding all characteristic arcs intersecting $\overline{\mathscr{O}}$ ([Po2003], Lemma 1; [26], Lemma 3.5). Then by Theorem 4.12(V), continuous CR functions on $M \backslash K$ extend holomorphically to a one-sided neighborhood $\mathscr{V}^{b}(M \backslash K)$.

For later application of the continuity principle, similarly as in [MP2002, Po2003, 26], we deform $M \backslash K$ in $\mathscr{V}^{b}(M \backslash K)$, so that the functions are holomorphic in some ambient neighborhood $\mathscr{U}$ of $M \backslash K$ in $\mathbb{C}^{2}$.

The first key idea is to construct an embedded 2 -sphere containing a neighborhood of $K$ in $D$ and to apply the filling Theorem 5.17. This will give us a Levi flat 3 -ball foliated by analytic discs, which by translations, will enable us to fill in a one-sided neighborhood of $K$.

In the case where $M=\partial \Omega$ is a strictly pseudoconvex boundary, the construction of the 2 -sphere is quite direct: we pick an open 2 -disc $D^{\prime}$ having $\mathscr{C}^{\infty}$ boundary $\partial D^{\prime} \simeq S^{1}$ with $K \subset D^{\prime} \Subset D$; translating it slightly and smoothly within $\partial \Omega$, we obtain an almost parallel copy $D^{\prime \prime} \subset \partial \Omega$; then we construct the 2 -sphere $S^{\prime}$ by gluing (inside $\partial \Omega$ ) a thin closed strip $\simeq[-\varepsilon, \varepsilon] \times S^{1}$ to $\partial D^{\prime} \simeq S^{1}$ and to $\partial D^{\prime \prime} \simeq S^{1}$; finally, we perturb the strip part of $S^{\prime}$ in a generic way to assure that $S^{\prime}$ has only (a finite number of) isolated complex tangencies of elliptic or of hyperbolic type ${ }^{31}$. Then Theorem 5.17 yields a Levi-flat 3-ball $B^{\prime} \subset \Omega$ with $\partial B^{\prime}=S^{\prime}$.

If $M$ is not strongly pseudoconvex, the filling of $S^{\prime}$ by a Levi-flat ball $B^{\prime}$ may fail, because of a known counter-example [FM1995]. As a trick, we modify the construction. Using the fact that the squared distance function $\operatorname{dist}\left(\cdot, D^{\prime}\right)^{2}$ is strictly plurisubharmonic in a neighborhood of $\bar{D}^{\prime}$ (by total reality), for $\varepsilon>0$ small, the sublevel sets

$$
\Omega_{\varepsilon}^{\prime}:=\left\{q \in \mathbb{C}^{2}: \operatorname{dist}\left(q, \overline{D^{\prime}}\right)<\varepsilon\right\}
$$

are strongly pseudoconvex neighborhoods of $\overline{D^{\prime}}$ intersecting $M$ transversally along the 2 -spheres $\partial \Omega_{\epsilon}^{\prime} \cap M$.


[^30]Furthermore, a given fixed $\Omega_{\epsilon}^{\prime}$ can be slightly isotoped (translated) to a domain $\Omega^{\prime}$ still strongly pseudoconvex and having boundary transverse to $M$ so that $D^{\prime}$ is precisely contained in the isotoped 2-sphere $\partial \Omega^{\prime} \cap M$. After a very slight generic perturbation, we may insure that $S^{\prime}$ has only elliptic or hyperbolic complex tangencies (a part of $\partial \Omega^{\prime}$ has also to be perturbed). In sum:

Lemma 5.21. ([Po2003]) There exists a bounded domain $\Omega^{\prime} \subset \mathbb{C}^{2}$ such that:

- $\partial \Omega^{\prime}$ is $\mathscr{C}^{\infty}$, strongly pseudoconvex and diffeomorphic to a 3 -sphere;
- $\partial \Omega^{\prime}$ intersects $M$ transversally in a two-sphere $S^{\prime \prime}:=\partial \Omega^{\prime} \cap M$;
- $S^{\prime}$ has $k$ hyperbolic and $k+2$ elliptic points;
- $\partial \Omega^{\prime}$ contains the open 2 -disc $D^{\prime} \supset K$.

Then Theorem 5.17 applies in the strongly pseudoconvex boundary $\partial \Omega^{\prime}$, yielding a Levi-flat 3 -sphere $B^{\prime} \subset \Omega^{\prime}$ with $\partial B^{\prime}=S^{\prime}$. However, the nonpseudoconvexity of $M$ obstructs further insights in the position of $B^{\prime}$ with respect to $M$. In fact, $B^{\prime}$ may change sides or even be partly contained in $M$.

In the (simpler) case where $M=\partial \Omega$ is a strongly pseudoconvex boundary, we introduce a foliation of a neighborhood of $S^{\prime}$ in $M$ by $\mathscr{C}^{\infty} 2$-spheres $S_{t}^{\prime}$ with $S_{0}^{\prime}=S^{\prime}$. By filling them, we get a family of Levi-flat 3-balls $B_{t}^{\prime}$ with $\partial B_{t}^{\prime}=S_{t}^{\prime}$. Denote $B_{t}^{\prime}=\cup_{s} \Delta_{t, s}^{\prime}$ the foliation of $B_{t}^{\prime}$ by holomorphic discs. For $t \neq 0$, each $\Delta_{t, s}^{\prime}$ has boundary $\partial \Delta_{t, s}^{\prime} \subset S_{t}^{\prime} \subset M \backslash K$. Thus, by means of the continuity principle, we may extend holomorphic functions in the neighborhood $\mathscr{U}$ of $M \backslash K$ to a neighborhood of $B_{t}^{\prime}$ in $\mathbb{C}^{n}$, for all small $t \neq 0$. A final simple check shows that Theorem 2.30 (rm5) applies to remove $B_{0}^{\prime}$, and we get holomorphic extension to the union $\cup_{t} B_{t}^{\prime}$, a set containing the strongly pseudoconvex open local side of $\Omega$ at every point of $K$.

Without pseudoconvexity assumption on $M$, we can still consider a foliation $S_{t}^{\prime}$, but now the global geometry of $B_{t}^{\prime}$ is no longer clear. If for instance $M$ is Levi-flat near $K$ and the $S_{t}^{\prime}$ are contained in the Levi-flat part, then the $B_{t}^{\prime}$ just form an increasing family whose union is just a subdomain of $M$. Therefore it seems necessary to deform $S^{\prime}$ once again in order to gain transversality of $B^{\prime}$ and $M$. Since the global behavior of $B^{\prime}$ is hard to control, a further localization is advisable.

As in [Me1997], we consider the set $K_{\mathrm{nr}}$ of points $q \in K$ such that $\mathscr{O}(\mathscr{V}(M \backslash K))$ does not extend holomorphically to a one-sided neighborhood of $q$. So $\mathscr{O}(\mathscr{V}(M \backslash K))$ extends holomorphically to a one-sided neighorhood $\mathscr{V}^{b}\left(K \backslash K_{\mathrm{nr}}\right)$. By deforming $M$ at points of $K \backslash K_{\mathrm{nr}}$, we come down to the same situation with $K$ replaced with $K_{\text {nr }}$, except that no point of
$K_{\mathrm{nr}}$ should be removable. Assuming $K_{\mathrm{nr}} \neq \emptyset$, to conclude by contradiction, it then suffices to remove only one point of $K_{\mathrm{nr}}$.

To begin with, assume that $K_{\mathrm{nr}}$ is contained in finitely many of the disc boundaries $\partial \Delta_{0, s}^{\prime}$ which foliate $S^{\prime}=S_{0}^{\prime}$. Then we claim that no $\partial \Delta_{0, s}^{\prime}$ can be contained in $K_{\mathrm{nr}}$. Otherwise, $\partial \Delta_{0, s}^{\prime} \subset K_{\mathrm{nr}} \subset D^{\prime} \subset D$ and the 2 -disc enclosed by $\partial \Delta_{0, s}$ in $S_{0}^{\prime}$ inside the totally real 2-disc $D^{\prime}$ contain no complex tangencies, but the filling provided by Theorem 7.17 excludes such a topological possibility. So $K_{\text {nr }}$ is properly contained in a finite union of arcs, and hence removable by Theorem 4.9.

Therefore we may assume that $K_{\mathrm{nr}}$ has nonvoid intersection with infinitely many of $\partial \Delta_{0, s}^{\prime}$. Since there is only finitely many complex tangencies, there exists a $\partial \Delta_{0, s_{0}}^{\prime}$ with $\partial \Delta_{0, s_{0}}^{\prime} \cap K_{\mathrm{nr}} \neq \emptyset$ not encountering them. The same argument as above shows that $\partial \Delta_{0, s_{0}}^{\prime} \not \subset K_{\mathrm{nr}}$. Let $p_{0} \in \partial \Delta_{0, s_{0}}^{\prime} \cap K_{\mathrm{nr}}$.

If $\Delta_{0, s_{0}}^{\prime}$ and $M$ meet transversally at $p_{0}^{\prime}$, holomorphic extension to a onesided neighborhood at $p_{0}^{\prime}$ proceeds as in the strongly pseudoconvex case, by applying the continuity principle with discs $\Delta_{t, s}^{\prime} \subset B_{t}^{\prime}$ for $t \neq O$.

Assume now that $\overline{\Delta_{0, s_{0}}^{\prime}}$ is tangential to $M$ in $p_{0}^{\prime}$ or equivalently, that $\partial \Delta_{0, s_{0}}^{\prime}$ is tangential to the characteristic leaf in $p_{0}^{\prime}$. The idea is to change the angle of the discs close to $\Delta_{0, s_{0}}^{\prime}$, and to apply the above argument to the deformed disc passing through $p_{0}^{\prime}$. Since $\partial \Delta_{0, s_{0}}^{\prime} \not \subset K_{\mathrm{nr}}$, we may deform slightly $S^{\prime}$ near some point $q_{0}^{\prime} \in \partial \Delta_{0, s_{0}}^{\prime} \backslash K_{\mathrm{nr}}$ in the direction normal to $B^{\prime}$. More precisely, one deforms $S^{\prime}$ slightly, so that Theorem 5.17 still applies, and then picks up the disc of the deformed Levi-flat 3-ball that passes through $p_{0}^{\prime}$. In view of known results about normal deformations of small discs (Proposition 2.21(V); [Trp1990, BRT1994, Tu1994a]), the turning of the angle for large discs ([Fo1986, Gl1994]) may also be established in such a way (see [Po2003, Po2004]).

There is one final point to be handled carefully. We have to be sure that after turning the discs, the deformed disc boundary passing through the point $p_{0}^{\prime} \in K_{\mathrm{nr}}$ is not entirely contained in $K_{\mathrm{nr}}$.


This can be assured by replacing $p_{0}^{\prime}$ by another special nearby point $\widehat{p}_{0} \in$ $K_{\mathrm{nr}}$ with a good transversality property as illustrated above.
Theorem 5.20 is not yet the complete generalization of Theorem 5.15 to nonpseudoconvex hypersurfaces, since $D$ is assumed to be totally real at every point. If $D$ has hyperbolic complex tangencies, it is not clear whether a sphere $S^{\prime}$ together with a strongly pseudoconvex boundary $\partial \Omega^{\prime} \supset S^{\prime}$ as in the above key lemma can be constructed. The recent Theorem 5.13 indicates that this is possible if hyperbolic complex tangencies are holomorphically flat, an assumption which would be rather $a d$ hoc for the removal of compact sets $K \subset D$.

In fact, assuming generally that $M$ is an arbitrary globally minimal hypersurface, that a given surface $S \subset M$ has arbitrary topology (not necessarily diffeomorphic to an open 2-disc) and possesses complex tangencies, the reduction to the filling Theorem 5.17 seems to be impossible. Indeed, Fornæss-Ma ([FM1995]) constructed an unknotted nonfillable 2sphere $S \subset \mathbb{C}^{2}$ having only two elliptic complex tangencies. To the authors' knowledge, the possibility of filling by Levi-flat 3 -spheres some 2 -spheres lying in a nonpseudoconvex hypersurface is a delicate open problem. In addition, for the higher codimensional generalization of Theorem 1.2, the idea of global filling seems to be irrelevant at present times, because no analog of the filling Theorem 5.17 is known in dimension $n \geqslant 3$.
5.22. Beyond this survey. In the research article [26] placed in direct continuation to this survey, we consider surfaces $S$ having arbitrary topology and we generalize Theorem 5.20 to arbitrary codimension, localizing the removability arguments and using only small analytic discs.

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## Characteristic foliations

# on maximally real submanifolds of $\mathbb{C}^{n}$ 

## and removable singularities for CR functions

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#### Abstract

In present-day multidimensional complex analysis, the existing cohomological or functional characterizations of removable singularities (for holomorphic or CR functions) do only seldom provide adequate insights into the geometrical structures. Nonetheless, in the theory of CR functions, some geometric criteria are accessible for $L^{p}$-removability in the spirit of the classical Denjoy theorem (cf. the Painlevé problem), especially in the case of CR dimension 1, where, as in the complex plane, a single $\bar{\partial}_{b}$ operator is concerned.

We consider closed or compact singularities a priori contained in some surface $S$ embedded into a globally minimal hypersurface $M \subset \mathbb{C}^{2}$ (geometric assumptions). If $S$ is totally real except at finitely many complex tangencies that are hyperbolic in the sense of Bishop, and if the union of the separatrices of its characteristic foliation is a tree of curves having no cycles, we show that every compact set $K \subset S$ is removable. Already in the hypersurface case, we endeavor a new localization procedure yielding substantial generalizations of this statement, for the removability of closed sets $C \subset M^{1} \subset M$ contained in a totally real 1-codimensional submanifold $M^{1}$ embedded in some $\mathscr{C}^{2, \alpha}(0<\alpha<1)$ generic submanifold $M \subset \mathbb{C}^{n}(n \geqslant 2)$ that has CR dimension 1 . We establish that every characteristically pseudoconcave subset $C \subset M^{1} \subset M$ closed both in $M^{1}$ and in $M$ is removable.


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Table of contents
Introduction .....  1.

1. Characteristic foliation and removability: main results .....  3.
2. Description of the proof of Theorem 1.3 and organization ..... 10.
3. Strategy per absurdum for the proof of Proposition 1.13 ..... 18.
4. Construction of a semi-local half-wedge ..... 24.
5. Choice of a special point of $C_{\mathrm{nr}}$ to be removed locally ..... 36.
6. Three preparatory lemmas on Hölder spaces ..... 52.
7. Families of analytic discs half-attached to maximally real submanifolds . ..... 54.
8. Geometric properties of families of half-attached analytic discs ..... 63.
9. End of proof of Proposition 1.13: application of the continuity principle ..... 71.
10. $\mathscr{W}$-removability implies $L^{\text {p }}$-removability ..... 10.
11. Proofs of Theorem 1.2 and of Corollary 1.5 ..... 83.
12. Polynomial convexity of certain real 2-dises ..... 92.
13. Proof of Theorem 1.9 ..... 95.

## INTRODUCTION

The study of removable singularities for solutions of (partial) differential equations, deeply rooted in the classical theory of holomorphic functions of one complex variable, plays a major rôle in contemporary Analysis. Historically, the subject was initiated by Riemann's basic removability theorem (1854) - stating that $\mathscr{O}(\mathbb{C} \backslash\{\mathrm{pt}\}) \cap L_{\text {loc }}^{\infty}(\mathbb{C} \backslash\{\mathrm{pt}\})=\mathscr{O}(\mathbb{C})$ - and in the last few years, the research field has enjoyed quite spectacular advances. For instance, Painlevé's long outstanding problem (see the Bourbaki survey [Pa2005]) about characterizing geometrically the compact sets $K \subset \mathbb{C}$ for which $\mathscr{O}(\mathbb{C} \backslash K) \cap L_{\text {loc }}^{\infty}(\mathbb{C} \backslash K)=\mathscr{O}(\mathbb{C})$, is nowadays considered to be essentially solved ([To2003]) in terms of the average Menger curvature of Radon measures supported on $K$.

For functions of several complex variables, the subject is even richer, because in higher dimensions, existing geometrical concepts and refined cohomological tools broaden considerably the research perspectives. Also, an adequate approach to removable singularities for operators of multidimensional complex analysis must certainly take account of the compulsory Hartogs-type extension phenomena that are widely known and still deeply studied in contemporary Cauchy-Riemann Geometry.

It is worth mentioning that since the 1990 's, singularities of CR functions on boundaries of domains in complex manifolds attracted much attention. An intensively studied question was to provide criterions insuring that the Hartogs-Kneser extension theorem still holds true, when considering CR functions that are defined only in the complement $\partial \Omega \backslash K$ of a compact subset $K$ of a connected boundary $\partial \Omega \Subset \mathbb{C}^{n}(n \geqslant 2)$. For $\mathscr{C}^{2}$ strongly pseudoconvex boundaries contained in two-dimensional Stein manifolds, a satisfactory function-theoretical characterization was obtained by Stout ([Stu1993]), namely $K$ is removable if and only if it is $\mathscr{O}(\bar{\Omega})$-convex. Slightly after, for
arbitrary complex dimension $n \geqslant 3$, a complete cohomological characterization of different nature was obtained by Lupacciolu ([Lu1994]). A survey of Chirka-Stout ([6]), a monograph of Kytmanov ([Ky1995]) and lecture notes by Laurent-Thiébaut ([Lt1997]) already restitute these aspects excellently and provide a valuable introduction to $\bar{\partial}$ techniques in removable singularities.

In 1988, opening a broad new geometric trend with totally different techniques, Jöricke established an outstanding removability theorem: closed maximally real $\mathscr{C}^{2}$ discs in strongly pseudoconvex boundaries $\partial \Omega \Subset \mathbb{C}^{2}$ are removable. This was the first CR version of Denjoy's approach to Painlevé's problem, where the singularity is assumed to be one-codimensional. Compared to other results, it was particularly satisfactory to devise the geometrical structure of removable sets, often invisible in functional-theoretic and in cohomological characterizations.

In the late 1990's, within the general framework of CR extension theory that reached a considerable degree of achievement thanks to the works of Trépreau and of Tumanov, it became mathematically accessible to endeavour the general study of (geometrically) removable singularities, for CR functions defined on embedded CR manifolds $M \subset \mathbb{C}^{n}$ that have arbitrary CR dimension and arbitrary codimension. In recent years, rather (almost) complete removability results have been published by Jöricke for hypersurfaces and by the two authors for general generic submanifolds. Usually, the given generic $M \subset \mathbb{C}^{n}$ is assumed to be globally minimal, i.e. to consist of a single CR orbit (a very weak assumption which allows $M$ to possess quite large Levi-flat regions); in fact, such orbits are the intrinsic objects adequately linked to CR extension; also, they appear to be bricks that are essentially independent; and in the technical details, proofs do in fact proceed orbitwise, so that known corollaries valuable for not globally minimal M's follow from elementary arguments. Since it is wiser to refrain from formulating superficial corollaries, one usually assumes global minimality everywhere.

Towards a general unified theory of removable singularities for CR functions, our finest joint result ([MP2002]) states that closed sets whose ( $\operatorname{dim} M-2$ )-dimensional Hausdorff measure vanishes are always removable on such globally minimal $M$ 's. Also, we obtained previously several positive results ([MP1999]) in the case where the illusory singularity is assumed to be a priori contained in a given submanifold $N$ of $M$. Thanks to the guiding ideas of Jöricke, complete results were obtained for $M$ of CR dimension $\geqslant 2$ (and of codimension $\geqslant 1$ ), with $\operatorname{codim}_{M} N=1,2$ or $\geqslant 3$, but in the much more delicate case where $M$ has only CR dimension $=1$, in the existing literature, the codimension of the singularity is assumed to be
$\geqslant 2$ ([Me1997, Po1997, Jö1999a, Jö1999b, MP1999]), except notably when $M$ is a hypersurface ([Jö1988, FS1991, Du1993]).

Thus, in the subject, there remained essentially one single principal (daring and difficult) open question raised by Jöricke in [Jö1999b], namely to study the (possibly very massive) singularities that are contained in a onecodimensional maximally real submanifold $M^{1}$ of a given generic submanifold $M \subset \mathbb{C}^{n}$ having CR dimension equal to 1 and codimension $\geqslant 2$ (whence $n \geqslant 3$ ). As already mentioned briefly, the original motivation was to elaborate CR versions of a celebrated characterization asserted by Denjoy in 1909, who obtained a partial solution to the Painlevé problem that was correct only in the case of a singularity contained in a real analytic curve; nowadays, the best generalization (solution of Denjoy's conjecture) says that a compact set $K \subset \mathbb{C}$ contained a priori in some Lipschitz curve is removable for bounded holomorphic functions if and only if it has zero length, viz. zero one-dimensional Hausdorff measure (see [Pa2005] for a precise historical account; recent results go far beyond Denjoy's original approach). In the expected CR generalization, $M$ plays the rôle of a domain in $\mathbb{C}$ and $M^{1}$ plays the rôle of the curve.

As discovered by Jöricke in [Jö1988], unlike in the complex plane and thanks to the freedom offered by the various Hartogs-type extension phenomena, removability of illusory singularities may hold true even if they have nonempty interior (in $M^{1}$ ), and without requiring neither the vanishing of some metrical (Lebesgue, Minkowski, Hausdorff) content, nor of some auxiliary capacity. Jöricke also cleverly emphasized that the classical removability theorems enjoyed by general linear partial differential operators that were unified by Harvey and Polking in [HP1970] do only provide restricted insight into the nature of CR singularities. In fact, because these results are based on elementary metrical estimates showing that the singularity becomes innocuous through integration by parts, the formulation of these theorems does depend on the class (e.g. $L_{l o c}^{\mathrm{p}}$ or $\mathscr{C}^{\kappa, \alpha}$ ) and also, it seems impossible to get $L^{1}$-removability without a strong assumption of growth near the singularity. On the contrary, Jöricke ([Jö1999a, Jö1999b]) and the two authors ([MP1999, MP2002]) obtained results formulated geometrically that are uniform with respect to the class - including $L_{l o c}^{1}$ and that require no growth tameness.

Following this trend of thought, our main objective in the present research paper is to answer completely the first Problem 2.1 raised in [Jö1999b] (and mentioned above), with $M \subset \mathbb{C}^{n}(n \geqslant 2)$ of CR dimension 1 and $M^{1} \subset M$ maximally real, both of class $\mathscr{C}^{2, \alpha}(0<\alpha<1)$. Since $M^{1} \subset M$ has null CR dimension, the standard processus of sweeping out by wedges becomes void, because small Bishop discs attached to $M^{1}$ are not available.

Accordingly, the proof of the main Proposition 1.13 below relies upon a new localization device, based on families of analytic discs that are only half-attached to $M^{1}$ (following Bishop and Pinchuk), the very gist of the argument being the selection of a special point to be removed. In the hypersurface case (only), previously known approaches relied upon a global Kontinuitätssatz, or upon a global filling of 2-spheres by Levi-flat 3-balls (following Bedford-Klingenberg and Kruzhilin), but both tools have no known controllable counterpart in higher codimension. So, as a final comment, we point out that it is satisfactory to bring in this paper a purely local framework for the treatment of one-codimensional singularities, even when $M$ is a hypersuface of $\mathbb{C}^{2}$.

The results presented here are entirely new in codimension $\geqslant 2$.

## §1. Characteristic foliation and removability: MAIN RESULTS

1.1. Removability of totally real discs having hyperbolic complex tangencies. By means of a global Kontinuitätssatz, Jöricke ([Jö1988]) showed removability of closed maximally real smooth discs contained in strongly pseudoconvex boundaries $\partial \Omega \Subset \mathbb{C}^{2}$. Applying the filling of 2 -spheres by Levi-flat 3-balls ([BK1991, Kr1991]), Forstnerič -Stout ([FS1991]) allowed finitely many complex tangencies of hyperbolic type (in the sense of Bishop) in the disc and established both its removability and its $\mathscr{O}(\bar{\Omega})$-convexity. Reasoning with Rossi's local maximum modulus principle and with Oka's criterion for holomorphic convexity, Duval ([Du1993]) re-obtained these result differently and generalized them to arbitrary surfaces $S \subset \partial \Omega$.

More recently, suppressing convexity hypotheses, the second author ([Po2004]) showed removability of closed maximally real discs contained in globally minimal hypersurfaces of $\mathbb{C}^{2}$. We point out that obtaining theorems without any assumption of (pseudo)convexity on CR manifolds leads to substantial difficulties, because one loses almost all of the strong interweavings between function-theoretic tools and geometric arguments which are valid in the pseudoconvex realm, for instance: Hopf lemma, plurisubharmonic exhaustions, envelopes of function spaces, local maximum modulus principle, Stein neighborhood basis and semi-global control of hulls.

Our first statement unifies the mentioned results, still without pseudoconvexity; importantly, we also establish removability of certain compact subsets of arbitrary surfaces, instead of plain discs, see Corollary 1.5 below. Throughout this article, all (sub)manifolds are assumed to be embedded.
Theorem 1.2. Let $M$ be a globally minimal $\mathscr{C}^{2, \alpha}(0<\alpha<1)$ hypersurface in $\mathbb{C}^{2}$ and let $D \subset M$ be a $\mathscr{C}^{2, \alpha}$ surface which is:

- diffeomorphic to the open unit 2 -disc of $\mathbb{R}^{2}$ and:
- totally real outside a discrete subset of isolated complex tangencies which are hyperbolic in the sense of Bishop.
Then every compact subset $K$ of $D$ is $C R$-, $\mathscr{W}$ - and $L^{\mathrm{p}}$-removable ${ }^{32}$.
The theorem holds with exactly the same proof if $\mathbb{C}^{2}$ is replaced by any two-dimensional complex manifold, not necessarily Stein. As a direct corollary, with $M=\partial \Omega \Subset \mathbb{C}^{2}$ being a $\mathscr{C}^{2, \alpha}$ compact boundary, hence automatically globally minimal ([Jö1999a, 29]), we obtain a Hartogs-Kneser extension theorem from $\partial \Omega \backslash K$. Also, the characterization of removable sets due to Stout yields that if $\partial \Omega \Subset \mathbb{C}^{2}$ is a $\mathscr{C}^{2, \alpha}$ boundary such that $\bar{\Omega}$ has a Stein neighborhood basis, then every $K \subset D \subset \partial \Omega$ as in the theorem is $\mathscr{O}(\bar{\Omega})$ convex.

As a more substantial application, reminding that satisfactory geometric criteria for polynomial convexity of general surfaces in $\mathbb{C}^{2}$ are far to be known, we derive new examples of polynomially convex sets contained in weakly pseudoconvex boundaries. The arguments of proof are postponed to Section 12.

Corollary 1.3. Let $\Omega \Subset \mathbb{C}^{2}$ be a domain with $\mathscr{C}^{2, \alpha}$ boundary. Suppose that $\bar{\Omega}$ is polynomially convex (whence $\Omega$ is weakly pseudoconvex). Let $D \subset \partial \Omega$ be an embedded 2-disc of class $\mathscr{C}^{2, \alpha}$ which is totally real outside a discrete subset of hyperbolic complex tangencies. Then each compact set $K \subset D$ is polynomially convex.

To describe briefly some aspects of the geometrical machinery underlying Theorem 1.2, we remind ([Jö1988, FS1991, Du1993]) that the totally real part of the 2-disc $D$ is equipped with a so-called characteristic foliation $\mathrm{F}_{D}^{c}$, obtained by integrating the line distribution $D \ni p \longmapsto T_{p}^{c} M \cap T_{p} D$, hence canonical. Then $\mathrm{F}_{D}^{c}$ has singularities exactly at the complex hyperbolic tangencies of $D$. If $D$ is totally real at every point, the Poincaré-Bendixson theorem assures the inexistence of limit cycles as well as of foci, of centers and of saddle points, so that all characteristic curves must go from a point of the boundary of $D$ to another boundary point; in the left diagram below, they are simply drawn as horizontal lines.

In [Du1993], Duval delineated a crucial, immediately seen geometric property: for every compact set $K \subset D$ (hence also trivially for every subcompact $K^{\prime} \subset K \subset D$ ), there exists at least one characteristic curve

[^31]$\gamma$ touching $K$ (resp. $K^{\prime}$ ) such that $K$ (resp. $K^{\prime}$ ) is located in one closed side of $\gamma$ in some thin, elongated neighborhood of $\gamma$. Also, the tips of $\gamma$ being close to the boundary of $D$, they must lay at a positive distance from $K$. Moving then such a curve $\gamma$ slightly up and down, one sees an intuitive processus of "erasing" the (bottom-left part in the picture) part of $K$ which is very similar to the classical Kontinuitätssatz, alias Continuity Principle, in which one moves an analytic disc, keeping its boundary inside some (usually pseudoconcave) (Riemann) domain, in order to describe a part of an envelope of holomorphy.

Strikingly, this informal analogy underlies a true removability fact, which we formulate as an independent, main technical proposition, directly useful to the proof of Theorem 1.2. All subsets $C$ of a submanifold $S$ of a manifold $M$ that are called closed are assumed to be closed both in $M$ and in $S$. We point out that now $D$ is replaced by a 2 -surface $S$ which may have arbitrary topology and that the removed set $C$ is not necessarily compact, which will be needed.

Proposition 1.4. Let $M$ be a $\mathscr{C}^{2, \alpha}$ globally minimal hypersurface in $\mathbb{C}^{2}$, let $S \subset M$ be a $\mathscr{C}^{2, \alpha}$ surface, open or closed, with or without boundary, which is totally real at every point. Let $C$ be a proper closed subset of $S$ and assume that the following topological condition holds, meaning that $C$ is nontransversal to $\mathrm{F}_{S}^{c}$ :

- for every closed subset $C^{\prime} \subset C$, there exists a simple $\mathscr{C}^{2, \alpha}$ curve $\gamma:[-1,1] \rightarrow S$, whose range is contained in a single leaf of the characteristic foliation $\mathrm{F}_{S}^{c}$ (obtained by integrating the characteristic line field $\left.T^{c} M\right|_{S} \cap T S$ ), with $\gamma(-1) \notin C^{\prime}, \gamma(0) \in C^{\prime}$ and $\gamma(1) \notin C^{\prime}$, such that $C^{\prime}$ lies completely in one closed side of $\gamma[-1,1]$ with respect to the topology of $S$ in a neighborhood of $\gamma[-1,1]$.

Then $C$ is $C R$-, $\mathscr{W}$ - and $L^{\mathrm{p}}$-removable.
In case $S=D$ is a 2 -disc and $C=K$ is compact, the left diagram provides an illustration.


Fig. 1: Nontransversality to the characteristic foliation

In the absence of a Poincaré-Bendixson theorem taming the topology of the characteristic foliation ( $S$ is not a disc), this nontransversality condition appears to be the most adequate cause of removability. In fact, it is well known that boundaries of analytic discs or of Riemann surfaces attached to a 2 -surface $S \subset \partial \Omega$ contained in some compact strongly pseudoconvex boundary $\partial \Omega \Subset \mathbb{C}^{2}$, are embedded circles everywhere transversal to $\mathrm{F}_{S}^{c}$ (because of Hopf's lemma), and it is clear that such holomorphic curves $\Lambda=\{g=0\}$, with $g \in \mathscr{O}(\Omega) \cap \mathscr{C}^{1}(\bar{\Omega})$, are never removable: it suffices to set $K:=\{g=0\} \cap \partial \Omega \subset S$ and to consider $\left.\frac{1}{g}\right|_{\partial \Omega \backslash K}$. It is thus remarkable that the nontransversality of $C$ to $\mathrm{F}_{S}^{c}$ appears again on globally minimal CR structures, where the distribution $p \mapsto T_{p}^{c} M$ is allowed to be very far from contact.

Let us now briefly explain why the main Proposition 1.4 is necessary to Theorem 1.2 (we recommend that the reader simultaneously watches Figure 22 in $\S 11.3$ below). At each hyperbolic point $p$, the phase diagram of $\mathbf{F}_{D}^{c}$ is saddle-like and contains two local separatrices intersecting at $p$ which are smooth and transversal (cross-like). Hence we can decompose the 2 -disc $D$ as a union $D=T_{D} \cup D_{o}$, where $T_{D}$ consists of the union of the hyperbolic points of $D$ together with the separatrices issuing from them, and where $D_{o}:=D \backslash T_{D}$ is the remaining open submanifold of $D$, obviously contained in the totally real part of $D$. By the theory of Poincaré-Bendixson ([HS1974, FS1991, Du1993]), since $D$ is a disc, $T_{D}$ must be a tree of $\mathscr{C}^{1, \alpha}$ curves which contains no subset homeomorphic to the unit circle. Accordingly, we set $K_{T_{D}}:=K \cap T_{D}$ and $C_{o}:=K \cap D_{o}$, so that $K=K_{T_{D}} \cup C_{o}$ decomposes in two parts. Then $C_{o}$ is a relatively closed subset of $D_{o}$, and importantly, it is also closed in $M_{o}:=M \backslash T_{D}$. Again thanks to PoincaréBendixson, $C_{o}$ is nontransversal to the characteristic foliation of $D_{o}$. So Proposition 1.4 applies: we may remove $C_{o}$ with respect to $M_{o}$, namely we get holomorphic extension to a global one-sided neighborhood $\omega_{M_{o}}$ of $M_{o}$ in $\mathbb{C}^{2}$.

Deforming $M$ slightly inside $\omega_{M_{o}}$, we are left with the much thinner singularity $K_{T_{D}}$, of codimension $\leqslant 2$ in $M$. Since $K_{T_{D}}$ contains no circle, its removal will follow from known theorems ([6, Jö1999a, MP1999]; however, a technical investigation of the behavior of the CR orbits near $T_{D}$ will be required).

Section 2 describes and summarizes the proof of the main Proposition 1.4 in geometric and in conceptional terms.

The nontransversality assumption is a common condition on $C$ and on the characteristic foliation $\mathrm{F}_{S}^{c}$, namely on the relative disposition of $C$ with respect to $\mathrm{F}_{S}^{c}$; Figure 3 below provides a second illustration of it. As already mentioned, if $S$ is diffeomorphic to a real 2-disc or if $S=D_{o}$ as above, then
nontransversality holds true. Generally, it also holds when the characteristic foliation is given by the level sets of some $\mathscr{C}^{1, \alpha}$ real-valued function defined on $S$. More interestingly, to conclude with removal in $\mathbb{C}^{2}$, we formulate a consequence of Proposition 1.4 that is more general than Theorem 1.2, since it holds without the restricted assumption that $S$ be diffeomorphic to a real 2 -disc. We believe that this application shows well the strength of our new localization procedure.
Corollary 1.5. Let $M$ be a $\mathscr{C}^{2, \alpha}$ globally minimal hypersurface in $\mathbb{C}^{2}$, let $S \subset M$ be a $\mathscr{C}^{2, \alpha}$ real 2-surface, open or closed, with or without boundary which is totally real outside a discrete subset of isolated hyperbolic complex tangencies and let $K \subset S$ be a compact set. If $T_{S}$ denotes the union of hyperbolic points of $S$ together with all separatrices, assume that:

- $K \cap\left(S \backslash T_{S}\right)$ is nontransversal to $\mathrm{F}_{S \backslash T_{S}}^{c}$;
- $K \cap T_{S}$ does not contain any subset homeomorphic to the unit circle.

Then $K$ is $C R$-, $\mathscr{W}$ - and $L^{\mathrm{p}}$-removable.
1.6. Passage to arbitrary codimension. Our principal motivation for the present work was to devise a purely local strategy of proof for Theorem 1.2 in order to obtain higher codimensional removability results in the most delicate case of CR dimension 1. Accordingly, let $M$ be a $\mathscr{C}^{2, \alpha}$ globally minimal generic submanifold of codimension $(n-1)$ in $\mathbb{C}^{n}$, with $n \geqslant 2$ arbitrary. Let $M^{1}$ be a $\mathscr{C}^{2, \alpha}$ one-codimensional submanifold of $M$ which is generic in $\mathbb{C}^{n}$, hence maximally real. As in the $\mathbb{C}^{2}$ case, $M^{1}$ carries a characteristic foliation $\mathrm{F}_{M^{1}}^{c}$, whose leaves are the integral curves of the canonical line distribution $M^{1} \ni p \longmapsto T_{p} M^{1} \cap T_{p}^{c} M$.

Next, let $K \subset M^{1}$ be a compact set. Of course, the assumption that $K$ locally lies in one closed side of some characteristic curve is meaningless inside $M^{1}$, when its dimension $n$ is $\geqslant 3$. Taking inspiration from (pseudo)convexity theory, the appropriate condition requires that every compact $K^{\prime} \subset K$ has at least one boundary point at which $M^{1} \backslash K^{\prime}$ becomes concave with respect to characteristic segments.
Definition 1.7. The complement $M^{1} \backslash K$ is called characteristically pseudoconcave if for every subcompact $K^{\prime} \subset K$, there is a $\mathscr{C}^{1}$ embedding $\Phi:[-1,1] \times\left[0, c_{1}\right] \rightarrow M^{1}, c_{1}>0$, such that:

- each horizontal leaf $\Phi([-1,1] \times\{\mathrm{cst}\})$ is contained in a single characteristic curve;
- for $0 \leqslant$ cst $<c_{1}$, the intersection $\Phi([-1,1] \times\{$ cst $\}) \cap K^{\prime}=\emptyset$ is void;
- $\Phi\left(\{-1\} \times\left[0, c_{1}\right]\right) \cap K^{\prime}=\emptyset$ and $\Phi\left(\{1\} \times\left[0, c_{1}\right]\right) \cap K^{\prime}=\emptyset$;
- $\Phi\left((-1,1) \times\left\{c_{1}\right\}\right) \cap K^{\prime} \neq \emptyset$ is nonempty (but the two endpoints $\Phi\left(-1, c_{1}\right)$ and $\Phi\left(1, c_{1}\right)$ must, by the above item, lie at a positive distance from $K^{\prime}$ ).

In the left Figure 1 above, $\Phi$ amounts to translating $\gamma$ downward. The reader will have noticed the close similarity with the classical continuity principle. In our setting, embedded families of holomorphic discs are replaced by families of characteristic segments, the endpoints of the segments corresponding to the boundary circles of the discs. We remind that to verify that a domain $\Omega \subset \mathbb{C}^{n}$ is pseudoconvex in the sense of Hartogs, one has to establish the Kontinuitätssatz for all appropriately embedded families of holomorphic discs. In our case the geometry is much more rigid because the directions of the embedded segments are already prescribed by the characteristic foliation.

Unexpectedly ${ }^{33}$, our principal result in this paper establishes a deep link between the characteristic pseudoconcavity of $M^{1} \backslash K$ in the real sense and the fact that the (partial) envelope of holomorphy of $M \backslash K$ is pseudoconcave enough to cover $M$.
Theorem 1.8. Let $M \subset \mathbb{C}^{n}$ be generic, $\mathscr{C}^{2, \alpha}$, of codimension $(n-1)$ and globally minimal, let $M^{1} \subset M$ be one-codimensional, $\mathscr{C}^{2, \alpha}$ and maximally real in $\mathbb{C}^{n}$, and let $K \subset M^{1}$ be a compact set. If $M^{1} \backslash K$ is characteristically pseudoconcave, then it is is $C R$-, $\mathscr{W}$ - and $L^{\mathrm{p}}$-removable.

In fact, we recall from [Me1997, MP1999, MP2002] that the removability of $K$ means (essentially) that the (partial) envelope of holomorphy of any wedgelike domain attached to $M \backslash K$ contains a complete wedge attached to $M$. Technically speaking, the proof is of high level and before launching the attack, we will formulate a more general main proposition, analogous to Proposition 1.4 and valid for certain closed sets that are nontransversal to $\mathrm{F}_{M^{1}}^{c}$ in a certain sense.

Meanwhile, we would like to mention that in the last Section 13 below (which may be read independently), we will exhibit a crucial example of a compact set $K \subset M^{1} \subset M \subset \mathbb{C}^{3}$ diffeomorphic to a two-dimensional torus and everywhere transversal to $\mathrm{F}_{M^{1}}^{c}$, namely $T_{p} K \oplus \mathrm{~F}_{M^{1}}^{c}(p)=T_{p} M^{1}$ for every $p \in K$, which is truly nonremovable. Since the embedded 2-torus $K$ has no boundary, the complement $M^{1} \backslash K$ cannot be characteristically pseudoconcave. This shows that the main geometrical assumption of the theorem above is adequate. In addition, similarly as in [JS2000], we may require (almost for free) that $M$ and $M^{1}$ have the simplest possible topology. Recall that, according to a classical definition, type 4 at a point $p \in M$ means

[^32]that the Lie brackets of the complex tangent bundle $T^{c} M$ up to length 4 generate $T_{p} M$.
Theorem 1.9. There exists a triple $\left(M, M^{1}, K\right)$, where
(i) $M$ is a $\mathscr{C}^{\infty}$ generic submanifold in $\mathbb{C}^{3}$ of $C R$ dimension 1, diffeomorphic to a real 4-ball;
(ii) $M^{1}$ is a $\mathscr{C}^{\infty}$ one-codimensional submanifold of $M$ which is maximally real in $\mathbb{C}^{n}$ and diffeomorphic to a real 3-ball;
(iii) $K$ is a compact subset of $M^{1}$ diffeomorphic to a real 2-torus which is everywhere transversal to the characteristic foliation $\mathrm{F}_{M^{1}}^{c}$, hence $M^{1} \backslash K$ cannot be characteristically pseudoconcave;
(iv) $M$ of finite type 4 at every point, hence globally minimal,
such that $K$ is neither $C R$ - nor $\mathscr{W}$ - nor $L^{\mathrm{p}}$-removable with respect to $M$.
1.10. Characteristic nontransversality and main proposition. Let $M$, $M^{1}, \mathrm{~F}_{M^{1}}^{c}$ be as before and let $C$ be a proper subset of $M^{1}$, closed in $M^{1}$ and closed in $M$. Here is the higher dimensional notion of characteristic nontransversality, already illustrated by the right diagram above.

Definition 1.11. The closed set $C \subset M^{1} \subset M$ is called nontransversal to the characteristic foliation if:

- for every closed subset $C^{\prime} \subset C$, there exists a simple $\mathscr{C}^{2, \alpha}$ curve $\gamma:[-1,1] \rightarrow M^{1}$ whose range $\gamma[-1,1]$ is contained in a single leaf of the characteristic foliation $\boldsymbol{F}_{M^{1}}^{c}$ with $\gamma(-1) \notin C^{\prime}, \gamma(0) \in C^{\prime}$ and $\gamma(1) \notin C^{\prime}$, there exists a local $(n-1)$-dimensional transversal $R^{1} \subset M^{1}$ to $\gamma$ passing through $\gamma(0)$ and there exists a thin elongated open neighborhood $V_{1}$ of $\gamma[-1,1]$ in $M^{1}$ such that if $\pi_{\mathrm{F}^{c}}$ : $V_{1} \rightarrow R^{1}$ denotes the semi-local projection parallel to the leaves of the characteristic foliation $\mathbf{F}_{M^{1}}^{c}$, then $\gamma(0)$ lies on the boundary, relatively to the topology of $R^{1}$, of $\pi_{\mathrm{F}^{c}}\left(C^{\prime} \cap V_{1}\right)$.

Clearly, in the case $n=2$, this amounts to say that $C^{\prime} \cap V_{1}$ lies completely in one side of $\gamma[-1,1]$, as written in Proposition 1.4.
Lemma 1.12. The two conditions introduced so far are in fact equivalent:

- $M^{1} \backslash C$ is characteristically pseudoconcave if and only if
- $C$ is nontransversal to $\mathrm{F}_{M^{1}}^{c}$.

Furthermore, for every nonempty closed $C^{\prime} \subset C$, there exists $p_{1} \in C^{\prime}$, there exists a characteristic embedded $\mathscr{C}^{2, \alpha}$ curve $\gamma:=[-1,1] \rightarrow M^{1}$ with $\gamma(-1) \notin C^{\prime}, \gamma(0)=p_{1}$ and $\gamma(1) \notin C^{\prime}$ and there exists a thin $\mathscr{C}^{1, \alpha}$ support hypersurface $H^{1} \subset M^{1}$ containing $\gamma$, foliated by characteristic curves and
elongated along $\gamma$ such that $C^{\prime}$ is contained in one closed side of $H^{1}$ inside $M^{1}$, locally in a neighborhood of $\gamma$.

The second assertion, proved in Proposition 5.2, entails immediately the equivalence. We may now formulate our main technical proposition (generalizing Proposition 1.4) upon which Theorem 1.8 relies.

Proposition 1.13. If the closed set $C \subset M^{1} \subset M$ is nontransversal to the characteristic foliation, it is $C R$-, $\mathscr{W}$ - and $L^{\mathrm{p}}$-removable.

We emphasize that this concise statement constitutes the essential core of the present article. Sections $3,4,5,6,7,8$ and 9 are integrally devoted to its proof.
1.14. Comparison with a third hypothesis sufficient for removability. With $C=K$ compact, in $\S 2.17$ below, we compare our main nontransversality assumption to the following condition, suggested by a referee.
$\mathrm{H}\{K\}$ : there is an open neighborhood $U$ of $K$ in $M^{1}$ and a $\mathscr{C}^{1}$ submersion $\rho: U \rightarrow V$ with values in a $\mathscr{C}^{1}$ not necessarily connected $(n-$ 1)-dimensional manifold $V$ without boundary and without compact components such that every level set $\rho^{-1}(q), q \in V$, is a union of leaves of $\mathrm{F}_{U}^{c}=\left.\mathrm{F}_{M^{1}}^{c}\right|_{U}$.
We first verify that $\mathrm{H}\{K\}$ implies that $K$ is nontransversal to $\mathrm{F}_{M^{1}}^{c}$. However, the reverse implication does not hold, so that for Theorem 1.8, $\mathrm{H}\{K\}$ is a strictly less general assumption than the characteristic pseudoconcavity of $M^{1} \backslash K$. This was forseeable, since global foliations are rarely induced by a submersion. In dimension $n=3$, we thus construct an example of ( $M, M^{1}, K$ ) with $M^{1} \backslash K$ characteristically pseudoconcave for which $\mathrm{H}\{K\}$ fails (see §2.17).
1.16. Application. For completeness, we formulate a higher codimensional version of Corollary 1.5.

Corollary 1.17. Let $M \subset \mathbb{C}^{n}$ be generic, $\mathscr{C}^{2, \alpha}$, of codimension $(n-1)$ and minimal at every point, let $\Lambda^{1} \subset M$ be a $\mathscr{C}^{2, \alpha}$ one-codimensional submanifold, totally real outside $\Sigma \cup \Lambda^{2}$, where $\Sigma \subset M$ is closed with vanishing (dim $M-2$ )-dimensional Hausdorff measure and where

$$
\begin{equation*}
\Lambda^{2}=\bigcup_{j \in J} \Lambda_{j}^{2} \tag{1.18}
\end{equation*}
$$

is a countable, locally finite union of disjoint connected 2-codimensional $\mathscr{C}^{2, \alpha}$ submanifolds $\Lambda_{j}^{2} \subset \Lambda^{1}$, and let $K \subset \Lambda^{1}$ be a compact set. Assume that

- $K \cap\left(\Lambda^{1} \backslash \Lambda^{2}\right)$ is nontransversal to $\mathrm{F}_{\Lambda^{1} \backslash \Lambda^{2}}^{c}$;
- $K \cap \Lambda_{j}^{2}$ is a proper subset of $\Lambda_{j}^{2}$ for every $j \in J$.

Then $K$ is $C R-, \mathscr{W}$ - and $L^{\mathrm{p}}$-removable.
Since $M$ is (locally) minimal at every point, $M \backslash E$ is globally minimal, for every $E \subset M$. The removal of $K \cap\left(\Lambda^{1} \backslash \Lambda^{2}\right)$ follows from the main Proposition 1.13, the removal of each $K \cap \Lambda_{j}^{2}$ is proved in [MP1999], and the removal of $\Sigma$ is established in [MP2002].
1.19. Acknowledgments. We would like to express our sincere gratitude to Burglind Jöricke for her kind interest and for her clever suggestions that incited us to improve substantially the presentation of our results.

## §2. DESCRIPTION OF THE PROOF OF PROPOSITION 1.4 AND ORGANIZATION

In this preliminary section, we summarize the hypersurface version Proposition 1.4. Our goal is to provide a conceptional description of the basic geometric constructions, which should be helpful to read the proof of the general Proposition 1.13. Because precise, complete and rigorous formulations will be developed in the next sections, we allow here the use of a slightly informal language.
2.1. Strategy per absurdum. Let $M, S$, and $C$ be as in Proposition 1.4. It is known that both the CR- and the $L^{\mathrm{p}}$-removability of $C$ are a (relatively mild) consequence of the $\mathscr{W}$-removability of $C$ (see $\S 3.14$ and Section 10 below). Thus, we shall describe in this section only the $\mathscr{W}$-removability of $C$.

First of all, as $M$ is globally minimal, it may be proved that for every closed subset $C^{\prime} \subset C$, the complement $M \backslash C^{\prime}$ is also globally minimal (see Lemma 3.5 below). As $M$ is of codimension one in $\mathbb{C}^{2}$, a wedge attached to $M \backslash C$ is simply a connected one-sided neighborhood of $M \backslash C$ in $\mathbb{C}^{2}$. Let us denote such a one-sided neighborhood by $\omega_{1}$. The goal is to prove that there exists a (bigger) one-sided neighborhood $\omega$ attached to $M$ to which holomorphic functions in $\omega_{1}$ extend holomorphically. By the definition of $\mathscr{W}$-removability, this will show that $C$ is $\mathscr{W}$-removable.

Reasoning by contradiction, we shall denote by $C_{\mathrm{nr}}$ the smallest nonremovable subpart of $C$. By this we mean that holomorphic functions in $\omega_{1}$ extend holomorphically to a one-sided neighborhood $\omega_{2}$ of $M \backslash C_{\mathrm{nr}}$ in $\mathbb{C}^{2}$ and that $C_{\mathrm{nr}}$ is the smallest subset of $C$ such that this extension property holds. If $C_{\mathrm{nr}}$ is empty, the conclusion of Proposition 1.4 holds, gratuitously: nothing has to be proved. If $C_{\mathrm{nr}}$ is nonempty, to come to an absurd, it suffices to show that at least one point of $C_{\mathrm{nr}}$ is locally removable. By this, we mean that there exists a local one-sided neighborhood $\omega_{3}$ of at least one point of $C_{\mathrm{nr}}$ such that holomorphic functions in $\omega_{2}$ extend holomorphically
to $\omega_{3}$. In fact, the choice of such a point will be the most delicate and the most tricky part of the proof.

In order to apply the continuity principle, as stated in [Me1997, 29], we now deform slightly $M$ inside the one-sided neighborhood $\omega_{2}$, keeping $C_{\mathrm{nr}}$ fixed, getting a hypersurface $M^{d}$ (with $d$ like "deformed") satisfying $M^{d} \backslash C_{\mathrm{nr}} \subset \omega_{2}$. We point out that a local one-sided neighborhood of $M^{d}$ at one point $p$ of $C_{\mathrm{nr}}$ always contains a local one-sided neighborhood of $M$ at $p$ (the reader may draw a figure), so we may well work on $M^{d}$ instead of working on $M$ (however, the analogous property about wedges over deformed generic submanifolds is untrue in codimension $\geqslant 2$, see $\S 3.16$ below, where supplementary arguments are required).

Replacing the notation $C_{\mathrm{nr}}$ by the notation $C$, the notation $M^{d}$ by the notation $M$ and the notation $\omega_{2}$ by the notation $\Omega$, we see that Proposition 1.4 is reduced to the following main proposition, whose formulation is essentially analogous to that of Proposition 1.4, except that it suffices to remove at least one special point.

Proposition 2.2. Let $M$ be a $\mathscr{C}^{2, \alpha}$ globally minimal hypersurface in $\mathbb{C}^{2}$, let $S \subset M$ be a $\mathscr{C}^{2, \alpha}$ surface which is totally real at every point. Let $C$ be a nonempty proper closed subset of $S$ and assume that it is nontransversal to $\mathrm{F}_{S}^{c}$. Let $\Omega$ be an arbitrary neighborhood of $M \backslash C$ in $\mathbb{C}^{n}$. Then there exists a special point $p_{\mathrm{sp}} \in C$ and there exists a local one-sided neighborhood $\omega_{p_{\mathrm{sp}}}$ of $M$ in $\mathbb{C}^{2}$ at $p_{\mathrm{sp}}$ such that holomorphic functions in $\Omega$ extend holomorphically to $\omega_{p_{\mathrm{sp}}}$.
2.3. Holomorphic extension to a half-one-sided neighborhood of $M$. The choice of the special point $p_{\text {sp }}$ will be achieved in two main steps. According to the nontransversality assumption, there exists a characteristic segment $\gamma:[-1,1] \rightarrow S$ with $\gamma(-1) \notin C$, with $\gamma(0) \in C$ and with $\gamma(1) \notin C$ such that $C$ lies in one (closed, semi-local) side of $\gamma$ in $S$. As $\gamma$ is a Jordan arc, we may orient $S$ in $M$ along $\gamma$, hence we may choose a semi-local open side $\left(S_{\gamma}\right)^{+}$of $S$ in $M$ along $\gamma$. In the first main step (to be conducted in Section 4 in the context of the general Proposition 1.13), we shall construct what we call a semi-local half-wedge $\mathscr{H} \mathscr{W}_{\gamma}^{+}$attached to $\left(S_{\gamma}\right)^{+}$along $\gamma$. By this, we mean the "half part" of a wedge attached to a neighborhood of the characteristic segment $\gamma$ in $M$, which yields a wedge attached to the semilocal one-sided neighborhood $\left(S_{\gamma}\right)^{+}$. For an illustration, see Figure 8 below, in which one should replace the notation $M^{1}$ by the notation $S$. Such a halfwedge may also be interpreted as a wedge attached to a neighborhood of $\gamma$ in $S$, but it should not be arbitrary, it should satisfy a further property: locally in a neighborhood of every point of $\gamma$, either the half-wedge contains $\left(S_{\gamma}\right)^{+}$ or one of its two ribs contains $\left(S_{\gamma}\right)^{+}$, as illustrated in Figure 8 below. Most
importantly, the cones of this attached half-wedge should vary continuously as we move along $\gamma$, cf. again Figure 8.

The way how we will construct this half-wedge $\mathscr{H} \mathscr{W}_{\gamma}^{+}$is as follows. As illustrated in Figure 2 just below, we shall first construct a string of analytic discs $Z_{r: s}(\zeta)$, where $r$ is the approximate radius of $Z_{r: s}(\partial \Delta)$, whose boundaries are contained in $\left(S_{\gamma}\right)^{+} \subset M$ and which touch the curve $\gamma$ only at the point $\gamma(s)$, for every $s \in[-1,1]$, namely $Z_{r: s}(1)=\gamma(s)$ and $Z_{r: s}(\partial \Delta \backslash\{1\}) \subset\left(S_{\gamma}\right)^{+}$.


Fig. 2: String of analytic discs attached to $\left(M_{\gamma}^{1}\right)^{+}$

Next, we fix a small radius $r_{0}$. By deforming the discs $Z_{r_{0}: s}(\zeta)$ in $\Omega$ near their opposite points $Z_{r_{0}: s}(-1)$, which lie at a positive distance from the singularity $C$, we construct in Section 4 an extended family of analytic discs $Z_{r_{0}, t: s}(\zeta)$, where $t \in \mathbb{R}$ is a small parameter, so that the disc boundaries $Z_{r_{0}, t: s}(\partial \Delta)$ are pivoting tangentially to $S$ at the point $\gamma(s) \equiv Z_{r_{0}, t: s}(1)$, which is assumed to remain fixed as $t$ varies. Precisely, we mean that $\frac{\partial Z_{r_{0}, t, s}}{\partial \theta}(1) \in T_{\gamma(s)} S$ and that the mapping $t \longmapsto \frac{\partial Z_{r_{0}, t: s}}{\partial \theta}(1)$ is of rank 1 at $t=0$. This construction and the next ones will be achieved thanks to perturbations of the Bishop equation, as in [Tu1994, MP1999]. Furthermore, we add a small parameter $\chi \in \mathbb{R}$ corresponding to vertical translations of the circles along $S$ near $\gamma$, getting a family $Z_{r_{0}, t, \chi: s}(\zeta)$ with the property that the mapping $(\chi, s) \longmapsto Z_{r_{0}, t, \chi: s}(1) \in S$ is a diffeomorphism onto a neighborhood of $\gamma([-1,1])$ in $S$, still with the property that the point $Z_{r_{0}, t, \chi: s}(1)$ is fixed equal to the point $Z_{r_{0}, 0, \chi: s}(1)$ as $t$ varies. Finally, we add a small parameter $\nu \in \mathbb{R}$ with $\nu>0$ corresponding to horizontal translations of the circles inside $\left(S_{\gamma}\right)^{+}$, getting a family $Z_{r_{0}, t, \chi, \nu: s}(\zeta)$ with $Z_{r_{0}, t, \chi, 0: s}(\zeta) \equiv Z_{r_{0}, t, \chi: s}(\zeta)$, such that the mapping $(\chi, \nu, s) \longmapsto Z_{r_{0}, t, \chi, \nu: s}(1)$ is a diffeomorphism onto the semi-local one-sided neighborhood $\left(S_{\gamma}\right)^{+}$of $S$ along $\gamma$ in $M$, provided
$\nu>0$. Then the semi-local attached half-wedge may be defined as

$$
\begin{align*}
\mathscr{H} \mathscr{W}_{\gamma}^{+}:=\left\{Z_{r_{0}, t, \chi, \nu: s}(\rho):\right. & |t|<\varepsilon,|\chi|<\varepsilon, 0<\nu<\varepsilon \\
& 1-\varepsilon<\rho<1,-1 \leqslant s \leqslant 1\} \tag{2.4}
\end{align*}
$$

for some small $\varepsilon>0$. In the first main technical step (Section 4), we shall show that every holomorphic function $f \in \mathscr{O}(\Omega)$ extends holomorphically to $\mathscr{H} \mathscr{W}_{\gamma}^{+}$. Then to prove Proposition 2.2, we must find ${ }^{34}$ a special point $p_{\mathrm{sp}} \in C$ such that there exists a local one-sided neighborhood $\omega_{p_{\mathrm{sp}}}$ at $p_{\mathrm{sp}}$ such that holomorphic functions in $\Omega \cup \mathscr{H} \mathscr{W}_{\gamma}^{+}$extend holomorphically to $\omega_{p_{\mathrm{sp}}}$.
2.5. Field of cones on $S$. We have to keep memory of the geometric disposal, of the orientation and of the size of $\mathscr{H} \mathscr{W}_{\gamma}^{+}$. The way how $\mathscr{H} \mathscr{W}_{\gamma}^{+}$passes continuously above and under the half hypersurface $\left(S_{\gamma}\right)^{+}$ (denoted $\left(M_{\gamma}^{1}\right)^{+}$in Figure 8) can be read off the full family of analytic $\operatorname{discs} Z_{r_{0}, t, \chi, \nu: s}(\zeta)$.

Thanks to a technical application of the implicit function theorem, we can arrange from the beginning that the vectors $\frac{\partial Z_{r_{0}, t, \chi, 0: s}}{\partial \theta}(1)$ are tangent to $S$ at the point $Z_{r_{0}, 0, \chi, 0: s}(1) \in S$ when $t$ varies, for all fixed $s$. Then by construction, when $t$ varies, the disc boundaries $Z_{r_{0}, t, \chi, 0: s}(\partial \Delta)$ are pivoting tangentially to $S$ at the point $Z_{r_{0}, t, \chi, 0: s}(1) \equiv Z_{r_{0}, 0, \chi, 0: s}(1)$. It follows that when $t$ varies, the oriented half-lines $\mathbb{R}^{+} \cdot \frac{\partial Z_{r_{0}, t, x, 0: s}}{\partial \theta}(1)$ describe an open infinite oriented cone in the tangent space to $S$ at the point $Z_{r_{0}, 0, \chi, 0: s}(1)$. Consequently, we may define a field of cones $p \mapsto C_{p}$ as

$$
\begin{equation*}
\mathrm{C}_{p}:=\left\{\mathbb{R}^{+} \cdot \frac{\partial Z_{r_{0}, t, \chi, 0: s}}{\partial \theta}(1):|t|<\varepsilon\right\} \tag{2.6}
\end{equation*}
$$

at every point $p=Z_{r_{0}, 0, \chi, 0: s}(1) \in S$ of a neighborhood of $\gamma$ in $S$. The following figure provides an illustration. One should intuitively think that the small cones $\mathrm{C}_{p}$ are generated when the small discs boundaries of Figure 2 pivote tangentially to $S$.

[^33]

Fig. 3: Field of cones on $T S$ and choice of a special point $p_{\mathrm{sp}}$
After having defined this field of cones, we shall fill all the cones as follows. The motivation is to describe precisely what kinds of small Bishop discs half-attached to $S$ (in the sense of Pinchuk) will surely have the other half of their boundaries contained in $\mathscr{H}^{\mathscr{W}}{ }_{\gamma}^{+}$, so that a version of the continuity principle will be applicable to get Proposition 2.2.

Remind that a neighborhood of $\gamma$ in $S$ is foliated by characteristic segments, which are approximatively parallel to $\gamma$. In Figure 3 above, one should think that the characteristic leaves are all horizontal. So there exists a nowhere vanishing vector field $p \mapsto X_{p}$ defined in a neighborhood of $\gamma$ whose integral curves are characteristic segments. We then define the filled cone $\mathrm{FC}_{p}$ by

$$
\begin{equation*}
\mathrm{FC}_{p}:=\left\{\lambda \cdot X_{p}+(1-\lambda) \cdot v_{p}: 0 \leqslant \lambda<1, v_{p} \in \mathrm{C}_{p}\right\} . \tag{2.7}
\end{equation*}
$$

Geometrically, we rotate every half-line $\mathbb{R}^{+} \cdot v_{p}$ towards the characteristic half-line $\mathbb{R}^{+} . X_{p}$ and we call the result the filling of $\mathrm{C}_{p}$. In Figure 3 above, the cone drawn near $\gamma(0)$ coincides with its filling. Thus we have constructed a field of filled cones $p \longmapsto \mathrm{FC}_{p}$ over a neighborhood of $\gamma$ in $S$.
2.8. Small analytic discs half-attached to $S$. The next main observation is that small analytic discs which are half-attached to $S$ are essentially contained in the half-wedge $\mathscr{H} \mathscr{W}_{\gamma}^{+}$, provided that they are approximatively directed by the cone $\mathrm{C}_{p}$ at the corresponding point $p \in S$. More is true: a similar property holds with the filled cone $\mathrm{FC}_{p}$ instead, and this fact will be used in an essential way, since we will need discs close to the characteristic direction.

Let us be more precise. Let $\partial^{+} \Delta:=\{\zeta \in \partial \Delta: \operatorname{Re} \zeta \geqslant 0\}$ denote the positive half part of the unit circle $\partial \Delta$. We say that an analytic disc $A: \bar{\Delta} \rightarrow \mathbb{C}^{2}$ is half-attached to $S$ if $A\left(\partial^{+} \Delta\right)$ is contained in $S$. Here, $A$ is at least of class $\mathscr{C}^{1}$ over $\bar{\Delta}$ and holomorphic in $\Delta$. In addition, we shall always assume that our discs $A$ are embeddings of $\bar{\Delta}$ into $\mathbb{C}^{2}$. We shall say that $A$
is approximatively straight (in an informal sense) if $A(\Delta)$ is close in $\mathscr{C}^{1}$ norm to an open subset of the complex line generated by the complex vector $\frac{\partial A}{\partial \zeta}(1)$. Finally, we say that $A$ is approximatively directed by the filled cone $\mathrm{FC}_{p}$ at $p=A(1)$, if the vector $\frac{\partial A}{\partial \theta}(1) \in T_{p} S$ belongs to $\mathrm{FC}_{p}$. Although this terminology will not be re-employed in the next sections, we may formulate a crucial geometric observation as follows.

Lemma 2.9. A sufficiently small approximatively straight analytic disc $A$ : $\bar{\Delta} \rightarrow \mathbb{C}^{2}$ of class at least $\mathscr{C}^{1}$ which is half-attached to $S$ and which is approximatively directed by the filled cone $\mathrm{FC}_{p}$ at $p=A(1) \in S$, necessarily satisfies

$$
\begin{equation*}
A\left(\bar{\Delta} \backslash \partial^{+} \Delta\right) \subset \mathscr{H} \mathscr{W}_{\gamma}^{+} . \tag{2.10}
\end{equation*}
$$

In the context of the general Proposition 1.13, this property (with more precisions) will be established in Section 8 below. Intuitively, the supplementary freedom offered by the filling $\mathrm{FC}_{p}$ comes from the fact that the half-wedge $\mathscr{H} \mathscr{W}_{\gamma}^{+}$is constructed by translating the discs horizontally (toward us in the two above figures) in ( $S_{\gamma}^{+}$), the distribution of horizontal planes being approximatively equal to $T_{p}^{c} M$ in the illustrations. In Figure 8, one should think that a vector which varies in a vertical cone drawn there, when it is multiplied by $i$, will cover the whole aperture of the filled cone $\mathrm{FC}_{p}$ (not only of $\mathrm{C}_{p}$ ).
2.11. Choice of a special point. In the second main step of the proof (to be conducted in Section 5 for the general Proposition 1.13), we shall choose the desired special point $p_{\text {sp }}$ of Proposition 2.2 to be removed locally as follows. Since we shall use half-attached analytic discs (applying the continuity principle), we want to find a special point $p_{\mathrm{sp}} \in C$ so that the following two conditions hold true:
(i) there exists a small approximatively straight analytic disc $A: \bar{\Delta} \rightarrow$ $\mathbb{C}^{2}$ with $A(1)=p_{\mathrm{sp}}$ which is half-attached to $S$ such that $A$ is approximatively directed by the filled cone $\mathrm{FC}_{p_{\mathrm{sp}}}$ (so that the conclusion of Lemma 2.9 above holds true);
(ii) the same disc satisfies $A\left(\partial^{+} \Delta \backslash\{1\}\right) \subset S \backslash C$.

In particular, since $M \backslash C$ is contained in $\Omega$, it follows from these two conditions that the (excised) disc boundary $A(\partial \Delta \backslash\{1\})$ is contained in the open subset $\Omega \cup \mathscr{H} \mathscr{W}_{\gamma}^{+}$, a property that will be appropriate for the application of the continuity principle, as we shall explain in Section 9 below.

To fulfill conditions (i) and (ii) above, we first construct a supporting real segment at a special point of the nonempty closed subset $C \subset S$.


Fig. 4: Half-boundary of a disc directed by a cone and touching $C_{\mathrm{nr}}$
Lemma 2.12. There exists at least one special point $p_{\mathrm{sp}} \in C$ arbitrarily close to $\gamma$ in a neighborhood of which the following two properties hold true:
(i') there exists a small $\mathscr{C}^{2, \alpha}$ open segment $H_{p_{\mathrm{sp}}} \subset S$ passing through $p_{\mathrm{sp}}$ such that an oriented tangent half-line to $H_{p_{\mathrm{sp}}}$ at $p_{\mathrm{sp}}$ is contained in the filled cone $\mathrm{FC}_{p_{\mathrm{sp}}}$, as illustrated in Figure 4 below;
(ii') the same segment is a supporting segment in the following sense: locally in a neighborhood of $p_{\mathrm{sp}}$, the set $C \backslash\left\{p_{\mathrm{sp}}\right\}$ is contained in one open side $\left(H_{p_{\mathrm{sp}}}\right)^{-}$if $H_{p_{\mathrm{sp}}}$ in $S$.

The way how we prove Lemma 2.12 is illustrated intuitively in Figure 3 above. For $\lambda \in \mathbb{R}$ with $0 \leqslant \lambda<1$ very close to 1 , the vector field $p \longmapsto$ $v_{p}^{\lambda}:=\lambda \cdot X_{p}+(1-\lambda) \cdot v_{p}$ is very close to the characteristic vector field $p \mapsto X_{p}$. By construction, this vector field runs into the filled field of cones $p \mapsto \mathrm{FC}_{p}$. In Figure 3, the integral curves of $p \mapsto v_{p}^{\lambda}$ are almost horizontal if $\lambda$ is very close to 1 . If we choose the first integral curve (the bold one) from the lower part of Figure 3 which touches $C$ at one special point $p_{\text {sp }} \in$ $C$ and if we choose for $H_{p_{\mathrm{sp}}}$ a small segment of this first integral curve, we may check that properties ( $\mathbf{i}^{\prime}$ ) and (ii') are satisfied, modulo some mild technicalities. A rigorous complete proof of Lemma 2.12 will be provided in Section 5 below.
2.13. Construction of analytic discs half-attached to $S$. Small analytic discs which are half-attached to a $\mathscr{C}^{2, \alpha}$ maximally real submanifold $M^{1}$ of $\mathbb{C}^{n}$ and which are approximatively straight will be constructed in Section 7 below. In fact, it is known ([Pi1974]) that one can prescribe arbitrarily the first order jet of a half-attached disc. However, prescribing $p_{\text {sp }}=A(1)$ and $T_{p_{\mathrm{sp}}} H_{p_{\mathrm{sp}}}=\mathbb{R} \cdot \frac{\partial A}{\partial \theta}(1)$ does not suffices: it may well occur that $A\left(\partial^{+} \Delta\right)$ intersects the singularity $C$ at several other points than $p_{\mathrm{sp}}$. Hopefully, $H_{p_{\mathrm{sp}}}$ being
totally real, we may curve it much in advance in some good holomorphic system of coordinates so that the singularity $C$ lies in the (closure of the) convex side of $H_{p_{\mathrm{sp}}}$, denoted by $\left(H_{p_{\mathrm{sp}}}\right)^{-}$in Figure 4. Thanks to this trick, provided only that the half boundary $A\left(\partial^{+} \Delta\right) \subset S$ is small and tangential to the convex side $\left(H_{p_{\mathrm{sp}}}\right)^{-}$at $p_{\mathrm{sp}}=A(1)$, it will follow just by a Taylor series argument that $A\left(\partial^{+} \Delta \backslash\{1\}\right)$ is contained in the good open side $\left(H_{p_{\mathrm{sp}}}\right)^{+}$not meeting $C$. In Figure 4 above, $H_{p_{\mathrm{sp}}}$ is straight and $A\left(\partial^{+} \Delta\right)$ is curved, which is equivalent. Thanks to this trick, we avoid having to construct discs with prescribed second order jet. Thus, the two geometric properties (i') and (ii') satisfied by the real segment $H_{p_{\mathrm{sp}}}$ may be realized by the half-boundary of a half-attached analytic disc.
2.14. Translation of half-attached discs and continuity principle. By means of the results of Section 7, we shall see that we may include the disc $A(\zeta)$ in a parametrized family $A_{x, v}(\zeta)$ of analytic discs half-attached to $S$, where $x \in \mathbb{R}^{2}$ and $v \in \mathbb{R}$ are small, so that the mapping $x \mapsto A_{x, 0}(1) \in S$ is a local diffeomorphism onto a neighborhood of $p_{\mathrm{sp}}$ in $S$ and so that the mapping $v \mapsto \frac{\partial A_{0, v}}{\partial \theta}(1)$ is of rank 1 at $v=0$. Furthermore, we introduce a new parameter $u \in \mathbb{R}$ in order to "translate" the totally real surface $S$ in $M$ by means of a family $S_{u} \subset M$ with $S_{0}=S$ and $S_{u} \subset\left(S_{\gamma}\right)^{+}$for $u>0$. Thanks to the flexibility of Bishop's equation, we deduce that there exists a deformed family of analytic discs $A_{x, v, u}(\zeta)$ which are half-attached to $S_{u}$ and which satisfy $A_{x, v, 0}(\zeta) \equiv A_{x, v}(\zeta)$. In particular, this family covers a local one-sided neighborhood $\omega_{p_{\mathrm{sp}}}$ of $M$ at $p_{\mathrm{sp}}$ defined by

$$
\begin{equation*}
\omega_{p_{\mathrm{sp}}}:=\left\{A_{x, v, u}(\rho):|x|<\varepsilon,|v|<\varepsilon,|u|<\varepsilon, 1-\varepsilon<\rho<1\right\} \tag{2.15}
\end{equation*}
$$

for some $\varepsilon>0$.
In the third and last main step of the proof (to be conducted in Section 9 below), we shall prove that every disc $A_{x, v, u}(\zeta)$ with $u \neq 0$ is analytically isotopic to a point with the boundary of every disc of the isotopy being contained in $\Omega \cup \mathscr{H} \mathscr{W}_{\gamma}^{+}$. In fact, for $u \neq 0$, the half-boundary $A_{x, v, u}\left(\partial^{+} \Delta\right)$ is contained in $S_{u} \subset \Omega$; the other half $A_{x, v, u}\left(\partial^{-} \Delta\right)$ remains stably inside $\mathscr{H} \mathscr{W}_{\gamma}^{+}$, as was arranged in advance thanks to (2.10) (also for $u=0$ ); and when $u>0$, the whole disc $A_{x, v, u}(\bar{\Delta})$ is contained in $\Omega \cup \mathscr{H}_{\mathscr{W}_{\gamma}^{+}}^{+}$, hence analytically isotopic to a point there (just shrink its radius).

Thanks to the continuity principle, we will deduce that every holomorphic function $f \in \mathscr{O}\left(\Omega \cup \mathscr{H} \mathscr{W}_{\gamma}^{+}\right)$extends holomorphically to $\omega_{p_{\mathrm{sp}}}$ minus a certain thin closed subset $\mathscr{C}_{p_{\text {sp }}}$ of $\omega_{p_{\mathrm{sp}}}$. Finally, we shall conclude both the proof of Proposition 2.2 and the proof of Proposition 1.4 by checking that the thin closed set $\mathscr{C}_{p_{\text {sp }}}$ is in fact removable for holomorphic functions defined in $\omega_{p_{\mathrm{sp}}} \backslash \mathscr{C}_{p_{\mathrm{sp}}}$.
2.16. Organization. In Sections $3,4,5,6,7,8$ and 9 , the proof of Proposition 1.13 will be endeavoured directly in arbitrary codimension, without any further reference to the hypersurface version. We point out that the crucial geometric argument which enables us to choose the desired special point will be conducted in the central Section 5 below.

In Section 10, we check that both the CR- and the $L^{\mathrm{p}}$-removability of $C$ are a consequence of the $\mathscr{W}$-removability of $C$. In Section 11, we provide the proofs of Theorem 1.2 and of Corollary 1.5. Section 12 treats the criterion of polynomial convexity stated as Corollary 1.3. Finally, Section 13 proves Theorem 1.9.
2.17. Comparison of the nontransversality assumption with $\mathrm{H}\{K\}$. Firstly, we claim that $\mathrm{H}\{K\}$ implies that $K$ is nontransversal to $\mathrm{F}_{M^{1}}^{c}$. Indeed, given a subcompact $K^{\prime} \subset K$, we look at the compact $\rho\left(K^{\prime}\right) \subset V$. Considering a family of spheres of increasing radius centered at some point $r_{1} \in V \backslash \rho\left(K^{\prime}\right)$ close to $\rho\left(K^{\prime}\right)$, we may find a first touched point $q_{1} \in \rho\left(K^{\prime}\right)$ at which a small spherical cap of the limit sphere constitutes a local support hypersurface $N^{1} \subset V$ with $q_{1} \in N^{1}$; indeed, the interior of the limit ball being contained in $V \backslash \rho\left(K^{\prime}\right)$ by construction, it follows that $\rho\left(K^{\prime}\right)$ is situated only in the closed side exterior to the cap $N^{1}$. Then $H^{1}:=\rho^{-1}\left(N^{1}\right)$ constitutes a $\mathscr{C}^{1, \alpha}$ support hypersurface as in Lemma 1.12 which is foliated by characteristic curves whose endpoints lie in $\partial U$, at a positive distance from $K$, q.e.d.

Example 2.18. We produce an example contradicting the reverse implication.
a) Let $T:=\mathbb{R}^{2} / \mathbb{Z}^{2}$ be the standard 2-dimensional real torus and let $\pi_{T}$ : $\mathbb{R}^{2} \rightarrow T$ denote the quotient map. For any slope $\alpha$, the straight lines $\{y=$ $\alpha x+b\}$ descend to a foliation $\mathrm{F}_{\alpha}$ of $T$. We fix $\alpha \in(4,8)$ and we set

$$
K_{T}:=\pi_{T}\left(\left\{(x+t, \alpha t) \in \mathbb{R}^{2}: 1 / 2 \leqslant x \leqslant 3 / 4,0 \leqslant t \leqslant 1 / \alpha\right\}\right)
$$

Geometrically, $K_{T}$ is a closed parallelogram wrapped in $y$-direction once around $T$. Its long sides (of length $\sqrt{1+\alpha^{-2}}$ ) are contained in two leaves and its short sides (of length $1 / 4$ ) do meet along a segment of (small) length $1 / 4-1 / \alpha>0$.
Since $4<\alpha<8$, the intersections of $K_{T}$ with the leaves are closed segments of length equal to $\sqrt{1+\alpha^{-2}}$ for $1 / 2+1 / 4-1 / \alpha<x \leqslant 3 / 4$, or equal to $2 \sqrt{1+\alpha^{-2}}$ for $1 / 2 \leqslant x \leqslant 1 / 2+1 / 4-1 / \alpha$, as $e . g$. the green bold leaf in the left diagram. Note in particular that $K_{T}$ does not contain any whole characteristic leaf.


Fig. ${ }^{2}$ : A nontransversal compact $K$ for which $H\{K\}$ fails
b) As was intended, we point out that for every (connected) neighborhood $U_{T}$ of $K_{T}$ in $T$, the restricted foliation $\left.\mathrm{F}_{\alpha}\right|_{U_{T}}$ cannot be parametrized by some submersion $\rho_{T}: U_{T} \rightarrow V_{T}$. Indeed, for dimensional reasons, $V_{T}$ should necessarily be a real interval and then, the restriction of $\rho_{T}$ to the transversal $\pi_{T}([1 / 2,3 / 4] \times\{0\})$ should be strongly monotonous. But this is impossible, because the green bold leaf intersects twice this transversal at two different points.
c) Unfortunately, the parallelogram $K_{T}$ is transversal to $\mathrm{F}_{\alpha}$, since for instance, the circle (of length 1) $\pi_{T}(\{3 / 4\} \times[0,1]) \subset K^{T}$ occurs to be everywhere transversal. Hopefully, we may increase the dimension by a unity. So we set $\widetilde{M}^{1}:=T \times(-\varepsilon, \varepsilon)$, and we embed it as a maximally real manifold $M^{1} \subset \mathbb{C}^{3}$ by means of the obvious quotient of the map $\phi: \mathbb{R}^{2} \times(-\varepsilon, \varepsilon) \rightarrow$ $\mathbb{C}^{3}$ defined by $\phi\left(t_{1}, t_{2}, t_{3}\right):=\left(\exp \left(2 \pi i t_{1}\right), \exp \left(2 \pi i t_{2}\right), t_{3}\right)$. Equipping the fibers $T \times\left\{t_{3}\right\}$ with parallel copies of $\mathrm{F}_{\alpha}$ yields on $\widetilde{M}^{1}$ a foliation by (quotiented) lines. We define $\mathrm{F}_{M^{1}}^{c}$ as its pushforward and we let $K$ be the image of $\widetilde{K}:=K_{T} \times[-\varepsilon / 2, \varepsilon / 2]$.
d) Let $T \mathrm{~F}_{M^{1}}^{c} \subset T M^{1} \subset T \mathbb{C}^{3}$ be the bundle of real lines tangent to $\mathrm{F}_{M^{1}}^{c}$. Since $M^{1}$ is totally real, the bundle $J T \mathrm{~F}_{M^{1}}^{c}=\bigcup_{p \in M^{1}} J_{p} T_{p} \mathrm{~F}_{M^{1}}^{c}$ obtained by complex multiplication is nowhere tangent to $M^{1}$. We choose a $\mathscr{C}^{\infty}$ manifold $M$ containing $M^{1}$ such that at every $p \in M^{1}, T_{p} M$ is spanned by $T_{p} M^{1}$ and $J_{p} T_{p} \mathrm{~F}_{M^{1}}^{c}$. By construction, $M$ is generic (provided it is defined to be a sufficiently thin strip along $M^{1}$ ) and the characteristic foliation of $M^{1}$ coincides with $\mathrm{F}_{M^{1}}^{c}$. Proceeding as in the the proof of Theorem 1.9(iv), we can even arrange that $M$ is of type 4 at every point, hence globally minimal.
e) We claim that $M^{1} \backslash K$ is characteristically pseudoconcave. Indeed, let $K^{\prime} \subset K$ be compact and let $h \in[-\varepsilon / 2, \varepsilon / 2]$ be maximal such that $\phi(T \times$ $\{h\}) \cap K^{\prime} \neq \emptyset$. Through any point $p \in \phi(T \times\{h\}) \cap K^{\prime}$ there passes a compact characteristic segment $I$ whose endpoints are not contained in
$K^{\prime}$. Translating $I$ upwards (with respect to $t_{3}$ ), we derive the characteristic pseudoconcavity of $M^{1} \backslash K$.
f) Finally, we claim that $K$ does not satisfy $\mathrm{H}\{K\}$. Assume on the contrary that there exists a submersion $\rho: U \rightarrow V$ as required in $\mathrm{H}\{K\}$. For $\delta>0$ very small, the $\delta$-neighborhood $U_{\delta}$ of $K$ in $M^{1}$ is contained in $U$. Restricting $\rho$ to $U_{\delta} \cap \phi(T \cap\{0\})$ and pulling back via $\phi$, we obtain a parametrization $\rho_{T}$ of $F_{\alpha}$ in a very thin connected neighborhood $U_{T}$ of $K_{T}$ with values in a 1-dimensional manifold $V_{T}$ without boundary. But b) already contradicted this.

## §3. Strategy per absurdum for the proof of Proposition 1.13

3.1. Preliminary. As in [6, Me1997, MP1999, MP2002, Po2000], we shall proceed by contradiction. This strategy possesses a considerable advantage: it will not be necessary to control the size of the local subsets of $C$ that are progressively removed, which will simplify substantially the presentation and the understandability of the reasonings. We shall explain how to reduce CR- and $L^{\text {p}}$-removability of $C$ to its $\mathscr{W}$-removability. Also, will show that the $\mathscr{W}$-removability of $C$ can be reduced to the simpler case where the functions which we have to extend are even holomorphic in a neighborhood of $M \backslash C$ in $\mathbb{C}^{n}$. Although such a strategy is essentially carried out in detail in previous references (with some variations), we shall for completeness recall the complete reasonings briefly here, in $\S 3.2$ and in $\S 3.16$ below.
3.2. Global minimality of $M \backslash C$. Background about CR orbits may be found in [Jö1999a, 29]. Using the characteristic nontransversality, we shall apply the following two Lemmas 3.3 and 3.5 about the CR structure of the complement $M \backslash C^{\prime}$, where $C^{\prime} \subset C \subset M^{1}$ is an arbitrary proper closed subset of $C$.

Lemma 3.3. Let $M$ be a $\mathscr{C}^{2, \alpha}$ generic submanifold of $\mathbb{C}^{n}(n \geqslant 2)$ of codimension ( $n-1$ ) and of CR dimension 1 , let $M^{1}$ be a $\mathscr{C}^{2, \alpha}$ one-codimensional submanifold of $M$ which is maximally real in $\mathbb{C}^{n}$ and let $C^{\prime}$ be an arbitrary proper closed subset of $M^{1}$. If $C^{\prime}$ is nontransversal to the characteristic foliation, then for every point $p^{\prime} \in C^{\prime}$, there exists a $\mathscr{C}^{2, \alpha}$ curve $\gamma:[0,1] \rightarrow M^{1}$ satisfying $d \gamma(s) / d s \in T_{\gamma(s)} M^{1} \cap T_{\gamma(s)}^{c} M \backslash\{0\}$ at every $s \in[0,1]$, such that $\gamma(0)=p^{\prime}$ and $\gamma(1)$ does not belong to $C^{\prime}$.
Proof. We proceed by contradiction and we suppose that there exists a point $p^{\prime} \in C^{\prime}$ such that all $\mathscr{C}^{2, \alpha}$ curves $\gamma:[0,1] \rightarrow M^{1}$ with $d \gamma(s) / d s \in$ $T_{\gamma(s)} M^{1} \cap T_{\gamma(s)}^{c} M \backslash\{0\}$ which have origin $p^{\prime}$ are entirely contained in $C^{\prime}$. It follows immediately that all such $\gamma$ are contained in a single characteristic leaf, and that the whole leaf is contained in $C^{\prime}$, contradicting the nontransversality assumption.

Lemma 3.5. With $M, M^{1}, C$ and $C^{\prime}$ as in the preceding lemma, assume that for every point $q^{\prime} \in C^{\prime}$, there exists a $\mathscr{C}^{2, \alpha}$ curve $\gamma:[0,1] \rightarrow M^{1}$ with $d \gamma(s) / d s \in T_{\gamma(s)} M^{1} \cap T_{\gamma(s)}^{c} M \backslash\{0\}$ at every $s \in[0,1]$, such that $\gamma(0)=q^{\prime}$ and $\gamma(1)$ does not belong to $C^{\prime}$. Then the $C R$ orbit in $M \backslash C^{\prime}$ of every point $p \in M \backslash C^{\prime}$ coincides with its CR orbit in $M$ minus $C^{\prime}$, namely

$$
\begin{equation*}
\mathscr{O}_{C R}\left(M \backslash C^{\prime}, p\right)=\mathscr{O}_{C R}(M, p) \backslash C^{\prime} . \tag{3.6}
\end{equation*}
$$

In particular, if $M$ is globally minimal, then $M \backslash C^{\prime}$ is also globally minimal.
Proof. We formulate a preliminary lemma.
Lemma 3.7. Under the assumptions of Lemma 3.5, for every point $q^{\prime} \in$ $C^{\prime} \subset M^{1}$, there exists a $\mathscr{C}^{1, \alpha}$ locally embedded submanifold $\Omega_{q^{\prime}}$ of $M$ passing through $q^{\prime}$ satisfying $T_{q^{\prime}} \Omega_{q^{\prime}}+T_{q^{\prime}} M^{1}=T_{q^{\prime}} M$, such that
(1) $\Omega_{q^{\prime}}$ is a $T^{c} M$-integral submanifold, namely $T_{p}^{c} M \subset T_{p} \Omega_{q^{\prime}}$, for every point $p \in \Omega_{q^{\prime}}$;
(2) $\Omega_{q^{\prime}} \backslash C^{\prime}$ is contained in a single CR orbit of $M$;
(3) $\Omega_{q^{\prime}} \backslash C^{\prime}$ is also contained in a single $C R$ orbit of $M \backslash C^{\prime}$.

Proof. So, let $q^{\prime} \in C^{\prime} \subset M^{1}$. Since $M^{1}$ is generic in $\mathbb{C}^{n}$, there exists a $\mathscr{C}^{1, \alpha}$ vector field $Y$ defined in a neighborhood of $q^{\prime}$ which is complex tangential to $M$ and locally transversal to $M^{1}$, see Figure 6 just below (for easier readability, we have erased the hatching of $C^{\prime}$ in a neighborhood of $q^{\prime}$ ).


Following the integral curve of $Y$ issued from $q^{\prime}$, we can define a point $q_{\epsilon}^{\prime}$ in an $\epsilon$-neighborhood of $q^{\prime}$ which does not belong to $M^{1}$. By assumption, there exists a $\mathscr{C}^{2, \alpha}$ curve $\gamma:[0,1] \rightarrow M^{1}$ with $d \gamma(s) / d s \in T_{\gamma(s)} M^{1} \cap T_{\gamma(s)}^{c} M \backslash\{0\}$ such that $\gamma(0)=q^{\prime}$ and $\gamma(1)$ does not belong to $C^{\prime}$. Furthermore, there exists a vector field $X$ defined in a neighborhood of $\gamma([0,1])$ in $M$ which is complex tangential to $M$, whose restriction to $M^{1}$ is a semi-local section of $\left.T M^{1} \cap T^{c} M\right|_{M^{1}}$, such that $\gamma$ is an integral curve of $X$ and such that $\gamma(1)=$
$\exp (X)\left(q^{\prime}\right) \in M^{1} \backslash C^{\prime}$. We can assume that the vector field $Y$ is defined in the same neighborhood of $\gamma([0,1])$ in $M$ and everywhere transversal to $M^{1}$. If $\epsilon$ is sufficiently small, i.e. if $q_{\epsilon}^{\prime}$ is sufficiently close to $q^{\prime}$, the point $r_{\epsilon}^{\prime}:=\exp (X)\left(q_{\epsilon}^{\prime}\right)$ is still very close to $M^{1}$. Thus, we can define a new point $r^{\prime} \in M^{1}$ to be the unique intersection with $M^{1}$ of the integral curve of $Y$ issued from $r_{\epsilon}^{\prime}$. By choosing $\epsilon$ small enough, the point $r_{\epsilon}^{\prime}$ will be arbitrarily close to $\gamma(1) \notin C^{\prime}$, and consequently, we can assume that $r^{\prime}$ also does not belong to $C^{\prime}$, as drawn in Figure 6. Notice that the integral curve of $X$ from $q_{\epsilon}^{\prime}$ to $r_{\epsilon}^{\prime}$ is contained in $M \backslash M^{1}$, since the flow of $X$ stabilizes $M^{1}$. We deduce that the two points $r_{\epsilon}^{\prime}$ and $r^{\prime}$ belong to the CR orbit $\mathscr{O}_{C R}\left(M \backslash C^{\prime}, q_{\epsilon}^{\prime}\right)$.

Let $\Omega_{r^{\prime}}$ denote a small piece of the orbit (an immersed submanifold) $\mathscr{O}_{C R}\left(M \backslash C^{\prime}, r^{\prime}\right)$ passing through $r^{\prime}$. By standard properties of CR orbits, $\Omega_{r^{\prime}}$ is an embedded $\mathscr{C}^{1, \alpha}$ submanifold of $M \backslash C^{\prime}$ of the same CR dimension as $M \backslash C^{\prime}$. Of course, $r_{\epsilon}^{\prime}$ belongs to $\Omega_{r^{\prime}}$. Since $Y$ is complex tangential to $M$, the submanifold $\Omega_{r^{\prime}}$ is necessarily stretched along the flow lines of $Y$, hence it is transversal to $M^{1}$.

We then define the submanifold $\exp (-X)\left(\Omega_{r^{\prime}}\right)$, close to the point $q^{\prime}$ (we shall argue in a while that it passes in fact through $q^{\prime}$ ). Since the flow of $X$ stabilizes $M^{1}$, it follows that $\exp (-X)\left(\Omega_{r^{\prime}}\right)$ is transversal to $M^{1}$ and that $\exp (-X)\left(\Omega_{r^{\prime}}\right)$ is divided in two parts by its one-codimensional $\mathscr{C}^{1, \alpha}$ submanifold $M^{1} \cap \exp (-X)\left(\Omega_{r^{\prime}}\right)$. Furthermore, we observe that the flow of $X$ stabilizes the two sides of $M^{1}$ in $M$, semi-locally in a neighborhood of $\gamma([0,1])$, since it stabilizes $M^{1}$. Consequently, every integral curve of $X$ issued from every point in $\Omega_{r^{\prime}} \backslash M^{1}$ stays in $M \backslash M^{1}$, hence in $M \backslash C^{\prime}$ and it follows that the submanifold

$$
\begin{equation*}
\exp (-X)\left(\Omega_{r^{\prime}}\right) \backslash M^{1} \tag{3.8}
\end{equation*}
$$

consisting of two connected pieces, is contained in the single CR orbit $\mathscr{O}_{C R}\left(M \backslash C^{\prime}, r^{\prime}\right)$. By the characteristic property of a CR orbit, this means that the two connected pieces of $\exp (-X)\left(\Omega_{r^{\prime}}\right) \backslash M^{1}$ are CR submanifolds of $M \backslash C^{\prime}$ of the same CR dimension as $M \backslash C^{\prime}$. Furthermore, since the intersection $M^{1} \cap \exp (-X)\left(\Omega_{r^{\prime}}\right)$ is one-codimensional, it follows by continuity that the $\mathscr{C}^{1, \alpha}$ submanifold $\exp (-X)\left(\Omega_{r^{\prime}}\right)$ is in fact a CR submanifold of $M$ of the same CR dimension as $M$.

Since $q_{\epsilon}^{\prime}$ belongs to $\exp (-X)\left(\Omega_{r^{\prime}}\right)$ and since the flow of the complex tangent vector field $Y$ necessarily stabilizes the $T^{c} M$-integral submanifold $\exp (-X)\left(\Omega_{r^{\prime}}\right)$, the point $q^{\prime}$ which belongs to an integral curve of $Y$ issued from $q_{\epsilon}^{\prime}$, must belong to the submanifold $\exp (-X)\left(\Omega_{r^{\prime}}\right)$, which we can now denote by $\Omega_{q^{\prime}}:=\exp (-X)\left(\Omega_{r^{\prime}}\right)$, as in Figure 6. This finishes to prove property (1).

Observe that locally in a neighborhood of $q^{\prime}$, the integral curves of $Y$ are transversal to $M^{1}$ and meet $M^{1}$ only at one point. Shrinking if necessary
$\Omega_{q^{\prime}}$ a little bit and using integral curves of $Y$ from both sides of $M^{1}$, we attain points in $\left(M^{1} \backslash C^{\prime}\right) \cap \Omega_{q^{\prime}}$. Hence $\Omega_{q^{\prime}} \backslash C^{\prime}$ is contained in the single CR orbit $\mathscr{O}_{C R}\left(M \backslash C^{\prime}, r^{\prime}\right)$, which proves property (3). Using again $Y$ to attain points of $C^{\prime} \cap \Omega_{q^{\prime}}$, we deduce also that $\Omega_{q^{\prime}}$ is contained in the single CR orbit $\mathscr{O}_{C R}\left(M, r^{\prime}\right)$, which proves property (2).

The proof of Lemma 3.7 is complete.
We can now prove Lemma 3.5. It suffices to establish that for every two points $p \in M \backslash C^{\prime}$ and $q \in \mathscr{O}_{C R}(M, p) \backslash C^{\prime}$, the point $q$ belongs in fact to $\mathscr{O}_{C R}\left(M \backslash C^{\prime}, p\right)$.

Since $q \in \mathscr{O}_{C R}(M, p)$, there exists a piecewise $\mathscr{C}^{2, \alpha}$ curve $\lambda:[0,1] \rightarrow M$ with $\lambda(0)=p, \lambda(1)=q$ and $d \lambda(s) / d s \in T_{\lambda(s)}^{c} M \backslash\{0\}$ at every $s \in[0,1]$ at which $\lambda$ is differentiable. For every $s$ with $0 \leqslant s \leqslant 1$, we define a local $\mathscr{C}^{1, \alpha}$ submanifold $\Omega_{\lambda(s)}$ of $M$ passing through $\lambda(s)$ as follows:

- if $\lambda(s)$ does not belong to $C^{\prime}$, choose for $\Omega_{\lambda(s)}$ a piece of the CR orbit of $\lambda(s)$ in $M \backslash C^{\prime}$;
- if $\lambda(s)$ belongs to $C^{\prime}$, choose for $\Omega_{\lambda(s)}$ the submanifold constructed in Lemma 3.7 above.

Then for each $s$, the complement $\Omega_{\lambda(s)} \backslash C^{\prime}$ is contained in a single CR orbit of $M \backslash C^{\prime}$. Since each $\Omega_{\lambda(s)}$ is a $T^{c} M$-integral submanifold, a neighborhood of $\lambda(s)$ in the arc $\lambda([0,1])$ is necessarily contained $\Omega_{\gamma(s)}$. By compactness of $[0,1]$, we can therefore find an integer $k \geqslant 1$ and real numbers (3.9)
$0=s_{1}<r_{1}<t_{1}<s_{2}<r_{2}<t_{2}<\cdots \cdots<s_{k-1}<r_{k-1}<t_{k-1}<s_{k}=1$, such that $\lambda([0,1])$ is covered by $\Omega_{\lambda(0)} \cup \Omega_{\lambda\left(s_{2}\right)} \cup \cdots \cup \Omega_{\lambda\left(s_{k-1}\right)} \cup \Omega_{\lambda(1)}$ and such that in addition, $\lambda\left(\left[r_{j}, t_{j}\right]\right) \subset \Omega_{\lambda\left(s_{j}\right)} \cap \Omega_{\lambda\left(s_{j+1}\right)}$ for $j=1, \ldots, k-1$.
Lemma 3.10. The following union minus $C^{\prime}$

$$
\begin{equation*}
\left(\Omega_{\lambda(0)} \cup \Omega_{\lambda\left(s_{2}\right)} \cup \cdots \cdots \cup \Omega_{\lambda\left(s_{k-1}\right)} \cup \Omega_{\lambda(1)}\right) \backslash C^{\prime} \tag{3.11}
\end{equation*}
$$

is contained in a single $C R$ orbit of $M \backslash C^{\prime}$.
Proof. It suffices to prove that for $j=1, \ldots, k-1$, the union $\left(\Omega_{\lambda\left(s_{j}\right)} \cup\right.$ $\left.\Omega_{\lambda\left(s_{j+1}\right)}\right) \backslash C^{\prime}$ minus $C^{\prime}$ is contained in a single CR orbit of $M \backslash C^{\prime}$.

Two cases are to be considered. Firstly, assume that $\lambda\left(\left[r_{j}, t_{j}\right]\right)$ is not contained in $C^{\prime}$, namely there exists $u_{j}$ with $r_{j} \leqslant u_{j} \leqslant t_{j}$ such that

$$
\begin{equation*}
\gamma\left(u_{j}\right) \in\left(\Omega_{\lambda\left(s_{j}\right)} \cap \Omega_{\lambda\left(s_{j+1}\right)}\right) \backslash C^{\prime} . \tag{3.12}
\end{equation*}
$$

Because $\Omega_{\lambda\left(s_{j}\right)} \backslash C^{\prime}$ and $\Omega_{\lambda\left(s_{j+1}\right)} \backslash C^{\prime}$ are both contained in a single CR orbit of $M \backslash C^{\prime}$, it follows from (3.12) that they are contained in the same CR orbit of $M \backslash C^{\prime}$, as desired.

Secondly, assume that $\lambda\left(\left[r_{j}, t_{j}\right]\right)$ is contained in $C^{\prime}$. Choose $u_{j}$ arbitrary with $r_{j} \leqslant u_{j} \leqslant t_{j}$. By construction, $\lambda\left(u_{j}\right)$ belongs to $\Omega_{\lambda\left(s_{j}\right)} \cap \Omega_{\lambda\left(s_{j+1}\right)}$ and both $\Omega_{\lambda\left(s_{j}\right)}$ and $\Omega_{\lambda\left(s_{j+1}\right)}$ are $T^{c} M$-integral submanifolds of $M$ passing through the point $\lambda\left(u_{j}\right)$. Let $Y$ be a local section of $T^{c} M$ defined in a neighborhood of $\lambda\left(u_{j}\right)$ which is not tangent to $M^{1}$ at $\lambda\left(u_{j}\right)$. On the integral curve of $Y$ issued from $\lambda\left(u_{j}\right)$, we can choose a point $\lambda\left(u_{j}\right)_{\epsilon}$ arbitrarily close to $\lambda\left(u_{j}\right)$ which does not belong to $C^{\prime}$. Since $Y$ is a section of $T^{c} M$, it is tangent to both $\Omega_{\lambda\left(s_{j}\right)}$ and $\Omega_{\lambda\left(s_{j+1}\right)}$, hence we deduce that

$$
\begin{equation*}
\gamma\left(u_{j}\right)_{\epsilon} \in\left(\Omega_{\lambda\left(s_{j}\right)} \cap \Omega_{\lambda\left(s_{j+1}\right)}\right) \backslash C^{\prime} . \tag{3.13}
\end{equation*}
$$

Consequently, as in the first case, it follows that $\Omega_{\lambda\left(s_{j}\right)} \backslash C^{\prime}$ and $\Omega_{\lambda\left(s_{j+1}\right)} \backslash C^{\prime}$ are both contained in the same CR orbit of $M \backslash C^{\prime}$, as desired.

Since $p$ and $q$ belong to the set (3.11), we deduce that $p=\lambda(0) \in M \backslash C^{\prime}$ and $q=\lambda(1) \in \mathscr{O}_{C R}(M, p) \backslash C^{\prime}$ belong to the same CR orbit of $M \backslash C^{\prime}$, which completes the proof of Lemma 3.5.
3.14. Reduction of CR- and of $L^{\mathrm{p}}$-removability to $\mathscr{W}$-removability. Thus, in Proposition 1.13, $M \backslash C$ is globally minimal. It follows ([Me1994, Jö1996]) that there exists a wedgelike domain attached to $M \backslash C$ to which $\mathscr{C}_{C R}^{0}(M)$ extends holomorphically. Consequently, the CR-removability of $C \subset M^{1}$ claimed in Proposition 1.13 is an immediate consequence of its $\mathscr{W}$-removability. Based on the construction of analytic discs half-attached to $M^{1}$ which will be achieved in Section 7, we shall also be able to settle the reduction of $L^{\text {p}}$-removability in Section 10.

Lemma 3.15. Under the assumptions of Proposition 1.13, if the closed subset $C \subset M^{1}$ is $\mathscr{W}$-removable, then it is $L^{\mathrm{p}}$-removable, for all p with $1 \leqslant p \leqslant \infty$.
3.16. Strategy per absurdum: removal of a single point of the residual non-removable subset. Thus, it suffices to establish that $C$ is $\mathscr{W}$ removable. Let us fix a (nonempty) wedgelike domain $\mathscr{W}_{1}$ attached to $M \backslash C$. Our precise goal is to establish that there exists a wedgelike domain $\mathscr{W}_{2}$ attached to $M$ (including $C$ ) and a wedgelike domain $\mathscr{W}_{3} \subset \mathscr{W}_{1} \cap \mathscr{W}_{2}$ attached to $M \backslash C$ such that for every holomorphic function $f \in \mathscr{O}\left(\mathscr{W}_{1}\right)$, there exists a holomorphic function $F \in \mathscr{O}\left(\mathscr{W}_{2}\right)$ which coincides with $f$ in $\mathscr{W}_{3}$. At first, we need some more definitions.

Let $C^{\prime}$ be an arbitrary closed subset of $C$. We shall say that $M \backslash C^{\prime}$ enjoys the wedge extension property if there exist a wedgelike domain $\mathscr{W}_{2}^{\prime}$ attached to $M \backslash C^{\prime}$ and a wedgelike subdomain $\mathscr{W}_{3}^{\prime} \subset \mathscr{W}_{1} \cap \mathscr{W}_{2}^{\prime}$ attached to $M \backslash C$ such that, for every function $f \in \mathscr{O}\left(\mathscr{W}_{1}\right)$, there exists a function $F^{\prime} \in \mathscr{O}\left(\mathscr{W}_{2}^{\prime}\right)$ which coincides with $f$ in $\mathscr{W}_{3}^{\prime}$.

The notion of wedge removability can be localized as follows. Let again $C^{\prime} \subset C$ be arbitrary. We shall say that a point $p^{\prime} \in C^{\prime}$ is locally $\mathscr{W}$ removable with respect to $C^{\prime}$ if for every wedgelike domain $\mathscr{W}_{1}^{\prime}$ attached to $M \backslash C^{\prime}$, there exists a neighborhood $U^{\prime}$ of $p^{\prime}$ in $M$, there exists a wedgelike domain $\mathscr{W}_{2}^{\prime}$ attached to $\left(M \backslash C^{\prime}\right) \cup U^{\prime}$ and there exists a wedgelike subdomain $\mathscr{W}_{3}^{\prime} \subset \mathscr{W}_{1}^{\prime} \cap \mathscr{W}_{2}^{\prime}$ attached to $M \backslash C^{\prime}$ such that for every holomorphic function $f \in \mathscr{O}\left(\mathscr{W}_{1}^{\prime}\right)$, there exists a holomorphic function $F^{\prime} \in \mathscr{O}\left(\mathscr{W}_{2}^{\prime}\right)$ which coincides with $f$ in $\mathscr{W}_{3}^{\prime}$.

Supppose now that $M \backslash C_{1}^{\prime}$ and $M \backslash C_{2}^{\prime}$ enjoy the wedge extension property, for some two closed subsets $C_{1}^{\prime}, C_{2}^{\prime} \subset C$. Using the CR edge-of-thewedge theorem ([Tu1994]), the two wedgelike domains attached to $M \backslash C_{1}^{\prime}$ and to $M \backslash C_{2}^{\prime}$ can be glued together (after appropriate shrinking) to produce a wedgelike domain $\mathscr{W}_{1}$ attached to $M \backslash\left(C_{1}^{\prime} \cap C_{2}^{\prime}\right)$ in such a way that $M \backslash\left(C_{1}^{\prime} \cap C_{2}^{\prime}\right)$ enjoys the $\mathscr{W}$-extension property. Also, if $M \backslash C^{\prime}$ enjoys the wedge extension property and if $p^{\prime} \in C^{\prime}$ is locally $\mathscr{W}$-removable with respect to $C^{\prime}$, then again by means of the CR edge of the wedge theorem, it follows that there exists a neighborhood $U^{\prime}$ of $p^{\prime}$ in $M$ such that $\left(M \backslash C^{\prime}\right) \cup U^{\prime}$ enjoys the wedge extension property.

Based on these preliminary remarks, we define the following set of closed subsets of $C$ :

$$
\begin{equation*}
\mathscr{C}:=\left\{C^{\prime} \subset C \text { closed } ; M \backslash C^{\prime} \text { enjoys the } \mathscr{W} \text {-extension property }\right\} . \tag{3.17}
\end{equation*}
$$

Then the residual set

$$
\begin{equation*}
C_{\mathrm{nr}}:=\bigcap_{C^{\prime} \in \mathscr{C}} C^{\prime} \tag{3.18}
\end{equation*}
$$

is a closed subset of $M^{1}$ contained in $C$. It follows from the above (abstract nonsense) considerations that $M \backslash C_{\mathrm{nr}}$ enjoys the wedge extension property and that no point of $C_{\mathrm{nr}}$ is locally $\mathscr{W}$-removable with respect to $C_{\mathrm{nr}}$. Here, we may think that the letters "nr" abbreviate "non-removable", because by the very definition of $C_{\mathrm{nr}}$, none of its points should be locally $\mathscr{W}$-removable. Notice also that $M \backslash C_{\mathrm{nr}}$ is globally minimal, thanks to Lemma 3.5.

Clearly, to establish Proposition 1.4, it is enough to show that $C_{\mathrm{nr}}=\emptyset$.
We shall argue indirectly (by contradiction) and assume that $C_{\mathrm{nr}} \neq \emptyset$. In order to derive a contradiction, it clearly suffices to show that there exists at least one point $p \in C_{\mathrm{nr}}$ which is in fact locally $\mathscr{W}$-removable with respect to $C_{\mathrm{nr}}$.

At this point, we notice that the main assumption that $C$ is nontransversal to $\mathrm{F}_{M^{1}}^{c}$ in Proposition 1.13 implies trivially that every closed subset $C^{\prime}$ of $C$ is also nontransversal to $\mathrm{F}_{M^{1}}^{c}$. In particular $C_{\mathrm{nr}}$ is nontransversal to $\mathrm{F}_{M^{1}}^{c}$. Consequently, by following a per absurdum strategy, we are led to prove a statement wich is totally similar to Proposition 1.13 except that we now
have only to establish that a single point of $C_{\mathrm{nr}}$ is locally $\mathscr{W}$-removable with respect to $C_{\mathrm{nr}}$. This preliminary logical consideration will simplify substantially the whole architecture of the proof. Another important advantage of this strategy is that we are even allowed to select a special point $p_{\mathrm{sp}}$ of $C_{\mathrm{nr}}$ by requiring some nice geometric disposition of $C_{\mathrm{nr}}$ in a neighborhood of $p_{\mathrm{sp}}$ before removing it. Sections 4 and 5 below are devoted to such a selection.

So we are led to show that for every wedgelike domain $\mathscr{W}_{1}$ attached to $M \backslash C_{\mathrm{nr}}$, there exists a special point $p_{\mathrm{sp}} \in C_{\mathrm{nr}}$, there exists a neighborhood $U_{p_{\mathrm{sp}}}$ of $p_{\mathrm{sp}}$ in $M$, there exists a wedgelike domain $\mathscr{W}_{2}$ attached to $\left(M \backslash C_{\mathrm{nr}}\right) \cup$ $U_{p_{\mathrm{sp}}}$ and there exists a wedgelike domain $\mathscr{W}_{3} \subset \mathscr{W}_{1} \cap \mathscr{W}_{2}$ attached to $M \backslash C_{\mathrm{nr}}$ such that for every holomorphic function $f \in \mathscr{O}\left(\mathscr{W}_{1}\right)$, there exists a function $F \in \mathscr{O}\left(\mathscr{W}_{2}\right)$ which coincides with $f$ in $\mathscr{W}_{3}$.

A further convenient simplification of the task may be achieved by deforming slightly $M$ inside the wedge $\mathscr{W}_{1}$ attached to $M \backslash C_{\mathrm{nr}}$. Indeed, by means of a partition of unity, we may perform arbitrarily small $\mathscr{C}^{2, \alpha}$ deformations $M^{d}$ of $M$ leaving $C_{\mathrm{nr}}$ fixed and moving $M \backslash C_{\mathrm{nr}}$ inside the wedgelike domain $\mathscr{W}_{1}$. Furthermore, we can make $M^{d}$ to depend on a single small real parameter $d \geqslant 0$ with $M^{0}=M$ and $M^{d} \backslash C_{\mathrm{nr}} \subset \mathscr{W}_{1}$ for all $d>0$. Now, the wedgelike domain $\mathscr{W}_{1}$ becomes a neighborhood of $M^{d}$ in $\mathbb{C}^{n}$. Let us denote by $\Omega$ this neighborhood. After some substantial technical work has been performed, at the very end of the proof of Proposition 1.13 (Section 9), we shall construct a local wedge $\mathscr{W}_{p_{\mathrm{sp}}}^{d}$ of edge $M^{d}$ at $p_{\mathrm{sp}}$ by means of small Bishop analytic discs glued to $M^{d}$, to $\Omega$ and to another subset (which we will call a half-wedge, see Section 4 below) such that every holomorphic function $f \in \mathscr{O}(\Omega)$ extends holomorphically to $\mathscr{W}_{p_{\mathrm{sp}}}^{d}$. Using the stability of Bishop's equation under perturbations, we shall argue in $\S 9.23$ below that all our constructions are stable under such small deformations ${ }^{35}$, whence in the limit $d \rightarrow 0$, the wedges $\mathscr{W}_{p_{\mathrm{sp}}}^{d}$ tend smoothly to a local wedge $\mathscr{W}_{p_{\mathrm{sp}}}:=\mathscr{W}_{p_{\mathrm{sp}}}^{0}$ of edge a neighborhood $U_{p_{\mathrm{sp}}}$ of $p_{\mathrm{sp}}$ in $M^{0} \equiv M$. In addition, we shall derive univalent holomorphic extension to $\mathscr{W}_{p_{\mathrm{sp}}}$. Finally, using again the edge of the wedge theorem to fill in the space between $\mathscr{W}_{1}$ and $\mathscr{W}_{p_{\mathrm{sp}}}$, possibly after appropriate contractions of these two wedgelike domains, we may construct a wedgelike domain $\mathscr{W}_{2}$ attached to $(M \backslash C) \cup U_{p_{\mathrm{sp}}}$ and a wedgelike domain $\mathscr{W}_{3} \subset \mathscr{W}_{1} \cap \mathscr{W}_{p_{\mathrm{sp}}}$ attached to $M \backslash C$ such that for every holomorphic function $f \in \mathscr{O}\left(\mathscr{W}_{1}\right)$, there exists a function $F \in \mathscr{O}\left(\mathscr{W}_{2}\right)$ which coincides with $f$ in $\mathscr{W}_{3}$. In conclusion, we will thus reach the desired contradiction to the definition of $C_{\mathrm{nr}}$.

[^34]To summarize, we have essentially shown that it suffices to prove Proposition 1.13 with two extra simplifying assumptions.

- Instead of functions which are holomorphic in a wedgelike domain attached to $M \backslash C_{\mathrm{nr}}$, we consider functions which are holomorphic in a neighborhood $\Omega$ of $M \backslash C_{\mathrm{nr}}$ in $\mathbb{C}^{n}$.
- Proceeding by contradiction, it suffices to remove at least one point of $C_{\mathrm{nr}}$.

After replacing $C_{\mathrm{nr}}$ by $C$ and $M^{d}$ by $M$, we are led to establish the following main assertion, to which Proposition 1.13 is reduced.
Theorem 3.19. Let $M$ be a $\mathscr{C}^{2, \alpha}$ globally minimal generic submanifold of $\mathbb{C}^{n}$ of codimension $(n-1)$ hence of $C R$ dimension 1 , let $M^{1} \subset M$ be a $\mathscr{C}^{2, \alpha}$ one-codimensional submanifold which is maximally real in $\mathbb{C}^{n}$, and let $C$ be a nonempty proper closed subset of $M^{1}$. Assume that $C$ is nontransversal to the characteristic foliation $\mathrm{F}_{M^{1}}^{c}$. Let $\Omega$ be an arbitrary neighborhood of $M \backslash C$ in $\mathbb{C}^{n}$. Then there exist a special point $p_{\mathrm{sp}} \in C$, there exists a local wedge $\mathscr{W}_{p_{\mathrm{sp}}}$ of edge $M$ at $p_{\mathrm{sp}}$ and there exists a subneighborhood $\Omega^{\prime} \subset \Omega$ of $M \backslash C$ in $\mathbb{C}^{n}$ with $\mathscr{W}_{p_{\mathrm{sp}}} \cap \Omega^{\prime}$ connected such that for every holomorphic function $f \in \mathscr{O}(\Omega)$, there exists a holomorphic function $F \in \mathscr{O}\left(\mathscr{W}_{p_{\mathrm{sp}}} \cup \Omega^{\prime}\right)$ which coincides with $f$ in $\Omega^{\prime}$.

## §4. Construction of a Semi-Local half wedge

4.1. Preliminary. Later, in Section 5 below, we will analyze the assumption of characteristic nontransversality, but in the present Section 4, we shall not at all take account of it. With $M$ and $M^{1}$ as above, let $\gamma:[-1,1] \rightarrow M^{1}$ be a $\mathscr{C}^{2, \alpha}$ curve, embedding the segment $[-1,1]$ into $M$, but not necessarily characteristic. In the present section, our goal is to construct a semi-local half-wedge attached to a one-sided neighborhood of $M^{1}$ along $\gamma$ with the property that holomorphic functions in the neighborhood $\Omega$ of $M \backslash C$ in $\mathbb{C}^{n}$ do extend holomorphically to this half-wedge. First of all, we need to define what we understand by the term "half-wedge".
4.2. Three equivalent definitions of attached half-wedges. We shall denote by $\Delta_{n}(p, \delta)$ the open polydisc centered at $p \in \mathbb{C}^{n}$ of radius $\delta>0$. Let $p_{1} \in M^{1}$, and let $C_{1}$ be an open infinite cone in the normal space $T_{p_{1}} \mathbb{C}^{n} / T_{p_{1}} M$. Classically, a local wedge of edge $M$ at $p_{1}$ is a set of the form: $\mathscr{W}_{p_{1}}:=\left\{p+\mathrm{c}_{1}: p \in M, \mathrm{c}_{1} \in \mathrm{C}_{1}\right\} \cap \Delta_{n}\left(p_{1}, \delta_{1}\right)$, for some $\delta_{1}>0$. Sometimes, we shall use the following terminology ([Tu1994, Me1994]): if $v_{1}$ is a nonzero vector in $T_{p_{1}} \mathbb{C}^{n} / T_{p_{1}} M$, we shall say that $\mathscr{W}_{p_{1}}$ is a local wedge at $\left(p_{1}, v_{1}\right)$. Thus, the positive half-line $\mathbb{R}^{+} \cdot v_{1}$ generated by the vector $v_{1}$ is locally contained in the wedge $\mathscr{W}_{p_{1}}$.

For us, a local half-wedge of edge $M$ at $p_{1}$ will be a set of the form

$$
\begin{equation*}
\mathscr{H} \mathscr{W}_{p_{1}}^{+}:=\left\{p+\mathrm{c}_{1}: p \in U_{1} \cap\left(M^{1}\right)^{+}, \mathrm{c}_{1} \in \mathrm{C}_{1}\right\} \cap \Delta_{n}\left(p_{1}, \delta_{1}\right) . \tag{4.3}
\end{equation*}
$$

This yields a first definition and we shall formulate two further equivalent definitions.

Let $\Delta$ denote the unit disc in $\mathbb{C}$, let $\partial \Delta$ denote its boundary, the unit circle and let $\bar{\Delta}=\Delta \cup \partial \Delta$ denote its closure. Throughout this article, we shall denote by $\zeta=\rho e^{i \theta}$ the variable of $\bar{\Delta}$ with $0 \leqslant \rho \leqslant 1$ and with $|\theta| \leqslant \pi$.

Concretely, our real local half-wedges (as to be constructed in this section) will be defined by means of a $\mathbb{C}^{n}$-valued map $(t, \chi, \nu, \rho) \longmapsto \mathscr{Z}_{t, \chi, \nu}(\rho)$ of class $\mathscr{C}^{2, \alpha-0}=\bigcap_{\beta<\alpha} \mathscr{C}^{2, \beta}$ which comes from a parametrized family of analytic discs of the form $\zeta \mapsto \mathscr{Z}_{t, \chi, \nu}(\zeta)$, where the parameters $t \in \mathbb{R}^{n-1}$, $\chi \in \mathbb{R}^{n}, \nu \in \mathbb{R}$ satisfy $|t|<\varepsilon,|\chi|<\varepsilon,|\nu|<\varepsilon$ for some small $\varepsilon>0$, and where $\mathscr{Z}_{t, \chi, \nu}(\zeta)$ is holomorphic with respect to $\zeta$ in $\Delta$ and $\mathscr{C}^{2, \alpha-0}$ in $\bar{\Delta}$. This mapping will satisfy the following three properties:
(i) the map $\chi \mapsto \mathscr{Z}_{0, \chi, 0}(1)$ is a diffeomorphism onto a neighborhood of $p_{1}$ in $M^{1}$, the map $(\chi, \nu) \mapsto \mathscr{Z}_{0, \chi, \nu}(1)$ is a diffeomorphism onto a neighborhood of $p_{1}$ in $M$, and $\left(M^{1}\right)^{+}$corresponds to $\nu>0$;
(ii) $\mathscr{Z}_{t, 0,0}(1)=p_{1}$ and the half-boundary $\mathscr{Z}_{t, \chi, \nu}\left(\left\{e^{i \theta}:|\theta| \leqslant \frac{\pi}{2}\right\}\right)$ is contained in $M$ for all $t$, all $\chi$ and all $\nu$;
(iii) the vector $v_{1}:=\frac{\partial \mathscr{R}_{0,0,0}}{\partial \theta}(1) \in T_{p_{1}} \mathbb{C}^{n}$ is nonzero and belongs to $T_{p_{1}} M^{1}$. Furthermore, the rank of the $\mathbb{R}^{n-1}$-valued $\mathscr{C}^{1, \alpha-0}$ mapping

$$
\begin{equation*}
\mathbb{R}^{n-1} \ni t \longmapsto \frac{\partial \mathscr{Z}_{t, 0,0}}{\partial \theta}(1) \in T_{p_{1}} M^{1} \bmod \left(T_{p_{1}} M^{1} \cap T_{p_{1}}^{c} M\right) \cong \mathbb{R}^{n-1} \tag{4.4}
\end{equation*}
$$

is maximal equal to $(n-1)$ at $t=0$.
By holomorphicity of the map $\zeta \mapsto \mathscr{Z}_{t, \chi, \nu}(\zeta)$, we have $\frac{\partial \mathscr{A}_{t, \chi, \nu}}{\partial \theta}(1)=J$. $\frac{\partial \mathscr{H}_{t, x, \nu}}{\partial \rho}(1)$, where $J$ denotes the complex structure of $T \mathbb{C}^{n}$. Since $J$ induces an isomorphism $T_{p_{1}} M / T_{p_{1}}^{c} M \xrightarrow{\sim} T_{p_{1}} \mathbb{C}^{n} / T_{p_{1}} M$, it follows from property (iii) above that the vectors $\frac{\partial \mathscr{F}_{t, 0,0}}{\partial \rho}(1)$ cover an open cone containing $J v_{1}$ in the quotient space $T_{p_{1}} M / T_{p_{1}}^{c} M$, as $v$ varies. Then a local half-wedge of edge $\left(M^{1}\right)^{+}$at $p_{1}$ will be a set of the form
$\mathscr{H} \mathscr{W}_{p_{1}}^{+}:=\left\{\mathscr{Z}_{t, \chi, \nu}(\rho) \in \mathbb{C}^{n}:|t|<\varepsilon,|\chi|<\varepsilon, 0<\nu<\varepsilon, 1-\varepsilon<\rho<1\right\}$.
We mention that a complete local wedge of edge $M$ at $p_{1}$ can also be produced by such a family $\mathscr{Z}_{t, \chi, \nu}(\zeta)$ and may be defined as $\mathscr{W}_{p_{1}}:=\left\{\mathscr{Z}_{t, \chi, \nu}(\rho)\right.$ : $|t|<\varepsilon,|\chi|<\varepsilon,|\nu|<\varepsilon, 1-\varepsilon<\rho<1\}$, the parameter $\nu$ being allowed to be negative (the points $\mathscr{Z}_{0, \chi, \nu}(1)$ then lie behind the "wall" $M^{1}$, namely in $\left.\left(M^{1}\right)^{-}\right)$.

As may be checked, this second definition of a half-wedge is essentially equivalent to the first one, in the sense that a half-wedge in the first sense always contains a half-wedge in the second sense, and vice versa, after appropriate shrinkings.

Furthermore, we may distinguish two cases: either the vector $v_{1}=$ $\frac{\partial \mathscr{O}, 0,0}{\partial \theta}(1)$ is not complex-tangential to $M$ at $p_{1}$ (generically true) or it is. In the first case, after possibly shrinking $\varepsilon>0$, it may be checked that a local half-wedge of edge $\left(M^{1}\right)^{+}$coincides with the intersection of a (full) local wedge $\mathscr{W}_{p_{1}}$ of edge $M$ at $p_{1}$ with a one-sided neighborhood $\left(N^{1}\right)^{+}$of a local hypersurface $N^{1}$ which intersects $M$ locally transversally along $M^{1}$ at $p_{1}$, as drawn in the left hand side of the following figure, where $M$ is of codimension two.


In the second case, $v_{1}=\frac{\partial \mathscr{E}_{0,0,0}}{\partial \theta}(1)$ belongs to the characteristic direction $T_{p_{1}} M^{1} \cap T_{p_{1}}^{c} M$, so the vector $-J v_{1}$ which is interiorly tangent to the disc $\mathscr{Z}_{0,0,0}(\Delta)$, is tangent to $M$ at $p_{1}$, is not tangent to $M^{1}$ at $p_{1}$, but points towards $\left(M^{1}\right)^{+}$at $p_{1}$. It may then be checked that a local half-wedge of edge $\left(M^{1}\right)^{+}$coincides with a local wedge $\mathscr{W}_{p_{1}}^{1}$ of edge $M^{1}$ at $\left(p_{1},-J v_{1}\right)$ containing the side $\left(M^{1}\right)^{+}$in its interior, as drawn in the right hand side of Figure 7 above, in which $M$ is of codimension one. This provides the third and the most intuitive definition of the notion of local half-wedge.

Finally, we may define the desired notion of a semi-local attached halfwedge. Let $\gamma:[-1,1] \rightarrow M^{1}$ be an embedded $\mathscr{C}^{2, \alpha}$ segment in $M^{1}$. We fix a coherent family of one-sided neighborhoods $\left(M_{\gamma}^{1}\right)^{+}$of $M^{1}$ in $M$ along $\gamma$. A half-wedge attached to a one-sided neighborhood $\left(M_{\gamma}^{1}\right)^{+}$of $M^{1}$ along $\gamma$ is a domain $\mathscr{H} \mathscr{W}_{\gamma}^{+}$which contains a local half-wedge of edge $\left(M^{1}\right)^{+}$at $\gamma(s)$ for every $s \in[-1,1]$. Another essentially equivalent definition is to require that we have a family $\mathscr{Z}_{t, \chi, \nu: s}(\rho)$ of maps smoothly varying with the parameter $s$ such that at each point $\gamma(s)=\mathscr{Z}_{t, \chi, \nu: s}(1)$, the three conditions
(i), (ii) and (iii) introduced above to define a local half-wedge are satisfied. Intuitively speaking, the direction of the cone defining the local half wedge at the point $\gamma(s)$ varies smoothly with respect to $s$.


Fig. 8: Semi-local half-wedge attached to a hypersurface
Proposition 4.6. Let $M, M^{1}, C, \Omega$ be as in Theorem 3.19 and let $\gamma:$ $[-1,1] \rightarrow M^{1}$ be an embedded $\mathscr{C}^{2, \alpha}$ curve. Then there exist a neighborhood $V_{\gamma}$ of $\gamma[-1,1]$ in $M$, there exists a semi-local one-sided neighborhood $\left(M_{\gamma}^{1}\right)^{+}$of $M^{1}$ in $M$ along $\gamma$ and there exists a semi-local half-wedge $\mathscr{H}^{2} \mathscr{W}_{\gamma}^{+}$ attached to $\left(M_{\gamma}^{1}\right)^{+} \cap V_{\gamma}$ with $\Omega \cap \mathscr{H} \mathscr{W}_{\gamma}^{+}$connected (shrinking $\Omega$ if necessary) such that for every holomorphic function $f \in \mathscr{O}(\Omega)$, there exists a holomorphic function $F \in \mathscr{O}\left(\mathscr{H} \mathscr{W}_{\gamma}^{+} \cup \Omega\right)$ with $\left.F\right|_{\Omega}=f$.

To build $\mathscr{H} \mathscr{W}_{\gamma}^{+}$, we shall construct families of analytic discs with boundaries in $\left(M_{\gamma}^{1}\right)^{+}$. First of all, we need to formulate a special, adapted version of the so-called approximation theorem ([BT1981]).
4.7. Local approximation theorem. As observed in [Me1997, MP1999, MP2002], when dealing with natural geometric assumptions on the singularity to be removed - for instance, a two-codimensional singularity $N \subset M$ with $T_{p} N \supset T_{p}^{c} M$ at some points $p \in N$ or metrically thin singularities $E \subset M$ with $\mathrm{H}^{\operatorname{dim} M_{-2}}(E)=0-$ it is impossible to show a priori that continuous CR functions on $M$ minus the singularity are approximable by polynomials, which justifies the introduction of deformations of $M$ and the use of the continuity principle. But in the present situation, the genericity of $M^{1}$ helps much.

Lemma 4.8. Let $p_{1} \in M^{1}$ and denote by $\left(M^{1}\right)^{ \pm}$the two sides in which $M$ is divided by $M^{1}$ near $p_{1}$. Then there exist two neighborhoods $U_{1}$ and $V_{1}$ of $p_{1}$ in $M$ with $V_{1} \subset \subset U_{1}$ such that for every continuous $C R$ function $f \in \mathscr{C}_{C R}^{0}\left(\left(M^{1}\right)^{+} \cap U_{1}\right)$, there exists a sequence of holomorphic polynomials $\left(P_{\nu}\right)_{\nu \in \mathbb{N}}$ wich converges uniformly to $f$ on $\left(M^{1}\right)^{+} \cap V_{1}$.

Proof. We adapt [BT1981]. In coordinates $z=\left(z_{1}, \ldots, z_{n}\right)=x+i y \in \mathbb{C}^{n}$ vanishing at $p_{1}$, we can assume that the tangent plane to $M^{1}$ at $p_{1}$ is $\mathbb{R}^{n}=$ $\{y=0\}$. We include $M^{1}$ in a one-parameter family of maximally real submanifolds $M_{u}^{1} \subset M$, where $u \in \mathbb{R}^{d}$ is small, with $M_{0}^{1}=M^{1}$, such that $M_{u}^{1} \cap V_{1}$ makes a foliation of $M \cap V^{1}$, for some neighborhood $V_{1} \subset \subset U^{1}$ of $p^{1}$ in $M$ and such that $M_{u}^{1} \cap V_{1}$ is contained in $\left(M^{1}\right)^{+}$for $u>0$. In addition, we can assume that all the $M_{u}^{1}$ coincide with $M_{0}^{1}$ in a neighborhood of $\partial U_{1}$.

Assume to simplify that the CR function $f$ is of class $\mathscr{C}^{1}$ on $\left(M^{1}\right)^{+} \cap U_{1}$, let $\tau \in \mathbb{R}$ with $\tau>0$, fix $u>0$, whence $M_{u}^{1} \cap V^{1}$ is contained in $\left(M^{1}\right)^{+}$, let $\widehat{z} \in\left(M^{1}\right)^{+} \cap V^{1}$ be an arbitrary point and consider the following convolution integral of $f$ with the Gaussian kernel:

$$
\begin{equation*}
G_{\tau} f(\widehat{z}):=\left(\frac{\tau}{\pi}\right)^{n / 2} \int_{U_{1} \cap M^{1}} e^{-\tau(z-\bar{z})^{2}} f(z) d z \tag{4.9}
\end{equation*}
$$

where $(z-\widehat{z})^{2}:=\left(z_{1}-\widehat{z}_{1}\right)^{2}+\cdots+\left(z_{n}-\widehat{z}_{n}\right)^{2}$ and $d z:=d z_{1} \wedge \cdots \wedge d z_{n}$. We claim that the value of $G_{\tau} f(\widehat{z})$ is the same if we replace integration on $U_{1} \cap M^{1}$ by integration on $U_{1} \cap M_{\widehat{u}}^{1}$, where $M_{\widehat{u}}^{1}$ is the unique maximally real leaf to which $\widehat{z}$ belong. Indeed, the region between $M^{1}$ and $M_{\widehat{u}}^{1}$ is an open diaphragm-like subset $\Sigma \subset M$ whose boundary $\partial \Sigma=M^{1}-M_{\widehat{t}}^{1}$ is entirely contained in $\left(M^{1}\right)^{+} \cap U_{1}$ and then Stokes' theorem gives:

$$
\begin{align*}
G_{\tau} f(\widehat{z}) & =\left(\frac{\tau}{\pi}\right)^{n / 2} \int_{U_{1} \cap M_{\hat{t}}} e^{-\tau(z-\widehat{z})^{2}} f(z) d z+\left(\frac{\tau}{\pi}\right)^{n / 2} \int_{\Sigma} d\left(e^{-\tau(z-\widehat{z})^{2}} f(z) d z\right)  \tag{4.10}\\
& =\left(\frac{\tau}{\pi}\right)^{n / 2} \int_{U_{1} \cap M_{\hat{t}}} e^{-\tau(z-\widehat{z})^{2}} f(z) d z
\end{align*}
$$

where the second integral vanishes, because $f$ and $e^{-(z-\bar{z})^{2}}$ are $\mathscr{C}_{C R}^{1}$.
Analyzing the real and the imaginary part of the phase function $-\tau(z-\widehat{z})^{2}$ on $M_{\widehat{u}}^{1}$, one verifies ([BT1981]) that the integral over $U_{1} \cap M_{\widehat{u}}^{1}$ tends to $f(\widehat{z})$ as $\tau$ tends to $\infty$, provided that the submanifold $U_{1} \cap M_{\widehat{z}}$ is sufficiently close to the real plane $\mathbb{R}^{n}$ in $\mathscr{C}^{1}$ norm (Gauss' kernel is an approximation of Dirac's measure). Finally, developing in power series, truncating the exponential in the first expression (4.9) which defines $G_{\tau} f(\widehat{z})$ and integrating termwise, we get a sequence of polynomials $\left(P_{\nu}(z)\right)_{\nu \in \mathbb{N}}$.
4.11. A family of straightenings. Our main goal is to construct a semilocal half-wedge attached to a one-sided neighborhood $\left(M_{\gamma}^{1}\right)^{+}$of $M^{1}$ in $M$ along $\gamma$, which shall consist of analytic discs attached to $\left(M_{\gamma}^{1}\right)^{+}$. First of all, we need a convenient family of normalizations of the local geometries of $M$ and of $M^{1}$ along the points $\gamma(s)$ of our characteristic curve $\gamma$, for all $s$ with $-1 \leqslant s \leqslant 1$.

Let $\Omega$ be a thin neighborhood of $\gamma([-1,1])$ in $\mathbb{C}^{n}$, say a union of polydiscs of fixed radius centered at the points $\gamma(s)$. Then there exists $n$ real valued $\mathscr{C}^{2, \alpha}$ functions $r_{1}(z, \bar{z}), \ldots, r_{n}(z, \bar{z})$ defined in $\Omega$ such that $M \cap \Omega$ is given by the $(n-1)$ Cartesian equations $r_{2}(z, \bar{z})=\cdots=r_{n}(z, \bar{z})=0$ and such that moreover, $M^{1} \cap \Omega$ is given by the $n$ Cartesian equations $r_{1}(z, \bar{z})=$ $r_{2}(z, \bar{z})=\cdots=r_{n}(z, \bar{z})=0$. We first center the coordinates at $\gamma(s)$ by setting $z^{\prime}:=z-\gamma(s)$. Then the defining functions centered at $z^{\prime}=0$ become

$$
\begin{equation*}
r_{j}\left(z^{\prime}+\gamma(s), \bar{z}^{\prime}+\overline{\gamma(s)}\right)-r_{j}(\gamma(s), \overline{\gamma(s)})=: r_{j}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}: s\right) \tag{4.12}
\end{equation*}
$$

for $j=1, \ldots, n$, and they are parametrized by $s \in[-1,1]$. Now, we drop the primes on coordinates and we denote by $r_{j}(z, \bar{z}: s), j=1, \ldots, n$, the defining equations for the new $M_{s}$ and $M_{s}^{1}$, which correspond to the old $M$ and $M^{1}$ locally in a neighborhood of $\gamma(s)$. Next, we straighten the tangent planes by using the linear change of coordinates $z^{\prime}=A_{s} \cdot z$, where the $n \times n$ matrix $A_{s}$ is defined by $A_{s}:=2 i\left(\frac{\partial r_{j}}{\partial z_{k}}(0,0: s)\right)_{1 \leqslant j, k \leqslant n}$. Then the defining equations for the two transformed $M_{s}^{\prime}$ and for $M_{s}^{1^{\prime}}$ are given by

$$
\begin{equation*}
r_{j}^{\prime}\left(z^{\prime}, \bar{z}^{\prime}: s\right):=r_{j}\left(A_{s}^{-1} \cdot z^{\prime}, \bar{A}_{s}^{-1} \cdot \bar{z}^{\prime}: s\right) \tag{4.13}
\end{equation*}
$$

and we check immediately that the matrix $\left(\frac{\partial r_{j}^{\prime}}{\partial z_{k}}(0,0: s)\right)_{1 \leqslant j, k \leqslant n}$ is equal to $2 i$ times the $n \times n$ identity matrix, whence $T_{0} M_{s}^{\prime}=\left\{y_{2}^{\prime} \stackrel{1 \leqslant j, k \leqslant n}{=}=y_{n}^{\prime}=0\right\}$ and $T_{0} M_{s}^{\prime 1}=\left\{y_{1}^{\prime}=y_{2}^{\prime}=\cdots=y_{n}^{\prime}=0\right\}$. It is important to notice that the matrix $A_{s}$ is only $\mathscr{C}^{1, \alpha}$ with respect to $s$. Consequently, if we now drop the primes on coordinates, the defining equations for $M_{s}$ and for $M_{s}^{1}$ are of class $\mathscr{C}^{2, \alpha}$ with respect to $(z, \bar{z})$ and only of class $\mathscr{C}^{1, \alpha}$ with respect to $s$.

Applying then the $\mathscr{C}^{2, \alpha}$ implicit function theorem, we deduce that there exist $(n-1)$ functions $\varphi_{j}\left(x, y_{1}: s\right), j=2, \ldots, n$, which are all of class $\mathscr{C}^{2, \alpha}$ with respect to $\left(x, y_{1}\right)$ in a real cube $\mathbb{I}_{n+1}\left(2 \rho_{1}\right):=\left\{\left(x, y_{1}\right) \in \mathbb{R}^{n} \times \mathbb{R}:\right.$ $\left.|x|<2 \rho_{1},\left|y_{1}\right|<2 \rho_{1}\right\}$, for some $\rho_{1}>0$, which are uniformly bounded in $\mathscr{C}^{2, \alpha}$-norm as the parameter $s$ varies in $[-1,1]$, which are of class $\mathscr{C}^{1, \alpha}$ with respect to $s$, such that $M_{s}$ may be represented in the polydisc $\Delta_{n}\left(\rho_{1}\right)$ by the ( $n-1$ ) graphed equations

$$
\begin{equation*}
y_{2}=\varphi_{2}\left(x, y_{1}: s\right), \ldots \ldots, y_{n}=\varphi_{n}\left(x, y_{1}: s\right) \tag{4.14}
\end{equation*}
$$

or more concisely by $y^{\prime}=\varphi^{\prime}\left(x, y_{1}: s\right)$, if we denote the coordinates $\left(z_{2}, \ldots, z_{n}\right)$ simply by $z^{\prime}=x^{\prime}+i y^{\prime}$. Here, by construction, we have the normalization conditions $\varphi_{j}(0: s)=\partial_{x_{k}} \varphi_{j}(0: s)=\partial_{y_{1}} \varphi_{j}(0: s)=0$, for $j=2, \ldots, n$ and $k=1, \ldots, n$. Sometimes in the sequel, we shall use the notation $\varphi_{j}\left(z_{1}, x^{\prime}: s\right)$ instead of $\varphi_{j}\left(x, y_{1}: s\right)$. Similarly, again by means of the implicit function theorem, we obtain $n$ functions $h_{k}(x: s)$, for
$k=1, \ldots, n$, which are of class $\mathscr{C}^{2, \alpha}$ in the cube $\mathbb{I}_{n}\left(2 \rho_{1}\right)$ (after possibly shrinking $\rho_{1}$ ) enjoying the same regularity property with respect to $s$, such that $M_{s}^{1}$ is represented in the polydisc $\Delta_{n}\left(\rho_{1}\right)$ by the $n$ graphed equations

$$
\begin{equation*}
y_{1}=h_{1}(x: s), y_{2}=h_{2}(x: s), \ldots \ldots, y_{n}=h_{n}(x: s) . \tag{4.15}
\end{equation*}
$$

In addition, we can assume that

$$
\begin{equation*}
h_{j}(x: s) \equiv \varphi_{j}\left(x, h_{1}(x: s): s\right), \quad j=2, \ldots, n \tag{4.16}
\end{equation*}
$$

Here, by construction, we have the normalization conditions $h_{k}(0: s)=$ $\partial_{x_{l}} h_{k}(0: s)=0$ for $k, l=1, \ldots, n$.

In the sequel, we shall denote by $\widehat{z}=\Phi_{s}(z)$ the final change of coordinates which is centered at $\gamma(s)$ and which straightens simultaneously the tangent planes to $M$ at $\gamma(s)$ and to $M^{1}$ at $\gamma(s)$ and we shall denote by $M_{s}$ and by $M_{s}^{1}$ the transformations of $M$ and of $M^{1}$.

Also, we must remind that the following regularity properties hold for the functions $\varphi_{j}\left(x, y_{1}: s\right)$ and $h_{k}(x: s)$.
(a) For fixed $s$, they are of class $\mathscr{C}^{2, \alpha}$ with respect to their principal variables, namely excluding the parameter $s$.
(b) They are of class $\mathscr{C}^{1, \alpha}$ with respect to all their variables, including the parameter $s$.
(c) Each of their first order partial derivative with respect to one of their principal variables is of class $\mathscr{C}^{1, \alpha}$ with respect to all their variables, including the parameter $s$.
Indeed, these properties are clearly satisfied for the functions (4.13) and they are inherited after the two applications of the implicit function theorem which yielded the functions $\varphi_{j}\left(x, y_{1}: s\right)$ and $h_{k}(x: s)$.
4.17. Contact of a small "round" analytic disc with $M^{1}$. Let $r \in \mathbb{R}$ with $0 \leqslant r \leqslant r_{1}$, where $r_{1}$ is small in comparison with $\rho_{1}$. Then the "round" analytic disc $\bar{\Delta} \ni \zeta \rightarrow \widehat{Z}_{1 ; r}(\zeta):=\operatorname{ir}(1-\zeta) \in \mathbb{C}$ with values in the complex plane equipped with the coordinate $z_{1}=x_{1}+i y_{1}$ is centered at the point ir of the $y_{1}$-axis, is of radius $r$ and is contained in the open upper half plane $\left\{z_{1} \in \mathbb{C}: y_{1}>0\right\}$, except its boundary point $\widehat{Z}_{1 ; r}(1)=0$. In addition, the tangent direction $\frac{\partial}{\partial \theta} \widehat{Z}_{1 ; r}(1)=r$ is directed along the positive $x_{1}$-axis, see in advance Figure 9 below.

We denote by $T_{1}$ the Hilbert transform ${ }^{36}$ on $\partial \Delta$ vanishing at 1 , namely $\left(T_{1} X\right)(1)=0$, whence $T_{1}\left(T_{1}(X)\right)=-X+X(1)$. Thanks to a standard processus, we may lift this scalar disc $\operatorname{ir}(1-\zeta)$ as disc attached to $M$ of the form

$$
\begin{equation*}
\widehat{Z}_{r: s}(\zeta)=\left(i r(1-\zeta), \widehat{Z}_{r: s}^{\prime}(\zeta)\right) \in \mathbb{C} \times \mathbb{C}^{n-1} \tag{4.18}
\end{equation*}
$$

[^35]where the real part $\widehat{X}_{r: s}^{\prime}(\zeta)$ of $\widehat{Z}_{r: s}^{\prime}(\zeta)$ satisfies the following Bishop-type equation on $\partial \Delta$
\[

$$
\begin{equation*}
\widehat{X}_{r: s}^{\prime}(\zeta)=-\left[T_{1} \varphi^{\prime}\left(\widehat{Z}_{1 ; r}(\cdot), \widehat{X}_{r: s}^{\prime}(\cdot): s\right)\right](\zeta), \quad \zeta \in \partial \Delta \tag{4.19}
\end{equation*}
$$

\]

By [Tu1994, Tu1996, 29], if $r_{1}$ is sufficiently small, there exists a solution which is $\mathscr{C}^{2, \alpha-0}$ with respect to $(r, \zeta)$, but only $\mathscr{C}^{1, \alpha-0}$ with respect to $(r, \zeta, s)$. Notice that for $r=0$, the disc $\widehat{Z}_{1 ; 0}\left(e^{i \theta}\right)$ is constant equal to 0 and by uniqueness of the solution of (4.19), it follows that $\widehat{Z}_{0: s}^{\prime}\left(e^{i \theta}\right) \equiv 0$. It follows trivially that $\partial_{\theta} \widehat{X}_{0: s}\left(e^{i \theta}\right) \equiv 0$ and that $\partial_{\theta} \partial_{\theta} \widehat{X}_{0: s}\left(e^{i \theta}\right) \equiv 0$, which will be used in a while. Notice also that $\widehat{X}_{r: s}(1)=0$ for all $r$ and all $s$.

On the other hand, since by assumption, we have $h_{1}(0: s)=0$ and $\partial_{x_{k}} h_{1}(0: s)=0$ for $k=1, \ldots, n$, it follows from the chain rule that if we set

$$
\begin{equation*}
F(r, \theta: s):=h_{1}\left(\widehat{X}_{r: s}\left(e^{i \theta}\right): s\right) \tag{4.20}
\end{equation*}
$$

where $\theta$ satisfies $0 \leqslant|\theta| \leqslant \pi$, then the following four equations hold (4.21)
$F(0, \theta: s) \equiv 0, \quad F(r, 0: s) \equiv 0, \quad \partial_{\theta} F(r, 0: s) \equiv 0, \quad \partial_{\theta} F(0, \theta: s) \equiv 0$.

We deduce that there exists a constant $C>0$ such that the following five inequalities hold for $0 \leqslant|\theta| \leqslant \pi$, for $0 \leqslant r \leqslant r_{1}$, for $s \in[-1,1]$ and for $|x| \leqslant \rho_{1}$ :

$$
\left\{\begin{array}{l}
\left|\widehat{X}_{r: s}\left(e^{i \theta}\right)\right| \leqslant C \cdot r  \tag{4.22}\\
\left|\partial_{\theta} \widehat{X}_{r: s}\left(e^{i \theta}\right)\right| \leqslant C \cdot r \\
\left|\partial_{\theta} \partial_{\theta} \widehat{X}_{r: s}\left(e^{i \theta}\right)\right| \leqslant C \cdot r^{\frac{\alpha}{2}}, \\
\sum_{k=1}^{n}\left|\partial_{x_{k}} h_{1}(x)\right| \leqslant C \cdot|x|, \\
\sum_{k_{1}, k_{2}=1}^{n}\left|\partial_{x_{k_{1}}} \partial_{x_{k_{2}}} h_{1}(x)\right| \leqslant C .
\end{array}\right.
$$

As in Lemma 6.4 (see below), the third inequality comes from $\partial_{\theta} \partial_{\theta} \widehat{X}_{0: s}\left(e^{i \theta}\right) \equiv 0$ and $\widehat{X}_{r: s}\left(e^{i \theta}\right) \in \mathscr{C}^{2, \alpha / 2}$.

Computing now the second derivative of $F(r, \theta: s)$ with respect to $\theta$, we obtain
(4.23)

$$
\begin{aligned}
\partial_{\theta} \partial_{\theta} F(r, \theta: s) & =\sum_{k=1}^{n} \partial_{x_{k}} h_{1}\left(\widehat{X}_{r: s}\left(e^{i \theta}\right): s\right) \cdot \partial_{\theta} \partial_{\theta} \widehat{X}_{k ; r: s}\left(e^{i \theta}\right)+ \\
& +\sum_{k_{1}, k_{2}=1}^{n} \partial_{x_{k_{1}}} \partial_{x_{k_{2}}} h_{1}\left(\widehat{X}_{r: s}\left(e^{i \theta}\right)\right) \cdot \partial_{\theta} \widehat{X}_{k_{1} ; r, s}\left(e^{i \theta}\right) \cdot \partial_{\theta} \widehat{X}_{k_{2} ; r, s}\left(e^{i \theta}\right),
\end{aligned}
$$

and we may apply the majorations (4.22) to get

$$
\begin{align*}
\left|\partial_{\theta} \partial_{\theta} F(r, \theta: s)\right| & \leqslant C \cdot\left|\widehat{X}_{r: s}\left(e^{i \theta}\right)\right| \cdot C \cdot r^{\frac{\alpha}{2}}+C \cdot(C \cdot r)^{2} \\
& \leqslant r \cdot C^{3}\left[r^{\frac{\alpha}{2}}+r^{2}\right] \tag{4.24}
\end{align*}
$$

Lemma 4.25. If $r_{1} \leqslant \min \left(1,\left(\frac{1}{4 C^{3} \pi^{2}}\right)^{\frac{2}{\alpha}}\right)$, then $\widehat{Z}_{r: s}(\partial \Delta \backslash\{1\})$ is contained in $\left(M_{s}^{1}\right)^{+}$for all $r$ with $0<r \leqslant r_{1}$ and all $s$ with $-1 \leqslant s \leqslant 1$.

Proof. In the polydisc $\Delta_{n}\left(\rho_{1}\right)$, the positive half-side $\left(M_{s}^{1}\right)^{+}$in $M$ is represented by the single equation $y_{1}>h_{1}(x: s)$, hence we have to check that $\widehat{Y}_{1 ; r}\left(e^{i \theta}\right)>\left|h_{1}\left(\widehat{X}_{r: s}\left(e^{i \theta}\right): s\right)\right|$, for all $\theta$ with $0<|\theta| \leqslant \pi$.

The $y_{1}$-component $\widehat{Y}_{1 ; r}\left(e^{i \theta}\right)$ of $\widehat{Z}_{r: s}\left(e^{i \theta}\right)$ is equal to $r(1-\cos \theta)$. We have the elementary minoration $r(1-\cos \theta) \geqslant r \cdot \theta^{2} \cdot \frac{1}{\pi^{2}}$, valuable for $0 \leqslant|\theta| \leqslant \pi$. Also, taking account of the second and of the fourth relations (4.21), Taylor's integral formula yields

$$
\begin{equation*}
F(r, \theta: s)=\int_{0}^{\theta}\left(\theta-\theta^{\prime}\right) \cdot \partial_{\theta} \partial_{\theta} F\left(r, \theta^{\prime}: s\right) \cdot d \theta^{\prime} \tag{4.26}
\end{equation*}
$$

Observing that $r^{2} \leqslant r^{\frac{\alpha}{2}}$, since $0<r \leqslant r_{1} \leqslant 1$, and using the majoration (4.24), we may estimate, taking account of the assumption on $r_{1}$ written in the statement of the lemma:

$$
\begin{equation*}
|F(r, \theta: s)| \leqslant r \cdot \frac{\theta^{2}}{2} \cdot C^{3}\left[2 r^{\frac{\alpha}{2}}\right] \leqslant r \cdot \theta^{2} \cdot \frac{1}{4 \pi^{2}} \tag{4.27}
\end{equation*}
$$

This yields the desired inequality $r(1-\cos \theta)>|F(r, \theta: s)|$.
We now fix once for all a radius $r_{0}$ with $0<r_{0} \leqslant r_{1}$. In the remainder of the present Section 4, we shall deform the disc $\widehat{Z}_{r_{0}: s}(\zeta)$ by adding many more parameters. We notice that for all $\theta$ with $0 \leqslant|\theta| \leqslant \frac{\pi}{4}$, we have the trivial minoration $\partial_{\theta} \partial_{\theta} \widehat{Y}_{1 ; r_{0}}\left(e^{i \theta}\right)=r_{0} \cos \theta \geqslant \frac{r_{0}}{\sqrt{2}}$. Also, by (4.24) and by the inequality on $r_{1}$ written in Lemma 4.25 , we deduce $\left|\partial_{\theta} \partial_{\theta} h_{1}\left(\widehat{X}_{r_{0}: s}\left(e^{i \theta}\right)\right)\right| \leqslant \frac{r_{0}}{2 \pi^{2}}$ for all $\theta$ with $0 \leqslant|\theta| \leqslant \pi$. Since we shall need a generalization of Lemma 4.25 in Lemma 4.51 below, let us remember
these two interesting inequalities, valid for $0 \leqslant|\theta| \leqslant \frac{\pi}{4}$ :

$$
\begin{equation*}
\left|\partial_{\theta} \partial_{\theta} h_{1}\left(\widehat{X}_{r_{0}: s}\left(e^{i \theta}\right)\right)\right| \leqslant \frac{r_{0}}{2 \pi^{2}}<\frac{r_{0}}{\sqrt{2}} \leqslant \partial_{\theta} \partial_{\theta} \widehat{Y}_{1 ; r_{0}}\left(e^{i \theta}\right) \tag{4.28}
\end{equation*}
$$

4.29. Normal deformations of the disc $\widehat{Z}_{r: s}(\zeta)$. So, we fix $r_{0}$ small with $0<r_{0} \leqslant r_{1}$ and we consider the disc $\widehat{Z}_{r_{0}: s}(\zeta)$ for $\zeta \in \bar{\Delta}$. Then the point $\widehat{Z}_{r_{0}: s}(-1)$ belongs to $\left(M_{s}^{1}\right)^{+}$for each $s$ and stays at a positive distance from $M_{s}^{1}$ as $s$ varies in $[-1,1]$. It follows that we can choose a subneighborhood $\omega_{s}$ of $\widehat{Z}_{r_{0}: s}(-1)$ in $\mathbb{C}^{n}$ which is contained in $\Omega$ and whose diameter is uniformly bounded from below.


Following [Tu1994, 29], we introduce normal deformations of the analytic discs $\widehat{Z}_{r_{0}: s}(\zeta)$. Let $\kappa: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ be a $\mathscr{C}^{2, \alpha}$ mapping fixing the origin and satisfying $\partial_{x_{k}} \kappa_{j}(0)=\delta_{k}^{j}$ (Kronecker's symbol). For $j=2, \ldots, n$, let $\eta_{j}=\eta_{j}\left(z_{1}, x^{\prime}: s\right)$ be a real-valued $\mathscr{C}^{2, \alpha}$ function compactly supported in a neighborhood of the point of $\mathbb{R}^{n+1}$ with coordinates $\left(\widehat{Z}_{1 ; r_{0}: s}(-1), \widehat{X}_{r_{0}: s}^{\prime}(-1)\right)$ and equal to 1 at this point. We then define the $\mathscr{C}^{2, \alpha}$ deformed generic submanifold $M_{s, t}$ of equations

$$
\begin{align*}
y^{\prime} & =\varphi^{\prime}\left(z_{1}, x^{\prime}: s\right)+\kappa(t) \cdot \eta^{\prime}\left(z_{1}, x^{\prime}: s\right) \\
& =: \Phi^{\prime}\left(z_{1}, x^{\prime}, t: s\right) . \tag{4.30}
\end{align*}
$$

Notice that $M_{s, 0} \equiv M_{s}$ and that $M_{s, t}$ coincides with $M_{s}$ in a small neighborhood of the origin, for all $t$. If $\mu=\mu\left(e^{i \theta}: s\right)$ is a real-valued nonnegative $\mathscr{C}^{2, \alpha}$ function defined for $e^{i \theta} \in \partial \Delta$ and for $s \in[-1,1]$ whose support is concentrated near the segment $\{-1\} \times[-1,1]$, then ([Tu1996, 29]), for each fixed $s \in[-1,1]$, there exists a $\mathscr{C}^{2, \alpha-0}$ solution of the Bishop-type equation

$$
\begin{equation*}
\widehat{X}_{r_{0}, t: s}^{\prime}\left(e^{i \theta}\right)=-\left[T_{1} \Phi^{\prime}\left(\widehat{Z}_{1 ; r_{0}: s}(\cdot), \widehat{X}_{r_{0}, t: s}^{\prime}(\cdot), t \mu(\cdot: s): s\right)\right]\left(e^{i \theta}\right), \tag{4.31}
\end{equation*}
$$

which produces the family of analytic discs

$$
\begin{equation*}
\widehat{Z}_{r_{0}, t: s}\left(e^{i \theta}\right):=\left(\widehat{Z}_{1 ; r_{0}: s}\left(e^{i \theta}\right), \widehat{X}_{r_{0}, t: s}^{\prime}\left(e^{i \theta}\right)+i T_{1}\left[\widehat{X}_{r_{0}, t: s}^{\prime}(\cdot)\right]\left(e^{i \theta}\right)\right) \tag{4.32}
\end{equation*}
$$

having boundaries contained in $M \cup \omega_{s}$. Taking account of the regularity properties (a), (b) and (c) stated after (4.16), the general solution $\widehat{Z}_{r_{0}, t: s}(\zeta)$ enjoys similar regularity properties.
(a) For fixed $s$, it is of class $\mathscr{C}^{2, \alpha-0}$ with respect to $(t, \zeta)$.
(b) It is of class $\mathscr{C}^{1, \alpha-0}$ with respect to all the variables $(t, \zeta, s)$.
(c) Each of its first order partial derivative with respect to the principal variables $(t, \zeta)$ is of class $\mathscr{C}^{1, \alpha-0}$ with respect to all the variables $(t, \zeta, s)$.
Since the solution is $\mathscr{C}^{1, \alpha-0}$ with respect to $s$, it crucially follows that the vector

$$
\begin{equation*}
v_{1: s}:=-\frac{\partial \widehat{Z}_{r_{0}, t: s}}{\partial \rho}(1), \tag{4.33}
\end{equation*}
$$

which points inside the analytic disc, varies continuously with respect to $s$. The next key proposition may be established as in [Tu1994, MP1999], taking account of the uniformity with respect to $s$.
Lemma 4.34. There exists a real-valued nonnegative $\mathscr{C}^{2, \alpha}$ function $\mu=$ $\mu\left(e^{i \theta}: s\right)$ defined for $e^{i \theta} \in \partial \Delta$ and $s \in[-1,1]$ whose support is concentrated near $\{-1\} \times[-1,1]$ such that the mapping

$$
\begin{equation*}
\left.\mathbb{R}^{n-1} \ni t \longmapsto \frac{\partial \widehat{X}_{r_{0}, t: s}^{\prime}}{\partial \theta}\left(e^{i \theta}\right)\right|_{\theta=0} \in \mathbb{R}^{n-1} \tag{4.35}
\end{equation*}
$$

is maximal equal to $(n-1)$ at $t=0$.
Geometrically speaking, since the vector $\left.\frac{\partial \widehat{X}_{1 ; r_{0}: s}}{\partial \theta}\left(e^{i \theta}\right)\right|_{\theta=0}$ is nonzero, it follows that when the parameter $t$ varies, the set of lines generated by the vectors $\left.\frac{\partial \widehat{X}_{r_{0}, t: s}}{\partial \theta}\left(e^{i \theta}\right)\right|_{\theta=0}$ covers an open cone in the space $T_{p_{1}} M^{1} \equiv \mathbb{R}^{n}$ equipped with coordinates $\left(x_{1}, x^{\prime}\right)$, see again Figure 9 above for an illustration.
4.36. Adding pivoting and translation parameters. Let $\chi=\left(\chi_{1}, \chi^{\prime}\right) \in$ $\mathbb{R} \times \mathbb{R}^{n-1}$ and $\nu \in \mathbb{R}$ satisfying $|\chi|<\varepsilon$ and $|\nu|<\varepsilon$ for some small $\varepsilon>0$. Then the mapping

$$
\begin{align*}
\mathbb{R}^{n+1} \ni\left(\chi_{1}, \chi^{\prime}, \nu\right) \longmapsto & \left(\chi_{1}+i\left[h_{1}(\chi: s)+\nu\right], \chi^{\prime}+i \varphi^{\prime}\left(\chi, h_{1}(\chi: s)+\nu: s\right)\right)  \tag{4.37}\\
& =: \widehat{p}(\chi, \nu: s) \in M_{s}
\end{align*}
$$

is a $\mathscr{C}^{2, \alpha}$ diffeomorphism onto a neighborhood of the origin in $M_{s}$ with:
(a) $\nu>0$ if and only if $\widehat{p}(\chi, \nu: s) \in\left(M_{s}^{1}\right)^{+}$;
(b) $\nu=0$ if and only if $\widehat{p}(\chi, \nu: s) \in M_{s}^{1}$;
(c) $\nu<0$ if and only if $\widehat{p}(\chi, \nu: s) \in\left(M_{s}^{1}\right)^{-}$.

If $\tau \in \mathbb{R}$ with $|\tau|<\varepsilon$ is a supplementary parameter, we now define a crucial deformation of the first component $\widehat{Z}_{1 ; r_{0}: s}\left(e^{i \theta}\right)$ by setting

$$
\begin{equation*}
\widehat{Z}_{1 ; r_{0}, \tau, \chi, \nu: s}\left(e^{i \theta}\right):=i r_{0}\left(1-e^{i \theta}\right)[1+i \tau]+\chi_{1}+i\left[h_{1}(\chi: s)+\nu\right] \tag{4.38}
\end{equation*}
$$

Of course, we have $\widehat{Z}_{1 ; r_{0}, 0,0,0: s}\left(e^{i \theta}\right) \equiv \widehat{Z}_{1 ; r_{0}: s}\left(e^{i \theta}\right)$. Geometrically speaking, this perturbation corresponds to add firstly a small "rotation parameter" $\tau$ which rotates (and slightly dilates) the disc $i r_{0}\left(1-e^{i \theta}\right)$ passing through the origin in $\mathbb{C}_{z_{1}}$, to add secondly a small "translation parameter" $\left(\chi_{1}, \chi^{\prime}\right)$ which will enable to cover a neighborhood of the origin in $M_{s}^{1}$ and to add thirdly a small translation parameter $\nu$ along the $y_{1}$-axis. Consequently, with this first $\mathbb{C}$-valued component $\widehat{Z}_{1 ; r_{0}, \tau, \chi, \nu: s}\left(e^{i \theta}\right)$, we can construct a $\mathbb{C}^{n}$-valued analytic disc $\widehat{Z}_{r_{0}, t, \tau, \chi, \nu: s}(\zeta)$ satisfying

$$
\begin{equation*}
\widehat{Z}_{r_{0}, t, \tau, \chi, \nu: s}(1)=\widehat{p}(\chi, \nu: s), \tag{4.39}
\end{equation*}
$$

simply by solving the perturbed Bishop-type equation which extends (4.31) (4.40)
$\widehat{X}_{r_{0}, t, \tau, \chi, \nu: s}^{\prime}\left(e^{i \theta}\right)=-\left[T_{1}\left(\Phi^{\prime}\left(\widehat{Z}_{1 ; r_{0}, \tau, \chi, \nu: s}(\cdot), \widehat{X}_{r_{0}, t, \tau, \chi, \nu: s}^{\prime}(\cdot), t \mu(\cdot: s): s\right)\right)\right]\left(e^{i \theta}\right)$.
Of course, thanks to the sympathetic stability of Bishop's equation under perturbation, the solution exists and satisfies smoothness properties entirely similar to the ones stated after (4.32). To summarize, we list the seven variables upon which our final family of analytic discs depends.
$\widehat{Z}_{r_{0}, t, \tau,, \nu: s}(\zeta):\left\{\begin{aligned} & r_{0}=\text { approximate radius } \\ & t=\text { normal deformation parameter } . \\ & \tau=\text { pivoting parameter } . \\ & \chi=\text { parameter of translation along } M^{1} . \\ & \nu=\text { parameter of translation in } M \text { transversally to } M^{1} . \\ & s=\text { parameter of the characteristic curve } \gamma . \\ & \zeta=\text { unit disc variable } .\end{aligned}\right.$
For every $t$ and every $\chi$, we now want to adjust the pivoting parameter $\tau$ in order that the disc boundary $\widehat{Z}_{r_{0}, t, \tau, \chi, 0: s}\left(e^{i \theta}\right)$ for $\nu=0$ is tangent to $M_{s}^{1}$. This tangency condition will be useful in order to derive the crucial Lemma 4.51 below.
Lemma 4.42. Shrinking $\varepsilon$ if necessary, there exists a unique $\mathscr{C}^{1, \alpha-0}$ map $(t, \chi, s) \mapsto \tau(t, \chi: s)$ defined for $|t|<\varepsilon$, for $|\chi|<\varepsilon$ and for $s \in[-1,1]$
satisfying $\tau(0,0: s)=\partial_{t_{j}} \tau(0,0: s)=\partial_{\chi_{k}} \tau(0,0: s)=0$ for $j=$ $1, \ldots, n-1$ and $k=1, \ldots, n$, such that the vector

$$
\begin{equation*}
\left.\frac{\partial}{\partial \theta}\right|_{\theta=0} \widehat{Z}_{r_{0}, t, \tau(t, \chi: s), \chi, 0: s}\left(e^{i \theta}\right) \tag{4.43}
\end{equation*}
$$

is tangent to $M_{s}^{1}$ at the point $\widehat{Z}_{r_{0}, t, \tau(t, \chi: s), \chi, 0: s}(1)=\widehat{p}(\chi, 0: s) \in M_{s}^{1}$.
Proof. We remind that $M_{s}$ is represented by the $(n-1)$ scalar equations $y^{\prime}=\varphi^{\prime}\left(x, y_{1}: s\right)$ and that $M_{s}^{1}$ is represented by the $n$ equations $y_{1}=$ $h_{1}(x: s)$ and $y^{\prime}=\varphi^{\prime}\left(x, h_{1}(x: s): s\right) \equiv h^{\prime}\left(x^{\prime}: s\right)$. We can therefore compute the Cartesian equations of the tangent plane to $M_{s}^{1}$ at the point $\widehat{p}(\chi, 0: s)=\chi+i h(\chi: s):$
(4.44)

$$
\left\{\begin{aligned}
\mathrm{Y}_{1}-h_{1}(\chi: s) & =\sum_{k=1}^{n} \partial_{x_{k}} h_{1}(\chi: s)\left[\mathrm{X}_{x}-\chi_{k}\right] \\
\mathrm{Y}^{\prime}-\varphi^{\prime}\left(\chi, h_{1}(\chi: s): s\right) & =\sum_{k=1}^{n}\left(\partial_{x_{k}} \varphi^{\prime}+\partial_{y_{1}} \varphi^{\prime} \cdot \partial_{x_{k}} h_{1}\right)\left[\mathrm{X}_{k}-\chi_{k}\right]
\end{aligned}\right.
$$

On the other hand, we observe that the tangent vector

$$
\begin{equation*}
\left.\frac{\partial}{\partial \theta}\right|_{\theta=0} \widehat{Z}_{r_{0}, t, \tau, \chi, 0: s}\left(e^{i \theta}\right)=\left(r_{0}[1+i \tau],\left.\frac{\partial}{\partial \theta}\right|_{\theta=0} \widehat{Z}_{r_{0}, t, \tau, \chi, 0: s}^{\prime}\left(e^{i \theta}\right)\right) \tag{4.45}
\end{equation*}
$$

is already tangent to $M_{s}$ at the point $\widehat{p}(\chi, 0: s)$, because $M_{s, t} \equiv M_{s}$ in a neighborhood of the origin. More precisely, since $\Phi^{\prime} \equiv \varphi^{\prime}$ in a neighborhood of the origin, we may differentiate with respect to $\theta$ at $\theta=0$ the relation

$$
\begin{equation*}
\widehat{Y}_{r_{0}, t, \tau, \chi, 0: s}^{\prime}\left(e^{i \theta}\right) \equiv \varphi^{\prime}\left(\widehat{X}_{r_{0}, \tau, \chi, 0: s}\left(e^{i \theta}\right), \widehat{Y}_{1 ; r_{0}, \tau, \chi, 0: s}\left(e^{i \theta}\right): s\right) \tag{4.46}
\end{equation*}
$$

which is valid for $|\theta| \leqslant \frac{\pi}{2}$, noticing in advance that it follows immediately from (4.38) that

$$
\begin{equation*}
\left.\frac{\partial}{\partial \theta}\right|_{\theta=0} \widehat{X}_{1 ; r_{0}, \tau, \chi, 0: s}\left(e^{i \theta}\right)=r_{0} \quad \text { and }\left.\quad \frac{\partial}{\partial \theta}\right|_{\theta=0} \widehat{Y}_{1 ; r_{0}, \tau, \chi, 0: s}\left(e^{i \theta}\right)=r_{0} \tau \tag{4.47}
\end{equation*}
$$

hence we obtain by a direct application of the chain rule

$$
\begin{align*}
\left.\frac{\partial}{\partial \theta}\right|_{\theta=0} \widehat{Y}_{r_{0}, t, \tau, \chi, 0: s}^{\prime}\left(e^{i \theta}\right)= & \partial_{y_{1}} \varphi^{\prime} \cdot r_{0} \tau+  \tag{4.48}\\
& +\sum_{k=1}^{n} \partial_{x_{k}} \varphi^{\prime} \cdot\left(\left.\frac{\partial}{\partial \theta}\right|_{\theta=0} \widehat{X}_{k ; r_{0}, t, \tau, \chi, 0: s}\left(e^{i \theta}\right)\right)
\end{align*}
$$

By (4.44), the vector (4.45) belongs to the tangent plane to $M_{s}^{1}$ if and only if
(4.49)
$\left\{\begin{aligned} r_{0} \tau & =\sum_{k=1}^{n} \partial_{x_{k}} h_{1}(\chi: s)\left[\left.\frac{\partial}{\partial \theta}\right|_{\theta=0} \widehat{X}_{k ; r_{0}, t, \tau, \chi, 0: s}\left(e^{i \theta}\right)\right], \\ \left.\frac{\partial}{\partial \theta}\right|_{\theta=0} \widehat{Y}_{r_{0}, t, \tau, \chi, 0: s}^{\prime}\left(e^{i \theta}\right) & =\sum_{k=1}^{n}\left(\partial_{x_{k}} \varphi^{\prime}+\partial_{y_{1}} \varphi^{\prime} \cdot \partial_{x_{k}} h_{1}\right) \cdot\left[\left.\frac{\partial}{\partial \theta}\right|_{\theta=0} \widehat{X}_{k ; r_{0}, t, \tau, \chi, 0: s}\left(e^{i \theta}\right)\right] .\end{aligned}\right.$
We observe that the first line of (4.49) together with the relation (4.48) already obtained implies the second line of (4.49) by an obvious linear combination. Consequently, the vector (4.45) belongs to the tangent plane to $M_{s}^{1}$ at $\widehat{p}(\chi, 0: s)$ if and only if the first line of (4.49) is satisfied. As $r_{0}$ is nonzero, as the first order derivatives $\partial_{x_{k}} h_{1}(\chi: s)$ are of class $\mathscr{C}^{1, \alpha}$ and vanish at $x=0$ and as $\left.\frac{\partial}{\partial \theta}\right|_{\theta=0} \widehat{X}_{k ; r_{0}, t, \tau, \chi, 0: s}\left(e^{i \theta}\right)$ is of class $\mathscr{C}^{1, \alpha-0}$ with respect to all variables $(t, \tau, \chi, s)$, it follows from the implicit function theorem that there exists a unique solution $\tau=\tau(t, \chi: s)$ of the first line of (4.49) which satisfies in addition the normalization conditions $\tau(0,0: s)=\partial_{t_{j}} \tau(0,0: s)=\partial_{\chi_{k}} \tau(0,0: s)=0$ for $j=1, \ldots, n-1$ and $k=1, \ldots, n$. This completes the proof of Lemma 4.42.

We now define the analytic disc

$$
\begin{equation*}
\widehat{\mathscr{Z}}_{t, \chi, \nu: s}(\zeta):=\widehat{Z}_{r_{0}, t, \tau(t, \chi: s), \chi, \nu: s}(\zeta) . \tag{4.50}
\end{equation*}
$$

Lemma 4.51. Shrinking $\varepsilon$ if necessary, the following two properties are satisfied:
(1) $\widehat{\mathscr{Z}}_{t, \chi, 0: s}(\partial \Delta \backslash\{1\}) \subset\left(M_{s}^{1}\right)^{+}$for all $t, \chi, \nu$ and $s$ with $|t|<\varepsilon$, with $|\chi|<\varepsilon$, with $|\nu|<\varepsilon$ and with $-1 \leqslant s \leqslant 1$.
(2) If $\nu$ satisfies $0<\nu<\varepsilon$, then $\widehat{\mathscr{Z}}_{t, \chi, \nu: s}(\partial \Delta) \subset\left(M_{s}^{1}\right)^{+}$for all $t, \chi$ and $s$ with $|t|<\varepsilon$, with $|\chi|<\varepsilon$ and with $-1 \leqslant s \leqslant 1$.

Proof. To establish (1), we first observe that the disc $\widehat{\mathscr{Z}}_{0,0,0: s}\left(e^{i \theta}\right)$ identifies with the disc $\widehat{Z}_{r_{0}: s}\left(e^{i \theta}\right)$ defined in $\S 4.29$. According to Lemma 4.25, we know that $\widehat{\mathscr{Z}}_{0,0,0: s}(\partial \Delta \backslash\{1\})$ is contained in $\left(M_{s}^{1}\right)^{+}$. By continuity, if $\varepsilon$ is sufficiently small, we can assume that for all $t$ with $|t|<\varepsilon$, for all $\chi$ with $|\chi|<\varepsilon$ and for all $\theta$ with $\frac{\pi}{4} \leqslant|\theta| \leqslant \pi$, the point $\widehat{\mathscr{Z}}_{t, \chi, 0: s}\left(e^{i \theta}\right)$ is contained in $\left(M_{s}^{1}\right)^{+}$. It remains to control the part of $\partial \Delta$ which corresponds to $|\theta| \leqslant \frac{\pi}{4}$.

Since the disc $\widehat{\mathscr{P}_{t, \chi, \nu: s}}\left(e^{i \theta}\right)$ is of class $\mathscr{C}^{2}$ with respect to all its principal variables $\left(t, \chi, \nu, e^{i \theta}\right)$, if $|t|<\varepsilon$, if $|\chi|<\varepsilon$ and if $0 \leqslant|\theta| \leqslant \frac{\pi}{4}$, for sufficiently small $\varepsilon$, then the inequalities (4.28) are just perturbed a little bit, so
we can assume that

$$
\begin{equation*}
\partial_{\theta} \partial_{\theta} \widehat{\mathscr{Y}}_{1 ; t, \chi, 0: s}\left(e^{i \theta}\right) \geqslant r_{0}>\frac{r_{0}}{2} \geqslant\left|\partial_{\theta} \partial_{\theta} h_{1}\left(\widehat{\mathscr{X}}_{t, \chi, 0: s}\left(e^{i \theta}\right)\right)\right| . \tag{4.52}
\end{equation*}
$$

We claim that the inequality

$$
\begin{equation*}
\widehat{\mathscr{Y}}_{1 ; t, \chi, 0: s}\left(e^{i \theta}\right)>\left|h_{1}\left(\widehat{\mathscr{X}}_{t, \chi, 0: s}\left(e^{i \theta}\right)\right)\right| \tag{4.53}
\end{equation*}
$$

holds for all $0<|\theta| \leqslant \frac{\pi}{4}$, which will complete the proof of property (1).
Indeed, we first remind that the tangency to $M_{s}^{1}$ of the vector $\left.\frac{\partial}{\partial \theta}\right|_{\theta=0} \widehat{\mathscr{\mathscr { P }}} t, \chi, 0: s,\left(e^{i \theta}\right)$ at the point $\widehat{p}(\chi, 0: s)$ is equivalent to the first relation (4.49), which may be rewritten in terms of the components of the disc $\widehat{\mathscr{Z}}_{t, \chi, 0: s}\left(e^{i \theta}\right)$ as follows

$$
\begin{equation*}
\partial_{\theta} \widehat{\mathscr{Y}}_{1 ; t, \chi, 0: s}(1)=\sum_{k=1}^{n}\left[\partial_{x_{k}} h_{1}\right]\left(\widehat{\mathscr{X}_{t, \chi, 0: s}}(1)\right) \cdot \partial_{\theta} \widehat{\mathscr{X}}_{k ; t, \chi, 0: s}(1) . \tag{4.54}
\end{equation*}
$$

Substracting this relation from (4.53) and substracting also the relation $\widehat{Y}_{1 ; t, \chi, 0: s}(1)=h_{1}\left(\mathscr{X}_{t, \chi, 0: s}(1)\right)$, we see that it suffices to establish that for all $\theta$ with $0<|\theta| \leqslant \frac{\pi}{4}$, we have the strict inequality
(4.55)

$$
\begin{aligned}
& \widehat{\mathscr{Y}}_{1 ; t, \chi, 0: s}\left(e^{i \theta}\right)-\widehat{\mathscr{Y}}_{1 ; t, \chi, 0: s}(1)-\theta \cdot \partial_{\theta} \widehat{\mathscr{Y}}_{1 ; t, \chi, 0: s}(1)> \\
& \quad>\left|h_{1}\left(\widehat{\mathscr{X}_{t, \chi, 0: s}}\left(e^{i \theta}\right)\right)-h_{1}\left(\widehat{\mathscr{X}_{t, \chi, 0: s}}(1)\right)-\sum_{k=1}^{n}\left[\partial_{x_{k}} h_{1}\right]\left(\widehat{\mathscr{X}_{t, \chi, 0: s}}(1)\right) \cdot \partial_{\theta} \widehat{\mathscr{X}_{k ; t, \chi, 0: s}}(1)\right|
\end{aligned}
$$

However, by means of Taylor's integral formula, this last inequality may be rewritten as
(4.56)
$\int_{0}^{\theta}\left(\theta-\theta^{\prime}\right) \cdot \partial_{\theta} \partial_{\theta} \widehat{\mathscr{Y}_{1 ;} ;, \chi, 0: s}\left(e^{i \theta^{\prime}}\right) \cdot d \theta^{\prime}>\left|\int_{0}^{\theta}\left(\theta-\theta^{\prime}\right) \cdot \partial_{\theta} \partial_{\theta}\left[h_{1}\left(\widehat{\mathscr{X}_{t, \chi, 0}}\left(e^{i \theta^{\prime}}\right)\right)\right] \cdot d \theta^{\prime}\right|$
and it follows immediately by means of (4.52).
Secondly, to check property (2), we observe that by the definition (4.37), the parameter $\nu$ corresponds to a translation of the $z_{1}$-component of the disc boundary $\widehat{\mathscr{P}_{t, \chi, 0: s}}(\partial \Delta)$ along the $y_{1}$ axis. More precisely, we have

$$
\begin{equation*}
\frac{\partial}{\partial \nu} \widehat{\mathscr{Y}}_{1 ; t, \chi, \nu: s}(\zeta) \equiv 1 \quad \text { and } \quad \frac{\partial}{\partial \nu} \widehat{\mathscr{X}}_{1 ; t, \chi, \nu: s}(\zeta) \equiv 0 . \tag{4.57}
\end{equation*}
$$

On the other hand, differentiating Bishop's equation (4.40), and using the smallness of the function $\Phi^{\prime}$, it may be checked that

$$
\begin{equation*}
\left|\frac{\partial}{\partial \nu} \widehat{Z}_{r_{0}, t, \tau, \chi, \nu: s}^{\prime}\left(e^{i \theta}\right)\right| \ll 1 \tag{4.58}
\end{equation*}
$$

if $r_{0}$ and $\varepsilon$ are sufficiently small. It follows that the disc boundary $\widehat{\mathscr{Z}}_{t, \chi, \nu: s}(\partial \Delta)$ is globally moved in the direction of the $y_{1}$-axis as $\nu>0$
increases, hence is contained in $\left(M_{s}^{1}\right)^{+}$. The proof of Lemma 4.51 is complete.
4.59. Holomorphic extension to a semi-local attached half-wedge. As a consequence of Lemma 4.34, of (4.39) and of property (2) of Lemma 4.51, we conclude that for every $s \in[-1,1]$, our discs $\widehat{\mathscr{L}_{t, \chi, \nu: s}}(\zeta)$ satisfy all the requirements (i), (ii) and (iii) of $\S 4.2$ insuring that the set defined by

$$
\begin{equation*}
\mathscr{H} \mathscr{W}_{s}^{+}:=\left\{\widehat{\mathscr{Z}}_{t, \chi, \nu: s}(\rho):|t|<\varepsilon,|\chi|<\varepsilon, 0<\nu<\varepsilon, 1-\varepsilon<\rho<1\right\} \tag{4.60}
\end{equation*}
$$

is a local half-wedge of edge $\left(M_{s}^{1}\right)^{+}$at the origin in the $\widehat{z}$-coordinates, which corresponds to the point $\gamma(s)$ in the $z$-coordinates. Coming back to the coordinates $z=\Phi_{s}^{-1}(\widehat{z})$, we define the family of analytic discs

$$
\begin{equation*}
\mathscr{Z}_{t, \chi, \nu: s}(\zeta):=\Phi_{s}^{-1}\left(\widehat{\mathscr{Z}_{t, \chi, \nu: s}}(\zeta)\right) \tag{4.61}
\end{equation*}
$$

Given an arbitrary $f \in \mathscr{O}(\Omega)$ as in Proposition 4.6, through the change of coordinates $\widehat{z}=\Phi_{s}(z)$ and by restriction to $\left(M_{s}^{1}\right)^{+}$, we get a CR function $\widehat{f}_{s} \in \mathscr{C}_{C R}^{0}\left(\left(M_{s}^{1}\right)^{+} \cap U_{1}\right)$, for some small neighborhood $U_{1}$ of the origin in $\mathbb{C}^{n}$, whose size is uniform with respect to $s$. Thanks to an obvious generalization of the approximation Lemma 4.8 with a supplementary parameter $s \in[-1,1]$, we know that there exists a second uniform neighborhood $V_{1} \subset \subset U_{1}$ of the origin in $\mathbb{C}^{n}$ such that $\widehat{f}_{s}$ is uniformly approximable by polynomials on $\left(M_{s}^{1}\right)^{+} \cap V_{1}$. Furthermore, choosing $r_{0}$ and $\varepsilon$ sufficiently small, we can insure that all the discs $\widehat{\mathscr{Z}}_{t, \chi, \nu: s}(\zeta)$ are attached to $\left(M_{s}^{1}\right)^{+} \cap V_{1}$. As in [Trp1990, Tu1994, Me1994, Jö1996], it then follows from the maximum principle applied to the approximating sequence of polynomials that for each $s \in[-1,1]$, the function $\widehat{f}_{s}$ extends holomorphically to the halfwedge defined by (4.60). Finally, we deduce that the holomorphic function $f \in \mathscr{O}(\Omega)$ extends holomorphically to the semi-local half-wedge attached to the one-sided neighborhood $\left(M_{\gamma}^{1}\right)^{+}$defined by
(4.62)
$\mathscr{H} \mathscr{W}_{\gamma}^{+}:=\left\{\mathscr{Z}_{t, \chi, \nu: s}(\rho):|t|<\varepsilon,|\chi|<\varepsilon, 0<\nu<\varepsilon, 1-\varepsilon<\rho<1,-1 \leqslant s \leqslant s\right\}$.
Without shrinking $\Omega$ near the points $\mathscr{Z}_{t, \chi, \nu: s}(-1)$ (otherwise, the crucial rank property of Lemma 4.34 would degenerate), we can shrink the open set $\Omega$ in a very thin neighborhood of the characteristic segment $\gamma$ in $M$ and we can shrink $\varepsilon>0$ if necessary in order that the intersection $\Omega \cap \mathscr{H} \mathscr{W}_{\gamma}^{+}$ is connected. By the principle of analytic continuation, this implies that there exists a well-defined holomorphic function $F \in \mathscr{O}\left(\Omega \cup \mathscr{H} \mathscr{W}_{\gamma}^{+}\right)$with $\left.F\right|_{\Omega}=f$.

The proof of Proposition 4.6 is complete.

## §5. Choice of a special point of $C_{\mathrm{nr}}$ TO BE REMOVED LOCALLY

5.1. Choice of a first supporting hypersurface. Continuing with the proof of Theorem 3.19, we now analyze the assumption that $C$ is nontransversal to $\mathrm{F}_{M^{1}}^{c}$. We first construct a foliated support hypersurface $H^{1}$.
Lemma 5.2. Under the assumptions of Theorem 3.19, there exists a $\mathscr{C}^{2, \alpha}$ embedded characteristic curve $\gamma:[-1,1] \rightarrow M^{1}$ with $\gamma(-1) \notin C, \gamma(0) \in$ $C, \gamma(1) \notin C$, and there exists a $\mathscr{C}^{1, \alpha}$ hypersurface $H^{1}$ of $M^{1}$ with $\gamma \subset H^{1}$ which is foliated by characteristic segments close to $\gamma$, such that locally in a neighborhood of $H^{1}$, the closed subset $C$ is contained in $\gamma \cup\left(H^{1}\right)^{-}$, where $\left(H^{1}\right)^{-}$denotes an open one-sided neighborhood of $H^{1}$ in $M^{1}$.

Proof. By the nontransversality assumption, there exists a first characteristic curve $\widetilde{\gamma}:[-1,1] \rightarrow M^{1}$ with $\widetilde{\gamma}(-1) \notin C, \widetilde{\gamma}(0) \in C$ and $\widetilde{\gamma}(1) \notin C$, there exists a neighborhood $V_{\widetilde{\gamma}}^{1}$ of $\widetilde{\gamma}$ in $M^{1}$ and there exists a local $(n-1)$ dimensional submanifold $R^{1}$ passing through $\widetilde{\gamma}(0)$ which is transversal to $\widetilde{\gamma}$ such that the semi-local projection $\pi_{\mathrm{F}_{M^{1}}^{c}}: V_{\widetilde{\gamma}}^{1} \rightarrow R^{1}$ parallel to the characteristic curves maps $C$ onto the closed subset $\pi_{\mathrm{F}^{c}}(C)$ with the property that $\pi_{\mathrm{F}^{c}}(\widetilde{\gamma})$ lies on the boundary of $\pi_{\boldsymbol{F}_{M^{1}}^{c}}(C)$ with respect to the topology of $R^{1}$. This property is illustrated in the right hand side of the following figure.

However, we want in addition a foliated supporting hypersurface $H^{1}$, which does not necessarily exist in a neighborhood of $\widetilde{\gamma}$. To construct $H^{1}$, let us first straighten the characteristic lines in a neighborhood of $\widetilde{\gamma}$, getting a product $[-1,1] \times\left[-\delta_{1}, \delta_{1}\right]^{n-1}$, for some $\delta_{1}>0$, equipped with coordinates $(s, \chi)=\left(s, \chi_{2}, \ldots, \chi_{n}\right) \in \mathbb{R} \times \mathbb{R}^{n-1}$, so that level-set $\{\chi=\operatorname{cst}\}$ correspond to characteristic lines. Such a straightening is only of class $\mathscr{C}^{1, \alpha}$, because the line distribution $\left.T^{c} M\right|_{M^{1}} \cap T M^{1}$ is only of class $\mathscr{C}^{1, \alpha}$. Clearly, we may assume that $\delta_{1}$ is so small that there exists $s_{1}$ with $0<s_{1}<1$ such that the two cubes $\left[-1,-s_{1}\right] \times\left[-\delta_{1}, \delta_{1}\right]^{n-1}$ and $\left[s_{1}, 1\right] \times\left[-\delta_{1}, \delta_{1}\right]^{n-1}$ do not meet the singularity $C$.


We may identify the transversal $R^{1}$ with $\left[-\delta_{1}, \delta_{1}\right]^{n-1}$; then the projection of $\widetilde{\gamma}$ is the origin of $R^{1}$. By assumption, $\pi_{F_{M^{1}}^{c}}(C)$ is a proper closed subset of $R^{1}$ with the origin lying on its boundary. We can therefore choose a point $\chi_{0}$ in the interior of $R^{1}$ lying outside $\pi_{F_{M^{1}}^{c}}(C)$. Also, we can choose a small open $(n-1)$-dimensional ball $Q_{0}$ centered at this point which is contained in the complement $R^{1} \backslash \pi_{F_{M^{1}}}(C)$. Furtermore, we can include this ball in a one parameter family of $\mathscr{C}^{1, \alpha}$ domains $Q_{\tau} \subset R^{1}$, for $\tau \geqslant 0$, which are parts of ellipsoids stretched along the segment which joins the point $\chi_{0}$ with the origin of $R^{1}$.

We then consider the tube domains $[-1,1] \times Q_{\tau}$ in $[-1,1] \times\left[-\delta_{1}, \delta_{1}\right]^{n-1}$. Clearly, there exists the smallest $\tau_{1}>0$ such that the tube $[-1,1] \times Q_{\tau_{1}}$ meets the singularity $C$ on its boundary $[-1,1] \times \partial Q_{\tau_{1}}$. In particular, there exists a point $\chi_{1} \in \partial Q_{\tau_{1}}$ such that the characteristic segment $[-1,1] \times\left\{\chi_{1}\right\}$ intersects $C$. Increasing a little bit the curvature of $\partial Q_{\tau_{1}}$ in a neighborhood of $\chi_{1}$ if necessary, we can assume that $\pi_{F_{M^{1}}^{c}}(C) \cap \overline{Q_{\tau_{1}}}=\left\{\chi_{1}\right\}$ in a neighborhood of $\chi_{1}$. Moreover, since by construction the two segments $\left[-1,-s_{1}\right] \times\left\{\chi_{1}\right\} \cup\left[s_{1}, 1\right] \times\left\{\chi_{1}\right\}$ do not meet $C$, we can reparametrize the characteristic segment $[-1,1] \times\left\{\chi_{1}\right\}$ as $\gamma:[-1,1] \rightarrow M^{1}$ with $\gamma(-1) \notin C$, $\gamma(0) \in C$ and $\gamma(1) \notin C$. Since all characteristic lines are $\mathscr{C}^{2, \alpha}$, we can choose the parametrization to be of class $\mathscr{C}^{2, \alpha}$. For the supporting hypersurface $H^{1}$, it suffices to choose a piece of $[-1,1] \times \partial Q_{\tau_{1}}$ near $[-1,1] \times\left\{\chi_{1}\right\}$. By construction, this supporting hypersurface is only of class $\mathscr{C}^{1, \alpha}$ and we have that $C$ is contained in $\gamma \cup\left(H^{1}\right)^{-}$semi-locally in a neighborhood of $\gamma$, as desired.
5.3. Field of cones on $M^{1}$. With the characteristic segment $\gamma$ constructed in Lemma 5.2, by an application of Proposition 4.6, we deduce that there exists a semi-local half-wedge $\mathscr{H} \mathscr{W}_{\gamma}^{+}$attached to $\left(M_{\gamma}^{1}\right)^{+} \cap V_{\gamma}$, for some neighborhood $V_{\gamma}$ of $\gamma$ in $M$, to which $\mathscr{O}(\Omega)$ extends holomorphically.

Then, we remind that by (4.37), (4.39) and (4.50), for all $t$ with $|t|<\varepsilon$, the point $\widehat{\mathscr{L}_{t, \chi, 0: s}}(1)$ identifies with the point $\widehat{p}(\chi, 0: s) \in M_{s}^{1}$ defined in (4.37) (which is independent of $t$ ) and the mapping $\chi \mapsto \widehat{\mathscr{Z}}_{t, \chi, 0: s}(1) \in M_{s}^{1}$ is a local diffeomorphism.

Sometimes in the sequel, we shall denote the disc $\mathscr{Z}_{t, \chi, \nu: s}(\zeta) \equiv$ $\Phi_{s}^{-1}\left(\widehat{\mathscr{Z}}_{t, \chi, \nu: s}(\zeta)\right)$ defined in (4.61) by $\mathscr{Z}_{t, \chi_{1}, \chi^{\prime}, \nu: s}(\zeta)$, where $\chi^{\prime}=\left(\chi_{2}, \ldots, \chi_{n}\right) \in \mathbb{R}^{n-1}$. Since the characteristic curve is directed along the $x_{1}$-axis, which is transversal in $T_{0} M_{s}^{1}$ to the space $\left\{\left(0, \chi^{\prime}\right)\right\}$, it follows that the mapping $\left(s, \chi^{\prime}\right) \longmapsto \mathscr{Z}_{t, 0, \chi^{\prime}, 0: s}(1)=\Phi_{s}^{-1}\left(\widehat{p}\left(0, \chi^{\prime}, 0: s\right)\right)$ is, independently of $t$, a diffeomorphism onto its image for $s \in[-1,1]$ and for $\chi^{\prime}$ close to the origin in $\mathbb{R}^{n-1}$. To fix ideas, we shall let $\chi^{\prime}$ vary in the closed
cube $[-\varepsilon, \varepsilon]^{n-1}$ (analogously to the fact that $s$ runs in the closed interval $[-1,1]$ ) and we shall denote by $V_{\gamma}^{1}$ the closed image of this diffeomorphism.
At every point $p:=\mathscr{Z}_{t, 0, \chi^{\prime}, 0: s}(1)=\mathscr{Z}_{0,0, \chi^{\prime}, 0: s}(1)$ of this neighborhood $V_{\gamma}^{1}$, we define an open infinite oriented cone contained in the $n$-dimensional linear space $T_{p} M^{1}$ by

$$
\begin{equation*}
\mathrm{C}_{p}:=\mathbb{R}^{+} \cdot\left\{\frac{\partial \mathscr{Z}_{t, 0, \chi^{\prime}, 0: s}}{\partial \theta}(1):|t|<\varepsilon\right\} . \tag{5.4}
\end{equation*}
$$

The fact that $C_{p}$ is indeed an open cone follows from Lemma 4.34, from (4.61) and from the fact that $\Phi_{s}^{-1}$ is a biholomorphism. This cone contains in its interior the nonzero vector

$$
\begin{equation*}
v_{p}^{0}:=\frac{\partial \mathscr{Z}_{0,0, \chi^{\prime}, 0: s}}{\partial \theta}(1) \in \mathrm{C}_{p} \subset T_{p} M^{1} \backslash\{0\} . \tag{5.5}
\end{equation*}
$$

We shall say that $\mathrm{C}_{p}$ is the cone created at $p$ by the semi-local attached halfwedge $\mathscr{H} \mathscr{W}_{\gamma}^{+}$(more precisely, by the family of analytic discs which covers this semi-local half-wedge).

As $p$ varies, $p \mapsto \mathrm{C}_{p}$ constitutes a field of cones over $V_{\gamma}^{1}$, as illustrated by Figures 3 and 11.


Fig. 11: Field of cones on $T M^{1}$ associated to the family $\mathscr{Z}_{t, 0, \chi^{\prime}, 0, s}(\zeta)$

The map $p \mapsto v_{p}^{0}$ is a $\mathscr{C}^{1, \alpha-0}$ vector field tangent to $M^{1}$, contained in the field of cones $p \mapsto \mathrm{C}_{p}$. Over $V_{\gamma}^{1}$, we also introduce a nowhere zero characteristic vector field $X$ which satisfies $\exp (s X)(\gamma(0))=\gamma(s)$ for all $s \in[-1,1]$. As in Section 2, for every $p \in V_{\gamma}^{1}$, we define the filled cone

$$
\begin{equation*}
\mathrm{FC}_{p}:=\mathbb{R}^{+} \cdot\left\{\lambda \cdot X_{p}+(1-\lambda) \cdot v_{p}: 0 \leqslant \lambda<1, v_{p} \in \mathrm{C}_{p}\right\} \tag{5.6}
\end{equation*}
$$

In $T_{p} M^{1}$ equipped with linear coordinates $\left(x_{1}, \ldots, x_{n}\right)$ such that the characteristic direction $T_{p}^{c} M \cap T_{p} M^{1}$ isx the $x_{1}$-axis, we draw $\mathrm{C}_{p}$, its filling $\mathrm{FC}_{p}$ and its projection $\pi^{\prime}\left(\mathrm{C}_{p}\right)$ onto the $\left(x_{2}, \ldots, x_{n}\right)$-space parallelly to the $x_{1}$-axis.


Fig. 12: A cone in $T_{p} M^{1}$ and its filling along the characteristic direction

For every nonzero vector $v_{p} \in \mathrm{C}_{p}$, it may be checked that a small neighborhood of the origin in the positive half-line $\mathbb{R}^{+} \cdot J v_{p}$ generated by $J v_{p}$ is contained in the attached half-wedge $\mathscr{H} \mathscr{W}_{\gamma}^{+}$.

Lemma 5.7. Fix a point $p \in V_{\gamma}^{1}$ and a vector $v_{p}$ in the cone $\mathrm{C}_{p}$ created by the semi-local attached half-wedge $\mathscr{H}^{\mathscr{W}}{ }_{\gamma}^{+}$at $p$. Suppose that there exist two constants $c_{1}>0$ and $\Lambda_{1}>1$ such that for every $c$ with $0<c \leqslant c_{1}$, there exists a $\mathscr{C}^{2, \alpha-0}$ analytic disc $A_{c}(\zeta)$ with $A_{c}\left(\partial^{+} \Delta\right) \subset M^{1}$, such that:
(i) the positive half-line generated by the boundary of $A_{c}$ at $\zeta=1$ coincides with the positive half-line generated by $v_{p}$, namely $\mathbb{R}^{+}$. $\frac{\partial A_{c}}{\partial \theta}(1) \equiv \mathbb{R}^{+} \cdot v_{p} ;$
(ii) $\left|A_{c}(\zeta)\right| \leqslant c^{2} \cdot \Lambda_{1}$ for all $\zeta \in \bar{\Delta}$ and $c \cdot \frac{1}{\Lambda_{1}} \leqslant\left|\frac{\partial A_{c}}{\partial \theta}(1)\right| \leqslant c \cdot \Lambda_{1}$;
(iii) $\left|\frac{\partial A_{c}}{\partial \theta}\left(\rho e^{i \theta}\right)-\frac{\partial A_{c}}{\partial \theta}(1)\right| \leqslant c^{2} \cdot \Lambda_{1}$ for all $\zeta=\rho e^{i \theta} \in \bar{\Delta}$.

If $c_{1}$ is sufficiently small, then for every $c$ with $0<c \leqslant c_{1}$, the closed disc minus its half-boundary $A_{c}\left(\bar{\Delta} \backslash \partial^{+} \Delta\right)$ is contained in the semi-local halfwedge $\mathscr{H}^{\mathscr{W}}{ }_{\gamma}^{+}$.

Furthermore, the same conclusion holds if the nonzero vector $v_{p}$ belongs to the filled cone $\mathrm{FC}_{p}$.

Details will be provided later. In fact, the reason why we introduce filled cones $\mathrm{FC}_{p}$ is because, for the selection of a special, locally removable point of $C$, we shall see that the corresponding direction $\mathbb{R}^{+} \cdot \frac{\partial A_{c}}{\partial \theta}(1) \equiv \mathbb{R}^{+} \cdot v_{p}$ of half-boundary $A\left(\partial^{+} \Delta\right) \subset M^{1}$ must unavoidably be almost parallel to the characteristic direction, and hopefully, vectors $v_{p} \in \mathrm{FC}_{p}$ may approach the characteristic direction arbitrarily. Indeed, not only we will have to assure that $A_{c}\left(\bar{\Delta} \backslash \partial^{+} \Delta\right) \subset \mathscr{H} \mathscr{W}_{\gamma}^{+}$(which works already for $v_{p} \in \mathrm{C}_{p}$ ), but also, we will have to insure that the disc $A_{c}$ with $A_{c}(1) \in C$ satisfies $A_{c}\left(\partial^{+} \Delta \backslash\{1\}\right)$, as drawn in Figure 4, in order to be able to apply the continuity principle.
5.8. Choice of the special point $p_{\mathrm{sp}}$. We can now answer the question implicitly left inside Theorem 3.19: how to choose the special point $p_{\text {sp }}$ to be removed locally?
Lemma 5.9. Let $\gamma$ be the characteristic segment constructed in Lemma 5.2, let $\mathscr{H} \mathscr{W}_{\gamma}^{+}$be the semi-local attached half-wedge of edge $\left(M_{\gamma}^{1}\right)^{+} \cap V_{\gamma}$ constructed in Proposition 4.6, and let $p \mapsto \mathrm{FC}_{p}$ be the filled field of cones created by $\mathscr{H} \mathscr{W}_{\gamma}^{+}$. Then there exists a special point $p_{\mathrm{sp}} \in C \cap V_{\gamma}^{1}$ such that:
(i) there exists a $\mathscr{C}^{2, \alpha}$ local supporting hypersurface $H_{\text {sp }}$ of $M^{1}$ passing through $p_{\text {sp }}$ such that, locally in a neighborhood of $p_{\mathrm{sp}}$, the closed subset $C$ is contained in $\left(H_{\mathrm{sp}}\right)^{-} \cup\left\{p_{\mathrm{sp}}\right\}$, where $\left(H_{\mathrm{sp}}\right)^{-}$denotes an open one-sided neighborhood of $H_{\mathrm{sp}}$ in $M^{1}$; and:
(ii) there exists a nonzero vector $v_{\mathrm{sp}} \in T_{p_{\mathrm{sp}}} H_{\mathrm{sp}}$ which belongs to the filled cone $\mathrm{FC}_{p_{\mathrm{sp}}}$.

Proof. According to Lemma 5.2, the singularity $C$ is contained in $\gamma \cup\left(H^{1}\right)^{-}$, where $H^{1}$ is a $\mathscr{C}^{1, \alpha}$ hypersurface containing $\gamma$ which is foliated by characteristic segments. If $\lambda \in[0,1)$ is very close to 1 , the vector field over $V_{\gamma}^{1}$ defined by

$$
\begin{equation*}
p \longmapsto v_{p}^{\lambda}:=\lambda \cdot X_{p}+(1-\lambda) \cdot v_{p} \in T_{p} M^{1} \tag{5.10}
\end{equation*}
$$

is very close to the characteristic vector field $X_{p}$, so the integral curves of $p \mapsto v_{p}^{\lambda}$ are very close to the integral curves of $p \mapsto X_{p}$. If $\lambda$ is sufficiently close to 1 , we can choose a subneighborhood $V_{\gamma}^{\lambda} \subset V_{\gamma}^{1}$ of $\gamma$ which is foliated by integral curves of $p \mapsto v_{p}^{\lambda}$. As in Lemma 5.2, let us fix an ( $n-1$ )-dimensional submanifold $R^{1}$ transversal to $\gamma$ and passing through $\gamma(0)$. Since the vector field $p \mapsto v_{p}^{\lambda}$ is very close to the characteristic vector field, it follows that after projection onto $R^{1}$ parallelly to the integral curves of $p \mapsto v_{p}^{\lambda}$, the closed set $C \cap V_{\gamma}^{\lambda}$ is again a proper closed subset of $R^{1}$. We notice that, by its very definition, the vector $v_{p}^{\lambda}$ belongs to the filled cone $\mathrm{FC}_{p}$ for all $p \in V_{\gamma}^{\lambda}$.

We can proceed exactly as in the proof of Lemma 5.2 with the foliation of $V_{\gamma}^{\lambda}$ induced by the integral curves of the vector field $p \mapsto v_{p}^{\lambda}$, instead of the characteristic foliation, except that we want a supporting hypersurface $H_{\text {sp }}$ which is of class $\mathscr{C}^{2, \alpha}$. Consequently, we first approximate the vector field $p \mapsto v_{p}^{\lambda}$ by a new vector field $p \mapsto \widetilde{v}_{p}^{\lambda}$ whose coefficients are of class $\mathscr{C}^{2, \alpha}$ (with respect to every local graphing function of $M^{1}$ ) and which is very close to the vector field $p \mapsto v_{p}^{\lambda}$ in $\mathscr{C}^{1, \alpha}$-norm. Again, we get a subneighborhood $\widetilde{V}_{\gamma}^{\lambda} \subset V_{\gamma}^{\lambda}$ of $\gamma$ which is foliated by integral curves of $p \mapsto \widetilde{v}_{p}^{\lambda}$ and a projection of $C \cap \widetilde{V}_{\gamma}^{\lambda}$ which is a proper closed subset of $R^{1}$. Moreover, if the approximation is sufficiently sharp, we still have $\widetilde{v}_{p}^{\lambda} \in \mathrm{FC}_{p}$ for all $p \in \widetilde{V}_{\gamma}^{\lambda}$.

Then by repeating the reasoning which yielded Lemma 5.2, we deduce that there exists an integral curve $\widetilde{\gamma}$ of the vector field $p \mapsto \widetilde{v}_{p}^{\lambda}$ satisfying (after reparametrization) $\widetilde{\gamma}(-1) \notin C, \widetilde{\gamma}(0) \in C$ and $\widetilde{\gamma}(1) \notin C$, together with a $\mathscr{C}^{2, \alpha}$ supporting hypersurface $\widetilde{H}$ of $\widetilde{V}_{\gamma}^{\lambda}$ which contains $\widetilde{\gamma}$ such that $C$ is contained in $\widetilde{\gamma} \cup(\widetilde{H})^{-}$.


Fig. 13: Dotted integral curve of the vector field $p \mapsto v_{p}^{\lambda}$ and choice of $p_{\text {sp }}$
To conclude the proof of Lemma 5.9, for the desired special point $p_{\mathrm{sp}}$, it suffices to choose $\widetilde{\gamma}(0)$. For the desired local supporting hypersurface $H_{p_{s p}}$, we cannot choose directly a piece of $\widetilde{H}$ passing through $p_{\text {sp }}$, because an open interval contained in $C \cap \widetilde{\gamma}$ may well be contained in $\widetilde{H}$. Fortunately, since we know that locally in a neighborhood of $p_{\text {sp }}$, the closed subset $C$ is contained in $(\widetilde{H})^{-} \cup \widetilde{\gamma}$, it suffices to choose for the desired supporting hypersurface $H_{p_{\mathrm{sp}}} \subset M^{1}$ a piece of a $\mathscr{C}^{2, \alpha}$ hypersurface passing through $p_{1}$, tangent to $\widetilde{H}$ at $p_{1}$ and satisfying $H_{p_{\mathrm{sp}}} \backslash\left\{p_{\mathrm{sp}}\right\} \subset(\widetilde{H})^{+}$in a neighborhood of $p_{\mathrm{sp}}$. Finally, for the nonzero vector $v_{\mathrm{sp}}$, it suffices to choose any positive multiple of the vector $\widetilde{v}_{p_{\text {sp }}}^{\lambda}$. This completes the proof of Lemma 5.9.
5.11. Main removability proposition. We can now formulate the main removability proposition to which Theorem 3.19 is now fully reduced. We localize the situation at $p_{\mathrm{sp}}$, we denote this point simply by $p_{1}$, we denote its supporting hypersurface simply by $H^{1}$ and we denote its associated vector simply by $v_{1} \in T_{p_{1}} H^{1}$. At $p_{1}$, we have a local half-wedge $\mathscr{H} \mathscr{W}_{p_{1}}^{+} \subset \mathscr{H} \mathscr{W}_{\gamma}^{+}$.
Proposition 5.12. Let $M \subset \mathbb{C}^{n}$ be a $\mathscr{C}^{2, \alpha}$ generic submanifold of codimension $n-1 \geqslant 1$, hence of $C R$ dimension 1 , let $M^{1} \subset M$ be a $\mathscr{C}^{2, \alpha}$ onecodimensional submanifold which is maximally real in $\mathbb{C}^{n}$, let $p_{1} \in M^{1}$, let $H^{1} \subset M^{1}$ be a $\mathscr{C}^{2, \alpha}$ one-codimensional submanifold of $M^{1}$ passing through $p_{1}$ and let $\left(H^{1}\right)^{-}$denote an open local one-sided neighborhood of $H^{1}$ in $M^{1}$. Let $C \subset M^{1}$ be a nonempty proper closed subset of $M^{1}$ with $p_{1} \in C$ which is situated, locally in a neighborhood of $p_{1}$, only in one side of $H^{1}$, namely $C \subset\left(H^{1}\right)^{-} \cup\left\{p_{1}\right\}$. Let $\Omega$ be a neighborhood of $M \backslash C$ in $\mathbb{C}^{n}$, let $\mathscr{H} W_{p_{1}}^{+}$be a local half-wedge of edge $\left(M^{1}\right)^{+}$at $p_{1}$ generated by a
family of analytic discs $\mathscr{Z}_{t, \chi, \nu}(\zeta)$ satisfying the properties $\mathbf{( i ) , ~ ( i i ) ~ a n d ~ ( i i i ) ~ o f ~}$ §4.2, let $\mathrm{C}_{p_{1}} \subset T_{p_{1}} M^{1}$ be the cone created by $\mathscr{H} \mathscr{W}_{p_{1}}^{+}$at $p_{1}$ and let $\mathrm{FC}_{p_{1}}$ be its filling. As a main hypothesis, assume that there exists a nonzero vector $v_{1} \in T_{p_{1}} H^{1}$ which belongs to the filled cone $\mathrm{FC}_{p_{1}}$.
(I) If $v_{1}$ does not belong to $T_{p_{1}}^{c} M$, then there exists a local wedge $\mathscr{W}_{p_{1}}$ of edge $M$ at $\left(p_{1}, J v_{1}\right)$ with $\mathscr{W}_{p_{1}} \cap\left[\Omega \cup \mathscr{H} \mathscr{W}_{p_{1}}^{+}\right]$connected (shrinking $\Omega \cup \mathscr{H} \mathscr{W}_{p_{1}}^{+}$if necessary) such that for every holomorphic function $f \in \mathscr{O}\left(\Omega \cup \mathscr{H} W_{p_{1}}^{+}\right)$, there exists a holomorphic function $F \in \mathscr{O}\left(\mathscr{W}_{1} \cup\left[\Omega \cup \mathscr{H} \mathscr{W}_{p_{1}}^{+}\right]\right)$with $\left.F\right|_{\Omega \cup \mathscr{C} \mathscr{W}_{p_{1}}^{+}}=f$.
(II) If $v_{1}$ belongs to $T_{p_{1}}^{c} M$, then there exists a neighborhood $\omega_{p_{1}}$ of $p_{1}$ in $\mathbb{C}^{n}$ with $\omega_{p_{1}} \cap\left[\Omega \cup \mathscr{H} \mathscr{W}_{p_{1}}^{+}\right]$connected (shrinking $\Omega \cup$ $\mathscr{H} \mathscr{W}_{p_{1}}^{+}$if necessary) such that for every holomorphic function $f \in \mathscr{O}\left(\Omega \cup \mathscr{H} W_{p_{1}}^{+}\right)$, there exists a holomorphic function $F \in$ $\mathscr{O}\left(\omega_{p_{1}} \cup\left[\Omega \cup \mathscr{H} \mathscr{W}_{p_{1}}^{+}\right]\right)$with $\left.F\right|_{\Omega \cup \mathscr{H} W_{p_{1}}^{+}}=f$.

The remainder of Section 5, and then Sections 6, 7, 8 and 9 are entirely devoted to the proof of this proposition.
5.13. A dichotomy. We shall indeed distinguish two cases:
(I) the nonzero vector $v_{1}$ does not belong to the characteristic direction $T_{p_{1}} M^{1} \cap T_{p_{1}}^{c} M ;$
(II) the nonzero vector $v_{1}$ belongs to the characteristic direction $T_{p_{1}} M^{1} \cap$ $T_{p_{1}}^{c} M$.
We must clarify the main assumption that $v_{1}$ belongs to the filling $\mathrm{FC}_{p_{1}}$ of the cone $\mathrm{C}_{p_{1}} \subset T_{p_{1}} M^{1}$ created by the local half-wedge $\mathscr{H} \mathscr{W}_{p_{1}}^{+}$. As we have observed in $\S 4.2$, in the (generic) situation of Case (I), a local half-wedge may be represented geometrically as the intersection of a (complete) local wedge of edge $M$ at $p_{1}$, with a local one-sided neighborhood $\left(N^{1}\right)^{+}$of a hypersurface $N^{1}$ passing through $p_{1}$, which is transversal to $M$ and which satisfies $N^{1} \cap M \equiv M^{1}$ in a neighborhood of $p_{1}$. The slope of the tangent space $T_{p_{1}} N^{1}$ to $N^{1}$ at $p_{1}$ with respect to the tangent space $T_{p_{1}} M$ to $M$ at $p_{1}$ may be understood in terms of the cone $\mathrm{C}_{p_{1}}$, as we will now explain. Afterwards, we shall consider Case (II) separately.

### 5.14. Cones, filled cones, subcones and local description of half-wedges

in Case (I). For the sake of concreteness, it will be convenient to work in a holomorphic coordinate system $z=\left(z_{1}, \ldots, z_{n}\right)=\left(x_{1}+i y_{1}, \ldots, x_{n}+i y_{n}\right)$ centered at $p_{1}$ in which $T_{p_{1}} M=\left\{y_{2}=\cdots=y_{n}=0\right\}$ and $T_{p_{1}} M^{1}=$ $\left\{y_{1}=y_{2}=\cdots=y_{n}=0\right\}$ (the existence of such a coordinate system which
straightens both $T_{p_{1}} M$ and $T_{p_{1}} M^{1}$ is a direct consequence of the considerations of §4.11). Let $\pi^{\prime}: T_{p_{1}} M^{1} \rightarrow T_{p_{1}} M^{1} /\left(T_{p_{1}} M^{1} \cap T_{p_{1}}^{c} M\right)$ denote the canonical projection, namey $\pi^{\prime}\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\left(x_{2}, \ldots, x_{n}\right)$. Sometimes, we shall denote the coordinates by $\left(z_{1}, z^{\prime}\right)=\left(x_{1}+i y_{1}, x^{\prime}+i y^{\prime}\right) \in \mathbb{C} \times \mathbb{C}^{n-1}$. In these coordinates, the characteristic direction at $p_{1}$ is the $x_{1}$-axis and we may assume that the tangent plane at $p_{1}$ of the one-sided neighborhood $\left(M^{1}\right)^{+}$is given by $T_{p_{1}}\left(M^{1}\right)^{+}=\left\{y^{\prime}=0, y_{1}>0\right\}$.
Let $\mathrm{C}_{p_{1}} \subset T_{p_{1}} M^{1}$ be the cone created by $\mathscr{H} \mathscr{W}_{p_{1}}^{+}$and let $\mathrm{C}_{p_{1}}^{\prime}:=\pi^{\prime}\left(\mathrm{C}_{p_{1}}\right)$ be its projection onto the $x^{\prime}$-space, which yields an $(n-1)$-dimensional infinite cone in the $x^{\prime}$-space, open in this space. Notice that, by the definition (5.6) of the filling (along the characteristic direction), the two projections $\pi^{\prime}\left(\mathrm{C}_{p_{1}}\right)$ and $\pi^{\prime}\left(\mathrm{FC}_{p_{1}}\right)$ are identical. We must now explain how these three cones $\mathrm{C}_{p_{1}}, \mathrm{FC}_{p_{1}}, \mathrm{C}_{p_{1}}^{\prime}$ and the nonzero vector $v_{1} \in \mathrm{FC}_{p_{1}}$ are disposed, geometrically.


Fig. 14: Cone created by the half-wedge and its relation to the half-wedge
Because the discs $\mathscr{Z}_{t, \chi, \nu}$ of Proposition 5.12 (constructed in Section 4) are small, the tangent vector $\frac{\partial \mathscr{F}_{0,0,0}}{\partial \theta}(1)$ is necessarily close to the complex tangent plane $T_{p_{1}}^{c} M$ : this may be checked directly by differentiating Bishop's equation (4.40) with respect to $\theta$, using the fact that the $\mathscr{C}^{1}$-norm of $\Phi^{\prime}$ is small. Moreover, since this vector $\frac{\partial \mathscr{F}_{0,0,0}}{\partial \theta}(1)$ also belongs to $T_{p_{1}} M^{1}$, it is in fact close to the positive $x_{1}$-axis. Furthermore, since the vector $v_{1}$ belongs to $\mathrm{FC}_{p_{1}}$ which contains the vector $\frac{\partial \mathscr{E}_{0,0,0}}{\partial \theta}(1)$, and since in the proof of Lemma 5.9 above we have chosen the special point, the supporting hypersurface and the vector $v_{1}$ with a parameter $\lambda$ very close to 1 , it follows that the vector $v_{1} \equiv \widetilde{v}_{p_{\mathrm{sp}}}^{\lambda}$ is even closer to the positive $x_{1}$-axis. However, we suppose in Case (I) that $v_{1}$ is not directed along the $x_{1}$-axis, so $v_{1}$ has coordinates $\left(v_{1 ; 1}, v_{2 ; 1}, \ldots, v_{n ; 1}\right) \in \mathbb{R}^{n}$ with $v_{1 ; 1}>0$, with $\left|v_{j ; 1}\right| \ll v_{1 ; 1}$ for $j=2, \ldots, n$ and with at least one $v_{j ; 1}$ being nonzero.

We need some general terminology. Let $C$ be an open infinite cone in a real linear subspace $E$ of dimension $q \geqslant 1$. We say that $\mathrm{C}^{\prime}$ is a proper subcone and we write $\mathrm{C}^{\prime} \subset \subset C$ (see the left hand side of Figure 12 above for
an illustration) if the intersection of $\mathrm{C}^{\prime}$ with the unit sphere of $E$ is a relatively compact subset of the intersection of $C$ with the unit sphere of $E$, this property being independent of the choice of a norm on $E$. We say that C is a linear cone if it may be defined by $\mathrm{C}=\left\{x \in E: \ell_{1}(x)>0, \ldots, \ell_{q}(x)>0\right\}$ for some $q$ linearly independent real linear forms $\ell_{1}, \ldots, \ell_{q}$ on $E$.

In the $x^{\prime}$-space, we now choose an open infinite strictly convex linear proper subcone $\mathrm{C}_{1}^{\prime} \subset \subset \mathrm{C}_{p_{1}}^{\prime}$ with the property that $v_{1}$ belongs to its filling $\mathrm{FC}_{1}^{\prime}$, cf. Figure 14. Here, we may assume that $\mathrm{C}_{1}^{\prime}$ is described by $(n-1)$ strict inequalities $\ell_{1}^{\prime}\left(x^{\prime}\right)>0, \ldots, \ell_{n-1}^{\prime}\left(x^{\prime}\right)>0$, where the $\ell_{j}^{\prime}\left(x^{\prime}\right)$ are linearly independent linear forms. It then follows that there exists a linear form $\sigma\left(x_{1}, x^{\prime}\right)$ of the form $\sigma\left(x_{1}, x^{\prime}\right)=x_{1}+a_{2} x_{2}+\cdots+a_{n} x_{n}$ such that the original filled cone $\mathrm{FC}_{p_{1}}$ is contained in the linear cone (5.15)

$$
\mathrm{C}_{1}:=\left\{\left(x_{1}, x^{\prime}\right) \in \mathbb{R}^{n}: \ell_{1}^{\prime}\left(x^{\prime}\right)>0, \ldots, \ell_{n-1}^{\prime}\left(x^{\prime}\right)>0, \sigma\left(x_{1}, x^{\prime}\right)>0\right\},
$$

which contains the vector $v_{1}$. This cone is automatically filled, namely $\mathrm{C}_{1} \equiv$ $\mathrm{FC}_{1}$.

We remind that by genericity of $M$, the complex structure $J$ of $T \mathbb{C}^{n}$ induces an isomorphim $T_{p_{1}} M / T_{p_{1}}^{c} M \rightarrow T_{p_{1}} \mathbb{C}^{n} / T_{p_{1}} M$. Hence $J C_{p_{1}}^{\prime}$ and $J \mathrm{C}_{1}^{\prime}$ are open infinite strictly convex linear proper cones in $T_{p_{1}} \mathbb{C}^{n} / T_{p_{1}} M \cong$ $\left\{\left(0, y^{\prime}\right) \in \mathbb{C}^{n}\right\}$. Since $J C_{1}^{\prime}$ is a proper subcone of $J C_{p_{1}}^{\prime}$ and since in the classical definition of a wedge, only the projection of the cone onto the quotient space $T_{p_{1}} M / T_{p_{1}}^{c} M$ has a contribution to the wedge, it then follows that the complete wedge $\mathscr{W}_{p_{1}}$ associated to the family $\mathscr{Z}_{t, \chi, \nu}(\zeta)$ (cf. the paragraph after (4.5)) contains a wedge of the form

$$
\begin{equation*}
\mathscr{W}_{1}:=\left\{p+\mathrm{c}_{1}^{\prime}: p \in M, \mathrm{c}_{1}^{\prime} \in J \mathrm{C}_{1}^{\prime}\right\} \cap \Delta_{n}\left(p_{1}, \delta_{1}\right), \tag{5.16}
\end{equation*}
$$

for some $\delta_{1}$ with $0<\delta_{1}<\varepsilon$, where $\varepsilon$ is as in $\S 4.2$. Furthermore, as observed in $\S 4.2$, there exists a $\mathscr{C}^{2, \alpha}$ hypersurface $N^{1}$ of $\mathbb{C}^{n}$ passing through $p_{1}$ with the property that $N^{1} \cap M \equiv M^{1}$ locally in a neighborhood of $p_{1}$ such that, shrinking $\delta_{1}>0$ if necessary, the local half-wedge $\mathscr{H} \mathscr{W}_{p_{1}}^{+}$contains a local half-wedge $\mathscr{H} \mathscr{W}_{1}^{+}$of edge $\left(M^{1}\right)^{+}$at $p_{1}$ which is described as the geometric intersection of the complete wedge $\mathscr{W}_{p_{1}}$ with a one-sided neighborhood $\left(N^{1}\right)^{+}$, namely

$$
\begin{equation*}
\mathscr{H} \mathscr{W}_{1}^{+}:=\mathscr{W}_{1} \cap\left(N^{1}\right)^{+} . \tag{5.17}
\end{equation*}
$$

An illustration for the case $n=2$ where $M \subset \mathbb{C}^{2}$ is a hypesurface is provided in the left hand side of Figure 14. In addition, it follows from the definition of $\mathscr{H} \mathscr{W}_{p_{1}}^{+}$by means of the segments $\mathscr{Z}_{t, \chi, \nu}((1-\varepsilon, 1))$ that we can assume that

$$
\begin{equation*}
T_{p_{1}}\left(N^{1}\right)^{+}=T_{p_{1}} M \oplus J\left(\Sigma^{1}\right)^{+}, \tag{5.18}
\end{equation*}
$$

where $\left(\Sigma_{1}\right)^{+}$is the hyperplane one-sided neighborhood $\left\{\left(x_{1}, x^{\prime}\right)\right.$ : $\left.\sigma\left(x_{1}, x^{\prime}\right)>0\right\} \subset T_{p_{1}} M^{1}$. Equivalently, $T_{p_{1}}\left(N^{1}\right)^{+}$is represented by the inequality $y_{1}+a_{2} y_{2}+\cdots+a_{n} y_{n}>0$. Consequently, there exists a $\mathscr{C}^{2, \alpha}$ function $\psi\left(x, y^{\prime}\right)$ with $\psi(0)=\partial_{x_{k}} \psi(0)=\partial_{y_{j}} \psi(0)=0$ for $k=1, \ldots, n$ and $j=2, \ldots, n$ such that $N^{1}$ is represented by the equation $y_{1}+a_{2} y_{2}+\cdots+a_{n} y_{n}=\psi\left(x, y^{\prime}\right)$ and $\left(N^{1}\right)^{+}$by the inequation $y_{1}+a_{2} y_{2}+\cdots+a_{n} y_{n}>\psi\left(x, y^{\prime}\right)$.
5.19. Cones, filled cones, subcones and local description of half-wedges in Case (II). Secondly, we assume that the nonzero vector $v_{1}$ of Proposition 5.12 belongs to the characteristic direction $T_{p_{1}} M^{1} \cap T_{p_{1}}^{c} M$. In this case, as observed in $\S 4.2$, the half-wedge $\mathscr{H} \mathscr{W}_{p_{1}}^{+}$coincides with a local wedge of edge $M^{1}$ at $\left(p_{1}, J v_{1}\right)$. After a real dilation of the $z_{1}$-axis, we can assume that $v_{1}=(1,0, \ldots, 0)$. Choosing an open infinite strictly convex linear proper subcone $\mathrm{C}_{2} \subset \subset \mathrm{C}_{p_{1}} \subset T_{p_{1}} M^{1}=\mathbb{R}_{x}^{n}$ defined by $n$ strict inequalities $\ell_{1}(x)>0, \ldots, \ell_{n}(x)>0$, where the $\ell_{j}(x)$ are linearly independent real linear forms - of course with $\mathrm{C}_{2}$ containing the vector $v_{1}$ - it follows that there exists $\delta_{1}>0$ such that the half-wedge $\mathscr{H} \mathscr{W}_{p_{1}}^{+}$contains the following local wedge of edge $M^{1}$ at $p_{1}$ :

$$
\begin{equation*}
\mathscr{W}_{2}:=\left\{p+\mathrm{c}_{2}: p \in M^{1}, \mathrm{c}_{2} \in J \mathrm{C}_{2}\right\} \cap \Delta_{n}\left(p_{1}, \delta_{1}\right) . \tag{5.20}
\end{equation*}
$$

We remind that it was observed in $\S 4.2$ ( $c f$. especially the right hand side of Figure 7) that $\mathscr{W}_{2}$ contains $\left(M^{1}\right)^{+}$locally in a neighborhood of $p_{1}$. In $\S 5.22$ below, we shall provide a more concrete representation of $\mathscr{W}_{2}$ in an appropriate system of coordinates.
5.21. A trichotomy. Let us pursue this discussion more concretely by introducing further normalizations. Our goal will now be to construct appropriate normalized coordinate systems. Analyzing further the dichotomy introduced in $\S 5.13$ by taking account of the presence of the one-codimensional submanifold $H^{1} \subset M^{1}$, we shall distinguish three cases by dividing Case (I) in two subcases ( $\mathbf{I}_{1}$ ) and ( $\mathbf{I}_{2}$ ).
( $\mathbf{I}_{1}$ ) The nonzero vector $v_{1}$ does not belong to the characteristic direction $T_{p_{1}} M^{1} \cap T_{p_{1}}^{c} M$ and $\operatorname{dim}_{\mathbb{R}}\left(T_{p_{1}} H^{1} \cap T_{p_{1}}^{c} M\right)=0$.
$\left(\mathbf{I}_{2}\right)$ The nonzero vector $v_{1}$ does not belong to the characteristic direction $T_{p_{1}} M^{1} \cap T_{p_{1}}^{c} M$ and $\operatorname{dim}_{\mathbb{R}}\left(T_{p_{1}} H^{1} \cap T_{p_{1}}^{c} M\right)=1$ (this possibility can only occur when $n \geqslant 3$ ).
(II) The nonzero vector $v_{1}$ belongs to the characteristic direction $T_{p_{1}} M^{1} \cap T_{p_{1}}^{c} M$.
In case ( $\mathbf{I}_{1}$ ), we notice that the assumption $T_{p_{1}} H^{1} \cap T_{p_{1}}^{c} M=\{0\}$ implies that $v_{1}$ does not belong to the characteristic direction, because $v_{1} \in T_{p_{1}} H^{1}$. Also, in case (II), we notice that $\operatorname{dim}_{\mathbb{R}}\left(T_{p_{1}} H^{1} \cap T_{p_{1}}^{c} M\right)=1$ because $v_{1} \in$
$T_{p_{1}} H^{1}$, because $T_{p_{1}} H^{1} \subset T_{p_{1}} M^{1}$ and because the characteristic direction $T_{p_{1}} M^{1} \cap T_{p_{1}}^{c} M$ is one-dimensional.

In each of the above three cases, it will be convenient in Section 8 below to work with simultaneously normalized defining (in)equations for $M$, for $M^{1}$, for $\left(M^{1}\right)^{+}$, for $H^{1}$, for $\left(H^{1}\right)^{+}$, for $\mathrm{C}_{1}^{\prime}$, for $v_{1}$, for $\mathrm{C}_{1}$, for $\left(N^{1}\right)^{+}$ and for $\mathscr{H} \mathscr{W}_{p_{1}}^{+}$, in a single coordinate system centered at $p_{1}$. In the next paragraphs, we shall set up further elementary normalization lemmas in a common system of coordinates, firstly for Case ( $\mathbf{I}_{1}$ ), secondly for Case ( $\mathbf{I}_{\mathbf{2}}$ ) and thirdly for Case (II). This technical work is unavoidable and it will be achieved rigorously.

First of all, in the above coordinate system $\left(z_{1}, z^{\prime}\right)$ with $T_{p_{1}} M=\left\{y_{2}=\right.$ $\left.\cdots=y_{n}=0\right\}$ and with $T_{p_{1}} M^{1}=\left\{y_{1}=y_{2}=\cdots=y_{n}=0\right\}$, by means of the implicit function theorem, we can represent locally $M$ by $(n-1)$ graphed equations of the form $y_{2}=\varphi_{2}\left(x, y_{1}\right), \ldots, y_{n}=\varphi_{n}\left(x, y_{1}\right)$, where the $\varphi_{j}$ are $\mathscr{C}^{2, \alpha}$ functions satisfying $\varphi_{j}(0)=\partial_{x_{k}} \varphi_{j}(0)=\partial_{y_{1}} \varphi_{j}(0)=0$ for $j=2, \ldots, n, k=1, \ldots, n$ and we can represent $M^{1}$ by $n$ graphed equations $y_{1}=h_{1}(x), y_{2}=h_{2}(x), \ldots, y_{n}=h_{n}(x)$, where the $h_{j}$ are $\mathscr{C}^{2, \alpha}$ functions satisfying $h_{j}(0)=\partial_{x_{k}} h_{j}(0)=0$ for $j, k=1, \ldots, n$.
5.22. First order normalizations in Case ( $\mathbf{I}_{1}$ ). Thus, let us deal first with Case ( $\mathbf{I}_{1}$ ). After a possible permutation of coordinates, we can assume that $T_{p_{1}} H^{1}$, which is a one-codimensional subspace of $T_{p_{1}} M^{1}$, is given by the equations

$$
\begin{equation*}
x_{1}=b_{2} x_{2}+\cdots+b_{n} x_{n}, \quad y_{1}=0, \quad y^{\prime}=0, \tag{5.23}
\end{equation*}
$$

for some real numbers $b_{2}, \ldots, b_{n}$. If we define the linear invertible transformation $\widehat{z}_{1}:=z_{1}-b_{2} z_{2}-\cdots-b_{n} z_{n}, \widehat{z}^{\prime}:=z^{\prime}$, then the plane $T_{p_{1}} H^{1}$ written in (5.23) clearly transforms to the plane $\widehat{x}_{1}=\widehat{y}_{1}=\widehat{y}^{\prime}=0$, and (fortunately) $T_{p_{1}} M$ and $T_{p_{1}} M^{1}$ are left unchanged, namely $T_{p_{1}} \widehat{M}=\left\{\widehat{y}^{\prime}=0\right\}$ and $T_{p_{1}} \widehat{M}^{1}=\left\{\widehat{y}_{1}=\widehat{y}^{\prime}=0\right\}$.

Dropping the hats on coordinates, we have $T_{p_{1}} M=\left\{y^{\prime}=0\right\}, T_{p_{1}} M^{1}=$ $\left\{y_{1}=y^{\prime}=0\right\}, T_{p_{1}} H^{1}=\left\{x_{1}=y_{1}=y^{\prime}=0\right\}$. Let $\mathrm{C}_{1}^{\prime} \subset \subset \mathrm{C}_{p_{1}}^{\prime}$ be the open infinite strictly convex linear cone introduced in $\S 5.14$, which is contained in the real $(n-1)$-dimensional space $\left\{\left(0, x^{\prime}\right)\right\}$ and which is defined by $(n-1)$ strict inequalities $\ell_{1}^{\prime}\left(x^{\prime}\right)>0, \ldots, \ell_{n-1}^{\prime}\left(x^{\prime}\right)>0$. By means of a real linear invertible transformation of the form $\widehat{z}_{1}:=z_{1}, \widehat{z}^{\prime}:=A^{\prime} \cdot z^{\prime}$, where $A^{\prime}$ is an $(n-1) \times(n-1)$ real matrix, we can transform $\mathrm{C}_{1}^{\prime}$ to a cone $\widehat{\mathrm{C}}_{1}^{\prime}$ defined by the simpler inequalities $\widehat{x}_{2}>0, \ldots, \widehat{x}_{n}>0$. Fortunately, this transformation stabilizes $T_{p_{1}} M, T_{p_{1}} M^{1}$ and $T_{p_{1}} H^{1}$.

Dropping the hats on coordinates, we now have $T_{p_{1}} M=\left\{y^{\prime}=0\right\}$, $T_{p_{1}} M^{1}=\left\{y_{1}=y^{\prime}=0\right\}, T_{p_{1}} H^{1}=\left\{x_{1}=y_{1}=y^{\prime}=0\right\}$ and $\mathrm{C}_{1}^{\prime}=\left\{\left(0, x^{\prime}\right): x_{2}>0, \ldots, x_{n}>0\right\}$. Then the nonzero vector $v_{1} \in T_{p_{1}} H^{1}$
which belongs to $\mathrm{C}_{1}^{\prime}$ has coordinates $v_{1}=\left(0, v_{2 ; 1}, \ldots, v_{n ; 1}\right) \in \mathbb{R}^{n}$, where $v_{2 ; 1}>0, \ldots, v_{n ; 1}>0$. By means of real dilations or real contractions of the real axes $\mathbb{R}_{x_{2}}, \ldots, \mathbb{R}_{x_{n}}$ (a transformation which does not perturb the previously achieved normalizations), we can arrange that $v_{1}=(0,1, \ldots, 1)$ and that $T_{p_{1}}\left(M^{1}\right)^{+}=\left\{y^{\prime}=0, y_{1}>0\right\}, T_{p_{1}}\left(H^{1}\right)^{+}=\left\{y=0, x_{1}>0\right\}$.

Finally, the linear one-codimensional subspace $\Sigma^{1} \subset T_{p_{1}} M^{1}$ introduced in $\S 5.14$ which does not contain the characteristic direction $T_{p_{1}} M^{1} \cap T_{p_{1}}^{c} M \equiv$ $\mathbb{R}_{x_{1}}$ may be represented by an equation of the form $\sigma\left(x_{1}, x^{\prime}\right):=x_{1}+a_{2} x_{2}+$ $\cdots+a_{n} x_{n}=0$, for some real numbers $a_{2}, \ldots, a_{n}$. The vector $v_{1}$ belongs to the cone $\mathrm{C}_{1}$ defined by (5.15), hence $a_{2}+\cdots+a_{n}>0$. After a dilation of the $x_{1}$-axis, we can even assume that $a_{2}+\cdots+a_{n}=1$. We remind that by (5.18), the half-space $T_{p_{1}}\left(N^{1}\right)^{+}$is given by $y_{1}+a_{2} y_{2}+\cdots+a_{n} y_{n}>0$, hence there exists a $\mathscr{C}^{2, \alpha}$ function $\psi\left(x, y^{\prime}\right)$ with $\psi(0)=\partial_{x_{k}} \psi(0)=\partial_{y_{j}} \psi(0)=0$ for $k=1, \ldots, n$ and $j=2, \ldots, n$ such that $\left(N^{1}\right)^{+}$is represented by the inequation $y_{1}+a_{2} y_{2}+\cdots+a_{n} y_{n}>\psi\left(x, y^{\prime}\right)$. Consequently, in this coordinate system, we may represent concretely the local half-wedge $\mathscr{H}_{W_{1}^{+}}^{+} \subset \mathscr{H}_{W_{1}}^{+}$ constructed in §5.14 as

$$
\left\{\begin{align*}
\mathscr{H} \mathscr{W}_{1}^{+}= & \left\{\left(z_{1}, z^{\prime}\right) \in \mathbb{C}^{n}:\left|z_{1}\right|<\delta_{1},\left|z^{\prime}\right|<\delta_{1}\right.  \tag{5.24}\\
& y_{1}+a_{2} y_{2}+\cdots+a_{n} y_{n}-\psi\left(x, y^{\prime}\right)>0 \\
& \left.y_{2}-\varphi_{2}\left(x, y_{1}\right)>0, \ldots, y_{n}-\varphi_{n}\left(x, y_{1}\right)>0\right\}
\end{align*}\right.
$$

For the continuation of the proof of Proposition 5.12, it will also be convenient to proceed to further second order normalizations of the totally real submanifolds $M^{1}$ and $H^{1}$. These normalizations will all be tangent to the identity tranformation, hence they will leave the previously achieved normalizations unchanged.
5.25. Second order normalizations in Case ( $\mathbf{I}_{1}$ ). Let us then perform a second order Taylor development of the defining equations of $M^{1}$

$$
\begin{equation*}
y=h(x)=\sum_{k_{1}, k_{2}=1}^{n} a_{k_{1}, k_{2}} x_{k_{1}} x_{k_{2}}+o\left(|x|^{2}\right), \tag{5.26}
\end{equation*}
$$

where the $a_{k_{1}, k_{2}}=\frac{1}{2} \partial_{x_{k_{1}}} \partial_{x_{k_{2}}} h(0)$ are vectors of $\mathbb{R}^{n}$. If we define the quadratic invertible transformation

$$
\begin{equation*}
\widehat{z}:=z-i \sum_{k_{1}, k_{2}=1}^{n} a_{k_{1}, k_{2}} z_{k_{1}} z_{k_{2}}=\Phi(z) \tag{5.27}
\end{equation*}
$$

which is tangent to the identity mapping at the origin, then for $x+i y=$ $x+i h(x) \in M^{1}$, we have by replacing (5.26) in the imaginary part of $\widehat{z}$
given by (5.27)

$$
\begin{align*}
\widehat{y} & =y-\sum_{k_{1}, k_{2}=1}^{n} a_{k_{1}, k_{2}} x_{k_{1}} x_{k_{2}}+\sum_{k_{1}, k_{2}=1}^{n} a_{k_{1}, k_{2}} y_{k_{1}} y_{k_{2}}  \tag{5.28}\\
& =\mathrm{o}\left(|x|^{2}\right) \\
& =\mathrm{o}\left(\left|\operatorname{Re} \Phi^{-1}(\widehat{z})\right|^{2}\right)=\mathrm{o}\left(|(\widehat{x}, \widehat{y})|^{2}\right),
\end{align*}
$$

whence by applying the $\mathscr{C}^{2, \alpha}$ implicit function theorem to solve (5.28) in terms of $\widehat{y}$, we find that $\widehat{M^{1}}:=\Phi\left(M^{1}\right)$ may be represented by an equation of the form $\widehat{y}=\widehat{h}(\widehat{x})$, for some $\mathbb{R}^{n}$-valued local $\mathscr{C}^{2, \alpha}$ mapping $\widehat{h}$ which satisfies $\widehat{h}(\widehat{x})=\mathrm{o}\left(|\widehat{x}|^{2}\right)$.

Dropping the hats on coordinates, we can assume that the functions $h_{1}, \ldots, h_{n}$ vanish at the origin to second order. Since $T_{p_{1}} H^{1}=\{y=$ $\left.0, x_{1}=0\right\}$, there exists a $\mathscr{C}^{2, \alpha}$ function $g\left(x^{\prime}\right)$ with $g(0)=\partial_{x_{k}} g(0)=0$ for $k=2, \ldots, n$ such that $\left(H^{1}\right)^{+}$is given by the equation $x_{1}>g\left(x^{\prime}\right)$. We want to normalize also the defining equation $x_{1}=g\left(x^{\prime}\right)$ of $H^{1}$. Instead of requiring, similarly as for $h_{1}, \ldots, h_{n}$, that $g$ vanishes to second order at the origin (which would be possible), we shall normalize $g$ in order that $g\left(x^{\prime}\right)=-x_{1}^{2}-\cdots-x_{n}^{2}+\mathrm{o}\left(\left|x^{\prime}\right|^{2+\alpha}\right)$ (which will also be possible, thanks to the total reality of $H^{1}$ ).

The reason why we want $\left(H^{1}\right)^{+}=\left\{x_{1}>g\left(x^{\prime}\right)\right\}$ to be strictly concave is a trick to avoid having to construct discs half-attached to $M^{1}$ with prescribed second order jet, in order that their half-boundary does almost not touch the singularity $C$, which lies behing the wall $H^{1} \subset M^{1}$, namely $C \subset p_{1} \cup\left(H^{1}\right)^{-}$. In Section 8 below, we shall construct such discs whose half-boundary is almost tangent to $\left(H^{1}\right)^{-}$at $p_{1}$, and by arranging in advance strong geometric convexity of $\left(H^{1}\right)^{-}$, it will suffice that the half boundaries are tangent to $H^{1}$. in Figure 18, the half boundaries are the vertical lines slightly rotated and indeed, they do not enter much $\left(H^{1}\right)^{-}$.

Thus, we perform a second order Taylor development of the defining equations of $H^{1}$

$$
\left\{\begin{array}{l}
x_{1}=g\left(x^{\prime}\right)=\sum_{k_{1}, k_{2}=2}^{n} b_{k_{1}, k_{2}} x_{k_{1}} x_{k_{2}}+o\left(\left|x^{\prime}\right|^{2}\right),  \tag{5.29}\\
y=h\left(g\left(x^{\prime}\right), x^{\prime}\right)=: k\left(x^{\prime}\right)=o\left(\left|x^{\prime}\right|^{2}\right)
\end{array}\right.
$$

where the $b_{k_{1}, k_{2}}=\frac{1}{2} \partial_{x_{k_{1}}} \partial_{x_{k_{2}}} g(0)$ are real numbers. If we define the quadratic invertible transformation

$$
\left\{\begin{array}{l}
\widehat{z}_{1}:=z_{1}-\sum_{k_{1}, k_{2}=2}^{n} b_{k_{1}, k_{2}} z_{k_{1}} z_{k_{2}}-z_{2}^{2}-\cdots-z_{n}^{2}  \tag{5.30}\\
\widehat{z}^{\prime}:=z^{\prime}
\end{array}\right.
$$

which is tangent to the identity mapping, then for $\left(g\left(x^{\prime}\right)+i k_{1}\left(x^{\prime}\right), x^{\prime}+\right.$ $\left.i k^{\prime}\left(x^{\prime}\right)\right) \in H^{1}$, we have by replacing (5.29) in the real part of $\widehat{z}_{1}$, given by (5.30):
(5.31)

$$
\begin{aligned}
\widehat{x}_{1} & =x_{1}-\sum_{k_{1}, k_{2}=2}^{n} b_{k_{1}, k_{2}} x_{k_{1}} x_{k_{2}}+\sum_{k_{1}, k_{2}=2}^{n} b_{k_{1}, k_{2}} y_{k_{1}} y_{k_{2}}-\sum_{k=2}^{n} x_{k}^{2}+\sum_{k=2}^{n} y_{k}^{2}, \\
& =-x_{2}^{2}-\cdots-x_{n}^{2}+\mathrm{o}\left(\left|x^{\prime}\right|^{2}\right) \\
& =-\widehat{x}_{2}^{2}-\cdots-\widehat{x}_{n}^{2}+\mathrm{o}\left(|(\widehat{x}, \widehat{y})|^{2}\right) .
\end{aligned}
$$

Similarly (dropping the elementary computations), we may obtain for the imaginary part of $\widehat{z}_{1}$ and for the imaginary part of $\widehat{z}^{\prime}$

$$
\begin{equation*}
\widehat{y}_{1}=\mathrm{o}\left(|(\widehat{x}, \widehat{y})|^{2}\right) \quad \text { and } \quad \widehat{y}^{\prime}=\mathrm{o}\left(|(\widehat{x}, \widehat{y})|^{2}\right) \tag{5.32}
\end{equation*}
$$

whence by applying the $\mathscr{C}^{2, \alpha}$ implicit function theorem to solve the system (5.31), (5.32) in terms of $\widehat{x}_{1}, \widehat{y}_{1}$ and $\widehat{y}^{\prime}$, we find that $\widehat{H}^{1}:=\Phi\left(H^{1}\right)$ may be represented by equations of the form

$$
\left\{\begin{array}{rl}
\widehat{x}_{1} & =\widehat{g}\left(\widehat{x}^{\prime}\right)  \tag{5.33}\\
=-\widehat{x}_{2}^{2}-\cdots-\widehat{x}_{n}^{2}+\mathrm{o}\left(\left|\widehat{x}^{\prime}\right|^{2}\right), \\
\widehat{y} & =\widehat{k}\left(\widehat{x}^{\prime}\right)
\end{array}=o\left(\left|\widehat{x}^{\prime}\right|^{2}\right) . ~ \$\right.
$$

It remains to check that the above transformation has not perturbed the previous second order normalizations of $h_{1}, \ldots, h_{n}$ (this is important), which is easy: replacing $y$ by $h(x)=\mathrm{o}\left(|x|^{2}\right)$ in the imaginary parts of $\widehat{z}_{1}$ and of $\widehat{z}^{\prime}$ defined by the transformation (5.30), we get firstly

$$
\begin{align*}
\widehat{y}_{1} & =y_{1}-\sum_{k_{1}, k_{2}}^{n} b_{k_{1}, k_{2}}\left(x_{k_{1}} y_{k_{2}}+y_{k_{1}} x_{k_{2}}\right)-2 \sum_{k=2}^{n} x_{k} y_{k}  \tag{5.34}\\
& =\mathrm{o}\left(|x|^{2}\right) \\
& =\mathrm{o}\left(\left|\operatorname{Re} \Phi^{-1}(\widehat{z})\right|^{2}\right)=\mathrm{o}\left(|(\widehat{x}, \widehat{y})|^{2}\right)
\end{align*}
$$

and similarly

$$
\begin{equation*}
\widehat{y}^{\prime}=o\left(|(\widehat{x}, \widehat{y})|^{2}\right) \tag{5.35}
\end{equation*}
$$

whence by applying the $\mathscr{C}^{2, \alpha}$ implicit function theorem to solve the system (5.34), (5.35) in terms of $\widehat{y}$, we find that $\widehat{M}^{1}:=\Phi\left(M^{1}\right)$ may be represented by equations of the form $\widehat{y}=\widehat{h}(\widehat{x})=\mathrm{o}\left(|\widehat{x}|^{2}\right)$. Thus, after dropping the hats on coordinates, all the desired normalizations are satisfied. We now summarize these normalizations and we formulate just afterwards the analogous normalizations for Cases ( $\mathbf{I}_{\mathbf{2}}$ ) and (II).
5.36. Simultaneous normalizations. In the following lemma, the final choice of sufficiently small radii $\rho_{1}>0$ and $\delta_{1}>0$ is made after that all the biholomorphic changes of coordinates are performed.
Lemma 5.37. Let $M, M^{1}, p_{1}, H^{1}, v_{1},\left(H^{1}\right)^{+}, \mathscr{H} \mathscr{W}_{p_{1}}^{+}, \mathrm{C}_{p_{1}}$ and $\mathrm{FC}_{p_{1}}$ be as in Proposition 5.12. Then there exists a sub-half-wedge $\mathscr{H} \mathscr{W}_{1}^{+}$contained in $\mathscr{H}_{p_{1}}^{+}$such that the following normalizations hold in each of the three cases $\left(\mathbf{I}_{1}\right),\left(\mathbf{I}_{\mathbf{2}}\right)$ and (II):
( $\mathbf{I}_{\mathbf{1}}$ ) If $\operatorname{dim}_{\mathbb{R}}\left(T_{p_{1}} H^{1} \cap T_{p_{1}}^{c} M\right)=0$ (whence $v_{1} \notin T_{p_{1}}^{c} M$ ), then there exists a system of holomorphic coordinates $z=\left(z_{1}, \ldots, z_{n}\right)=$ $\left(x_{1}+i y_{1}, \ldots, x_{n}+i y_{n}\right)$ vanishing at $p_{1}$ with the vector $v_{1}$ equal to $(0,1, \ldots, 1)$, there exists positive numbers $\rho_{1}$ and $\delta_{1}$ with $0<$ $\delta_{1}<\rho_{1}$, there exist $\mathscr{C}^{2, \alpha}$ functions $\varphi_{2}, \ldots, \varphi_{n}, h_{1}, \ldots, h_{n}, g$, $k_{1}, \ldots, k_{n}, \psi$, all defined in real cubes of edge $2 \rho_{1}$ and of the appropriate dimension, and there exist real numbers $a_{1}, \ldots, a_{n}$ with $a_{2}+\cdots+a_{n}=1$, such that, if we denote $z^{\prime}:=\left(z_{2}, \ldots, z_{n}\right)=$ $x^{\prime}+i y^{\prime}$, then $M, M^{1},\left(M^{1}\right)^{+}, H^{1},\left(H^{1}\right)^{+}$and $N^{1}$ are represented in the polydisc of radius $\rho_{1}$ centered at $p_{1}$ by the following graphed (in)equations and the sub-half-wedge $\mathscr{H} \mathscr{W}_{1}^{+} \subset \mathscr{H} W_{p_{1}}^{+}$is represented in the polydisc of radius $\delta_{1}$ centered at $p_{1}$ by the following inequations

$$
\left\{\begin{align*}
M: & y_{2}=\varphi_{2}\left(x, y_{1}\right), \ldots \ldots, y_{n}=\varphi_{n}\left(x, y_{1}\right)  \tag{5.38}\\
M^{1}: & y_{1}=h_{1}(x), y_{2}=h_{2}(x), \ldots \ldots, y_{n}=h_{n}(x), \\
\left(M^{1}\right)^{+}: & y_{1}>h_{1}(x), y_{2}=\varphi_{2}\left(x, y_{1}\right), \ldots \ldots, y_{n}=\varphi_{n}\left(x, y_{1}\right), \\
H^{1}: & x_{1}=g\left(x^{\prime}\right), y_{1}=k_{1}\left(x^{\prime}\right), \ldots \ldots, y_{n}=k_{n}\left(x^{\prime}\right), \\
\left(H^{1}\right)^{+}: & x_{1}>g\left(x^{\prime}\right), y_{1}=h_{1}(x), y_{2}=h_{2}(x), \ldots \ldots, y_{n}=h_{n}(x), \\
N^{1}: & y_{1}+a_{2} y_{2}+\cdots+a_{n} y_{n}=\psi\left(x, y^{\prime}\right), \\
\mathscr{H} \mathscr{W}_{1}^{+}: & y_{1}+a_{2} y_{2}+\cdots+a_{n} y_{n}>\psi\left(x, y^{\prime}\right), \\
& y_{2}>\varphi_{2}\left(x, y_{1}\right), \ldots, y_{n}>\varphi_{n}\left(x, y_{1}\right),
\end{align*}\right.
$$

where we can assume that $M^{1}$ coincides with the intersection $M \cap$ $\left\{y_{1}=h_{1}(x)\right\}$, that $H^{1}$ coincides with the intersection $M^{1} \cap\left\{x_{1}=\right.$
$\left.g\left(x^{\prime}\right)\right\}$ and that $N^{1}$ contains $M^{1}$, which yields at the level of defining equations the following three collections of identities

$$
\left\{\begin{align*}
h_{2}(x) & \equiv \varphi_{2}\left(x, h_{1}(x)\right), \ldots \ldots, h_{n}(x) \equiv \varphi_{n}\left(x, h_{1}(x)\right)  \tag{5.39}\\
k_{1}\left(x^{\prime}\right) & \equiv h_{1}\left(g\left(x^{\prime}\right), x^{\prime}\right), \ldots \ldots, k_{n}\left(x^{\prime}\right) \equiv h_{n}\left(g\left(x^{\prime}\right), x^{\prime}\right), \\
\psi\left(x, h^{\prime}(x)\right) & \equiv h_{1}(x)+a_{2} h_{2}(x)+\cdots+a_{n} h_{n}(x),
\end{align*}\right.
$$

and where the following normalizations hold:

$$
\left\{\begin{array}{l}
\varphi_{j}(0)=\partial_{x_{k}} \varphi_{j}(0)=\partial_{y_{1}} \varphi_{j}(0)=0, \quad j=2, \ldots, n, k=1, \ldots, n,  \tag{5.40}\\
h_{j}(0)=\partial_{x_{k}} h_{j}(0)=\partial_{x_{k_{1}}} \partial_{x_{k_{2}}} h_{j}(0)=0, \quad j, k, k_{1}, k_{2}=1, \ldots, n, \\
g(0)=\partial_{x_{k}} g(0)=k_{j}(0)=\partial_{x_{k}} k_{j}(0)=0, \quad j=1, \ldots, n, k=2, \ldots, n, \\
\partial_{x_{k_{1}}} \partial_{x_{k_{2}}} g(0)=-\delta_{k_{1}}^{k_{2}}, \quad k_{1}, k_{2}=2, \ldots, n, \\
\psi(0)=\partial_{x_{k}} \psi(0)=\partial_{y_{j}} \psi(0)=0, \quad k=1, \ldots, n, j=2, \ldots, n
\end{array}\right.
$$

In other words, $T_{0} M=\left\{y^{\prime}=0\right\}$ (hence $T_{0}^{c} M$ coincides with the complex $z_{1}$-axis), $T_{0} N^{1}=\left\{y_{1}+a_{2} y_{2}+\cdots+a_{n} y_{n}=0\right\}$ and the second order Taylor approximations of the defining equations of $M^{1}$, of $H^{1}$ and of $\left(H^{1}\right)^{+}$are the quadrics

$$
\left\{\begin{align*}
T_{p_{1}}^{(2)} M^{1}: & y_{1}=0, \ldots \ldots, y_{n}=0  \tag{5.41}\\
T_{p_{1}}^{(2)} H^{1}: & x_{1}=-x_{2}^{2}-\cdots-x_{n}^{2}, y_{1}=0, \ldots \ldots, y_{n}=0 \\
T_{p_{1}}^{(2)}\left(H^{1}\right)^{+}: & x_{1}>-x_{2}^{2}-\cdots-x_{n}^{2}, y_{1}=0, \ldots \ldots, y_{n}=0 .
\end{align*}\right.
$$

( $\mathbf{I}_{2}$ ) Similarly, if $\operatorname{dim}_{\mathbb{R}}\left(T_{p_{1}} H^{1} \cap T_{p_{1}}^{c} M\right)=1$ and if $v_{1} \notin T_{p_{1}}^{c} M$ (this possibility can only occur in the case $n \geqslant 3$ ), then there exists a system of holomorphic coordinates $z=\left(z_{1}, \ldots, z_{n}\right)=\left(x_{1}+\right.$ $\left.i y_{1}, \ldots, x_{n}+i y_{n}\right)$ vanishing at $p_{1}$ with $v_{1}$ equal to $(1, \ldots, 1,0)$, there exists positive numbers $\rho_{1}$ and $\delta_{1}$ with $0<\delta_{1}<\rho_{1}$, there exist $\mathscr{C}^{2, \alpha}$-smoooth functions $\varphi_{2}, \ldots, \varphi_{n}, h_{1}, \ldots, h_{n}, g, k_{1}, \ldots, k_{n}, \psi$ all defined in real cubes of edge $2 \rho_{1}$ and of the appropriate dimension, such that if we denote $z^{\prime \prime}:=\left(z_{1}, \ldots, z_{n-1}\right)=x^{\prime \prime}+i y^{\prime \prime}$ and $z^{\prime}=\left(z_{2}, \ldots, z_{n}\right)=x^{\prime}+i y^{\prime}$, then $M, M^{1},\left(M^{1}\right)^{+}, H^{1},\left(H^{1}\right)^{+}$ and $N^{1}$ are represented in the polydisc of radius $\rho_{1}$ centered at $p_{1}$ by the following graphed (in)equations and the sub-half-wedge
$\mathscr{H} \mathscr{W}_{1}^{+} \subset \mathscr{H} \mathscr{W}_{p_{1}}^{+}$is represented in the polydisc of radius $\delta_{1}$ centered at $p_{1}$ by the following inequations

$$
\left\{\begin{align*}
M: & y_{2}=\varphi_{2}\left(x, y_{1}\right), \ldots \ldots, y_{n}=\varphi_{n}\left(x, y_{1}\right),  \tag{5.42}\\
M^{1}: & y_{1}=h_{1}(x), y_{2}=h_{2}(x), \ldots \ldots, y_{n}=h_{n}(x), \\
\left(M^{1}\right)^{+}: & y_{1}>h_{1}(x), y_{2}=\varphi_{2}\left(x, y_{1}\right), \ldots \ldots, y_{n}=\varphi_{n}\left(x, y_{1}\right), \\
H^{1}: & x_{n}=g\left(x^{\prime \prime}\right), y_{1}=k_{1}\left(x^{\prime \prime}\right), \ldots \ldots, y_{n}=k_{n}\left(x^{\prime \prime}\right), \\
\left(H^{1}\right)^{+}: & x_{n}>g\left(x^{\prime \prime}\right), y_{1}=h_{1}(x), y_{2}=h_{2}(x), \ldots \ldots, y_{n}=h_{n}(x), \\
N^{1}: & y_{2}+\cdots+y_{n-1}-y_{n}=\psi\left(x, y^{\prime}\right), \\
\mathscr{H} \mathscr{W}_{1}^{+}: & y_{2}+\cdots+y_{n-1}-y_{n}>\psi\left(x, y^{\prime}\right), \\
& y_{1}>\varphi_{1}\left(x, y_{1}\right), \ldots, y_{n-1}>\varphi_{n-1}\left(x, y_{1}\right),
\end{align*}\right.
$$

where we can assume that $M^{1}$ coincides with the intersection $M \cap$ $\left\{y_{1}=h_{1}(x)\right\}$, that $H^{1}$ coincides with the intersection $M^{1} \cap\left\{x_{1}=\right.$ $\left.g\left(x^{\prime}\right)\right\}$ and that $N^{1}$ contains $M^{1}$, which yields at the level of defining equations the following three collections of identities

$$
\left\{\begin{align*}
h_{2}(x) & \equiv \varphi_{2}\left(x, h_{1}(x)\right), \ldots \ldots, h_{n}(x) \equiv \varphi_{n}\left(x, h_{1}(x)\right),  \tag{5.43}\\
k_{1}\left(x^{\prime \prime}\right) & \equiv h_{1}\left(x^{\prime \prime}, g\left(x^{\prime \prime}\right)\right), \ldots \ldots, k_{n}\left(x^{\prime \prime}\right) \equiv h_{n}\left(x^{\prime \prime}, g\left(x^{\prime \prime}\right)\right), \\
\psi\left(x, h^{\prime}(x)\right) & \equiv h_{1}(x)+h_{2}(x)+\cdots+h_{n-1}(x)-h_{n}(x),
\end{align*}\right.
$$

and where the following normalizations hold:

$$
\left\{\begin{array}{l}
\varphi_{j}(0)=\partial_{x_{k}} \varphi_{j}(0)=\partial_{y_{1}} \varphi_{j}(0)=0, \quad j=2, \ldots, n, k=2, \ldots, n,  \tag{5.44}\\
h_{j}(0)=\partial_{x_{k}} h_{j}(0)=\partial_{x_{k_{1}}} \partial_{x_{k_{2}}} h_{j}(0)=0, \quad j, k, k_{1}, k_{2}=1, \ldots, n, \\
g(0)=\partial_{x_{k}} g(0)=k_{j}(0)=\partial_{x_{k}} k_{j}(0)=0, \quad j=1, \ldots, n, k=1, \ldots, n-1, \\
\quad \partial_{x_{k_{1}}} \partial_{x_{k_{2}}} g(0)=-\delta_{k_{1},}^{k_{2}}, \quad k_{1}, k_{2}=1, \ldots, n-1, \\
\psi(0)=\partial_{x_{k}} \psi(0)=\partial_{y_{j}} \psi(0)=0, \quad k=1, \ldots, n, j=2, \ldots, n .
\end{array}\right.
$$

In other words, $T_{0} M=\left\{y^{\prime}=0\right\}$ (hence $T_{0}^{c} M$ coincides with the complex $z_{1}$-axis), $T_{0} N^{1}=\left\{y_{1}+y_{2}+\cdots+y_{n-1}-y_{n}=0\right\}$ and the second order Taylor approximations of the defining equations of $M^{1}$, of $H^{1}$ and of $\left(H^{1}\right)^{+}$are the quadrics

$$
\left\{\begin{align*}
T_{p_{1}}^{(2)} M^{1}: & y_{1}=0, \ldots \ldots, y_{n}=0  \tag{5.45}\\
T_{p_{1}}^{(2)} H^{1}: & x_{n}=-x_{1}^{2}-\cdots-x_{n-1}^{2}, y_{1}=0, \ldots \ldots, y_{n}=0 \\
T_{p_{1}}^{(2)}\left(H^{1}\right)^{+}: & x_{n}>-x_{1}^{2}-\cdots-x_{n-1}^{2}, y_{1}=0, \ldots \ldots, y_{n}=0
\end{align*}\right.
$$

(II) Finally, if $\operatorname{dim}_{\mathbb{R}}\left(T_{p_{1}} H^{1} \cap T_{p_{1}}^{c} M\right)=1$ and if $v_{1} \in T_{p_{1}}^{c} M$ (this possibility can occur in all cases $n \geqslant 2$ ), then there exists a system of
holomorphic coordinates $z=\left(z_{1}, \ldots, z_{n}\right)=\left(x_{1}+i y_{1}, \ldots, x_{n}+i y_{n}\right)$ vanishing at $p_{1}$ with $v_{1}$ equal to $(1,0, \ldots, 0)$, there exist positive numbers $\rho_{1}$ and $\delta_{1}$ with $0<\delta_{1}<\rho_{1}$, there exist $\mathscr{C}^{2, \alpha}$-smoooth functions $\varphi_{2}, \ldots, \varphi_{n}, h_{1}, \ldots, h_{n}, g, k_{1}, \ldots, k_{n}$ all defined in real cubes of edge $2 \rho_{1}$ and of the appropriate dimension, such that if we denote $z^{\prime \prime}:=\left(z_{1}, \ldots, z_{n-1}\right)=x^{\prime \prime}+i y^{\prime \prime}$ and $z^{\prime}=\left(z_{2}, \ldots, z_{n}\right)=x^{\prime}+i y^{\prime}$, then $M, M^{1},\left(M^{1}\right)^{+}, H^{1}$ and $\left(H^{1}\right)^{+}$are represented in the polydisc of radius $\rho_{1}$ centered at $p_{1}$ by the first five (in)equations of (5.42) together with the normalizations (5.45) and such that the local wedge $\mathscr{W}_{2} \subset \mathscr{H}^{W_{p}}+{ }_{1}$ of edge $M^{1}$ at $p_{1}$ is represented in the polydisc of radius $\delta_{1}$ centered at $p_{1}$ by the following inequations

$$
\left\{\begin{align*}
\mathscr{W}_{2}: & y_{1}-h_{1}(x)>-\left[y_{2}-h_{2}(x)\right], \ldots \ldots, y_{1}-h_{1}(x)>-\left[y_{n}-h_{n}(x)\right],  \tag{5.46}\\
& y_{1}-h_{1}(x)>y_{2}-h_{2}(x)+\cdots+y_{n}-h_{n}(x) .
\end{align*}\right.
$$

5.47. Summarizing figure and proof of Lemma 5.37. To illustrate this technical lemma, by specifying the value $n=3$, we draw the cones $\mathrm{C}_{1}$ and $\mathrm{C}_{2}$ together with the vector $v_{1}$, the tangent plane $T_{p_{1}} H^{1}$ and the hyperplane $\Sigma^{1}$ in the three cases $\left(\mathbf{I}_{1}\right),\left(\mathbf{I}_{2}\right)$ and (II). In the left part of this figure, the cone $\mathrm{C}_{1}$ is given by $x_{2}>0, x_{3}>0, x_{1}>-\frac{1}{2} x_{2}-\frac{1}{2} x_{3}$, namely we choose the values $a_{2}=a_{3}=\frac{1}{2}$ for the drawing; in the central part, the cone $\mathrm{C}_{1}$ is given by $x_{1}>0, x_{2}>0, x_{2}>x_{3}$; in the right part, the cone $\mathrm{C}_{2}$ is given by $x_{1}>-x_{2}, x_{1}>-x_{3}, x_{1}>x_{2}+x_{3}$.


Fig. 15: The subcone $C_{1}$ in the three cases $\left(I_{1}\right),\left(I_{2}\right)$ and (II)
Proof. Case ( $\mathbf{I}_{\mathbf{1}}$ ) has been completed before the statement of Lemma 5.37.
For Case ( $\mathbf{I}_{2}$ ), we reason similarly, as follows. We start with the normalizations $T_{p_{1}} M=\left\{y^{\prime}=0\right\}$ and $T_{p_{1}} M^{1}=\{y=0\}$ as in the end of §5.21. By assumption, $T_{p_{1}} H^{1}$ contains the characteristic direction, which coincides with the $x_{1}$-axis. By means of an elementary real linear transformation of the form $\widehat{z}_{1}:=z_{1}, \widehat{z}^{\prime}=A^{\prime} \cdot z^{\prime}$, we may first normalize $T_{p_{1}} H^{1}$ to be the hyperplane (after dropping the hats on coordinates) $\left\{x_{n}=0, y=0\right\}$. Similarly, we may normalize $v_{1}$ to be the vector $(1,1, \ldots, 1,0)$. Let again
$\pi^{\prime}:\left(x_{1}, x^{\prime}\right) \mapsto x^{\prime}$ denote the canonical projection on the $x^{\prime}$-space. Then $\pi^{\prime}\left(v_{1}\right)=(1, \ldots, 1,0)$. Using again a real linear transformation of the form $\widehat{z}_{1}:=z_{1}, \widehat{z}^{\prime}=A^{\prime} \cdot z^{\prime}$, we can assume that the proper subcone $\mathrm{C}_{1}^{\prime} \subset \subset \mathrm{C}_{p_{1}}^{\prime} \equiv \pi^{\prime}\left(\mathrm{C}_{p_{1}}\right)$ which contains the vector $v_{1}$ is given (after dropping the hats on coordinates) by

$$
\begin{equation*}
\mathrm{C}_{1}^{\prime}: \quad x_{2}>0, \ldots, x_{n-1}>0, x_{2}+\cdots+x_{n-1}>x_{n} . \tag{5.48}
\end{equation*}
$$

Following $\S 5.14$ (cf. Figure 14), we choose a linear cone $\mathrm{C}_{1} \subset \subset \mathrm{FC}_{p_{1}}$ defined by the $(n-1)$ inequations of $C_{1}^{\prime}$ plus one inequation of the form $x_{1}>a_{2} x_{2}+\cdots+a_{n} x_{n}$ with $1>a_{2}+\cdots+a_{n-1}$, since $v_{1}$ belongs to $\mathrm{C}_{1}$. Then by means of a real linear transformation of the form $\widehat{z}_{1}:=z_{1}+$ $a_{2} z_{2}+\cdots+a_{n} z_{n}, \widehat{z}^{\prime}:=z^{\prime}$, which stabilizes $\pi^{\prime}\left(v_{1}\right)$ and the inequations (5.48) of $C_{1}^{\prime}$, we can assume that the supplementary inequation for $C_{1}$, namely the inequation for $\left(\Sigma^{1}\right)^{+}$, is simply (after dropping the hats on coordinates) $x_{1}>$ 0 . Then the vector $v_{1}$ is mapped to the vector of coordinates $\left(1-a_{2}-\cdots-\right.$ $\left.a_{n}, 1, \ldots, 1,0\right)$, which we map to the vector of coordinates $(1,1, \ldots, 1,0)$ by an obvious positive scaling of the $x_{1}$-axis. In conclusion, in the final system of coordinates, the cone $\mathrm{C}_{1}$ is given by

$$
\begin{equation*}
\mathrm{C}_{1}: \quad x_{1}>0, x_{2}>0, \ldots, x_{n-1}>0, x_{2}+\cdots+x_{n-1}-x_{n}>0 \tag{5.49}
\end{equation*}
$$

This implies that the half-wedge $\mathscr{H} \mathscr{W}_{1}^{+} \subset \mathscr{H} \mathscr{W}_{p_{1}}^{+}$may be represented by the inequations of the last two line of (5.42). To conclude the proof of Case ( $\mathbf{I}_{2}$ ) of Lemma 5.37, it suffices to observe that, as in Case ( $\mathbf{I}_{1}$ ), the further second order normalizations do not perturb the previously achieved first order normalizations, because the transformations are tangent to the identity mapping at the origin.

Finally, we treat Case (II) of Lemma 5.37, starting with the system of coordinates $\left(z_{1}, \ldots, z_{n}\right)$ of the end of $\S 5.21$. After an elementary real linear transformation stabilizing the characteristic $x_{1}$-axis, we can assume that $v_{1}=(1,0, \ldots, 0)$ and that the convex infinite linear cone $\mathrm{C}_{2}$ introduced in $\S 5.19$ which contains $v_{1}$ is given by the inequations

$$
\begin{equation*}
x_{1}>-x_{2}, \ldots \ldots, x_{1}>-x_{n}, x_{1}>x_{2}+\cdots+x_{n} . \tag{5.50}
\end{equation*}
$$

This implies that the local wedge $\mathscr{W}_{2} \subset \mathscr{H} \mathscr{W}_{p_{1}}^{+}$of edge $M^{1}$ at $p_{1}$ introduced in $\S 5.19$ may be represented by the inequations (5.46). Finally, the second order normalizations, which are tangent to the identity mapping, are achieved as in the two previous cases $\left(\mathbf{I}_{1}\right)$ and $\left(\mathbf{I}_{2}\right)$.

The proof of Lemma 5.37 is complete.

## §6. Three preparatory lemmas in Hölder spaces

We first collect a few very elementary lemmas that will be useful in our geometric construction of half-attached analytic discs (Section 7). The index notation $g_{x_{k}}$ denotes partial derivative.
6.1. Local growth of $\mathscr{C}^{2, \alpha}$ mappings. Let $n \in \mathbb{N}$ with $n \geqslant 1$ and let $x=$ $\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{n}$. We consider the norm $|x|:=\max _{1 \leqslant k \leqslant n}\left|x_{k}\right|$. If $g=$ $g(x)$ is an $\mathbb{R}^{n}$-valued $\mathscr{C}^{1}$ map defined in the real cube $\left\{x \in \mathbb{R}^{n}:|x|<\right.$ $\left.2 \rho_{1}\right\}$, for some $\rho_{1}>0$, and if $\left|x^{\prime}\right|,\left|x^{\prime \prime}\right| \leqslant \rho$, for some $\rho<2 \rho_{1}$, then for $j=1, \ldots, n$, we have the mean value inequality

$$
\begin{equation*}
\left|g_{j}\left(x^{\prime}\right)-g_{j}\left(x^{\prime \prime}\right)\right| \leqslant\left|x^{\prime}-x^{\prime \prime}\right| \cdot\left(\sum_{k=1}^{n} \sup _{|x| \leqslant \rho}\left|g_{j, x_{k}}(x)\right|\right) . \tag{6.2}
\end{equation*}
$$

By the definition of the norm $|\cdot|$, we deduce $\left|g\left(x^{\prime}\right)-g\left(x^{\prime \prime}\right)\right| \leqslant\|g\|_{\mathscr{G} 1} \cdot\left|x^{\prime}-x^{\prime \prime}\right|$.
Let $\alpha$ with $0<\alpha<1$ and let $h=h(x)=\left(h_{1}(x), \ldots, h_{n}(x)\right)$ be an $\mathbb{R}^{n}$-valued $\mathscr{C}^{2, \alpha}$ map defined in $\left\{x \in \mathbb{R}^{n}:|x|<2 \rho_{1}\right\}$. For every $\rho<2 \rho_{1}$, we define:

$$
\begin{align*}
\|h\|_{\mathscr{C}^{2}, \alpha}^{2}(\{|x| \leqslant \rho\}) & :=\sup _{|x| \leqslant \rho}|h(x)|+\sum_{k=1}^{n} \sup _{|x| \leqslant \rho}\left|h_{x_{k}}(x)\right|+\sum_{k_{1}, k_{2}=1}^{n}\left|h_{x_{k_{1}} x_{k_{2}}}(x)\right|+ \\
& +\sum_{k_{1}, k_{2}=1}^{n}\left|x^{\prime}\right|,\left|x^{\prime \prime}\right| \leqslant \rho, x^{\prime} \neq x^{\prime \prime} \tag{6.3}
\end{align*} \frac{\left|h_{x_{k_{1}} x_{k_{2}}}\left(x^{\prime}\right)-h_{x_{k_{1}} x_{k_{2}}}\left(x^{\prime \prime}\right)\right|}{\left|x^{\prime}-x^{\prime \prime}\right|^{\alpha}}<\infty . ~ \$
$$

Lemma 6.4. Under the above assumptions, let

$$
\begin{equation*}
K_{1}:=\|h\|_{\mathscr{C}^{2}, \alpha}\left(\left\{|x| \leqslant \rho_{1}\right\}\right)<\infty \tag{6.5}
\end{equation*}
$$

be the $\mathscr{C}^{2, \alpha}$ norm of $h$ over $\left\{|x| \leqslant \rho_{1}\right\}$ and assume that $h_{j}(0)=0$, $h_{j, x_{k}}(0)=0$ and $h_{j, x_{k_{1}} x_{k_{2}}}(0)=0$, for all $j, k, k_{1}, k_{2}=1, \ldots, n$. Then for $|x| \leqslant \rho_{1}$ we have:

$$
\begin{cases}{[\mathbf{1}]:} & |h(x)| \leqslant|x|^{2+\alpha} \cdot K_{1},  \tag{6.6}\\ {[\mathbf{2}]:} & \sum_{k=1}^{n}\left|h_{x_{k}}(x)\right| \leqslant|x|^{1+\alpha} \cdot K_{1}, \\ {[\mathbf{3}]:} & \sum_{k_{1}, k_{2}=1}^{n}\left|h_{x_{k_{1}} x_{k_{2}}}(x)\right| \leqslant|x|^{\alpha} \cdot K_{1} .\end{cases}
$$

6.7. A $\mathscr{C}^{1, \alpha}$ estimate for composition of mappings. Recall that $\Delta$ is the open unit disc in $\mathbb{C}$ and that $\partial \Delta$ is its boundary, namely the unit circle. We shall constantly denote the complex variable in $\bar{\Delta}:=\Delta \cup \partial \Delta$ by $\zeta=\rho e^{i \theta}$, where $0 \leqslant \rho \leqslant 1$ and where $|\theta| \leqslant \pi$, except when we consider two points $\zeta^{\prime}=e^{i \theta^{\prime}}, \zeta^{\prime \prime}=e^{i \theta^{\prime \prime}}$, in which case we may obviously choose $\left|\theta^{\prime}\right|,\left|\theta^{\prime \prime}\right| \leqslant 2 \pi$ with $0 \leqslant\left|\theta^{\prime}-\theta^{\prime \prime}\right| \leqslant \pi$. Let now $X(\zeta)=\left(X_{1}(\zeta), \ldots, X_{n}(\zeta)\right)$ be an $\mathbb{R}^{n}$ valued mapping which is $\mathscr{C}^{1, \alpha}$ on $\partial \Delta$. We define its $\mathscr{C}^{1, \alpha}$-norm precisely
by
$\|X\|_{\mathscr{C}^{1, \alpha}(\partial \Delta)}:=\sup _{|\theta| \leqslant \pi}\left|X\left(e^{i \theta}\right)\right|+\sup _{|\theta| \leqslant \pi}\left|\frac{d X\left(e^{i \theta}\right)}{d \theta}\right|+\sup _{0<\left|\theta^{\prime}-\theta^{\prime \prime}\right| \leqslant \pi} \frac{\left|\frac{d X\left(e^{\left.i \theta^{\prime}\right)}\right)}{d \theta}-\frac{d X\left(e^{i \theta^{\prime \prime}}\right)}{d \theta}\right|}{\left|\theta^{\prime}-\theta^{\prime \prime}\right|^{\alpha}}$,
and its $\mathscr{C}^{1}$-norm $\|X\|_{\mathscr{C}^{1}(\partial \Delta)}$ by keeping only the first two terms.
Lemma 6.9. If $h$ is as in Lemma 6.5, and if moreover $\left|X\left(e^{i \theta}\right)\right| \leqslant \rho$ for all $\theta$ with $|\theta| \leqslant \pi$, withe $\rho \leqslant \rho_{1}$, then we have the following composition norm estimates:

$$
\begin{aligned}
\|h(X)\|_{\mathscr{C}^{1, \alpha}(\partial \Delta)} \leqslant & \sup _{|x| \leqslant \rho}|h(x)|+\left(\sum_{k=1}^{n} \sup _{|x| \leqslant \rho}\left|h_{x_{k}}(x)\right|\right) \cdot\|X\|_{\mathscr{C}^{1}(\partial \Delta)}+ \\
& +\left(\sum_{k_{1}, k_{2}=1}^{n} \sup _{|x| \leqslant \rho}\left|h_{x_{k_{1} x_{k_{2}}}}(x)\right|\right) \cdot \pi^{1-\alpha} \cdot\left[\|X\|_{\mathscr{C}^{1}(\partial \Delta)}\right]^{2}+ \\
& +\left(\sum_{k=1}^{n} \sup _{|x| \leqslant \rho}\left|h_{x_{k}}(x)\right|\right) \cdot\|X\|_{\mathscr{C}^{1, \alpha}(\partial \Delta)}
\end{aligned}
$$

(6.10)

$$
\begin{aligned}
& \sum_{k=1}^{n}\left\|h_{x_{k}}(X)\right\|_{\mathscr{C}^{\alpha}(\partial \Delta)} \leqslant \sum_{k=1} \sup _{|x| \leqslant \rho}\left|h_{x_{k}}(x)\right|+ \\
&+\left(\sum_{k_{1}, k_{2}=1}^{n} \sup _{|x| \leqslant \rho}\left|h_{x_{k_{1}} x_{k_{2}}}(x)\right|\right) \cdot \pi^{1-\alpha} \cdot\|X\|_{\mathscr{C}^{1}(\partial \Delta)} \\
& \sum_{k_{1}, k_{2}=1}^{n} \| h_{x_{k_{1} x_{k_{2}}}(X) \|_{\mathscr{C}^{\alpha}(\partial \Delta)} \leqslant} \sum_{k_{k_{1}, k_{2}=1}^{n}} \sup _{|x| \leqslant \rho}\left|h_{x_{k_{1}} x_{k_{2}}}(x)\right|+ \\
& \quad+\|h\|_{\mathscr{C}^{2}, \alpha}(\{|x| \leqslant \rho\}) \cdot\left(\|X\|_{\mathscr{C}^{1}(\partial \Delta)}\right)^{\alpha}
\end{aligned}
$$

Proof. We summarize the computations. Applying the definition (6.8), using the chain rule for the calculation of $d h\left(X\left(e^{i \theta}\right)\right) / d \theta$, and using the trivial inequality $\left|a^{\prime} b^{\prime}-a^{\prime \prime} b^{\prime \prime}\right| \leqslant\left|a^{\prime}\right| \cdot\left|b^{\prime}-b^{\prime \prime}\right|+\left|b^{\prime \prime}\right| \cdot\left|a^{\prime}-a^{\prime \prime}\right|$, we may majorize (6.11)

$$
\begin{aligned}
\|h(X)\|_{\mathscr{C} 1, \alpha}(\partial \Delta) \leqslant & \sup _{|\theta| \leqslant \pi}\left|h\left(X\left(e^{i \theta}\right)\right)\right|+\left(\sum_{k=1}^{n} \sup _{|\theta| \leqslant \pi}\left|h_{x_{k}}\left(X\left(e^{i \theta}\right)\right)\right|\right) \cdot \max _{1 \leqslant k \leqslant n} \sup _{|\theta| \leqslant \pi}\left|\frac{d X_{k}\left(e^{i \theta}\right)}{d \theta}\right|+ \\
& \sup _{0<\left|\theta^{\prime}-\theta^{\prime \prime}\right| \leqslant \pi} \sum_{k=1}^{n} \frac{\left|h_{x_{k}}\left(X\left(e^{i \theta^{\prime}}\right)\right)-h_{x_{k}}\left(X\left(e^{i \theta^{\prime \prime}}\right)\right)\right|}{\left|\theta^{\prime}-\theta^{\prime \prime}\right| \alpha} \cdot \max _{1 \leqslant k \leqslant n} \sup _{\left|\theta^{\prime}\right| \leqslant \pi}\left|\frac{d X_{k}\left(e^{i \theta^{\prime}}\right)}{d \theta}\right|+ \\
& \left(\sum_{k=1}^{n} \sup _{\left|\theta^{\prime \prime}\right| \leqslant \pi}\left|h_{x_{k}}\left(e^{i \theta^{\prime \prime}}\right)\right|\right) \cdot\left(\max _{1 \leqslant k \leqslant n} \sup _{0<\left|\theta^{\prime}-\theta^{\prime \prime}\right| \leqslant \pi} \frac{\left|\frac{d X_{k}\left(e^{i \theta^{\prime}}\right)}{d \theta}-\frac{d X_{k}\left(e^{i \theta^{\prime \prime}}\right)}{d \theta}\right|}{\left|\theta^{\prime}-\theta^{\prime \prime}\right|^{\alpha}}\right),
\end{aligned}
$$

which yields the first inequality of (6.10) after using (6.2) for the second line of (6.11) and the trivial majoration $\left|\theta^{\prime}-\theta^{\prime \prime}\right|^{1-\alpha} \leqslant \pi^{1-\alpha}$. The second and the third inequalities of (6.10) are established similarly, which completes the proof.

Lemma 6.12. With $h$ as in Lemma 6.4, suppose that there exist constants $c_{1}>0, K_{2}>0$ with $c_{1} K_{2} \leqslant \rho_{1}$ such that for each $c \in \mathbb{R}$ with $0 \leqslant c \leqslant c_{1}$, there exists $X_{c} \in \mathscr{C}^{1, \alpha}\left(\partial \Delta, \mathbb{R}^{n}\right)$ with $\left\|X_{c}\right\|_{\mathscr{C}^{1, \alpha}(\partial \Delta)} \leqslant c \cdot K_{2}$. Then there exists a constant $K_{3}>0$ such that the following three estimates hold:

$$
\left\{\begin{align*}
\left\|h\left(X_{c}\right)\right\|_{\mathscr{C} 1, \alpha}(\partial \Delta) & \leqslant c^{2+\alpha} \cdot K_{3}  \tag{6.13}\\
\sum_{k=1}^{n}\left\|h_{x_{k}}\left(X_{c}\right)\right\|_{\mathscr{C}_{\alpha}(\partial \Delta)} & \leqslant c^{1+\alpha} \cdot K_{3} \\
\sum_{k_{1}, k_{2}=1}^{n}\left\|h_{x_{k_{1}} x_{k_{2}}}\left(X_{c}\right)\right\|_{\mathscr{C}^{\alpha}(\partial \Delta)} & \leqslant c^{\alpha} \cdot K_{3} .
\end{align*}\right.
$$

Proof. Applying Lemmas 6.4 and 6.9, we see that it suffices to choose

$$
\begin{equation*}
K_{3}:=\max \left(K_{1} K_{2}^{2+\alpha}\left(3+\pi^{1-\alpha}\right), K_{1} K_{2}^{1+\alpha}\left(1+\pi^{1-\alpha}\right), 2 K_{1} K_{2}^{\alpha}\right) \tag{6.14}
\end{equation*}
$$

which completes the proof.
Up to now, we have introduced three positive constants $K_{1}, K_{2}, K_{3}$. In Sections 7, 8 and 9 below, we shall introduce further positive constants $K_{4}$, $K_{5}, K_{6}, K_{7}, K_{8}, K_{9}, K_{10}, K_{11}, K_{12}, K_{13}, K_{14}, K_{15}, K_{16}, K_{17}, K_{18}$ and $K_{19}$, whose precise value will not be important.

## §7. FAMILIES OF ANALYTIC DISCS HALF-ATTACHED TO MAXIMALLY REAL SUBMANIFOLDS

7.1. Preliminary. If $\partial^{+} \Delta:=\{\zeta \in \partial \Delta: \operatorname{Re} \zeta \geqslant 0\}$ denotes the positive half-boundary of $\Delta$, we say that an analytic disc $A \in \mathscr{O}\left(\Delta, \mathbb{C}^{n}\right) \cap \mathscr{C}^{0}(\bar{\Delta})$ is half-attached to a set $E \subset \mathbb{C}^{n}$ if $A\left(\partial^{+} \Delta\right) \subset E$.

We will construct local families of analytic discs $Z_{c, x, v}^{1}(\zeta): \bar{\Delta} \rightarrow \mathbb{C}^{n}$, where $c \in \mathbb{R}^{+}$is small, where $x \in \mathbb{R}^{n}$ is small and where $v \in \mathbb{R}^{n}$ is small, which are half-attached to a $\mathscr{C}^{2, \alpha}$ maximally real submanifold $M^{1}$ of $\mathbb{C}^{n}$, which satisfy $Z_{c, 0, v}^{1}(1) \equiv p_{1} \in M^{1}$, such that the boundary point $Z_{c, x, v}^{1}(1)$ covers a neighborhood of $p_{1}$ in $M^{1}$ when $x$ varies ( $c$ and $v$ being fixed) and such that the tangent vector $\frac{\partial Z_{c, 0, v}^{1}}{\partial \theta}(1)$ at the fixed point $p_{1}$ covers a cone in $T_{p_{1}} M^{1}$ when $v$ varies. With this choice, when $x$ varies, $v$ varies and $\zeta$ varies (but $c$ is fixed), the set of points $Z_{c, x, v}^{1}(\zeta)$, covers a thin wedge of edge $M^{1}$ at $p_{1}$. By maximal reality of $M^{1}$, the tangent vector $\frac{\partial Z_{c, 0, v}^{1}}{\partial \theta}(1) \in T_{p_{1}} M^{1}$ will be arbitrary, hence the associated wedge can have arbitrary orientation.

To summarize symbolically the structure of the desired family:

$$
Z_{c, x, v}^{1}(\zeta):\left\{\begin{array}{l}
c=\text { small scaling factor }  \tag{7.2}\\
x=\text { translation parameter } \\
v=\text { rotation parameter } \\
\zeta=\text { unit disc variable }
\end{array}\right.
$$

We begin our constructions in the "flat" case where the maximally real submanifold $M^{1}$ coincides with $\mathbb{R}^{n}$. Afterwards, we perform a pertubation argument, using the scaling parameter $c$ in an essential way.
7.3. A family of analytic discs sweeping $\mathbb{R}^{n} \subset \mathbb{C}^{n}$ with prescribed first order jets. We denote the coordinates over $\mathbb{C}^{n}$ by $z=x+i y=\left(x_{1}+\right.$ $\left.i y_{1}, \ldots, x_{n}+i y_{n}\right)$. Let $c \in \mathbb{R}$ with $c \geqslant 0$ be a "scaling factor", let $n \geqslant 2$, let $x=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{n}$, let $v=\left(v_{1}, \ldots, v_{n}\right) \in \mathbb{R}^{n}$ and consider the algebraically parametrized family of analytic discs defined by

$$
\begin{equation*}
B_{c, x, v}(s+i t):=\left(x_{1}+c v_{1}(s+i t), \ldots, x_{n}+c v_{n}(s+i t)\right) \tag{7.4}
\end{equation*}
$$

where $s+i t \in \mathbb{C}$ is the holomorphic variable. For $c \neq 0$, the map $B_{c, x, v}$ embeds the complex line $\mathbb{C}$ into $\mathbb{C}^{n}$ and sends $\mathbb{R}$ into $\mathbb{R}^{n}$ with arbitrary first order jet at 0 : center point $B_{c, x, v}(0)=x$ and tangent direction $\partial B_{c, x, v}(s) /\left.\partial s\right|_{s=0}=c v$.

To localize our family of analytic discs, we restrict the map (7.4) to the following specific set of values: $0 \leqslant c \leqslant c_{0}$ for some $c_{0}>0 ;|x| \leqslant c$; $|v| \leqslant 2$; and $|s+i t| \leqslant 4$. To localize $\mathbb{R}^{n}$, we shall denote $M^{0}:=\left\{x \in \mathbb{R}^{n}\right.$ : $\left.|x| \leqslant \rho_{0}\right\}$, where $\rho_{0}>0$, and we notice that $B_{c, x, v}(\{|s+i t| \leqslant 4\}) \subset M^{0}$ for all $c$, all $x$ and all $v$ provided that $c_{0} \leqslant \rho_{0} / 9$.

We then consider the mapping $(s+i t) \longmapsto B_{c, x, v}(s+i t)$ as a local (nonsmooth) analytic disc defined in the rectangle $\{s+i t \in \mathbb{C}:|s| \leqslant 4,0 \leqslant$ $t \leqslant 4\}$ whose bottom boundary part $B_{c, x, v}([-4,4])$ is a small real segment contained in $\mathbb{R}^{n}$.
7.5. A useful conformal equivalence. To get rid of the corners of the rectangle, we proceed as follows. In the complex plane equipped with coordinates $s+i t$, let $\mathscr{D}(i \sqrt{3}, 2)$ be the open disc of center $i \sqrt{3}$ and of radius 2 . Let $\mu:(-2,2) \rightarrow[0,1]$ be an even $\mathscr{C}^{\infty}$ function satisfying $\mu(s)=0$ for $0 \leqslant s \leqslant 1 ; \mu(s)>0$ and $d \mu(s) / d s>0$ for $1<s<2$; and $\mu(s)=\sqrt{3}-\sqrt{4-s^{2}}$ for $\sqrt{3} \leqslant s<2$. The simply connected domain $C^{+} \subset\{t>0\}$ which is represented in Figure 16 may be formally defined as

$$
\left\{\begin{array}{c}
C^{+} \cap\{t \geqslant \sqrt{3}-1\}:=\mathscr{D}(i \sqrt{3}, 2) \cap\{t \geqslant \sqrt{3}-1\},  \tag{7.6}\\
C^{+} \cap\{0<t<\sqrt{3}-1\}:=\{s+i t \in \mathbb{C}: t>\mu(s)\} .
\end{array}\right.
$$



Let $\Psi: \Delta \rightarrow C^{+}$be a conformal equivalence (Riemann's theorem). Since the boundary $\partial C^{+}$is $\mathscr{C}^{\infty}$, the mapping $\Psi$ extends as a $\mathscr{C}^{\infty}$ diffeomorphism $\partial \Delta \rightarrow \partial C^{+}$. After a reparametrization of $\Delta$, we can (and we shall) assume that $\Psi\left(\partial^{+} \Delta\right)=[-1,1], \Psi(1)=0$ and $\Psi( \pm i)= \pm 1$. It follows that $d \Psi\left(e^{i \theta}\right) / d \theta$ is a positive real number for all $e^{i \theta} \in \partial^{+} \Delta$.
7.7. Flat families of half-attached analytic discs. Thanks to $\Psi$, we can define a family of small analytic discs which are half-attached to the flat maximally real manifold $M^{0} \equiv\left\{x \in \mathbb{R}^{n}:|x| \leqslant \rho_{0}\right\}$ as follows

$$
\begin{equation*}
Z_{c, x, v}^{0}(\zeta):=B_{c, x, v}(\Psi(\zeta))=(x+c v \Psi(\zeta)) \tag{7.8}
\end{equation*}
$$

We then have $Z_{c, x, v}^{0}\left(\partial^{+} \Delta\right) \subset M^{0}$ and $Z_{c, x, v}^{0}(1)=x$. Notice that every disc $Z_{c, x, v}(\bar{\Delta})$ is contained in a single complex line. Starting with a maximally real submanifold of $\mathbb{C}^{n}$ as in Proposition 5.12, but dealing with the flat maximally real submanifold $M^{0} \equiv \mathbb{R}^{n}$, we first construct a flat model of the desired family of analytic disc.

Lemma 7.9. Let $p_{0} \equiv 0 \in M^{0}$ denote the origin and let $v_{0} \in T_{p_{0}} M^{0}$ be a tangent vector with $\left|v^{0}\right|=1$. Then there exists a constant $\Lambda_{0}>0$ and there exists a $\mathscr{C}^{\infty}$ family $A_{c, x, v}^{0}(\zeta)$ of analytic discs defined for $c \in \mathbb{R}$ with $0 \leqslant c \leqslant c_{0}$ for some $c_{0}>0$ with $c_{0} \leqslant \rho_{0} / 9$, for $x \in \mathbb{R}^{n}$ with $|x| \leqslant c$ and for $v \in \mathbb{R}^{n}$ with $|v| \leqslant c$, which enjoys the following six properties:
$\left(1_{0}\right) A_{c, 0, v}^{0}(1)=p_{0}=0$ for all $c$ and all $v$.
$\left(2_{0}\right) A_{c, x, v}^{0}: \bar{\Delta} \rightarrow \mathbb{C}^{n}$ is an embedding and $\left|A_{c, x, v}^{0}(\zeta)\right| \leqslant c \cdot \Lambda_{0}$ for all $c$, all $x$, all $v$ and all $\zeta$.
$\left(3_{0}\right) \quad A_{c, x, v}^{0}\left(\partial^{+} \Delta\right) \subset M^{0}$ for all $c$, all $x$ and all $v$.
( $4_{0}$ ) $\frac{\partial A_{c, 0,0}^{0}}{\partial \theta}(1)$ is a positive multiple of $v_{0}$ for all $c \neq 0$.
(50) For all $c$, all $v$ and all $e^{i \theta} \in \partial^{+} \Delta$, the mapping $x \longmapsto A_{c, x, v}^{0}\left(e^{i \theta}\right) \in$ $M^{0}$ is of rank $n$.
( $\mathbf{6}_{\mathbf{0}}$ ) For all $e^{i \theta} \in \partial^{+} \Delta$, all $c \neq 0$ and all $x$, the mapping $v \longmapsto$ $\frac{\partial A_{c, x, v}^{0}}{\partial \theta}\left(e^{i \theta}\right)$ is of rank $n$ at $v=0$. Consequently, when $v$ varies, the positive half-lines $\mathbb{R}^{+} \cdot \frac{\partial A_{c, 0, v}}{\partial \theta}(1)$ describe an open infinite cone containing $v_{0}$ with vertex $p_{0}$ in $T_{p_{0}} M^{0}$.

Proof. Proceeding similarly as in the proof of Lemma 5.37, we can find a new complex affine coordinate system centered at $p_{0}$ and stabilizing $\mathbb{R}^{n}$, which we shall still denote by $\left(z_{1}, \ldots, z_{n}\right)$, in which the vector $v_{0}$ has coordinates $(0, \ldots, 0,1)$. In this coordinate system, we then construct the family $Z_{c, x, v}^{0}(\zeta)$ as in (7.8) above and we define the desired family simply as follows:

$$
\begin{equation*}
A_{c, x, v}^{0}(\zeta):=Z_{c, x, v_{0}+v}^{0}(\zeta) \tag{7.10}
\end{equation*}
$$

where we restrict the variations of the parameter $v$ to $|v| \leqslant c$. Notice that every disc $A_{c, x, v}^{0}(\bar{\Delta})$ is contained in a single complex line. All the properties are then elementary consequences of the explicit expression (7.8) of $Z_{c, x, v}^{0}(\zeta)$.

Finally, we notice that it follows from properties $\left(\mathbf{5}_{\mathbf{0}}\right)$ and $\left(\mathbf{6}_{\mathbf{0}}\right)$ that the set of points $A_{c, x, v}^{0}(\zeta)$, where $c>0$ is fixed, where $x$ varies, where $v$ varies and where $\zeta$ varies covers a local wedge of edge $M^{0}$ at $p_{0}$.
7.11. Curved families of half-attached analytic discs. Our main goal in this section is to obtain a statement similar to Lemma 7.9 after replacing the flat maximally real submanifold $M^{0} \cong \mathbb{R}^{n}$ by a curved $\mathscr{C}^{2, \alpha}$ maximally real submanifold $M^{1}$. We set up a formulation which will be appropriate for the achievement of the proof of Proposition 5.12 (Sections 8 and 9).

We will first construct a family $Z_{c, x, v}^{1}(\zeta)$ as a perturbation of the family $Z_{c, x, v}^{0}(\zeta)$, and then shrink the domain of variation of $x$, requiring $|x| \leqslant c^{2}$, in order to insure small disc size $\leqslant c^{2} \cdot \Lambda_{1}$ (instead of $\leqslant c \cdot \Lambda_{1}$, which would be the property analogous to $\left(2_{0}\right)$ ). Then $c$ will not be considered as a parameter, so we denote by $A_{x, v: c}^{1}(\zeta)$ the desired family, putting $c$ after a semicolon. In fact, in our construction, we unavoidably loose the $\mathscr{C}^{2, \alpha-0}$ smoothness with respect to $c$, and the family degenerates to a constant for $c=0$.

Lemma 7.12. Let $M^{1}$ be $\mathscr{C}^{2, \alpha}$ maximally real submanifold of $\mathbb{C}^{n}$, let $p_{1} \in$ $M^{1}$ and let $v_{1} \in T_{p_{1}} M^{1}$ be a tangent vector with $\left|v_{1}\right|=1$. Then there exists a positive constant $\Lambda_{1}>0$ and there exists $c_{1} \in \mathbb{R}$ with $c_{1}>0$ such that for every $c \in \mathbb{R}$ with $0<c \leqslant c_{1}$, there exists a family $A_{x, v: c}^{1}(\zeta)$ of analytic discs defined for $x \in \mathbb{R}^{n}$ with $|x| \leqslant c^{2}$ and for $v \in \mathbb{R}^{n}$ with $|v| \leqslant c$ which is $\mathscr{C}^{2, \alpha-0}$ with respect to $(x, v, \zeta)$ and which enjoys the following six properties:
(11) $A_{0, v: c}^{1}(1)=p_{1}$ for all $v$.
(2 $\mathbf{1}_{1}$ ) $A_{x, v: c}^{1}: \bar{\Delta} \rightarrow \mathbb{C}^{n}$ is an embedding and $\left|A_{x, v: c}^{1}(\zeta)\right| \leqslant c^{2} \cdot \Lambda_{1}$ for all $x$, all $v$ and all $\zeta$.
(31) $A_{x, v: c}^{1}\left(\partial^{+} \Delta\right) \subset M^{1}$ for all $x$ and all $v$.
$\left(4_{1}\right) \frac{\partial A_{0,0: c}^{1}}{\partial \theta}(1)$ is a positive multiple of $v_{1}$.
( $5_{1}$ ) The mapping $x \longmapsto A_{x, 0: c}^{1}(1) \in M^{1}$ is of rank $n$.
$\left(6_{1}\right)$ The mapping $v \longmapsto \frac{\partial A_{0, v: c}^{1}}{\partial \theta}\left(e^{i \theta}\right)$ is of rank $n$ at $v=0$. Consequently, as $v$ varies, the positive half-lines $\mathbb{R}^{+} \cdot \frac{\partial A_{0, v: c}^{1}}{\partial \theta}(1)$ describe an open infinite cone containing $v_{1}$ with vertex $p_{1}$ in $T_{p_{1}} M^{1}$ and the set of points $A_{x, v: c}^{1}(\zeta)$, as $|x| \leqslant c^{2},|v| \leqslant c$ and $\zeta \in \Delta$ vary, covers a wedge of edge $M^{1}$ at $\left(p_{1}, J v_{1}\right)$.

In Figure 18 below, we represent the cone property ( $\mathbf{6}_{\mathbf{1}}$ ). The remainder of this Section 7 is entirely devoted to complete the proof of Proposition 7.12.
7.13. Perturbed family of analytic discs half-attached to a maximally real submanifold. Thus, let $M^{1} \subset \mathbb{R}^{n}$ be a locally defined maximally real $\mathscr{C}^{2, \alpha}$ submanifold passing through the origin. We can assume it to be represented by $n$ Cartesian equations

$$
\begin{equation*}
y_{1}=h_{1}\left(x_{1}, \ldots, x_{n}\right), \cdots \cdots, y_{n}=h_{n}\left(x_{1}, \ldots, x_{n}\right), \tag{7.14}
\end{equation*}
$$

where $|x| \leqslant \rho_{1}$ for some $\rho_{1}>0$, where $h=h(x)$ is of class $\mathscr{C}^{2, \alpha}$ in $\left\{|x|<2 \rho_{1}\right\}$, and where, importantly, $h_{j}(0)=h_{j, x_{k}}(0)=h_{j, x_{k_{1}} x_{k_{2}}}(0)=0$, for all $j, k, k_{1}, k_{2}=1, \ldots, n$. We set $K_{1}:=\|h\|_{\mathscr{C}^{2, \alpha}\left(\left\{|x| \leqslant \rho_{1}\right\}\right)}$. Also, we can assume that $v_{1}=(0, \ldots, 0,1)$.

Our first goal is to produce a $\mathscr{C}^{2, \alpha-0}$ family of analytic discs $Z_{c, x, v}^{1}(\zeta)$ which are half-attached to $M^{1}$ and which are $\mathscr{C}^{2}$-close to the original family $Z_{c, x, v}^{0}(\zeta)$. After having constructed the family $Z_{c, x, v}^{1}(\zeta)$, we shall define the desired family $A_{x, v: c}^{1}(\zeta)$.

Let $d \in \mathbb{R}$ with $0 \leqslant d \leqslant 1$ and let the maximally real submanifold $M^{d}$ (like " $M$ deformed") be defined precisely as the set of $z=x+i y \in \mathbb{C}^{n}$ with $|x| \leqslant \rho_{1}$ which satisfy the $n$ Cartesian equations

$$
\begin{equation*}
y_{1}=d \cdot h_{1}\left(x_{1}, \ldots, x_{n}\right), \cdots \cdots, y_{n}=d \cdot h_{n}\left(x_{1}, \ldots, x_{n}\right) \tag{7.15}
\end{equation*}
$$

Of course, $\left.M^{d}\right|_{d=0} \equiv\left\{x \in \mathbb{R}^{n}:|x| \leqslant \rho_{1}\right\}$ contains the $M^{0}$ of Lemma 7.9 if we choose $\rho_{0} \leqslant \rho_{1}$, and moreover, $\left.M^{d}\right|_{d=1} \equiv M^{1}$. Adding $d \in[0,1]$ as a parameter, we will construct a family of analytic dics $Z_{c, x, v}^{d}(\zeta)$ half-attached to $M^{d}$ which is of class $\mathscr{C}^{2, \alpha-0}$ with respect to all its variables $(c, x, v, d, \zeta)$.

The disc $Z_{c, x, v}^{d}(\zeta)=: X_{c, x, v}^{d}(\zeta)+i Y_{c, x, v}^{d}(\zeta)$ is half-attached to $M^{d}$ if and only if

$$
\begin{equation*}
Y_{c, x, v}^{d}(\zeta)=d \cdot h\left(X_{c, x, v}^{d}(\zeta)\right), \quad \text { for } \zeta \in \partial^{+} \Delta \tag{7.16}
\end{equation*}
$$

and in addition, $Y_{c, x, v}^{d}$ should be a harmonic conjugate of $X_{c, x, v}^{d}$. However, the condition (7.16) does not give any relation between $X_{c, x, v}^{d}$ and $Y_{c, x, v}^{d}$ on the negative part $\partial^{-} \Delta$ of the unit circle. To fix this point, we assign the following complete equation on the unit circle

$$
\begin{equation*}
Y_{c, x, v}^{d}(\zeta)=d \cdot h\left(X_{c, x, v}^{d}(\zeta)\right)+Y_{c, x, v}^{0}(\zeta), \quad \text { for all } \quad \zeta \in \partial \Delta \tag{7.17}
\end{equation*}
$$

which coincides with (7.16) for $\zeta \in \partial^{+} \Delta$, since we have $Z_{c, x, v}^{0}\left(\partial^{+} \Delta\right) \subset \mathbb{R}^{n}$ by construction. Also, we require that $X_{c, x, v}^{d}(1)=x$, whence $Y_{c, x, v}^{d}(1)=$ $d \cdot h(x)$.

By a theorem due to Privalov (see e.g. [29]), the Hilbert transform $T_{1}$ has bounded norm $\left\|T_{1}\right\|_{\kappa, \alpha} \simeq \frac{\text { cst }}{\alpha(1-\alpha)}$ as a linear operator $\mathscr{C}^{\kappa, \alpha}\left(\partial \Delta, \mathbb{R}^{n}\right) \rightarrow$ $\mathscr{C}^{\kappa, \alpha}\left(\partial \Delta, \mathbb{R}^{n}\right)$ for $\kappa \in \mathbb{N}$ and $0<\alpha<1$, where cst is an absolute constant.

Thus, the mapping $\zeta \mapsto Y_{c, x, v}^{d}(\zeta)$ should necessarily coincide with the harmonic conjugate $\zeta \mapsto\left[T_{1} X_{c, x, v}^{d}\right](\zeta)+d \cdot h(x)$ (this property is already satisfied for $d=0$ ) and we deduce that $X_{c, x, v}^{d}(\zeta)$ should satisfy the Bishoptype equation
(7.18) $X_{c, x, v}^{d}(\zeta)=-T_{1}\left[d \cdot h\left(X_{c, x, v}^{d}\right)\right](\zeta)+X_{c, x, v}^{0}(\zeta), \quad$ for all $\zeta \in \partial \Delta$.

Conversely, if $X_{c, x, v}^{d}$ is a solution of this functional equation, then setting $Y_{c, x, v}^{d}(\zeta):=T_{1} X_{c, x, v}^{d}(\zeta)+d \cdot h(x)$, the analytic disc $Z_{c, x, v}^{d}(\zeta):=X_{c, x, v}^{d}(\zeta)+$ $i Y_{c, x, v}^{d}(\zeta)$ is half-attached to $M^{d}$ and more precisely, it satisfies (7.17).

Thanks to the solvability of Bishop's equation ([Tu1996, 29]), if $c$ satisfies $0 \leqslant c \leqslant c_{1}$ with $c_{1}>0$ sufficiently small, there exists a unique solution $X_{c, x, v}^{d}(\zeta)$ to (7.18) which is $\mathscr{C}^{2, \alpha}$ with respect to $\zeta$ and $\mathscr{C}^{2, \alpha-0}$ with respect to all the variables $(c, x, v, d, \zeta)$, where $0 \leqslant c \leqslant c_{1},|x| \leqslant c,|v| \leqslant 2$ and $\zeta \in \bar{\Delta}$. We shall now estimate the difference $\left\|Z_{c, x, v}^{d}-Z_{c, x, v}^{0}\right\|_{\mathscr{C} 1, \alpha}(\partial \Delta)$ and prove that it is bounded by a constant times $c^{2+\alpha}$. In particular, if $c_{1}$ is sufficiently small, this will imply that $Z_{c, x, v}^{d}$ is nonconstant.
7.19. Size of the solution $X_{c, x, v}^{d}(\zeta)$ in $\mathscr{C}^{1, \alpha}$-norm. Following the beginning of the proof of the existence theorem in [Tu1996, 29], we introduce the map

$$
\begin{equation*}
F: X(\zeta) \longmapsto X_{c, x, v}^{0}(\zeta)-T_{1}[d \cdot h(X)](\zeta) \tag{7.20}
\end{equation*}
$$

from a neighborhood of 0 in $\mathscr{C}^{1, \alpha}\left(\partial \Delta, \mathbb{R}^{n}\right)$ to $\mathscr{C}^{1, \alpha}\left(\partial \Delta, \mathbb{R}^{n}\right)$, and then we perform a Picard iteration scheme, setting firstly $X\{0\}_{c, x, v}^{d}(\zeta):=X_{c, x, v}^{0}(\zeta)$ and then inductively

$$
\begin{equation*}
X\{\nu+1\}_{c, x, v}^{d}(\zeta):=F\left(X\{\nu\}_{c, x, v}^{d}(\zeta)\right), \tag{7.21}
\end{equation*}
$$

for every integer $\nu \geqslant 0$. According to [Tu1996, 29], the sequence $\left(X\{\nu\}_{c, x, v}^{d}(\zeta)\right)_{\nu \in \mathbb{N}}$ converges in $\mathscr{C}^{1, \alpha}(\partial \Delta)$ towards the unique solution $X_{c, x, v}^{d}(\zeta)$ of (7.18). We want to extract the supplementary information that $\left\|X_{c, x, v}^{d}\right\|_{\mathscr{C} 1, \alpha(\partial \Delta)} \leqslant c \cdot K_{2}$ for some positive constant $K_{2}$, which will play the rôle of the constant $K_{2}$ of Lemma 6.12.

By construction (cf. (7.8)) there exists a constant $K_{4}>0$ such that

$$
\begin{equation*}
\left\|X_{c, x, v}^{0}\right\|_{\mathscr{C}^{2}, \alpha(\partial \Delta)} \leqslant c \cdot K_{4} . \tag{7.22}
\end{equation*}
$$

Lemma 7.23. Setting $K_{5}:=K_{1}\left(3+\pi^{1-\alpha}\right)\left\|T_{1}\right\|_{\mathscr{F}_{1, \alpha}(\partial \Delta)}$, if

$$
\begin{equation*}
c_{1} \leqslant \min \left(\frac{\rho_{1}}{2 K_{4}},\left(\frac{1}{2^{2+\alpha} K_{4}^{1+\alpha} K_{5}}\right)^{\frac{1}{1+\alpha}}\right) \tag{7.24}
\end{equation*}
$$

then $X_{c, x, v}^{d}$ satisfies $\left|X_{c, x, v}^{d}\left(e^{i \theta}\right)\right| \leqslant \rho_{1}$ for all $e^{i \theta} \in \partial \Delta$ and there exists $K_{2}>0$ such that

$$
\begin{equation*}
\left\|X_{c, x, v}^{d}\right\|_{\mathscr{C}_{1, \alpha}(\partial \Delta)} \leqslant c \cdot K_{2} . \tag{7.25}
\end{equation*}
$$

In fact, it suffices to choose $K_{2}:=2 K_{4}$.
Proof. Indeed, applying Lemmas 6.4 and 6.9, if $X \in \mathscr{C}{ }^{1, \alpha}\left(\partial \Delta, \mathbb{R}^{n}\right)$ satisfies $\left|X\left(e^{i \theta}\right)\right| \leqslant \rho_{1}$ for all $e^{i \theta} \in \partial \Delta$ and $\|X\|_{\mathscr{G} 1, \alpha(\partial \Delta)} \leqslant c \cdot 2 K_{4}$ for all $c \leqslant c_{1}$, where $c_{1}$ is as in (7.24), we may estimate (remind $0 \leqslant d \leqslant 1$ ):
(7.26)

$$
\begin{aligned}
\|F(X)\|_{\mathscr{C} 1, \alpha}(\partial \Delta) & \leqslant\left\|X_{c, x, v}^{0}\right\|_{\mathscr{C}_{1, \alpha}(\partial \Delta)}+\left\|T_{1}\right\|_{\mathscr{C} 1, \alpha}(\partial \Delta) \cdot\|h(X)\|_{\mathscr{C} 1, \alpha}(\partial \Delta) \\
& \leqslant c \cdot K_{4}+\left\|T_{1}\right\|_{\mathscr{C} 1, \alpha}(\partial \Delta) \cdot K_{1}\left(c \cdot 2 K_{4}\right)^{2+\alpha}\left(3+\pi^{1-\alpha}\right) \\
& =c \cdot\left(K_{4}+c^{1+\alpha} 2^{2+\alpha} K_{4}^{2+\alpha} K_{5}\right) \\
& \leqslant c \cdot\left(K_{4}+c_{1}^{1+\alpha} 2^{2+\alpha} K_{4}^{2+\alpha} K_{5}\right) \\
& \leqslant c \cdot 2 K_{4} .
\end{aligned}
$$

From the last inequality, it also follows that $\left|F\left(X\left(e^{i \theta}\right)\right)\right| \leqslant \rho_{1}$ for all $e^{i \theta} \in$ $\partial \Delta$. Consequently, the iteration (7.21) is well defined for all $\nu \in \mathbb{N}$ and from the inequality (7.26), we deduce that the limit $X_{c, x, v}^{d}$ satisfies the desired estimate $\left\|X_{c, x, v}^{d}\right\|_{\mathscr{C}^{1, \alpha}(\partial \Delta)} \leqslant c \cdot 2 K_{4}$.

Corollary 7.27. Under the above assumptions, there exists a constant $K_{6}>$ 0 such that

$$
\begin{equation*}
\left\|X_{c, x, v}^{d}-X_{c, x, v}^{0}\right\|_{\mathscr{C}_{1, \alpha}(\partial \Delta)} \leqslant c^{2+\alpha} \cdot K_{6} . \tag{7.28}
\end{equation*}
$$

Proof. We estimate
(7.29)

$$
\begin{aligned}
\left\|X_{c, x, v}^{d}-X_{c, x, v}^{0}\right\|_{\mathscr{C}^{1, \alpha}(\partial \Delta)} & \leqslant\left\|T_{1}\right\|_{\mathscr{C}^{1, \alpha}(\partial \Delta)} \cdot\left\|h\left(X_{c, x, v}^{d}\right)\right\|_{\mathscr{C}^{1, \alpha}(\partial \Delta)} \\
& \leqslant\left\|T_{1}\right\|_{\mathscr{C}^{1, \alpha}(\partial \Delta)} \cdot K_{1}\left(c \cdot 2 K_{4}\right)^{2+\alpha}\left(3+\pi^{1-\alpha}\right) \\
& \leqslant c^{2+\alpha} \cdot K_{5}\left(2 K_{4}\right)^{2+\alpha} .
\end{aligned}
$$

so that it suffices to set $K_{6}:=K_{5}\left(2 K_{4}\right)^{2+\alpha}$.
7.30. Smallness of the deformation in $\mathscr{C}^{2}$-norm. As already mentioned, the solution $X_{c, x, v}^{d}(\zeta)$ is in fact $\mathscr{C}^{2, \alpha}$ with respect to $\zeta$ and $\mathscr{C}^{2, \alpha-0}$ with respect to all variables $(d, c, x, v, \zeta)$. We can therefore differentiate twice Bishop's equation (7.18). First of all, if $X \in \mathscr{C}^{2, \alpha-0}\left(\partial \Delta, \mathbb{R}^{n}\right)$, we remind the commutation relation $\frac{\partial}{\partial \theta}(T X)=T\left(\frac{\partial X}{\partial \theta}\right)$, whence

$$
\begin{equation*}
\frac{\partial}{\partial \theta}\left(T_{1} X\right)=T\left(\frac{\partial X}{\partial \theta}\right) \tag{7.31}
\end{equation*}
$$

since $T_{1} X=T X-T X(1)$. We may then compute the first order derivative of (7.18):

$$
\begin{equation*}
\frac{\partial}{\partial \theta} X_{c, x, v}^{d}\left(e^{i \theta}\right)-\frac{\partial}{\partial \theta} X_{c, x, v}^{0}\left(e^{i \theta}\right)=-T\left[d \cdot \sum_{l=1}^{n} \frac{\partial h}{\partial x_{l}}\left(X_{c, x, v}^{d}\right) \frac{\partial X_{l ; c, x, v}^{d}}{\partial \theta}\right]\left(e^{i \theta}\right) . \tag{7.32}
\end{equation*}
$$

and then its second order partial derivatives $\partial^{2} / \partial v_{k} \partial \theta$, for $k=1, \ldots, n$ :
(7.33)

$$
\begin{aligned}
\frac{\partial^{2} X_{c, x, v}^{d}}{\partial v_{k} \partial \theta}-\frac{\partial^{2} X_{c, x, v}^{0}}{\partial v_{k} \partial \theta}=-T[d & \cdot \sum_{l_{1}, l_{2}=1}^{n} \frac{\partial^{2} h}{\partial x_{l_{1}} \partial x_{l_{2}}}\left(X_{c, x, v}^{d}\right) \frac{\partial X_{l_{1} ; c, x, v}^{d}}{\partial v_{k}} \frac{\partial X_{l_{2} ; c, x, v}^{d}}{\partial \theta}+ \\
& \left.+d \cdot \sum_{l=1}^{n} \frac{\partial h_{j}}{\partial x_{l}}\left(X_{c, x, v}^{d}\right) \frac{\partial^{2} X_{l ; c, x, v}^{d}}{\partial v_{k} \partial \theta}\right]
\end{aligned}
$$

Let now $K_{2}$ be as in (7.25) and let $K_{3}$ be as in Lemma 6.12, applied to $X_{c, x, v}^{d}(\zeta)$.
Lemma 7.34. If in addition to the inequality (7.24), the constant $c_{1}$ satisfies

$$
\begin{equation*}
c_{1} \leqslant\left(\frac{1}{2 K_{3}\|T\|_{\mathscr{C}^{\alpha}(\partial \Delta)}}\right)^{\frac{1}{1+\alpha}} \tag{7.35}
\end{equation*}
$$

then there exists $K_{7}>0$ such that for all d, all c, all $x$, all $v$, and for $k=1, \ldots, n$ :

$$
\left\{\begin{array}{l}
\left\|\frac{\partial^{2} X_{c, x, v}^{d}}{\partial v_{k} \partial \theta}-\frac{\partial^{2} X_{c, x, v}^{0}}{\partial v_{k} \partial \theta}\right\|_{\mathscr{C}^{\alpha}(\partial \Delta)} \leqslant c^{2+\alpha} \cdot K_{7},  \tag{7.36}\\
\left\|\frac{\partial^{2} X_{c, x, v}^{d}}{\partial \theta^{2}}-\frac{\partial^{2} X_{c, x, v}^{0}}{\partial \theta^{2}}\right\|_{\mathscr{C}^{\alpha}(\partial \Delta)} \leqslant c^{2+\alpha} \cdot K_{7} .
\end{array}\right.
$$

Proof. We check only the first inequality, the second being similar. Introducing for the second line of (7.33) a new simplified notation $\mathscr{R}:=$ $-T\left[d \cdot \sum_{l_{1}, l_{2}=1}^{n} \frac{\partial^{2} h}{\partial x_{1} \partial x_{l_{2}}}\left(X_{c, x, v}^{d}\right) \frac{\partial X_{l_{1}, c, x, v}^{d}}{\partial v_{k}} \frac{\partial X_{l_{2}, c, x, v}^{d}}{\partial \theta}\right]$ and setting further obvious simplifying changes of notation, we can rewrite (7.33) as

$$
\begin{equation*}
\mathscr{X}^{d}-\mathscr{X}^{0}=\mathscr{R}-T\left[d \cdot \mathscr{H} \mathscr{X}^{d}\right] . \tag{7.37}
\end{equation*}
$$

Thanks to the inequality $\left\|X_{c, x, v}^{d}\right\|_{\mathscr{C} 1, \alpha(\partial \Delta)} \leqslant c \cdot K_{2}$ already established in Lemma 7.23 and thanks to Lemma 6.12, we know that the vector $\mathscr{R} \in$ $\mathscr{C}^{\alpha}\left(\partial \Delta, \mathbb{R}^{n}\right)$ and the matrix $\mathscr{H} \in \mathscr{C}^{1, \alpha}\left(\partial \Delta, \mathscr{M}_{n \times n}(\mathbb{R})\right)$ are small:

$$
\left\{\begin{align*}
\|\mathscr{R}\|_{\mathscr{C}^{\alpha}(\partial \Delta)} & \leqslant c^{2+\alpha} \cdot\|T\|_{\mathscr{C}^{\alpha}(\partial \Delta)} K_{3}\left(K_{2}\right)^{2}  \tag{7.38}\\
\|\mathscr{H}\|_{\mathscr{C}^{\alpha}(\partial \Delta)} & \leqslant c^{1+\alpha} \cdot K_{3} .
\end{align*}\right.
$$

We then rewrite (7.37) under the form

$$
\begin{equation*}
\mathscr{X}^{d}-\mathscr{X}^{0}=\mathscr{S}-T\left[d \cdot \mathscr{H}\left(\mathscr{X}^{d}-\mathscr{X}^{0}\right)\right] \tag{7.39}
\end{equation*}
$$

with $\mathscr{S}:=\mathscr{R}-T\left[d \cdot \mathscr{H} \mathscr{X}^{0}\right]$. Using the inequality $\left\|\mathscr{X}^{0}\right\|_{\mathscr{C}^{\alpha}(\partial \Delta)} \leqslant c \cdot K_{4}$ which is a direct consequence of (7.22) and taking (7.38) into account, we deduce:

$$
\begin{equation*}
\|\mathscr{S}\|_{\mathscr{C}^{\alpha}(\partial \Delta)} \leqslant c^{2+\alpha} \cdot\|T\|_{\mathscr{C}^{\alpha}(\partial \Delta)}\left[K_{3}\left(K_{2}\right)^{2}+K_{3} K_{4}\right] . \tag{7.40}
\end{equation*}
$$

Taking the $\mathscr{C}^{\alpha}(\partial \Delta)$ norm of both sides of (7.39), we deduce the estimate

$$
\begin{align*}
\left\|\mathscr{X}^{d}-\mathscr{X}^{0}\right\|_{\mathscr{C}^{\alpha}(\partial \Delta)} & \leqslant c^{2+\alpha} \cdot \frac{\|T\|_{\mathscr{C}^{\alpha}(\partial \Delta)}\left[K_{3}\left(K_{2}\right)^{2}+K_{3} K_{4}\right]}{1-c^{1+\alpha} \cdot\|T\|_{\mathscr{C}^{\alpha}(\partial \Delta)} K_{3}}  \tag{7.41}\\
& \leqslant c^{2+\alpha} \cdot 2\|T\|_{\mathscr{C}^{\alpha}(\partial \Delta)}\left[K_{3}\left(K_{2}\right)^{2}+K_{3} K_{4}\right],
\end{align*}
$$

where we use (7.35). It suffices to set $K_{7}:=2\|T\|_{\mathscr{C}^{\alpha}(\partial \Delta)}\left[K_{3}\left(K_{2}\right)^{2}+K_{3} K_{4}\right]$.
7.42. Adjustment of the tangent vector. Let $v_{1} \in T_{p_{1}} M^{1}$ with $\left|v_{1}\right|=1$, as in Lemma 7.12. Coming back to the first family $Z_{c, x, v}^{0}(\zeta)$ defined by (7.8), we observe that

$$
\left\{\begin{align*}
\frac{\partial Z_{j ; c, 0, v_{1}}^{0}}{\partial x_{k}}(1) & =\delta_{k}^{j}, \quad j, k=1, \ldots, n  \tag{7.43}\\
\frac{\partial^{2} Z_{j ; c, 0, v_{1}}^{0}}{\partial v_{k} \partial \theta}(1) & =c \frac{\partial \Psi}{\partial \theta}\left(e^{i \theta}\right) \delta_{k}^{j}, \quad j, k=1, \ldots, n
\end{align*}\right.
$$

From now on, we shall set $d=1$ and we shall only consider the family $Z_{c, x, v}^{1}(\zeta)$. Thanks to the estimates (7.28) and (7.36), we deduce that if $c_{1}$ is sufficiently small, then for all $c$ with $0<c \leqslant c_{1}$, the two Jacobian matrices

$$
\begin{equation*}
\left(\frac{\partial Z_{j ; c, 0, v_{1}}^{1}}{\partial x_{k}}(1)\right)_{1 \leqslant j, k \leqslant n} \quad \text { and } \quad\left(\frac{\partial^{2} Z_{j ; c, 0, v_{1}}^{1}}{\partial v_{k} \partial \theta}(1)\right)_{1 \leqslant j, k \leqslant n} \tag{7.44}
\end{equation*}
$$

are invertible. It would follow that if set $A_{x, v: c}^{1}(\zeta):=Z_{c, x, v_{1}+v}^{1}(\zeta)$, then the disc $A_{x, v: c}^{1}(\zeta)$ would satisfy the two rank properties $\left(5_{1}\right)$ and $\left(\mathbf{6}_{\mathbf{1}}\right)$ of Lemma 7.12. However, the tangency condition ( $4_{1}$ ) would certainly not be satisfied, because as $d$ varies from 0 to 1 , the disc $Z_{c, x, v}^{d}(\zeta)$ undergoes a nontrivial deformation.

Consequently, for every $c$ with $0<c \leqslant c_{1}$, we have to adjust the "cone parameter" $v$ in order to maintain the tangency condition.
Lemma 7.45. For every $c$ with $0<c \leqslant c_{1}$, there exists a vector $v(c) \in \mathbb{R}^{n}$ such that

$$
\begin{equation*}
\frac{\partial Z_{c, 0, v_{1}+v(c)}^{1}}{\partial \theta}(1)=\frac{\partial Z_{c, 0, v_{1}}^{0}}{\partial \theta}(1)=c \cdot \frac{\partial \Psi}{\partial \theta}(1) \cdot v_{1} . \tag{7.46}
\end{equation*}
$$

Furthermore, there exists a constant $K_{8}>0$ such that $|v(c)| \leqslant c^{1+\alpha} \cdot K_{8}$.
Proof. Unfortunately, we cannot apply the implicit function theorem, because the mapping $Z_{c, x, v}^{1}$ is identically zero when $c=0$, so we have to proceed differently. First, we set

$$
\begin{equation*}
C_{1}:=\frac{\partial \Psi}{\partial \theta}(1), \quad \text { and } \quad C_{2}:=\|\Psi\|_{\mathscr{C}^{2}(\bar{\Delta})} \tag{7.47}
\end{equation*}
$$

The constant $C_{2}$ will be used only in Section 8 below. Choose $K_{8} \geqslant \frac{2 K_{6}}{C_{1}}$. According to the explicit expression (7.8), the set of points

$$
\begin{equation*}
\left\{\frac{\partial X_{c, 0, v_{1}+v}^{0}}{\partial \theta}(1) \in \mathbb{R}^{n}:|v| \leqslant c^{1+\alpha} \cdot K_{8}\right\} \tag{7.48}
\end{equation*}
$$

covers a cube in $\mathbb{R}^{n}$ centered at the point $\frac{\partial X_{c, 0, v_{1}}^{0}}{\partial \theta}(1)$ of radius $c^{2+\alpha} \cdot C_{1} K_{8}$. Thanks to the estimate (7.28), we deduce that the (deformed) set of points

$$
\begin{equation*}
\left\{\frac{\partial X_{c, 0, v_{1}+v}^{1}}{\partial \theta}(1) \in \mathbb{R}^{n}:|v| \leqslant c^{1+\alpha} \cdot K_{8}\right\} \tag{7.49}
\end{equation*}
$$

covers a cube in $\mathbb{R}^{n}$ centered at the same point $\frac{\partial X_{c, 0, v_{1}}^{0}}{\partial \theta}(1)$, but of radius

$$
\begin{equation*}
c^{2+\alpha} \cdot C_{1} K_{8}-c^{2+\alpha} \cdot K_{6} \geqslant c^{2+\alpha} \cdot K_{6} . \tag{7.50}
\end{equation*}
$$

Consequently, there exists at least one $v(c) \in \mathbb{R}^{n}$ with $|v(c)| \leqslant c^{1+\alpha} \cdot K_{8}$ such that (7.46) holds, which completes the proof.
7.51. Construction of the family $A_{x, v: c}^{1}(\zeta)$. We can now complete the proof of the main Lemma 7.12. First of all, with $\Psi(\zeta)$ as in $\S 7.5$, we consider the composed conformal mapping

$$
\begin{equation*}
\zeta \longmapsto c \Psi(\zeta) \longmapsto \frac{i-c \Psi(\zeta)}{i+c \Psi(\zeta)}=: \Phi_{c}(\zeta) . \tag{7.52}
\end{equation*}
$$

The image $\Phi_{c}(\zeta)$ of the unit disc is a small domain contained in $\Delta$ and concentrated near 1.


More precisely, assuming that $c$ satifies $0<c \leqslant c_{1}$ with $c_{1} \ll 1$ as in the previous paragraphs, and taking account of the definition of $\Psi(\zeta)$, it can be checked easily that $\Phi_{c}(1)=1$, that $\Phi_{c}\left(\partial^{+} \Delta\right)$ is contained in $\left\{e^{i \theta} \in \partial^{+} \Delta\right.$ : $|\theta|<10 c\}$, and that
(7.53)
$\Phi_{c}\left(\bar{\Delta} \backslash \partial^{+} \Delta\right) \subset\{\zeta \in \Delta:|\zeta-1|<8 c\} \subset\left\{\rho e^{i \theta} \in \Delta:|\theta|<10 c, 1-10 c<\rho<1\right\}$.
the second inclusion being trivial.
We can finally define the desired family of analytic discs, writing the parameter $c$ after a semi-colon, since we have lost the $\mathscr{C}^{2, \alpha-0}$-smoothness with respect to it after the application of Lemma 7.45, and since $c$ will be fixed afterwards anyway:

$$
\begin{equation*}
A_{x, v: c}^{1}(\zeta):=Z_{c, x, v_{1}+v(c)+v}^{1}\left(\Phi_{c}(\zeta)\right) . \tag{7.54}
\end{equation*}
$$

We restrict the variation of the parameters $x$ to $|x| \leqslant c^{2}$ and $v$ to $|v| \leqslant c$. Property ( $4_{1}$ ) holds immediately, thanks to the choice of $v(c)$. Properties $\left(1_{1}\right),\left(3_{1}\right),\left(5_{1}\right)$ and $\left(6_{1}\right)$ as well as the embedding property in $\left(2_{1}\right)$ are direct consequences of the immersive properties (7.44) satisfied by $Z_{c, x, v_{1}+v(c)+v}^{1}(\zeta)$, using the chain rule and the nonvanishing of the partial derivative $\frac{\partial \Phi_{c}}{\partial \theta}(1)$. The size estimate in $\left(\mathbf{2}_{1}\right)$ follows from (7.25), from (7.28), from the restriction of the domains of variation of $x$ and of $v$ and from (7.53). This completes the proof of Lemma 7.12.

## §8. GEOMETRIC PROPERTIES OF FAMILIES OF HALF-ATTACHED ANALYTIC DISCS

8.1. Preliminary. By Lemma 7.12, for every $c$ with $0<c \leqslant c_{1}$, the family of half-attached analytic discs $A_{x, v: c}^{1}(\zeta)$ covers a local wedge of edge $M^{1}$ at $p_{1}$. However, not only we want the family $A_{x, v: c}^{1}$ to cover a local wedge of edge $M^{1}$ at $p_{1}$, but we certainly want to remove the point $p_{1}$ of Proposition 5.12 by means of the continuity principle. Consequently, in each one
of the three geometric situations $\left(\mathbf{I}_{\mathbf{1}}\right),\left(\mathbf{I}_{\mathbf{2}}\right)$ and (II) which we have normalized in Lemma 5.37 above, we shall firstly deduce from the tangency condition ( $\mathbf{4}_{1}$ ) of Lemma 7.12 that the (excised) half-boundary $A_{0,0: c}^{1}\left(\partial^{+} \Delta \backslash\{1\}\right)$ is contained in the open side $\left(H^{1}\right)^{+}$(this is why we have normalized in Lemma 5.37 the second order terms of the supporting hypersurface $H^{1}$ in order that $\left(H^{1}\right)^{+}$is strictly concave; we also want that $A_{0,0: c}^{1}\left(\partial^{+} \Delta \backslash\{1\}\right)$ is contained in $\left(H^{1}\right)^{+}$in order to apply the continuity principle). Secondly, we shall show that for all $x$ with $|x| \leqslant c^{2}$, the disc interior $A_{x, 0: c}(\Delta)$ is contained in the local half-wedge $\mathscr{H} \mathscr{W}_{1}^{+}$in the cases $\left(\mathbf{I}_{\mathbf{1}}\right),\left(\mathbf{I}_{\mathbf{2}}\right)$ and is contained in the wedge $\mathscr{W}_{2}$ in case (II).
8.2. Geometric disposition of the dises with respect to $H^{1}$ and to $\mathscr{H} \mathscr{W}_{1}^{+}$ or to $\mathscr{W}_{2}$. We remember that the positive $c_{1}$ of Lemmas 7.12, 7.23 and 7.34 was shrunk explicitely, in terms of the constants $K_{1}, K_{2}, K_{3}, \ldots$ In this section, we shall again shrink $c_{1}$ a finite number of times, but without mentioning all the similar explicit inequalities which will appear. The precise statement of the main lemma of this section, which is a continuation of Lemma 7.12, is as follows; whereas we can essentially gather the three cases in the formal statement of the lemma, it is necessary to treat them separately in the proof, because the normalizations of Lemma 5.37 differ.

Lemma 8.3. Let $M$, let $M^{1}$, let $p_{1}$, let $H^{1}$, let $v_{1}$, let $\left(H^{1}\right)^{+}$, let $\mathscr{H} \mathscr{W}_{1}^{+}$ (or let $\mathscr{H} \mathscr{W}_{2}$ ) and let a coordinate system $z=\left(z_{1}, \ldots, z_{n}\right)$ vanishing at $p_{1}$ be as in Case ( $\mathbf{I}_{\mathbf{1}}$ ), as in Case ( $\mathbf{I}_{\mathbf{2}}$ ) or as in Case (II) of Lemma 5.37. Aa a local one-dimensional submanifold $T^{1} \subset M^{1}$ transversal to $H^{1}$ in $M^{1}$ and passing through $p_{1}$, choose $\left.T_{1}:=\left\{\left(x_{1}, 0, \ldots, 0\right)+i h\left(x_{1}, 0, \ldots, 0\right)\right)\right\}$ in Case $\left(\mathbf{I}_{1}\right)$ and $\left.T_{1}:=\left\{\left(0, \ldots, 0, x_{n}\right)+i h\left(0, \ldots, 0, x_{n}\right)\right)\right\}$ in Cases $\left(\mathbf{I}_{2}\right)$ and (II). For every $c$ with $0<c \leqslant c_{1}$, let $A_{x, v: c}^{1}(\zeta)$ be the family of analytic discs satisfying properties $\left(\mathbf{1}_{1}\right),\left(\mathbf{2}_{1}\right),\left(\mathbf{3}_{1}\right),\left(\mathbf{4}_{1}\right),\left(\mathbf{5}_{1}\right)$ and $\left(\mathbf{6}_{1}\right)$ of Lemma 7.12. Shrinking $c_{1}$ if necessary, then for every $c$ with $0<c \leqslant c_{1}$, the following three further properties hold:
$\left(7_{1}\right) A_{0,0: c}^{1}\left(\partial^{+} \Delta \backslash\{1\}\right) \subset\left(H^{1}\right)^{+} ;$
$\left(8_{1}\right) A_{x, 0: c}^{1}\left(\partial^{+} \Delta\right)$ is contained in $\left(H^{1}\right)^{+}$for all $x$ such that the point $A_{x, 0: c}^{1}(1)$ belongs to $T^{1} \cap\left(H^{1}\right)^{+}$;
$\left(9_{1}\right) A_{x, v: c}^{1}\left(\bar{\Delta} \backslash \partial^{+} \Delta\right)$ is contained in the half-wedge $\mathscr{H} \mathscr{W}_{1}^{+}$or in the wedge $\mathscr{W}_{2}$ for all $x$ and all $v$.

Proof. We treat only Case ( $\mathbf{I}_{1}$ ), the other two cases being similar. Figure 18 just below illustrates properties $\left(7_{1}\right)$ and $\left(8_{1}\right)$ and also properties $\left(1_{1}\right),\left(5_{1}\right)$ and ( $\boldsymbol{6}_{1}$ ) of Lemma 7.12.


By construction of $A_{x, v: c}^{1}$, if the scaling parameter $c_{1}$ is small enough, the $\operatorname{disc} A_{x, v: c}^{1}(\bar{\Delta})$ is only a slightly deformed small part of the straight complex line $\mathbb{C} \cdot\left(v_{1}+J v_{1}\right)$, where $v_{1} \in T_{p_{1}} H^{1}$ is as in Lemma 7.12. Intuitively speaking, the reason why property $\left(7_{1}\right)$ holds true then becomes clear: the open set $\left(H^{1}\right)^{+}$is strictly concave and the small, almost straight curve $A_{0,0: c}^{1}\left(\partial^{+} \Delta\right)$ is tangent to $H^{1}$ at $p_{1}$. Concerning ( $\mathbf{8}_{\mathbf{1}}$ ), when $x$ varies, the small segments $A_{x, 0: c}^{1}\left(\partial^{+} \Delta\right)$ are essentially translated to the right (inside $M^{1}$ ) by the vector $x \in \mathbb{R}^{n}$. Also, $\left(9_{1}\right)$ should hold because the half-wedge $\mathscr{H} \mathscr{W}_{1}^{+}$(or the wedge $\mathscr{W}_{2}$ ) is directed by $J v_{1}$. The next paragraphs will establish these properties rigorously.

Firstly, let us prove property $\left(\mathbf{7}_{1}\right)$ in Case $\left(\mathbf{I}_{1}\right)$. According to Lemma 5.37, the vector $v_{1}$ is given by $(0,1, \ldots, 1)$ and the side $\left(H^{1}\right)^{+} \subset M^{1}$ is defined by $x_{1}>g\left(x^{\prime}\right)=-x_{2}^{2}-\cdots-x_{n}^{2}+\widehat{g}\left(x^{\prime}\right)$, with $\widehat{g}\left(x^{\prime}\right)=\mathrm{o}\left(\left|x^{\prime}\right|^{2}\right)$ by $(5.40)_{3}$. According to Lemma 6.4, we then have $\left|\widehat{g}\left(x^{\prime}\right)\right| \leqslant K_{9} \cdot\left(\sum_{j=2}^{n} x_{j}^{2}\right)^{\frac{\alpha+2}{2}}$, for some constant $K_{9}>0$. Since the strictly concave open subset $\left(\widetilde{H}^{1}\right)^{+}$of $M^{1}$ with $\mathscr{C}^{2, \alpha}$ boundary defined by $x_{1}>-x_{1}^{2}-\cdots-x_{n}^{2}+K_{9} \cdot\left(\sum_{j=2}^{n} x_{j}^{2}\right)^{\frac{2+\alpha}{2}}$ is contained in $\left(H^{1}\right)^{+}$, it suffices to prove property $\left(7_{1}\right)$ with $\left(H^{1}\right)^{+}$replaced by $\left(\widetilde{H}^{1}\right)^{+}$.

By construction, the disc boundary $A_{0,0: c}(\partial \Delta)$ is tangent at $p_{1}$ to $H^{1}$, hence also to $\widetilde{H}^{1}$. Intuitively, it is again clear that the (excised) halfboundary $A_{0,0: c}\left(\partial^{+} \Delta \backslash\{1\}\right)$ should then be contained in the strictly concave side $\left(\widetilde{H}^{1}\right)^{+}$, see again Figure 18 above.

To proceed rigorously, we come back to the definition $A_{0,0: c}^{1}(\zeta) \equiv$ $Z_{c, 0, v_{1}+v(c)}^{1}\left(\Phi_{c}(\zeta)\right)$, with the tangency condition (7.46) satisfied. First of all,
denoting $v(c)=\left(v_{1}(c), \ldots, v_{n}(c)\right)$, we compute the second order derivatives of the similar discs attached to $M^{0}$ :
(8.4)

$$
\left\{\begin{array}{l}
\frac{\partial^{2} Z_{1 ; c, 0, v_{1}+v(c)}^{0}}{\partial \theta^{2}}(1)=c \cdot \frac{\partial^{2} \Psi}{\partial \theta^{2}}\left(e^{i \theta}\right) \cdot v_{1}(c), \\
\frac{\partial^{2} Z_{j ; c, 0, v_{1}+v(c)}^{0}}{\partial \theta^{2}}(1)=c \cdot \frac{\partial^{2} \Psi}{\partial \theta^{2}}\left(e^{i \theta}\right) \cdot\left(1+v_{j}(c)\right), \quad j=2, \ldots, n .
\end{array}\right.
$$

Using the definition (7.47), the inequality $|v(c)| \leqslant c^{1+\alpha} \cdot K_{8}$ and the second estimate (7.36), we deduce that

$$
\left\{\begin{array}{l}
\left|\frac{\partial^{2} Z_{1, c, 0, v_{1}+v(c)}^{1}}{\partial \theta^{2}}(1)\right| \leqslant c^{2+\alpha} \cdot K_{7}+c^{2+\alpha} \cdot C_{2} K_{8}=: c^{2+\alpha} \cdot 2 K_{10}  \tag{8.5}\\
\left|\frac{\partial^{2} Z_{j ; c, 0, v_{1}+v(c)}^{1}}{\partial \theta^{2}}(1)\right| \leqslant c \cdot 2 C_{2}, \quad j=2, \ldots, n
\end{array}\right.
$$

Applying then Taylor's integral formula $F(\theta)=F(0)+\theta \cdot F^{\prime}(0)+$ $\int_{0}^{\theta}\left(\theta-\theta^{\prime}\right) \cdot \partial_{\theta} \partial_{\theta} F\left(\theta^{\prime}\right) \cdot d \theta^{\prime}$ to $F(\theta):=X_{1 ; c, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right)$ and afterwards to $F(\theta):=X_{j ; c, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right)$ for $j=2, \ldots, n$, taking account of the tangency conditions
(8.6)
$\frac{\partial X_{1 ; c, 0, v_{1}+v(c)}^{1}}{\partial \theta}(1)=0, \quad \frac{\partial X_{j ; c, 0, v_{1}+v(c)}^{1}}{\partial \theta}(1)=c \cdot C_{1}, \quad j=2, \ldots, n$,
(a simple rephrasing of (7.46)) and using the inequalities (8.5), we deduce

$$
\left\{\begin{align*}
\left|X_{1 ;, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right)\right| & \leqslant \theta^{2} \cdot c^{2+\alpha} \cdot K_{10}  \tag{8.7}\\
\left|X_{j ; c, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right)-\theta \cdot c \cdot C_{1}\right| & \leqslant \theta^{2} \cdot c \cdot C_{2}, \quad j=2, \ldots, n
\end{align*}\right.
$$

Recall the equation of $\left(\widetilde{H}^{1}\right)^{+}$:

$$
\begin{equation*}
x_{1}>\widetilde{g}\left(x^{\prime}\right):=-x_{2}^{2}-\cdots-x_{n}^{2}+K_{9}\left(\sum_{j=2}^{n} x_{j}^{2}\right)^{\frac{2+\alpha}{2}} \tag{8.8}
\end{equation*}
$$

We now claim that if $c_{1}$ is sufficiently small, then for every $\theta$ with $0<|\theta|<$ $10 c$, we have
(8.9) $X_{1 ; c, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right)>\widetilde{g}\left(X_{2 ; c, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right), \ldots \ldots, X_{n ; c, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right)\right)$.

Since $\Phi_{c}\left(\partial^{+} \Delta\right)$ is contained in $\left\{e^{i \theta} \in \partial^{+} \Delta:|\theta|<10 c\right\}$, this will imply the inclusion proving $\left(\mathbf{7}_{1}\right)$ :

$$
\begin{align*}
A_{x, v: c}^{1}\left(\partial^{+} \Delta \backslash\{1\}\right) & =Z_{c, 0, v_{1}+v(c)}^{1}\left(\Phi_{c}\left(\partial^{+} \Delta \backslash\{1\}\right)\right) \subset \\
& \subset Z_{c, 0, v_{1}+v(c)}^{1}\left(\left\{e^{i \theta} \in \partial^{+} \Delta: 0<|\theta| \leqslant 10 c\right\}\right)  \tag{8.10}\\
& \subset\left(\widetilde{H}^{1}\right)^{+}
\end{align*}
$$

To prove the claim, using (8.7), we get a minoration of the left hand side of (8.9):

$$
\begin{equation*}
X_{1 ; c, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right) \geqslant-\theta^{2} \cdot c^{2+\alpha} \cdot K_{10} \tag{8.11}
\end{equation*}
$$

On the other hand, using two inequalities which are direct consequences of the second line of (8.7), provided that $c_{1}$ satisfies $10 c_{1} \cdot C_{2} \leqslant \frac{C_{1}}{2}$, we have:

$$
\begin{align*}
\left|X_{j ; c, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right)\right| \leqslant|\theta| \cdot c \cdot\left(C_{1}+|\theta| \cdot C_{2}\right) \leqslant|\theta| \cdot c \cdot \frac{3 C_{1}}{2} \\
{\left[X_{j ; c, 0, v_{1}+v(c)}^{1}\right]^{2} \geqslant \theta^{2} \cdot c^{2} \cdot\left(C_{1}-|\theta| \cdot C_{2}\right)^{2} \geqslant \theta^{2} \cdot c^{2} \cdot \frac{C_{1}^{2}}{4} } \tag{8.12}
\end{align*}
$$

for $j=2, \ldots, n$. We deduce the following majoration of the right hand side of (8.9):

$$
\begin{align*}
& \widetilde{g}\left(X_{2 ; c, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right), \ldots \ldots, X_{n ; c, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right)\right)= \\
&=-\sum_{j=2}^{n}\left[X_{j ; c, 0, v_{1}+v(c)}^{1}\right]^{2}+K_{9}\left(\sum_{j=2}^{n}\left[X_{j ; c, 0, v_{1}+v(c)}\left(e^{i \theta}\right)\right]^{2}\right)^{\frac{2+\alpha}{2}}  \tag{8.13}\\
& \leqslant-\theta^{2} \cdot c^{2} \cdot \frac{C_{1}^{2}}{4}(n-1)+|\theta|^{2+\alpha} \cdot c^{2+\alpha} \cdot\left(\frac{(n-1) 9 C_{1}^{2}}{4}\right)^{\frac{2+\alpha}{2}} K_{9} \\
& \leqslant-\theta^{2} \cdot c^{2}\left(\frac{C_{1}^{2}}{4}(n-1)-c^{\alpha} \cdot\left(\frac{(n-1) 9 C_{1}^{2}}{4}\right)^{\frac{2+\alpha}{2}} K_{9}\right) .
\end{align*}
$$

Thanks to the minoration (8.11) and to the majoration (8.13), in order that the inequality (8.9) holds for all $\theta$ with $0<|\theta| \leqslant 10 c$, it suffices that the right hand side of (8.11) be greater than the last line of (8.13). Writing this (strict) inequality and clearing the factor $\theta^{2} \cdot c^{2}$, we see that it suffices that

$$
\begin{equation*}
-K_{10} \cdot c^{\alpha}>-\left(\frac{C_{1}^{2}}{4}(n-1)-c^{\alpha} \cdot\left(\frac{(n-1) 9 C_{1}^{2}}{4}\right)^{\frac{2+\alpha}{2}} K_{9}\right), \tag{8.14}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
c_{1}<\left(\frac{\frac{C_{1}^{2}}{4}(n-1)}{K_{10}+\left(\frac{(n-1) 9 C_{1}^{2}}{4}\right)^{\frac{2+\alpha}{2}} K_{9}}\right)^{\frac{1}{\alpha}} . \tag{8.15}
\end{equation*}
$$

This completes the proof of property $\left(7_{1}\right)$.
Secondly, let us prove property ( $\mathbf{8}_{1}$ ) in Case ( $\mathbf{I}_{1}$ ). As above, we come back to the definition $A_{x, 0: c}^{1}(\zeta):=Z_{c, x, v_{1}+v(c)}^{1}\left(\Phi_{c}(\zeta)\right)$ and we remind that $A_{x, 0: c}^{1}(1)=Z_{c, x, v_{1}+v(c)}^{1}(1)=x+i h(x)$, which follows by putting $d=1$ and $\zeta=1$ in (7.18). Thanks to the inclusion $\Phi_{c}\left(\partial^{+} \Delta\right) \subset\left\{e^{i \theta} \in \partial^{+} \Delta:|\theta|<\right.$ $10 c\}$, it suffices to prove that the segment $Z_{c, x, v_{1}+v(c)}\left(\left\{e^{i \theta}:|\theta|<10 c\right\}\right)$ is contained in the open side $\left(\widetilde{H}^{1}\right)^{+} \subset\left(H^{1}\right)^{+}$defined by the inequation (8.8), if the point $x+i h(x)$ belongs to the transverse half-submanifold $T^{1} \cap\left(H^{1}\right)^{+}$, namely if $x=\left(x_{1}, 0, \ldots, 0\right)$ with $x_{1}>0$. In the sequel, we shall denote
the disc $Z_{c, x, v_{1}+v(c)}^{1}(\zeta)$ by $Z_{c, x_{1}, x^{\prime}, v_{1}+v(c)}^{1}(\zeta)$, emphasizing the decomposition $x=\left(x_{1}, x^{\prime}\right) \in \mathbb{R} \times \mathbb{R}^{n-1}$, and we shall also use the convenient notation (8.16)
$Z_{c, x_{1}, x^{\prime}, v_{1}+v(c)}^{\prime 1}\left(\rho e^{i \theta}\right):=\left(Z_{2 ;,, x_{1}, x^{\prime}, v_{1}+v(c)}^{1}\left(\rho e^{i \theta}\right), \ldots \ldots, Z_{n ; c, x_{1}, x^{\prime}, v_{1}+v(c)}^{1}\left(\rho e^{i \theta}\right)\right)$.
So, we have to show that for all $c$ with $0<c \leqslant c_{1}$, all $x_{1}$ with $0<x_{1} \leqslant c^{2}$ and all $\theta$ with $|\theta|<10 c$, then the following strict inequality holds true:

$$
\begin{equation*}
X_{1 ;, x_{1}, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right)>\widetilde{g}\left(X_{c ; x_{1}, 0, v_{1}+v(c)}^{1^{\prime}}\left(e^{i \theta}\right)\right) . \tag{8.17}
\end{equation*}
$$

First of all, coming back to the family of discs attached to $M^{0}$, we see by
 Next, by differentiating twice Bishop's equation (7.18) with respect to $x_{1}$ and by reasoning as in Lemma 7.34, we get the estimate

$$
\begin{equation*}
\left\|\frac{\partial^{2} Z_{c, x_{1}, 0, v_{1}+v(c)}^{1}}{\partial x_{1}^{2}}\right\|_{\mathscr{C}_{\alpha}(\partial \Delta)} \leqslant c^{2+\alpha} \cdot K_{7}, \tag{8.18}
\end{equation*}
$$

with, say, the same constant $K_{7}>0$ as in Lemma 7.34, after enlarging it if necessary. Applying then Taylor's integral formula $F\left(x_{1}\right)=F(0)+x_{1}$. $\partial_{x_{1}} F(0)+\int_{0}^{x_{1}}\left(x_{1}-\widetilde{x}_{1}\right) \cdot \partial_{x_{1}} \partial_{x_{1}} F\left(\widetilde{x}_{1}\right) \cdot d \widetilde{x}_{1}$ to $F\left(x_{1}\right):=X_{1 ; c, x_{1}, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right)$, we deduce the minoration

$$
\begin{equation*}
X_{1 ; c, x_{1}, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right) \geqslant X_{1 ; c, 0,0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right)+x_{1} \cdot \frac{\partial X_{1 ; c, 0,0, v_{1}+v(c)}^{1}}{\partial x_{1}}\left(e^{i \theta}\right)-x_{1}^{2} \cdot c^{2+\alpha} \cdot \frac{K_{7}}{2} . \tag{8.19}
\end{equation*}
$$

On the other hand, by differentiating Bishop's equation (7.18) with respect to $x_{1}$ at $x=0$, the derivative $\partial_{x_{1}} x$ yields the vector $(1,0, \ldots, 0)$ and we obtain

$$
\begin{align*}
\frac{\partial X_{c, 0,0, v_{1}+v(c)}}{\partial x_{1}}\left(e^{i \theta}\right)= & -T_{1}\left[\sum_{l=1}^{n} \frac{\partial h}{\partial x_{l}}\left(X_{c, 0,0, v_{1}+v(c)}^{1}(\cdot)\right) \frac{\partial X_{l, c, 0,0, v_{1}+v(c)}^{1}}{\partial x_{1}}(\cdot)\right]\left(e^{i \theta}\right)+  \tag{8.20}\\
& +(1,0, \ldots, 0)
\end{align*}
$$

Using then (6.13) ${ }_{2}$ and (7.25), we deduce from (8.20)

$$
\begin{align*}
\left\|\frac{\partial X_{1 ; c, 0,0, v_{1}+v(c)}^{1}}{\partial x_{1}}(\cdot)-1\right\|_{\mathscr{C}^{\alpha}(\partial \Delta)} & \leqslant c^{2+\alpha} \cdot\left\|T_{1}\right\|_{\mathscr{C}^{\alpha}(\partial \Delta)} K_{2} K_{3}  \tag{8.21}\\
\left\|\frac{\partial X_{j ; c, 0,0, v_{1}+v(c)}^{1}}{\partial x_{1}}(\cdot)\right\|_{\mathscr{C}^{\alpha}(\partial \Delta)} & \leqslant c^{2+\alpha} \cdot\left\|T_{1}\right\|_{\mathscr{C}^{\alpha}(\partial \Delta)} K_{2} K_{3} .
\end{align*}
$$

Thanks to (8.21) ${ }_{1}$, we can work out the minoration (8.19) by replacing the first order partial derivative $\left.\frac{\partial X_{1 ;,, 0,0, v_{1}+v(c)}^{\partial x_{1}}}{\partial{ }^{i \theta}}\right)>0$ in the right hand side
of (8.19) by the constant 1 , and applying trivial minoration $-x_{1}^{2} \geqslant-x_{1}$, which yields:

$$
\begin{equation*}
X_{1 ; c, x_{1}, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right) \geqslant X_{1 ; c, 0,0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right)+x_{1}-x_{1} \cdot c^{2+\alpha} \cdot K_{11}, \tag{8.22}
\end{equation*}
$$

for some constant $K_{11}>0$. On the other hand, using the inequalities $\left|\partial_{x_{j}} \widetilde{g}\left(x^{\prime}\right)\right| \leqslant\left|x^{\prime}\right|+K_{9} \cdot\left|x^{\prime}\right|^{1+\alpha} \cdot\left(1+\frac{\alpha}{2}\right)(n-1)^{\frac{\alpha}{2}}$ for $j=2, \ldots, n$, using the estimate (7.25) and using (6.2), we deduce an inequality of the form

$$
\begin{equation*}
\widetilde{g}\left(X_{c, x_{1}, 0, v_{1}+v(c)}^{\prime 1}\left(e^{i \theta}\right)\right) \leqslant \widetilde{g}\left(X_{c, 0,0, v_{1}+v(c)}^{\prime}\left(e^{i \theta}\right)\right)+x_{1} \cdot c \cdot K_{12} \tag{8.23}
\end{equation*}
$$

for some constant $K_{12}>0$. Finally, putting together the two inequalities (8.22) and (8.23), and using the following immediate consequence of (8.9):

$$
\begin{equation*}
X_{1 ; c, 0,0, v_{1}+v(c)}\left(e^{i \theta}\right) \geqslant \widetilde{g}\left(X_{c, 0,0, v_{1}+v(c)}^{\prime 1}\left(e^{i \theta}\right)\right) \tag{8.24}
\end{equation*}
$$

valuable for all $\theta$ with $|\theta|<10 c$, we deduce the desired inequality (8.17):
(8.25)

$$
\left\{\begin{aligned}
X_{1 ; c, x_{1}, 0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right) & \geqslant X_{1 ; c, 0,0, v_{1}+v(c)}^{1}\left(e^{i \theta}\right)+x_{1}-x_{1} \cdot c^{2+\alpha} \cdot K_{11} \\
& \geqslant \widetilde{g}\left(X_{c, 0,0, v_{1}+v(c)}^{\prime}\left(e^{i \theta}\right)\right)+x_{1}-x_{1} \cdot c^{2+\alpha} \cdot K_{11} \\
& \geqslant \widetilde{g}\left(X_{c, x_{1}, 0, v_{1}+v(c)}^{\prime}\left(e^{i \theta}\right)\right)+x_{1}-x_{1} \cdot c \cdot K_{11}-x_{1} \cdot c \cdot K_{12} \\
& >\widetilde{g}\left(X_{c, x_{1}, 0, v_{1}+v(c)}^{\prime 1}\left(e^{i \theta}\right)\right)
\end{aligned}\right.
$$

for all $x_{1}$ with $0<x_{1} \leqslant c^{2}$, all $\theta$ with $|\theta|<10 c$ and all $c$ with $0<c \leqslant c_{1}$, provided

$$
\begin{equation*}
c_{1} \leqslant \frac{1 / 2}{K_{11}+K_{12}} \tag{8.26}
\end{equation*}
$$

This completes the proof of property ( $\mathbf{8}_{\mathbf{1}}$ ).
Thirdly, let us prove property $\left(\mathbf{9}_{1}\right)$ in Case ( $\mathbf{I}_{1}$ ). The half-wedge $\mathscr{H}^{\mathscr{W}}{ }_{1}^{+}$ is defined by the $n$ inequalities of the last two lines of (5.38), where $a_{2}+$ $\cdots+a_{n}=1$. For notational convenience, we set $a_{1}:=1$ and we write the first inequality defining $\mathscr{H} \mathscr{W}_{1}^{+}$simply as $\sum_{j=1}^{n} a_{j} y_{j}>\psi\left(x, y^{\prime}\right)$.

Because $\Phi_{c}\left(\bar{\Delta} \backslash \partial^{+} \Delta\right)$ is contained in the open sector $\left\{\rho e^{i \theta} \in \bar{\Delta}\right.$ : $|\theta|<10 c, 1-10 c<\rho<1\}$, taking account of the definition (7.54) of $A_{x, v: c}^{1}(\zeta)$, in order to check property $\left(9_{1}\right)$, it clearly suffices to show that $Z_{c, x, v_{1}+v(c)+v}^{1}\left(\left\{\rho e^{i \theta} \in \Delta: 1-10 c<\rho<1,|\theta|<10 c\right\}\right)$ is contained in $\mathscr{H} \mathscr{W}_{1}^{+}$, which amounts to establish that for all $x$ with $|x| \leqslant c^{2}$, all $v$ with $|v| \leqslant c$, all $\rho e^{i \theta}$ with $1-10 c<\rho<1$ and with $|\theta|<10 c$, the following two
collections of strict inequalities hold true
(8.27)

$$
\begin{aligned}
\sum_{k=1}^{n} a_{k} Y_{j ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right) & >\psi\left(X_{c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right), Y_{c, x, v_{1}+v(c)+v}^{\prime 1}\left(\rho e^{i \theta}\right)\right) \\
Y_{j ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right) & >\varphi_{j}\left(X_{c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right), Y_{1 ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right)\right)
\end{aligned}
$$

for $j=2, \ldots, n$, provided $c_{1}$ is sufficiently small, where we use the notation (8.16).

We first treat the collection of $(n-1)$ strict inequalities in the second line of (8.27). First of all, by differentiating (7.8) twice with respect to $\theta$, we obtain

$$
\begin{equation*}
\frac{\partial^{2} Z_{c, x, v_{1}+v(c)+v}^{0}}{\partial \theta^{2}}\left(e^{i \theta}\right)=c \cdot \frac{\partial^{2} \Psi}{\partial \theta^{2}}\left(e^{i \theta}\right) \cdot\left[v_{1}+v(c)+v\right] \tag{8.28}
\end{equation*}
$$

Using (7.36) $)_{2}$, we deduce that there exists a constant $K_{13}>0$ such that

$$
\begin{equation*}
\left|\frac{\partial^{2} Z_{c, x, v_{1}+v(c)+v}^{1}}{\partial \theta^{2}}\left(e^{i \theta}\right)\right| \leqslant c \cdot K_{13} \tag{8.29}
\end{equation*}
$$

Using the inequality (6.2), using (8.29), and then taking account of the inequalities $|\theta|<10 c,|x| \leqslant c^{2}$ and $|v|<c$, we deduce the inequality

$$
\begin{align*}
\left|\frac{\partial Z_{c, x, v_{1}+v(c)+v}^{1}}{\partial \theta}\left(e^{i \theta}\right)-\frac{\partial Z_{c, 0, v_{1}+v(c)}^{1}}{\partial \theta}(1)\right| & \leqslant c \cdot(|\theta|+|x|+|v|)  \tag{8.30}\\
& \leqslant c^{2} \cdot K_{14}
\end{align*}
$$

for some constant $K_{14}>0$. On the other hand, differentiating (7.8) with respect to $\theta$ at $\theta=0$ and applying the inequality (7.28), we obtain

$$
\begin{equation*}
\left|\frac{\partial Z_{c, 0, v_{1}+v(c)}^{1}}{\partial \theta}(1)-c \cdot C_{1} \cdot(0,1, \ldots, 1)\right| \leqslant c^{2+\alpha} \cdot K_{6} \tag{8.31}
\end{equation*}
$$

where $C_{1}=\frac{\partial \Psi}{\partial \theta}(1)$, as defined in (7.47). We remind that for every $\mathscr{C}^{1}$ function $Z$ on $\bar{\Delta}$ which is holomorphic in $\Delta$, we have $i \frac{\partial}{\partial \theta} Z\left(e^{i \theta}\right)=-\frac{\partial}{\partial \rho} Z\left(e^{i \theta}\right)$. Consequently, we deduce from (8.30) the following first (among three) interesting inequality

$$
\begin{equation*}
\left|-\frac{\partial Z_{c, x, v_{1}+v(c)+v}^{1}}{\partial \rho}\left(e^{i \theta}\right)-c \cdot C_{1} \cdot(0, i, \ldots, i)\right| \leqslant c^{2} \cdot K_{15} \tag{8.32}
\end{equation*}
$$

for some $K_{15}>0$. Next, according to the definition (7.8), we have

$$
\begin{equation*}
\frac{\partial^{2} Z_{c, x, v_{1}+v(c)+v}^{0}}{\partial \rho^{2}}\left(\rho e^{i \theta}\right)=c \cdot \frac{\partial^{2} \Psi}{\partial \rho^{2}}\left(\rho e^{i \theta}\right) \cdot\left(v_{1}+v(c)+v\right) \tag{8.33}
\end{equation*}
$$

Reasoning as in the proof of Lemma 7.34, we may obtain an inequality similar to (7.36), with the second order partial derivative $\partial^{2} / \partial \theta^{2}$ replaced by
the second order partial derivative $\partial^{2} / \partial \rho^{2}$. Putting this together with (8.33), we deduce that there exists a constant $K_{16}>0$ such that

$$
\begin{equation*}
\left|\frac{\partial^{2} Z_{c, x, v_{1}+v(c)+v}^{1}}{\partial \rho^{2}}\left(\rho e^{i \theta}\right)\right| \leqslant c \cdot 2 K_{16} . \tag{8.34}
\end{equation*}
$$

Applying then Taylor's integral formula $F(\rho)=F(1)+(\rho-1) \cdot \partial_{\rho} F(1)+$ $\int_{1}^{\rho}(\rho-\widetilde{\rho}) \cdot \partial_{\rho} \partial_{\rho} F(\widetilde{\rho}) \cdot d \widetilde{\rho}$ to the functions $F(\rho):=Y_{k ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right)$ for $k=1, \ldots, n$, we deduce the second interesting collection of inequalities, for $k=1, \ldots, n$ :

$$
\begin{align*}
\mid Y_{k ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right) & -Y_{k ; c, x, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right)-  \tag{8.35}\\
& \left.-(\rho-1) \cdot \frac{\partial Y_{k ; c, x, v_{1}+v(c)+v}^{1}}{\partial \rho}\left(e^{i \theta}\right) \right\rvert\, \leqslant(1-\rho)^{2} \cdot c \cdot K_{16}
\end{align*}
$$

On the other hand, thanks to the normalizations of the functions $\varphi_{j}\left(x, y_{1}\right)$ given in (5.40), we get (increasing possibly $K_{1}>0$ ) two inequalities:
(8.36)

$$
\begin{aligned}
& \sum_{k=1}^{n}\left|\varphi_{j, x_{k}}\left(x, y_{1}\right)\right|+\left|\varphi_{j, y_{1}}\left(x, y_{1}\right)\right| \leqslant\left(|x|+\left|y_{1}\right|\right) \cdot K_{1} \\
& \left|\varphi_{j}\left(x, y_{1}\right)-\varphi_{j}\left(\widetilde{x}, \widetilde{y}_{1}\right)\right| \leqslant\left(|x-\widetilde{x}|+\left|y_{1}-\widetilde{y}_{1}\right|\right) \cdot\left(\sum_{k=1}^{n} \sup _{|x|,\left|y_{1}\right| \leqslant c \cdot K_{2}}\left|\varphi_{j, x_{k}}\left(x, y_{1}\right)\right|+\right. \\
& \left.\quad+\sup _{|x|,\left|y_{1}\right| \leqslant c \cdot K_{2}}\left|\varphi_{j, y_{1}}\left(x, y_{1}\right)\right|\right)
\end{aligned}
$$

for $j=2, \ldots, n$, provided $|x|,|\widetilde{x}|,\left|y_{1}\right|,\left|\widetilde{y_{1}}\right| \leqslant c \cdot K_{2}$. On the other hand, computing $\frac{\partial Z_{c,,, v_{1}+\nu(c)+v}^{0}}{\partial \rho}\left(\rho e^{i \theta}\right)$ in (7.8), using (7.25), (7.28) and an inequality of the form (6.2), we deduce that there exists a constant $K_{17}>0$ such that

$$
\begin{equation*}
\left|Z_{c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right)-Z_{c, x, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right)\right| \leqslant(1-\rho) \cdot c \cdot K_{17} . \tag{8.37}
\end{equation*}
$$

Finally, using the inequality $\left|Z_{c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right)\right| \leqslant c \cdot K_{2}$ obtained in (7.25), using the collection of inequalities (8.36) and using the inequality (8.37), we
may deduce the third (and last) interesting inequality for $j=2, \ldots, n$ :
(8.38)

$$
\begin{aligned}
& \mid \varphi_{j}\left(X_{c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right), Y_{1 ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right)\right)- \\
& \quad-\varphi_{j}\left(X_{c, x, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right), Y_{1 ; c, x, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right)\right) \mid \leqslant \\
& \left(\left|X_{c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right)-X_{c, x, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right)\right|+\right. \\
& \left.\quad+\left|Y_{1 ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right)-Y_{1 ; c, x, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right)\right|\right) . \\
& \quad \cdot\left(\sum_{k=1}^{n} \sup _{|x|,\left|y_{1}\right| \leqslant c \cdot K_{2}}\left|\varphi_{j, x_{k}}\left(x, y_{1}\right)\right|+\sup _{|x|,\left|y_{1}\right| \leqslant c \cdot K_{2}}\left|\varphi_{j, y_{1}}\left(x, y_{1}\right)\right|\right) \leqslant \\
& \leqslant(1-\rho) \cdot c^{2} \cdot K_{18},
\end{aligned}
$$

for some constant $K_{18}>0$.
We can now complete the proof of the collection of inequalities in the second line of (8.27). As before, let $c$ with $0<c \leqslant c_{1}$, let $\rho$ with $10 c<\rho<1$, let $\theta$ with $|\theta|<10 c$, let $x$ with $|x| \leqslant c^{2}$, let $v$ with $|v| \leqslant c$ and let $j=2, \ldots, n$. Starting with (8.35), using (8.32), using the fact that $Z_{c, x, v_{1}+v(c)+v}^{1}\left(\partial^{+} \Delta\right) \subset M^{1} \subset M$ and using (8.38), we have

$$
Y_{j ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right) \geqslant
$$

$$
\geqslant Y_{j ; c, c, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right)+(\rho-1) \cdot \frac{\partial Y_{j ; c, x, v_{1}+v(c)+v}^{1}}{\partial \rho}\left(e^{i \theta}\right)-(1-\rho)^{2} \cdot c \cdot K_{16} \geqslant
$$

$$
\geqslant Y_{j ; c, x, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right)+(1-\rho) \cdot c \cdot C_{1}-(1-\rho) \cdot c^{2} \cdot K_{15}-(1-\rho)^{2} \cdot c \cdot K_{16}
$$

$$
=\varphi_{j}\left(X_{1 ; c, x, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right), Y_{1 ; c, x, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right)\right)+(1-\rho) \cdot c \cdot C_{1}-
$$

$$
\begin{equation*}
-(1-\rho) \cdot c^{2} \cdot K_{15}-(1-\rho)^{2} \cdot c \cdot K_{16} \tag{8.39}
\end{equation*}
$$

$$
\begin{aligned}
& \geqslant \varphi_{j}\left(X_{1 ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right), Y_{1 ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right)\right)+(1-\rho) \cdot c \cdot C_{1}- \\
& \quad-(1-\rho) \cdot c^{2} \cdot K_{15}-(1-\rho)^{2} \cdot c \cdot K_{16}-(1-\rho) \cdot c^{2} \cdot K_{18} \\
& \geqslant \varphi_{j}\left(X_{1 ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right), Y_{1 ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right)\right)+(1-\rho) \cdot c \cdot\left[C_{1}-\right. \\
& \left.\quad-c \cdot K_{15}-10 c \cdot K_{16}-c \cdot K_{18}\right] \\
& \geqslant \varphi_{j}\left(X_{1 ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right), Y_{1 ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right)\right)+(1-\rho) \cdot c \cdot \frac{C_{1}}{2} \\
& >\varphi_{j}\left(X_{1 ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right), Y_{1 ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right)\right),
\end{aligned}
$$

provided that

$$
\begin{equation*}
c_{1} \leqslant \frac{C_{1} / 2}{K_{15}+10 K_{16}+K_{18}} . \tag{8.40}
\end{equation*}
$$

This yields the collection of inequalities in the second line of (8.27).
For the first inequality (8.27), we proceed similarly. Recall that $v_{1}=(0,1, \ldots, 1)$, that $a_{1}=1$ and that $a_{2}+\cdots+a_{n}=1$. Since $Z_{c, x, v_{1}+v(c)+v}^{1}\left(\partial^{+} \Delta\right) \subset M^{1} \subset N^{1}$, we have for all $\theta$ with $|\theta| \leqslant \frac{\pi}{2}$ the following relation
(8.41)

$$
\sum_{k=1}^{n} a_{k} Y_{k ; c, x, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right)=\psi\left(X_{c, x, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right), Y_{c, x, v_{1}+v(c)+v}^{\prime 1}\left(e^{i \theta}\right)\right) .
$$

Using that $\psi$ vanishes to order one at the origin by the normalization conditions (5.40) and proceeding as in the previous paragraph concerning the functions $\varphi_{j}$, we obtain an inequality similar to (8.38):
(8.42)

$$
\begin{aligned}
& \mid \psi\left(X_{c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right), Y_{c, x, v_{1}+v(c)+v}^{\prime}\left(\rho e^{i \theta}\right)\right)- \\
& \quad-\psi\left(X_{c, x, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right), Y_{c, x, v_{1}+v(c)+v}^{\prime 1}\left(e^{i \theta}\right)\right) \mid \leqslant(1-\rho) \cdot c^{2} \cdot K_{19},
\end{aligned}
$$

for some constant $K_{19}>0$.
As before, let $c$ with $0<c \leqslant c_{1}$, let $\rho$ with $10 c<\rho<1$, let $\theta$ with $|\theta|<10 c$, let $x$ with $|x| \leqslant c^{2}$ and let $v$ with $|v| \leqslant c$. Using then (8.35), (8.32), (8.41) and (8.42), we deduce the desired strict inequality

$$
\begin{aligned}
& \sum_{k=1}^{n} a_{k} Y_{k ; c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right) \geqslant \sum_{k=1}^{n} a_{k} Y_{k ; c, c, v, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right)+ \\
& \quad+(1-\rho)\left[\sum_{k=1}^{n} a_{k}\left(-\frac{\partial Y_{k ; c, x, v_{1}+v(c)+v}^{1}}{\partial \rho}\left(e^{i \theta}\right)\right)\right]-(1-\rho)^{2} \cdot c \cdot\left(\sum_{k=1}^{n} a_{k}\right) K_{16} \\
& \quad \geqslant \sum_{k=1}^{n} a_{k} Y_{k ; c, x, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right)+(1-\rho)\left[\sum_{j=2}^{n} a_{j} \cdot c \cdot C_{1}-\sum_{k=1}^{n} a_{k} \cdot c^{2} \cdot K_{15}\right]-
\end{aligned}
$$

$$
\begin{gather*}
-(1-\rho)^{2} \cdot c \cdot 2 K_{16}  \tag{8.43}\\
\geqslant \sum_{k=1}^{n} a_{k} Y_{k ; c, x, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right)+(1-\rho) \cdot c \cdot C_{1}- \\
-(1-\rho) \cdot c^{2} \cdot 2 K_{15}-(1-\rho)^{2} \cdot c \cdot 2 K_{16} \\
=\psi\left(X_{c, x, v_{1}+v(c)+v}^{1}\left(e^{i \theta}\right), Y_{c, x, v_{1}+v(c)+v}^{\prime}\left(e^{i \theta}\right)\right)+ \\
\quad+(1-\rho) \cdot c \cdot\left[C_{1}-c \cdot 2 K_{15}-10 c \cdot 2 K_{16}\right]
\end{gather*}
$$

$$
\begin{aligned}
& \geqslant \psi\left(X_{c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right), Y_{c, x, v_{1}+v(c)+v}^{\prime \prime}\left(\rho e^{i \theta}\right)\right)+ \\
& \quad+(1-\rho) \cdot c \cdot\left[C_{1}-c \cdot 2 K_{15}-10 c \cdot 2 K_{16}-c \cdot K_{19}\right] \\
& \geqslant \psi\left(X_{c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right), Y_{c, x, v_{1}+v(c)+v}^{\prime 1}\left(\rho e^{i \theta}\right)\right)+(1-\rho) \cdot c \cdot \frac{C_{1}}{2} \\
& >\psi\left(X_{c, x, v_{1}+v(c)+v}^{1}\left(\rho e^{i \theta}\right), Y_{c, x, v_{1}+v(c)+v}^{\prime}\left(\rho e^{i \theta}\right)\right)
\end{aligned}
$$

provided $c_{1} \leqslant \frac{C_{1} / 2}{2 K_{15}+20 K_{16}+K_{19}}$. This yields the first inequality of (8.27) and completes the proof of $\left(\mathbf{9}_{\mathbf{1}}\right)$ in Case ( $\left.\mathbf{I}_{\mathbf{1}}\right)$.

## §9. END OF PROOF OF PROPOSITION 1.13: APPLICATION OF THE CONTINUITY PRINCIPLE

9.1. Preliminary. In this section, we complete the proof of Proposition 5.12, hence the proof of Theorem 3.19, hence also the proof of the main Proposition 1.13 (at last!).

Translating $M^{1}$ inside $M$, we will introduce a supplementary small real parameter $u$, getting a family $A_{x, v, u: c}^{1}(\zeta)$ of analytic discs partially attached to the translate $M_{u}^{1}$. Applying the continuity principe to this family of discs, we shall show that, in Cases $\left(\mathbf{I}_{1}\right)$ and $\left(\mathbf{I}_{2}\right)$, there exists a local wedge $\mathscr{W}_{p_{1}}$ of edge $M$ at $p_{1}$ to which $\mathscr{O}\left(\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}\right)$extends holomorphically; in Case (II), there will exist a whole (small) neighborhood $\omega_{p_{1}}$ of $p_{1}$ in $\mathbb{C}^{n}$ to which $\mathscr{O}\left(\Omega \cup \mathscr{W}_{2}\right)$ extends holomorphically. To organize well this last main step of the proof of Proposition 5.12, we shall consider jointly Cases $\left(\mathbf{I}_{1}\right),\left(\mathbf{I}_{2}\right)$ and then afterwards Case (II) separately.
9.2. Translations of $M^{1}$ in $M$. According to Lemma 5.37, in Case ( $\mathbf{I}_{\mathbf{1}}$ ), the one-codimensional submanifold $M^{1} \subset M$ is given by the equations $y^{\prime}=\varphi^{\prime}\left(x, y_{1}\right)$ and $x_{1}=g\left(x^{\prime}\right)$. If $u \in \mathbb{R}$ is a small real parameter, we may define a "translation" $M_{u}^{1}$ of $M^{1}$ in $M$ by the $n$ equations

$$
\begin{equation*}
M_{u}^{1}: \quad y^{\prime}=\varphi^{\prime}\left(x, y_{1}\right), \quad x_{1}=g\left(x^{\prime}\right)+u \tag{9.3}
\end{equation*}
$$

Clearly, we have $M_{u}^{1} \subset\left(M^{1}\right)^{+}$if $u>0$ and $M_{u}^{1} \subset\left(M^{1}\right)^{-}$if $u<0$. We may perturb the family of analytic discs $Z_{c, x, v}^{d}(\zeta)$ half-attached to $M^{1}$ satisfying Bishop's equation (7.18) by requiring that it is attached to $M_{u}^{1}$. Thanks to the stability under perturbation of the solutions to Bishop's equation, we then obtain a new family of analytic discs $Z_{c, x, v, u}^{d}(\zeta)$ which is halfattached to $M_{u}^{1}$ and which is of class $\mathscr{C}^{2, \alpha-0}$ with respect to all variables $(c, x, v, u, \zeta)$. For $u=0$, this solution coincides with the family $Z_{c, x, v}^{d}(\zeta)$ constructed in $\S 7.13$. Using a similar definition as in (7.54), namely setting $A_{x, v, u: c}^{1}(\zeta):=Z_{c, x, v_{1}+v(c)+v, u}^{1}\left(\Phi_{c}(\zeta)\right)$, we obtain a new family of analytic discs which coincides, for $u=0$, with the family $A_{x, v: c}^{1}(\zeta)$ of Lemmas 7.12 and 8.3.

In Case ( $\mathbf{I}_{\mathbf{2}}$ ), taking account of the normalizations stated in Lemma 5.37, we may also construct a similar family of analytic discs $A_{x, v, u: c}^{1}(\zeta)$. From now on, we fix the scaling parameter $c$ with $0<c \leqslant c_{1}$, so that the nine properties $\left(\mathbf{1}_{\mathbf{1}}\right)$ to $\left(\mathbf{9}_{\mathbf{1}}\right)$ of Lemmas 7.12 and 8.3 are satisfied by $A_{x, v, 0: c}^{1}(\zeta)$.
9.4. Definition of a local wedge of edge $M$ at $p_{1}$ in Cases $\left(\mathbf{I}_{1}\right)$ and ( $\left.\mathbf{I}_{2}\right)$. First of all, in Cases $\left(\mathbf{I}_{1}\right)$ and $\left(\mathbf{I}_{2}\right)$, we restrict the variation of the parameter $v$ to a certain $(n-2)$-dimensional linear subspace $V_{2}$ of $T_{p_{1}} \mathbb{R}^{n} \simeq \mathbb{R}^{n}$ as follows. By hypothesis, the vector $v_{1}$ does not belong to the characteristic direction $T_{p_{1}} M^{1} \cap T_{p_{1}}^{c} M$, so the real vector space $\left(\mathbb{R} \cdot v_{1}\right) \oplus$ $\left(T_{p_{1}} M^{1} \cap T_{p_{1}}^{c} M\right) \subset T_{p_{1}} M^{1}$ is 2-dimensional. We choose an arbitrary ( $n-2$ )-dimensional real vector subspace $V_{2} \subset T_{p_{1}} M^{1}$ which is a supplementary in $T_{p_{1}} M^{1}$ to $\left(\mathbb{R} \cdot v_{1}\right) \oplus\left(T_{p_{1}} M^{1} \cap T_{p_{1}}^{c} M\right)$ and we shall let the parameter $v$ vary only in $V_{2}$. Also, we choose a local $(n-1)$-dimensional submanifold $X_{1} \subset M^{1}$ passing through $p_{1}$ with $\mathbb{R} \cdot v_{1} \oplus T_{p_{1}} X_{1}=T_{p_{1}} M^{1}$.

From the rank properties $\left(5_{1}\right)$ and $\left(6_{1}\right)$ of Lemma 7.12 and from the definitions of $V_{2}$ and of $X_{1}$, it may then be verifed (as in [MP1999, MP2002]) that, for $\varepsilon>0$ small enough with $\varepsilon \ll c^{2}$, the mapping

$$
\begin{equation*}
(x, v, u, \rho, \theta) \longmapsto A_{x, v, u: c}^{1}\left(\rho e^{i \theta}\right) \tag{9.5}
\end{equation*}
$$

is a one-to-one immersion ${ }^{37}$ from the open set $\left\{(x, v, u, \rho) \in X_{1} \times V_{2} \times \mathbb{R} \times\right.$ $\mathbb{R} \times \mathbb{R}:|x|<\varepsilon,|v|<\varepsilon,|u|<\varepsilon, 1-\varepsilon<\rho<1,|\theta|<\varepsilon\}$ onto its image (9.6)

$$
\begin{aligned}
& \mathscr{W}_{p_{1}}:=\left\{A_{x, v, u: c}^{1}\left(\rho e^{i \theta}\right) \in \mathbb{C}^{n}:(x, v, u, \rho, \theta) \in X_{1} \times V_{2} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\right. \\
&|x|<\varepsilon,|v|<\varepsilon,|u|<\varepsilon, 1-\varepsilon<\rho<1,|\theta|<\varepsilon\}
\end{aligned}
$$

which is a local wedge of edge $M$ at $\left(p_{1}, J v_{1}\right)$, with $\mathscr{W}_{p_{1}} \cap M=\emptyset$.
Let the singularity $C$ with $p_{1} \in C$ and $C \backslash\left\{p_{1}\right\} \subset\left(H^{1}\right)^{-}$, let the neighborhood $\Omega$ of $M \backslash C$ in $\mathbb{C}^{n}$, let the half-wedge $\mathscr{H} \mathscr{W}_{p_{1}}^{+}$be as in Proposition 5.12, and let the sub-half-wedge $\mathscr{H} \mathscr{W}_{1}^{+} \subset \mathscr{H} \mathscr{W}_{p_{1}}^{+}$be as in $\S 5.14$ and Lemma 5.37. In Cases ( $\mathbf{I}_{\mathbf{1}}$ ) and ( $\mathbf{I}_{\mathbf{2}}$ ), we shall prove that a (sufficiently thick) part of the envelope of holomorphy of $\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}$contains the wedge $\mathscr{W}_{p_{1}}$ and is schlicht over it.
9.7. Boundaries of analytic discs. Since we want to apply the continuity principle, we must verify that most discs $A_{x, v, u: c}^{1}(\zeta)$ have their boundaries in $\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}$. To this aim, we decompose the boundary $\partial \Delta$ in three closed

[^36]parts $\partial \Delta=\partial^{1} \Delta \cup \partial^{2} \Delta \cup \partial^{3} \Delta$, where
\[

\left\{$$
\begin{array}{l}
\partial^{1} \Delta:=\left\{e^{i \theta} \in \partial \Delta:|\theta| \leqslant \pi / 2-\varepsilon\right\} \subset \partial^{+} \Delta  \tag{9.8}\\
\partial^{2} \Delta:=\left\{e^{i \theta} \in \partial \Delta: \pi / 2+\varepsilon \leqslant|\theta| \leqslant \pi\right\} \subset \partial^{-} \Delta \\
\partial^{3} \Delta:=\left\{e^{i \theta} \in \partial \Delta: \pi / 2-\varepsilon \leqslant|\theta| \leqslant \pi / 2+\varepsilon\right\} \subset \partial \Delta
\end{array}
$$\right.
\]

where $\varepsilon$ with $0<\varepsilon \ll c^{2}$ is as in $\S 9.4$ just above. This decomposition is illustrated in the left Figure 19 below. Next, we observe that the two points $A_{0,0,0: c}^{1}(i)$ and $A_{0,0,0: c}^{1}(-i)$ belong to $\left(H^{1}\right)^{+} \subset M \backslash C \subset \Omega$, hence there exists a fixed open neighborhood of these two points which is contained in $\Omega$. We shall denote by $\omega^{3}$ such a (disconnected) neighborhood, for instance the union of two small open polydiscs centered at these two points. To proceed further, we need a crucial geometric information about the boundaries of the analytic discs $A_{x, v, u: c}^{1}(\zeta)$ with $u \neq 0$.

Lemma 9.9. In Cases $\left(\mathbf{I}_{\mathbf{1}}\right)$ and $\left(\mathbf{I}_{\mathbf{2}}\right)$, after shrinking $\varepsilon>0$ if necessary, then

$$
\begin{equation*}
A_{x, v, u: c}^{1}(\partial \Delta) \subset \Omega \cup \mathscr{H} \mathscr{W}_{1}^{+} \tag{9.10}
\end{equation*}
$$

for all $x$ with $|x|<\varepsilon$, for all $v$ with $|v|<\varepsilon$ and for all nonzero $u \neq 0$ with $|u|<\varepsilon$.

Proof. Firstly, since $A_{0,0,0: c}^{1}( \pm i) \in \omega^{3}$, it follows just by continuity of the family $A_{x, v, u: c}^{1}(\zeta)$ that, after possibly shrinking $\varepsilon>0$, the closed arc $A_{x, v, u: c}^{1}\left(\partial^{3} \Delta\right)$ is contained in $\omega^{3}$, for all $x$ with $|x|<\varepsilon$, for all $v$ with $|v|<\varepsilon$ and for all $u$ with $|u|<\varepsilon$. Secondly, since $A_{0,0,0: c}^{1}\left(\partial^{2} \Delta\right) \subset$ $A_{0,0,0: c}^{1}\left(\partial^{-} \Delta \backslash\{i,-i\}\right) \subset \mathscr{H} \mathscr{W}_{1}^{+}$, then by property $\left(\mathbf{9}_{1}\right)$ of Lemma 8.3 , it follows just thanks to continuity of the family $A_{x, v, u: c}^{1}(\zeta)$ that, after possibly shrinking $\varepsilon>0$, the closed arc $A_{x, v, u: c}^{1}\left(\partial^{2} \Delta\right)$ is contained in $\mathscr{H} \mathscr{W}_{1}^{+}$, for all $x$ with $|x|<\varepsilon$, for all $v$ with $|v|<\varepsilon$ and for all $u$ with $|u|<\varepsilon$. Thirdly, it follows from the inclusion $A_{x, v, u: c}^{1}\left(\partial^{1} \Delta\right) \subset A_{x, v, u: c}^{1}\left(\partial^{+} \Delta\right) \subset M_{u}^{1}$ and from the inclusion $M_{u}^{1} \subset \Omega$ for all $u \neq 0$ that, after possibly shrinking $\varepsilon>0$, the closed arc $A_{x, v, u: c}^{1}\left(\partial^{1} \Delta\right)$ is contained in $\Omega$, for all $x$ with $|x|<\varepsilon$, for all $v$ with $|v|<\varepsilon$ and for all $u$ with $|u|<\varepsilon$ and $u \neq 0$.


Fig. 19: Decomposition of $\partial \Delta$ and isotopies of the analytic dises $A_{x, v, u: c}(\zeta)$
9.11. Analytic isotopies. Following [Me1997], two analytic discs $A^{\prime}, A^{\prime \prime} \in$ $\mathscr{O}\left(\Delta, \mathbb{C}^{n}\right) \cap \mathscr{C}^{1}(\bar{\Delta})$ which are both embeddings of $\bar{\Delta}$ into $\mathbb{C}^{n}$ are said to be analytically isotopic if there exists a $\mathscr{C}^{1}$ family of embedded analytic discs $A_{\tau} \in \mathscr{O}\left(\Delta, \mathbb{C}^{n}\right) \cap \mathscr{C}^{1}(\bar{\Delta}), \tau \in[0,1]$, with $A_{0}=A^{\prime}$ and $A_{1}=A^{\prime \prime}$. If $\mathscr{D} \subset \mathbb{C}^{n}$ is a domain, a disc $A^{\prime}$ is analytically isotopic to a point with boundaries inside $\mathscr{D}$ if $A^{\prime \prime}(\bar{\Delta}) \equiv p^{\prime \prime} \in \mathscr{D}$ is a constant disc, if each $A_{\tau}$ is embedded, for $0 \leqslant \tau<1$, and if $A_{\tau}(\partial \Delta) \subset \mathscr{D}$ for $0 \leqslant \tau \leqslant 1$.

In Case ( $\mathbf{I}_{1}$ ), we fix some $x_{0}=\left(x_{1 ; 0}, 0, \ldots, 0\right) \in \mathbb{R}^{n}$ with $0<x_{1 ; 0}<\varepsilon$. Then $A_{x_{0}, 0,0: c}^{1}(1)=x_{0}+i h\left(x_{0}\right)$ belongs to $T^{1} \cap\left(H^{1}\right)^{+}$. Analogously, in Cases ( $\mathbf{I}_{\mathbf{2}}$ ), we fix some $x_{0}=\left(0, \ldots, 0, x_{n ; 0}\right) \in \mathbb{R}^{n}$ with $0<x_{n ; 0}<\varepsilon$. Then in this second case, the point $A_{x_{0}, 0,0: c}^{1}(1)=x_{0}+i h\left(x_{0}\right)$ also belongs to $T^{1} \cap\left(H^{1}\right)^{+}$. We fix this reference disc $A_{x_{0}, 0,0: c}^{1}(\zeta)$, which satisfies $A_{x_{0}, 0,0: c}\left(\partial^{+} \Delta\right) \subset\left(H^{1}\right)^{+}$.
Lemma 9.12. In Cases $\left(\mathbf{I}_{\mathbf{1}}\right)$ and $\left(\mathbf{I}_{\mathbf{2}}\right)$, every disc $A_{x, v, u: c}^{1}(\zeta)$ with $|x|<\varepsilon$, $|v|<\varepsilon,|u|<\varepsilon$ and $u \neq 0$ is analytically isotopic to the disc $A_{x_{0}, 0,0: c}^{1}(\zeta)$, with the boundaries of the analytic discs of the isotopy being all contained in $\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}$. The same property is enjoyed by every disc $A_{x, v, 0: c}^{1}$ such that $A_{x, v, 0: c}^{1}\left(\partial^{+} \Delta\right) \subset\left(H^{1}\right)^{+}$.

Furthermore, $A_{x_{0}, 0,0: c}^{1}(\bar{\Delta}) \subset \Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}$, hence $A_{x_{0,0,0: c}}^{1}$ is analytically isotopic to a point with boundaries inside $\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}$(just shrink its radius).

Consequently, all discs $A_{x, v, u: c}^{1}$ with $u \neq 0$ and all discs $A_{x, v, 0: c}^{1}$ with $A_{x, v, 0: c}^{1}\left(\partial^{+} \Delta\right) \subset\left(H^{1}\right)^{+}$are analytically isotopic to a point with boundaries inside $\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}$.
Proof. Since $\{u=0\}$ is a hyperplane of the whole parameter space, there exists a $\mathscr{C}^{2, \alpha-0}$ curve $\tau \mapsto(x(\tau), v(\tau), u(\tau))$ in the parameter space which joins a given arbitrary point $\left(x^{*}, v^{*}, u^{*}\right)$ with $u^{*} \neq 0$ to the point $\left(x_{0}, 0,0\right)$ without meeting the hyperplane $\{u=0\}$, except at its endpoint ( $x_{0}, 0,0$ ). According to the previous Lemma 9.9, each boundary $A_{x(\tau), v(\tau), u(\tau): c}^{1}(\partial \Delta)$ is then automatically contained in $\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}$.

Also, if $A_{x, v, 0: c}^{1}\left(\partial^{+} \Delta\right) \subset\left(H^{1}\right)^{+}$, whence in particular $A_{x, v, 0: c}^{1}(1)=$ $x+i h(x) \in\left(H^{1}\right)^{+}$, we first isotope $A_{x, v, 0: c}^{1}$ to $A_{x, 0,0: c}^{1}$ just by moving in the $v$-parameter space along straight segment $[0, v]$. Thanks to the strong convexity of $\left(H^{1}\right)^{-}$and to the almost straighteness of the half boundaries (Figure 18) which rotate slightly as $v^{\prime} \in[0, v]$ varies, the half boundary $A_{x, v^{\prime}, 0: c}^{1}\left(\partial^{+} \Delta\right)$ stays in $\left(H^{1}\right)^{+}$, while the remainder part of the boundary $A_{x, v^{\prime}, 0: c}^{1}\left(\partial^{-} \Delta \backslash\{ \pm i\}\right)$ stays in $\mathscr{H}^{\mathscr{W}} \mathscr{W}_{1}^{+}$. Then $A_{x, 0,0: c}^{1}$ is trivially isotopic to $A_{x_{0}, 0,0: c}^{1}$.

Finally, since $A_{x_{0}, 0,0: c}^{1}(1)$ belongs to $T^{1} \cap\left(H^{1}\right)^{+}$, property $\left(\mathbf{8}_{\mathbf{1}}\right)$ of Lemma 8.3 insures that $A_{x_{0}, 0,0: c}^{1}\left(\partial^{+} \Delta\right)$ is contained in $\left(H^{1}\right)^{+}$, hence in $\Omega$. Then property $\left(\mathbf{9}_{1}\right)$ says that $A_{x_{0}, 0,0: c}^{1}(1)\left(\bar{\Delta} \backslash \partial^{+} \Delta\right)$ is contained in $\mathscr{H}^{W} \mathscr{W}_{1}^{+}$, which completes the proof.
9.13. holomorphic extension to a local wedge of edge $M$ at $p_{1}$. In Cases ( $\mathbf{I}_{1}$ ) and ( $\mathbf{I}_{\mathbf{2}}$ ), we define a $\mathscr{C}^{2, \alpha-0}$ connected hypersurface of $\mathscr{W}_{p_{1}}$ :

$$
\begin{array}{r}
\mathscr{M}_{p_{1}}:=\left\{A_{x, v, 0: c}^{1}\left(\rho e^{i \theta}\right):(x, v, \rho, \theta) \in X_{1} \times V_{2} \times \mathbb{R} \times \mathbb{R},\right.  \tag{9.14}\\
|x|<\varepsilon,|v|<\varepsilon, 1-\varepsilon<\rho<\varepsilon,|\theta|<\varepsilon\},
\end{array}
$$

together with a proper subset of $\mathscr{M}_{p_{1}}$ :
(9.15)

$$
\begin{aligned}
\mathscr{C}_{p_{1}}:=\{ & A_{x, v, 0: c}^{1}\left(\rho e^{i \theta}\right):(x, v, \rho, \theta) \in \mathbb{R}^{n} \times V_{2} \times \mathbb{R} \times \mathbb{R}, \\
& \left.A_{x, v, 0: c}^{1}\left(\partial^{+} \Delta\right) \not \subset\left(H^{1}\right)^{+},|x|<\varepsilon,|v|<\varepsilon, 1-\varepsilon<\rho<\varepsilon,|\theta|<\varepsilon\right\} .
\end{aligned}
$$



Fig. 20: The proper closed subset $\mathscr{C}_{p_{1}}$ of the hypersurface $\mathscr{M}_{p_{1}} \subset \mathscr{W}_{p_{1}}$
We can now state the main lemma of this section, completing the proof of Proposition 5.12.

Lemma 9.16. In Cases $\left(\mathbf{I}_{1}\right)$ and $\left(\mathbf{I}_{\mathbf{2}}\right)$, after possibly shrinking $\Omega$ in a small neighborhood of $p_{1}$ and after possibly shrinking $\varepsilon>0$, the set $\mathscr{W}_{p_{1}} \cap\left[\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}\right]$is connected and for every holomorphic function $f \in$
$\mathscr{O}\left(\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}\right)$, there exists a holomorphic function $F \in \mathscr{O}\left(\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+} \cup\right.$ $\left.\mathscr{W}_{p_{1}}\right)$ such that $\left.F\right|_{\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}}=f$.

Proof. Remind that $\varepsilon \ll c^{2}$ and remind that the wedge $\mathscr{W}_{p_{1}}$ with $\mathscr{W}_{p_{1}} \cap M=$ $\emptyset$ in the two cases is of size $\mathrm{O}(\varepsilon)$. Since the singularity $C$ is contained in $\left(H^{1}\right)^{-} \cup\left\{p_{1}\right\} \subset M^{1}$, its complement $M \backslash C$ is locally connected near $p_{1}$. The half-wedge $\mathscr{H} \mathscr{W}_{1}^{+}$defined in Lemma 5.37 by simple inequalities is of size $\mathrm{O}\left(\delta_{1}\right)$. If $\varepsilon \ll \delta_{1}$, after shrinking $\Omega$ if necessary in a smal neighborhood of $p_{1}$ whose size is $\mathrm{O}(\varepsilon)$, it follows that we can assume that $\mathscr{W}_{p_{1}} \cap\left[\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}\right]$ is connected.

Let $f$ be an arbitrary holomorphic function in $\mathscr{O}\left(\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}\right)$. Thanks to the isotopy Lemma 9.12, applying the continuity principle ([Me1997]), we deduce that $f$ extends holomorphically to a (very, very thin) neighborhood in $\mathbb{C}^{n}$ of every disc $A_{x, v, u: c}^{1}(\bar{\Delta})$ with $u \neq 0$ and also, to neighborhood in $\mathbb{C}^{n}$ of every disc $A_{x, v, 0: c}^{1}(\bar{\Delta})$ such that $A_{x, v, 0: c}^{1}\left(\partial^{+} \Delta\right) \subset\left(H^{1}\right)^{+}$.

Using the fact that the mapping (9.5) is one-to-one onto $\mathscr{W}_{p_{1}}$, we deduce that $f$ extends uniquely at all such points $A_{x, v, u: c}^{1}\left(\rho e^{i \theta}\right) \in \mathscr{W}_{p_{1}}$ simply by means of Cauchy's formula:

$$
\begin{equation*}
f\left(A_{x, v, u: c}^{1}\left(\rho e^{i \theta}\right)\right):=\int_{\partial \Delta} \frac{f\left(A_{x, v, u: c}^{1}(\widetilde{\zeta})\right)}{\widetilde{\zeta}-\rho e^{i \theta}} d \widetilde{\zeta} . \tag{9.17}
\end{equation*}
$$

Consequently, $f$ extends holomorphically and uniquely to the domain $\mathscr{W}_{p_{1}} \backslash \mathscr{C}_{p_{1}}$. Let $F \in \mathscr{O}\left(\mathscr{W}_{p_{1}} \backslash \mathscr{C}_{p_{1}}\right)$ denote this holomorphic extension. Since $\mathscr{W}_{p_{1}} \cap\left[\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}\right]$is connected, it follows that $\left[\mathscr{W}_{p_{1}} \backslash \mathscr{C}_{p_{1}}\right] \cap\left[\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}\right]$ is also connected. From the principle of analytic continuation, we deduce that there exists a well-defined function, still denoted by $F$, which is holomorphic in $\left[\mathscr{W}_{p_{1}} \backslash \mathscr{C}_{p_{1}}\right] \cup\left[\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}\right]$and which extends $f$, namely $\left.F\right|_{\Omega \cup \mathscr{H} \mathscr{W}_{1}^{+}}=f$.

We remind that $A_{0,0,0: c}^{1}\left(\partial^{+} \Delta\right)$ is tangent to $\left(H^{1}\right)^{-}$at $p_{1}$. By continuity, for small enough $\varepsilon$, it follows that $A_{x, v, u: c}^{1}\left(\partial^{+} \Delta\right) \cap C$ is contained in $A_{x, v, u: c}^{1}\left(\left\{e^{i \theta}:|\theta| \leqslant \frac{\pi}{4}\right\}\right)$ for all $|x|<\varepsilon,|v|<\varepsilon,|u|<\varepsilon$. Thus $A_{x, v, u: c}^{1}\left(\partial^{+} \Delta\right) \cap \Omega$ is always nonempty. The $\mathscr{C}^{2, \alpha-0}$ hypersurface $\mathscr{M}_{p_{1}} \subset \mathscr{W}_{p_{1}}$ is foliated by small pieces of analytic discs. Each such piece is necessarily contained in a single CR orbit of $\mathscr{M}_{p_{1}}$. The residual singularity of the holomorphic function $F$ can only be $\mathscr{C}_{p_{1}}^{\prime}:=\mathscr{C}_{p_{1}} \backslash \Omega$. Since $A_{x, v, u: c}^{1}\left(\partial^{+} \Delta \backslash\left\{e^{i \theta}:\right.\right.$ $\left.\left.|\theta| \leqslant \frac{\pi}{4}\right\}\right)$ is contained in $\Omega$, it follows that $\mathscr{C}_{p_{1}}^{\prime} \subset \mathscr{M}_{p_{1}}$ cannot contain any CR orbit of $\mathscr{M}_{p_{1}}$. According to Lemma 2.10 of [MP1999], $F$ then extends holomorphically and uniquely through $\mathscr{C}_{p_{1}}^{\prime}$.

The proofs of Lemma 9.16 and of Proposition 5.12 in Cases $\left(\mathbf{I}_{1}\right)$ and $\left(\mathbf{I}_{\mathbf{2}}\right)$ are complete.
9.18. End of proof of Proposition 5.12 in Case (II). According to Lemma 5.37, in Case (II), the one-codimensional totally real submanifold $M^{1} \subset M$ is given by the equations $y^{\prime}=\varphi^{\prime}\left(x, y_{1}\right)$ and $x_{n}=g\left(x^{\prime \prime}\right)$. If $u \in \mathbb{R}$ is a small real parameter, we may define a "translation" $M_{u}^{1}$ of $M^{1}$ in $M$ by the equations

$$
\begin{equation*}
y^{\prime}=\varphi^{\prime}\left(x, y_{1}\right), \quad x_{n}=g\left(x^{\prime \prime}\right)+u \tag{9.19}
\end{equation*}
$$

Similarly as in $\S 9.2$, we may construct a family of analytic discs $A_{x, v, u: c}^{1}(\zeta)$ half-attached to $M_{u}^{1}$. We then we fix a small scaling parameter $c$ with $0<$ $c \leqslant c_{1}$ so that properties $\left(\mathbf{1}_{1}\right)$ to $\left(\mathbf{9}_{1}\right)$ of Lemmas 7.12 and 8.3 hold true for $A_{x, v, 0: c}^{1}$.

We restrict the variation of the parameter $v$ to an arbitrary $(n-1)$ dimensional subspace $V_{1}$ of $T_{p_{1}} M^{1} \simeq \mathbb{R}^{n}$ which is supplementary to the real line $\mathbb{R} \cdot v_{1}$ in $T_{p_{1}} M^{1}$ (this makes a difference with §9.4). Also, we choose a local ( $n-1$ )-dimensional submanifold $X_{1} \subset M^{1}$ passing through $p_{1}$ with $\mathbb{R} \cdot v_{1} \oplus T_{p_{1}} X_{1}=T_{p_{1}} M^{1}$. If $\varepsilon>0$ is small enough with $\varepsilon \ll c^{2}$, then for every fixed $u$, the mapping

$$
\begin{equation*}
(x, v, u, \rho) \longmapsto A_{x, v, u: c}^{1}(\rho) \tag{9.20}
\end{equation*}
$$

is a one-to-one immersion from the open set $\left\{(x, v, \rho) \in \mathbb{R}^{n} \times V_{1} \times \mathbb{R} \times \mathbb{R}\right.$ : $|x|<\varepsilon,|v|<\varepsilon, 1-\varepsilon<\rho<1\}$ into $\mathbb{C}^{n}$ onto its image

$$
\begin{align*}
\mathscr{W}_{u}^{1}:=\left\{A_{x, v, u: c}^{1}\left(\rho e^{i \theta}\right) \in \mathbb{C}^{n}\right. & :(x, v, \rho) \in \mathbb{R}^{n} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R},  \tag{9.21}\\
& |x|<\varepsilon,|v|<\varepsilon, 1-\varepsilon<\rho<1,|\theta|<\varepsilon\}
\end{align*}
$$

is a local wedge of edge $M_{u}^{1}$. Clearly, this wedge $\mathscr{W}_{u}^{1}$ is $\mathscr{C}^{2, \alpha-0}$ with respect to $u$.

Using the fact that in Case (II) we have

$$
\begin{equation*}
\frac{\partial A_{0,0,0: c}^{1}}{\partial \theta}(1)=v_{1}=(1,0, \ldots, 0) \in T_{p_{1}} M^{1} \cap T_{p_{1}}^{c} M \tag{9.22}
\end{equation*}
$$

one can prove that Lemma 9.9 holds true with $\mathscr{H} \mathscr{W}_{1}^{+}$replaced by $\mathscr{W}_{2}$ in (9.10) and also that Lemma 9.12 holds true, again with $\mathscr{H} \mathscr{W}_{1}^{+}$replaced by $\mathscr{W}_{2}$. Similarly as in the proof of Lemma 9.16, applying then the continuity principle and using the fact that the mapping (9.20) is one-to-one, after possibly shrinking $\Omega$ in a neighborhood of $p_{1}$, and shrinking $\varepsilon>0$, we deduce that for each $u \neq 0$, there exists a holomorphic function $F \in \mathscr{O}\left(\Omega \cup \mathscr{W}_{2} \cup \mathscr{W}_{u}^{1}\right)$ with $\left.F\right|_{\Omega \cup W_{2}}=f$.

To conclude, it suffices to observe that for every fixed small $u$ with $-\varepsilon<$ $<u<0$, the wedge $\mathscr{W}_{u}^{1}$ contains in fact a neighborhood $\omega_{p_{1}}$ of $p_{1}$ in $\mathbb{C}^{n}$ (the reader may draw a figure).

The proofs of Proposition 5.12 and of Theorem 3.19 are complete now.
9.23. End of proof of Proposition 1.13. In order to derive Proposition 1.13 from Theorem 3.19, we now remind the necessity of supplementary arguments about the stability of our constructions under deformation.

Coming back to the strategy developped in $\S 3.16$, we had a first wedge $\mathscr{W}_{1}$ attached to $M \backslash C_{\mathrm{nr}}$. Using a partition of unity, we introduce a one-parameter $\mathscr{C}^{2, \alpha}$ family of generic submanifolds $M^{d}, d \in \mathbb{R}, d \geqslant 0$, with $M^{0} \equiv M$, with $M^{d}$ containing $C_{\mathrm{nr}}$ and with $M^{d} \backslash C_{\mathrm{nr}}$ contained in $\mathscr{W}_{1}$. In the proof of Theorem 3.19, thanks to this deformation, the wedge $\mathscr{W}_{1}$ was replaced by a neighborhood $\Omega$ of $M \backslash C_{\mathrm{nr}}$ in $\mathbb{C}^{n}$.

In Sections 4 and 5, we constructed a semi-local half-wedge $\left(\mathscr{H} \mathscr{W}_{\gamma}^{+}\right)^{d}$ attached to a one-sided neighborhood of $\left(M^{1}\right)^{d}$ in $M^{d}$ along a characteristic segment $\gamma^{d}$ of $M^{d}$. Now, we crucially claim that by arranging well this deformation $M^{d}$, we may achieve that the geometric extent of this semi-local half-wedge is uniform as $d>0$ tends to zero, namely $\left(\mathscr{H} \mathscr{W}_{\gamma}^{+}\right)^{d}$ tends to a nonvoid semi-local half-wedge $\left(\mathscr{H} \mathscr{W}_{\gamma}^{+}\right)^{0}$ attached to a one-sided neighborhood of $M^{1}$ in $M$ along $\gamma$, as $d$ tends to zero. Indeed, in Section 4 we have constructed a family of analytic discs $\left(\mathscr{Z}_{t, \chi, \nu: s}(\zeta)\right)^{d}(c f .(4.61))$ which covers the half-wedge $\left(\mathscr{H}^{\mathscr{W}}{ }_{\gamma}^{+}\right)^{d}$. Thanks to the stability of Bishop's equation under $\mathscr{C}^{2, \alpha}$ perturbations, the deformed family $\left(\mathscr{Z}_{t, \chi, \nu: s}(\zeta)\right)^{d}=: \mathscr{Z}_{t, \chi, \nu: s}^{d}(\zeta)$ is also of class $\mathscr{C}^{2, \alpha-0}$ with respect to the parameter $d$. We remind that for every $d>0$, the family $\mathscr{Z}_{t, \chi, \nu: s}^{d}(\zeta)$ was in fact constructed by means of a family $\widehat{Z}_{r_{0}, t, \tau,, \nu: s}^{d}(\zeta)$ obtained by solving Bishop's equation (4.40), where we now add the parameter $d$ in the function $\Phi^{\prime}$. In order to construct the semi-local attached half-wedge, we have used the rank property stated in Lemma 4.34. This rank property relied on the possibility of deforming the disc $\widehat{Z}_{r_{0}, t: s}(\zeta)$ near the point $\widehat{Z}_{r_{0}, t: s}^{d}(-1)$ in the open neighborhood $\Phi_{s}(\Omega) \equiv \Phi_{s}\left(\mathscr{W}_{1}\right)$ of $\Phi_{s}\left(M^{d}\right)$. As $d>0$ tends to zero, if $M^{d}$ tends to $M$, the size of the neighborhood $\Phi_{s}\left(\mathscr{W}_{1}\right)$ shrinks to zero, hence it could seem that the we have no control on the semi-local attached half-wege $\left(\mathscr{H} \mathscr{W}_{\gamma}^{+}\right)^{d}$ as $d>0$ tends to zero. Fortunately, since the points $\widehat{Z}_{r_{0}, 0: s}^{d}(-1)$ in a neighborhood of which we introduce the deformations (4.30) stay at a uniformly positive distance $\delta>0$ from the characteristic segment $\gamma$, we may choose the deformation $M^{d}$ of $M$ to tend to $M$ as $d$ tends to zero only in some thin, elongated tubular neighborhood of $\gamma$, whose width is small in comparison to this distance $\delta$. By smoothness with respect to $d$ of the family $\mathscr{Z}_{t, \chi, \nu: s}^{d}(\zeta)$, we then deduce that the semi-local half-wedge $\left(\mathscr{H} \mathscr{W}_{\gamma}^{+}\right)^{d}$ tends to a nontrivial semi-local half-wedge $\left(\mathscr{H} \mathscr{W}_{\gamma}^{+}\right)^{0}$ as $d$ tends to zero, which proves the claim.

Next, again thanks to the stability of Bishop's equation under perturbations, all the constructions of Sections 5, 6, 7, 8 and 9 above may be achieved to depend in a $\mathscr{C}^{2, \alpha-0}$ way with respect to $d$, hence uniformly. Importantly, we observe that if the deformation $M^{d}$ is chosen so that $M^{d}$ tends to $M$ only in a small neighborhood of $p_{1}$ of size $\ll \varepsilon$, then the shrinking of $\varepsilon$ which occurs in Lemma 9.9 may be achieved to be uniform as $d$ tends to zero, because the part $A_{x, v, u: c}\left(\partial^{3} \Delta\right)$ stays in a uniform compact subset of $\Omega$, as $d$ tends to zero. At the end of the proof of Proposition 5.12, we then obtain univalent holomorphic extension to a local wedge $\mathscr{W}_{p_{1}}^{d}$ of edge $M^{d}$ or to a neighborhood $\omega_{p_{1}}^{d}$ of $M^{d}$ in $\mathbb{C}^{n}$, and they tend smoothly to a wedge $\mathscr{W}_{p_{1}}^{0}$ of edge $M$ at $p_{1}$ or to a neighborhood $\omega_{p_{1}}$ of $p_{1}$ in $\mathbb{C}^{n}$.

The proof of Proposition 1.13 is complete.

## $\S 10 \mathscr{W}$-REMOVABILITY IMPLIES $L^{\text {p}}$-REMOVABILITY

10.1. Preliminary. From [Me1994, Jö1996], we remind that if $M^{\prime}$ is a globally minimal $\mathscr{C}^{2, \alpha}$ generic submanifold of $\mathbb{C}^{n}$ of CR dimension $m \geqslant 1$ and of codimension $d=n-m \geqslant 1$, there exists a wedge $\mathscr{W}^{\prime}$ attached to $M^{\prime}$ constructed by means of analytic discs glued progressively to $M^{\prime}$ and to some intermediate conelike submanifolds attached to $M^{\prime}$. Classically, one deduces that continuous CR functions on $M^{\prime}$ extend holomorphically to $\mathscr{W}^{\prime}$, and continuously to $M^{\prime} \cup \mathscr{W}^{\prime}$.

For $L_{l o c, C R}^{\mathrm{p}}$ functions, some supplementary, routine, though not straightforward, work has to be achieved. First of all, on a $\mathscr{C}^{2}$ generic submanifold $M^{\prime}$ of $\mathbb{C}^{n}$, the approximation theorem states that every $L_{l o c, C R}^{\mathrm{p}}$ function on $M^{\prime}$ is locally the limit, in the $L^{\mathrm{p}}$ norm, of a sequence of polynomials ( $c f$. Lemma 3.3 in [Jö1999b]). In the case where $M^{\prime}$ is a hypersurface, studied in [Jö1999b], the wedge is in fact a one-sided neighborhood of $M^{\prime}$, which we will denote by $\mathscr{S}^{\prime}$. The theory of Hardy spaces on the unit disc transfers to parameterized families of small analytic discs glued to $M^{\prime}$, provided the boundaries of these discs foliate an open subset of $M^{\prime}$. Using Carleson's imbedding theorem and the $L^{\mathrm{p}}$ approximation theorem, Jöricke established in [Jö1999b] that every $L_{l o c, C R}^{\mathrm{p}}$ function defined in a globally minimal $\mathscr{C}^{2}$ hypersurface $M^{\prime}$ extends holomorphically in the Hardy space $H^{\mathrm{p}}\left(\mathscr{S}^{\prime}\right)$ of holomorphic functions defined in $\mathscr{S}^{\prime}$ which enjoy $L^{\mathrm{p}}$ boundary values on $M^{\prime}$. In [Po2000, Po2004], the theory was built in higher codimension, introducing and studying the Hardy space $H^{\mathrm{p}}\left(\mathscr{W}^{\prime}\right)$ (see also [29]).
10.2. $L^{\text {p }}$-removability of nullsets. Let us say that a subset $\Phi$ of a $\mathscr{C}^{2, \alpha}$ generic submanifold is stably $\mathscr{W}$-removable if it is $\mathscr{W}$-removable with respect to every compactly supported sufficiently small $\mathscr{C}^{2, \alpha}$ deformation $M^{d}$
of $M$ leaving $\Phi$ fixed. Just by abstract nonsense, the singularity $C$ of Proposition 1.13 (in which it only remains to show $L^{p}$-removability) is seen to be stably removable.
Proposition 10.3. Let $M$ be a $\mathscr{C}^{2, \alpha}$ generic submanifold of $\mathbb{C}^{n}$ of $C R$ dimension $m \geqslant 1$ and of codimension $d=n-m \geqslant 1$, hence of dimension $(2 m+d)$, let $\Phi \subset M$ be a nonempty proper closed subset whose $(2 m+d)$ dimensional Hausdorff measure is equal to zero. Assume that $M \backslash \Phi$ is globally minimal and let $\mathscr{W}$ be a wedge attached to $M \backslash \Phi$ such that every function in $L_{l o c}^{\mathrm{p}}(M) \cap C R(M \backslash \Phi)$ extends holomorphically as a function in the Hardy space $H^{\mathrm{p}}(\mathscr{W})$. If $\Phi$ is stably $\mathscr{W}$-removable, then $\Phi$ is $L^{\mathrm{p}}$-removable.

Let us summarize informally the arguments. Fix $f \in L_{\text {loc }}^{\mathrm{p}}(M) \cap$ $C R(M \backslash \Phi)$. As soon as wedge extension over points of $\Phi$ is known, we may deform $M$ over $\Phi$ in the wedgelike domain, thus erasing the singularity $\Phi$. We get a $L_{l o c, C R}^{\mathrm{p}}$ function $f^{d}$ on the deformed manifold $M^{d}$, without singularities anymore. As a crucial fact, when the deformation $M^{d}$ tend to $M$, we shall have a uniform $L^{\mathrm{p}}$ control of the extension $f^{d}$, and this will insure that $f^{d}$ tends to a CR extension of $f$ through $\Phi$.
Proof. We claim that $\Phi$ is $L^{\mathrm{p}}$-removable for every p with $1 \leqslant \mathrm{p} \leqslant \infty$ if and only if $\Phi$ is $L^{1}$-removable. Indeed, suppose that for every function $f \in L_{l o c}^{1}(M) \cap \operatorname{CR}(M \backslash \Phi)$, and every smooth ( $n, m-1$ )-form with compact support, we have $\int_{M} f \cdot \bar{\partial} \psi=0$. Since $L_{l o c}^{\mathrm{p}}$ is contained in $L_{l o c}^{1}$ (by Hölder's inequality), this property holds in particular for every $g \in L_{l o c}^{\mathrm{p}}(M) \cap C R(M \backslash \Phi)$, hence $\Phi$ is $L^{\mathrm{p}}$-removable, as claimed.

Let $f \in L_{l o c, C R}^{1}(M \backslash \Phi) \cap L^{1}(M)$ be an arbitrary function. The goal is to show that $f$ is in fact CR on $\Phi$. Of course, it suffices to show that $f$ is CR locally at every point of $\Phi$. So, we fix an arbitrary point $q \in \Phi$. If $\psi$ is an arbitrary $(n, m-1)$-form of class $\mathscr{C}^{1}$ supported in a sufficiently small neighborhood of $q$, we have to prove that $\int_{M} f \cdot \bar{\partial} \psi=0$.

We also fix a small open polydisc $\mathscr{V}_{q}$ centered at $q$. We first claim that we can assume that the $L_{l o c}^{1}$ function $f$ is holomorphic in a neighborhood of $(M \backslash \Phi) \cap \mathscr{V}_{q}$ in $\mathbb{C}^{n}$. Indeed, since $M \backslash \Phi$ is globally minimal, there exists a wedge $\mathscr{W}$ attached to $M \backslash \Phi$ such that every $L_{l o c, C R}^{1}$ function on $M \backslash \Phi$, and in particular $f$, extends holomorphically as a function which belongs to the Hardy space $H^{1}(\mathscr{W})$. By slightly deforming $(M \backslash \Phi) \cap \mathscr{V}_{q}$ into $\mathscr{W}$ along Bishop discs glued to $M \backslash \Phi$, keeping $\Phi$ fixed, using the theory of Hardy spaces in wedges developed in [Po1997, MP1999, Po2000, Po2004, 29], we may obtain the following deformation result with $L^{1}$ control.
Proposition 10.4. For every $\varepsilon>0$, there exists a small $\mathscr{C}^{2, \alpha-0}$ deformation $M^{d}$ of $M$ with support contained in $\overline{\mathscr{V}}_{q}$ and there exists a function $f^{d} \in$ $L_{\text {loc }}^{1}\left(M^{d}\right) \cap C R\left(M^{d} \backslash \Phi\right)$, such that
(1) $M^{d} \cap \mathscr{V}_{q} \supset \Phi \cap \mathscr{V}_{q} \ni q$.
(2) $\left(M^{d} \backslash \Phi\right) \cap \mathscr{V}_{q} \subset \mathscr{W} \cap \mathscr{V}_{q}$.
(3) $f^{d}$ is holomorphic in the neighborhood $\mathscr{W} \cap \mathscr{V}_{q}$ of $\left(M^{d} \backslash \Phi\right) \cap \mathscr{V}_{q}$ in $\mathbb{C}^{n}$.
(4) $M \cap \mathscr{V}_{q}$ and $M^{d} \cap \mathscr{V}_{q}$ are graphed over the same $(2 m+d)$ linear real subspace and $\left\|M^{d} \cap \mathscr{V}_{q}-M \cap \mathscr{V}_{q}\right\|_{\mathscr{G}^{2}, \beta} \leqslant \varepsilon$.
(5) The volume forms of $M \cap \mathscr{V}_{q}$ and of $M^{d} \cap \mathscr{V}_{q}$ may be identified and $\left|f-f^{d}\right|_{L^{1}\left(M \cap \mathscr{V}_{q}\right)} \leqslant \varepsilon$.

Let us be more explicit about conditions (4) and (5). Without loss of generality, we can assume that in coordinates $(z, w)=(x+i y, u+i v) \in$ $\mathbb{C}^{m} \times \mathbb{C}^{d}$ centered at $q$, we have $T_{q} M=\{v=0\}$, hence the generic submanifolds $M$ and $M^{d}$ are represented locally by vectorial equations $v=\varphi(x, y, u)$ and $v=\varphi^{d}(x, y, u)$, where $\varphi$ and $\varphi^{d}$ are defined in the real cube $\mathbb{I}_{2 m+d}\left(2 \rho_{1}\right)$, for some small $\rho_{1}>0$ and that $\mathscr{V}_{q}$ is the polydisc $\Delta_{n}\left(\rho_{1}\right)$ of radius $\rho_{1}$. Then condition (4) simply means that $\left\|\varphi^{d}-\varphi\right\|_{\mathscr{C}^{2}\left(\mathbb{I}_{2 m+d}\left(\rho_{1}\right)\right)} \leqslant \varepsilon$ and condition (5) is clear if we choose $d x d y d u$ as the volume form on $M$ and on $M^{d}$.

Suppose that for every $\varepsilon>0$ and for every deformation $M^{d}$, we can show that the function $L_{l o c}^{1}$ function $f^{d}$ on $M^{d}$ is in fact CR over $M^{d} \cap \Delta_{n}\left(\rho_{1}\right)$. Then we claim that $f$ is CR in a neighborhood of $q$.

Indeed, to begin with, let us denote by $\bar{L}_{1}, \ldots, \bar{L}_{m}$ a basis of $(0,1)$ vector fields tangent to $M$, having coefficients depending on the first order derivatives of $\varphi$. More precisely, in slightly abusive matrix notation, we can choose the basis $\bar{L}:=\frac{\partial}{\partial \bar{z}}+2\left(i-\varphi_{u}\right)^{-1} \varphi_{\bar{z}} \frac{\partial}{\partial \bar{w}}$. Let us denote this basis vectorially by $\bar{L}=\frac{\partial}{\partial \bar{z}}+A \frac{\partial}{\partial \bar{w}}$. To compute the formal adjoint of $\bar{L}$ with respect to the local Lebesgue measure $d x d y d u$ on $M$, we choose two $\mathscr{C}^{1}$ functions $\psi$, $\chi$ of $(x, y, u)$ with compact support in $\mathbb{I}_{2 m+d}\left(\rho_{1}\right)$. Then the integration by part $\int_{T} \bar{L}(\psi) \cdot \chi \cdot d x d y d u=\int \psi \cdot{ }^{T} \bar{L}(\chi) \cdot d x d y d u$ yields the explicit expression ${ }^{T} \bar{L}(\chi):=-\bar{L}(\chi)-A_{\bar{w}} \cdot \chi$ of the formal adjoint of $\bar{L}$.

It follows immediately that if we denote by ${ }^{T}\left(\bar{L}^{d}\right)$ the formal adjoint of the basis of CR vector fields tangent to $M^{d}$, then we have an estimate of the form $\left\|^{T}\left(\bar{L}^{d}\right)-{ }^{T}(\bar{L})\right\|_{\mathscr{C}^{1}} \leqslant \mathrm{C} \cdot \varepsilon$, for some constant $C>0$. Recall that $f^{d}$ is assumed to be CR in $M^{d} \cap \Delta_{n}\left(\rho_{1}\right)$. Equivalently, we have $\int f^{d} \cdot{ }^{T}\left(\bar{L}^{d}\right)(\psi)$. $d x d y d u=0$ for every $\mathscr{C}^{1}$ function $\psi$ with compact support in the cube
$\mathbb{I}_{2 m+d}\left(\rho_{1}\right)$. Then we deduce that (some explanation follows) (10.5)

$$
\begin{aligned}
& \left|\int f \cdot{ }^{T} \bar{L}(\psi) \cdot d x d y d u\right|=\left|\int\left[f \cdot{ }^{T} \bar{L}(\psi)-f^{d} \cdot{ }^{T}\left(\bar{L}^{d}\right)(\psi)\right] \cdot d x d y d u\right| \\
& \leqslant\left|\int\left[f \cdot{ }^{T} \bar{L}(\psi)-f \cdot{ }^{T}\left(\bar{L}^{d}\right)(\psi)+f \cdot{ }^{T}\left(\bar{L}^{d}\right)(\psi)-f^{d} \cdot{ }^{T}\left(\bar{L}^{d}\right)(\psi)\right] \cdot d x d y d u\right| \\
& \leqslant C_{1}(\psi) \cdot \varepsilon \cdot \int_{\mathbb{I}_{2 m+d}\left(\rho_{1}\right)}|f| \cdot d x d y d u+C_{2}(\psi) \cdot \int_{\mathbb{I}_{2 m+d}\left(\rho_{1}\right)}\left|f-f^{d}\right| \cdot d x d y d u \\
& \leqslant C\left(\psi, f, \rho_{1}\right) \cdot \varepsilon,
\end{aligned}
$$

taking account of property (5) of Proposition 10.4 for the passage from the third to the fourth line, where $C\left(\psi, f, \rho_{1}\right)$ is a positive constant. As $\varepsilon$ was arbitrarily small, it follows that $\int f \cdot{ }^{T} \bar{L}(\psi) \cdot d x d y d u=0$ for every $\psi$, namely $f$ is CR on $M \cap \Delta_{n}\left(\rho_{1}\right)$, as was claimed.

It remains to show that $f^{d}$ is CR on $M^{d} \cap \Delta_{n}\left(\rho_{1}\right)$. For every deformation $M^{d}$ stabilizing $\Phi$ as in Proposition 10.4, the wedge $\mathscr{W}$ attached to $M \backslash \Phi$ is still a wedge attached to $M^{d} \backslash \Phi$ and it contains a neighborhood of $\left(M^{d} \backslash \Phi\right) \cap$ $\Delta_{n}\left(\rho_{1}\right)$ in $\mathbb{C}^{n}$. As $\Phi$ was supposed to be stably removable, it follows that there exists a wedge $\mathscr{W}_{1}$ attached to $M^{d}$ (including points of $\Phi$ ) to which holomorphic functions in $\mathscr{W}$ extend holomorphically.

Consequently, replacing $M^{d} \cap \Delta_{n}\left(\rho_{1}\right)$ by $M$, we are led to prove the following lemma, which, on the geometric side, is totally similar to Proposition 10.3 , except that the wedge $\mathscr{W}$ attached to $M \backslash \Phi$ appearing in the formulation of Proposition 10.3 is now replaced by a neighborhood $\Omega$ of $M \backslash \Phi$ in $\mathbb{C}^{n}$.

Lemma 10.6. Let $M$ be a $\mathscr{C}^{2, \alpha}$ generic submanifold of $\mathbb{C}^{n}$ of $C R$ dimension $m \geqslant 1$ and of codimension $d=n-m \geqslant 1$, let $\Phi \subset M$ be a nonempty proper closed subset whose $(2 m+d)$-dimensional Hausdorff measure is equal to zero. Let $\Omega$ be a neighborhood of $M \backslash \Phi$ in $\mathbb{C}^{n}$ and let $\mathscr{W}_{1}$ be a wedge attached to $M$, including points of $\Phi$. Let $f \in L_{l o c}^{1}(M)$ and assume that its restriction to $M \backslash \Phi$ extends as a holomorphic function $f^{\prime} \in \mathscr{O}\left(\Omega \cup \mathscr{W}_{1}\right)$. Then $f$ is CR all over $M$.

Proof. It suffices to prove that $f$ is CR at every point of $\Phi$. Let $q \in \Phi$ be arbitrary and let $\mathscr{W}_{q}$ be a local wedge of edge $M$ at $q$ which is contained in $\mathscr{W}_{1}$. Without loss of generality, we can assume that in coordinates $(z, w)=$ $(x+i y, u+i v) \in \mathbb{C}^{m} \times \mathbb{C}^{d}$ vanishing at $q$ with $T_{q} M=\{v=0\}$, the generic submanifold $M$ is represented locally in the polydisc $\Delta_{n}\left(\rho_{1}\right)$ by $v=\varphi(x, y, u)$ for some $\mathscr{C}^{2, \alpha} \mathbb{R}^{d}$-valued mapping $\varphi$ defined on the real cube on $\mathbb{I}_{2 m+d}\left(\rho_{1}\right)$. First of all, we construct a family of analytic discs half attached to $M$ whose interior is contained in the local wedge $\mathscr{W}_{q} \subset \mathscr{W}_{1}$.

Lemma 10.7. There exists a family of analytic discs $A_{s}(\zeta)$, with $s \in$ $\mathbb{R}^{2 m+d-1},|s| \leqslant 2 \delta$ for some $\delta>0$, and $\zeta \in \bar{\Delta}$, which is of class $\mathscr{C}^{2, \alpha-0}$ with respect to all variables, such that
(1) $A_{0}(1)=q$.
(2) $A_{s}(\bar{\Delta}) \subset \Delta_{n}\left(\rho_{1}\right)$.
(3) $A_{s}(\Delta) \subset \mathscr{W}_{q} \cap \Delta_{n}\left(\rho_{1}\right)$.
(4) $A_{s}\left(\partial^{+} \Delta\right) \subset M$.
(5) $A_{s}(i) \in M \backslash \Phi$ and $A_{s}(-i) \in M \backslash \Phi$ for all $s$.
(6) The mapping $[-2 \delta, 2 \delta]^{2 m+d-1} \times[-\pi / 2, \pi / 2] \ni(s, \theta) \longmapsto A_{s}\left(e^{i \theta}\right) \in$ $M$ is an embedding onto a neighborhood of $q$ in $M$.
(7) There exists $\rho_{2}>0$ such that the image of $[-\delta, \delta]^{2 m+d-1} \times$ $[-\pi / 4, \pi / 4]$ through this mapping contains $M \cap \Delta_{n}\left(\rho_{2}\right)$.

Proof. Let $M^{1}$ be a $\mathscr{C}^{2, \alpha}$ maximally real submanifold of $M$ passing through $q$ such that $M^{1} \cap \Phi$ is of zero measure with respect to the Lebesgue measure of $M^{1}$. Let $t \in \mathbb{R}^{d}$ and include $M^{1}$ in a parametrized family of maximally real submanifolds $M_{t}^{1}$ which foliates a neighborhood of $q$ in $M$. Starting with a family of analytic discs $A_{c, x, v}^{1}(\zeta)$ which are half-attached to $M^{1}$ as constructed in Lemma 7.12 above, we first choose the rotation parameter $v_{0}$ and a sufficiently small scaling factor $c_{0}$ in order that $A_{c_{0}, 0, v_{0}}^{1}( \pm i)$ does not belong to $\Phi$. In fact, this can be done for almost every $\left(c_{0}, v_{0}\right)$, because the mapping $(c, v) \mapsto A_{c, 0, v}^{1}( \pm i)$ is of rank $n$ at every point $(c, v)$ with $c \neq 0$ and $v \neq 0$. In addition, we adjust the rotation parameter $v_{0}$ in order that the vector $J v_{0}$ points inside a proper subcone of the cone which defines the wedge $\mathscr{W}_{q}$. If the scaling parameter $c$ is sufficiently small, this implies that $A_{c_{0}, 0, v_{0}}^{1}(\Delta)$ is contained in $\mathscr{W}_{q} \cap \Delta_{n}\left(\rho_{1}\right)$, as in Lemma 8.3 above. The translation parameter $x$ runs in $\mathbb{R}^{n}$ and we may select a $(n-1)$ dimensional parameter subspace $x^{\prime}$ which is transversal in $M^{1}$ to the half boundary $A_{c_{0}, 0, v_{0}}^{1}\left(\partial^{+} \Delta\right)$. With such a choice, there exists $\delta>0$ such that the mapping $[-2 \delta, 2 \delta]^{n-1} \times[-\pi / 2, \pi / 2] \ni\left(x^{\prime}, \theta\right) \longmapsto A_{c_{0}, x^{\prime}, v_{0}}^{1}\left(e^{i \theta}\right)$ is a diffeomorphism onto a neighborhood of $q$ in $M^{1}$. Finally, using the stability of Bishop's equation under perturbations, we can deform this family of discs by requiring that it is half attached to $M_{t}^{1}$, thus obtaining a family $A_{s}(\zeta):=A_{c_{0}, x^{\prime}, v_{0}, t}^{1}(\zeta)$ with $s:=\left(x^{\prime}, t\right) \in \mathbb{R}^{2 m+d-1}$. Shrinking $\delta$ if necessary, we can check as in the proof of Lemma 8.3 (91) that property (3) holds. This completes the proof.

Let now $f \in L_{l o c}^{1}(M)$ and let $f^{\prime} \in \mathscr{O}\left(\Omega \cup \mathscr{W}_{1}\right)$. Thanks to the foliation propery (6) of Lemma 10.7, it follows from Fubini's theorem that for almost every translation parameter $s$, the mapping $e^{i \theta} \mapsto f\left(A_{s}\left(e^{i \theta}\right)\right)$ defines a $L^{1}$ function on $\partial^{+} \Delta$. In addition, the restriction of the function $f^{\prime} \in \mathscr{O}\left(\Omega \cup \mathscr{W}_{1}\right)$
to the disc $A_{s}(\Delta) \subset \mathscr{W}_{q} \subset \mathscr{W}_{1}$ yields a holomorphic function $f^{\prime}\left(A_{s}(\zeta)\right)$ in $\Delta$.

Lemma 10.8. For almost every $s$ with $|s| \leqslant 2 \delta$, the function $f^{\prime}\left(A_{s}(\zeta)\right)$ belongs to the Hardy space $H^{1}(\Delta)$.
Proof. Indeed, for almost every $s$, the intersection $\Phi \cap A_{s}\left(\partial^{+} \Delta\right)$ is of zero one-dimensional measure. By the assumption of Lemma 10.6, the restriction of $f \circ A_{s}$ and of $f^{\prime} \circ A_{s}$ to $\partial^{+} \Delta \backslash \Phi$ coincide. Recall that $\partial^{-} \Delta=\{\zeta \in \partial \Delta$ : $\operatorname{Re} \zeta \leqslant 0\}$. Since $A_{s}( \pm i)$ does not belong to $\Phi$ and since $A_{s}\left(e^{i \theta}\right)$ belongs to $\mathscr{W}_{q}$ for all $\theta$ with $\pi / 2<|\theta| \leqslant \pi$, it follows that $\left.f \circ A_{s}\right|_{\partial+\Delta}$ and $\left.f^{\prime} \circ A_{s}\right|_{\partial-\Delta}$ (which is holomorphic in a neighborhood of $\partial^{-} \Delta$ in $\mathbb{C}$ ) match together in a function which is $L^{1}$ on $\partial \Delta$. Let us denote this function by $f_{s}$. Furthermore, $f_{s}$ extends holomorphically to $\Delta$ as $\left.f^{\prime} \circ A_{s}\right|_{\Delta}$. Consequently, $\left.f^{\prime} \circ A_{s}\right|_{\Delta}$ belongs to the Hardy space $H^{1}(\Delta)$.

Since the boundary value of $f^{\prime}$ on $M \backslash \Phi$ along the family of discs $A_{s}(\zeta)$ coincides with $f$, we can now denote both functions by the same letter $f$.

For $\varepsilon \geqslant 0$ small, let now $\chi_{\varepsilon}\left(s, e^{i \theta}\right)$ be a $\mathscr{C}^{2}$ function on $[-2 \delta, 2 \delta] \times \partial \Delta$ which equals $\varepsilon$ for $|s| \leqslant \delta$ and for $\theta \in[-\pi / 4, \pi / 4]$ and which equals 0 if either $\pi / 2 \leqslant|\theta| \leqslant \pi$ or $|s| \geqslant 2 \delta / 3$. We may require in addition that $\left\|\chi_{\varepsilon}\right\|_{\mathscr{C}^{2}} \leqslant \varepsilon$. We define a deformation $M^{\varepsilon}$ of $M$ compactly supported in a neighborhood of $q$ by pushing $M$ inside $\mathscr{W}_{q}$ along the family of discs $A_{s}(\zeta)$ as follows:

$$
\begin{equation*}
M^{\varepsilon}:=\left\{A_{s}\left(\left[1-\chi_{\varepsilon}\left(s, e^{i \theta}\right)\right] e^{i \theta}\right):|\theta| \leqslant \pi / 2,|s| \leqslant 2 \delta\right\} . \tag{10.9}
\end{equation*}
$$

Notice that $M^{\varepsilon}$ coincides with $M$ outside a small neighborhood of $q$. Then we have $\left\|M^{\varepsilon}-M\right\|_{\mathscr{C}_{2}} \leqslant C \cdot \varepsilon$, for some constant $C>0$ which depends only on the $\mathscr{C}^{2}$ norms of $A_{s}(\zeta)$ and of $\chi_{\varepsilon}\left(s, e^{i \theta}\right)$. If the radius $\rho_{2}$ is as in Property (7) of Lemma 10.7 above, the deformation $M^{\varepsilon} \cap \Delta_{n}\left(\rho_{2}\right)$ is entirely contained in $\mathscr{W}_{q}$ and since $f$ is holomorphic in $\mathscr{W}_{q}$, its restriction to $M^{\varepsilon} \cap \Delta_{n}\left(\rho_{2}\right)$ is obviously CR.


Fig. 21: The arc $\Gamma_{\varepsilon, s}$ and the deformation $M^{\varepsilon}$

As in [Jö1999b, MP1999, Po2000], we notice that for every $s$ and every $\varepsilon$, the one-dimensional Lebesgue measure on the arc

$$
\begin{equation*}
\Gamma_{\varepsilon, s}:=\left\{\left[1-\chi_{\varepsilon}\left(s, e^{i \theta}\right)\right] e^{i \theta} \in \Delta:|\theta| \leqslant \pi\right\} \tag{10.10}
\end{equation*}
$$

is a Carleson measure. Thanks to the geometric uniformity of these arcs $\Gamma_{\varepsilon, s}$, it follows from an inspection of the proof of Carleson's imbedding theorem that there exists a (uniform) constant $C$ such that for all $s$ with $|s| \leqslant 2 \delta$ and all $\varepsilon$, one has the estimate

$$
\begin{equation*}
\int_{\Gamma_{\varepsilon, s}}\left|f\left(A_{s}\left(\left[1-\chi_{\varepsilon}\left(s, e^{i \theta}\right)\right] e^{i \theta}\right)\right)\right| \cdot d \theta \leqslant C \int_{\partial \Delta}|f| \cdot d \theta . \tag{10.11}
\end{equation*}
$$

We are now ready to complete the proof of Lemma 10.6. Let $\pi_{x, y, u}$ denote the projection parallel to the $v$-space from $\mathbb{C}^{n}$ onto the $(x, y, u)$ space. The mapping $(s, \theta) \mapsto \pi_{x, y, u}\left(A_{s}(\theta)\right)$ may be used to define new coordinates in a neighborhood of the origin in $\mathbb{C}^{m} \times \mathbb{R}^{d}$, an open subset above which $M$ and $M^{\varepsilon}$ are graphed. We shall now work with these coordinates. With respect to the coordinates $(s, \theta)$, on $M$ and on $M^{\varepsilon}$, we have formal adjoints ${ }^{T} \bar{L}$ and ${ }^{T}\left(\bar{L}^{\varepsilon}\right)$ of the basis of CR vector fields with an estimation of the form $\left\|^{T}\left(\bar{L}^{\varepsilon}\right)-{ }^{T} \bar{L}\right\|_{\mathscr{C}_{1}} \leqslant C \cdot \varepsilon$, for some constant $C>0$. Let now $\psi=\psi(s, \theta)$ be $\mathscr{C}^{1}$ function with compact support in the set $\{|s|<\delta,|\theta| \leqslant \pi / 4\}$. By construction, the subpart of $M^{\varepsilon}$ defined by $\widetilde{M}^{\varepsilon}:=\left\{A_{s}\left(\left[1-\chi_{\varepsilon}\left(s, e^{i \theta}\right)\right] e^{i \theta}\right):|\theta| \leqslant \pi / 4,|s| \leqslant \delta\right\}$ is contained in the wedge $\mathscr{W}_{q}$, hence the restriction of the holomorphic function $f \in \mathscr{W}_{q}$ to $\widetilde{M}^{\varepsilon}$ is obviously CR on $\widetilde{M}^{\varepsilon}$.

For simplicity of notation, we shall denote $f\left(A_{s}\left(e^{i \theta}\right)\right)$ by $f_{s}(\theta)$ and $f\left(A_{s}\left(\left[1-\chi_{\varepsilon}\left(s, e^{i \theta}\right)\right] e^{i \theta}\right)\right)$ by $f_{s}^{\varepsilon}(\theta)$. Since by construction for every $\varepsilon>0$, the $L^{1}$ function $(s, \theta) \mapsto f_{s}^{\varepsilon}(\theta)$ is annihilated in the distributional sense by the CR vector fields $\bar{L}^{\varepsilon}$ on $\widetilde{M}^{\varepsilon}$, we may compute (not writing the arguments $(s, \theta)$ of $\psi$ )

$$
\begin{aligned}
& \left|\int_{|s| \leqslant \delta} \int_{|\theta| \leqslant \pi / 4} f_{s}(\theta) \cdot{ }^{T} \bar{L}(\psi) \cdot d s d \theta\right|= \\
& =\left|\int_{|s| \leqslant \delta} \int_{|\theta| \leqslant \pi / 4}\left[f_{s}(\theta) \cdot{ }^{T} \bar{L}(\psi)-f_{s}^{\varepsilon}(\theta) \cdot{ }^{T}\left(\bar{L}^{\varepsilon}\right)(\psi)\right] \cdot d s d \theta\right| \\
& \leqslant \mid \int_{|s| \leqslant \delta}\left(\int _ { | \theta | \leqslant \pi / 4 } \left[f_{s}(\theta) \cdot{ }^{T} \bar{L}(\psi)-f_{s}(\theta) \cdot{ }^{T}\left(\bar{L}^{\varepsilon}\right)(\psi)+\right.\right.
\end{aligned}
$$

$$
\begin{equation*}
\left.\left.+f_{s}(\theta) \cdot{ }^{T}\left(\bar{L}^{\varepsilon}\right)(\psi)-f_{s}^{\varepsilon}(\theta) \cdot{ }^{T}\left(\bar{L}^{\varepsilon}\right)(\psi)\right] \cdot d \theta\right) \cdot d s \tag{10.12}
\end{equation*}
$$

$$
\begin{aligned}
& \leqslant C_{1}(\psi) \cdot \varepsilon \cdot \int_{|s| \leqslant \delta} \int_{|\theta| \leqslant \pi / 4}\left|f_{s}(\theta)\right| \cdot d s d \theta+ \\
& \quad+C_{2}(\psi) \cdot \int_{|s| \leqslant \delta} \int_{|\theta| \leqslant \pi / 4}\left|f_{s}(\theta)-f_{s}^{\varepsilon}(\theta)\right| \cdot d s d \theta \\
& \leqslant C_{1}(\psi, f, \delta) \cdot \varepsilon+C_{2}(\psi, \delta) \cdot \max _{|s| \leqslant \delta} \int_{|\theta| \leqslant \pi / 4}\left|f_{s}(\theta)-f_{s}^{\varepsilon}(\theta)\right| \cdot d s d \theta .
\end{aligned}
$$

Thanks to the estimate (10.11) and thanks to Lebesgue's dominated convergence theorem, the last integral tends to zero as $\varepsilon$ tends to zero. It follows that the integral in the first line can be made arbitrarily small, hence it vanishes. This proves that $f$ is CR in a neighborhood of $q$ and completes the proof of Lemma 10.6.

The proof of Proposition 10.3 is complete.

## §11. Proofs of Theorem 1.2 and of Corollary 1.5

11.1. Tree of separatrices linking hyperbolic points. Let $M \subset \mathbb{C}^{2}$ be a globally minimal $\mathscr{C}^{2, \alpha}$ hypersurface, let $S \subset M$ be a $\mathscr{C}^{2, \alpha}$ surface and let $K \subset S$ be a proper compact subset of $S$. Assume that $S$ is totally real outside a discrete subset of complex tangencies which are hyperbolic in the sense of Bishop. Since we aim to remove the compact subset $K$ of $S$, we can shrink the open surface $S$ around $K$ in order that $S$ contains only finitely many such hyperbolic complex tangencies, which we shall denote by $\left\{h_{1}, \ldots, h_{\lambda}\right\}$, where $\lambda$ is some integer, possibly zero. Furthermore, we can assume that $\partial S$ is $\mathscr{C}^{2, \alpha}$. As a corollary of the qualitative theory of planar vector fields, due to Poincaré-Bendixson ([HS1974]), we know that
(i) the hyperbolic points $h_{1}, \ldots, h_{\lambda}$ are singularities of the characteristic foliation $\mathrm{F}_{S}^{c}$;
(ii) through every hyperbolic point $h_{1}, \ldots, h_{\lambda}$, there are exactly four $\mathscr{C}^{2, \alpha}$ open separatrices;
(iii) after perturbing slightly the boundary $\partial S$ if necessary, these separatrices are all transversal to $\partial S$ and the union of all separatrices together with all hyperbolic points makes a finite tree without cycles in $S$.

Precisely, by an (open) separatrix, we mean a $\mathscr{C}^{2, \alpha}$ curve $\tau:(0,1) \rightarrow S$ with $\frac{d \tau}{d s}(s) \in T_{\tau(s)} S \cap T_{\tau(s)}^{c} M \backslash\{0\}$ for every $s \in(0,1)$, namely its tangent vectors are all nonzero and characteristic, such that one limit point, say $\lim _{s \rightarrow 0} \tau(s)$ is a hyperbolic point, and the other $\lim _{s \rightarrow 1} \tau(s)$ either belong to the boundary $\partial S$ or is a second hyperbolic point.

From the local study of saddle phase diagrams ([Ha1982]), we get in addition:
(iv) there exists $\varepsilon>0$ and for every $l=1, \ldots, \lambda$, there exist two curves $\gamma_{l}^{1}, \gamma_{l}^{2}:(-\varepsilon, \varepsilon) \rightarrow S$ which are of class $\mathscr{C}^{1, \alpha}$, not more, with $\gamma_{l}^{i}(0)=h_{l}$ and $\frac{d \gamma_{l}^{i}}{d t}(s) \in T_{\gamma_{l}^{i}(s)} S \cap T_{\gamma_{l}^{i}(s)}^{c} M \backslash\{0\}$ for every $s \in(-\varepsilon, \varepsilon)$ and for $i=1,2$, such that the four open segments $\gamma_{l}^{1}(-\varepsilon, 0), \gamma_{l}^{1}(0, \varepsilon), \gamma_{l}^{2}(-\varepsilon, 0)$ and $\gamma_{l}^{2}(0, \varepsilon)$ cover the four pieces of open separatrices incoming at $h_{l}$.

Let $\tau_{1}, \ldots, \tau_{\mu}:(0,1) \rightarrow S$ denote all the separatrices of $S$, where $\mu$ is some integer, possibly equal to zero. By the finite hyperbolic tree $T_{S}$ of $S$, we mean:

$$
\begin{equation*}
T_{S}:=\left\{h_{1}, \ldots, h_{\lambda}\right\} \bigcup_{1 \leqslant k \leqslant \mu} \tau_{k}(0,1) . \tag{11.2}
\end{equation*}
$$

We say that $T_{S}$ has no cycle if it does not contain any subset homeomorphic to the unit circle. For instance, in the case where $S \equiv D$ is diffeomorphic to a real disc (as in the assumptions of Proposition 1.4), its hyperbolic tree $T_{D}$ necessarily has no cycle. However, in the case where $S$ is an annulus (for instance), there is a trivial example of a characteristic foliation with one (or two, or more) hyperbolic point(s) and a circle in the hyperbolic tree.
11.3. Hyperbolic decomposition in the disc case. Let the real disc $D$ and the compact subset $K \subset D$ be as in Theorem 1.2. We shrink $D$ slightly and smooth out its boundary, so that its hyperbolic tree $T_{D}$ is finite and has no cycle. We may decompose $D$ as the disjoint union

$$
\begin{equation*}
D=T_{D} \cup D_{o} \tag{11.4}
\end{equation*}
$$

where the complement of the hyperbolic tree $D_{o}:=D \backslash T_{D}$ is an open subset of $D$ entirely contained in the totally real part of $D$. Then $D_{o}$ has finitely many connected components $D_{1}, \ldots, D_{\nu}$, the hyperbolic sectors of $D$. Also, for $j=1, \ldots, \nu$, we define the proper closed subsets $C_{j}:=D_{j} \cap K$ of $D_{j}$.


Fig. 22: Removal of hyperbolic sectors and removal of a tree without cycles

Again from the Poincaré-Bendixson theory, we know that for every component $D_{j}$ (in which the characteristic foliation is nonsingular), the proper closed subset $C_{j}$ is nontransversal to $\mathrm{F}_{D_{j}}^{c}$. In the figure, we have drawn the characteristic curves only in the two sectors $D_{4}$ and $D_{6}$.
11.5. Global minimality of some complements. We state a generalization of Lemma 3.5 to the case where some hyperbolic complex tangencies are allowed. Its proof is not immediate.
Proposition 11.6. Let $M$ be a $\mathscr{C}^{2, \alpha}$ hypersurface in $\mathbb{C}^{2}$ and let $S \subset M$ be $\mathscr{C}^{2, \alpha}$ surface which is totally real outside a discrete subset of hyperbolic complex tangencies. Assume that the hyperbolic tree $T_{S}$ of $S$ has no cycle. Then for every compact subset $K \subset S$ and for every point $p \in M \backslash K$ :

$$
\begin{equation*}
\mathscr{O}_{C R}(M \backslash K, p)=\mathscr{O}_{C R}(M, p) \backslash K \tag{11.7}
\end{equation*}
$$

As a corollary, $M \backslash K$ is globally minimal if $M$ is so.
Proof. As above, we may assume that $S$ coincides with the shrinking of a slightly larger surface and has finitely many hyperbolic points $\left\{h_{1}, \ldots, h_{\lambda}\right\}$. Let $K_{T_{S}}:=K \cap T_{S}$ be the track of $K$ on the hyperbolic tree $T_{S}$. Since $K_{T_{S}}$ may in general coincide with any arbitrary closed (e.g. Cantor) subset of $T_{S}$, in order to fix ideas, it will be convenient to deal with an enlargement $\underline{K}$ of $K_{T_{S}}$, simply defined by filling the possible holes of $K_{T_{S}}$ in $T_{S}$ : more precisely, $\underline{K}$ should contain all hyperbolic points of $S$ together with all separatrices joining them and for every separatrix $\tau_{k}(0,1)$ with right limit point $\lim _{s \rightarrow 1} \tau_{k}(s)$ belonging to the boundary of $S$, we require that $\underline{K}$ contains the segment $\tau_{k}\left[0, r_{1}\right]$, where $r_{1}<1$ is close enough to 1 in order that $\underline{K}$ effectively contains $K_{T_{S}}$. Equivalently, $\underline{K}$ is a small shrinking of $T_{S}$, still compact in $S$.


Fig. 23: the disposition of $K, K_{T_{S}}$ and $\bar{K}$
The main step in the proof of Proposition 11.6 is as follows.
Lemma 11.8. We have $\mathscr{O}_{C R}(M \backslash \underline{K}, q)=\mathscr{O}_{C R}(M, q) \backslash \underline{K}$, for every $q \in$ $M \backslash \underline{K}$.

Before pursuing, we establish the proposition. The inclusion $\mathscr{O}_{C R}(M \backslash K, p) \subset \mathscr{O}_{C R}(M, p) \backslash K$ is trivial. Reversely, for $q \in \mathscr{O}_{C R}(M, p) \backslash K$ arbitrary, we must find a piecewise smooth complextangential curve joining $q$ to $p$ and running entirely in $M \backslash K$. We first join $p$ to some $p^{\prime} \in M \backslash S$ and $q$ to some $q^{\prime} \in M \backslash S$ as follows.

Lemma 11.9. The $C R$ orbit $\mathscr{O}_{C R}(M \backslash K, r)$ of every $r \in M \backslash K$ contains points $r^{\prime} \in M \backslash S$ arbitrarily close to $r$.

Proof. If $r \in M \backslash S$, the claim is gratuitous. If $r \in S \backslash\left\{h_{1}, \ldots, h_{\lambda}\right\}$, whence $S$ is totally real in a neighborhood of $r$, we just choose a local section $Y$ of $T^{c} M$ defined near $r$ which is transversal to $S$ at $r$ and we follow the integral curve of $Y$ to escape from $S$, as shown by the figure.

If $r=h_{l}$ is a hyperbolic point, we may use one of the four separatrices to join $r$ to some point $r^{\prime \prime} \in T_{S}$ close to $r$ and $\neq h_{l}$, hence in the totally real part of $S$. Then we join $r^{\prime \prime}$ to some $r^{\prime} \in M \backslash S$ as above by means of some vector field $Y^{\prime \prime}$ transversal to $S$ at $r^{\prime \prime}$.

Necessarily, both $p^{\prime}$ and $q^{\prime}$ belong to $\mathscr{O}_{C R}(M, p) \backslash K$ and hence, in order to get (11.7), it suffices to produce a piecewise smooth complex-tangential curve joining $q^{\prime}$ to $p^{\prime}$ which runs in $M \backslash K$.

Taking for granted Lemma 11.8, we first get a piecewise smooth complextangential curve joining $q^{\prime}$ to $p^{\prime}$ and running in $M^{\prime}:=M \backslash \underline{K}$. Equivalently, $q^{\prime} \in \mathscr{O}_{C R}\left(M^{\prime}, p^{\prime}\right)$. The set $C^{\prime}:=K \cap M^{\prime}$ is closed in $M^{\prime}$, is closed in $S^{\prime}:=S \backslash \underline{K}$ and is nontransversal to $\mathrm{F}_{S^{\prime}}^{c}$. Lemmas 3.3 and 3.5 showed that $\mathscr{O}_{C R}\left(M^{\prime} \backslash C^{\prime}, r^{\prime}\right)=\mathscr{O}_{C R}\left(M^{\prime}, r^{\prime}\right) \backslash C^{\prime}$, for every $r^{\prime} \in M^{\prime} \backslash C^{\prime}$. Consequently, there exists a piecewise smooth complex-tangential curve joining $q^{\prime}$ to $p^{\prime}$ which runs in $M^{\prime} \backslash C^{\prime}$, hence in $M \backslash K$. Thus $q^{\prime} \in \mathscr{O}_{C R}\left(M \backslash K, p^{\prime}\right)$ and hence in conclusion, $q \in \mathscr{O}_{C R}(M \backslash K, p)$, which completes the proof of Proposition 11.6.

It remains to establish Lemma 11.8. As $M$ is a hypersurface of $\mathbb{C}^{2}$, its CR orbits are of dimension either 2 or 3. We state an analog to Lemma 3.7.

Lemma 11.10. Let $M, S, T_{S}$ and $\underline{K} \subset T_{S}$ be as above. There exists a connected embedded submanifold $\Omega \subset M$ containing the hyperbolic tree $T_{S}$ such that:
(1) $\Omega$ is a $T^{c} M$-integral manifold, namely $T_{p}^{c} M \subset T_{p} \Omega$ for every $p \in \Omega$;
(2) $\Omega$ is contained in a single CR orbit of $M$;
(3) $\Omega \backslash \underline{K}$ is also contained in a single $C R$ orbit of $M \backslash \underline{K}$.

More precisely, $\Omega$ is an open neighborhood of $T_{S}$ if it is of real dimension 3 and a complex curve surrounding $T_{S}$ if it is of dimension 2.

Reasoning as in Lemma 3.10, to get Lemma 11.8, starting with a piecewise smooth complex tangential curve joining $q$ to some arbitrary $r \in$ $\mathscr{O}_{C R}(M, q) \backslash \underline{K}$, if it meets $\underline{K}$, we can modify the trajectory by running only inside $\Omega \backslash \underline{K}$ (surrounding the obstacle), whence $r \in \mathscr{O}_{C R}(M \backslash \underline{K}, q)$. This completes the proof of Lemma 11.8, granted Lemma 11.10, which we now prove.
Proof. We shall construct $\Omega$ by means of a complex-tangential flowing procedure, stretching and enlarging local pieces of it.

Fix any point $p_{0} \in \underline{K} \backslash\left\{h_{1}, \ldots, h_{\lambda}\right\}$. Since $S$ is totally real near $p_{0}$, there exists a locally defined $T^{c} M$-tangent vector field $Y$ which is transversal to $S$ at $p_{0}$. Consequently, for $\delta>0$ small enough, the small curve $I_{0}:=$ $\left\{\exp (s Y)\left(p_{0}\right):-\delta<s<\delta\right\}$ is transversal to $S$ at $p_{0}$ and moreover, the two half-curves

$$
\begin{equation*}
I_{0}^{ \pm}:=\left\{\exp (s Y)\left(p_{0}\right): 0< \pm s<\delta\right\} \tag{11.11}
\end{equation*}
$$

lie in $M \backslash S$.
Since $p_{0} \in \underline{K}$ belongs to some open separatrix $\tau_{k}(0,1)$, there exists a $\mathscr{C}^{1, \alpha}$ complex-tangential vector field $X$ defined in a neighborhood of $p_{0}$ in $M$ which is tangent to $S$ and whose integral curve passing through $p_{0}$ is a piece of $\tau_{k}(0,1)$. Since $Y$ is transversal to $S$ at $p_{0}$, it follows that the set

$$
\begin{equation*}
\omega_{0}:=\left\{\exp \left(s_{2} X\right)\left(\exp \left(s_{1} Y\right)\left(p_{0}\right)\right):-\delta<s_{1}, s_{2}<\delta\right\} \tag{11.12}
\end{equation*}
$$

is a small $\mathscr{C}^{1, \alpha}$ one-codimensional submanifold of $M$ passing through $p_{0}$ which is transversal to $S$ at $p_{0}$. Clearly, $T_{p_{0}} \omega_{0}=T_{p_{0}}^{c} M$. Thanks to the fact that the flow of $X$ stabilizes $S$, we see that the integral curves $s_{2} \mapsto \exp \left(s_{2} X\right)\left(\exp \left(s_{1} Y\right)\left(p_{0}\right)\right)$ are contained in $M \backslash S$ for every starting point $\exp \left(s_{1} Y\right)\left(p_{0}\right) \in I_{0}$ with $s_{1} \neq 0$, namely for all $s_{1} \neq 0$. We deduce that each one of the two open halves

$$
\begin{equation*}
\omega_{0}^{ \pm}:=\left\{\exp \left(s_{2} X\right)\left(\exp \left(s_{1} Y\right)\left(p_{0}\right)\right): 0< \pm s_{1}<\delta,-\delta<s_{2}<\delta\right\} \tag{11.13}
\end{equation*}
$$

is contained in a single CR orbit of $M \backslash \underline{K}$.
To pursue, abandoning the consideration of $\omega_{0}^{-}$, we shall assume that the piece of CR orbit $\omega_{0}^{+}$is of real dimension 2 , whence it is a complex curve. Afterwards, we shall treat the (simpler) case where $\omega_{0}^{+}$is 3 -dimensional.

By the $S$-boundary of a set $E \subset M \backslash S$, we shall mean the intersection $\partial E \cap S$ of the boundary of $E$ in $M$ with $S$. Thus, the $S$-boundary of $\omega_{0}^{+}$is just the piece of characteristic curve $\left\{\exp \left(s_{2} X\right):-\delta \leqslant s_{2} \leqslant \delta\right\}$, contained in $\tau_{k}(0,1)$.

Since $\tau_{k}(0,1)$ is an embedded segment, we may suppose from the beginning that the vector field $X$ is defined in a neighborhood of $\tau_{k}(0,1)$ in $M$. Using then the flow of $X$, we may prolong the small piece $\omega_{0}^{+}$to get a semilocal $\mathscr{C}^{1, \alpha}$ submanifold $\omega_{k}^{+}$stretched along $\tau_{k}(0,1)$. Again, this piece $\omega_{k}^{+}$is
(by construction) contained in a single CR orbit of $M \backslash \underline{K}$. By the fundamental stability property of CR orbits under flows, we deduce that $\omega_{k}^{+}$is in fact a long thin complex curve in $M \backslash \underline{K}$ whose $S$-boundary contains $\tau_{k}(0,1)$.


Remind that by definition of separatrices, the point $\tau_{k}(0)$ is always a hyperbolic point. There is a dichotomy: either $\tau_{k}(1)$ is also a hyperbolic point or it lies in $\partial S$. If $\tau_{k}(1)$ is a hyperbolic point, then by the definition of $\underline{K}$, the complete closed separatrix $\tau_{k}[0,1]$ is contained in $\underline{K}$, hence it may not be crossed by means of a CR curve running in $M \backslash \underline{K}$.

So we must again prolong $\omega_{k}^{+}$and in the neighborhood of the hyperbolic point $h_{l}=\tau_{k}(0)$, the geometric situation is different. As in the figure, let $\tau_{j}(0,1)$ be the separatrix issued from $h_{l}$ next to $\tau_{l}[0,1]$.
Lemma 11.14. There exists a long thin complex curve $\omega_{j}^{+}$whose $S$ boundary contains $\tau_{j}(0,1)$ which is contained in the same CR orbit as $\omega_{k}^{+}$ and which matches up with $\omega_{k}^{+}$near $h_{l}$. Geometrically, $\omega_{k}^{+}$and $\omega_{j}^{+}$coincide near $h_{l}$ and constitute a piece of cornered complex curve.

Proof. We introduce local holomorphic coordinates $(z, w)=(x+i y, u+$ $i v) \in \mathbb{C}^{2}$ vanishing at $h_{l}$ in which the hypersurface $M$ is given as the graph $v=\varphi(x, y, u)$, where $\varphi$ is a $\mathscr{C}^{2, \alpha}$ function. Since $M$ contains the complex curve $\omega_{k}^{+}$, it is Levi degenerate at $h_{l} \in \bar{\omega}_{k}^{+}$. Thus, we may assume that $|\varphi(x, y, u)| \leqslant C \cdot(|x|+|y|+|u|)^{2+\alpha}$.

The surface $S$, as a subset of $M$, is represented by one supplementary $\mathscr{C}^{2, \alpha}$ equation of the form $u=h(x, y)$. According to Bishop ([Bi1965]), a suitable change of holomorphic coordinates normalizes

$$
\begin{align*}
h(x, y) & =z \bar{z}+\gamma\left(z^{2}+\bar{z}^{2}\right)+\mathrm{O}\left(|z|^{2+\alpha}\right) \\
& =(2 \gamma+1) x^{2}-(2 \gamma-1) y^{2}+\mathrm{O}\left(|z|^{2+\alpha}\right), \tag{11.15}
\end{align*}
$$

where $\gamma \in \mathbb{R}^{+}$is a biholomorphic invariant satisfying $\gamma>\frac{1}{2}$ by the hyperbolicity assumption. Then the tangents at $h_{l}$ to the two half-separatrices $\tau_{k}$ and $\tau_{j}$ are given respectively by the linear (in)equations $x>0, y=-\frac{2 \gamma+1}{2 \gamma-1} x$,
$u=0$ and $x<0, y=-\frac{2 \gamma+1}{2 \gamma-1} x, u=0$. In the figure below, where we do not draw the axes, the $u$-axis is vertical, the $y$ axis points behind $h_{l}$ and the $x$-axis is horizontal, from the left to the right.


Fig. 25: Behavior of the complex curve orbit around a hyperbolic point
The saddle-looking surface $S$ is represented in the 3 -dimensional space $M$; the horizontal plane passing through $h_{l}$ is thought to be the complex tangent plane $T_{h_{l}}^{c} M$.

We introduce two $T^{c} M$-tangent vector fields $X_{1}$ and $X_{2}$ defined in a neighborhood of $h_{l}=\tau_{k}(0)=\tau_{j}(0)$ with $X_{1}\left(h_{l}\right)$ directed along $\tau_{k}$ in the sense of increasing $s$ and $X_{2}\left(h_{l}\right)$ directed along $\tau_{j}$ in the sense of increasing $s$, defined by

$$
\left\{\begin{array}{l}
X_{1}=\frac{\partial}{\partial x}-\left(\frac{2 \gamma+1}{2 \gamma-1}\right) \frac{\partial}{\partial y}+A_{1}(x, y, u) \frac{\partial}{\partial u}  \tag{11.16}\\
X_{2}=-\frac{\partial}{\partial x}-\left(\frac{2 \gamma+1}{2 \gamma-1}\right) \frac{\partial}{\partial y}+A_{2}(x, y, u) \frac{\partial}{\partial u}
\end{array}\right.
$$

with $A_{1}$ and $A_{2}$ being certain rational functions of the first order jet of $\varphi$. Since $\varphi$ vanishes to second order at $h_{l}$, the two $\mathscr{C}^{1, \alpha}$ coefficients $A_{1}$ and $A_{2}$ satisfy

$$
\begin{equation*}
\left|A_{1}, A_{2}(x, y, u)\right|<C \cdot(|x|+|y|+|u|)^{1+\alpha} . \tag{11.17}
\end{equation*}
$$

Let $Z$ denote the vector field $X_{1}+X_{2}$, as shown in the top of the left Figure 25 . Using the flow of $Z$ we begin by extending the banana-looking piece $\omega_{k}^{+}$of complex curve by introducing the submanifold $\omega$ consisting of points

$$
\begin{equation*}
\exp \left(s_{2} Z\right)\left(\tau_{k}\left(s_{1}\right)\right) \tag{11.18}
\end{equation*}
$$

where $0<s_{1}<\delta$ and $0<s_{2}<\delta$, for some small $\delta>0$. One checks that all these points stay in $M \backslash S$, hence are contained in the same CR orbit as $\omega_{k}^{+}$ in $M \backslash \underline{K}$. By the stability property of CR orbits, it follows that $\omega$ is a piece
of complex curve contained in $M \backslash \underline{K}$. Roughly, the (horizontal) projection of $\omega$ onto $T_{h_{l}}^{c} M$ covers $\sim \frac{1}{8}$ of a neighborhood of $h_{l}$ in $T_{h_{l}}^{c} M$.

For $0<s<\delta$, let $\mu(s):=\exp (s Z)\left(h_{l}\right)$ denote the CR curve lying "between" $\tau_{k}$ and $\tau_{j}$ and which constitutes a part of the boundary of $\omega$. Let $p$ be an arbitrary point of this curve, close to $h_{l}$.

Lemma 11.19. The integral curve $s \mapsto \exp \left(-s X_{1}\right)(p)$ of $-X_{1}$ issued from $p$ necessarily intersects $S$ at a point $q$ close to $h_{l}$ and close to $\tau_{j}$ (cf. Figure 25).

Proof. This integral curve is contained in the real 2-surface passing through $h_{l}$ defined by

$$
\begin{equation*}
\Sigma:=\left\{\exp \left(-s_{2} X_{1}\right)\left(\exp \left(s_{1} Z\right)\left(h_{l}\right)\right):-\delta<s_{1}, s_{2}<\delta\right\} \tag{11.20}
\end{equation*}
$$

for some $\delta>0$. Because the vector fields $X_{1}, X_{2}$ and $Z=X_{1}+X_{2}$ have $\mathscr{C}^{1, \alpha}$ coefficients, the surface $\Sigma$ is only $\mathscr{C}^{1, \alpha}$ in general. In $M$ equipped with the three real coordinates $(x, y, u)$, we may parametrize $\Sigma$ by a mapping of the form

$$
\begin{equation*}
\left(s_{1}, s_{2}\right) \longmapsto\left(s_{2}-2 s_{1}\left(\frac{2 \gamma+1}{2 \gamma-1}\right), s_{2}\left(\frac{2 \gamma+1}{2 \gamma-1}\right), u\left(s_{1}, s_{2}\right)\right), \tag{11.21}
\end{equation*}
$$

where $u$ is of class $\mathscr{C}^{1, \alpha}$. It is clear that $u(0)=u_{s_{1}}(0)=u_{s_{2}}(0)=0$, so that there is a constant $C$ such that

$$
\begin{equation*}
\left|u\left(s_{1}, s_{2}\right)\right|<C \cdot\left(\left|s_{1}\right|+\left|s_{2}\right|\right)^{1+\alpha} \tag{11.22}
\end{equation*}
$$

Furthermore, we claim that $u$ satisfies the better estimate

$$
\begin{equation*}
\left|u\left(s_{1}, s_{2}\right)\right|<C \cdot\left(\left|s_{1}\right|+\left|s_{2}\right|\right)^{2+\alpha} \tag{11.23}
\end{equation*}
$$

for some constant $C>0$. In other words, $\Sigma$ osculates the complex tangent plane $T_{h_{l}}^{c} M$ to second order at $h_{l}: \Sigma$ is more flat than $S$ at $h_{l}$. Since the second order terms $z \bar{z}+\gamma\left(z^{2}+\bar{z}^{2}\right)$ of the graphing function $h$ of $S$ are nonvanishing, the curve $s \mapsto \exp \left(-s X_{1}\right)(p)$ must necessarily intersect the saddle $S$, whence Lemma 11.19 follows.

To verify the remaining claim, we will reason with the two linear combinations $L_{1}:=\frac{\partial}{\partial x}+B_{1}(x, y, u) \frac{\partial}{\partial u}$ and $L_{2}:=\frac{\partial}{\partial y}+B_{2}(x, y, u) \frac{\partial}{\partial u}$ of $X_{1}$ and $X_{2}$. This will lighten the computations (with $X_{1}$ and $X_{2}$, the principle is the same).

Here, $B_{1}$ and $B_{2}$ are $\mathscr{C}^{1, \alpha}$ and satisfy (11.17). Denote by $s_{1} \longmapsto$ $\left(s_{1}, \lambda\left(s_{1}\right), \mu\left(s_{1}\right)\right)$ the integral curve of $L_{1}$ passing through the origin. It is $\mathscr{C}^{2, \alpha}$ and we have

$$
\begin{equation*}
\left|\lambda\left(s_{1}\right)\right|<C \cdot\left|s_{1}\right|^{2+\alpha} \quad \text { and } \quad\left|\mu\left(s_{1}\right)\right|<C \cdot\left|s_{1}\right|^{2+\alpha} \tag{11.24}
\end{equation*}
$$

for some $C>0$. In the definition of $\Sigma$, we replace $X_{1}$ and $X_{2}$ by $L_{1}$ and $L_{2}$ (with $X_{1}$ and $X_{2}$, the principle is the same, although the obtained graphing function $u^{L}\left(s_{1}, s_{2}\right)$ differs). Considering the composition of flows
$\exp \left(s_{2} L_{2}\right)\left(\exp \left(s_{1} L_{1}\right)(0)\right)$, we have to solve the system of ordinary differential equations

$$
\begin{equation*}
\frac{d x}{d s_{2}}=0, \quad \frac{d y}{d s_{2}}=1, \quad \frac{d u}{d s_{2}}=B_{2}(x, y, u) \tag{11.25}
\end{equation*}
$$

with initial conditions

$$
\begin{equation*}
x(0)=s_{1}, \quad y(0)=\lambda\left(s_{1}\right), \quad u(0)=\mu\left(s_{1}\right) . \tag{11.26}
\end{equation*}
$$

This yields $x\left(s_{1}, s_{2}\right)=s_{1}, y\left(s_{1}, s_{2}\right)=\lambda\left(s_{1}\right)+s_{2}$ and the integral equation

$$
\begin{equation*}
u^{L}\left(s_{1}, s_{2}\right)=\mu\left(s_{1}\right)+\int_{0}^{s_{2}} B_{2}\left(s_{1}, s_{2}^{\prime}+\lambda\left(s_{1}\right), u^{L}\left(s_{1}, s_{2}^{\prime}\right)\right) d s_{2}^{\prime} \tag{11.27}
\end{equation*}
$$

Since $u^{L}$ already satisfies (11.22), inserting the estimate (11.17) satisfied by $B_{2}$ and integrating, it is now elementary to obtain $\left|u^{L}\left(s_{1}, s_{2}\right)\right| \leqslant C \cdot\left|s_{1}\right|+$ $\left.\left|s_{2}\right|\right)^{2+\alpha}$.

We can now achieve the proof of Lemma 11.14. So, for various points $p=\mu(s)$ close to $h_{l}$ the intersection points $q \in S$ exist. If all points $q$ belong to $\tau_{j}$, we are done: the piece $\omega$ extends as a cornered (roughly $\frac{1}{4}$ ) piece of complex curve with $S$-boundary $\tau_{k} \cup \tau_{j}$ near $h_{l}$.

Assume therefore that one such point $q$ does not belong to $\tau_{j}$, as drawn in the left hand side of Figure 25. Suppose that $q$ lies above $\tau_{j}$, the case where $q$ lies under $\tau_{j}$ being similar. The characteristic curve $\gamma^{\prime} \subset S$ passing through $q$ stays above $\tau_{j}$ and is nonsingular. Prolongating the complex curve $\omega$ in $M \backslash \underline{K}$ by means of the flow of $-X_{1}$, we deduce that there exists at $q$ a local piece $\omega_{q}^{+}$of complex curve with $S$-boundary contained in $\gamma^{\prime}$ which is contained in the same CR orbit as $\omega$. Using then the flow of a CR vector having $\gamma^{\prime}$ as an integral curve, we can prolong $\omega_{q}^{+}$along $\gamma^{\prime}$, which yields a long thin banana-looking complex curve with boundary in $\gamma^{\prime}$. However, this piece may remain too thin. Fortunately, thanks to the flow of $X_{1}-X_{2}$, we can extend it as a piece $\omega^{\prime}$ of complex curve with boundary $\gamma^{\prime}$ which then goes over $h_{l}$, with respect to a complex projection onto $T_{h_{l}}^{c} M$, as illustrated in Figure 25 above. We claim that this yields a contradiction.

Indeed, as $\omega$ and $\omega^{\prime}$ are complex curves, they are locally defined as graphs of holomorphic functions $g$ and $g^{\prime}$ defined in domains $D$ and $D^{\prime}$ in the complex line $T_{h_{l}}^{c} M$. By construction, there exists a point in $r \in D \cap D^{\prime}$ at which the values of $g$ and $g^{\prime}$ are distinct. However, since by construction $g$ and $g^{\prime}$ coincide in a neighborhood of the CR curve joining $p$ to $q$, they must coincide at $r$ because of the principle of analytic continuation: this is a contradiction. In conclusion, the CR orbit passes through the hyperbolic point $h_{l}$, in a neighborhood of which it consists of a cornered complex curve with boundary $\tau_{k} \cup \tau_{j}$. This completes the proof of Lemma 11.14.

We can now conclude Lemma 11.10. Again, we may prolong $\omega_{j}^{+}$all along $\tau_{j}(0,1)$. If $\tau_{j}(1)$ is a new hyperbolic point $h_{m}$, again we prolong, etc.
Since the hyperbolic tree $T_{S}$ does not contain any cycle, after some steps, an endpoint $\tau_{k}(1)$ will not be a hyperbolic point, hence belong to $\partial S$. But we arranged at the beginning that $\underline{K} \cap \tau_{k}(0,1)=\tau_{k}\left(0, r_{1}\right]$, where $r_{1}<1$. It is then crucial that when a limit point $\tau_{k}(1)$ belongs to $\partial S$, we escape from $\underline{K}$ and using a local CR vector field $Y$ transversal to $S$, we may cross the separatrix $\tau_{k}(0,1)$ at some point $\tau_{k}\left(r_{2}\right)$ where $r_{2}$ satisfies $r_{1}<r_{2}<$ 1. Hence, we pass to the other side of $S$ in $M$ and then, by means of a further prolongation, we turn around to the other side of $\tau_{k}(0,1)$. Also, the two pieces in either side of $\tau_{k}(0,1)$ match up at least in a $\mathscr{C}^{1, \alpha}$ way. Then thanks to the stability property of orbits under flows, we deduce that these two pieces match up as a piece of complex curve containing $\tau_{k}(0,1)$ in its interior.

Continuing the prolongation, we construct the complex curve $\Omega$ surrounding $T_{S}$, which is obviously contained in a single CR orbit of $M$. Furthermore, by construction, $\Omega \backslash \underline{K}$ is contained in a single CR orbit of $M \backslash \underline{K}$. Thus, we have established Lemma 3.12 under the assumption that the initial CR orbit of $q_{0}^{+}$is two-dimensional.

Assume finally that the CR orbit of $q_{0}^{+}$is 3 -dimensional. By a similar (and in fact easier) propagation procedure, we may construct a neighborhood $\Omega$ in $M$ of the hyperbolic tree satisfying conditions (1), (2) and (3) of Lemma 11.10.
11.28. Proofs of Theorem 1.2 and of Corollary 1.5. We treat directly the more general Corollary 1.5. As already known, it suffices to establish the $\mathscr{W}$-removability of $K$.

Let $\omega_{1}$ be a one-sided neighborhood of $M \backslash K$ in $\mathbb{C}^{2}$. Let $\underline{K} \subset T_{S}$ be a filling of $K_{T_{S}}=K \cap S$, as in Lemma 11.8. By this lemma, $M \backslash \underline{K}$ is globally minimal. Because $K \cap\left(S \backslash T_{S}\right)$ is nontransversal to $\mathbf{F}_{S \backslash T_{S}}^{c}$ by assumption, we may apply Proposition 1.4 to the totally real surface $S \backslash T_{S}$ in the globally minimal $M \backslash \underline{K}$ to remove $K \cap\left(S \backslash T_{S}\right)$. We deduce that there exists a one-sided neighborhood $\omega_{2}$ of $M \backslash \underline{K}$ in $\mathbb{C}^{2}$ such that (after shrinking $\omega_{1}$ if necessary), holomorphic functions in $\omega_{1}$ extend holomorphically to $\omega_{2}$. Then we slightly deform $M$ inside $\omega_{2}$ over points of $K \cap\left(S \backslash T_{S}\right)$. We obtain a $\mathscr{C}^{2, \alpha}$ hypersurface $M^{d}$ with $M^{d} \backslash \underline{K} \subset \omega_{2}$. Also, by stability of global minimality under small perturbations, we can assume that $M^{d}$ is also globally minimal.

Since $M$ and $M^{d}$ are of codimension 1, the union of a one-sided neighborhood $\omega^{d}$ of $M^{d}$ in $\mathbb{C}^{2}$ together with $\omega_{2}$ constitutes a complete one-sided neighborhood of $M$ in $\mathbb{C}^{2}$. To conclude the proof of Corollary 1.5 , it suffices therefore to show the following.

Lemma 11.29. $\underline{K}$ is $C R$-, $\mathscr{W}$ - and $L^{\mathrm{p}}$-removable.
Proof. Let $K_{\mathrm{nr}} \subset \underline{K}$ denote the smallest nonremovable subset. Reasoning by contradiction, assume $K_{\mathrm{nr}}$ is nonempty.

Let $T^{\prime}$ be a connected component of the minimal subtree of $T$ containing $K_{\mathrm{nr}}$. By a subtree of a tree $T$ defined as in (11.2) above, we mean of course a finite union of some of the separatrices $\tau_{1}(0,1)$ together with all hyperbolic points which are endpoints of separatrices. Since $T^{\prime}$ does not contain any cycle, there exists at least one extremal branch of $T^{\prime}$, say $\tau_{1}(0,1)$ after renumbering.

At first, suppose to simplify that the subtree $T^{\prime}$ consists only of a single branch $\tau_{1}[0,1]$. Thanks to properties (iii) and (iv) of §11.1, we can enlarge a little bit this branch by prolongating the curve $\tau_{1}(0,1)$ as an open $\mathscr{C}^{2, \alpha}$ Jordan arc $\tau_{1}[0,1+\varepsilon)$, for some $\varepsilon>0$. But then by [Me1997, MP1999], every proper closed subset of $\tau_{1}[0,1+\varepsilon)$ is $\mathscr{W}$-removable, hence $K_{\mathrm{nr}}$ is removable, a contradiction.

If $T^{\prime}$ consists of at least two branches, with $\tau_{1}(1)$ being an extremal point, since the hyperbolic point $\tau_{1}(0)$ belongs to another separatrix $\tau_{k}[0,1] \subset T^{\prime}$, it follows from the assumption that $T^{\prime}$ is the smallest subtree containing $K_{\mathrm{nr}}$ that $K_{\mathrm{nr}} \cap \tau_{1}(0,1]$ must be nonempty. But then since we may prolong $\tau_{1}$ to $\left(0,1+\varepsilon\right.$ ], by [Me1997, MP1999] again, $K_{\mathrm{nr}} \cap \tau_{1}(0,1]$ is $\mathscr{W}$-removable, a contradiction to its definition.

The proofs of Theorem 1.2 and of Corollary 1.5 are complete.

## §12. Polynomial convexity of certain real 2-discs

12.1. Convexity and removability. Let $K \Subset D \subset \partial \Omega \Subset \mathbb{C}^{2}$ be as in Corollary 1.3. Recall that $K$ is polynomially convex if it coincides with its polynomial hull

$$
\begin{equation*}
\widehat{K}:=\left\{z \in \mathbb{C}^{2}:|p(z)| \leqslant \max _{w \in K}|p(w)|, \text { for every } p \in \mathbb{C}\left[z_{1}, z_{2}\right]\right\} . \tag{12.2}
\end{equation*}
$$

In complex dimension $n=2$ (only), removability is closely related to convexity properties ([Jö1988, FS1991, Stu1993, Du1993]). Indeed, for strictly pseudoconvex domains $\Omega \Subset \mathbb{C}^{2}$, a structural result due to Stout shows that a compact set $K \subset \partial \Omega$ is removable if and only if it is $\mathscr{O}(\bar{\Omega})$-convex ${ }^{38}$. If in addition $\bar{\Omega}$ is polynomially convex, removability of $K$ holds if and only if $K$ is polynomially convex. In such a situation, removability yields information about polynomial convexity as a byproduct.

Corollary 1.3 improves these results, passing to weakly pseudoconvex boundaries. For totally real discs in convex boundaries, it was already shown

[^37]in [Po2004]. So far, it seems to be the best available insight into polynomial convexity of discs in $\mathbb{C}^{2}$ (more generally, one may also consider surfaces $S$ as in Corollary 1.5 instead of discs as in Theorem 1.2). In fact, the general question of characterizing polynomial convexity of arbitrary surfaces (not contained in pseudoconvex boundaries), even for totally real discs in $\mathbb{C}^{2}$, is still mostly open ([HN1994], pp. 353-355).

Proof of Corollary 1.3. Since $\bar{\Omega}$ (containing $K$ ) is assumed to be polynomially convex, we deduce:

$$
\begin{equation*}
\widehat{K} \cap\left(\mathbb{C}^{2} \backslash \bar{\Omega}\right)=\emptyset, \quad \text { or equivalently: } \quad \widehat{K} \subset \bar{\Omega} . \tag{12.3}
\end{equation*}
$$

Firstly, a general fact of independent interest will yield $\widehat{K} \cap \partial \Omega=K$.
Lemma 12.4. Let $\Omega \Subset \mathbb{C}^{2}$ be a domain with $\mathscr{C}^{2, \alpha}$ boundary whose closure is polynomially convex and let $K \subset \bar{\Omega}$ be an arbitrary compact set. Then $\widehat{K} \cap(\partial \Omega \backslash K)$ coincides with the union of all the complex-curve CR orbits of $\partial \Omega \backslash K$.

Indeed, in the situation of Corollary $1.3, K \Subset D$ lies in a globally minimal boundary $\partial \Omega$ and we already verified in Proposition 11.6 that $\partial \Omega \backslash K$ is also globally minimal, namely it contains no complex-curve CR orbit, whence $\widehat{K} \cap \partial \Omega=K$.

Secondly, we will control $\widehat{K} \cap \Omega$ thanks to the pseudoconvexity of $\Omega \backslash \widehat{K}$.
Lemma 12.5. Let $\Omega \Subset \mathbb{C}^{2}$ be an arbitrary pseudoconvex domain. Then for any compact set $K \Subset \mathbb{C}^{2} \backslash \Omega$, the open set $\Omega \backslash \widehat{K}$ is a union of pseudoconvex domains.

Granted these two lemmas and Theorem 1.2, we may conclude the proof of Corollary 1.3. Indeed, since $\widehat{K}$ does not meet $\partial \Omega \backslash K$, there is a domain contained in $\Omega \backslash \widehat{K}$ whose closure contains the connected hypersurface $\partial \Omega \backslash K$, namely a one-sided neighborhood $\mathscr{V}(\partial \Omega \backslash K)$. The $\mathscr{W}$ removability of $K$ in Theorem 1.2 yields univalent holomorphic extension from $\mathscr{V}(\partial \Omega \backslash K)$ to $\Omega$. Thus, there can exist only one pseudoconvex component of $\Omega \backslash \widehat{K}$, the domain $\Omega$ itself! Thus $\Omega \backslash \widehat{K}=\Omega$, which, together with $\widehat{K} \subset \bar{\Omega}$ and $\widehat{K} \cap \partial \Omega=K$, gives $\widehat{K}=K$.

Proof of Lemma 12.4. We assume $K \neq \emptyset$ throughout.
Let $\mathscr{O}$ be a complex-curve CR orbit of $\partial \Omega \backslash K$. Its closure $\overline{\mathscr{O}}^{\partial \Omega \backslash K}$ in $\partial \Omega \backslash K$ is a relatively closed subset of $\partial \Omega \backslash K$ laminated by complex curves, in which $\mathscr{O}$ (as well as every other maximal connected complex curve) is dense, for the topology induced from $\partial \Omega \backslash K$ (see [Jö1999a, 29]). The full closure $\overline{\mathscr{O}}^{\partial \Omega} \supset \overline{\mathscr{O}}^{\partial \Omega \backslash K}$ is compact and the complement $\overline{\mathscr{O}}^{\partial \Omega} \backslash \overline{\mathscr{O}}^{\partial \Omega \backslash K}$ is contained in $K$. Since $\mathbb{C}^{2}$ cannot contain any compact set laminated by complex curves
([Jö1999a, 29]) and since $K \neq \emptyset$, this complement $\overline{\mathscr{O}}^{\partial \Omega} \backslash \overline{\mathscr{O}}^{\partial \Omega \backslash K} \subset K$ must be nonempty.

Pick $z \in \mathscr{O}$ and let $p$ be an arbitrary holomorphic polynomial. To verify that $|p(z)| \leqslant \max _{K}|p|$, two cases occur.
(a) The maximum of $|p|$ on the compact set $\overline{\mathscr{O}}^{\partial \Omega}$ is attained at some point $w \in \overline{\mathscr{O}}^{\partial \Omega} \backslash \overline{\mathscr{O}}^{\partial \Omega \backslash K}$; then $w \in K$, whence obviously $|p(z)| \leqslant$ $|p(w)| \leqslant \max _{K}|p|$.
(b) The maximum of $|p|$ on $\overline{\mathscr{O}}^{\partial \Omega}$ is attained at some point $w \in \overline{\mathscr{O}}^{\partial \Omega \backslash K}$; then by the lamination property, $w$ belongs to (the interior of) some immersed complex curve $\mathscr{O}^{\prime}$, again dense in the sense that $\overline{\mathscr{O}^{\prime}} \partial \Omega \backslash K=$ $\overline{\mathscr{O}}^{\partial \Omega \backslash K}$. The maximum principle entails that $\left.p\right|_{O^{\prime}}$ is equal to a constant $\sigma \in \mathbb{C}$, whence by continuity $\left.p\right|_{\overline{\sigma^{\prime}} \partial \Omega \backslash K},\left.p\right|_{\bar{\sigma}^{\partial \Omega \backslash K}}$ and $\left.p\right|_{\bar{\sigma}^{\partial \Omega}}$ are all equal to the same constant $\sigma$ and since $\overline{\mathscr{O}}^{\partial \Omega} \cap K \neq \emptyset$, we conclude that $|p(z)| \leqslant \max _{\bar{\sigma}}{ }^{\circ \Omega}|p|=|p(w)|=|\sigma| \leqslant \max _{K}|p|$.

Thus, it remains to show that every point $p \in \partial \Omega \backslash K$ which belongs to a 3-dimensional (hence open in $\partial \Omega \backslash K$ ) CR orbit does not belong to $\widehat{K}$. The local Oka criterion ([Stu1971]) is suitable for that purpose. We state an adapted and simplified version, using discs.

If $K \subset \mathbb{C}^{2}$ is a compact set, a point $p_{0} \in \mathbb{C}^{2}$ does not belong to $\widehat{K}$ provided one can construct a continuous one-parameter family $\left\{A_{t}\right\}_{0 \leqslant t \leqslant 1}$ of analytic discs $A_{t}: \Delta \rightarrow \mathbb{C}^{2}$ such that

- $A_{t}(\bar{\Delta}) \cap K=\emptyset$ for every $t$ with $0 \leqslant t \leqslant 1$;
- $A_{t}(\partial \Delta) \cap \widehat{K}=\emptyset$ for every $t$ with $0 \leqslant t \leqslant 1$;
- $A_{1}(\bar{\Delta}) \cap \widehat{K}=\emptyset$;
- $p_{0} \in A_{0}(\Delta)$.

Since everything is biholomorphically invariant, we observe a direct analogy with the continuity principle.

Thus, let $\mathscr{O}$ be an open, 3 -dimensional CR orbit of $\partial \Omega \backslash K$. Since $\mathscr{O}$ has CR dimension 1 , it contains some strongly pseudoconvex points (otherwise $T^{c} \mathscr{O}$ would be Frobenius-integrale, hence $\mathscr{O}$ would be a complex-curve CR orbit). To establish $\mathscr{O} \cap \widehat{K}=\emptyset$, it is thus sufficient to show:
(i) no strictly pseudoconvex point $p_{0} \in \partial \Omega \backslash K$ can be contained in $\widehat{K}$; and:
(ii) the property $q \notin \widehat{K}$ propagates along CR curves running inside $\mathscr{O}$.

To show (i), we choose local holomorphic coordinates centered at $p_{0}$ in which $\Omega$ corresponds to

$$
\begin{equation*}
y_{2}>\left|z_{1}\right|^{2}+\mathrm{O}\left(\left|z_{1}\right|+\left|x_{2}\right|\right)^{2+\alpha} \tag{12.6}
\end{equation*}
$$

For small $\varepsilon>0$, the family $A_{t}(\zeta):=(\varepsilon \zeta,-\varepsilon i t)$, which translates downwards a small piece of the complex line $T_{p_{0}}^{c} \partial \Omega$, does satisfy the four items above, whence $p_{0} \notin \widehat{K}$.

Gratuitously, we deduce that all points $p_{0}^{\prime}$ belonging to some neighborhood $U_{p_{0}}$ of $p_{0}$ in $\partial \Omega \backslash K$ also avoid $\widehat{K}$.

To show (ii), we recall from [Tu1994, 29] that for every $q \in \mathscr{O}=$ $\mathscr{O}_{C R}\left(\partial \Omega \backslash K, p_{0}\right)$ and for every small $\varepsilon>0$, there exist $\ell \in \mathbb{N}$ with $\ell=\mathrm{O}(1 / \varepsilon)$ and a chain of $\mathscr{C}^{2, \alpha-0}$ analytic discs $A^{1}, A^{2}, \ldots, A^{\ell-1}, A^{\ell}$ attached to $M$ with the properties:

- $A^{1}(-1)=: p_{0}^{\prime} \in U_{p_{0}}$;
- $A^{1}(1)=A^{2}(-1), A^{2}(1)=A^{3}(-1), \ldots, A^{\ell-1}(1)=A^{\ell}(-1)$;
- $A^{\ell}(1)=q$;
- $\left\|A^{k}\right\|_{\mathscr{C}^{1}(\bar{\Delta})} \leqslant \varepsilon$, for $k=1,2, \ldots, \ell$;
- each $A^{k}$ is an embedding $\bar{\Delta} \rightarrow \mathbb{C}^{2}$.

By construction ([Tu1994, 29]), the projections onto $T_{A^{k}(1)}^{c} \partial \Omega$ of each $A^{k}(\zeta)$ are round discs $\Delta \ni \zeta \mapsto \lambda(1-\zeta) \in \mathbb{C}$, for some appropriate $\lambda \in \mathbb{C}$ satisfying $|\lambda|=\mathrm{O}(\varepsilon)$. Hence we are reduced to proving for a small round disc the implication $A(-1) \notin \widehat{K} \Longrightarrow A(1) \notin \widehat{K}$. Roundness is useful to control the geometry.

We first consider the case where $A$ is transverse to $\partial \Omega$ at $A(1)$, namely $-\frac{\partial A}{\partial \rho}(1) \notin T_{A(1)} \partial \Omega$. Since $\Omega$ is pseudoconvex, this vector $-\frac{\partial A}{\partial \rho}(1)$ points inside $\Omega$. We choose coordinates centered at $A(1)$ in which $\Omega$ is represented by $y_{2}>\varphi\left(x_{1}, y_{1}, x_{2}\right)$, with $\varphi(0)=0$ and $d \varphi(0)=0$. Perturbing slightly the base point $A(1)$ and solving an appropriate Bishop-type equation, we may as in [Tu1994, MP1999, MP2002] construct a 2-parameter family of analytic discs $A_{s_{1}, s_{2}}$ attached to $\partial \Omega \backslash K$ with $A_{0,0}=A$ whose boundaries $A_{s_{1}, s_{2}}(\partial \Delta)$ foliate a neighborhood of $A(1)$ in $\partial \Omega \backslash K$. Then the interiors $A_{s_{1}, s_{2}}(\Delta)$ foliate the pseudoconvex side of $\partial \Omega \backslash K$ near $A(1)$. Fixing a very small $\delta>0$, there exists a unique $A_{s_{1}^{\prime}, s_{2}^{\prime}}$ such that the point $A(1)=0$ is contained in the image of pushed-down disc $A_{s_{1}^{\prime}, s_{2}^{\prime}}+(0,-i \delta)$. For an $\eta>0$ which is approximately equal to twice the diameter of $A_{s_{1}^{\prime}, s_{2}^{\prime}}(\bar{\Delta})$, we look at the family $A_{t}:=A_{s_{1}^{\prime}, s_{2}^{\prime}}+(0,-i t)$, where $\delta \leqslant t \leqslant \eta$. Clearly the final disc $A_{\eta}(\bar{\Delta})$ lies in $\mathbb{C}^{2} \backslash \bar{\Omega}$, hence it does not meet $\widehat{K} \subset \bar{\Omega}$. Also, the boundaries $A_{t}(\partial \Delta)$ do not intersect $\widehat{K}$ since they lie in $\mathbb{C}^{2} \backslash \bar{\Omega}$ for all $t$ with $\delta \leqslant t \leqslant \eta$. Since the $A_{t}$ are round discs graphed over the $z_{1}$-axis, we find holomorphic
defining functions as required in the Oka criterion. Hence we deduce that $\bigcup_{\delta \leqslant t \leqslant \eta} A_{t}(\bar{\Delta}) \ni A(1)$ does not meet $\widehat{K}$.

The second case where $A$ is tangent to $M$ at $A(1)$ can be reduced to the above arguments by means of a preliminary slight normal deformation of the disc near the point $A(-1)$ ([Tu1994, MP1999, MP2002]) which produces a new disc $A^{d}$ nontangent to $\partial \Omega$ at $A(1)$.
Proof of Lemma 12.5. We verify that $\Omega \backslash \widehat{K}$ does satisfy the Kontinuitätssatz. Let $\Phi$ be an injective holomorphic map sending a neighborhood of $[0,1] \times$ $\left\{\left|z_{2}\right| \leqslant 1\right\}$ in $\mathbb{C}^{2}$ into $\mathbb{C}^{2}$. Assuming that $\Phi$ maps

$$
\begin{equation*}
\left(\{0\} \times\left\{\left|z_{2}\right| \leqslant 1\right\}\right) \bigcup\left([0,1] \times\left\{\left|z_{2}\right|=1\right\}\right) \tag{12.7}
\end{equation*}
$$

into $\Omega \backslash \widehat{K}$, we have to show $\Phi\left([0,1] \times\left\{\left|z_{2}\right| \leqslant 1\right\}\right) \subset \Omega \backslash \widehat{K}$ also.
Assume on the contrary that there exists a smallest $t^{*} \in(0,1]$ such that $\Phi\left(\left\{t^{*}\right\} \times\left\{\left|z_{2}\right|<1\right\}\right) \not \subset \Omega \backslash \widehat{K}$. Then the open analytic disc $\Phi\left(\left\{t^{*}\right\} \times\right.$ $\left.\left\{\left|z_{2}\right|<1\right\}\right)$ contains a point of $\partial \Omega$ or a point of $\widehat{K} \backslash \partial \Omega$. In the first case, we would contradict the pseudoconvexity of $\Omega$ and in the second case, we would contradict the Oka criterion for $\widehat{K}$.

## §13. Proof of Theorem 1.9

13.1. The geometric recipe. We first construct the 2 -torus $K=T^{2}$, then construct the maximally real $M^{1}$ and finally define $M$ as a certain thickening of $M^{1}$. The argument to insure global minimality of $M$ involves computations with Lie brackets and is postponed to the end.

Firstly, in $\mathbb{R}^{3}=\mathbb{R}^{3} \oplus i\{0\} \subset \mathbb{C}^{3}$ equipped with the coordinates $\left(x_{1}, x_{2}, x_{3}\right)$, where $x_{j}=\operatorname{Re} z_{j}$ for $j=1,2,3$, pick the "standard" 2 dimensional torus $T^{2}$ of Cartesian equation

$$
\begin{equation*}
\left(\sqrt{x_{1}^{2}+x_{2}^{2}}-2\right)^{2}+x_{3}^{2}=1 \tag{13.2}
\end{equation*}
$$

This torus is stable under the rotations directed by the $x_{3}$-axis; its intersection with the $\left(x_{1}, x_{3}\right)$-plane consists of two circles of radius 1 centered at the points $x_{1}=2$ and $x_{1}=-2$; it bounds a three-dimensional open "full" torus $T^{3}$; both $T^{2}$ and $T^{3}$ are contained in the ball $B^{3}$ of radius 5 centered at the origin.

It is better to drop the square root: one checks that the equations of $T^{2}$ and $T^{3}$ are equally given by $T^{2}:=\{\rho=0\}$ and $T^{3}:=\{\rho<0\}$, by means of the polynomial defining function

$$
\begin{equation*}
\rho\left(x_{1}, x_{2}, x_{3}\right):=\left(x_{1}^{2}+x_{2}^{2}+x_{3}^{2}+3\right)^{2}-16\left(x_{1}^{2}+x_{2}^{2}\right), \tag{13.3}
\end{equation*}
$$

which has nonvanishing differential at every point of $T^{2}$. Consequently, the extrinsinc complexification of $T^{2}$, namely the complex hypersurface defined
by

$$
\begin{equation*}
\Sigma:=\left\{\left(z_{1}, z_{2}, z_{3}\right) \in \mathbb{C}^{3}: \rho\left(z_{1}, z_{2}, z_{3}\right)=0\right\} \tag{13.4}
\end{equation*}
$$

cuts $\mathbb{R}^{3}$ along $T^{2}$ with the transversality property $T_{x} \mathbb{R}^{3} \cap T_{x} \Sigma=T_{x} T^{2}$ for every point $x \in T^{2}$.

Secondly, according to Reeb ([CLN1985]) see also the figures there), by considering the space $\mathbb{R}^{3} \equiv S^{3} \backslash\{\infty\}$ as a punctured three-dimensional sphere $S^{3}$, one may glue a second three-dimensional full torus $\widetilde{T}^{3}$ to $T^{3}$ along $T^{2}$ with $\infty \in \widetilde{T}^{3}$ and then construct a foliation of $S^{3}$ by 2 -dimensional surfaces all of whose leaves, except one, are diffeomorphic to $\mathbb{R}^{2}$, are contained in either $T^{3}$ or in $\widetilde{T}^{3}$ and are accumulating on $T^{2}$, and finally, whose single compact leaf is the above 2 -torus $T^{2}$. This yields the so-called Reeb foliation of $S^{3}$, which is $\mathscr{C}^{\infty}$ and orientable. Consequently, there exists a $\mathscr{C}^{\infty}$ smooth vector field $L=a_{1}(x) \partial_{x_{1}}+a_{2}(x) \partial_{x_{2}}+a_{3}(x) \partial_{x_{3}}$ of norm 1, namely $a_{1}(x)^{2}+a_{2}(x)^{2}+a_{3}(x)^{2}=1$ for every $x \in \mathbb{R}^{3}$, which is everywhere orthogonal (with respect to the standard Euclidean structure) to the leaves of the Reeb foliation. Geometrically, the integral curves of $L$ accumulate asymptotically on the two nodal (central) circles of $T^{3}$ and of $\widetilde{T}^{3}$.

The open ball $B^{3} \subset \mathbb{R}^{3}$ of radius 5 centered at the origin will be our maximally real submanifold $M^{1}$. The two-dimensionaly torus $T^{2}$ will be our nonremovable compact set $K$. The integral curves of the vector field $L$ will be our characteristic lines. Since $L$ is orthogonal to $T^{2}$, these characteristic lines will of course be everywhere transverse to $K$, so that $K=T^{2}$ is nontransversal to the integral curves of $L$.

Thirdly, it remains to construct the generic submanifold $M$ of CR dimension 1 containing $M^{1}$ and to check that $K$ will be nonremovable.

First of all, we notice that $L$ provides the characteristic directions of $M^{1}$ if and only if $T_{x} M=T_{p} \mathbb{R}^{3} \oplus \mathbb{R} J L(x)$ for every point $x \in M^{1} \equiv B^{3}$. Consequently, all submanifolds $M \subset \mathbb{C}^{3}$ obtained by slightly thickening $M^{1}$ in the direction of $J L(x)$ will be convenient; in other words, only the first jet of $M$ along $M^{1}$ is prescribed by our choice of the characterisctic vector field $L$. Notice that all such thin strips $M$ along $M^{1}$ will be diffeomorphic to a real 4-ball.

The fact that $K$ is nonremovable for all such generic submanifolds $M$ is now clear: the hypersurface $\Sigma=\left\{z \in \mathbb{C}^{3}: \rho(z)=0\right\}$ satisfying $T_{x} \Sigma=$ $T_{x} T^{2} \oplus \mathbb{R} J T_{x} T^{2}$ for all $x \in T^{2}$ and $L$ being transversal to $T^{2}$, we easily deduce the transversality property $T_{x} \Sigma+T_{x} M=T_{x} \mathbb{C}^{3}$ for all $x \in T^{2}$, a geometric property which insures that the holomorphic function $1 / \rho(z)$, which is CR on $M \backslash K$, does not extend holomorphically to any wedge of edge $M$ at any point of $K$. Intuitively, $T_{x} \Sigma / T_{x} M$ absorbs all the normal space $T_{x} \mathbb{C}^{3} / T_{x} M$ at every point $x \in T^{2}=K$, leaving no room for any open cone.

Finally, to fulfill all the hypotheses of Proposition 1.13 (except of course nontransversality of $K$ to $\mathrm{F}_{M^{1}}^{c}$ ), we have to insure that $M$ is globally minimal. We claim that by bending strongly the second and the fourth order jet of $M$ along $M^{1}$ (without modifying the first order jet which must be prescribed by $J L$ ), one may insure that $M$ is of type 4 in the sense of Definition 4.22 (III) in [29] at every point of $M^{1}$; since being of finite type is an open property, it follows that $M$ is finite type at every point provided that, as a strip, $M$ is sufficiently thin along $M^{1}$. As is known, finite-typeness at every point implies local minimality at every point which in turn implies global minimality. This completes the recipe.
13.5. Finite-typisation. To complete the arguments of Theorem 1.9, it remains to construct a generic submanifold $M \subset \mathbb{C}^{3}$ of CR dimension 1 satisfying $T_{x} M=T_{x} M^{1} \oplus \mathbb{R} J L(x)$ for every $x \in M^{1}$, which is of type 4 at every point $x \in M^{1}$.

First of all, let us denote by $L=a_{1}(x) \partial_{x_{1}}+a_{2}(x) \partial_{x_{2}}+a_{3}(x) \partial_{x_{3}}$ the unit vector field which was constructed as a field orthogonal to the Reeb foliation: it is defined over $\mathbb{R}^{3}$ and has $\mathscr{C}^{\infty}$ coefficients satisfying $a_{1}(x)^{2}+$ $a_{2}(x)^{2}+a_{3}(x)^{2}=1$ for all $x \in \mathbb{R}^{2}$. The two-dimensional quotient vector bundle $T \mathbb{R}^{3} /(\mathbb{R} L)$ with contractible base being necessarily trivial, it follows that we can complete $L$ by two other $\mathscr{C}^{\infty}$ unit vector fields $K^{1}$ and $K^{2}$ defined over $\mathbb{R}^{3}$ such that the triple $\left(L(x), K^{1}(x), K^{2}(x)\right)$ forms a direct orthonormal frame at every point $x \in \mathbb{R}^{3}$. Let us denote the coefficients of $K^{1}$ and of $K^{2}$ by

$$
\begin{align*}
& K^{1}=\rho_{1} \partial_{x_{1}}+\rho_{2} \partial_{x_{2}}+\rho_{3} \partial_{x_{3}}, \\
& K^{2}=r_{1} \partial_{x_{1}}+r_{2} \partial_{x_{2}}+r_{3} \partial_{x_{3}} \tag{13.6}
\end{align*}
$$

where $\rho_{j}$ and $r_{j}$ for $j=1,2,3$ are $\mathscr{C}^{\infty}$ functions of $x \in \mathbb{R}^{3}$ satisfying $\rho_{1}^{2}+\rho_{2}^{2}+\rho_{3}^{2}=1$ and $r_{1}^{2}+r_{2}^{2}+r_{3}^{2}=1$. In our case, $K^{1}$ and $K^{2}$ may even be constructed directly by means of a trivialization of the bundle tangent to the Reeb foliation.

Let $P>0$ be a constant, which will be chosen later to be large. Since by construction we have the two orthogonality relations $a_{1} \rho_{1}+a_{2} \rho_{2}+a_{3} \rho_{3}=0$ and $a_{1} r_{1}+a_{2} r_{2}+a_{3} r_{3}=0$, it follows that every generic submanifold $M_{P} \subset$ $\mathbb{C}^{3}$ defined by the two Cartesian equations

$$
\begin{align*}
& 0=\rho=y_{1} \rho_{1}(x)+y_{2} \rho_{2}(x)+y_{3} \rho_{3}(x)+P\left[y_{1}^{2}+y_{2}^{2}+y_{3}^{2}\right], \\
& 0=r=y_{1} r_{1}(x)+y_{2} r_{2}(x)+y_{3} r_{3}(x)+P^{3}\left[y_{1}^{4}+y_{2}^{4}+y_{3}^{4}\right] \tag{13.7}
\end{align*}
$$

enjoys the property that the vector field $J L(x)=a_{1}(x) \partial_{y_{1}}+a_{2}(x) \partial_{x_{2}}+$ $a_{3}(x) \partial_{x_{3}}$ is tangent to $M_{P}$ at every $x \in \mathbb{R}^{3}$. As desired, we deduce that $T_{x}^{c} M=\mathbb{R} L(x) \oplus J \mathbb{R} L(x)$ for every $x \in \mathbb{R}^{3}$, a property which insures that $\mathbb{R} L(x)$ is the characteristic direction of $M^{1}$ in $M_{P}$, independently of $P$.

To complete the final minimalization argument for the construction of a nonremovable compact set $C:=T^{2} \subset M^{1} \subset M$ which appears in the Introduction, it suffices now to apply the following lemma with $R=5$. Though calculatory, its proof is totally elementary.

Lemma 13.8. For every $R>0$, there exist $P>0$ sufficiently large such that $M_{P}$ is of type 4 at every point $x \in \mathbb{R}^{3}$ with $x_{1}^{2}+x_{2}^{2}+x_{3}^{2} \leqslant R^{2}$.
Proof. As above, let $M_{P}=\left\{z \in \mathbb{C}^{3}: \rho=r=0\right\}$. By writing the tangency condition, one checks immediately that the one-dimensional complex vector bundle $T^{1,0} M_{P}$ is generated over $\mathbb{C}$ by the vector field $\mathbb{L}:=A_{1} \partial_{z_{1}}+A_{2} \partial_{z_{2}}+$ $A_{3} \partial_{z_{3}}$, with the explicit expressions

$$
\begin{align*}
& A_{1}:=4 \rho_{z_{3}} r_{z_{2}}-4 \rho_{z_{2}} r_{z_{3}}, \\
& A_{2}:=4 \rho_{z_{1}} r_{z_{3}}-4 \rho_{z_{3}} r_{z_{1}},  \tag{13.9}\\
& A_{3}:=4 \rho_{z_{2}} r_{z_{1}}-4 \rho_{z_{1}} r_{z_{2}} .
\end{align*}
$$

Using the expressions (13.7) for $\rho$ and $r$, we see that these three components restrict on $\{y=0\}$ as the Plücker coordinates of the bivector $\left(K^{1}, K^{2}\right)$, namely

$$
\begin{align*}
& \left.A_{1}\right|_{y=0}=\rho_{2} r_{3}-\rho_{3} r_{2}=: \Delta_{2,3}, \\
& \left.A_{2}\right|_{y=0}=\rho_{3} r_{1}-\rho_{1} r_{3}=: \Delta_{3,1},  \tag{13.10}\\
& \left.A_{3}\right|_{y=0}=\rho_{1} r_{2}-\rho_{2} r_{1}=: \Delta_{1,2} .
\end{align*}
$$

As $K^{1}$ and $K^{2}$ are of norm 1 and orthogonal at every point, it follows by direct computation that $\Delta_{2,3}^{2}+\Delta_{3,1}^{2}+\Delta_{1,2}^{2}=1$ and that the vector of coordinates $\left(\Delta_{2,3}, \Delta_{3,1}, \Delta_{1,2}\right)$ is orthogonal to both $K^{1}$ and $K^{2}$. Moreover, as the orthonormal trihedron $\left(L(x), K^{1}(x), K^{2}(x)\right)$ is direct at every point, we deduce that necessarily

$$
\begin{equation*}
\Delta_{2,3} \equiv a_{1}, \quad \Delta_{3,1} \equiv a_{2}, \quad \Delta_{1,2} \equiv a_{3} . \tag{13.11}
\end{equation*}
$$

Next, we compute in length $A_{1}, A_{2}$ and $A_{3}$ using (13.7). As their complete explicit development will not be crucial for the sequel and as we shall perform with them differentiations and linear combinations yielding relatively complicated expressions, let us adopt the following notation: by $\mathscr{R}^{0}$, we denote various expressions which are polynomials in the jets of the functions $\rho_{1}, \rho_{2}, \rho_{3}$ and $r_{1}, r_{2}, r_{3}$. Similarly, by $\mathscr{R}^{I}$, by $\mathscr{R}^{I I}$, by $\mathscr{R}^{I I I}$ and by $\mathscr{R}^{I V}$, we denote polynomials in the transverse variables $\left(y_{1}, y_{2}, y_{3}\right)$ which are homogeneous of degree $1,2,3$ and 4 and have as coefficients various expressions $\mathscr{R}^{0}$.

Importantly, we make the convention that such expressions $\mathscr{R}^{0}, \mathscr{R}^{I}, \mathscr{R}^{I I}$, $\mathscr{R}^{I I I}$ and $\mathscr{R}^{I V}$ should be totally independent of the constant $P$. Consequently, if $P$ appears somehow, we shall write it as a factor, as for instance in $P \mathscr{R}^{I}$ or in $P^{3} \mathscr{R}^{I I I}$.

With this convention at hand, we may develope (13.9) using the expressions (13.7) by writing out only the terms which will be useful in the sequel and by treating the rest as controlled remainders. Let us detail the computation of $A_{1}$ :
(13.12)

$$
\begin{aligned}
A_{1}= & 4\left[-\frac{i}{2} \rho_{3}-i P y_{3}+\mathscr{R}^{I}\right]\left[-\frac{i}{2} r_{2}-2 i P^{3} y_{2}^{3}+\mathscr{R}^{I}\right]- \\
& -4\left[-\frac{i}{2} \rho_{2}-i P y_{2}+\mathscr{R}^{I}\right]\left[-\frac{i}{2} r_{3}-2 i P^{3} y_{3}^{3}+\mathscr{R}^{I}\right] \\
= & -\rho_{3} r_{2}-4 P^{3} \rho_{3} y_{2}^{3}+\mathscr{R}^{I}-2 P r_{2} y_{3}+P^{4} \mathscr{R}^{I V}+P \mathscr{R}^{I}+\mathscr{R}^{I}+P^{3} \mathscr{R}^{I V}+\mathscr{R}^{I I} \\
& +\rho_{2} r_{3}+4 P^{3} \rho_{2} y_{3}^{3}+\mathscr{R}^{I}+2 P r_{3} y_{2}+P^{4} \mathscr{R}^{I V}+P \mathscr{R}^{I}+\mathscr{R}^{I}+P^{3} \mathscr{R}^{I V}+\mathscr{R}^{I I} \\
= & \rho_{2} r_{3}-\rho_{3} r_{2}+2 P r_{3} y_{2}-2 P r_{2} y_{3}+4 P^{3} \rho_{2} y_{3}^{3}-4 P^{3} \rho_{3} y_{2}^{3}+ \\
& +\mathscr{R}^{I}+\mathscr{R}^{I I}+P \mathscr{R}^{I I}+P^{3} \mathscr{R}^{I V}+P^{4} \mathscr{R}^{I V} .
\end{aligned}
$$

In the development, before simplification, we firstly write out in lines 3 and 4 all the $9 \times 2$ terms of the two product: for instance, the third term of the first product, namely $4\left(-\frac{i}{2} \rho_{3}\right)\left(\mathscr{R}^{I}\right)$, yields a term $\mathscr{R}^{I}$ whereas the fifth term $4\left(-i P y_{3}\right)\left(-2 i P^{3} y_{2}^{3}\right)$ yields a term $P^{4} \mathscr{R}^{I V}$; secondly, we simplify the obtained sum: by our convention, $\mathscr{R}^{I}+\mathscr{R}^{I}=\mathscr{R}^{I}$, whereas $\mathscr{R}^{I}+P \mathscr{R}^{I}$ cannot be simplified, since the large constant $P$ will be chosen later. With these technical explanations at hand, we shall not provide any intermediate detail for the further computations, whose rules are totally analogous. For $A_{1}, A_{2}$ and $A_{3}$, we obtain

$$
\left\{\begin{align*}
A_{1}= & \rho_{2} r_{3}-\rho_{3} r_{2}+2 P r_{3} y_{2}-2 P r_{2} y_{3}+4 P^{3} \rho_{2} y_{3}^{3}-4 P^{3} \rho_{3} y_{2}^{3}+  \tag{13.13}\\
& +\mathscr{R}^{I}+\mathscr{R}^{I I}+P \mathscr{R}^{I I}+P^{3} \mathscr{R}^{I V}+P^{4} \mathscr{R}^{I V}, \\
A_{2}= & \rho_{3} r_{1}-\rho_{1} r_{3}+2 P r_{1} y_{3}-2 P r_{3} y_{1}+4 P^{3} \rho_{3} y_{1}^{3}-4 P^{3} \rho_{1} y_{3}^{3}+ \\
& +\mathscr{R}^{I}+\mathscr{R}^{I I}+P \mathscr{R}^{I I}+P^{3} \mathscr{R}^{I V}+P^{4} \mathscr{R}^{I V} \\
A_{3}= & \rho_{1} r_{2}-\rho_{2} r_{1}+2 P r_{2} y_{1}-2 P r_{1} y_{2}+4 P^{3} \rho_{1} y_{2}^{3}-4 P^{3} \rho_{2} y_{1}^{3}+ \\
& +\mathscr{R}^{I}+\mathscr{R}^{I I}+P \mathscr{R}^{I I}+P^{3} \mathscr{R}^{I V}+P^{4} \mathscr{R}^{I V} .
\end{align*}\right.
$$

Now that we have written the complex vector field $\mathbb{L}$ and its coefficients $A_{1}, A_{2}$ and $A_{3}$, in order to establish Lemma 13.8, it suffices to choose $P>0$ sufficiently large in order that the four complex vector fields

$$
\begin{equation*}
\left.\overline{\mathbb{L}}\right|_{y=0},\left.\quad \mathbb{L}\right|_{y=0},\left.\quad[\overline{\mathbb{L}}, \mathbb{L}]\right|_{y=0},\left.\quad[\overline{\mathbb{L}},[\overline{\mathbb{L}},[\overline{\mathbb{L}}, \mathbb{L}]]]\right|_{y=0} \tag{13.14}
\end{equation*}
$$

are linearly independent at every point $x \in \mathbb{R}^{3}$ with $x_{1}^{2}+x_{2}^{2}+x_{3}^{2} \leqslant R^{2}$. At the end of the proof, we shall explain why we cannot insure type 3 at every point, namely why the consideration of $\left.[\mathbb{L},[\mathbb{L}, \mathbb{L}]]\right|_{y=0}$ instead of the length four last Lie bracket in (13.14) would fail.

As promised, we shall now summarize all the subsequent computations. As we aim to restrict the last Lie bracket to $\{y=0\}$ which is of length four and whose coefficients involve derivatives of order at most three of the coefficients $A_{1}, A_{2}$ and $A_{3}$, we can already neglect the last two remainders $P^{3} \mathscr{R}^{I V}$ and $P^{4} \mathscr{R}^{I V}$ in (13.13). In other words, we can consider $A^{1}, A^{2}$ and $A^{3} \bmod (I V)$. Similarly, in the computation of the Lie bracket

$$
\begin{equation*}
[\overline{\mathbb{L}}, \mathbb{L}]=: C_{1} \partial_{z_{1}}+C_{2} \partial_{z_{2}}+C_{3} \partial_{z_{3}}-\overline{C_{1}} \partial_{\bar{z}_{1}}-\overline{C_{2}} \partial_{\bar{z}_{2}}-\overline{C_{3}} \partial_{\bar{z}_{3}} \tag{13.14}
\end{equation*}
$$

before restriction to $\{y=0\}$, we can restrict our task to developing the coefficients

$$
\begin{align*}
& C_{1}:=\overline{A_{1}} A_{1, \bar{z}_{1}}+\overline{A_{2}} A_{1, \bar{z}_{2}}+\overline{A_{3}} A_{1, \bar{z}_{3}}, \\
& C_{2}  \tag{13.16}\\
& :=\overline{A_{1}} A_{2, \bar{z}_{1}}+\overline{A_{2}} A_{2, \bar{z}_{2}}+\overline{A_{3}} A_{2, \bar{z}_{3}}, \\
& C_{3}:=\overline{A_{1}} A_{3, \bar{z}_{1}}+\overline{A_{2}} A_{3, \bar{z}_{2}}+\overline{A_{3}} A_{3, \bar{z}_{3}}
\end{align*}
$$

only modulo order (III), which yields by means of the expressions (13.13) (13.17)

$$
\begin{aligned}
C_{1} \bmod (I I I) \equiv & -i P \rho_{1}+6 i P^{3} a_{3} \rho_{2} y_{3}^{2}-6 i P^{3} a_{2} \rho_{3} y_{2}^{2}+\mathscr{R}^{0}+\mathscr{R}^{I}+ \\
& +P \mathscr{R}^{I}+P^{2} \mathscr{R}^{I}+\mathscr{R}^{I I}+P \mathscr{R}^{I I}+P^{2} \mathscr{R}^{I I}, \\
C_{2} \bmod (I I I) \equiv & -i P \rho_{2}+6 i P^{3} a_{1} \rho_{3} y_{1}^{2}-6 i P^{3} a_{3} \rho_{1} y_{3}^{2}+\mathscr{R}^{0}+\mathscr{R}^{I}+ \\
& +P \mathscr{R}^{I}+P^{2} \mathscr{R}^{I}+\mathscr{R}^{I I}+P \mathscr{R}^{I I}+P^{2} \mathscr{R}^{I I}, \\
C_{3} \bmod (I I I) \equiv & -i P \rho_{3}+6 i P^{3} a_{2} \rho_{1} y_{2}^{2}-6 i P^{3} a_{1} \rho_{2} y_{1}^{2}+\mathscr{R}^{0}+\mathscr{R}^{I}+ \\
& +P \mathscr{R}^{I}+P^{2} \mathscr{R}^{I}+\mathscr{R}^{I I}+P \mathscr{R}^{I I}+P^{2} \mathscr{R}^{I I} .
\end{aligned}
$$

We must mention the use of natural rule hold for computing the partial derivatives $A_{j, \bar{z}_{k}}$ : we have for instance $\partial_{\bar{z}_{k}}\left(\mathscr{R}^{I I}\right)=\mathscr{R}^{I}+\mathscr{R}^{I I}$. Also, we have used the hypothesis that $\left(L(x), K^{1}(x), K^{2}(x)\right)$ provides a direct orthonormal frame at every $x \in \mathbb{R}^{3}$, which yields in particular the three relations

$$
\begin{equation*}
a_{2} r_{3}-a_{3} r_{2}=-\rho_{1}, \quad a_{3} r_{1}-a_{1} r_{3}=-\rho_{2}, \quad a_{1} r_{2}-a_{2} r_{1}=-\rho_{3} . \tag{13.18}
\end{equation*}
$$

After mild computation, the coefficients $F_{1}, F_{2}$ and $F_{3}$ of the length four Lie bracket

$$
\stackrel{(13.19)}{[\overline{\mathbb{L}},[\overline{\mathbb{L}},[\overline{\mathbb{L}}, \mathbb{L}]]]=F_{1} \partial_{z_{1}}+F_{2} \partial_{z_{2}}+F_{3} \partial_{z_{3}}+G_{1} \partial_{\bar{z}_{1}}+G_{2} \partial_{\bar{z}_{2}}+G_{3} \partial_{\bar{z}_{3}}}
$$

are given, after restriction to $\{y=0\}$, by

$$
\begin{align*}
& \left.F_{1}\right|_{y=0}=3 i P^{3} a_{2}^{3} \rho_{3}-3 i P^{3} a_{3}^{3} \rho_{2}+\mathscr{R}^{0}+P \mathscr{R}^{0}+P^{2} \mathscr{R}^{0}, \\
& \left.F_{2}\right|_{y=0}=3 i P^{3} a_{3}^{3} \rho_{1}-3 i P^{3} a_{1}^{3} \rho_{3}+\mathscr{R}^{0}+P \mathscr{R}^{0}+P^{2} \mathscr{R}^{0},  \tag{13.20}\\
& \left.F_{3}\right|_{y=0}=3 i P^{3} a_{1}^{3} \rho_{2}-3 i P^{3} a_{2}^{3} \rho_{1}+\mathscr{R}^{0}+P \mathscr{R}^{0}+P^{2} \mathscr{R}^{0},
\end{align*}
$$

We can now complete the proof of Lemma 13.8. In the basis $\left(\partial_{z_{1}}, \partial_{z_{2}}, \partial_{z_{3}}, \partial_{\bar{z}_{1}}, \partial_{\bar{z}_{2}}, \partial_{\bar{z}_{3}}\right)$, the $4 \times 6$ matrix associated with the four vector fields (13.14) (without mentioning $\left.\right|_{y=0}$ )

$$
\left(\begin{array}{cccccc}
0 & 0 & 0 & a_{1} & a_{2} & a_{3}  \tag{13.21}\\
a_{1} & a_{2} & a_{3} & 0 & 0 & 0 \\
C_{1} & C_{2} & C_{3} & -\overline{C_{1}} & -\overline{C_{2}} & -\overline{C_{3}} \\
F_{1} & F_{2} & F_{3} & G_{1} & G_{2} & G_{3}
\end{array}\right)
$$

has rank four at a point $x \in \mathbb{R}^{3}$ if and only if the $3 \times 3$ determinant in the left low corner is nonvanishing, namely if and only if the developped expression (13.22)

$$
\begin{aligned}
& \left|\begin{array}{ccc}
a_{1} & a_{2} & a_{3} \\
-i P \rho_{1}+\mathscr{R}^{0} & -i P \rho_{2}+\mathscr{R}^{0} & -i P \rho_{3}+\mathscr{R}^{0} \\
3 i P^{3} a_{2}^{3} \rho_{3}-3 i P^{3} a_{3}^{3} \rho_{2}+ & 3 i P^{3} a_{3}^{3} \rho_{1}-3 i P^{3} a_{1}^{3} \rho_{3}+ & 3 i P^{3} a_{1}^{3} \rho_{2}-3 i P^{3} a_{2}^{3} \rho_{1}+ \\
+\mathscr{R}^{0}+P \mathscr{R}^{0}+P^{2} \mathscr{R}^{0} & +\mathscr{R}^{0}+P \mathscr{R}^{0}+P^{2} \mathscr{R}^{0} & +\mathscr{R}^{0}+P \mathscr{R}^{0}+P^{2} \mathscr{R}^{0}
\end{array}\right| \\
& =3 P^{4}\left(r_{3}\left[a_{1}^{3} \rho_{2}-a_{2}^{3} \rho_{1}\right]+r_{2}\left[a_{3}^{3} \rho_{1}-a_{1}^{3} \rho_{3}\right]+r_{1}\left[a_{2}^{3} \rho_{3}-a_{3}^{3} \rho_{2}\right]\right)+ \\
& +\mathscr{R}^{0}+P \mathscr{R}^{0}+P^{2} \mathscr{R}^{0}+P^{3} \mathscr{R}^{0}+P^{4} \mathscr{R}^{0}
\end{aligned}
$$

is nonvanishing.
At this point, the conclusion of the lemma is now an immediate consequence of the following trivial assertion: Let $a_{1}, a_{2}$ and $a_{3}$ be $\mathscr{C}^{\infty}$ functions on $\mathbb{R}^{3}$ satisfying $a_{1}(x)^{2}+a_{2}(x)^{2}+a_{3}(x)^{2}=1$ for all $x \in \mathbb{R}^{3}$ and let $\mathscr{R}_{0}^{0}$, $\mathscr{R}_{1}^{0}, \mathscr{R}_{2}^{0}, \mathscr{R}_{3}^{0}$ and $\mathscr{R}_{4}^{0}$ be $\mathscr{C}^{\infty}$ functions on $\mathbb{R}^{3}$. For every $R>0$, there exists a constant $P>0$ large enough so that the function

$$
\begin{equation*}
3 P^{4}\left(a_{1}^{4}+a_{2}^{4}+a_{3}^{4}\right)+\mathscr{R}_{0}^{0}+P \mathscr{R}_{1}^{0}+P^{2} \mathscr{R}_{2}^{0}+P^{3} \mathscr{R}_{3}^{0}+P^{4} \mathscr{R}_{4}^{0} \tag{13.23}
\end{equation*}
$$

is positive at every $x \in \mathbb{R}^{3}$ with $x_{1}^{2}+x_{2}^{2}+x_{3}^{2} \leqslant R^{2}$.
If we had put $y_{1}^{3}+y_{2}^{3}+y_{3}^{3}$ instead of $y_{1}^{4}+y_{2}^{4}+y_{3}^{4}$ in the second equation (13.7), we would have considered the length three Lie bracket $\left.[\mathbb{L},[\mathbb{L}, \mathbb{L}]]\right|_{y=0}$ instead of the length four Lie bracket in (13.14), and hence instead of the quartic $a_{1}^{4}+a_{2}^{4}+a_{3}^{4}$ in (13.23), we would have obtained the cubic $a_{1}^{3}+a_{2}^{3}+a_{3}^{3}$, a function which (unfortunately) vanishes, for instance if $a_{1}(x)=\frac{1}{\sqrt{2}}, a_{2}(x)=-\frac{1}{\sqrt{2}}$ and $a_{3}(x)=0$. We notice that in our example, this value of $\left(a_{1}, a_{2}, a_{3}\right)$ is indeed attained at the point $x \in T^{2}$ of coordinates $\left(\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}}, 0\right)$, whence the necessity of passing to type 4 . The proof of Lemma 13.8 is complete.

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# A Morse-theoretical proof of the Hartogs extension theorem 

Joël Merker and Egmont Porten


#### Abstract

Hartogs published a celebrated extension phenomenon (birth of Several Complex Variables), whose global counterpart was understood later: holomorphic functions in a connected neighborhood $\mathscr{V}(\partial \Omega)$ of a connected boundary $\partial \Omega \Subset \mathbb{C}^{n}(n \geqslant 2)$ do extend holomorphically and uniquely to the domain $\Omega$. Martinelli in the early 1940's and Ehrenpreis in 1961 obtained a rigorous proof, using a new multidimensional integral kernel or a short $\bar{\partial}$ argument, but it remained unclear how to derive a proof using only analytic discs, as did Hurwitz (1897), Hartogs (1906) and E.E. Levi (1911) in some special, model cases. In fact, known attempts (e.g. Osgood 1929, Brown 1936) struggled for monodromy against multivaluations, but failed to get the general global theorem.

Moreover, quite unexpectedly, Fornæss in 1998 exhibited a topologically strange (nonpseudoconvex) domain $\Omega^{\mathrm{F}} \subset \mathbb{C}^{2}$ that cannot be filled in by holomorphic discs, when one makes the additional requirement that discs must all lie entirely inside $\Omega^{\mathrm{F}}$. However, one should point out that the standard, unrestricted disc method usually allows discs to go outsise the domain (just think of Levi pseudoconcavity).

Using the method of analytic discs for local extensional steps and Morsetheoretical tools for the global topological control of monodromy, we show that the Hartogs extension theorem can be established in such a way.


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Table of contents

1. The Hartogs extension theorem and the method of analytic discs . . . . . . . . . 407 .
2. Preparation of the boundary and unique extension . . . . . . . . . . . . . . . . . . . . . 412 .
3. Quantitative Hartogs-Levi extension by pushing analytic discs . . . . . . . . . . . 415.
4. Filling domains outside balls of decreasing radius ............................... . 420.
5. Creating domains, merging and suppressing connected components ...... 428.
6. The exceptional case $k_{\lambda}=1$.......................................................... 439.
[22 colored illustrations]

## §1. The Hartogs extension theorem AND THE METHOD OF ANALYTIC DISCS

100 years ago exactly, in 1906, the publication of Hartogs's thesis ([14] under the direction of Hurwitz) revealed what is now considered to be the most striking fact of multidimensional complex analysis: the automatic, compulsory holomorphic extension of functions of several complex variables to larger domains, especially for a class of "pot-looking" domains,
nowadays called Hartogs figures, that may be filled in up to their top. Soon after, E.E. Levi [25] applied the Hurwitz-Hartogs argument of Cauchy integration on complex affine circles moving in the domain (firstly discovered in [21]), in order to perform local holomorphic extension across strictly (Levi) pseudoconcave boundaries. The so-called method of analytic discs was born, historically.

Hartogs extension theorem. Let $\Omega \Subset \mathbb{C}^{n}$ be a bounded domain having connected boundary. If $n \geqslant 2$, every function holomorphic in some connected open neighborhood $\mathscr{V}(\partial \Omega)$ of $\partial \Omega$ extend holomorphically and uniquely inside $\Omega$, i.e.:

$$
\forall f \in \mathscr{O}(\mathscr{V}(\partial \Omega)), \quad \exists!F \in \mathscr{O}(\Omega \cup \mathscr{V}(\partial \Omega)) \text { s.t. }\left.\quad F\right|_{\mathscr{V}(\partial \Omega)}=f
$$

Classically, one also presents an alternative formulation, which is checked to be equivalent - think that $K=\Omega \backslash \mathscr{V}(\partial \Omega)$.
 compact such that $\Omega \backslash K$ connected, then $\mathscr{O}(\Omega \backslash K)=\left.\mathscr{O}(\Omega)\right|_{\Omega \backslash K}$.

Already in [14] (p. 231), Hartogs stated such a global theorem in the typical language of those days, without claiming single-valuedness however something that he consistently mentions in other places. Later in [32], Osgood (who gives the reference to Hartogs) "proves" unique holomorphic extension with discs, but what is written there is seriously erroneous, even when applied to a ball. In 1936, well before Milnor ([31]) had popularized Morse theory, using topological concepts and a language which are nowadays difficult to grasp, Brown ([5]) fixed somehow single-valuedness of the extension ${ }^{39}$ : discretizing $\Omega \backslash K$ to tame the topology, he exhausts $\mathbb{C}^{n}$ by spheres of decreasing radius (as we will do in this paper), but we believe that his proof still contains imprecisions, because the subtracting process that we encounter unavoidably when applying Morse theory does not appear in [5].

Since the 1940's, few complex analysts have seriously thought about testing the limit of the disc method probably because the motivation was gone, and in fact, the possible existence of an elementary rigorous proof of the global Hartogs extension theorem using only a finite number of Hartogs figures remained a folklore belief; for instance, in [35], p. 133, it is just left as an "exercise". But to the authors' knowledge, no reliable mathematical publication shows fully how to perform a rigorous proof of the global theorem, using only the original Hurwitz-Hartogs-Levi analytic discs as a tool.

[^38]On the other hand, thanks to the contributions of Kneser ([24]), of Fueter ([11]), of Martinelli ([27, 28]), of Bochner ([4]) and of Fichera ([9]), powerful multidimensional integral kernels were discovered that provided a complete proof, from the side of Analysis. Soon after, Ehrenpreis ([8]) found what is known to be the most concise proof, based on the vanishing of $\bar{\partial}$ cohomology with compact support. This proof was learnt by generations of complex analysts, thanks to Hörmander's book [18]. Range's Correction of the Historical Record [30] provides an excellent account of the very birth of integral formulas in $\mathbb{C}^{n}$. Since the 1960 's, $\bar{\partial}$ techniques, $L^{2}$ methods and integral kernels developed into a vast field of research in Several Complex Variables, c.f. $[18,2,16,15,29,6,7,22,21,26,16]$.

A decade ago, Fornæss [10] produced a topologically strange domain $\Omega^{F}$ that cannot be filled in by means of analytic discs, when one makes the additional requirement that discs must all lie entirely inside the domain. Possibly, one could interpret this example as a "defeat" of geometrical methods.

But in absence of pseudoconvexity, it is much more natural to allow discs to go outside the domain, because the local E.E. Levi extension theorem already needs that. In fact, as remarked by Bedford in his review [3] of [10], Hartogs' phenomenon for Fornæss' domain $\Omega^{\mathrm{F}}$ may be shown to hold straightforwardly by means of the usual, unrestricted disk method.

Furthermore, the study of envelopes of holomorphy (see the monograph of Jarnicki and Pflug [22] for an introduction to Riemann domains spread over $\mathbb{C}^{n}$ and [29] for applications in a CR context) shows well how natural it is to deal with sucessively enlarged (Riemann) domains. Bishop's constructive approach, especially his famous idea of gluing discs to real submanifolds, reveals to be adequate in such a widely open field of research. We hence may hope that, after the very grounding historical theorem of Hartogs has enjoyed a renewed proof, geometrical methods will undergo further developments, especially to devise fine holomorphic extension theorems that are unreachable by means of contemporary $\bar{\partial}$ techniques.

In this paper, we establish rigorously that the Hartogs extension theorem can be proved by means of a finite number of parameterized families of analytic discs (Theorems 2.7 and 5.4). The discs we use are all (tiny) pieces of complex lines in $\mathbb{C}^{n}$. The main difficulty is topological and we use the Morse machinery to tame multisheetedness.

At first, we shall replace the boundary $\partial \Omega$ by a $\mathscr{C}^{\infty}$ connected oriented hypersurface $M \Subset \mathbb{C}^{n}(n \geqslant 2)$ for which the restriction to $M$ of the Euclidean norm function $z \mapsto\|z\|$ is a good Morse function (Lemma 3.3), namely there exist only finitely many points $\widehat{p}_{\lambda} \in M, 1 \leqslant \lambda \leqslant \kappa$, with $\left\|\widehat{p}_{1}\right\|<\cdots<\left\|\widehat{p}_{\kappa}\right\|$ at which $z \mapsto\|z\|$ restricted to $M$ has vanishing differential. We also replace $\mathscr{V}(\partial \Omega)$ by a very thin tubular neighborhood $\mathscr{V}_{\delta}(M)$,
$0<\delta \ll 1$, and $\Omega$ by a domain $\Omega_{M} \Subset \mathbb{C}^{n}$ bounded by $M$. Next, we will introduce a modification of the Hartogs figure, called a Levi-Hartogs figure, which is more appropriate to produce holomorphic extension from the cut out domains $\{\|z\|>r\} \cap \Omega_{M}$, where the radius $r$ will decrease, inductively. Local Levi pseudoconcavity of the exterior of a ball then enables us to prolong the holomorphic functions to $\{\|z\|>r-\eta\} \cap \Omega_{M}$, for some uniform $\eta$ with $0<\eta \ll 1$, which depends on the dimension $n \geqslant 2$, on $\delta$, and on the diameter of $\bar{\Omega}$. We hence descend stepwise to lower radii until the domain is fully filled in.


Fig. 1: Filling the domain, creating, merging and suppressing components
However, this naive conclusion fails because of multivaluations and a crucial three-piece topological device is required. We begin by filling the top of the domain, which is simply diffeomorphic to a cut out piece of ball. Geometrically speaking, Morse points $\widehat{p}_{\lambda}, 1 \leqslant \lambda \leqslant \kappa$, are the only points of $M$ at which the family of spheres $(\{\|z\|=r\})_{0<r<\infty}$ are tangent to $M$. We denote $\left\|\widehat{p}_{\lambda}\right\|=$ : $\widehat{r}_{\lambda}$ with $\widehat{r}_{1}<\cdots<\widehat{r}_{\kappa}$. In Figure 1, we have $\kappa=6$. For an arbitrary fixed radius $r$ with $\widehat{r}_{\lambda}<r<\widehat{r}_{\lambda+1}$, and some fixed $\lambda$ with $1 \leqslant \lambda \leqslant \kappa-1$, we consider all connected components $M_{>r}^{c}, 1 \leqslant c \leqslant c_{\lambda}$, of the cut out hypersurface $M \cap\{\|z\|>r\}$. Their number $c_{\lambda}$ is the same for all $r \in\left(\widehat{r}_{\lambda}, \widehat{r}_{\lambda+1}\right)$. In Figure 1, when $\widehat{r}_{3}<r<\widehat{r}_{4}$, we see three such components.

By descending discrete induction $r \mapsto r-\eta$, we show that each such connected hypersurface $M_{>r}^{c} \subset\{\|z\|>r\}$ bounds a certain domain $\widetilde{\Omega}_{>r}^{c} \subset\{\|z\|>r\}$ which is relatively compact in $\mathbb{C}^{n}$ and that holomorphic functions in $\mathscr{V}_{\delta}(M)$ do extend holomorphically and uniquely to $\widetilde{\Omega}_{>r}^{c}$. While approaching a lower Morse point, three different topological processes will occur ${ }^{40}$ : creating a new component $\widetilde{\Omega} \widetilde{P r-\eta}_{c^{\prime}}$ to be filled in further; merging

[^39]two components $\widetilde{\Omega}_{\underset{\sim}{c} \widetilde{\Omega}_{1-\eta}^{\prime}}^{c_{1}^{\prime}}$ and $\widetilde{\Omega}_{>r-\eta}^{c_{2}^{\prime}}$ which meet; and suppressing one superfluous component $\widetilde{\Omega}_{>r-\eta}^{c_{1}^{\prime}}$.

The unavoidable multivaluation phenomenon will be tamed by the idea of separating ab initio the components $M_{>r}^{c}, 1 \leqslant c \leqslant c_{\lambda}$. Indeed, an advantageous topological property will be shown to be inherited through the induction $r$ $\mapsto r-\eta$, hence always true, namely that two different domains $\widetilde{\Omega}_{>r}^{c_{1}}$ and $\widetilde{\Omega}_{>r}^{c_{1}}$ are either disjoint or one is contained in the other. Consequently, the multivaluation aspect will only happen in the sense that the two uniquely defined and univalent holomorphic extensions $f_{r}^{c_{1}}$ to $\widetilde{\Omega}_{>r}^{c_{1}}$ and $f_{r}^{c_{2}}$ to $\widetilde{\Omega}_{>r}^{c_{2}}$ can be different on $\widetilde{\Omega}_{>r}^{c_{1}}$, in case $\widetilde{\Omega}_{>r}^{c_{1}} \subset \widetilde{\Omega}_{>r}^{c_{2}}$, or vice versa. In this way, we avoid completely to deal with Riemann domains spread over $\mathbb{C}^{n}$.

Some of the elements of our approach should be viewed in a broader context. In their celebrated paper [1] (see also [15]), Andreotti and Grauert observed that convenient exhaustion functions can be used to prove very general extension and finiteness results on $q$-concave complex varieties. Their arguments implicitly contained a geometrical proof of the Hartogs extension theorem in the case where the domain $\Omega \subset \mathbb{C}^{n}$ is pseudoconvex (whence Fornæss' counter-example must be nonpseudoconvex). However, in contrast to our finer method, the existence of an internal strongly pseudoconvex exhaustion function $\rho$ on a complex manifold $X$ excludes ab initio multisheetedness: indeed, in such a circumstance, extension holds stepwise from shells of the form $\{z \in X: a<\rho(z)<b\}$ just to deeper shells $\left\{a^{\prime}<\rho<b\right\}$ with $a^{\prime}<a$ (details are provided in [27]), namely the topology is controlled in advance by $\rho$ and multiple domains as $\widetilde{\Omega}_{>r}^{c}$ above cannot at all appear.

There is a nice alternative approach to the (singular) Hartogs extension theorem via a global continuity principle, realized in [21] by Jöricke and the second author, with the purpose of understanding removable singularities by means of (geometric) envelopes of holomorphy. The idea is to perform holomorphic extensions along one-parameter families of holomorphic curves (not suppose to be discs). A basic extension theorem on some appropriate Levi flat 3-manifolds, called Hartogs manifolds in [21], is shown via stepwise extension in the direction of an increasing real parameter. The geometrical scheme of this construction has a common topological element with our method: the simultaneous holomorphic extension to collections of domains that are pairwise either disjoint or one is contained in the other.

On the other hand, our technique only rely upon the existence of appropriate exhaustion functions, without requiring neither the existence of Levi-flat 3-manifolds nor the existence of global holomorphic functions in the ambient complex manifold. In addition, inspired by a definition formulated by

Fornæss in [10], we establish that only a finite number of Levi-Hartogs figures is needed in the filling process. Finally, we would like to mention that a straightforward adaptation of the proof developed here would yield a geometrical proof of the Hartogs-type extension theorem of Andreotti and Hill ([2]), which is valid for arbitrary domains in $(n-1)$-complete manifolds (in the sense of Andreotti-Grauert [10]).

Twenty-two colored illustrations appear, each one being inserted at the appropriate place in the text. Abstract geometrical thought being intrinsically pictural, we hope to address to a broad audience of complex analysts and geometers.

## §2. PREPARATION OF THE bOUNDARY AND UNIQUE EXTENSION

2.1. Preparation of a good $\mathscr{C}^{\infty}$ boundary. Denote by $\|z\|:=\left(\left|z_{1}\right|^{2}+\cdots+\right.$ $\left.\left|z_{n}\right|^{2}\right)^{1 / 2}$ the Euclidean norm of $z=\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{C}^{n}$ and by $\mathbb{B}^{n}(p, \delta):=$ $\{\|z-p\|<\delta\}$ the open ball of radius $\delta>0$ centered at a point $p$. If $E \subset \mathbb{C}^{n}$ is any set,

$$
\mathscr{V}_{\delta}(E):=\cup_{p \in E} \mathbb{B}^{n}(p, \delta)
$$

is a concrete open neighborhood of $E$.
As in the Hartogs theorem, assume that the domain $\Omega \Subset \mathbb{C}^{n}$ has connected boundary $\partial \Omega$ and let $\mathscr{V}(\partial \Omega)$ be an open neighborhood of $\partial \Omega$, also connected. Clearly, there exists $\delta_{1}$ with $0<\delta_{1} \ll 1$ such that $\partial \Omega \subset$ $\mathscr{V}_{\delta_{1}}(\partial \Omega) \subset \mathscr{V}(\partial \Omega)$; of course, $\mathscr{V}_{\delta_{1}}(\partial \Omega)$ is then also connected. Choose a point $p_{0} \in \mathbb{C}^{n}$ with dist $\left(p_{0}, \bar{\Omega}\right)=3$, center the coordinates $\left(z_{1}, \ldots, z_{n}\right)$ at $p_{0}$ and consider the distance function

$$
\begin{equation*}
r(z):=\left\|z-p_{0}\right\|=\|z\| . \tag{2.2}
\end{equation*}
$$

It is crucial to prepare as follows the boundary, replacing $(\Omega, \partial \Omega)$ by $\left(\Omega_{M}, M\right)$, thanks to some transversality arguments that are standard in Morse theory ([31] and [17], Ch. 6).

Lemma 2.3. There exists a $\mathscr{C}^{\infty}$ connected closed and oriented hypersurface $M \subset \mathscr{V}_{\delta_{1} / 2}(\partial \Omega)$ such that:
(i) $M$ bounds a unique bounded domain $\Omega_{M}$ with $\Omega \subset \Omega_{M} \cup \mathscr{V}(\partial \Omega)$;
(ii) the restriction $r_{M}(z):=\left.r(z)\right|_{M}$ of the distance function $r(z)=\|z\|$ to $M$ has only a finite number $\kappa$ of critical points $\widehat{p}_{\lambda} \in M, 1 \leqslant \lambda \leqslant$ $\kappa$, located on different sphere levels, namely

$$
2 \leqslant r\left(\widehat{p}_{1}\right)<\cdots<r\left(\widehat{p}_{\kappa}\right) \leqslant 5+\operatorname{diam}(\bar{\Omega}) ;
$$

(iii) all the $(2 n-1) \times(2 n-1)$ Hessian matrices $\mathrm{H}\left[r_{M}\right]\left(\widehat{p}_{1}\right), \ldots, \mathrm{H}\left[r_{M}\right]\left(\widehat{p}_{\kappa}\right)$ have a nonzero determinant.
$m n \partial \Omega$


Sometimes, $r_{M}$ satisfying (ii) and (iii) is called a good Morse function on $M$. We will shortly say that $M$ is a good boundary.

If $k_{\lambda}$ is the number of positive eigenvalues of the (symmetric) Hessian matrix $\mathrm{H}\left[r_{M}\right]\left(\widehat{p}_{\lambda}\right)$, the extrinsic Morse lemma ( $\left.[31,17]\right)$ shows that there exist $2 n$ real coordinates $\left(v, x_{1}, \ldots, x_{k_{\lambda}}, y_{1}, \ldots, y_{2 n-k_{\lambda}-1}\right)$ in a neighborhood of $\widehat{p}_{\lambda}$ in $\mathbb{C}^{n}$ such that

- the sets $\{v(z)=$ cst $\}$ simply correspond ${ }^{41}$ to the spheres $\{r(z)=$ cst $\}$ near $\widehat{p}_{\lambda}$;
- $\left(x_{1}, \ldots, x_{k_{\lambda}}, y_{1}, \ldots, y_{2 n-k_{\lambda}-1}\right)$ provide $(2 n-1)$ local coordinates on the hypersurface $M$, whose graphed equation is normalized to be the simple hyperquadric

$$
v=\sum_{1 \leqslant j \leqslant k_{\lambda}} x_{j}^{2}-\sum_{1 \leqslant j \leqslant 2 n-k_{\lambda}-1} y_{j}^{2} .
$$

Classically, the number $\left(2 n-k_{\lambda}-1\right)$ of negatives is called the Morse index of $\left.r(z)\right|_{M}$ at $\widehat{p}_{\lambda}$; we will call $k_{\lambda}$ its Morse coindex.

For rather general differential-geometric objects, Morse theory enables to control a significant part of homotopy groups and of (co)homologies, e.g. via Morse inequalities. In our case, we shall be able to control somehow the global topology of the cut-out domains $\Omega_{M} \cap\{\|z\|>r\}$ that re external to closed balls of radius $r$, filling them progressively by means of analytic discs contained in small (Levi-)Hartogs figures (Section 3). We start by checking rigorously that the Hartogs theorem can be reduced to some good boundary.
2.4. Unique holomorphic extension. If $\mathscr{U} \subset \mathbb{C}^{n}$ is open, $\mathscr{O}(\mathscr{U})$ denotes the ring of holomorphic functions in $\mathscr{U}$.

[^40]Definition 2.5. Given two connected open sets $\mathscr{U}_{1} \subset \mathbb{C}^{n}$ and $\mathscr{U}_{2} \subset \mathbb{C}^{n}$ with $\mathscr{U}_{1} \cap \mathscr{U}_{2}$ nonempty, we will say ${ }^{42}$ that $\mathscr{O}\left(\mathscr{U}_{1}\right)$ extends holomorphically to $\mathscr{U}_{1} \cup \mathscr{U}_{2}$ if :

- the intersection $\mathscr{U}_{1} \cap \mathscr{U}_{2}$ is connected;
- there exists an open nonempty set $\mathscr{V} \subset \mathscr{U}_{1} \cap \mathscr{U}_{2}$ such that for every $f_{1} \in \mathscr{O}\left(\mathscr{U}_{1}\right)$, there exist $f_{2} \in \mathscr{O}\left(\mathscr{U}_{2}\right)$ with $\left.f_{2}\right|_{\mathscr{V}}=\left.f_{1}\right|_{\mathscr{V}}$.

It then follows from the principle of analytic continuation that $\left.f_{1}\right|_{\mathscr{U}_{1} \cap \mathscr{Q}_{2}}=$ $\left.f_{2}\right|_{\mathscr{U}_{1} \cap \mathscr{U}_{2}}$, so that the joint function $F$, equal to $f_{j}$ on $\mathscr{U}_{j}$ for $j=1,2$, is well defined, is holomorphic in $\mathscr{U}_{1} \cup \mathscr{U}_{2}$ and extends $f_{1}$, namely $\left.F\right|_{\mathscr{U}_{1}}=f_{1}$.

In concrete extensional situations, the coincidence of $f_{1}$ with $f_{2}$ is controlled only in some small $\mathscr{V} \subset \mathscr{U}_{1} \cap \mathscr{U}_{2}$, so the connectedness of $\mathscr{U}_{1} \cap \mathscr{U}_{2}$ appears to be useful to insure monodromy. Sometimes also, we shall briefly write $\mathscr{O}\left(\mathscr{U}_{1}\right)=\left.\mathscr{O}\left(\mathscr{U}_{1} \cup \mathscr{U}_{2}\right)\right|_{\mathscr{U}_{1}}$, instead of spelling rigorously:

$$
\forall f_{1} \in \mathscr{O}\left(\mathscr{U}_{1}\right) \quad \exists F \in \mathscr{O}\left(\mathscr{U}_{1} \cup \mathscr{U}_{2}\right) \quad \text { such that }\left.\quad F\right|_{\mathscr{U}_{1}}=f_{1} .
$$

Lemma 2.6. Suppose that for some $\delta$ with $0<\delta \leqslant \delta_{1} / 2$ so small that $\mathscr{V}_{\delta}(M) \simeq M \times(-\delta, \delta)$ is a thin tubular neighborhood of the good boundary $M \subset \mathscr{V}_{\delta_{1} / 2}(\partial \Omega) \subset \mathscr{V}(\partial \Omega)$, the Hartogs theorem holds for the pair $\left(\Omega_{M}, \mathscr{V}_{\delta}(M)\right):$

$$
\mathscr{O}\left(\mathscr{V}_{\delta}(M)\right)=\left.\mathscr{O}\left(\Omega_{M} \cup \mathscr{V}_{\delta}(M)\right)\right|_{\mathscr{V}_{\delta}(M)} .
$$

Then the general Hartogs extension property holds:

$$
\mathscr{O}(\mathscr{V}(\partial \Omega))=\left.\mathscr{O}(\Omega \cup \mathscr{V}(\partial \Omega))\right|_{\mathscr{V}(\partial \Omega)} .
$$

Proof. Let $f \in \mathscr{O}(\mathscr{V}(\partial \Omega))$. By assumption, its restriction to $\mathscr{V}_{\delta}(M) \subset$ $\mathscr{V}(\partial \Omega)$ enjoys an extension $F_{\delta} \in \mathscr{O}\left(\Omega_{M} \cup \mathscr{V}_{\delta}(M)\right)$. To ascertain that $f$ and $F_{\delta}$ coincide in $\Omega_{M} \cap \mathscr{V}(\partial \Omega)$, connectedness of $\Omega_{M} \cap \mathscr{V}(\partial \Omega)$ is welcome.


Fig. 3: Checking connectedness of $\Omega_{M} \cap \mathscr{V}(\partial \Omega)$
Letting $p, q \in \Omega_{M} \cap \mathscr{V}(\partial \Omega)$, there exists a $\mathscr{C}^{\infty}$ curve $\gamma:[0,1] \rightarrow \mathscr{V}(\partial \Omega)$ connecting $p$ to $q$. If $\gamma$ meets $M$, let $p^{\prime}$ be the first point on $\gamma \cap M$ and let $q^{\prime}$ be the last one. We then modify $\gamma$, joining $p^{\prime}$ to $q^{\prime}$ by means of a curve

[^41]$\mu$ entirely contained in the connected hypersurface $M$. It suffices to push $\mu$ slightly inside $\Omega_{M}$ to get an appropriate curve running from $p$ to $q$ inside $\Omega_{M} \cap \mathscr{V}(\partial \Omega)$. Thus, $\Omega_{M} \cap \mathscr{V}(\partial \Omega)$ is connected. It follows, moreover, that the open set
$$
\left[\Omega_{M} \cup \mathscr{V}_{\delta}(M)\right] \cap \mathscr{V}(\partial \Omega)=\left[\Omega_{M} \cap \mathscr{V}(\partial \Omega)\right] \bigcup \mathscr{V}_{\delta}(M)
$$
is also connected, so the coincidence $f=F_{\delta}$, valid in $\mathscr{V}_{\delta}(M)$, propagates to $\left[\Omega_{M} \cap \mathscr{V}(\partial \Omega)\right] \cup \mathscr{V}_{\delta}(M)$. Finally, the function
\[

F:=\left\{$$
\begin{aligned}
F_{\delta} & \text { in } \Omega_{M} \cup \mathscr{V}_{\delta}(M), \\
f & \text { in } \mathscr{V}(\partial \Omega) \backslash \bar{\Omega}_{M},
\end{aligned}
$$\right.
\]

is well defined (since $F_{\delta}=f$ in $\left.\mathscr{V}_{\delta}(M) \backslash \bar{\Omega}_{M} \simeq M \times(0, \delta)\right)$, is holomorphic in

$$
\Omega_{M} \cup \mathscr{V}(\partial \Omega)=\Omega \cup \mathscr{V}(\partial \Omega)
$$

and coincides with $f$ in $\mathscr{V}(\partial \Omega)$.
Thus, we are reduced to establish global holomorphic extension with some good, geometrically controlled data.

Theorem 2.7. Let $M \Subset \mathbb{C}^{n}(n \geqslant 2)$ be a connected $\mathscr{C}^{\infty}$ hypersurface bounding a domain $\Omega_{M} \Subset \mathbb{C}^{n}$. Suppose to fix ideas that $2 \leqslant \operatorname{dist}\left(0, \bar{\Omega}_{M}\right) \leqslant$ 5 and assume that the restriction $r_{M}:=\left.r\right|_{M}$ of the distance function $r(z)=$ $\|z\|$ to $M$ is a Morse function having only a finite number $\kappa$ of critical points $\widehat{p}_{\lambda} \in M, 1 \leqslant \lambda \leqslant \kappa$, located on different sphere levels:

$$
2 \leqslant \widehat{r}_{1}:=r\left(\widehat{p}_{1}\right)<\cdots<\widehat{r}_{\kappa}:=r\left(\widehat{p}_{\kappa}\right) \leqslant 5+\operatorname{diam}\left(\bar{\Omega}_{M}\right) .
$$

Then there exists $\delta_{1}>0$ such that for every $\delta$ with $0<\delta \leqslant \delta_{1}$, the (tubular) neighborhood $\mathscr{V}_{\delta}(M)$ enjoys the global Hartogs extension property into $\Omega_{M}$ :

$$
\mathscr{O}\left(\mathscr{V}_{\delta}(M)\right)=\left.\mathscr{O}\left(\Omega_{M} \cup \mathscr{V}_{\delta}(M)\right)\right|_{\mathscr{V}_{\delta}(M)},
$$

by "pushing" analytic discs inside a finite number of Levi-Hartogs figures (§3.3), without using neither the Martinelli kernel, nor solutions of an auxiliary $\bar{\partial}$ problem.

## §3. Quantitative Hartogs-Levi extension bY PUSHING ANALYTIC DISCS

3.1. The classical Hartogs figure. Local Hartogs phenomena can now enter the scene. They involve translating ("pushing") analytic discs and they will provide small, elementary extensional steps to fill in $\Omega_{M}$.

Given $\varepsilon \in \mathbb{R}$ with $0<\varepsilon \ll 1$ and $a \in \mathbb{N}$ with $1 \leqslant a \leqslant n-1$, we split the coordinates $z \in \mathbb{C}^{n}$ as $\left(z_{1}, \ldots, z_{a}\right)$ together with $\left(z_{a+1}, \ldots, z_{n}\right)$, and we define the $(n-a)$-concave Hartogs figure by

$$
\begin{aligned}
\mathscr{H}_{\varepsilon}^{n-a}:= & \left\{\max _{1 \leqslant i \leqslant a}\left|z_{i}\right|<1, \max _{a+1 \leqslant j \leqslant n}\left|z_{j}\right|<\varepsilon\right\} \\
& \bigcup\left\{1-\varepsilon<\max _{1 \leqslant i \leqslant a}\left|z_{i}\right|<1, \max _{a+1 \leqslant j \leqslant n}\left|z_{j}\right|<1\right\} .
\end{aligned}
$$



Fig. 4: Two views of the standard Hartogs figure $\mathscr{H}_{\varepsilon}^{2-1} \subset \mathbb{C}^{2}$
Lemma 3.2. $\mathscr{O}\left(\mathscr{H}_{\varepsilon}^{n-a}\right)$ extends holomorphically to the unit polydisc

$$
\widehat{\mathscr{H}_{\varepsilon}^{n-a}}:=\left\{z \in \mathbb{C}^{n}: \max _{1 \leqslant i \leqslant n}\left|z_{i}\right|<1\right\}=\Delta^{n}
$$

Proof. As in the diagram, we consider only $n=2, a=1$, the general case being similar. Pick an arbitrary $f \in \mathscr{O}\left(\mathscr{H}_{\varepsilon}^{2-1}\right)$. Letting $\varepsilon^{\prime}$ with $0<\varepsilon^{\prime}<\varepsilon$, letting $z_{2} \in \mathbb{C}$ with $\left|z_{2}\right|<1$, the analytic disc

$$
\zeta \longmapsto\left(\left[1-\varepsilon^{\prime}\right] \zeta, z_{2}\right)=: A_{z_{2}}^{\varepsilon^{\prime}}(\zeta),
$$

where $\zeta$ belongs to the closed unit disc $\bar{\Delta}=\{|\zeta| \leqslant 1\}$, has its boundary $A_{z_{2}}^{\varepsilon^{\prime}}(\partial \Delta)=A_{z_{2}}^{\varepsilon^{\prime}}(\{|\zeta|=1\})$ contained in $\mathscr{H}_{\varepsilon}^{2-1}$, the set where $f$ is defined. Lowering dimensions by a unit, we draw discs as (green) segments and boundaries of discs as (green) bold points. Thus, we may compute the Cauchy integral

$$
F\left(z_{1}, z_{2}\right):=\frac{1}{2 \pi i} \int_{\partial \Delta} \frac{f\left(A_{z_{2}}^{\varepsilon^{\prime}}(\zeta)\right)}{\zeta-z_{1}} d \zeta .
$$

Differentiating under the sum, the function $F$ is seen to be holomorphic. In addition, for $\left|z_{2}\right|<\varepsilon$, it coincides with $f$, because the full closed disc $A_{z_{2}}^{\varepsilon^{\prime}}(\bar{\Delta})$ is contained in $\mathscr{H}_{\varepsilon}^{2-1}$ and thanks to Cauchy's formula. Clearly, the $A_{z_{2}}^{\varepsilon^{\prime}}(\Delta)$ all together fill in the bidisc $\Delta^{2}$. One may think that, as $z_{2}$ varies, discs are "pushed" gently by a virtual thumb.
3.3. Levi extension and the Levi-Hartogs figure. Geometrically, the standard Hartogs figure is not best suited to perform holomorphic extension from a strongly (pseudo)concave boundary. For instance, in the proof of Theorem 2.7 , we will encounter complements in $\mathbb{C}^{n}$ of some closed balls whose radius decreases step by step, and more generally spherical shells whose thickness increases interiorly. Thus, we delineate an appropriate set up.

For $r \in \mathbb{R}$ with $r>1$ and for $\delta \in \mathbb{R}$ with $0<\delta \ll 1$, the sphere $\mathrm{S}_{r}^{2 n-1}=\left\{z \in \mathbb{C}^{n}:\|z\|=r\right\}$ of radius $r$ is the interior (and strongly concave) boundary component of the spherical shell domain

$$
\mathscr{S}_{r}^{r+\delta}:=\{r<\|z\|<r+\delta\}=\bigcup_{p \in \mathrm{~S}_{r}^{2 n-1}} \mathbb{B}^{n}(p, \delta) \cap\{\|z\|>r\}
$$



Fig. 5: Relevance of the Levi-Hartogs figure
Near a point $p \in \mathbf{S}_{r}^{2 n-1}$ (left figure), all copies of $\mathbb{C}^{n-1}$ (in green) which are parallel to the complex tangent plane $T_{p}^{c} \mathrm{~S}_{r}^{2 n-1}$ and which lie above the real plane $T_{p} \mathrm{~S}_{r}^{2 n-1}$ are entirely contained in $\mathbb{C}^{n} \backslash \overline{\mathbb{B}}_{r}^{n}$. To remain inside the shell $\mathscr{S}_{r}^{r+\delta}$, we could (for instance) restraint our considerations to some half-cylinder of diameter $\approx \delta$, but it will be better to shape a convenient half parallelepiped. Accordingly, for two small $\varepsilon_{j}>0, j=1,2$, we introduce a geometrically relevant Levi-Hartogs figure (right illustration, reverse orientation):

$$
\begin{aligned}
\mathscr{L} \mathscr{H}_{\varepsilon_{1}, \varepsilon_{2}}:= & \left\{\max _{1 \leqslant i \leqslant n-1}\left|z_{i}\right|<\varepsilon_{1}, \quad\left|x_{n}\right|<\varepsilon_{1}, \quad-\varepsilon_{2}<y_{n}<0\right\} \\
& \bigcup\left\{\varepsilon_{1}-\left(\varepsilon_{1}\right)^{2}<\max _{1 \leqslant i \leqslant n-1}\left|z_{i}\right|<\varepsilon_{1}, \quad\left|x_{n}\right|<\varepsilon_{1} \quad\left|y_{n}\right|<\varepsilon_{2}\right\} .
\end{aligned}
$$

To fill in this (bed-like) figure, we just compute the Cauchy integral on appropriate analytic discs (the (green) horizontal ones) whose boundaries remain in $\mathscr{L}_{\mathscr{H}}^{\varepsilon_{1}, \varepsilon_{2}}$.
Lemma 3.4. $\mathscr{O}\left(\mathscr{L} \mathscr{H}_{\varepsilon_{1}, \varepsilon_{2}}\right)$ extends holomorphically to the full parallelepiped

$$
\widehat{\mathscr{L} \mathscr{H}_{\varepsilon_{1}, \varepsilon_{2}}}:=\left\{\max _{1 \leqslant i \leqslant n-1}\left|z_{i}\right|<\varepsilon_{1}, \quad\left|x_{n}\right|<\varepsilon_{1}, \quad\left|y_{n}\right|<\varepsilon_{2}\right\}
$$

Next, we must reorient and scale $\mathscr{L} \mathscr{H}_{\varepsilon_{1}, \varepsilon_{2}}$ in order to put it inside the shell. For every point $p \in \mathrm{~S}_{r}^{2 n-1}$, there exists some complex unitarian affine map

$$
\Phi_{p}: \quad z \longmapsto p+U z,
$$

with $U \in \operatorname{SU}(n, \mathbb{C})$, sending the origin $0 \in{\overline{\mathscr{L}} \mathscr{H}_{\varepsilon_{1}, \varepsilon_{2}}}$ to $p$ and $T_{0} \mathscr{L} \mathscr{H}_{\varepsilon_{1}, \varepsilon_{2}}$ to $T_{p} \mathrm{~S}_{r}^{2 n-1}$, which in addition sends the half-parallelepiped (open) part outside $\overline{\mathbb{B}}_{r}^{n}$. But we have to insure that $\Phi_{p}\left(\mathscr{L} \mathscr{H}_{\varepsilon_{1}, \varepsilon_{2}}\right)$ as a whole (including the thin walls) lies outside $\overline{\mathbb{B}}_{r}^{n}$.
Lemma 3.5. If $\varepsilon_{1}=c \delta$ and $\varepsilon_{2}=c \delta^{2}$ with some appropriate ${ }^{43}$ positive constant $c<1$, then $\Phi_{p}\left(\mathscr{L}_{\mathscr{H}}^{\varepsilon_{1}, \varepsilon_{2}} ⿵ 冂\right.$ is entirely contained in the shell $\mathscr{S}_{r}^{r+\delta}$. Furthermore, $\Phi_{p}\left(\widehat{\mathscr{L} \mathscr{H}_{\varepsilon_{1}}, \varepsilon_{2}}\right)$ contains a rind of thickness c $\frac{\delta^{2}}{r}$ around some region $\mathrm{R}_{p} \subset \mathrm{~S}_{r}^{2 n-1}$ whose $(2 n-1)$-dimensional area equals $\simeq c \delta^{2 n-1}$.


By a (radial) rind of thickness $\eta>0$ around an open region $\mathrm{R} \subset \mathrm{S}_{r}^{2 n-1}$, we mean

$$
\operatorname{Rind}(\mathrm{R}, \eta):=\{(1+s) z: z \in \mathrm{R},|s|<\eta / r\}
$$

We require that $|s|<\eta / r$ to insure that at every $z \in \mathrm{R}$, the half-line $(0 z)^{+}$ emanating from the origin intersects $\operatorname{Rind}(\mathrm{R}, \eta)$ along a symmetric segment of length $2 \eta$ centered at $z$.

In the diagram above, we draw (in green) only the lower part of the small region $\mathrm{R}_{p}$ got in Lemma 3.5. Its shape, when projected onto $T_{p} \mathrm{~S}_{r}^{2 n-1}$, can either be (approximately) a parallelepiped $\left\{\left|z^{\prime}\right|<c \delta,\left|x_{n}\right|<c \delta\right\}$, as in the figure, or say, a ball $\left\{\left(\left\|z^{\prime}\right\|^{2}+\left|x_{n}\right|^{2}\right)^{1 / 2}<c \delta\right\}$; only the scaling constant $c$ changes.

The rigorous proof of the lemma (not developed here) involves elementary reasonings with geometric inequalities and a dry explicit control of the constants that does not matter for the sequel. The main argument uses the fact that $\mathrm{S}_{r}^{2 n-1}$ detaches quadratically from $T_{p} \mathrm{~S}_{r}^{2 n-1}$, similarly as the parabola $\left\{y=-\frac{1}{r} x^{2}\right\}$ separates from the line $\{y=0\}$ in $\mathbb{R}_{x, y}^{2}$.

[^42]Since the area of $\mathrm{S}_{r}^{2 n-1}$ equals $\frac{2 \pi^{n}}{(n-1)!} r^{2 n-1}=C r^{2 n-1}$, by covering $\mathrm{S}_{r}^{2 n-1}$ with such adjusted $\mathrm{R}_{p} \subset \Phi_{p}\left(\widehat{\mathscr{L} \mathscr{H}_{\varepsilon_{1}}, \varepsilon_{2}}\right)$ of area $c \delta^{2 n-1}$ and by controlling monodromy (see rigorous arguments below) we deduce:
Corollary 3.6. By means of a finite number $\leqslant C\left(\frac{r}{\delta}\right)^{2 n-1}$ of Levi-Hartogs figures, $\mathscr{O}\left(\mathscr{S}_{r}^{r+\delta}\right)$ extends holomorphically to the slightly deeper spherical shell $\mathscr{S}_{r-c \frac{\delta^{2}}{r}}^{r+}$.

This application could seem superfluous, because large analytic discs with boundaries contained in $\mathscr{S}_{r}^{r+\delta}$ would yield holomorphic extension to the whole ball $\mathbb{B}_{r+\delta}^{n}$ in one single step. However, in our situation illustrated by Figure 1, when intersecting $\mathrm{S}_{r}^{2 n-1}$ with the neighborhood $\mathscr{V}_{\delta}(M)$, we shall only get small subregions of $\mathrm{S}_{r}^{2 n-1}$. Hopefully, thanks to our local LeviHartogs figures, we may obtain a suitable semi-global extensional statement, valuable for proper subsets of the shell $\mathscr{S}_{r}^{r+\delta}$ whose shape is arbitrary. The next statement, not available by means of large discs, will be used a great number of times in the sequel.

Proposition 3.7. Let $\mathrm{R} \subset \mathrm{S}_{r}^{2 n-1}$ (with $r>1$ and $n \geqslant 2$ ) be a relatively open set having $\mathscr{C}^{\infty}$ boundary $\mathrm{N}:=\partial \mathrm{R}$ and let $\delta>0$ with $0<\delta \ll 1$. Then holomorphic functions in the open piece of shell (a one-sided neighborhood of $\mathrm{R} \cup \mathrm{N}$ ):

$$
\begin{aligned}
\operatorname{Shell}_{r}^{r+\delta}(\mathrm{R} \cup \mathrm{~N}) & :=\left(\mathbb{C}^{n} \backslash \overline{\mathbb{B}}_{r}^{n}\right) \cap \mathscr{V}_{\delta}(\mathrm{R} \cup \mathrm{~N}) \\
& =\bigcup_{p \in \mathrm{R} \cup \mathrm{~N}} \mathbb{B}^{n}(p, \delta) \cap\{\|z\|>r\}
\end{aligned}
$$

do extend holomorphically to a rind of thickness $c \frac{\delta^{2}}{r}$ around R by means of a finite number $\leqslant C \frac{\operatorname{ara}(\mathrm{R})}{\delta^{2 n-1}}$ of Levi-Hartogs figures.


Fig. 7: Semi-global extension from a pseudoconcave piece of shell
Proof. We must control uniqueness of holomorphic extension (monodromy) into rinds covered by successively attached Levi-Hartogs figures. Noticing $c \delta^{2} r^{-1} \ll \delta$, the considered rinds are much thinner than the piece of shell.
Lemma 3.8. If $\mathrm{R}^{\prime} \subset \mathrm{R}$ is an arbitrary open subset and if $\mathrm{R}_{p^{\prime}} \subset$ $\Phi_{p^{\prime}}\left(\widehat{\mathscr{L} \mathscr{H}_{\varepsilon_{1}, \varepsilon_{2}}}\right)$ is a small Levi-Hartogs region centered at an arbitrary point
$p^{\prime} \in \mathrm{R}$, then the intersection

$$
\begin{equation*}
\operatorname{Rind}\left(\mathrm{R}_{p^{\prime}}, c \delta^{2} r^{-1}\right) \bigcap\left(\operatorname{Shell}_{r}^{r+\delta}(\mathrm{R} \cup \mathrm{~N}) \bigcup \operatorname{Rind}\left(\mathrm{R}^{\prime}, c \delta^{2} r^{-1}\right)\right) \tag{3.9}
\end{equation*}
$$

is connected.
Admitting the lemma for a while, we pick a finite number $m \leqslant C \frac{\operatorname{area}(\mathrm{R})}{\delta^{2 n-1}}$ of points $p_{1}, \ldots, p_{m} \in \mathrm{R} \cup \mathrm{N}$ such that the associated local regions $\mathrm{R}_{p_{k}}$ contained in the filled Levi-Hartogs figures $\Phi_{p_{k}}\left(\widehat{\mathscr{L} \mathscr{H}_{\varepsilon_{1}, \varepsilon_{2}}}\right)$ provided by Lemma 3.5 do cover $R \cup N$, namely $R_{p_{1}} \cup \cdots \cup R_{p_{m}} \supset R \cup N$.

Starting with $\mathrm{R}^{\prime}:=\emptyset$ and $p^{\prime}:=p_{1}$, unique holomorphic extension of $\mathscr{O}\left(\operatorname{Shell}_{r}^{r+\delta}(\mathrm{R} \cup \mathrm{N})\right)$ to $\operatorname{Rind}\left(\mathrm{R}_{p^{\prime}}, c \delta^{2} r^{-1}\right)$ holds by means of Lemma 3.4, monodromy being assured thanks to the connectedness of the intersection (3.9). Reasoning by induction, fixing some $k$ with $1 \leqslant k \leqslant m-1$, setting $\mathrm{R}^{\prime}:=\cup_{1 \leqslant j \leqslant k} \mathrm{R}_{p_{j}}, p^{\prime}:=p_{k+1}$ and assuming that unique holomorphic extension is got from $\operatorname{Shell}_{r}^{r+\delta}(\mathrm{R} \cup \mathrm{N})$ into

$$
\operatorname{Shell}_{r}^{r+\delta}(\mathrm{R} \cup \mathrm{~N}) \bigcup \operatorname{Rind}\left(\mathrm{R}^{\prime}, c \delta^{2} r^{-1}\right)=\operatorname{Shell}_{r}^{r+\delta}(\mathrm{R} \cup \mathrm{~N}) \bigcup_{1 \leqslant j \leqslant k} \operatorname{Rind}\left(\mathrm{R}_{p_{j}}, c \delta^{2} r^{-1}\right),
$$

we add the Levi-Hartogs figure $\Phi_{p_{k+1}}\left(\widehat{\mathscr{L} \mathscr{H}_{\varepsilon_{1}, \varepsilon_{2}}}\right)$ constructed in Lemma 3.5, and we get unique holomorphic extension to $\operatorname{Rind}\left(\mathrm{R}_{p_{k+1}}, c \delta^{2} r^{-1}\right)$, monodromy being assured again thanks to the connectedness of the intersection (3.9). Since $\operatorname{Rind}\left(\mathrm{R}, c \delta^{2} r^{-1}\right) \subset \bigcup_{1 \leqslant k \leqslant m} \operatorname{Rind}\left(\mathrm{R}_{p_{k}}, c \delta^{2} r^{-1}\right)$, the proposition is proved.
Proof of Lemma 3.8. To establish connectedness of the open set (3.9), picking two arbitrary points $q_{0}, q_{1}$ in it, we must produce a curve joining $q_{0}$ to $q_{1}$ inside (3.9). The two radial segments of length $2 c \delta^{2} r^{-1}$ passing through $q_{0}$ and $q_{1}$ that are centered at two appropriate points of $\mathrm{S}_{r}^{2 n-1}$ are by definition both entirely contained in $\operatorname{Rind}\left(\mathrm{R}_{p^{\prime}}, c \delta^{2} r^{-1}\right)$ as well as in $\operatorname{Rind}\left(\mathrm{R}^{\prime}, c \delta^{2} r^{-1}\right)$. Thus, moving radially, we may join inside (3.9) $q_{0}$ to a new point $q_{0}^{\prime}$ and $q_{1}$ to a new point $q_{1}^{\prime}$, which both belong to the upper half-rind

$$
\left\{(1+s) z: z \in \mathrm{R}_{p^{\prime}}, 0<s<c \delta^{2} r^{-1} / r\right\} .
$$

Since this upper half-rind is connected and contained in $\operatorname{Shell}_{r}^{r+\delta}(\mathrm{R} \cup \mathrm{N})$, we may finally join inside (3.9) the point $q_{0}^{\prime}$ to $q_{1}^{\prime}$.

In the sequel, in order to avoids several gaps and traps, we will put emphasis on rigourously checking univalence of holomorphic extensions.

## §4. Filling domains outside balls of decreasing radius

4.1. Global Levi-Hartogs filling from the farthest point. We can now launch the proof of Theorem 2.7. The $\delta_{1}$ is first chosen so small that $\mathscr{V}_{\delta}(M)$ is a true tubular neighborhood of $M$ for every $\delta$ with $0<\delta \leqslant \delta_{1}$. Shrinking
even more $\delta_{1}$, in balls of radius $\delta_{1}$ centered at its points, the hypersurface $M$ is well approximated by its tangent planes.

The farthest point of $\bar{\Omega}_{M}$ from the origin is unique and it coincides with $\widehat{p}_{\kappa}$ since by assumption $\widehat{p}_{\kappa}$ is the single critical point of $\left.r(z)\right|_{M}$ with $\left\|\widehat{p}_{\kappa}\right\|=\max _{1 \leqslant \lambda \leqslant \kappa}\left\|\widehat{p}_{\lambda}\right\|$. By assumption also, the Hessian matrix of $\left.r(z)\right|_{M}$ is nondegenerate at $\widehat{p}_{\kappa}$; this also follows automatically from the inclusion $\bar{\Omega}_{M} \subset \overline{\mathbb{B}}_{\widehat{r}_{\kappa}}^{n}$, which constrains strong convexity of $M$ at $\widehat{p}_{\kappa}$. Consequently, according to the Morse lemma ([31], [17], Ch. 6), there exist local coordinates $\left(\theta_{1}, \ldots, \theta_{2 n-1}\right)$ on $M$ centered at $\widehat{p}_{\kappa}$ such that the intersection $M \cap \mathrm{~S}_{r}^{2 n-1}$ is given by the equation

$$
-\theta_{1}^{2}-\cdots-\theta_{2 n-1}^{2}=r-\widehat{r}_{\kappa},
$$

for all $r$ close to $\widehat{r}_{\kappa}$. Thus $M \cap \mathrm{~S}_{r}^{2 n-1}$ is empty for $r>\widehat{r}_{\kappa}$; it reduces to $\left\{\widehat{p}_{\kappa}\right\}$ for $r=\widehat{r}_{\kappa}$; and it is diffeomorphic to a $(2 n-2)$-sphere for $r<\widehat{r}_{\kappa}$ close to $\widehat{r}_{\kappa}$.

Similarly, the nearest point of $\bar{\Omega}_{M}$ from the origin is unique and it coincides with $\widehat{p}_{1}$; notice that hence $\kappa \geqslant 2$. Also, the second farthest critical point $\widehat{p}_{\kappa-1}$ lies at a distance $\widehat{r}_{\kappa-1}<\widehat{r}_{\kappa}$ from 0 . If necessary, we shrink $\delta_{1}$ to insure

$$
\begin{equation*}
\delta_{1} \ll \min _{1 \leqslant \lambda \leqslant \kappa-1}\left\{\widehat{r}_{\lambda+1}-\widehat{r}_{\lambda}\right\} . \tag{4.2}
\end{equation*}
$$

Next, for every radius $r$ with $\widehat{r}_{\kappa-1}<r<\widehat{r}_{\kappa}$, we introduce the cut out domain

$$
\Omega_{>r}:=\Omega_{M} \cap\{\|z\|>r\}
$$

together with the cut out hypersurface

$$
M_{>r}:=M \cap\{\|z\|>r\} .
$$



Fig. 8: Filling the domain from the farthest point
Since there are no critical points of $\left.r(z)\right|_{M}$ in the interval $\left(\widehat{r}_{\kappa-1}, \widehat{r}_{\kappa}\right)$, Morse theory shows that $M_{>r}$ is a deformed spherical cap diffeomorphic to $\mathbb{R}^{2 n-1}$ for every $r$ with $\widehat{r}_{\kappa-1}<r<\widehat{r}_{\kappa}$. Also, $\Omega_{>r}$ is then a piece of deformed ball diffeomorphic to $\mathbb{R}^{2 n}$.

The boundary in $\mathbb{C}^{n}$ of $\Omega_{>r}$

$$
\partial \Omega_{>r}=M_{>r} \cup \mathrm{R}_{r} \cup \mathrm{~N}_{r}
$$

consists of $M_{>r}$ together with the open subregion $\mathrm{R}_{r}:=\Omega_{M} \cap\{\|z\|=r\}$ of $\mathrm{S}_{r}^{2 n-1}$ which is diffeomorphic to $\mathbb{R}^{2 n-1}$ and has boundary $\mathrm{N}_{r}:=M \cap\{\|z\|=$ $r\}$ diffeomorphic to the unit $(2 n-2)$-sphere. Thus, the global geometry of $\Omega_{>r}$ is understood.

We can also cut out $\mathscr{V}_{\delta}(M)$, getting $\mathscr{V}_{\delta}(M)_{>r}$. The central figure shows that when $r>\widehat{r}_{\kappa-1}$ is very close to $\widehat{r}_{\kappa-1}$, a parasitic connected component $\mathscr{W}_{>r}$ of $\mathscr{V}_{\delta}(M)_{>r}$ might appear near $\widehat{p}_{\kappa-1}$. After filling $\Omega_{>r}$ progressively by means of Levi-Hartogs figures (see below), because $\Omega_{>r} \cap \mathscr{V}_{\delta}(M)_{>r}$ is not connected in such a situation, no unique holomorphic extension can be assured, and in fact, multivalence might well occur.

A trick to erase such parasitic components $\mathscr{W}_{>r}$ is to consider instead the open set

$$
\mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}=\mathscr{V}_{\delta}\left(M_{>r}\right) \cap\{\|z\|>r\}
$$

putting a double " $>r$ ". It is drawn in the right figure and it is always diffeomorphic to $M_{>r} \times(-\delta, \delta)$.

From pieces of shells as in Proposition 3.7 which embrace spheres of varying radius $r$, holomorphic extension holds to (symmetric) rinds whose thickness $c \delta r^{-1}$ also varies. To simplify, we introduce the smallest appearing thickness

$$
\begin{equation*}
\eta:=\min _{\widehat{r}_{1} \leqslant r \leqslant \widehat{r}_{\kappa}} c \delta r^{-1}=c \delta{\widehat{r_{\kappa}}}^{-1} \tag{4.3}
\end{equation*}
$$

and we observe that it follows trivially from Proposition 3.7 (just by shrinking and by restricting) that holomorphic extension holds to some rind around R of arbitrary smaller thickness $\eta^{\prime}>0$ with $0<\eta^{\prime} \leqslant \eta$. In the sequel, our rinds shall most often have the uniform thickness $\eta$, and sometimes also, a smaller one $\eta^{\prime}$. Shrinking the constant $c$ of $\eta$ in (4.3), we insure $\eta \ll \delta_{1}$.

Summarizing, we list and we compare the quantities introduced so far:

$$
\left\{\begin{array}{lr}
0<\delta \leqslant \delta_{1} & \text { neighborhood } \mathscr{V}_{\delta}(M)  \tag{4.4}\\
2 \leqslant r\left(\widehat{p}_{1}\right)<\cdots<r\left(\widehat{p}_{k}\right) \leqslant 5+\operatorname{diam}\left(\bar{\Omega}_{M}\right) & \text { Morse radii } \\
\delta \leqslant \delta_{1} \ll \min _{1 \leqslant \lambda \leqslant \kappa-1}\left\{\widehat{r}_{\lambda+1}-\widehat{r}_{\lambda}\right\} & \text { smallness of } \mathscr{V}_{\delta}(M) \\
\eta:=c \delta^{2} \widehat{r}_{\kappa}^{-1} & \text { uniform useful rind thickness } \\
\eta \ll \delta & \text { thickness of extensional rinds is tiny }
\end{array}\right.
$$

Proposition 4.5. For every cutting radius $r$ with $\widehat{r}_{\kappa-1}<r<\widehat{r}_{\kappa}$ arbitrarily close to $\widehat{r}_{\kappa-1}$, holomorphic functions in the open set

$$
\mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}=\mathscr{V}_{\delta}\left(M_{>r}\right) \cap\{\|z\|>r\}
$$

do extend holomorphically and uniquely to $\Omega_{>r}$ by means of a finite number $\leqslant C\left(\frac{\widehat{r}_{\kappa}}{\delta}\right)^{2 n-1}\left[\frac{\widehat{\widehat{r}}_{k}-r}{\eta}\right]$ of Levi-Hartogs figures.
Proof. We fix such a radius $r$ with $\widehat{r}_{\kappa-1}<r<\widehat{r}_{\kappa}$. Putting a single LeviHartogs figure at $\widehat{p}_{\kappa}$ as in Proposition 3.7, we get unique holomorphic extension to $\Omega_{>\widehat{r}_{\kappa}-\eta}$. Since $\eta \ll \delta$, we have $\widehat{r}_{\kappa}-\eta>\widehat{r}_{\kappa-1}$. If the radius $\widehat{r}_{\kappa}-\eta$ is already $<r$, we just shrink to $\eta^{\prime}:=\widehat{r}_{\kappa}-r<\eta$ the thickness of our single rind, getting unique holomorphic extension to $\Omega_{>\widehat{r}_{\kappa}-\eta^{\prime}}=\Omega_{>r}$.

Performing induction on an auxiliary integer $k \geqslant 1$, we suppose that, by descending from $\widehat{r}_{\kappa}$ to a lower radius $r^{\prime}:=\widehat{r}_{\kappa}-k \eta$ assumed to be still $\geqslant r$, holomorphic functions in $\mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$ extend holomorphically and uniquely (remind Definition 2.5) to $\Omega_{>r^{\prime}}$.
Lemma 4.6. For every radius $r^{\prime}$ with $\widehat{r}_{\kappa-1}<r<r^{\prime}<\widehat{r}_{\kappa}$,

$$
\begin{equation*}
\text { Shell } r_{r^{\prime}}^{r^{\prime}+\delta}\left(\mathrm{R}_{r^{\prime}} \cup \mathrm{N}_{r^{\prime}}\right) \text { is contained in } \Omega_{>r^{\prime}} \bigcup \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r} \text {. } \tag{4.7}
\end{equation*}
$$



Fig. 9: A shell contained in the cap-shaped domain and the associated rind
Proof. Picking an arbitrary point $p \in \mathrm{R}_{r^{\prime}} \cup \mathrm{N}_{r^{\prime}}$, we must verify that

$$
\mathbb{B}^{n}(p, \delta) \cap\left\{\|z\|>r^{\prime}\right\}
$$

is contained in the right hand side of (4.7).
If $p \in \mathrm{~N}_{r^{\prime}} \subset M$, whence $p \in M_{>r}$, we get simply what we want:

$$
\begin{aligned}
\mathbb{B}^{n}(p, \delta) \cap\left\{\|z\|>r^{\prime}\right\} & \subset \mathscr{V}_{\delta}\left(M_{>r}\right) \cap\left\{\|z\|>r^{\prime}\right\} \\
& \subset \mathscr{V}_{\delta}\left(M_{>r}\right) \cap\{\|z\|>r\} \\
& =\mathscr{V}_{\delta}\left(M_{>r}\right)_{>r} .
\end{aligned}
$$

If $p \in \mathrm{R}_{r^{\prime}} \backslash \mathrm{N}_{r^{\prime}}$, whence $p \in \Omega_{M}$, reasoning by contradiction, we assume that there exists a point $q \in \mathbb{B}^{n}(p, \delta) \cap\left\{\|z\|>r^{\prime}\right\}$ in the cut out ball which does not belong to the right hand side of (4.7). Since $\Omega_{>r^{\prime}}=\Omega_{M} \cap\{\|z\|>$ $\left.r^{\prime}\right\}$, we have $q \notin \Omega_{M}$.

Reminding $\mathrm{R}_{r^{\prime}} \subset \mathrm{S}_{r^{\prime}}^{2 n-1}$, the tangent plane $T_{p} \mathrm{~S}_{r^{\prime}}^{2 n-1}=T_{p} \mathrm{R}_{r^{\prime}}$ divides $\mathbb{C}^{n}$ in two closed half-spaces, $\bar{T}_{p}^{+} \mathrm{S}_{r^{\prime}}^{2 n-1}$ exterior to $\mathbb{B}_{r^{\prime}}^{n}$ and the opposite one $\bar{T}_{p}^{-} S_{r^{\prime}}^{2 n-1}$. We distinguish two (nonexclusive) cases.


Firstly, suppose that the half-line $(p q)^{+}$is contained in $\bar{T}_{p}^{+} S_{r^{\prime}}^{2 n-1}$, as in the left figure. Since $p \in \Omega_{M}$ and $q \notin \Omega_{M}$, there exists at least one point $\widetilde{p}$ of the open segment $(p, q)$ which belongs to $M$, hence $\widetilde{p} \in M_{>r}$. Then

$$
\|q-\widetilde{p}\|<\|q-p\|<\delta
$$

whence $q \in \mathbb{B}^{n}(\widetilde{p}, \delta) \cap\{\|z\|>r\}$ and we deduce that $q \in \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$ belongs to the right hand side of (4.7), contradiction.

Secondly, suppose that the half-line $(p q)^{+}$is contained in $\bar{T}_{p}^{-} S_{r^{\prime}}^{2 n-1}$, as in the right figure. Let $\widetilde{q} \in(p, q)$ be the middle point. In the plane passing through $0, p$ and $q$, consider a circle passing through $p$ and $q$ and centered at some point close to 0 in the open segment $(0, \widetilde{q})$. It has radius $<r^{\prime}$ close to $r^{\prime}$. The open arc of circle between $p$ and $q$ is fully contained in $\left\{\|z\|>r^{\prime}\right\}$.

Since $p \in \Omega_{M}$ and $q \notin \Omega_{M}$, there exists at least one point $\widetilde{p}$ of the open arc of circle between $p$ and $q$ which belongs to $M$, hence $\widetilde{p} \in M_{>r}$. But then $(p, q)$ is the hypothenuse of the triangle $p q \widetilde{p}$ (remind $r^{\prime}>1$ and $\|q-p\|<$ $\delta \ll 1$ ), whence $\|q-\widetilde{p}\|<\|q-p\|<\delta$, hence again as in the first case, we deduce that $q \in \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$, contradiction.

If the slightly smaller radius

$$
r^{\prime \prime}:=r^{\prime}-\eta=\widehat{r}_{\kappa}-(k+1) \eta
$$

is already $<r$, we will shrink to $\eta^{\prime}:=\widehat{r}_{\kappa}-r-k \eta<\eta$ the thickness of the final extensional rind. Otherwise, in the generic case, $\widehat{r}_{\kappa}-(k+1) \eta$ is still $>r$. The final (exceptional) case being formally similar, we continue the proof with $r^{\prime}=\widehat{r}_{\kappa}-k \eta$ and $r^{\prime \prime}=r^{\prime}-\eta$, assuming that $r^{\prime \prime} \geqslant r$.

Setting $r^{\prime}:=\widehat{r}_{\kappa}-k \eta$ in the auxiliary Lemma 4.6, functions holomorphic in $\Omega_{>r^{\prime}} \cup \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$ restrict to Shell $r_{r^{\prime}}^{r^{\prime}+\delta}\left(\mathrm{R}_{r^{\prime}} \cup \mathrm{N}_{r^{\prime}}\right)$ and then, thanks to Proposition 3.7, these restricted functions extend holomorphically to Rind $\left(\mathrm{R}_{r^{\prime}}, \eta\right)$.

Lemma 4.8. The following intersection of two open sets is connected:

$$
\begin{equation*}
\operatorname{Rind}\left(\mathrm{R}_{r^{\prime}}, \eta\right) \bigcap\left(\Omega_{>r^{\prime}} \cup \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}\right) \tag{4.9}
\end{equation*}
$$

Furthermore, the union of the same two open sets contains

$$
\begin{equation*}
\Omega_{>r^{\prime}-\eta} \cup \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r} . \tag{4.10}
\end{equation*}
$$

Thus we get unique holomorphic extension to (4.10) and finally, by induction on $k$ and taking account of the final step where $\eta$ should be shrunk appropriately, we get unique holomorphic extension to $\Omega_{>r} \cup \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$.

The number of utilized Levi-Hartogs figures is majorated by the product of the number of needed rinds $\sim \frac{\widehat{r}_{k}-r}{\eta}$ times the maximal area of $\mathrm{R}_{r^{\prime}}$, which we roughly majorate by the area $C\left(\widehat{r}_{\kappa}\right)^{2 n-1}$ of the biggest sphere $\mathrm{S}_{\widehat{r}_{\kappa}}^{2 n-1}$, everything being divided by the area $c \delta^{2 n-1}$ covered by a small Levi-Hartogs figure. This yields the finite number claimed in Proposition 4.5, achieving its proof.

Proof of Lemma 4.8. [May be skipped in a first reading] To establish connectedness, we decompose the rind as

$$
\begin{aligned}
\operatorname{Rind}^{+} & :=\left\{(1+s) z: z \in \mathrm{R}_{r^{\prime}}, 0<s<\eta / r^{\prime}\right\} \\
\operatorname{Rind}^{0} & :=\mathrm{R}_{r^{\prime}}, \\
\operatorname{Rind}^{-} & :=\left\{(1-s) z: z \in \mathrm{R}_{r^{\prime}}, 0<s<\eta / r^{\prime}\right\}
\end{aligned}
$$

so that Rind $=$ Rind $^{-} \cup$ Rind $^{0} \cup$ Rind $^{+}$, without writing the common argument $\left(\mathrm{R}_{r^{\prime}}, \eta\right)$.

Obviously, the upper Rind ${ }^{+}$is diffeomorphic to $\mathrm{R}_{r^{\prime}} \times(0, \eta) \simeq \mathbb{R}^{2 n-1} \times$ $(0, \eta)$, hence is connected. We claim that, moreover, the full Rind ${ }^{+}$is contained in $\Omega_{>r^{\prime}} \cup \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$, whence

$$
\begin{equation*}
\operatorname{Rind}^{+}=\operatorname{Rind}^{+} \bigcap\left(\Omega_{>r^{\prime}} \cup \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}\right) . \tag{4.11}
\end{equation*}
$$

Indeed, let $q^{\prime} \in$ Rind $^{+}$, hence of the form $q^{\prime}=(1+s) p^{\prime}$ for some $p^{\prime} \in$ Rind ${ }^{0}=\mathrm{R}_{r^{\prime}}$ and some $s$ with $0<s<\eta / r^{\prime}$. If the half-open-closed segment ( $\left.p^{\prime}, q^{\prime}\right]$ is contained in $\Omega_{M}$, hence in $\Omega_{>r^{\prime}}=\Omega_{M} \cap\left\{\|z\|>r^{\prime}\right\}$, we get for free $q^{\prime} \in \Omega_{>r^{\prime}}$.

If on the contrary, $\left(p^{\prime}, q^{\prime}\right]$ is not contained in $\Omega_{M}$, then there exists a point $\widetilde{q}^{\prime} \in\left(p^{\prime}, q^{\prime}\right]$ with $\widetilde{q} \in M=\partial \Omega_{M}$, whence $\widetilde{q} \in M_{>r^{\prime}} \subset M_{>r}$ (remind $\left.r^{\prime}-\eta \geqslant r\right)$. The ball $\mathbb{B}^{n}\left(\widetilde{q}^{\prime}, \delta\right)$ then contains $q^{\prime}$, because $\left\|q^{\prime}-\widetilde{q}^{\prime}\right\|<\left\|q^{\prime}-p^{\prime}\right\| \leqslant$ $\eta \ll \delta$. This shows $q^{\prime} \in \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$, achieving the claim.

Thus, the (upper) subpart (4.11) of the intersection (4.9) is already connected.

To conclude the proof of connectedness, it suffices to show that every point $p^{\prime}$ of the remaining part

$$
\begin{equation*}
\left(\operatorname{Rind}^{0} \cup \operatorname{Rind}^{-}\right) \bigcap\left(\Omega_{>r^{\prime}} \cup \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}\right) \tag{4.12}
\end{equation*}
$$

can be joined, by means of some appropriate continuous curve running inside the intersection (4.9), to some point $q^{\prime}$ of the connected upper subpart (4.11). Thus, let $p^{\prime}$ in (4.12) be arbitrary.

If $p^{\prime} \in \operatorname{Rind}^{0} \cap\left(\Omega_{>r^{\prime}} \cup \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}\right)$, it suffices to join radially $p^{\prime}$ to $q^{\prime}=\left(1+s_{\varepsilon}\right) p^{\prime}$, for some $s_{\varepsilon}$ with $0<s_{\varepsilon} \ll \eta$. Indeed, such a $q^{\prime}$ then belongs to Rind ${ }^{+} \cap\left(\Omega_{>r^{\prime}} \cup \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}\right)$.

If $p^{\prime} \in \operatorname{Rind}^{-} \cap\left(\Omega_{>r^{\prime}} \cup \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}\right)$, then necessarily $p^{\prime} \in \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$, because by definition:

$$
\operatorname{Rind}^{-}\left(\mathrm{R}_{r^{\prime}}, \eta\right) \cap \Omega_{>r^{\prime}}=\emptyset
$$

So there is a point $q \in M_{>r}$ with $p^{\prime} \in \mathbb{B}^{n}(q, \delta)$.


Fig. 11: Joining a point $p^{\prime}$ of the lower rind to the connected upper rind

We then distinguish two exclusive cases: either $r(q) \geqslant r^{\prime}$ or $r(q)<r^{\prime}$.
Firstly, assume $r(q) \geqslant r^{\prime}$ (left diagram).
If $0, p^{\prime}$ and $q$ are aligned, we simply join $p^{\prime}$ to the point $q^{\prime}:=(1+$ $\left.s_{\varepsilon}\right) \frac{r^{\prime}}{r\left(p^{\prime}\right)} p^{\prime}$ which belongs to Rind ${ }^{+}$. The segment $\left[p^{\prime}, q^{\prime}\right]$ is then entirely contained in Rind $\cap \mathbb{B}^{n}(q, \delta)_{>r}$, hence in (4.9).

Otherwise, in the unique plane passing through $0, p^{\prime}$ and $q$, consider the point $q^{\prime \prime}:=\frac{r\left(p^{\prime}\right)}{r(q)} q$, satisfying $r\left(q^{\prime \prime}\right)=r\left(p^{\prime}\right)$ and belonging to $(0, q)$. Since $q^{\prime \prime}$ is the orthogonal projection of $q$ onto $\overline{\mathbb{B}^{n}\left(0, r\left(p^{\prime}\right)\right)}$, we get $\left\|q-q^{\prime \prime}\right\|<$ $\left\|q-p^{\prime}\right\|<\delta$, whence $q^{\prime \prime} \in \mathbb{B}^{n}(q, \delta)$. The circle of radius $r\left(p^{\prime}\right)$ centered at 0 joins $p^{\prime}$ to $q^{\prime \prime}$ by means of a small arc which is entirely contained in $\mathbb{B}^{n}(q, \delta)$. Denote by $\gamma:[0,1] \rightarrow \mathbb{B}^{n}(q, \delta)$ a parametrization of this arc of circle, with $\gamma(0)=p^{\prime}$ and $\gamma(1)=q^{\prime \prime}$.

If $\gamma[0,1]$ is entirely contained in Rind ${ }^{-}$, we conclude by joining $q^{\prime \prime}$ radially to the point $q^{\prime}:=\left(1+s_{\varepsilon}\right) \frac{r^{\prime}}{r\left(q^{\prime \prime}\right)} q^{\prime \prime}$.

If $\gamma[0,1]$ is not contained in Rind, let $t_{1} \in(0,1)$ satisfying $\gamma\left[0, t_{1}\right) \subset$ Rind ${ }^{-}$but $\gamma\left(t_{1}\right) \notin$ Rind $^{-}$. Then $\gamma\left(t_{1}\right)$ belongs to $\partial$ Rind $^{-}$and since $r\left(\gamma\left(t_{1}\right)\right)=r\left(p^{\prime}\right)$ still satisfies $r^{\prime}-\eta<r\left(p^{\prime}\right)<r^{\prime}$, necessarily $\gamma\left(t_{1}\right)$ belongs "vertical part" of $\partial \mathrm{Rind}^{-}$, namely to the strip $\{(1-s) z: z \in$ $\left.\mathrm{N}_{r^{\prime}}, 0 \leqslant s \leqslant \eta / r^{\prime}\right\}$. Hence the point $q^{\prime \prime \prime}:=\frac{r^{\prime}}{r\left(\gamma\left(t_{1}\right)\right)} \gamma\left(t_{1}\right)$ belongs to $\mathrm{N}_{r^{\prime}}$. We now modify $\gamma$ by constructing a curve which remains entirely inside $\mathbb{B}^{n}\left(q^{\prime \prime \prime}, \delta\right)_{>r} \subset \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$ as follows: choose $t_{2}<t_{1}$ very close to $t_{1}$, join $p^{\prime}$ to $\gamma\left(t_{2}\right) \in$ Rind $^{-}$through $\gamma$ and then $\gamma\left(t_{2}\right)$ radially to the point
$q^{\prime}:=\left(1+s_{\varepsilon}\right) \frac{r^{\prime}}{r\left(\gamma\left(t_{2}\right)\right)} \gamma\left(t_{2}\right) \in$ Rind $^{+}$. The resulting curve is entirely contained in (4.9). In conclusion, we have joined $p^{\prime}$ to a suitable point $q^{\prime}$, as announced.

Secondly, assume that $r(q)<r^{\prime}$. Consider the normalized gradient vector field $\frac{\nabla r_{M}}{\left\|\nabla r_{M}\right\|}$, defined and nowhere singular on $M \cap\left\{\widehat{r}_{\kappa-1}<\|z\|<\widehat{r}_{\kappa}\right\}$, hence on $M_{>r} \backslash\left\{\widehat{p}_{\kappa}\right\}$. For $t \in[0,2 \eta]$, denote by $t \mapsto q_{t}$ the integral curve of $\frac{\nabla r_{M}}{\left\|\nabla r_{M}\right\|}$ passing through $q$, satisfying $q_{0}=q, q_{t} \in M$ and $r\left(q_{t}\right)=r(q)+t$. Together with its center $q$, the ball is translated as $\mathbb{B}^{n}\left(q_{t}, \delta\right)$. Accordingly, the point $p^{\prime}$ is moved, yielding a curve $p_{t}^{\prime}$ such that $p_{t}^{\prime}$ occupies a fixed position with respect to the moving ball. Explicitly: $p_{t}^{\prime}=p^{\prime}+q_{t}^{\prime}-q$. Thanks to $r^{\prime}>1$ and $\delta \ll 1$, one may check ${ }^{44}$ that $\frac{d r\left(p_{t}^{\prime}\right)}{d t} \geqslant 1-c_{r^{\prime}, \delta}$, for some small positive constant $c_{r^{\prime}, \delta}<1$.

Thus, as $t$ increases, the point $p_{t}^{\prime}$ moves away from 0 at speed almost equal to 1 . Since $r^{\prime}-\eta<r\left(p_{0}^{\prime}\right)<r^{\prime}$, we deduce that for $t=2 \eta$, we have $r\left(p_{2 \eta}^{\prime}\right)>r^{\prime}$, namely $p_{2 \eta}^{\prime}$ has escaped from Rind ${ }^{-}$. Consequently, there exists $t_{1} \in(0,2 \eta)$ with $p_{t}^{\prime} \in$ Rind $^{-}$for $0 \leqslant t<t_{1}$ such that $p_{t_{1}}^{\prime} \in \partial$ Rind $^{-}$.

The boundary of Rind ${ }^{-}$has three parts: the top $\mathrm{R}_{r^{\prime}}$, the bottom $\{(1-$ $\left.\left.\eta / r^{\prime}\right) z: z \in \mathrm{R}_{r^{\prime}}\right\}$ and the (closed) strip $\left\{(1-s) z: z \in \mathbf{N}_{r^{\prime}}, 0 \leqslant s \leqslant \eta / r^{\prime}\right\}$. The limit point $p_{t_{1}}^{\prime}$ cannot belong to the bottom, since $r\left(p_{t_{1}}^{\prime}\right)>r\left(p_{0}^{\prime}\right)>$ $r^{\prime}-\eta$.

Since by construction $p_{t}^{\prime} \in \mathbb{B}^{n}\left(q_{t}, \delta\right)$ with $q_{t} \in M_{>r}$, we observe that $p_{t}^{\prime} \in \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$ for every $t \in[0,2 \eta]$. Consequently:

$$
p_{t}^{\prime} \in \operatorname{Rind}^{-} \bigcap \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}, \quad \forall t \in\left[0, t_{1}\right)
$$

Assuming that $p_{t_{1}}^{\prime} \in \partial$ Rind $^{-}$belongs to the top $\mathrm{R}_{r^{\prime}}=$ Rind ${ }^{0}$, we may join $p_{t_{1}}^{\prime}$ radially to $q^{\prime}:=\left(1+s_{\varepsilon}\right) p_{t_{1}}^{\prime}$. In this way, $p^{\prime}$ is joined, by means of a continuous curve running in the intersection (4.9), to the point $q^{\prime}=$ $\left(1+s_{\varepsilon}\right) p_{t_{1}}^{\prime}$ belonging to the connected upper subpart (4.11).

Finally, assume that $p_{t_{1}}^{\prime} \in \partial$ Rind $^{-}$belongs to the strip $\{(1-s) z: z \in$ $\left.\mathrm{N}_{r^{\prime}}, 0 \leqslant s \leqslant \eta / r^{\prime}\right\}$. The point $q^{\prime \prime}:=\frac{r^{\prime}}{r\left(p_{p_{1}}\right)} p_{t_{1}}^{\prime}$ belongs to $\mathrm{N}_{r^{\prime}} \subset M_{>r}$, and we will construct a small curve running entirely inside $\mathbb{B}^{n}\left(q^{\prime \prime}, \delta\right)_{>r} \subset$ $\mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$. Choose $t_{2} \in\left(0, t_{1}\right)$ very close to $t_{1}$, join $p^{\prime}$ to $p_{t_{2}}^{\prime} \in \operatorname{Rind}^{-}$as above (but do not go up to $p_{t_{1}}^{\prime}$ ) and then join $p_{t_{2}}^{\prime}$ radially to the point $q^{\prime}:=$ $\left(1+s_{\varepsilon}\right) \frac{r^{\prime}}{r\left(p_{t_{2}}^{\prime}\right)} p_{t_{2}}^{\prime}$, which belongs to Rind ${ }^{+}$. The small radial segment from $p_{t_{2}}^{\prime}$ to $q^{\prime}$ is entirely contained in $\mathbb{B}^{n}\left(q^{\prime \prime}, \delta\right)$ and in the full Rind. In conclusion, $p^{\prime}$ is joined, by means of a continuous curve running in the intersection (4.9),

[^43]to this point $q^{\prime}=\left(1+s_{\varepsilon}\right) \frac{r^{\prime}}{r\left(p_{t_{2}}^{\prime}\right)} p_{t_{2}}^{\prime}$ which belongs to the connected upper subpart (4.11).

The proof of the connectedness of the intersection (4.9) is complete.
We now show that the union, instead of the intersection in (4.9), contains (4.10).

Let $p^{\prime} \in \Omega_{>r^{\prime}-\eta} \backslash \Omega_{>r^{\prime}}$, whence $r^{\prime}-\eta<\left\|p^{\prime}\right\| \leqslant r^{\prime}$. The radial half line $\left\{t p^{\prime}: 0<t<\infty\right\}$ emanating from the origin and passing through $p^{\prime}$ meets $\mathrm{S}_{r^{\prime}}^{2 n-1}$ at the point $q^{\prime}=\frac{r^{\prime}}{\left\|p^{\prime}\right\|} p^{\prime}$.

If the closed segment $\left[p^{\prime}, q^{\prime}\right]$ is contained in $\Omega_{>r^{\prime}-\eta}$, then $q^{\prime} \in \Omega_{M}$. Since $\left\|q^{\prime}\right\|=r^{\prime}$ and since $\mathrm{R}_{r^{\prime}}=\Omega_{M} \cap\left\{\|z\|=r^{\prime}\right\}$, we get $q^{\prime} \in \mathrm{R}_{r^{\prime}}$, whence $p^{\prime} \in \operatorname{Rind}\left(\mathrm{R}_{r^{\prime}}, \eta\right)$.

If on the contrary, the closed segment $\left[p^{\prime}, q^{\prime}\right]$ is not contained in $\Omega_{>r^{\prime}-\eta}$, then there exists $\widetilde{q}^{\prime} \in\left(p^{\prime}, q^{\prime}\right]$ with $\widetilde{q}^{\prime} \in M=\partial \Omega_{M}$, whence $\widetilde{q} \in M_{>r^{\prime}-\eta} \subset$ $M_{>r}$. Since $\eta \ll \delta$, we deduce $p^{\prime} \in \mathbb{B}^{n}\left(\tilde{q}^{\prime}, \delta\right)$ and finally $p^{\prime} \in \mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$.

The proofs of Lemma 4.8 and hence also of Proposition 4.5 are complete.

## §5. CREATING DOMAINS, MERGING

AND SUPPRESSING CONNECTED COMPONENTS

### 5.1. Topological stability and global extensional geometry between reg-

 ular values of $r_{M}$. In the preceding Section 4 , for $r$ with $\widehat{r}_{\kappa-1}<r<\widehat{r}_{\kappa}$, we described the simple shape of the cut out domain $\Omega_{>r}=\Omega_{M} \cap\{\|z\|>r\}$, just diffeomorphic to a piece of ball. Decreasing the radius under $\widehat{r}_{\kappa-1}$, the topological picture becomes more complex. At least for radii comprised between two singular values of $\left.r(z)\right|_{M}$, Morse theory assures geometrical control together with constancy properties.Lemma 5.2. Fix a radius $r$ satisfying $\widehat{r}_{\lambda}<r<\widehat{r}_{\lambda+1}$ for some $\lambda$ with $1 \leqslant \lambda \leqslant \kappa-1$, hence noncritical for the distance function $\left.r(z)\right|_{M}$. Then:
(a) $T_{z} M+T_{z} \mathrm{~S}_{r}^{2 n-1}=T_{z} \mathbb{C}^{n}$ at every point $z \in M \cap \mathrm{~S}_{r}^{2 n-1}$;
(b) the intersection $M \cap \mathrm{~S}_{r}^{2 n-1}$ is a $\mathscr{C}^{\infty}$ compact hypersurface $\mathrm{N}_{r} \subset$ $\mathrm{S}_{r}^{2 n-1}$ of codimension 2 in $\mathbb{C}^{n}$, without boundary and having finitely many connected components;
(c) $\mathrm{N}_{r^{\prime \prime}}$ is diffeomorphic to $\mathbf{N}_{r^{\prime}}$, whenever $\widehat{r}_{\lambda}<r^{\prime \prime}<r^{\prime}<\widehat{r}_{\lambda+1}$;
(d) $M_{>r}=M \cap\{\|z\|>r\}$ has finitely many connected components $M_{>r}^{c}$, with $1 \leqslant c \leqslant c_{\lambda}$, for some $c_{\lambda}<\infty$ which is independent of $r$;
(e) $M_{>r^{\prime \prime}}^{c}$ is diffeomorphic to $M_{>r^{\prime}}^{c}$, whenever $\widehat{r}_{\lambda}<r^{\prime \prime}<r^{\prime}<\widehat{r}_{\lambda+1}$, for all $c$ with $1 \leqslant c \leqslant c_{\lambda}$;
(f) $M \cap\left\{r^{\prime \prime}<\|z\|<r^{\prime}\right\}$ is diffeomorphic to $\mathrm{N}_{r^{\prime}} \times\left(r^{\prime \prime}, r^{\prime}\right)$, hence also to $\mathrm{N}_{r^{\prime \prime}} \times\left(r^{\prime \prime}, r^{\prime}\right)$, whenever $\widehat{r}_{\lambda}<r^{\prime \prime}<r^{\prime}<\widehat{r}_{\lambda+1}$;

Proof. We summarize the known arguments of proof (cf. [31] and [17], Ch. 6). Equivalently, (a) says that $d r: T_{z} M \rightarrow T_{r(z)} \mathbb{R}$ is onto, and this holds true since by assumption $M \cap\left\{\widehat{r}_{\lambda}<\|z\|<\widehat{r}_{\lambda+1}\right\}$ contains no critical points of $\left.r(z)\right|_{M}$. Then (b) follows from this transversality (a).

Next, consider the Euclidean metric $(v, w):=\sum_{k=1}^{2 n} v_{k} w_{k}$ on $\mathbb{C}^{n} \simeq \mathbb{R}^{2 n}$, which induces a Riemannian metric $(\cdot, \cdot)_{M}$ on $M$, a nondegenerate positive bilinear form on $T M$. The gradient $\nabla\left(\left.r\right|_{M}\right)$ of $\left.r(z)\right|_{M}$ is the vector field on $M$ defined by requiring that $\left(\nabla\left(\left.r\right|_{M}\right), X\right)_{M}=d\left(\left.r\right|_{M}\right)(X)$ for all $\mathscr{C}^{\infty}$ (locally defined) vector fields $X$ on $M$. Let $\mathrm{D}:=2 \operatorname{Re} \sum_{k=1}^{n} z_{k} \frac{\partial}{\partial z_{k}}$ be the radial vector field which is obviously orthogonal to spheres and consider the orthogonal projection $X_{\mathrm{D}}$ of $\left.\mathrm{D}\right|_{M}$ on $T M$, a $\mathscr{C}^{\infty}$ vector field on $M$. We want to scale the gradient as $\mathrm{V}_{r, M}:=\lambda \cdot \nabla\left(\left.r\right|_{M}\right)$ so that its radial component is identically equal to one, namely, so that $\left(\mathrm{V}_{r, M}, \mathrm{D}\right) \equiv 1$, which gives the equation:

$$
1=\lambda\left(\nabla\left(\left.r\right|_{M}\right), \mathrm{D}\right)=\lambda\left(\nabla\left(\left.r\right|_{M}\right), X_{\mathrm{D}}\right)=\lambda\left(\nabla\left(\left.r\right|_{M}\right), X_{\mathrm{D}}\right)_{M}=\lambda d\left(\left.r\right|_{M}\right)\left(X_{\mathrm{D}}\right)
$$

To simply set $\lambda:=\frac{1}{d\left(\left.r\right|_{M}\right)\left(X_{\mathrm{D}}\right)}$, we must establish that $X_{\mathrm{D}}$ cannot belong to Ker $d\left(\left.r\right|_{M}\right)$ at any point $z \in M \cap\left\{\widehat{r}_{\lambda}<\|z\|<\widehat{r}_{\lambda+1}\right\}$ of a noncritical shell.

We check this. At such a point $z, \mathrm{D}(z)$ is not orthogonal to $T_{z} M$ (otherwise $T_{z} M$ would coincide with $T_{z} \mathrm{~S}_{\|z\|}^{2 n-1}$ ), whence its orthogonal projection $X_{\mathrm{D}}(z)$ is $\neq 0$. By definition, $\left(\mathrm{D}-X_{\mathrm{D}}\right)(z)$ is orthogonal to $T_{z} M \ni X_{\mathrm{D}}(z)$, hence it is orthogonal to $X_{\mathrm{D}}(z)$ inside the 2-dimensional plane $\Pi_{z}$ generated by $X_{\mathrm{D}}(z) \neq 0$ and by $\mathrm{D}(z) \neq 0$. If, contrary to what we want, $X_{\mathrm{D}}(z)$ would belong to $\operatorname{Ker} d\left(\left.r\right|_{M}\right)=T_{z} S_{\|z\|}^{2 n-1}$, then it would be orthogonal to $\mathrm{D}(z)$, and in the plane $\Pi_{z}$, we would have both $\mathrm{D}(z)$ and the hypothenuse $\left(\mathrm{D}-X_{\mathrm{D}}\right)(z)$ orthogonal to $X_{\mathrm{D}}(z)$, which is impossible.

Thus, in spherical coordinates $\left(r, \vartheta_{1}, \ldots, \vartheta_{2 n-1}\right)$ restricted to a noncritical shell, the $r$-component of the $\mathscr{C}^{\infty}$ scaled gradient vector field $\mathrm{V}_{r, M}:=\frac{\nabla\left(\left.r\right|_{M}\right)}{\left(\nabla\left(\left.r\right|_{M}\right), \mathrm{D}\right)}$ is $\equiv 1$. We deduce that the flow (wherever defined) $z_{s}:=\exp \left(s \bigvee_{r, M}\right)(z)$ simply increases the norm as $\left\|z_{s}\right\|=\|z\|+s$, whence $\exp \left(\left(r^{\prime}-r^{\prime \prime}\right) \mathrm{V}_{r, M}\right)(\cdot)$ induces a diffeomorphism from $\mathrm{N}_{r^{\prime \prime}}$ onto $\mathrm{N}_{r^{\prime}}$ : this yields (c). Also, $\left(z^{\prime \prime}, s\right) \longmapsto \exp \left(\left(r^{\prime \prime}+s\right) \bigvee_{r, M}\right)\left(z^{\prime \prime}\right)$ gives the diffeomorphism of $\mathrm{N}_{\mathrm{r}^{\prime \prime}} \times\left(r^{\prime}-r^{\prime \prime}\right)$ onto the strip $M \cap\left\{r^{\prime \prime}<\|z\|<r^{\prime}\right\}$, which is (f).

Next, the compact manifold with boundary $M_{>r} \cup \mathrm{~N}_{r}$ surely has finitely many connected components, whose number is constant for all $\widehat{r}_{\lambda}<r<$ $\widehat{r}_{\lambda+1}$, because when $r$ increases or decreases, the connected components of the slices $\mathrm{N}_{r}$ do slide smoothly in $\mathrm{S}_{r}^{2 n-1}$ without encountering each other: this is (d). Finally, (e) follows from (f) and the trivial fact that the two
segments $\left(r^{\prime \prime}, r^{0}\right)$ and $\left(r^{\prime}, r^{0}\right)$ are diffeomorphic, whenever $\widehat{r}_{\lambda}<r^{\prime \prime}<r^{\prime}<$ $r^{0}<\widehat{r}_{\lambda+1}$.

We can now state the very main technical proposition of this paper.
Proposition 5.3. Fix a radius $r$ satisfying $\widehat{r}_{\lambda}<r<\widehat{r}_{\lambda+1}$ for some $\lambda$ with $1 \leqslant \lambda \leqslant \kappa-1$ and let $M_{>r}^{c}, c=1, \ldots, c_{\lambda}$, denote the collection of connected components of $M \cap\{\|z\|>r\}$. Then:
(i) each $M_{>r}^{c}$ bounds in $\{\|z\|>r\}$ a unique domain $\widetilde{\Omega}_{>r}^{c}$ which is relatively compact in $\mathbb{C}^{n}$;
(ii) the boundary in $\mathbb{C}^{n}$ of each $\widetilde{\Omega}_{>r}^{c}$, namely:

$$
\partial \widetilde{\Omega}_{>r}^{c}=M_{>r}^{c} \cup \mathrm{~N}_{r}^{c} \cup \widetilde{\mathrm{R}}_{r}^{c}
$$

consists of $M_{>r}^{c}$ together with some appropriate union $\mathrm{N}_{r}^{c}$ of finitely many connected components of $\mathrm{N}_{r}=M \cap\{\|z\|=r\}$ and with an appropriate region $\widetilde{\mathrm{R}}_{r}^{c} \subset \mathrm{~S}_{r}^{2 n-1}$ delimited by $\mathrm{N}_{r}^{c}$;
(iii) two such domains $\widetilde{\Omega}_{>r}^{c_{1}}$ and $\widetilde{\Omega}_{>r}^{c_{2}}$, associated to two different connected components $M_{>r}^{c_{1}}$ and $M_{>r}^{c_{2}}$ of $M_{>r}$, are either disjoint or one is contained in the other;
(iv) for $c_{1} \neq c_{2}$, the regions $\widetilde{\mathrm{R}}_{r}^{c_{1}}$ and $\widetilde{\mathrm{R}}_{r}^{c_{2}}$ are either disjoint or one is contained in the other, while their boundaries $\mathrm{N}_{r}^{c_{1}}$ and $\mathrm{N}_{r}^{c_{2}}$ are always disjoint;
(v) for each $c=1, \ldots, c_{\lambda}$, every function $f$ holomorphic in $\mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$ has a restriction to $\mathscr{V}_{\delta}\left(M_{>r}^{c}\right)_{>r}$ which extends holomorphically and uniquely to $\widetilde{\Omega}_{>r}^{c}$ by means of a finite number of Levi-Hartogs figures.

We point out that in (i) and (ii), neither $\widetilde{\Omega}_{r}^{c}$ nor $\widetilde{\mathrm{R}}_{r}^{c}$ need be contained in our original domain $\Omega_{M}$ (as it was the case in Section 4 for $\widehat{r}_{\kappa-1}<r<$ $\widehat{r}_{\kappa}$ ): this is why we introduced a widetilde notation. We refer to the middle Figure 1 for an illustration. Similarly, neither $\widetilde{\Omega}_{r}^{c}$ nor $\widetilde{R}_{r}^{c}$ need be contained in $\mathbb{C}^{n} \backslash \bar{\Omega}_{M}$ : they both may intersect $\Omega_{M}$ and $\mathbb{C}^{n} \backslash \bar{\Omega}_{M}$. Also, the number of connected components of $\mathrm{N}_{r}^{c}$ is $\geqslant$ that of $\widetilde{\mathrm{R}}_{r}^{c}$ and may be $>$, as illustrated below.


Fig. 12: Possible topologies of the cut out hypersurfaces $M_{>r}$
As a direct application, we may achieve the proof of our principal result.

Theorem 5.4. Under the precise assumptions of Theorem 2.7, holomorphic functions in $\mathscr{V}_{\delta}(M)$ do extend holomorphically and uniquely to $\Omega_{M}$ by means of a finite number of Levi-Hartogs figures:

$$
\forall f \in \mathscr{O}\left(\mathscr{V}_{\delta}(M)\right) \quad \exists F \in \mathscr{O}\left(\Omega_{M} \cup \mathscr{V}_{\delta}(M)\right) \quad \text { s.t. }\left.F\right|_{\mathscr{V}_{\delta}(M)}=f .
$$

Proof. In the main Proposition 5.3, we choose $r=\widehat{r}_{1}+\varepsilon$ (where $\varepsilon>0$ satisfies $\varepsilon \ll \delta$ ) very close to the last, smallest singular radius. Then $M_{>r}$ has a single connected component, $M_{>r}$ itself, and it simply bounds $\left(\Omega_{M}\right)_{>r}$. The remainder part of $M$, namely $M \cap\left\{\|z\| \leqslant \widehat{r}_{1}+\varepsilon\right\}$ is diffeomorphic to a very small closed $(2 n-1)$-dimensional spherical cap and is entirely contained in $\mathscr{V}_{\delta}(M)$.

Fix an arbitrary function $f \in \mathscr{O}\left(\mathscr{V}_{\delta}(M)\right)$ and restrict it to $\mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$. Thanks to the proposition, $f$ extend holomorphically and uniquely to $\left(\Omega_{M}\right)_{>r}$ by means of a finite number of Levi-Hartogs figures. Since

$$
\mathscr{V}_{\delta}(M) \bigcap\left(\mathscr{V}_{\delta}\left(M_{>r}\right)_{>r} \cup\left(\Omega_{M}\right)_{>r}\right)
$$

is easily seen to be connected, we get a globally defined extended function which is holomorphic in

$$
\mathscr{V}_{\delta}(M) \bigcup\left(\mathscr{V}_{\delta}\left(M_{>r}\right)_{>r} \cup\left(\Omega_{M}\right)_{>r}\right)=\mathscr{V}_{\delta}(M) \cup \Omega_{M}
$$

This completes the proof.
Proof of Proposition 5.3. In (i), let us check the uniqueness of a relatively compact $\widetilde{\Omega}_{>r}^{c}$. Since $M_{>r}^{c}$ inherits an orientation from $M$, the complement $\{\|z\|>r\} \backslash M_{>r}^{c}$ has at most 2 connected components. As $M \Subset \mathbb{C}^{n}$ is bounded, at least one component contains the points at infinity, hence there can remain at most one component of $\{\|z\|>r\} \backslash M_{>r}^{c}$ that is relatively compact in $\mathbb{C}^{n}$.

If $r$ satisfies $\widehat{r}_{\kappa-1}<r<\widehat{r}_{\kappa}$, Proposition 4.5 already completes the proof.
Assume therefore that $r$ satisfies $\widehat{r}_{\mu}<r<\widehat{r}_{\mu+1}$, for some $\mu \in \mathbb{N}$ with $1 \leqslant \mu \leqslant \kappa-1$. For every $\lambda$ with $2 \leqslant \lambda \leqslant \kappa-1$, it will be convenient to flank each singular radius $\widehat{r}_{\lambda}$ by the following two very close nonsingular radii

$$
\begin{equation*}
\widehat{r}_{\lambda}:=\widehat{r}_{\lambda}-\eta / 2 \quad \text { and } \quad \widehat{r}_{\lambda}^{+}:=\widehat{r}_{\lambda}+\eta / 2 \text {, } \tag{5.5}
\end{equation*}
$$

with $\eta$ being the same uniform thickness of extensional rinds as before. We fix once for all an arbitrary function $f$ holomorphic in $\mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$. Letting $\lambda$ be arbitrary with $\mu \leqslant \lambda \leqslant \kappa-1$, the logic of the proof shows up two topologically distinct phenomena that we overview.
A: Filling domains through regular radii intervals. Assume that at the regular radius $\widehat{r}_{\lambda+1}=\widehat{r}_{\lambda+1}-\frac{\eta}{2}$, all domains $\widetilde{\Omega}_{>\widehat{r}_{\lambda+1}^{-}}^{c}, c=1, \ldots, c_{\lambda}$, as well
as the corresponding holomorphic extensions, have been constructed. Then prolong the domains (without topological change) as $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c}, c=1, \ldots, c_{\lambda}$, up to $\widehat{r}_{\lambda}^{+}=\widehat{r}_{\lambda}+\frac{\eta}{2}$ and fill in the conquered territory by means of a finite number of Levi-Hartogs figures.
B: Jumping across singular radii and changing the domains. Restarting at $\widehat{r}_{\lambda}^{+}$with the domains $\widetilde{\Omega}_{>r_{\lambda}^{+}}^{c}, c=1, \ldots, c_{\lambda}$, distinguish three cases as follows. Remind from $\S 2.3$ that $M$ is represented by $v=\sum_{1 \leqslant j \leqslant k_{\lambda}} x_{j}^{2}-$ $\sum_{1 \leqslant j \leqslant 2 n-k_{\lambda}-1} y_{j}^{2}$ in suitable coordinates $(x, y, v)$ centered at $\widehat{p}_{\lambda}$, where $k_{\lambda}$ is the Morse coindex of $\left.r(z)\right|_{M}$ at $\widehat{p}_{\lambda}$.
(I) Firstly, assume $k_{\lambda}=0$, namely $\left.z \mapsto r(z)\right|_{M}$ has a local maximum at $\widehat{p}_{\lambda}$, or inversely, assume $k_{\lambda}=2 n-1$, namely $\left.z \mapsto r(z)\right|_{M}$ has a local minimum at $\widehat{p}_{\lambda}$. This is the easiest case, the only one in which new domains can be born or die, locally.
(II) Secondly, assume $k_{\lambda}=1$. This is the most delicate case, because in a small neighborhood of $\widehat{p}_{\lambda}$, the cut out hypersurface $M_{>\widehat{r}_{\lambda}^{+}}$has exactly 2 connected components, so that two different enclosed domains $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{1}}$ and $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{2}}$ can meet here; it may also occur that the two parts near $\widehat{p}_{\lambda}$ belong to the same domain, i.e. that $c_{2}=c_{1}$. While descending down to $\widehat{r}_{\lambda}$, we must analyze the way how the two (maybe the single) component(s) merge. Three subcases will be distinguished, one of which showing a crucial trick of subtracting one growing component from a larger one which also grows (right Figure 1).
(III) Thirdly, assume that $2 \leqslant k_{\lambda} \leqslant 2 n-2$. In all these cases, locally in a neighborhood of $\widehat{p}_{\lambda}$, the cut out hypersurface $M_{>\widehat{r}_{\lambda}^{+}}$has exactly 1 connected component and the way how the corresponding single enclosed domain $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c}$ grows will be topologically constant.

Reasoning by induction on $\lambda$ and applying the filling processes A and $B$, we then descend progressively inside deeper spherical shells, checking all properties of Proposition 5.3. When approaching the bottom radius $r$ of Proposition 5.3, it will suffice to shortcut A or B appropriately in order to complete the proof.
5.6. Filling domains through regular radii intervals. Recall that $\widehat{r}_{\mu}<$ $r<\widehat{r}_{\mu+1}$, let $\lambda$ with $\mu \leqslant \lambda \leqslant \kappa-1$ and consider the regular radius interval $\left[\widehat{r}_{\lambda}^{+}, \widehat{r}_{\lambda+1}\right]$. We suppose first that $r \leqslant \widehat{r}_{\lambda}^{+}$, so that we may descend inside the whole spherical shell $\left\{\widehat{r}_{\lambda}^{+}<\|z\| \leqslant \widehat{r}_{\lambda+1}\right\}$. Afterwards, we explain how we stop in the case where $\lambda=\mu$ and $\widehat{r}_{\mu}^{+}<r<\widehat{r}_{\mu+1}$.

By descending induction on $\lambda$ through A and B , we may assume that at the superlevel set $(\cdot)_{>\widehat{r}_{\lambda+1}^{-}}$, the domains $\widetilde{\Omega}_{>\widehat{r}_{\lambda+1}^{c}}^{c}$ enclosed by $M_{>\widehat{r}_{\lambda+1}^{-}}^{c}$ for $1 \leqslant c \leqslant c_{\lambda}$ have been constructed and that each restriction $f_{\widehat{r}_{\lambda+1}^{-}}^{c}$ of $f \in$ $\mathscr{O}\left(\mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}\right)$ to $\mathscr{V}_{\delta}\left(M_{>\widehat{r}_{\lambda+1}^{-}}^{c}\right)_{>\widehat{r}_{\lambda+1}^{-}}$extends holomorphically and uniquely to the domain

$$
\begin{equation*}
\widetilde{\Omega}_{>\hat{r}_{\lambda+1}^{-}}^{c} \bigcup \mathscr{V}_{\delta}\left(M_{>\widehat{r}_{\lambda+1}^{-}}^{c}\right)_{>\widehat{r}_{\lambda+1}^{-}} . \tag{5.7}
\end{equation*}
$$

For every radius $r^{\prime}$ with $\widehat{r}_{\lambda}^{+} \leqslant r^{\prime}<\widehat{r}_{\lambda+1}$, the cut out hypersurface $M_{>r^{\prime}}=\bigcup_{1 \leqslant c \leqslant c_{\lambda}} M_{>r^{\prime}}^{c}$ has the same number of connected components, each $M_{>r^{\prime}}^{c}$ is diffeomorphic to $M_{>\widehat{r}_{\lambda+1}}^{c}$ and the difference $M_{>r^{\prime}}^{c} \backslash M_{>\widehat{r}_{\lambda+1}}^{c}$ is diffeomorphic to $N_{\widetilde{r}_{\lambda+1}}^{c} \times\left(r^{\prime}, \widehat{r}_{\lambda+1}\right]$. Furthermore, each prolongation $\widetilde{\Omega}_{>r^{\prime}}^{c}$ of $\widetilde{\Omega}_{>\widetilde{r}_{\lambda+1}}^{c}$ is obviously defined just by adding the tube domain surrounded by $M_{>r^{\prime}}^{c} \backslash M_{\widehat{r_{\lambda+1}}}^{c}$. Then each $N_{r^{\prime}}^{c}=\partial \widetilde{\mathrm{R}}_{r^{\prime}}^{c}$ has finitely many connected components $N_{r^{\prime}}^{c, j}$, with $1 \leqslant j \leqslant j_{\lambda, c}$, where $j_{\lambda, c}$ is independent of $r^{\prime}$.


Since $f$ was defined in $\mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$ and since $r \leqslant \widehat{r}_{\lambda}^{+}$, we claim that each restriction $f_{\widehat{r}_{\lambda+1}}^{c}$ may be extended holomorphically and uniquely to

$$
\begin{equation*}
\widetilde{\Omega}_{>\widetilde{r}_{\lambda+1}^{-}}^{c} \bigcup \mathscr{V}_{\delta}\left(M_{>\widetilde{r}_{\lambda}^{+}}^{c}\right)_{>\widehat{r}_{\lambda}^{+}} . \tag{5.8}
\end{equation*}
$$

Indeed, to the original domain of definition (5.7) of $\int_{\widehat{r}_{\lambda+1}}^{c}$ which was contained in $\left\{\|z\|>\widehat{r}_{\lambda+1}\right\}$, we add in the enlarged domain (5.8) a finite number $j_{\lambda, c}$ of tubular domains around the connected components of $M_{>r^{\prime}}^{c} \backslash M_{>r_{\lambda+1}^{-}}^{c}$. Because $\delta$ was chosen so small that $\mathscr{V}_{\delta}(M)$ is a small tubular neighborhood of $M$, and because $f \in \mathscr{O}\left(\mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}\right)$ is uniquely defined, we get a unique extension, still denoted by $f_{\widehat{r}_{\lambda+1}}^{c}$, to (5.8).

We can now apply the same reasoning as in Proposition 4.5, which consists of progressive holomorphic extension by means of thin rinds. Reproducing the proof of Lemma 4.6 (with changes of notation only), we get for
every radius $r^{\prime}$ with $\widehat{r}_{\lambda}^{+}<r^{\prime} \leqslant \widehat{r}_{\lambda+1}$ that

$$
\begin{equation*}
\text { Shell } r_{r^{\prime}}^{r^{\prime}+\delta}\left(\widetilde{\mathrm{R}}_{r^{\prime}}^{c} \cup \mathrm{~N}_{r^{\prime}}^{c}\right) \text { is contained in } \widetilde{\Omega}_{>r^{\prime}}^{c} \bigcup \mathscr{V}_{\delta}\left(M_{>\widetilde{r}_{\lambda}^{+}}^{c}\right)_{>\widetilde{r}_{\lambda}^{+}} \tag{5.9}
\end{equation*}
$$

Similarly, reproducing the proof of Lemma 4.8 yields the connectedness of

$$
\operatorname{Rind}\left(\mathrm{R}_{r^{\prime}}^{c}, \eta\right) \bigcap\left(\widetilde{\Omega}_{>r^{\prime}}^{c} \cup \mathscr{V}_{\delta}\left(M_{>\widetilde{r}_{\lambda}^{+}}^{c}\right)_{>\widetilde{r}_{\lambda}^{+}}\right),
$$

and furthermore, this yields that the union, instead of the intersection, contains

$$
\widetilde{\Omega}_{>r^{\prime}-\eta}^{c} \bigcup \mathscr{V}_{\delta}\left(M_{>\widetilde{r}_{\lambda}^{+}}^{c}\right)_{>\widetilde{r}_{\lambda}^{+}},
$$

whenever $r^{\prime}-\eta$ is still $\geqslant \widehat{r}_{\lambda}^{+}$(otherwise, shrink conveniently the thickness of the last extensional rind, as in the proof of Proposition 4.5). Thus, by piling up $\frac{\widehat{r}_{\lambda+1}^{-}-\widehat{r}_{\lambda}^{+}}{\eta}$ rinds and by using a finite number $\leqslant C\left(\frac{\widehat{r}_{\kappa}}{\delta}\right)^{2 n-1}\left[\frac{r_{\lambda+1}^{-}-\widehat{r}_{\lambda}^{+}}{\eta}\right]$ of Levi-Hartogs figures, we get unique holomorphic extension to

$$
\begin{equation*}
\mathscr{V}_{\delta}\left(M_{>r_{\lambda}^{+}}^{c}\right)_{>\overparen{r}_{\lambda}^{+}} \bigcup \widetilde{\Omega}_{>r_{\lambda}^{+}}^{c} \tag{5.10}
\end{equation*}
$$

Finally, if $r$ satisfies $\widehat{r}_{\mu}^{+}<r<\widehat{r}_{\mu+1}$, descending from $(\cdot)_{>\widehat{r}_{\mu+1}^{-}}$with $\lambda=\mu$ as above, we just stop the construction of rinds to $(\cdot)_{>r}$ by shrinking appropriately the thickness of the last extensional rind.

The property (iii) that enclosed domains $\widetilde{\Omega}_{>r}^{c}$ are either disjoint or one is contained in the other remains stable as $r$ decreases through the whole nonsingular interval $\left(\widehat{r}_{\lambda}, \widehat{r}_{\lambda+1}\right)$, because their (moving) boundaries always remain disjoint, so that property (iv) is also simultaneously transmitted to lower regular radii. This completes A.
5.11. Localizing (pseudo)cubes at Morse points. We now study B. Recall that $\widehat{r}_{\mu}<r<\widehat{r}_{\mu+1}$, let $\lambda$ with $\mu \leqslant \lambda \leqslant \kappa-1$ and suppose that $r \leqslant \widehat{r}_{\lambda}$, so that starting from $(\cdot)_{>\widehat{r}_{\lambda}^{+}}$, we may (and we must) continue the Hartogs-Levi filling inside the whole thin spherical shell $\left\{\widehat{r}_{\lambda}<\|z\| \leqslant \widehat{r}_{\lambda}^{+}\right\}$. Similarly as above, the way how we should stop the process in the case where $\lambda=\mu$ and $\widehat{r}_{\mu}<r<\widehat{r}_{\mu}^{+}$is obvious.

By descending induction on $\lambda$ through A and B , we may assume that at $\widehat{r}_{\lambda}^{+}$, the domains $\widetilde{\Omega}_{>r_{\lambda}^{+}}^{c}$ enclosed by $M_{>r_{\lambda}^{+}}^{c}$ for $1 \leqslant c \leqslant c_{\lambda}$ have been constructed and that each restriction $f_{\widehat{r}_{\lambda}^{+}}^{c}$ of $f \in \mathscr{O}\left(\mathscr{V}_{\delta}\left(M_{>r}^{c}\right)_{>r}\right)$ to $\mathscr{V}_{\delta}\left(M_{>\widehat{r}_{\lambda}^{+}}^{c}\right)_{>\widehat{r}_{\lambda}^{+}}$extends holomorphically to the domain (5.10) of the previous paragraph.

By an elementary analysis of the Morse normalizing quadric, we will see that in some small (pseudo)cube centered at $\widehat{p}_{\lambda}$, there passes in most cases only one component $M_{>r_{\lambda}^{+}}^{c}$, while in a single exceptional case, there can pass two (at most) different connected components $M_{>\widehat{r}_{\lambda}^{+}}^{c_{1}}$ and $M_{>\hat{r}_{\lambda}^{+}}^{c_{2}}$. We will consider only this single (or these two) component(s), because the other
components do pass regularly and without topological change accross $\widehat{p}_{\lambda}$, hence are filled in by Levi-Hartogs figures exactly as in A.

Shrinking the $\delta_{1}$ of Theorem 2.7 if necessary (remind $0<$ $\delta \leqslant \delta_{1}$ ), we may assume that the Morse normalizing coordinates $\left(v, x_{1}, \ldots, x_{k_{\lambda}}, y_{1}, \ldots, y_{2 n-1-k_{\lambda}}\right)$ near $\widehat{p}_{\lambda}$ are defined in the ball $\mathbb{B}^{n}\left(\widehat{p}_{\lambda}, \delta_{1}\right)$ and that the map

$$
z \longmapsto(v(z), x(z), y(z)), \quad \mathbb{B}^{n}\left(\widehat{p}_{\lambda}, \delta_{1}\right) \longrightarrow \mathbb{R}^{2 n}
$$

is close in $\mathscr{C}^{1}$ norm to its differential at $\widehat{p}_{\lambda}$, so that it is almost not distorting. Then $\delta_{1}$ shall not be shrunk anymore.

Because in the estimates of the (finite) number of Levi-Hartogs figures, $\eta$ only appears as a denominator in a factor $\frac{r^{\prime}-r^{\prime \prime}}{\eta}(c f$. Proposition 4.5), it is allowed to work with extensional rinds of smaller universal positive thickness, at the cost of spending a number of pushed analytic discs that is greater, of course, but still finite. If necessary, we shrink $\eta>0$ to insure that $\eta^{1 / 2} \ll \delta$. Then $\eta$ will not be shrunk anymore.

Thanks to these preliminaries, we may define a convenient (pseudo)cube centered at $\widehat{p}_{\lambda}$ by

$$
\begin{equation*}
\mathrm{C}_{\eta}:=\left\{z \in \mathbb{B}^{n}\left(\widehat{p}_{\lambda}, \delta_{1}\right):|v(z)|<\eta,\|x(z)\|<2 \eta^{1 / 2},\|y(z)\|<2 \eta^{1 / 2}\right\} \tag{5.12}
\end{equation*}
$$

It then follows that $\mathrm{C}_{\eta}$ is properly contained in $\mathscr{V}_{\delta}(M)$ and is relatively small. Reminding that $v(z)=r(z)-r\left(\widehat{p}_{\lambda}\right)$, the radial thickness of $\mathrm{C}_{\eta}$ is equal to $2 \eta$, twice the difference $\widehat{r}_{\lambda}^{+}-\widehat{r}_{\lambda}=\eta$. We draw a diagram assuming $k_{\lambda}=2 n-1$ (see only the left one).


Fig. 14: The radial (pseudo)cube $C_{\eta}$ centered at $\widehat{p}_{\lambda}$
5.13. Topology of horizontal super-level sets in the complement of quadrics. Simultaneously to the proof, we provide an auxiliary elementary study. Let $n \in \mathbb{N}$ with $n \geqslant 2$, let $k \in \mathbb{N}$ with $0 \leqslant k \leqslant 2 n-1$, let $x=\left(x_{1}, \ldots, x_{k}\right) \in \mathbb{R}^{k}$, let $y=\left(y_{1}, \ldots, y_{2 n-1-k}\right) \in \mathbb{R}^{2 n-1-k}$, let $v \in \mathbb{R}$, and in $\mathbb{R}^{2 n}$ equipped with the coordinates $(x, y, v)$, consider the quadric of
equation

$$
\begin{equation*}
v=\sum_{1 \leqslant j \leqslant k} x_{j}^{2}-\sum_{1 \leqslant j \leqslant 2 n-1-k} y_{j}^{2}, \tag{5.14}
\end{equation*}
$$

which we will denote by $\mathrm{Q}_{k}$. The coordinate $v$ playing the rôle of $r(z)-$ $r\left(\widehat{p}_{\lambda}\right)$ near a singular radius $\widehat{r}_{\lambda}$ having Morse coindex $k_{\lambda}$, we want to understand how the topology of the super-level sets

$$
\{v>\varepsilon\} \cap\left(\mathbb{R}^{2 n} \backslash Q_{k}\right)
$$

(which relate to the possible domains $\widetilde{\Omega}_{>r}^{c}$ for $r$ close to $\widehat{r}_{\lambda}$ ) do change when the parameter $\varepsilon$ descends from a small positive value to a small negative value.


Fig. 15: Growing of superlevel domains near a local maximum or minimum
In the case $k=0$ (left figure) the quadric looks like a spherical cap, its complement $\mathbb{R}^{2 n} \backslash Q_{0}$ having exactly two connected components. For positive values of $\varepsilon$, there is only one (green) super-level component $\{v>$ $\varepsilon\} \cap\left(\mathbb{R}^{2 n} \backslash Q_{0}\right)$. As $\varepsilon$ becomes negative, this component grows regularly, allowing a newly created hole to widen inside the slices $\{v=\varepsilon\}$. The (blue) holes then pile up to constitute a newly created, local component $M_{\widehat{r}_{\lambda}^{-}}^{c}$.

The (reverse) case $k=2 n-1$ exhibits the local end of some component $M_{>\widehat{r}_{\lambda}}^{c}$. In a while, we will see that there is a salient topological difference between the two remaining (less obvious) cases $2 \leqslant k \leqslant 2 n-2$ and $k=1$, the exceptional one. Before pursuing, we conclude the proof of B in case $\widehat{p}_{\lambda}$ is a local maximum or minimum.

We assume $k_{\lambda}=2 n-1$, the case $k_{\lambda}=0$ being already considered (essentially completely) in Section 4. Observe that $M_{>\widehat{r}_{\lambda}^{+}} \cap \mathrm{C}_{\eta}$ is diffeomorphic to $\mathrm{S}^{2 n-2} \times(c / 2, c)$, hence connected. Thus, let $M_{>\widetilde{r}_{\lambda}^{+}}^{c}$ denote the single component entering $C_{\eta}$. By descending induction through A and $\mathrm{B}, M_{>\widetilde{r}_{\lambda}^{+}}^{c}$ bounds a relatively compact domain of holomorphic extension $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c}$, with $\partial \widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c}=M_{>\widetilde{r}_{\lambda}^{+}}^{c} \cup N_{\widetilde{r}_{\lambda}^{+}}^{c} \cup \widetilde{R}_{\widetilde{r}_{\lambda}^{+}}^{c}$, as in property (ii) of Proposition 5.3, all the other properties also holding true on $(\cdot)_{>r_{\lambda}^{+}}$. Denote by $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c, k}, 1 \leqslant k \leqslant k_{\lambda, c}$, the connected components of $\widetilde{R}_{\widetilde{r}_{\lambda}^{+}}^{c}$ and by $\mathbb{N}_{\widetilde{r}_{\lambda}^{+}}^{c, j}, 1 \leqslant j \leqslant j_{\lambda, c}$, with $j_{\lambda, c} \geqslant k_{\lambda, c}$, the components of $\mathrm{N}_{\widetilde{r}_{\lambda}}^{c}$.


Fig. 16: Two distinct Hartogs-Levi fillings at a point of Morse coindex $2 n-1$ :
We do the numbering so that $C_{\eta}$ encloses the first (small) $N_{\widetilde{r}_{\lambda}^{+}}^{c, 1}$, which is diffeomorphic to a small $(2 n-2)$-dimensional sphere. Also, we number so that the boundary of $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c, 1}$ in ${\widehat{\widehat{r}_{\lambda}^{+}}}_{2 n-1}^{c}$ contains $\mathrm{N}_{\widetilde{r}_{\lambda}^{+}}^{c, 1}$, whence $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c, 1}$ meets $\mathrm{C}_{\eta}$. We do not draw $C_{\eta}$.

Observe that, by means of extensional rinds that are symmetric around the other components $\widetilde{R}_{\widetilde{r}_{\lambda}^{+}}^{c, 2}, \ldots, \widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c, k_{\lambda, c}}$, we may achieve the Hartogs-Levi filling exactly as in A , because $\left.r(z)\right|_{M}$ is regular in $\mathscr{V}_{\delta}\left(\mathrm{N}_{\widetilde{r}_{\lambda}}^{c, j}\right)$, for every $j$ such that $\mathrm{N}_{\widetilde{r}_{\lambda}^{+}}^{c, j}$ is contained in the boundary of each of these other components. Hence it remains only to discuss what is happening in a neighborhood of the single component $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c, 1}$, and especially near $\widehat{p}_{\lambda}$.

For the disposition of $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c} \cap \mathrm{C}_{\eta}$, or equivalently of $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c, 1} \cap \mathrm{C}_{\eta}$, two cases occur. Let $\left(v, x_{1}, \ldots, x_{2 n-1}\right)$ be the Morse coordinates centered at $\widehat{p}_{\lambda}$.
(a) As illustrated by the left figure above, $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c} \cap C_{\eta}$ consists of the space ${ }^{45}$ lying above $\{v=\eta / 2\}$ and above $\left\{v=x_{1}^{2}+\cdots+x_{2 n-1}^{2}\right\}$, a cap-shaped space which is clearly connected; the region $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c, 1}$ is then diffeomorphic to a small $(2 n-1)$-dimensional ball.
(b) As illustrated by the right figure above, $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c} \cap C_{\eta}$ consists of the space lying above $\{v=\eta / 2\}$ but below $\left\{v=x_{1}^{2}+\cdots+x_{2 n-1}^{2}\right\}$;
 a fact that a one-dimensional diagram cannot show adequately; then $\widetilde{\Omega}_{>r_{\lambda}^{+}}^{c} \cap \mathrm{C}_{\eta}$ is also connected.
In case (a), near $\widehat{p}_{\lambda}$, a piece of $\widetilde{\Omega}_{>r_{\lambda}^{+}}^{c}$ ends up while descending to the lower super-level set $(\cdot)_{>\widehat{r}_{\lambda}}$. We do not use any extensional rind there, we

[^44]just observe that unique holomorphic extension is got for free in
$$
\left[\mathscr{V}_{\delta}\left(M_{>\widehat{r}_{\lambda}^{-}}^{c}\right)_{>\widehat{r}_{\lambda}^{-}}\right] \cap \mathrm{C}_{\eta},
$$
since this domain is fully contained in $\mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}$.
In case (b), we apply Hartogs Levi extension to Rind $\left(\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{c}}^{c, 1}, \eta\right)$ and we get unique holomorphic extension from (5.10) to
$$
\left[\mathscr{V}_{\delta}\left(M_{>\widetilde{r}_{\lambda}^{-}}^{c}\right)_{>\widetilde{r}_{\lambda}^{-}}\right] \bigcup \operatorname{Rind}\left(\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c, 1}, \eta\right) .
$$

The union of this open set together with (5.10) contains a unique well defined domain $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{-}}^{c}$ with the property that the passage from $\widetilde{\mathrm{R}}_{>\widetilde{r}_{\lambda}^{+}}^{c, 1}$ to $\widetilde{\mathrm{R}}_{>\widehat{r}_{\lambda}^{-}}^{c, 1}$ fills a hole, as illustrated by the right diagram above, whence $\mathrm{N}_{>\widehat{r}_{\lambda}^{-}}^{c}$ has one less connected component, because the $(2 n-2)$-sphere $\mathrm{N}_{>\widehat{r}_{\lambda}+\varepsilon}^{c, 1}$ drops when $\varepsilon<0$.

The properties that two different domains $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{1}}$ and $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{2}}$ are either disjoint or one is contained in the other is easily seen to be inherited by $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{-}}^{c_{1}}$ and $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{-}}^{c_{2}}$ : it suffices to distinguish two cases: $c_{2} \neq c$ and $c_{1} \neq c$, or $c_{2} \neq c$ and $c_{1}=c$; to look at ( $\mathbf{a}$ ) or (b) and then to conclude.

The proof of B in case $k_{\lambda}=2 n-1$ is complete. The case $k_{\lambda}=0$ is similar: two subcases (a') - reverse (a) - and (b') - reverse (b) then appear; subcase ( $a^{\prime}$ ) exhibits the birth of a new component (blue left Figure 15), as already fully studied in Section 4 while subcase (b') (green left Figure 15) shows that an external component descends regularly as do clouds around a hill.
5.15. The regular cases $2 \leqslant k_{\lambda} \leqslant 2 n-2$. Let $k$ with $2 \leqslant k \leqslant 2 n-2$ and consider the quadric $\mathrm{Q}_{k}$ of (5.14). We claim that $\mathrm{Q}_{k} \cap\{v>\varepsilon\}$ has exactly one connected component for every $\varepsilon>0$. Indeed, $\mathbf{Q}_{k} \cap\{v>\varepsilon\}$ can be represented as

$$
\bigcup_{y_{1}, \ldots, y_{2 n-k-1}} \bigcup_{\varepsilon^{\prime}>\varepsilon}\left\{x_{1}^{2}+\cdots+x_{k}^{2}=\varepsilon^{\prime}+y_{1}^{2}+\cdots+y_{2 n-1-k}^{2}\right\} .
$$

Since $\varepsilon^{\prime}$ is always positive, we hence have a smoothly parameterized family of $(k-1)$-dimensional spheres that are all connected. Consequently, the union is also connected, as claimed.

To view the topology more adequately, in the case $n=2$, we draw a short movie consisting of the 3-dimensional slices $\left\{v=\varepsilon^{\prime}\right\} \cap\left(\mathbb{R}^{2 n} \backslash \mathrm{Q}_{k}\right)$, where $\varepsilon^{\prime}=\frac{2}{3} \eta, \frac{1}{2} \eta, 0,-\frac{1}{2} \eta$. To conceptualize (in case $n=2$ ) the super-level
sets

$$
\{v>\varepsilon\} \cap\left(\mathbb{R}^{2 n} \backslash \mathrm{Q}_{k}\right)=\bigcup_{\varepsilon^{\prime}>\varepsilon}\left\{v=\varepsilon^{\prime}\right\} \cap\left(\mathbb{R}^{2 n} \backslash \mathrm{Q}_{k}\right),
$$

it suffices to pile up intuitively the images of the corresponding movie.


Fig. 17: Sliced view of the growing of the two possible domains in case $2 \leqslant k_{\lambda} \leqslant 2 n-2$ :

So let $M_{>r_{\lambda}^{+}}^{c}$ be the single connected component of $M \cap\left\{\|z\|>\widehat{r}_{\lambda}^{+}\right\}$ that enters $C_{\eta}$. The corresponding domain $\widetilde{\Omega}_{>r_{\lambda}^{+}}^{c}$ can be located from one or the other side. Its prolongation up to the deeper sublevel set $(\cdot)_{>\widehat{r}_{\lambda}^{-}}$(viewed only inside $C_{\eta}$ ) consists of piling up the (blue) small symmetric regions or the (green) surrounding regions drawn above.

We do the numbering so that $\mathrm{N}_{\widetilde{r}_{\lambda}^{+}}^{c, 1}$ enters $\mathrm{C}_{\eta}$, being a (connected) hyperboloid as drawn in the first picture of Figure 17 and so that $\widetilde{R}_{\widetilde{r}_{\lambda}^{+}}^{c, 1}$ enters $\mathrm{C}_{\eta}$ as one (connected, blue or green) side of this hyperboloid. As previously in the two cases $k_{\lambda}=0$ and $k_{\lambda}=2 n-1$, the Hartogs-Levi filling goes through exactly as in the regular case $\mathbf{A}$ for all other $\widetilde{R}_{r_{+}^{+}}^{c, 2}, \ldots, \widetilde{R}_{\widetilde{r}_{\lambda}^{+}}^{c, k_{\lambda, c}}$. Next, by putting finitely many Levi-Hartogs figures in $\operatorname{Rind}\left(\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c, 1}, \eta\right)$ we get holomorphic extension from the domain (5.10) to

$$
\left[\mathscr{V}_{\delta}\left(M_{>\widetilde{r}_{\lambda}}^{c}\right)_{>\widetilde{r}_{\lambda}^{-}}\right] \bigcup \operatorname{Rind}\left(\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c, 1}, \eta\right) .
$$

The intersection of (5.10) with this open set is connected because $\widetilde{R}_{\widetilde{r}_{\lambda}}^{c, 1}$ is connected, and the union of both contains a well defined domain $\widetilde{\Omega}_{>\mathfrak{r}_{\lambda}}^{c}$ obtained by adding the (blue or green) slices of Figure 17.

## §6. The EXCEPTIONAL CASE $k_{\lambda}=1$

6.1. Illustration. To begin with the most delicate case, we draw a 3dimensional diagram showing a saddle-like $M$ localized in a (pseudo)cube $C_{\eta}$ centered at $\widehat{p}_{\lambda}$.


For every $\varepsilon$ satisfying $0<\varepsilon<\eta$, there are two connected components $M_{>\varepsilon}^{-}$and $M_{>\varepsilon}^{+}$of $M_{>\widehat{r}_{\lambda}+\varepsilon} \cap \mathrm{C}_{\eta}$, namely the two upper tips of the saddle, defined in equations by

$$
M_{>\varepsilon}^{ \pm}:=\left\{v=x^{2}-y_{1}^{2}-\cdots-y_{2 n-2}^{2}\right\} \cap\{ \pm x>0\} \cap\{v>\varepsilon\} .
$$

With $\varepsilon=\frac{1}{2} \eta$, we are simply looking at $M_{>\widehat{r}_{\lambda}^{+}} \cap \mathrm{C}_{\eta}$. By descending induction through $A$ and $B$, we are given two domains of holomorphic extension $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{1}}$ and $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{2}}$ whose boundary contains $M_{>\eta / 2}^{-}$and $M_{>\eta / 2}^{+}$, respectively.

Firstly, we assume that $c_{2} \neq c_{1}$. Since each one of the two pieces of hypersurfaces $M_{>\eta / 2}^{-}$and $M_{>\eta / 2}^{+}$has two sides, there are $2 \times 2=4$ subcases to be considered for the relative disposition of $\Omega_{>\eta / 2}^{-}:=\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{1}} \cap C_{\eta}$ and of $\Omega_{>\eta / 2}^{+}:=\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{2}} \cap C_{\eta}$, with $c_{2} \neq c_{1}$.
(a) $\Omega_{>\eta / 2}^{-}$(resp. $\Omega_{>\eta / 2}^{+}$) consists of the space lying above the hyperplane $\{v=\eta / 2\}$ and below the left (resp. right) tip of the saddle, namely in equations:

$$
\Omega_{>\eta / 2}^{ \pm}=\{v>\eta / 2\} \bigcap\{ \pm x>0\} \bigcap\left\{v<x^{2}-y_{1}^{2}-\cdots-y_{2 n-2}^{2}\right\} .
$$

(b) $\Omega_{>\eta / 2}^{-}$is the small nose as in (a) but $\Omega_{>\eta / 2}^{+}$consists of the other side, i.e. of the (rather bigger) space lying inside $\{v>\eta / 2\}$ left to $M_{>\eta / 2}^{+}$, namely in equations:

$$
\Omega_{>\eta / 2}^{+}=\{v>\eta / 2\} \backslash\left(\{x>0\} \bigcap\left\{v \leqslant x^{2}-y_{1}^{2}-\cdots-y_{2 n-2}^{2}\right\}\right) .
$$

(c) Symetrically to (b), $\Omega_{>\eta / 2}^{+}$is the small nose as in (a) but

$$
\Omega_{>\eta / 2}^{-}=\{v>\eta / 2\} \backslash\left(\{x<0\} \bigcap\left\{v \leqslant x^{2}-y_{1}^{2}-\cdots-y_{2 n-2}^{2}\right\}\right) .
$$

(d) Finally, $\Omega_{>\eta / 2}^{-}$is as in (c) and $\Omega_{>\eta / 2}^{+}$is as in (b).

The last subcase (d) cannot occur, because it is ruled out by property (iii) of Proposition 5.3, which holds on the super-level set $(\cdot)_{>\widehat{r}_{\lambda}^{+}}$by the inductive assumption.

Secondly, we assume that $c_{2}=c_{1}$. Then there can occur a subcase (a') very similar to (a), in which $c_{2}=c_{1}$, so that $\Omega_{>\eta / 2}^{-}$and $\Omega_{>\eta / 2}^{+}$belong to the same enclosed relatively compact domain. But with $c_{2}=c_{1}$, no subcase similar to (b) - or to (c) - can occur, because $M_{>\eta / 2}^{-} \subset \partial \Omega_{>\eta / 2}^{+}$- or $M_{\eta / 2}^{+} \subset \partial \Omega_{>\eta / 2}^{-}$- would then bound the same relatively compact domain from its both sides, but we already know from the beginning of the proof, that one side at least must always contain the points at infinity.

Finally, with $c_{2}=c_{1}=c$, there remains the following last subcase (unseen previously).
(e) $\Omega_{>\eta / 2}:=\widetilde{\Omega}_{>r_{\lambda}^{+}}^{c} \cap C_{\eta}$ consists of the space lying above $\{v=\eta / 2\}$ and above the saddle, namely

$$
\Omega_{>\eta / 2}=\{v>\eta / 2\} \bigcap\left\{v>x^{2}-y_{1}^{2}-\cdots-y_{2 n-2}^{2}\right\} .
$$

As $M=\partial \Omega_{M}$ lies in $\mathbb{C}^{n}$ with $n \geqslant 2$, whence $2 n-2 \geqslant 2$, there is at least one dimension of $y \in \mathbb{R}^{2 n-2}$ which is missing in the left figure above. To view the topology more adequately, coming back to the abstract quadric $\mathrm{Q}_{1}$ and assuming $n=2$, we plan to draw a short movie consisting of the 3-dimensional slices $\left\{v=\varepsilon^{\prime}\right\} \cap\left(\mathbb{R}^{2 n} \backslash \mathrm{Q}_{1}\right)$, where $\varepsilon^{\prime}=\frac{2}{3} \eta, \frac{1}{2} \eta, 0,-\frac{1}{2} \eta$.

Recall that we are interested in the connected components of the superlevel sets

$$
\{v>\varepsilon\} \cap\left(\mathbb{R}^{2 n} \backslash \mathrm{Q}_{1}\right)=\bigcup_{\varepsilon^{\prime}>\varepsilon}\left\{v=\varepsilon^{\prime}\right\} \cap\left(\mathbb{R}^{2 n} \backslash \mathrm{Q}_{1}\right) .
$$

As suggested by this sliced union, to conceptualize these 4-dimensional (in case $n=2$ ) super-level sets, it suffices to pile up intuitively the images of the corresponding movie.


Fig. 19: Sliced view of the merging of the two domains in subcase (a) of $k_{\lambda}=1$
Here, the second picture shows $\widetilde{R}_{\widetilde{r}_{\lambda}^{+}}^{c_{1}} \cap C_{\eta}$ (in blue, to the left) together with $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{1}} \cap \mathrm{C}_{\eta}$ (in black, to the right). Then the third picture shows how the two components do touch and the fourth one shows how they should be merged as $\varepsilon^{\prime}=-\frac{1}{2} \eta$ becomes negative. The complete discussion follows in a while.

We next offer the movie of (b), the movie of (c) being obtained from it just by a reflection across the hyperplane $\{x=0\}$.


Fig. 20: Sliced view of the substraction of the left domain in subcase (b) of $k_{\lambda}=1$
Here again, the second picture shows $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{1}} \cap \mathrm{C}_{\eta}$ (in blue, to the left) together with $\widetilde{R}_{\widetilde{r}_{\lambda}^{+}}^{c_{2}} \cap \mathrm{C}_{\eta}$ (the large (black) region, containing the small (blue) one). Then the third picture, namely the slice $\varepsilon^{\prime}=0$, shows a not allowed situation: the left cone does bound two regions from its two sides, contrary to the a priori unique relatively compact domain $\widetilde{\Omega}_{\widehat{r}_{\lambda}}^{c_{1}} \subset\left\{\|z\|>\widehat{r}_{\lambda}\right\}$ we are seeking to construct, when starting from $\widetilde{\Omega}_{\widetilde{r}_{\lambda}^{+}}^{c_{1}}$. The trick is then to suppress the (blue) small slice, or equivalently to subtract it from the (black) large slice which contains it. Then the black winning slice continues to grow up to $\{v=-\eta / 2\}$ (fourth picture). The complete discussion follows in a while.

Finally, here is the (simpler) movie of (e).


Fig. 21: Sliced view of the growing of the external domain in subcase (e) of $k_{\lambda}=1$
6.2. Jumping across the singular radius: merging process. Assuming $k_{\lambda}=1$, we can now complete B in subcase (a), postponing subcase (a'). We look at Figures 17 and 18.

Let $M_{>r_{\lambda}^{+}}^{c_{1}} \cap \mathrm{C}_{\eta}$ and $M_{>r_{\lambda}^{+}}^{c_{2}} \cap \mathrm{C}_{\eta}$ be the two "nose" components of $M_{>\widehat{r}_{\lambda}^{+}}$ entering $\mathrm{C}_{\eta}$. Here, $c_{2} \neq c_{1}$. By descending induction through A and B , $M_{>\tilde{r}_{\lambda}^{+}}^{c_{1}}$ and $M_{>\tilde{r}_{\lambda}^{+}}^{c_{2}}$ bound some two relatively compact domains of holomorphic extension $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{1}}$ and $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{2}}$ with $\partial \widetilde{\Omega}_{>\widetilde{\Omega}_{\lambda}^{+}}^{c_{1}}=M_{>\widetilde{r}_{\lambda}^{+}}^{c_{1}} \cup N_{\widetilde{r}_{\lambda}^{+}}^{c_{1}} \cup \widetilde{R}_{\widetilde{r}_{\lambda}^{+}}^{c_{1}}$ and $\partial \widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{2}}=M_{>\widetilde{r}_{\lambda}^{+}}^{c_{2}} \cup{\stackrel{N}{r_{\lambda}^{+}}}_{c_{2}}^{c_{2}} \cup \widetilde{R}_{\widetilde{r}_{\lambda}^{+}}^{c_{2}}$ as in property (ii) of Proposition 5.3, all the other properties also holding true on $(\cdot)_{>\widehat{r}_{\lambda}^{+}}$.

We remind that the other domains $\widetilde{\Omega}_{>\widehat{r}_{\lambda}^{+}}^{c}$ for $c \neq c_{1}$ and $c \neq c_{2}$ with $1 \leqslant c \leqslant c_{\lambda}$ do pass regularly through $\widehat{r}_{\lambda}$ up to $(\cdot)_{>\widehat{r}_{\lambda}^{-}}$, thanks to A .

For $i=1,2$, denote by $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}}^{c_{i}, k}, 1 \leqslant k \leqslant k_{\lambda, c_{i}}$, the connected components of $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{i}}$ and by $\mathrm{N}_{\widetilde{r}_{\lambda}^{+}}^{c_{i}, j}, 1 \leqslant j \leqslant j_{\lambda, c_{i}}$, with $j_{\lambda, c_{i}} \geqslant k_{\lambda, c_{i}}$, the components of $\mathrm{N}_{\widetilde{r}_{\lambda}^{+}}^{c_{i}}$. We do the numbering so that $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{1}, 1}$ (resp. $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{2}, 1}$ ) enters $\mathrm{C}_{\eta}$ to the left (resp. right), together with $N_{\widetilde{r}_{\lambda}^{+}}^{c_{1}, 1}$ (resp. $\mathrm{N}_{\widetilde{r}_{\lambda}^{+}}^{c_{2}, 1}$ ), as illustrated by Figure 17 .

As in the case $k_{\lambda}=2 n-1$, for $i=1,2$, by means of extensional rinds that are symmetric around the other components $\widetilde{R}_{\widetilde{r}_{\lambda}}^{c_{i}, 2}, \ldots, \widetilde{R}_{\widetilde{r}_{\lambda}}^{c_{i}, k_{\lambda, c_{i}}}$, we may achieve the Hartogs-Levi filling exactly as in A , because $\left.r(z)\right|_{M}$ is regular in $\mathscr{V}_{\delta}\left(\mathrm{N}_{\widetilde{r}_{\lambda}^{+}}^{c_{i, j}}\right)$, for every $j$ such that $\mathrm{N}_{\widetilde{r}_{\lambda}^{+}}^{c_{i, j}}$ is contained in the boundary of each of these other components. Hence it remains only to discuss what is happening in a neighborhood of the two components $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{i}, 1}, i=1,2$, and especially near the saddle point $\widehat{p}_{\lambda}$.

While descending from $\widehat{r}_{\lambda}^{+}$to $\widehat{r_{\lambda}}$, the two regions $\widetilde{\mathrm{R}}_{\widehat{r}_{\lambda}^{+}}^{c_{1}, 1} \subset \mathrm{~S}_{\widehat{r}_{\lambda}^{+}}^{2 n-1}$ and $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{2}, 1} \subset \mathrm{~S}_{\widetilde{r}_{\lambda}^{+}}^{2 n-1}$ do merge as a single connected region contained in $\mathrm{S}_{\widetilde{r}_{\lambda}^{-}}^{2 n-1}$ that we will denote by $\widetilde{\mathrm{R}}_{\widetilde{r}_{-}^{-}}^{*}$, see the right Figure 17. In Morse theory ( $[31,17]$ ), one speaks of attaching a one-cell, since in the merging process, the two regions are essentially joined by means of a (thickened) segment directed along the $x$-axis. It follows that the two hypersurfaces $M_{>\widehat{r}_{\lambda}^{+}}^{c_{1}}$ and $M_{>\widehat{r}_{\lambda}^{+}}^{c_{2}}$ do merge as a connected hypersurface $M_{>\overparen{r}_{\lambda}^{-}}^{*}$ containing them, and furthermore, that the two domains $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{1}}$ and $\widetilde{\Omega}_{>r_{\lambda}^{+}}^{c_{2}}$ do prolong uniquely up to the slightly deeper super-level set $(\cdot)_{>\widehat{r}_{\lambda}^{-}}$, merging as a uniquely defined domain $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{-}}^{*}$ which is relatively compact in $\mathbb{C}^{n}$ and which contains $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{*}$ in its boundary $\partial \widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{*}}^{*}$.

As $c_{2} \neq c_{1}$, the new number of domains in the interval $\left(\widehat{r}_{\lambda-1}, \widehat{r}_{\lambda}\right)$ is lowered by a unit, i.e. $c_{\lambda-1}=c_{\lambda}-1$ (if $c_{2}=c_{1}$ as in (a'), the number would not change, i.e. $c_{\lambda-1}=c_{\lambda}$ ).

For $i=1,2$, let $f_{\widetilde{r}_{\lambda}^{+}}^{c_{i}}$ denote the restriction of $f \in \mathscr{O}\left(\mathscr{V}_{\delta}\left(M_{>r}\right)_{>r}\right)$ to $\mathscr{V}_{\delta}\left(M_{>\widetilde{r}_{\lambda}^{+}}^{c_{i}}\right)_{>\widetilde{r}_{\lambda}^{+}}$. By descending induction through A and $\mathrm{B}, f_{\widehat{r}_{\lambda}^{+}}^{c_{i}}$ extends holomorphically and uniquely to $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{i}}$. Then both functions do extend holomorphically and uniquely to

$$
\mathscr{V}_{\delta}\left(M_{>\overparen{r}_{\lambda}^{-}}^{*}\right)_{>\overparen{r}_{\lambda}^{-}},
$$

since they coincide with $f$ near $\widehat{p}_{\lambda}$. We then introduce the two extensional rinds Rind $\left(\widetilde{R}_{r_{\lambda}^{+}}^{c_{i}}, \eta\right)$, drawn in the right Figure 17. Two applications of Proposition 3.7 together with a geometrically clear connectedness property yield
unique holomorphic extension to

$$
\operatorname{Rind}\left(\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{1}} \eta\right) \bigcup \operatorname{Rind}\left(\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{2}} \eta\right) \bigcup \mathscr{V}_{\delta}\left(M_{>\widetilde{r}_{\lambda}^{-}}^{*}\right)_{\widehat{r}_{\lambda}^{-}} \bigcup \widetilde{\Omega}_{>\widetilde{\Omega}_{\lambda}^{+}}^{c_{1}} \bigcup \widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{2}} .
$$

In sum, we have got unique holomorphic extension to

$$
\mathscr{V}_{\delta}\left(M_{>\widehat{r}_{\lambda}^{-}}^{*}\right)_{\widehat{r}_{\lambda}^{-}} \bigcup \widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{-}}^{*} .
$$

To establish (iv) of Proposition 5.3 at $(\cdot)_{>\widehat{r_{\lambda}}}$, it suffices to show (iii), which is checked to be equivalent. We observe that, for logical reasons only, a given region $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c}$ for $c \neq c_{1}$ and $c \neq c_{2}$ can:

- be disjoint from $\widetilde{R}_{r_{\lambda}^{+}}^{c_{1}}$ and also disjoint from $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{2}}$;
- be contained in $\widetilde{R}_{\widetilde{r}_{\lambda}^{+}}^{c_{1}}$ or (exclusive "or") in $\widetilde{R}_{\widetilde{r}_{\lambda}^{+}}^{c_{2}}$;
- contain $\widetilde{R}_{\widetilde{r}_{\lambda}^{+}}^{c_{1}}$ or (inclusive "or") $\widetilde{R}_{\widetilde{r}_{\lambda}^{+}}^{c_{2}}$.

But we claim that in the latter case, $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c}$ necessarily contains both regions $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{1}}$ and $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{2}}$. Indeed, otherwise the boundary $\mathrm{N}_{\widetilde{r}_{\lambda}^{+}}^{c}$ of $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c}$ should separate $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{1}} \cap \mathrm{C}_{\eta}$ from $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{2}} \cap \mathrm{C}_{\eta}$ in the level set $\left\{v=\frac{\eta}{2}\right\} \cap \mathrm{C}_{\eta}$, which is impossible since $\mathrm{N}_{\widetilde{r}_{\lambda}^{+}}^{c} \cap \mathrm{C}_{\eta}$ is exactly equal to $\left(\mathrm{N}_{\widehat{r}_{\lambda}^{+}}^{c_{1}, 1} \cap \mathrm{C}_{\eta}\right) \cup\left(\mathrm{N}_{\widetilde{r}_{\lambda}^{+}}^{c_{2}, 1} \cap \mathrm{C}_{\eta}\right)$, not more.

It follows in all cases that $\mathrm{N}_{\widetilde{r}_{\lambda}^{+}}^{c}=\partial \widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c}$ is disjoint from $\mathrm{C}_{\eta}$, hence it lies in $\left\{\widehat{r_{\lambda}} \leqslant\|z\| \leqslant \widehat{r}_{\lambda}^{+}\right\} \backslash C_{\eta}$. Consequently, the regular flow of $\frac{\nabla r_{M}}{\left\|\nabla r_{M}\right\|}$ on

$$
\left[M \cap\left\{\widehat{r}_{\lambda} \leqslant\|z\| \leqslant \widehat{r}_{\lambda}^{+}\right\}\right] \backslash C_{\eta}
$$

pushes down regularly $\mathrm{N}_{\widetilde{r}_{\lambda}^{+}}^{c}$, as a uniquely defined compact 2-codimensional $\mathrm{N}_{\widehat{r}_{\lambda}}^{c} \subset \mathrm{~S}_{\widehat{r}_{\lambda}^{+}}^{2 n-1}$, disjointly from the newly created merged boundary $\mathrm{N}_{\widehat{r}_{\lambda}}^{*}=$ $\partial \widetilde{\Omega}_{>\widehat{r}_{\lambda}^{-}}^{*} \subset \mathrm{~S}_{\widetilde{r}_{\lambda}^{-}}^{2 n-1}$. This information suffices now to check that (iii) and (iv) of Proposition 5.3 are transmitted to $(\cdot)_{>\widehat{r}_{\lambda}}$, just for logical reasons.

The proof of B in case $k_{\lambda}=1$, subcase (a) is complete. Subcase (a') involves only minor differences.
6.3. Subtracting process. We now summarize the discussion of subcase (b), focusing only on topological aspects and dropping the formal considerations about holomorphic extensions. For an adequate three-dimensional illustration, think of a smoothly cut cylindrical piece of modelling clay in which a thin finger drills a hole.


Fig. 22: Three-dimensional view of subcase (b) at a point of Morse coindex $k_{\lambda}=1$

As in $\S 5.6$, in $C_{\eta}$, there enter exactly two domains $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{i}}, i=1,2$, with $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{1}} \subset \widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{2}}$ by the induction assumption. Also, there enter two connected regions $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{1}, 1} \subset \mathrm{~S}_{\widetilde{r}_{\lambda}^{+}}^{2 n-1}, i=1,2$, with $\widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{1}, 1} \subset \widetilde{\mathrm{R}}_{\widetilde{r}_{\lambda}^{+}}^{c_{2}, 1}$. Their boundaries contain two connected hypersurfaces $\mathrm{N}_{\widetilde{r}_{\lambda}}^{c_{i}, 1}$ of $\mathrm{S}_{\widetilde{r}_{\lambda}^{+}}^{2 n-1}, i=1,2$, which enter $\mathrm{C}_{\eta}$ as the two caps of the third pic of Figure 19.

By descending the interval $\left(\widehat{r}_{\lambda}, \widehat{r}_{\lambda}^{+}\right)$up to $(\cdot)_{>\widehat{r}_{\lambda}}$, we get two regions $\widetilde{\mathrm{R}}_{\mathrm{r}_{\lambda}}^{c_{,}, 1}$, $i=1,2$, that touch at $\widehat{p}_{\lambda}$, namely the left cone and the exterior of the right cone in the second pic of Figure 19.

While descending further to $(\cdot)_{>\widehat{r}_{\lambda}-\varepsilon}$, with $\varepsilon>0$ very small, the left cone does merge with the right (white) cone. Observe that the points of this (white) cone may be joined continuously to points of the (white) right cap of the first pic, which by hypothesis lies outside $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{2}}$, hence in the same connected component as the points at infinity. Consequently, we cannot prolong the left domain $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{+}}^{c_{1}}$ so that its prolongation contains the left cone in the slice $\{v=0\}$ (third pic), because no admissible prolongation would enjoy the relative compactness (i) of Proposition 5.3. Hence we have no other choice except to suppress $\widetilde{\Omega}_{>\widehat{r}_{\lambda}}^{c_{1}}$ when attaining $(\cdot)_{>\widehat{r}_{\lambda}}$. We then get a new domain $\widetilde{\Omega}_{>\widehat{r}_{\lambda}}^{*}$ defined as $\widetilde{\Omega}_{>\widehat{r}_{\lambda}}^{c_{2}}$ minus the closure of $\widetilde{\Omega}_{>\widehat{r}_{\lambda}}^{c_{1}}$ (subtraction process), which is checked to be relatively compact in $\mathbb{C}^{n}$. This domain then descends as a uniquely defined domain $\widetilde{\Omega}_{>\widetilde{r}_{\lambda}^{-}}^{*}$ at $(\cdot)_{>\widetilde{r}_{\lambda}}$. We also get a corresponding connected region $\widetilde{R}_{\widetilde{r}_{\lambda}}^{*}$ approximately equal to $\widetilde{R}_{\widetilde{r}_{\lambda}}^{c_{2}, 1}$ minus the closure of $\widetilde{R}_{\widetilde{r}_{\lambda}}^{c_{1}, 1}$ whose boundary contains a connectedd $N_{\widetilde{r}_{\lambda}^{-}}^{*}$ (bottom right Figure 21), obtained by merging $N_{\widetilde{r}_{\lambda}}^{c_{1}, 1}$ with $N_{\widetilde{r}_{\lambda}}^{c_{2}, 1}$.

The last subcase (e) above is topologically similar to what happens in $\S 5.15$, hence the proof of Proposition 5.3 is complete.

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## The Hartogs extension theorem

# on ( $n-1$ )-complete complex spaces 

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#### Abstract

Performing local extension from pseudoconcave boundaries along LeviHartogs figures and building a Morse-theoretical frame for the global control of monodromy, we establish a version of the Hartogs extension theorem which is valid in singular complex spaces (and currently not available by means of $\bar{\partial}$ techniques), namely: for every domain $\Omega$ of an ( $n-1$ )-complete normal complex space of pure dimension $n \geqslant 2$, and for every compact set $K \subset \Omega$ such that $\Omega \backslash K$ is connected, holomorphic or meromorphic functions in $\Omega \backslash K$ extend holomorphically or meromorphically to $\Omega$. Assuming that $X$ is reduced and globally irreducible, but not necessarily normal, and that the regular part $[\Omega \backslash K]_{\text {reg }}$ is connected, we also show that meromorphic functions on $\Omega \backslash K$ extend meromorphically to $\Omega$.



[ 5 colored illustrations]

## §1. Introduction

The goal of the present article is to establish a Hartogs extension theorem - known until now on complex manifolds - in normal singular complex spaces which enjoy appropriate convexity conditions. For local extensional steps, we apply geometric constructions that are closely related to the Andreotti-Grauert bump method developed since the 1980's by Henkin and Leiterer ([15], cf. also [25]). For the global topological control of monodromy - a new feature, compared to the classical $k$-concavity theory we push forward the Morse-theoretical frame introduced recently in [27].

In its original form, the theorem states that in an arbitrary bounded domain $\Omega \Subset \mathbb{C}^{n}(n \geqslant 2)$, every compact set $K \subset \Omega$ with $\Omega \backslash K$ connected is an illusory singularity for holomorphic functions, namely $\mathscr{O}(\Omega \backslash K)=\left.\mathscr{O}(\Omega)\right|_{\Omega \backslash K}$ (for history, motivations and background, we refer e.g. to [16, 29, 30]). By now, the shortest proof, due to Ehrenpreis, follows easily from the simple proposition that $\bar{\partial}$-cohomology with compact support vanishes in bidegre
$(0,1)$ (see [18]). Along these lines and after results due to Kohn-Rossi, the Hartogs theorem was generalized to ( $n-1$ )-complete complex manifolds by Andreotti-Hill [2], i.e. manifolds exhausted by a $\mathscr{C}^{\infty}$ function whose Levi-form has at least 2 positive eigenvalues at every point. We also refer to [23] for an approach via the holomorphic Plateau boundary problem.

To endeavor the theory in general singular complex spaces $\left(X, \mathscr{O}_{X}\right)$, it is at present advisable to look for methods avoiding global $\bar{\partial}$ techniques, as well as global integral kernels, because such tools are not yet available. The geometric Hartogs theory was attacked long ago by Rothstein, who introduced the notion of $q$-convexity. On the other hand, within the modern sheaf-theoretic setting, the so-called Andreotti-Grauert theory ( $[1,15,24]$ ) allows to perform extension (of holomorphic functions, of differentials forms, of coherent sheaves, etc.) from shell-like regions of the form $\{z \in X: a<\rho(z)<b\}$ into their inside $\{z \in X: \rho(z)<b\}$, where $\rho$ is a fixed $(n-1)$-convex exhaustion function for $X$. Geometrically speaking, one performs holomorphic extension by means of the Grauert bump method through the level sets of $\rho$ in the direction of decreasing values, jumping finitely many times across the critical points of $\rho$.

However, a satisfying, complete generalization of the Hartogs theorem should apply to general excised bounded domains $\Omega \backslash K$ lying in an $(n-1)$ complete complex space $\left(X, \mathscr{O}_{X}\right)$, not only to shells $\{a<\rho<b\}$ relative to the $(n-1)$-convex exhaustion function. But then, after perturbing and smoothing out $\partial \Omega$, one must unavoidably take account of the critical points of $\left.\rho\right|_{\partial \Omega}$ and also of the possible multi-sheetedness of the intermediate stepwise extensions. This causes considerably more delicate topological problems than in the well known Grauert bump method, in which monodromy of the holomorphic (or meromorphic, or sheaf-theoretic) extensions from $\{a<\rho<b\}$ to $\left\{a^{\prime}<\rho<b\right\}$ with $a^{\prime}<a$ is almost freely assured ${ }^{46}$, even across critical points of $\rho$. Considering simply a domain $\Omega \Subset \mathbb{C}^{n}(n \geqslant 2)$, with obvious exhaustion $\rho(z):=\|z\|$, the classical Hartogs theorem based on analytic discs and on Morse theory was worked out in [27], where emphasis was put on rigor in order to provide with firm grounds the subsequent works on the subject. The essence of the present article is to transfer such an approach to ( $n-1$ )-complete general complex spaces, where $\bar{\partial}$ techniques are still lacking, with some new difficulties due to the singularities. Another current, active direction of research studies extension phenomena related to the geometry of the target space, see e.g. [19].

[^45]
## §2. Statement of the results

Thus, let $\left(X, \mathscr{O}_{X}\right)$ be a reduced complex analytic space of pure dimension $n \geqslant 2$. We will use open covers $X=\bigcup_{j \in J} U_{j}$ together with holomorphic isomorphisms $\varphi_{j}: U_{j} \rightarrow \mathrm{~A}_{j}$ onto some closed complex analytic sets $\mathrm{A}_{j}$ contained in balls $\widetilde{\mathrm{B}}_{j} \subset \mathbb{C}^{N_{j}}$, some $N_{j} \geqslant 2$. By definition ( $[8,13]$ ), a $\mathscr{C}^{\infty}$ function $f: X \rightarrow \mathbb{C}$ is locally represented as $\left.f\right|_{U_{j}}=\widetilde{\mathfrak{f}}_{j} \circ \varphi_{j}$ for some collection of $\mathscr{C} \infty$ "ambient" functions $\widetilde{\mathrm{f}}_{j}: \widetilde{\mathrm{B}}_{j} \rightarrow \mathbb{C}, j \in J$. A realvalued continuous function $\rho$ on $X$ is an exhaustion function if sublevel sets $\{z \in X: \rho(z)<c\}$ are relatively compact in $X$ for every $c \in \mathbb{R}$. A $\mathscr{C}^{\infty}$ function $\rho: X \rightarrow \mathbb{R}$ is called strongly $q$-convex if the $\mathscr{C}^{\infty}$ ambient $\widetilde{\rho}_{j}: \widetilde{\mathrm{B}}_{j} \rightarrow \mathbb{R}$ can be chosen to be strongly $q$-convex, i.e. their Levi-forms $i \partial \bar{\partial}\left(\widetilde{\rho}_{j}\right)$ have at least $N_{j}-q+1$ positive eigenvalues at every point, for all $j \in J$. Finally ${ }^{47}, X$ is called $q$-complete if it possesses a $\mathscr{C}^{\infty}$ strongly $q$-convex exhaustion function. Note that the 1 -complete spaces are precisely the Stein spaces.

We will mainly work with a normal $(n-1)$-complete $X$, and we recall that a reduced complex space $\left(X, \mathscr{O}_{X}\right)$ is normal if the sheaf of weakly holomorphic functions, namely functions defined and holomorphic on the regular part $X_{\text {reg }}=X \backslash X_{\text {sing }}$ which are $L_{\text {loc }}^{\infty}$ on $X$, coincides with the complete sheaf $\mathscr{O}_{X}$ of holomorphic functions on $X$. Then $X_{\text {sing }}$ is of codimension $\geqslant 2$ at every point of $X([8,13])$ and for every open set $U \subset X$, both restriction maps

$$
\begin{equation*}
\mathscr{O}_{X}(U) \longrightarrow \mathscr{O}_{X}\left(U \backslash X_{\text {sing }}\right) \quad \text { and } \quad \mathscr{M}_{X}(U) \longrightarrow \mathscr{M}_{X}\left(U \backslash X_{\text {sing }}\right) \tag{2.1}
\end{equation*}
$$

are bijective ${ }^{48}$, where $\mathscr{M}_{X}$ denotes the meromorphic sheaf. To generalize Hartogs extension, normality of $X$ is an unavoidable assumption, because there are examples of Stein surfaces $S$ having a single singular point $\widehat{p}$ which are not normal ([13], vol. II, p. 196), whence $K:=\{\widehat{p}\}$ fails to be removable for holomorphic functions defined in a neighborhood of $K$.

We can now state our main result.
Theorem 2.2. Let $X$ be a connected ( $n-1$ )-complete normal complex space of pure dimension $n \geqslant 2$. Then for every domain $\Omega \subset X$ and every compact set $K \subset \Omega$ with $\Omega \backslash K$ connected, holomorphic or meromorphic functions on $\Omega \backslash K$ extend holomorphically or meromorphically and uniquely to $\Omega$ :

$$
\mathscr{O}_{X}(\Omega \backslash K)=\left.\mathscr{O}_{X}(\Omega)\right|_{\Omega \backslash K} \quad \text { or } \quad \mathscr{M}_{X}(\Omega \backslash K)=\left.\mathscr{M}_{X}(\Omega)\right|_{\Omega \backslash K} .
$$

[^46]Some comments on the hypotheses are in order. Firstly, connectedness of $X$ is not a restriction, since otherwise, $\Omega$ would be contained in a single component of $X$. Secondly, as $X$ is $(n-1)$-complete, $i \partial \bar{\partial}\left(\left.\rho\right|_{X_{\text {reg }}}\right)$ has at least 2 positive eigenvalues at every point $z \in X_{\mathrm{reg}}$, and consequently, each super-level set

$$
\{z \in X: \rho(z)>c\}
$$

has a pseudoconcave boundary at every smooth point $z \in X_{\text {reg }}$ with $d \rho(z) \neq 0$ and in fact, the Levi-form of this boundary has at least one negative eigenvalue at $z$. Thirdly, by a theorem of Ohsawa ([28]), every (connected) $n$-dimensional noncompact complex manifold is $n$-complete, and in fact, easy examples show that Hartogs extension may fail: take the product $X:=R \times S$ of two Riemann surfaces, with $R$ compact and $S$ noncompact, take a point $s \in S$ and set $K:=R \times\{s\}$; by [9], there exists a meromorphic function function having a pole of order 1 at $s$, whence $\mathscr{O}(X)$ does not extend through $K$. Consequently, in the category of strong Levi-form assumptions, $(n-1)$-convexity is sharp.

For the theorem, the main strategy of proof consists of performing holomorphic or meromorphic extension entirely within the regular part of $X$.

Proposition 2.3. With $X, \Omega$ and $K$ as in Theorem 2.2, holomorphic or meromorphic functions on $[\Omega \backslash K]_{\mathrm{reg}}$ extend holomorphically or meromorphically to $\Omega_{\mathrm{reg}}$.

Notice that both $[\Omega \backslash K]_{\mathrm{reg}}$ and $\Omega_{\mathrm{reg}}$ are connected (footnote 3). Then by (2.1), extension immediately holds to $\Omega$. This yields Theorem 2.2 if one takes the proposition for granted; Sections 3 and 4 below are devoted to prove this proposition.

For meromorphic extension, one could in principle well avoid the assumption of normality. In the case of meromorphic extension, we get a general result valid for reduced spaces without further local assumptions.

Theorem 2.4. Let $X$ be a globally irreducible $(n-1)$-complete reduced complex space of pure dimension $n \geqslant 2$. Then for every domain $\Omega \subset X$ and every compact set $K \subset \Omega$ with $[\Omega \backslash K]_{\text {reg }}$ connected, meromorphic functions on $\Omega \backslash K$ extend meromorphically and uniquely to $\Omega$ :

$$
\mathscr{M}_{X}(\Omega \backslash K)=\left.\mathscr{M}_{X}(\Omega)\right|_{\Omega \backslash K} .
$$

If moreover the data lie in $\mathscr{O}_{X}(\Omega \backslash K)$, the extension is weakly holomorphic.
The proof, also relying upon an application of Proposition 2.3 , is postponed to Section 5 ; an example in $\S 5.1$ shows that requiring only that $\Omega \backslash K$ is connected does not suffices.

For the proposition, the main difficulty is that $X_{\text {sing }}$ can in general cross $\Omega \backslash K$. We will approach $X_{\text {sing }}$ from the regular part and fill in progressively $\Omega_{\mathrm{reg}}$ by means of the super-level sets of a suitable modification $\mu$ of the exhaustion $\rho$, such that $\mu$ is still strongly $(n-1)$-convex but exhausts only $X_{\text {reg }}$ in a neighborhood of $\bar{\Omega}$. To verify that the extension procedure devised in [27] can be performed, preparational constructions are required.

## §3. Geometrical preparations

3.1. Smoothing out the boundary. To launch the filling procedure, we want to view the connected open set $\Omega \backslash K$ as a neighborhood of some convenient connected hypersurface $M$ contained in $(\Omega \backslash K) \cap X_{\text {reg }}$.
Lemma 3.2. Let $X, \Omega$ and $K$ be as in Theorem 2.2. Then there is a domain $D \Subset \Omega$ containing $K$ such that $M:=\partial D \cap X_{\text {reg }}$ is a $\mathscr{C}^{\infty}$ connected hypersurface of $X_{\text {reg }}$.

Proof. Suppose first that $X=\mathbb{C}^{n}$. Let d be a regularized distance function ([31]) for $K$, i.e. a $\mathscr{C}^{\infty}$ real-valued function with $K=\{\mathrm{d}=0\}$ and $\frac{1}{c} \operatorname{dist}(x, K) \leqslant \mathrm{d}(x) \leqslant c$ dist $(x, K)$ for some constant $c>1$, where dist is the Euclidean distance in $\mathbb{R}^{2 n}$. By Sard's theorem, there are arbitrarily small $\varepsilon>0$ such that $\widehat{M}:=\{\mathrm{d}=\varepsilon\}$ is a $\mathscr{C}^{\infty}$ hypersurface of $\mathbb{R}^{2 n}$ bounding the open set $\widehat{\Omega}:=\{\mathrm{d}<\varepsilon\}$ which satisfies $K \subset \widehat{\Omega} \Subset \Omega$. However, since $\widehat{M}$ need not be connected, we must modify it.

To this aim, we pick finitely many disjoint closed simple $\mathscr{C}^{\infty}$ arcs $\gamma_{1}, \ldots, \gamma_{r}$ which meet $\widehat{M}$ transversally only at their endpoints such that $\widehat{M} \cup \gamma_{1} \cup \cdots \cup \gamma_{r}$ is connected. Since $\Omega \backslash K$ is connected, we can insure that each $\gamma_{k}$ is contained in $\Omega \backslash K$.


We can then modify $\widehat{M}$ in the following way: we cut out a very small ball in $\widehat{M}$ around each endpoint of every $\gamma_{k}$, and we link up the connected components of the excised hypersurface with $r$ thin tubes $\simeq \mathbb{R} \times S^{2 n-2}$ almost parallel to the $\gamma_{k}$, smoothing out the corners appearing near the endpoints. The resulting hypersurface $M$ is $\mathscr{C}^{\infty}$ and connected. Since each $\gamma_{k}$ is either
contained in $\widehat{\Omega} \cup \widehat{M}$ or in $\mathbb{R}^{2 n} \backslash \widehat{\Omega}$, a new open set $D$ with $\partial D=M$ is obtained by either deleting from $\widehat{\Omega}$ or adding to $\widehat{\Omega}$ the thin tube around each $\gamma_{k}$. All the tubes around the $\gamma_{k}$ which are contained in $\mathbb{R}^{2 n} \backslash \widehat{\Omega}$ constitute thin open tunnels between the components of $\widehat{\Omega}$, whence $D$ is connected.

On a general complex space $X$, the idea is to embed a neighborhood of $\bar{\Omega}$ smoothly into some Euclidean space $\mathbb{R}^{N}$ and then to proceed similarly.

We can assume that the holomorphic isomorphisms $\phi_{j}: U_{j} \rightarrow \mathrm{~A}_{j} \subset$ $\widetilde{\mathrm{B}}_{j} \subset \mathbb{C}^{N_{j}}$ are defined in slightly larger open sets $U_{j}^{\prime} \ni U_{j}$, for all $j \in J$. Pick $\mathscr{C}^{\infty}$ functions $\lambda_{j}$ having compact support in $U_{j}^{\prime}$ and satisfying $\lambda_{j}=1$ on $\bar{U}_{j}$; prolong them to be 0 on $X$ outside $U_{j}^{\prime}$. By compactness, there is a finite open cover:

$$
\bar{\Omega} \subset U_{j_{1}} \cup \cdots \cup U_{j_{m}} .
$$

Consider the $\mathscr{C}^{\infty}$ map, valued in $\mathbb{R}^{N}$ with $N:=2\left(N_{j_{1}}+\cdots+N_{j_{m}}\right)+m$, which is defined by:

$$
\Psi:=\left(\lambda_{j_{1}} \cdot \phi_{j_{1}}, \ldots, \lambda_{j_{m}} \cdot \phi_{j_{m}}, \quad \lambda_{j_{1}}, \ldots, \lambda_{j_{m}}\right) .
$$

It is an immersion at every point $x$ of $U_{j_{1}} \cup \cdots \cup U_{j_{m}}$, because $x$ belongs to some $U_{j_{k}}$, whence the $j_{k}$-th component $\lambda_{j_{k}} \cdot \phi_{j_{k}} \equiv \phi_{j_{k}}$ of $\Psi$ is even an embedding of $U_{k} \ni x$. Furthermore, we claim that $\Psi$ separates points. Indeed, if we set:

$$
W_{j_{k}}:=\left\{z \in X: \lambda_{j_{k}}(z)=1\right\},
$$

then clearly $U_{j_{k}} \subset W_{j_{k}} \subset U_{j_{k}}^{\prime}$. Pick two distinct points $x, y \in U_{j_{1}} \cup \cdots \cup$ $U_{j_{m}}$. Then $x$ belongs to some $U_{j_{k}}$, so $\lambda_{j_{k}}(x)=1$. If $\lambda_{j_{k}}(y) \neq 1$, then $\Psi(y) \neq \Psi(x)$ and we are done. If $\lambda_{j_{k}}(y)=1$, i.e. if $y \in W_{j_{k}}$, then the $j_{k}$-th component of $\Psi$ distinguishes $x$ from $y$, since $\lambda_{j_{k}} \cdot \phi_{j_{k}}(y)=\phi_{j_{k}}(y)$ differs from $\phi_{j_{k}}(x)$ because $\phi_{j_{k}}$ embeds $U_{j_{k}}^{\prime}$ into $\mathbb{R}^{2 N_{j_{k}}}$. So $\Psi$ embeds into $\mathbb{R}^{N}$ the neighborhood $U_{j_{1}} \cup \cdots \cup U_{j_{m}}$ of $\bar{\Omega}$.

We choose a regularized distance function $\mathrm{d}_{\Psi(K)}$ for $\Psi(K)$ in $\mathbb{R}^{N}$. We stratify $X$ so that $X_{\text {reg }}$ is the single largest stratum (remind it is connected) and then stratify $X_{\text {sing }}$ by listing all connected components of $\left[X_{\text {sing }}\right]_{\text {reg }}$, then continuing with $\left[X_{\text {sing }}\right]_{\text {sing }}$, and so on inductively. By Sard's theorem and the stratified transversality theorem ([17]), for almost every $\varepsilon>0$, the intersection

$$
\left\{x \in \mathbb{R}^{N}: \mathrm{d}_{\Psi(K)}(x)=\varepsilon\right\} \cap \Psi\left(\Omega_{\mathrm{reg}}\right)
$$

is a $\mathscr{C}^{\infty}$ real hypersurface of $\Psi\left(\Omega_{\mathrm{reg}}\right)$ having finitely many connected components which are contained in $\Psi\left([\Omega \backslash K]_{\text {reg }}\right)$. Importantly, we can construct the thin connecting tubes so that they lie all entirely inside $\Psi\left([\Omega \backslash K]_{\text {reg }}\right)$, thanks to the fact that $\Psi\left(\Omega_{\mathrm{reg}}\right)$ is locally (arcwise) connected, also near points of $\Psi\left(\Omega_{\text {sing }}\right)$. Then the remaining arguments are the same and we
put everything back to $X$ via $\Psi^{-1}$, getting a connected $\mathscr{C}^{\infty}$ hypersurface $M \subset[\Omega \backslash K]_{\mathrm{reg}}$ and a domain $D$ with $K \subset D \Subset \Omega$. (We remark that normality of $X$ was crucially used.)

As we said, we will perform the filling procedure entirely inside $X_{\text {reg }}$. This is possible thanks to an idea of Demailly which consists of modifying the initial exhaustion $\rho$ so that $X_{\text {sing }}$ is put at $-\infty$. A recent application of this idea also appears in [7].
3.3. Putting $X_{\text {sing }}$ into a well. By Lemma 5 in [4], there exists an almost plurisubharmonic function ${ }^{49} v$ on $X$ which is $\mathscr{C}^{\infty}$ on $X_{\text {reg }}$ and has poles along $X_{\text {sing }}$ :

$$
X_{\text {sing }}=\{v=-\infty\} .
$$

As in Section 2, if $\mathrm{A}_{j}=\varphi_{j}\left(U_{j}\right)$ is represented in a local ball $\widetilde{\mathrm{B}}_{j} \subset \mathbb{C}^{N_{j}}$ of radius $r_{j}>0$ centered at $z_{j} \in \mathbb{C}^{N_{j}}$ as the zero-set $\left\{g_{j, \nu}=0\right\}$ of finitely many functions $g_{j, \nu}$ holomorphic in a neighborhood of the closure of $\widetilde{\mathrm{B}}_{j}$, the local ambient $\widetilde{v}_{j}: \widetilde{\mathrm{B}}_{j} \rightarrow\{-\infty\} \cup \mathbb{R}$ is essentially of the form ${ }^{50}$ :

$$
\widetilde{v}_{j}=\log \left(\sum_{\nu}\left|g_{j, \nu}\right|^{2}\right)-\frac{1}{r_{j}^{2}-\left|z-z_{j}\right|^{2}} .
$$

Thus, locally on each $\widetilde{\mathrm{B}}_{j}$, the function $v$ we pick from [4] is of the form:

$$
\widetilde{v}_{j}=\widetilde{u}_{j}+\widetilde{r}_{j},
$$

with $\widetilde{u}_{j}$ strictly psh, $\mathscr{C}^{\infty}$ on $\widetilde{\mathrm{B}}_{j} \backslash\left[\mathrm{~A}_{j}\right]_{\text {sing }}$, equal to $\{-\infty\}$ on $\left[\mathrm{A}_{j}\right]_{\text {sing }}$ and with a remainder $\widetilde{\mathrm{r}}_{j}$ which is $\mathscr{C}^{\infty}$ on the whole of $\widetilde{\mathrm{B}}_{j}$. Notice that each $\widetilde{v}_{j}$ is $L_{\text {loc }}^{\infty}$.
3.4. Modified strongly $(n-1)$-convex exhaustion function $\mu$. Pick a constant $C>0$ such that $\max _{\bar{D}}(\rho)<C$.
Lemma 3.5. There exists $\varepsilon_{0}>0$ such that for all $\varepsilon$ with $0<\varepsilon \leqslant \varepsilon_{0}$, the function:

$$
\mu:=\rho+\varepsilon v
$$

is $\mathscr{C}^{\infty}$ on $X_{\text {reg }}$ and satisfies:
(a) $\max _{\bar{D}}(\mu)<C$;
(b) $X_{\text {sing }}=\{\mu=-\infty\}$;

[^47](c) $\mu$ is strongly $(n-1)$-convex in a neighborhood of $\{\rho \leqslant C\}$.

Proof. Property (b) holds provided only that $\varepsilon<\frac{C-\max _{\bar{D}}(\rho)}{\max _{\bar{D}}(v)}$. Furthermore, (a) is clear since $\rho$ is $\mathscr{C}^{\infty}$ and since $X_{\text {sing }}=\{v=-\infty\}$. To check (c), we compute Levi-forms as $(1,1)$-forms:

$$
\begin{align*}
i \partial \bar{\partial} \widetilde{\mu}_{j} & =i \partial \bar{\partial} \widetilde{\rho}_{j}+\varepsilon i \partial \bar{\partial} \widetilde{v}_{j}  \tag{3.6}\\
& =i \partial \bar{\partial} \widetilde{\rho}_{j}+\varepsilon i \partial \bar{\partial} \widetilde{u}_{j}+\varepsilon i \partial \bar{\partial} \widetilde{r}_{j}
\end{align*}
$$

Here, $\varepsilon i \partial \bar{\partial} \widetilde{u}_{j}$ adds positivity to $i \partial \bar{\partial} \widetilde{\rho}_{j}$ (since $\widetilde{u}_{j}$ is psh), whereas the negative contribution due to $i \partial \bar{\partial} \widetilde{\mathrm{r}}_{j}$ is bounded from below on $\{\rho \leqslant 2 C\}$, and consequently, $\varepsilon>0$ can be chosen small enough so that $i \partial \bar{\partial} \widetilde{\mu}_{j}$ still has 2 eigenvalues $>0$ at every point.

In the next section, while applying the holomorphic extension procedure of [27], we shall have to insure that the extensional domains attached to $M$ from either the outside or the inside cannot go beyond $\{\rho \leqslant C\}$. So we have to prepare in advance the curvature of the limit hypersurface $\{\rho=C\} \cap X_{\text {reg }}$.

Enlarging $C$ of an arbitrarily small increment if necessary, we can assume (thanks to Sard's theorem) that $C$ is a regular value of $\left.\rho\right|_{X_{\text {reg }}}$, so that

$$
\Lambda:=\{\rho=C\} \cap X_{\mathrm{reg}}
$$

is a $\mathscr{C}^{\infty}$ real hypersurface of $X_{\text {reg }}$.
Lemma 3.7. Lowering again $\varepsilon>0$ if necessary, the following holds:
(d) At every point $q$ of the $\mathscr{C}^{\infty}$ real hypersurface $\Lambda=\{\rho=C\} \cap X_{\text {reg }}$, one can find a complex line $E_{q} \subset T_{q}^{c} \Lambda$ on which the Levi-forms of both $\rho$ and $\mu$ are positive.

Here, $q \mapsto E_{q}$ might well be discontinuous, but this shall not cause any trouble in the sequel.
Proof. Each $p \in\{\rho=C\}$ is contained in some $U_{j(p)}$, whence $\rho$ is represented by an ambient function $\widetilde{\rho}_{j(p)}: \widetilde{\mathrm{B}}_{j(p)} \rightarrow \mathbb{R}$ whose Levi-form has at least $N_{j(p)}-n+2$ eigenvalues $>0$. By diagonalizing the Levi matrix $i \partial \bar{\partial} \widetilde{\rho}_{j(p)}$ at the central point of $\widetilde{\mathrm{B}}_{j(p)}$, we may easily define, in some small open sub-ball $\widetilde{\mathrm{C}}_{j(p)} \subset \widetilde{\mathrm{B}}_{j(p)}$ having the same center, a $\mathscr{C}$ (family $\widetilde{q} \mapsto \widetilde{F}_{\widetilde{q}}$ of complex $\left(N_{j(p)}-n+2\right)$-dimensional affine subspaces such that the Leviform of $\widetilde{\rho}_{j(p)}$ is positive definite on every $\widetilde{F}_{\widetilde{q}}$, for every $\widetilde{q} \in \widetilde{\mathrm{C}}_{j(p)}$.

Next, if we set $V_{j(p)}:=\varphi_{j(p)}^{-1}\left(\widetilde{\mathrm{C}}_{j(p)}\right)$, which is an open subset of $U_{j(p)}$, we can cover the compact set $\{\rho=C\}$ by finitely many $V_{j(p)}$, hence there is a finite number of points $p_{a}, a=1, \ldots, A$, such that

$$
\{\rho=C\} \subset V_{j\left(p_{1}\right)} \cup \cdots \cup V_{j\left(p_{A}\right)} .
$$

According to (3.6), on each $\widetilde{\mathrm{C}}_{j\left(p_{a}\right)}, a=1, \ldots A$, we have:

$$
i \partial \bar{\partial} \widetilde{\mu}_{j\left(p_{a}\right)}=i \partial \bar{\partial} \widetilde{\rho}_{j\left(p_{a}\right)}+\varepsilon i \partial \bar{\partial} \widetilde{u}_{j\left(p_{a}\right)}+\varepsilon i \partial \bar{\partial} \widetilde{r}_{j\left(p_{a}\right)}
$$

We choose $\varepsilon>0$ so small that the remainder $\varepsilon i \partial \bar{\partial} \widetilde{\mathrm{r}}_{j\left(p_{a}\right)}$ does not perturb positivity on $\widetilde{\mathrm{C}}_{j\left(p_{a}\right)}$ for every $a=1, \ldots A$, and we get that $i \partial \bar{\partial} \widetilde{\mu}_{j\left(p_{a}\right)}$ is still positive on $\widetilde{F}_{\widetilde{q}}$ for every $\widetilde{q} \in \widetilde{\mathrm{C}}_{j\left(p_{a}\right)}$, and every $a=1, \ldots A$.

Let $q \in\{\rho=C\} \cap X_{\text {reg }}$. Then $q \in V_{j\left(p_{a}\right)}$ for some $a$. We set $\widetilde{q}:=$ $\varphi_{j\left(p_{a}\right)}(q) \in \widetilde{\mathrm{C}}_{j\left(p_{a}\right)}$ and we define:

$$
F_{q}:=\left(d \varphi_{j\left(p_{a}\right)}\right)^{-1}\left(\widetilde{F}_{\widetilde{q}} \cap T_{\widetilde{q}} \mathrm{~A}_{j\left(p_{a}\right)}\right) .
$$

Then the complex linear spaces $\widetilde{F}_{\widetilde{q}}$ and $F_{q}$ are at least of dimension 2 and the Levi-form of $\mu$ is positive on any 1-dimensional subspace $E_{q} \subset F_{q} \cap T_{q}^{c} \Lambda$.

Next, applying Morse transversality theory, we may perturb $\mu$ in $X_{\text {reg }} \cap$ $\{\rho<2 C\}$ in an arbitrarily small way, so that ${ }^{51}$ :
(e) $\mu$ is a Morse function on $X_{\text {reg }} \cap\{\rho<2 C\}$ having finitely many or at most countably many critical points; moreover, different critical points of $\mu$ are located in different level sets $\{\mu=c\}$.

Of course, if they are infinite in number, critical values can only accumulate at $-\infty$. Similarly, we may perturb $\rho$ very slightly near $\{\rho=C\}$ so that:
(f) the $\mathscr{C}^{\infty}$ hypersurface $\{\rho=C\} \cap X_{\text {reg }}$ does not contain any critical point of $\mu$.

Finally, again thanks to Morse transversality theory, we may perturb the connected $\mathscr{C}^{\infty}$ hypersurface $M \subset \partial D$ of Lemma 3.2 in an arbitrarily small way so that ${ }^{52}$ :
(g) $M$ does not contain critical points of $\mu$, and $\left.\mu\right|_{M}$ is a Morse function on $M$ having finitely many or at most countably many critical points; moreover, any two different critical points of $\mu$ or of $\left.\mu\right|_{M}$ have different critical values.

We draw a diagram, where $X_{\text {sing }}$ is symbolically represented as a continuous broken line having spikes, with a level-set $\{\mu=\widehat{c}\}$ which is critical for $\left.\mu\right|_{M}$ and a single critical point $\widehat{p} \in M \cap\{\mu=\widehat{c}\}$.

[^48]

## §4. Holomorphic extension to $D_{\text {reg }}$

For $c \in \mathbb{R}$, we introduce

$$
X_{\mu>c}:=\{z \in X: \mu(z)>c\} .
$$

This open set is contained in $X_{\text {reg }}$, since $X_{\text {sing }}=\{\mu=-\infty\}$. For every connected component $M_{\mu>c}^{\prime}$ of

$$
M_{\mu>c}:=M \cap X_{\mu>c}=M \cap\{\mu>c\},
$$

we want to fill in (by means of a finite number of families of analytic discs) a certain domain $Q_{\mu>c}^{\prime}$ which is enclosed by $M_{\mu>c}^{\prime}$ inside $\{\mu>c\}$. Similarly as in Proposition 5.3 of [27], we must consider all the connected components $M_{\mu>c}^{\prime}$ and analyze the combinatorics of how they merge or disappear.

Let $\mathscr{V}(M)$ be a thin tubular neighborhood of $M$, whose thinness shrinks to zero while approaching $X_{\text {sing }}$. For every connected component $M_{\mu>c}^{\prime}$ of $M_{\mu>c}$, we denote by $\mathscr{V}\left(M_{\mu>c}^{\prime}\right)_{\mu>c}$ the part of $\mathscr{V}(M)$ around $M_{\mu>c}^{\prime}$ again intersected with $\{\mu>c\}$. It is a connected tubular neighborhood of $M_{\mu>c}^{\prime}$ inside $\{\mu>c\}$.
Proposition 4.1. Let $c \in \mathbb{R}$ with $c<\max _{M}(\mu)<C$ be any regular value of $\mu$ and of $\left.\mu\right|_{M}$. Let $M_{\mu>c}^{\prime}$ be any nonempty connected component of $M \cap X_{\mu>c}$. Then there is a unique connected component $Q_{\mu>c}^{\prime}$ of $X_{\mu>c} \backslash M_{\mu>c}^{\prime}$ which is relatively compact in $X_{\mathrm{reg}}$ and contained in $\{\rho<C\}$ with the property that two different domains $Q_{\mu>c}^{\prime}$ and $Q_{\mu>c}^{\prime \prime}$ are either disjoint or one is contained in the other. Furthermore, for every holomorphic or meromorphic function $f$ defined in the thin tubular neighborhood $\mathscr{V}(M)$ of $M$, there exists a unique holomorphic or meromorphic extension $F$, constructed by means of a finite number of $(n-1)$-concave Levi-Hartogs figures and defined in

$$
Q_{\mu>c}^{\prime} \bigcup \mathscr{V}\left(M_{\mu>c}^{\prime}\right)_{\mu>c},
$$

such that $F=f$ when both functions are restricted to $\mathscr{V}\left(M_{\mu>c}^{\prime}\right)_{\mu>c}$.

Proof. We only describe the modifications one must bring to the arguments of [27].

1) The Levi-form of the compact $\mathscr{C}^{\infty}$ boundary $\{\mu=c\}$ of the superlevel set $\{\mu>c\}$ (contained in $X_{\text {reg }}$ ) has 1 negative eigenvalue, so that the Levi extension theorem with analytic discs (cf. the survey [26]) applies at each point of $\{\mu=c\}$. In Section 3 of [27], we defined $(n-a)$-concave Hartogs figures for $1 \leqslant a \leqslant n-1$, but we used only 1-concave ones, because the Levi-form of exterior of spheres $\{\|z\|<r\}$ in $\mathbb{C}^{n} \operatorname{had}(n-1)$ negative eigenvalues. Here, we start from $(n-1)$-concave Hartogs figures, we modify them similarly as in Section 3 of [27] (details are skipped) and we call them $(n-1)$-concave Levi-Hartogs figures.

Next, we use a finite number of these figures, via some local charts of $X_{\text {reg }}$, to cover $\{\mu=c\}$ and to show that holomorphic ${ }^{53}$ (or meromorphic) functions in $\{\mu>c\}$ extend to a slightly deeper super-level set $\{\mu>c-\eta\}$ (provided no critical point of $\mu$ or of $\left.\mu\right|_{M}$ is encountered in the shell $\{c \geqslant$ $\mu>c-\eta\}$ ), for some $\eta>0$ which depends on $X$, on $n$, on $\mu$, but not on $c$.
2) Contrary to the $\mathbb{C}^{n}$ case treated in [27], $\mu$ may have critical points on $X_{\text {reg }}$. Grauert's theory shows how to jump across them with $\overline{\bar{\partial}}$ techniques, and we summarize how we can proceed here ${ }^{54}$, using only analytic discs in Levi-Hartogs figures.

Consider a point $\widehat{p} \in X_{\text {reg }}$ which is critical: $d \mu(\widehat{p})=0$, and set $\widehat{c}:=\mu(\widehat{p})$. The Morse lemma provides local real coordinates centered at $\widehat{p}$ in which $\mu=x_{1}^{2}+\cdots+x_{k}^{2}-y_{1}^{2}-\cdots-y_{2 n-k}^{2}$, for some $k$. Since $i \partial \bar{\partial} \mu$ has at least 2 positive eigenvalues everywhere, $k$ is $\geqslant 2$. This is a crucial fact, because this implies that super-level sets $\{\mu>\widehat{c}+\delta\}$ are all connected ${ }^{55}$ in a neighborhood of $\widehat{p}$, for every $\delta \in \mathbb{R}$ close to 0 , and moreover, that these domains grow regularly and continuously as $\delta$ decreases from positive values to negative values.

[^49]
mnFig. 3: Filling outside a neighborhood of $\widehat{p}$ and shifting $\widehat{p}$
Next, we fix a ball $\widehat{B}$ centered at $\widehat{p}$ and we cut out a small neighborhood $\widehat{U} \subset \widehat{B}$ of $\widehat{p}$. If $\widehat{V} \subset \widehat{U}$ is a small neighborhood, we consider the $\mathscr{C}^{\infty}$ hypersurface:
$$
\left\{\mu>\widehat{c}+\frac{\eta}{2}\right\} \backslash \widehat{V} .
$$

Placing finitely many ( $n-1$ )-concave Levi-Hartogs figures at points of this hypersurface, we get holomorphic or meromorphic extension to $\{\mu>\widehat{c}-$ $\left.\frac{\eta}{2}\right\} \backslash \widehat{V}_{1}$, where $\widehat{V}_{1} \subset \widehat{V}$ is slightly bigger than $\widehat{V}$. Repeating the filling process finitely many times until $\left\{\mu=\widehat{c}-\frac{k \eta}{2}\right\}$ does not intersect $\widehat{B}$, where $k$ is an odd integer, we fill in $\widehat{B} \backslash \widehat{U}$. At each step, monodromy of the extension is assured thanks to connectedness of $\{\mu>\widehat{c}+\delta\} \backslash \widehat{U}$, for every small $\delta \in \mathbb{R}$. However, we cannot fill in $\widehat{U}$ directly this way.

The trick is to shift $\widehat{p}$. One introduces a $\mathscr{C}^{\infty}$ perturbation $\mu^{\prime}$ of $\mu$ localized near $\widehat{p}$ (namely $\mu^{\prime}=\mu$ elsewhere) such that $\mu^{\prime}$ has another critical point $\hat{p}^{\prime}$ (having the same Morse index of course), with corresponding neighborhoods disjoint: $\widehat{U} \cap \widehat{U}^{\prime}=\emptyset$ and both contained in $\widehat{B} \cap \widehat{B}^{\prime}$. We repeat the Levi-Hartogs filling with $\mu^{\prime}$, getting holomorphic or meromorphic extension $\left\{\mu^{\prime}>\widehat{c}-k^{\prime} \frac{\eta}{2}\right\} \backslash \widehat{U}^{\prime}$, a domain which contains $\widehat{B}^{\prime} \backslash \widehat{U}^{\prime}$, hence contains $\widehat{U}$. Monodromy is again well controlled, just because topologically, $\widehat{B} \backslash \widehat{U}$ and $\widehat{B}^{\prime} \backslash \widehat{U}^{\prime}$ are complete shells.
3) We prove the proposition by decreasing $c$. Provided $c$ does not cross critical values of $\left.\mu\right|_{M}$, the domains $Q_{\mu>c}^{\prime}$ do grow regularly and continuously, even when $c$ crosses critical values of $\mu$, according to what has been said just above. At a critical value $\widehat{c}$ of $\left.\mu\right|_{M}$, for a domain $Q_{\mu>\widehat{c}}$ whose closure contains the corresponding unique critical point $\widehat{p} \in M$, similarly as in [27], three cases may occur:
(i) the domain $Q_{\mu>\widehat{c}+\delta}^{\prime}$ grows regularly and continuously as $\delta$ decreases in a neighborhood of 0 ;
(ii) precisely when $\delta$ becomes negative, the domain $Q_{\mu>\hat{c}+\delta}^{\prime}$ is merged with a second domain $Q_{\mu>\widehat{c}+\delta}^{\prime \prime}$ whose closure also contains $\widehat{p}$ for $\delta=$ 0 (the case of three domains or more never occurs);
(iii) the domain $Q_{\mu>\widehat{c}+\delta}^{\prime}$ is contained in a bigger domain $Q_{\mu>\widehat{c}+\delta}^{\prime \prime}$ for all small $\delta>0$, and exactly at $\delta=0$, the closure of the domain $Q_{\mu>\bar{c}}^{\prime}$ is subtracted from $Q_{\mu>\widehat{c}}^{\prime \prime}$, yielding a new domain $Q_{\mu>\widehat{c}}^{\prime \prime \prime}$ which starts to grow regularly and continuously as $Q_{\mu>\hat{c}+\delta}^{\prime \prime \prime}$ for small $\delta<0$.

We then check by decreasing induction on $c$ that such domains are relatively compact and are either disjoint or one is contained in the other, and we achieve extension by means of $(n-1)$-concave Levi-Hartogs figures similarly as in [27]. But here, a single fact remains to be established, namely that the domains $Q_{\mu>c}^{\prime}$ remain all contained inside the relatively compact region $\{\rho<C\}$.

This is true at the beginning of the filling process, namely for $c$ slightly smaller than $\max _{M}(\mu)$, because $M_{\mu>c}$ is then diffeomorphic to a small spherical cap (hence connected) and the relatively compact domain enclosed by $M_{\mu>c}$ in $X_{\mu>c} \backslash M_{\mu>c}$ is just the piece $D_{\mu>c}$ of $D$, which is diffeomorphic to a thin cut out piece of ball close to $M$ and clearly contained in $\{\rho<C\}$, since $D \cup M \subset\{\rho<C\}$ by (a).

To prove that all $Q_{\mu>c}^{\prime}$ are contained in $\{\rho<C\}$, we proceed by contradiction. Let $c^{*}$ be first $c$ (as $c$ decreases) for which some $Q_{\mu>c}^{\prime}$ is not contained in $\{\rho<C\}$. In the process described above of constructing the domains $Q_{\mu>c}^{\prime}$, the only discontinuity occurs in (iii) and it consists of a suppression. Consequently, the domains $Q_{\mu>c}^{\prime}$ cannot jump discontinuously across $\{\rho=C\}$, hence at $c=c^{*}$ (which might be either critical or noncritical), all $Q_{\mu>c^{*}}^{\prime}$ are still contained in $\{\rho \leqslant C\}$ and the boundary of at least one domain, say $Q_{\mu>c^{*}}^{*}$, touches the $\mathscr{C}^{\infty}$ border hypersurface $\{\rho=C\} \cap X_{\text {reg }}$.


On the other hand, by definition and by construction, for each $c$, the boundary of each $Q_{\mu>c}^{\prime}$ consists of two parts: $M_{\mu>c}^{\prime}$, which is contained in $M$, hence remains always in $\{\rho<C\}$, together with a certain closed region $R_{\mu=c}^{\prime} \cup N_{\mu=c}^{\prime}$ contained in $\{\mu=c\}$, with $R_{\mu=c}^{\prime}$ open and $N_{\mu=c}^{\prime}$ being
the boundary in $\{\mu=c\}$ of $R_{\mu=c}^{\prime}$. In fact, similarly as in Section 5 of [27], $R_{\mu=c}^{\prime}$ is always contained in $\{\mu=c\} \backslash M$ and $N_{\mu=c}^{\prime}$, always contained in $M \cap\{\mu=c\}$ is a $\mathscr{C}^{\infty}$ real submanifold of $X_{\text {reg }}$ of codimension 2 provided $c$ is noncritical for $\left.\mu\right|_{M}$, while $N_{\mu=c}^{\prime}$ may have as a single singular (corner) point $\widehat{p}$ for $c=\widehat{p}$ critical. But since $N_{\mu=c}^{\prime}$ is a subset of $M \cap\{\mu=c\}$, it is always contained in $\{\rho<C\}$.

Consequently, the boundary of $Q_{\mu>c^{*}}^{*}$ can touch $\{\rho=C\}$ only at some point $p^{*} \in R_{\mu=c^{*}}^{*}$. So we have $\mu\left(p^{*}\right)=c^{*}$ and $\rho\left(p^{*}\right)=C$, namely $p^{*}$ lies in $\left\{\mu=c^{*}\right\}$ and on the $\mathscr{C}^{\infty}$ hypersurface $\{\rho=C\}$.

By (f) above, $p^{*} \in\{\rho=C\}$ cannot be a critical point of $\mu$, whence $\left\{\mu=c^{*}\right\}$ and $\{\rho=C\}$ are both $\mathscr{C}^{\infty}$ real hypersurfaces passing through $p^{*}$. Furthermore, $\left\{\mu \geqslant c^{*}\right\}$ is still contained in $\{\rho \leqslant C\}$, by definition of $c^{*}$, whence $T_{p^{*}}\{\rho=C\}=T_{p^{*}}\left\{\mu=c^{*}\right\}$.

Thanks to (d), there is a complex line

$$
E_{p^{*}} \subset T_{p^{*}}^{c}\{\rho=C\}=T_{p^{*}}^{c}\left\{\mu=c^{*}\right\}
$$

on which the Levi-forms of both $\rho$ and $\mu$ are positive definite. On the other hand, since $\left\{-\mu<-c^{*}\right\}$ is contained in $\{\rho<C\}$, the Levi-form of $-\mu$ in the direction of $E_{p^{*}}$ should then be $\geqslant$ the Levi-form of $\rho$ in the same direction. This is a contradiction, and the proof that all $Q_{\mu>c}^{\prime}$ remain in $\{\rho<C\}$ is completed. This finishes our proof of Proposition 4.1.
4.2. End of proof of Proposition 2.3. As in Section 2 of [27], one checks that extension holds from $[\Omega \backslash K]_{\text {reg }}$ to $\Omega_{\text {reg }}$ provided holomorphic or meromorphic functions defined in the thin tubular neighborhood $\mathscr{V}(M)$ of $M \subset$ $X_{\text {reg }}$ do extend uniquely to $D_{\text {reg }} \bigcup \mathscr{V}(M)$. So we work with $M, \mathscr{V}(M)$ and $D_{\text {reg }}$, and since everything is exhausted as $c \rightarrow-\infty$, the conclusion of the proof of Proposition 2.3 is an immediate consequence of the following.

Proposition 4.3. For every regular value $c>-\infty$ of $\left.\mu\right|_{M}$, holomorphic or meromorphic functions defined in $\mathscr{V}(M)$ do extend holomorphically or meromorphically and uniquely to

$$
D_{\mu>c} \bigcup \mathscr{V}\left(M_{\mu>c}\right)_{\mu>c} .
$$

Proof. We set $c_{1}:=\max _{M}(\mu)=\max _{\bar{D}}(\mu)<C$. There is a unique " $\mu-$ farthest point" $p_{1} \in M$ with $\mu\left(p_{1}\right)=c_{1}$ and this point is obviously a critical point of Morse index equal to $-(2 n-1)$ for $\left.\mu\right|_{M}$, by virtue of ( $\mathbf{g}$ ). Consequently, for all $c<c_{1}$ close to $c_{1}$, there is a single connected component in $M_{\mu>c}$, namely $M_{\mu>c}$ itself, which is diffeomorphic to a small spherical cap and encloses the domain $D_{\mu>c}$, diffeomorphic to a thin cut out piece of ball. For such $c<c_{1}$ close to $c_{1}$, the proposition is thus a direct consequence of the previous Proposition 4.1.

For arbitrary noncritical $c$, there is a well defined connected component $M_{\mu>c}^{1}$ of $M_{\mu>c}$ with $p_{1} \in M_{\mu>c}^{1}$, and we denote by $M_{\mu>c}^{2}, \ldots, M_{\mu>c}^{k}$ the other connected components of $M_{\mu>c}$. Also, each connected component $D_{\mu>c}^{\sim}$ of $D_{\mu>c}$ is bounded by some of the $M_{\mu>c}^{j}$, inside $\{\mu>c\}$. The problem is that the various extensions provided by Proposition 4.1 need not stick together, but fortunately, we can go to deeper super-level sets $\left\{\mu>c^{\prime}\right\}$.

Lemma 4.4. For every $c^{\prime}$ with $-\infty<c^{\prime} \leqslant c$ which is noncritical for $\left.\mu\right|_{M}$, the $\mu$-farthest point $p_{1}$ belongs to a unique connected component $M_{\mu>c^{\prime}}^{\prime}$ of $M \cap\left\{\mu>c^{\prime}\right\}$ and the enclosed domain $Q_{\mu>c^{\prime}}^{\prime}$ constructed by Proposition 4.1 contains $D$ in a neighborhood of $p_{1}$.

Proof. Indeed, if this were not true, there would exist the first $c^{\prime}=c^{*}$ (as $c^{\prime} \leqslant c$ decreases) for which $Q_{\mu>c^{\prime}}^{\prime}$ switches to the other side of $M$ near $p_{1}$. According to the topological combinatorial processus (i), (ii), (iii) above, this could only occur in case (iii) with $c^{*}$ critical, where a component is suppressed from a bigger one $Q_{\mu>c^{*}}^{\prime \prime}$ bounded by some $M_{\mu>c^{*}}^{\prime \prime}$, the suppressed component necessarily being $Q_{\mu>c^{*}}^{\prime}$ itself. Then the bigger component $Q_{\mu>c^{*}}^{\prime \prime}$ would contain the side of $M$ which is exterior to $D$ near $\widehat{p}_{1}$, whence

$$
c_{1}^{\prime \prime}:=\max \left\{\mu(q): q \in M_{\mu>c^{*}}^{\prime \prime}\right\}
$$

would necessarily be $>c_{1}$, which contradicts $c_{1}=\max _{M}(\mu)$.


Next, since $M$ is connected (according to Lemma 3.2), we can pick a $\mathscr{C}^{\infty}$ Jordan arc $\gamma$ running in $M$ which starts at $p_{1}$ and visits every other connected component $M_{\mu>c}^{2}, \ldots, M_{\mu>c}^{k}$ of $M_{\mu>c}$. Since $\gamma$ is compact, there is a noncritical $c^{\prime}>-\infty$ such that $\gamma \subset\left\{\mu>c^{\prime}\right\}$. Fix such a $c^{\prime}$ and denote by $M_{\mu>c^{\prime}}^{\prime}$ the connected component of $M \cap\left\{\mu>c^{\prime}\right\}$ to which $p_{1}$ belongs. Then let $Q_{\mu>c^{\prime}}^{\prime}$ be as in Lemma 4.4.

Lemma 4.5. The domain $Q_{\mu>c^{\prime}}^{\prime}$ contains $D_{\mu>c}$.

Proof. Near $p_{1}$, this domain already contains a piece of $D$ thanks to Lemma 4.4. From the beginning, $M$ is oriented, since it bounds the domain $D$. Thus, we can push $\gamma$ slightly inside $D$, getting a curve $\gamma^{\sharp}$ almost parallel to $\gamma$ which is entirely contained in $D$, and also contained in $\left\{\mu>c^{\prime}\right\}$ if the push is sufficiently small. Furthermore, $\gamma^{\sharp}$ is also entirely contained in $Q_{\mu>c^{\prime}}^{\prime}$, because the extensional domain $Q_{\mu>c^{\prime}}^{\prime}$ is, at least near $p_{1}$, located on the same side (with respect to $M$ ) as $D$.

Let $D_{\mu>c}^{\sim}$ be any connected component of $D_{\mu>c}$. By construction, $\gamma^{\sharp}$ visits $D_{\mu>c}^{\sim}$. Thus, every point of $D_{\mu>c}^{\sim}$ may be joined to some point of $\gamma^{\sharp}$ by means of some auxiliary $\mathscr{C}^{\infty}$ curve running in $D_{\mu>c}^{\sim}$. All such auxiliary curves do not meet $M$, hence they do not meet $M_{\mu>c^{\prime}}^{\prime}$, whence they all run in $Q_{\mu>c^{\prime}}^{\prime}$. Consequently, by means of $\gamma^{\sharp}$ and of the auxiliary curves in each $D_{\mu>c}^{\sim}$, we may connect, without crossing $M$ even once, every point of $D_{\mu>c}$ with the starting point of $\gamma^{\sharp}$, contained in $Q_{\mu>c^{\prime}}^{\prime}$ near $p_{1}$. Thus $D_{\mu>c}$ is effectively contained in $Q_{\mu>c^{\prime}}^{\prime}$.

To conclude, an application of Proposition 4.1 yields unique extension to $Q_{\mu>c^{\prime}}^{\prime} \bigcup \mathscr{V}\left(M_{\mu>c^{\prime}}^{\prime}\right)_{\mu>c^{\prime}}$, and by plain restriction, we get unique extension to $D_{\mu>c} \bigcup \mathscr{V}\left(M_{\mu>c}\right)_{\mu>c}$.

This completes the proofs of Propositions 4.3 and 2.3.

## §5. MEROMORPHIC EXTENSION ON NONNORMAL COMPLEX SPACES

5.1. An example. To see that the weaker assumption that $\Omega \backslash K$ is connected does not suffice, we consider $X=\mathbb{C}^{2} /((-1,0) \sim(+1,0))$, the euclidean $\mathbb{C}^{2}$ with two points identified. If we define the structure sheaf by $\mathscr{O}_{\mathbb{C}^{2}, z}$ at all single points and by $\mathscr{O}_{\mathbb{C}^{2}, \pm}=\left\{(f, g) \in \mathscr{O}_{\mathbb{C}^{2},-1} \times \mathscr{O}_{\mathbb{C}^{2}, 1}: f(-1,0)=\right.$ $g(+1,0)\}$ at the double point $( \pm 1,0)$, the space $\left(X, \mathscr{O}_{X}\right)$ is reduced and modelled near $( \pm 1,0)$ on $\left\{(z, w) \in \mathbb{C}^{2} \times \mathbb{C}^{2}:\{z=0\} \cup\{w=0\}\right\}$. This makes it easy to check that the function $\left|z_{1}+1\right|^{2}+\left|z_{1}-1\right|^{2}+\left|z_{2}\right|^{2}$ descends to a 1-convex exhaustion of $X$ via the quotient projection $\pi: \mathbb{C}^{2} \rightarrow X$. Letting $\Omega:=X$ and $K:=\pi\left(\left\{\left|z_{1}+1\right|^{2}+\left|z_{2}\right|^{2}=1\right\}\right)$, we see that $\Omega \backslash K$ is connected. Furthermore, $\mathscr{O}(\Omega \backslash K)$ consists of all functions holomorphic in $\mathbb{C}^{2} \backslash\left\{\left|z_{1}+1\right|^{2}+\left|z_{2}\right|^{2}=1\right\}$ which satisfy $f(-1,0)=f(+1,0)$. Then obviously, the conclusion of Theorem 2.4 does not hold.
5.2. Proof of Theorem 2.4. To begin with, we observe that Proposition 2.3 carries over without change to the more general setting of Theorem 2.4: indeed, thanks to the connectedness of $[\Omega \backslash K]_{\text {reg }}$, we may construct $M$ and $D$ as in Lemma 3.2; the construction of an almost psh function $v$ with $X_{\text {sing }}=\{v=-\infty\}$ holds without assumption of normality ([4]), and then Propositions 4.1 and 4.3 do go through (notice that both $\Omega \backslash K$ and $\Omega_{\mathrm{reg}}$
are connected). Thus $\mathscr{M}_{X}(\Omega \backslash K)$ extends uniquely as $\mathscr{M}_{X}\left(\Omega_{\mathrm{reg}} \cup[\Omega \backslash K]\right)$, holomorphicity being preserved.

Extension across $\Omega_{\text {sing }} \cap K$ is slightly more complicated than in the normal case due to the fact that $\Omega_{\text {sing }}$ may have components of codimension one. Let $\pi: \widehat{X} \rightarrow X$ be the normalization of $X$. Let $X_{\text {norm }}$ be the set of the normal points of $X$. Recall that $\pi$ restricts to a biholomorphism on $\pi^{-1}\left(X_{\text {norm }}\right)$. Topologically, $\pi$ is a local homeomorphism over irreducible points of $X$ and separates the irreducible local components at reducible points. For every open $U \subset X$, setting $\widehat{U}=\pi^{-1}(U)$, we have a canonical isomorphism $\pi^{*}$ : $\mathscr{M}_{X}(U) \rightarrow \mathscr{M}_{\widehat{X}}(\widehat{U})$ ([11], p. 155). Hence it is enough to extend from $\mathscr{M}_{\widehat{X}}(\widehat{\Omega} \backslash L)$ to $\mathscr{M}_{\widehat{X}}(\widehat{\Omega})$, where $\widehat{\Omega}:=\pi^{-1}(\Omega)$ and $L:=\pi^{-1}\left(\Omega_{\text {sing }} \cap K\right)$.

By the Levi extension theorem, we can extend through all points of $z \in L$ with $\operatorname{dim}_{z} \pi^{-1}\left(\Omega_{\text {sing }}\right) \leqslant n-2$. Let $H$ be an irreducible component of $\Omega_{\text {sing }}$ of codimension one. Since $\operatorname{dim} \widehat{\Omega}_{\text {sing }} \leqslant n-2$, it follows that $\widehat{H}^{\prime}:=\pi^{-1}(H) \cap \widehat{\Omega}_{\text {reg }}$ is dense, open and connected in $\widehat{H}=\pi^{-1}(H)$. Because $X$ is ( $n-1$ )-convex, it cannot contain any compact analytic hypersurface according to Lemma 5.3 just below, and $H$ has to intersect $\Omega \backslash K$. For dimensional reasons, $\widehat{H}^{\prime}$ intersects $\left[\pi^{-1}(\Omega \backslash K)\right]_{\mathrm{reg}}$, and we can apply the following version of the Levi extension theorem for complex manifolds ([12]): Let $Y$ be an analytic subset of a complex manifold of $M$ of codimension at least one. If $U \subset M$ is a domain containing $M \backslash Y$ and intersecting each irreducible one-codimensional component of $Y$, then holo-/meromorphic functions on $U$ extend holo-/meromorphically to $M$.

The remaining part of the singularity lies in $\widehat{\Omega}_{\text {sing }}$ and can be removed by the Levi extension theorem. If the original function on $\Omega \backslash K$ is holomorphic, the extension on $\widehat{\Omega}$ is so too, and its push-forward to $\Omega$ is weakly holomorphic. The proof of Theorem 2.4 is complete.

Lemma 5.3. An (n-1)-convex complex space $X$ of pure dimension $n$ cannot contain any analytic hypersurface $Y$ which is compact.

Proof. Let $\rho$ be an $(n-1)$-convex exhaustion function. Let $\left(U_{j}\right)_{j \in J}$ be a locally finite covering of $X$ by open subsets which can be embedded onto analytic subsets $\mathrm{A}_{j}$ of euclidean domains $\widetilde{\mathrm{B}}_{j} \subset \mathbb{C}^{N_{j}}$ such that the pushforward of $\rho$ extends as an $(n-1)$-convex function $\widetilde{\rho}_{j} \in \mathscr{C}{ }^{\infty}\left(\widetilde{\mathrm{B}}_{j}\right)$. By an inductive deformation of $\rho$, we may arrange that all $\widetilde{\rho}_{j}$ can be chosen to be Morse functions without critical points on $\mathrm{A}_{j}$.

If there is a compact analytic hypersurface $Y \subset X$, then $\left.\rho\right|_{Y}$ attains a global maximum at some point $z_{0} \in Y$. We can assume that $z_{0}$ lies in some ball $\widetilde{\mathrm{B}}_{j}$, we denote by $\mathrm{E}_{j} \subset \mathrm{~A}_{j} \subset \widetilde{\mathrm{~B}}_{j} \subset \mathbb{C}^{N_{j}}$ the local representative of $Y$ and we drop the index $j$, because the rest of the argument is local. By
construction $\left\{z: \widetilde{\rho}(z)=\widetilde{\rho}\left(z_{0}\right)\right\}$ is a smooth $(n-1)$-convex real hypersurface such that $\mathrm{E} \subset\left\{\widetilde{\rho} \leqslant \widetilde{\rho}\left(z_{0}\right)\right\}$. Bending this hypersurface a little, we can arrange that E is in fact contained in $\left\{\widetilde{\rho}<\widetilde{\rho}\left(z_{0}\right)\right\} \cup\left\{z_{0}\right\}$ near $z_{0}$. By $(n-1)$ convexity of $\widetilde{\rho}$, there is a piece $\Lambda$ of a small $(N-n+1)$-dimensional complex plane passing through $z_{0}$ and contained in the complex tangent plane $T_{z_{0}}^{c}\left\{\widetilde{\rho}=\widetilde{\rho}\left(z_{0}\right)\right\}$ on which the Levi-form $i \partial \bar{\partial} \widetilde{\rho}$ is positive. Thus $\Lambda$ is contained in $\left\{\widetilde{\rho}>\widetilde{\rho}\left(z_{0}\right)\right\} \cup\left\{z_{0}\right\}$ and has a contact of order exactly two with $\left\{\widetilde{\rho}=\widetilde{\rho}\left(z_{0}\right)\right\}$ at $z_{0}$. Furthermore, if we pick a nonzero vector $v \in T_{z_{0}} \mathbb{C}^{N}$ which points into $\left\{\rho>\rho\left(z_{0}\right)\right\}$, the translates $\Lambda_{\epsilon}:=\Lambda+\varepsilon v$ do all lie in $\left\{\rho>\rho\left(z_{0}\right)\right\}$ for every small $\varepsilon>0$, whence $\Lambda_{\varepsilon} \cap \mathrm{E}$ is empty. But given that $\Lambda_{0} \cap Y=\left\{z_{0}\right\} \neq \emptyset$, this contradicts the persistence, under perturbation, of the intersection of two complex analytic sets of complementary dimensions in $\mathbb{C}^{N}$. The lemma is proved.

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# On wedge extendability of CR-meromorphic functions 

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#### Abstract

Performing local extension from In this article, we consider metrically thin singularities $E$ of the solutions of the tangential Cauchy-Riemann operators on a $\mathscr{C}^{2, \alpha}$-smooth embedded Cauchy-Riemann generic manifold $M$ (CR functions on $M \backslash E$ ) and more generally, we consider holomorphic functions defined in wedgelike domains attached to $M \backslash E$. Our main result establishes the wedge- and the $L^{1}$ removability of $E$ under the hypothesis that the (dim $M-2$ )-dimensional Hausdorff volume of $E$ is zero and that $M$ and $M \backslash E$ are globally minimal. As an application, we deduce that there exists a wedgelike domain attached to an everywhere locally minimal $M$ to which every CR-meromorphic function on $M$ extends meromorphically.


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## 1. Introduction and statement of results

In continuation with our previous works [MP1,2,3], we study the wedge removability of metrically thin singularities of CR functions and its application to the local extendability of CR-meromorphic functions defined on CR manifolds of arbitrary codimension.

First we need to recall some fundamental notions concerning CR manifolds. For a detailed presentation we refer to [Bo]. Let $M$ be a connected smooth CR generic manifold in $\mathbb{C}^{m+n}$ with CRdim $M=m \geq 1$, $\operatorname{codim}_{\mathbb{R}} M=n \geq 1$, and $\operatorname{dim}_{\mathbb{R}} M=2 m+n$. We denote sometimes $N:=m+n$. In suitable holomorphic coordinates $(w, z=x+i y) \in \mathbb{C}^{m+n}$, $M$ may be represented as the graph of a differentiable vector-valued mapping in the form $x=h(w, y)$ with $h(0)=0, d h(0)=0$. The manifold $M$ is called globally minimal if it consists of a single CR orbit. This notion generalizes the concept of local minimality in the sense of Tumanov, cf. [Trp], [Tu1,2], [J1,2], [M], [MP1]. A wedge $\mathscr{W}$ with edge $M^{\prime} \subset M$ is a set of the form $\mathscr{W}=\left\{p+c: p \in M^{\prime}, c \in C\right\}$, where $C \subset \mathbb{C}^{m+n}$ is a truncated open cone with vertex in the origin. By a wedgelike domain $\mathscr{W}$ attached to $M$ we mean a domain which contains for every point $p \in M$ a wedge with edge a neighborhood of $p$ in $M$ ( $c f$. [MP1,2,3]).

A closed subset $E$ of $M$ is called wedge removable (briefly $\mathscr{W}$-removable) if for every wedgelike domain $\mathscr{W}_{1}$ attached to $M \backslash E$, there is a wedgelike domain $\mathscr{W}_{2}$ attached to $M$ such that for every holomorphic function $f \in$ $\mathscr{O}\left(\mathscr{W}_{1}\right)$, there exists a holomorphic function $F \in \mathscr{O}\left(\mathscr{W}_{2}\right)$ which coincides with $f$ in some wedgelike open set $\mathscr{W}_{3} \subset \mathscr{W}_{1}$ attached to $M \backslash E$. We say that $E$ is $L^{1}$-removable if every locally integrable function $f$ on $M$ which is CR on $M \backslash E$ is CR on all of $M$ (here, CR is understood in the distributional sense).

Let $H^{\kappa}$ denote $\kappa$-dimensional Hausdorff measure, $\kappa \geq 0$. Our main result is :

Theorem 2.4. Suppose $M$ is $\mathscr{C}^{2, \alpha}$-smooth, $0<\alpha<1$. Then every closed subset $E$ of $M$ such that $M$ and $M \backslash E$ are globally minimal and such that $H^{2 m+n-2}(E)=0$ is $\mathscr{W}$ - and $L^{1}$-removable.
(We shall say sometimes that $E$ is of codimension $2^{+0}$ in M.) The hypersurface case of this statement follows from works of Lupacciolu, Stout, Chirka and others, with weaker regularity assumptions, $M$ being $\mathscr{C}^{2}$ smooth, $\mathscr{C}^{1}$-smooth or even a Lipschitz graph (see [LS], [CS]), so Theorem 1.1 is new essentially in codimension $n \geq 2$. Recently, many geometrical removability results have been established in case the singularity $E$ is a submanifold (see [St], [LS], [CS], [J2,3], [P1], [MP1,2,3], [JS], [P2], [MP4]) and Theorem 1.1 appears to answer one of the last open general questions in the subject (see also [J3], [MP4] for related open problems). As a rule $L^{1}$ removability follows once $\mathscr{W}$-removability being established (see especially Proposition 2.11 in [MP1]). In the case at hand we have already proved $L^{1}$ removability by different methods earlier (Theorem 3.1 in [MP3]) and also $\mathscr{W}$-removability if $M$ is real analytic (see [MP2, Theorem 5.1], with $M$ being $\mathscr{C}^{\omega}$-smooth and $H^{2 m+n-2}(E)=0$ ).

For the special case where $M$ is $\mathscr{C}^{3}$-smooth and Levi-nondegenerate (i.e. the convex hull of the image of the Levi-form has nonempty interior), Theorem 1.1 is due to Dinh and Sarkis [DS]. It is known that this assumption entails the dimensional inequality $m^{2} \geq n$. Especially, in the case of CR dimension $m=1$, the abovementioned authors recover only the known hypersurface case $(n=1)$. We also point out a general restriction: by assuming that $M$ is Levi-nondegenerate, or more generally that it is of Bloom-Graham finite type at every point of $M$, one would not take account of propagation aspects for the regularity of CR functions. For instance, it is well known that wedge extendability may hold despite of large Levi-flat regions in manifolds $M$ consisting of a single CR orbit (cf. [Trp], [Tu1,2], [J1], [M]). For the sake of generality, this is why we only assume that $M$ and $M \backslash E$ are globally minimal in Theorem 1.1.

A straightforward application is as follows. First, by [Trp], [Tu1,2], [J1], [ M , Theorem 3.4], as $M \backslash E$ is globally minimal, there is a wedgelike domain $\mathscr{W}_{0}$ attached to $M \backslash E$ to which every continuous CR function (resp. CR distribution) $f$ on $M \backslash E$ extends as a holomorphic function with continuous (resp. distributional) boundary value $f$. Then Theorem 1.1 entails that there exists a wedge $\mathscr{W}$ attached to $M$ such that every such $f$ extend holomorphically as an $F \in \mathscr{O}(\mathscr{W})$. There is a priori no growth control of $F$ up to $E$. However, as proved in [MP1, Proposition 2.11], in the case where $f$ is an element of $L^{1}(M)$ which is CR on $M \backslash E$, some growth control of Hardy-spaces type can be achieved on $F$ to show that it admits a boundary value $b(F)$ over $M$ (including $E$ ) which is $L^{1}$ and CR on $M$. This is how one may deduce $L^{1}$-removability from $\mathscr{W}$-removability in Theorem 1.1.

We now indicate a second application of Theorem 1.1 to the extension of CR-meromorphic functions. This notion was introduced for hypersurfaces by Harvey and Lawson [HL] and for generic CR manifolds by Dinh and Sarkis. Let $f$ be a CR-meromorphic function, namely: 1. $f: \mathscr{D}_{f} \rightarrow P_{1}(\mathbb{C})$ is a $\mathscr{C}^{1}$-smooth mapping defined over a dense open subset $\mathscr{D}_{f}$ of $M$ with values in the Riemann sphere; 2. The closure $\Gamma_{f}$ of its graph in $\mathbb{C}^{m+n} \times P_{1}(\mathbb{C})$ defines an oriented scarred $\mathscr{C}^{1}$-smooth CR manifold of CR dimension $m$ (i.e. CR outside a closed thin set) and 3. We assume that $d\left[\Gamma_{f}\right]=0$ in the sense of currents (see [HL], [Sa], [DS], [MP2] for further definition). According to an observation of Sarkis based on a counting dimension argument, the indeterminacy set $\Sigma_{f}$ of $f$ is a closed subset of empty interior in a two-codimensional scarred submanifold of $M$ and its scar set is always metrically thin : $H^{2 m+n-2}\left(S c\left(\Sigma_{f}\right)\right)=0$. Moreover, outside $\Sigma_{f}, f$ defines a CR current in some suitable projective chart, hence it enjoys all the extendability properties of an usual CR function or distribution. However, the complement $M \backslash \Sigma_{f}$ need not be globally minimal if $M$ is, and it is easy to construct manifolds $M$ and closed sets $E \subset M$ with $H^{2 m-1}(E)<\infty$ ( $m=\operatorname{dim}_{C R} M$ ) which perturb global minimality (see [MP1], p. 811). It is therefore natural to make the additional assumption that $M$ is minimal (locally, in the sense of Tumanov) at every point, which seems to be the weakest assumption which insures that $M \backslash E$ is globally minimal for arbitrary closed sets $E \subset M$ (even with a bound on their Hausdorff dimension). Finally, under these circumstances, the set $\Sigma_{f}$ will be $\mathscr{W}$-removable: for its regular part $\operatorname{Reg}\left(\Sigma_{f}\right)$, this already follows from Theorem 4 (ii) in [MP1] and for its scar set $S c\left(\Sigma_{f}\right)$, this follows from Theorem 1.1 above. The removability of $\Sigma_{f}$ means that the envelope of holomorphy of every wedge $\mathscr{W}_{1}$ attached to $M \backslash \Sigma_{f}$ contains a wedge $\mathscr{W}_{2}$ attached to $M$. As envelopes of meromorphy and envelopes of holomorphy of domains in $\mathbb{C}^{m+n}$ coincide by a theorem of Ivashkovich ([I]), we conclude :

Theorem 2.4. Suppose $M$ is $\mathscr{C}^{2, \alpha}$-smooth and locally minimal at every point. Then there exists a wedgelike domain $\mathscr{W}$ attached to $M$ to which all CR-meromorphic functions on $M$ extend meromorphically.

The remainder of the paper is devoted to the proof of Theorem 1.1. We combine the local and the global techniques of deformations of analytic discs, using in an essential way two important papers of Tumanov [Tu1] and of Globevnik [G1]. In Sections 2 and 3, we first set up a standard local situation (cf. [MP1,2,3]). These preliminaries provide the necessary background for an informal discussion of the techniques of deformations of analytic discs we have to introduce. After these motivating remarks, a detailed presentation of the main part of the proof is provided in Section 4 (see especially Main Lemma 4.3).
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## 2. Localization

The following section contains important preliminary steps for the proof of Theorem 1.1 ( $c f$. [MP1,2,3]).

As in [CS, p.96], we shall proceed by contradiction, since this strategy simplifies the general reasonings in the large. Also, in Section 3 below, we shall explain how to reduce the question to the simpler case where the functions which we have to extend are even holomorphic near $M \backslash E$. Whereas such a strategy is carried out in detail in [MP1] (with minor variations), we shall for completeness recall the complete reasonings briefly here, in Sections 2 and 3.

Thus, we fix $\mathscr{W}_{1}$ attached to $M \backslash E$ and say that an open submanifold $M^{\prime} \subset$ $M$ containing $M \backslash E$ enjoys the $\mathscr{W}$-extension property if there is a wedgelike domain $\mathscr{W}^{\prime}$ attached to $M^{\prime}$ and a wedgelike set $\mathscr{W}_{1}^{\prime} \subset \mathscr{W}^{\prime} \cap \mathscr{W}_{1}$ attached to $M \backslash E$ such that, for each function $f \in \mathscr{O}\left(\mathscr{W}_{1}\right)$, its restriction to $\mathscr{W}_{1}^{\prime}$ extends holomorphically to $\mathscr{W}^{\prime}$.

This notion can be localized as follows. Let $E^{\prime} \subset E$ be an arbitrary closed subset of $E$. We shall say that a point $p^{\prime} \in E^{\prime}$ is (locally) removable (with respect to $E^{\prime}$ ) if for every wedgelike domain $\mathscr{W}_{1}$ attached to $M \backslash E^{\prime}$, there exists a neighborhood $U$ of $p^{\prime}$ in $M$ and a wedgelike domain $\mathscr{W}_{2}$ attached to $\left(M \backslash E^{\prime}\right) \cup U$ such that for every holomorphic function $f \in \mathscr{O}\left(\mathscr{W}_{1}\right)$, there exists a holomorphic function $F \in \mathscr{O}\left(\mathscr{W}_{2}\right)$ which coincides with $f$ in some wedgelike open set $\mathscr{W}_{3} \subset \mathscr{W}_{1}$ attached to $M \backslash E^{\prime}$.

Next, we define the following set of closed subsets of $E$ :

$$
\begin{aligned}
\mathscr{E}:=\left\{E^{\prime} \subset E \text { closed } ; M \backslash E^{\prime}\right. & \text { is globally minimal } \\
& \text { and has the } \mathscr{W} \text {-extension property }\} .
\end{aligned}
$$

Then the residual set

$$
E_{\mathrm{nr}}:=\bigcap_{E^{\prime} \in \mathscr{E}} E^{\prime}
$$

is closed. Here, the letters "nr" abbreviate "non-removable", since one expects a priori that no point of $E_{\mathrm{nr}}$ should be removable in the above sense. Notice that for any two sets $E_{1}^{\prime}, E_{2}^{\prime} \in \mathscr{E}, M \backslash E_{1}^{\prime}$ and $M \backslash E_{2}^{\prime}$ consist of a single CR orbit and have nonempty intersection. Hence $\left(M \backslash E_{1}^{\prime}\right) \cup\left(M \backslash E_{2}^{\prime}\right)$ is globally minimal and it follows that $M \backslash E_{\mathrm{nr}}$ is globally minimal.

Using Ayrapetian's version of the edge of the wedge theorem (see also [Tu1, Theorem 1.2]), the different wedgelike domains attached to the sets $M \backslash E^{\prime}$ can be glued (after appropriate contraction of their cone) to a wedgelike domain $\mathscr{W}_{1}$ attached to $M \backslash E_{\text {nr }}$ in such a way that $M \backslash E_{\text {nr }}$ enjoys the $\mathscr{W}$-extension property. Clearly, to establish Theorem 1.1, it is enough to show that $E_{\mathrm{nr}}=\emptyset$.

Let us argue indirectly (by contradiction) and assume that $E_{\mathrm{nr}} \neq \emptyset$. With respect to the ordering of sets by the inclusion relation, $E_{\mathrm{nr}}$ is then the minimal non-removable subset of $E$. In order to derive a contradiction to the minimality of $E_{\mathrm{nr}}$, it suffices therefore to remove one single point $p \in E_{\mathrm{nr}}$. More precisely one has to look for a neighborhood $U_{p}$ of $p$ such that $U_{p} \cup\left(M \backslash E_{\text {nr }}\right)$ is globally minimal and has the $\mathscr{W}$-extension property.

In order to achieve the first required property, it is very convenient to choose the point $p$ such that locally the singularity $E_{\text {nr }}$ lies behind a "wall" through $p$. More precisely we shall construct a generic hypersurface $M_{1} \subset$ $M$ containing $p$ such that a neighborhood $V$ of $p$ in $M$ writes as the disjoint union $M^{+} \cup M^{-} \cup M_{1}$ of connected sets, where $M^{ \pm}$are two open "sides", and the inclusion $E_{\mathrm{nr}} \cap V \subset M^{-} \cup\{p\}$ holds true. Since $M_{1}$ is a generic CR manifold, there is a CR vector field $X$ on $M$ defined in a neighborhood of $p$ which is transverse to $M_{1}$. By integrating $X$, one easily finds a basis of neighborhoods $U$ of $p$ in $M$ such that $U \cup\left(M \backslash E_{\text {nr }}\right)$ is globally minimal. Hence it remains to establish the $\mathscr{W}$-extension property at $p$, which is the main task.

For sake of completeness, we recall from [MP1] how to construct the generic wall $M_{1}$.

Lemma 5.3. There is a point $p_{1} \in E_{\text {nr }}$ and a $\mathscr{C}^{2, \alpha}$-smooth generic hypersurface $M_{1} \subset M$ passing through $p_{1}$ so that $E_{\mathrm{nr}} \backslash\left\{p_{1}\right\}$ lies near $p_{1}$ on one side of $M_{1}$ (see Figure 1).

Proof. Let $p \in E_{\mathrm{nr}} \neq \emptyset$ be an arbitrary point and let $\gamma$ be a piecewise differentiable CR-curve linking $p$ with a point $q \in M \backslash E_{\text {nr }}$ (such a $\gamma$ exists because $M$ and $M \backslash E_{\mathrm{nr}}$ are globally minimal by assumption). After shortening $\gamma$, we may suppose that $\{p\}=E_{\mathrm{nr}} \cap \gamma$ and that $\gamma$ is a smoothly embedded segment. Therefore $\gamma$ can be described as a part of an integral curve of some nonvanishing $\mathscr{C}^{1, \alpha}$-smooth CR vector field (section of $T^{c} M$ ) $L$ defined in a neighborhood of $p$.


Let $H \subset M$ be a small ( $\operatorname{dim} M-1$ )-dimensional hypersurface of class $\mathscr{C}^{2, \alpha}$ passing through $p$ and transverse to $L$. Integrating $L$ with initial values in $H$ we obtain $\mathscr{C}^{1, \alpha}$-smooth coordinates $(t, s) \in \mathbb{R} \times \mathbb{R}^{\operatorname{dim} M-1}$ so that for fixed $s_{0}$, the segments $\left(t, s_{0}\right)$ are contained in the trajectories of $L$. After a translation, we may assume that $(0,0)$ corresponds to a point of $\gamma$ close to $p$ which is not contained in $E_{\mathrm{nr}}$, again denoted by $q$. Fix a small $\varepsilon>0$ and for real $\tau \geq 1$, define the ellipsoids (see Figure 1 above)

$$
Q_{\tau}:=\left\{(t, s):|t|^{2} / \tau+|s|^{2}<\varepsilon\right\} .
$$

There is a minimal $\tau_{1}>1$ with $\overline{Q_{\tau_{1}}} \cap E_{\mathrm{nr}} \neq \emptyset$. Then $\overline{Q_{\tau_{1}}} \cap E_{\mathrm{nr}}=\partial Q_{\tau_{1}} \cap E_{\mathrm{nr}}$ and $Q_{\tau_{1}} \cap E_{\mathrm{nr}}=\emptyset$. Observe that every $\partial Q_{\tau}$ is transverse to the trajectories of $L$ out off the equatorial set $\Upsilon:=\left\{(0, s):|s|^{2}=\varepsilon\right\}$ which is contained in $M \backslash E_{\mathrm{nr}}$. Hence $\partial Q_{\tau_{1}}$ is transverse to $L$ in all points of $\partial Q_{\tau_{1}} \cap E_{\mathrm{nr}}$. So $\partial Q_{\tau_{1}} \backslash \Upsilon$ is generic in $\mathbb{C}^{m+n}$, since $L$ is a CR field.

We could for instance choose a point $p_{1} \in \partial Q_{\tau_{1}} \cap E_{\text {nr }}$ and take for $M_{1}$ a neighborhood of $p_{1}$ in $\partial Q_{\tau_{1}}$, but such an $M_{1}$ would be only of class $\mathscr{C}^{1, \alpha}$ and we want $\mathscr{C}^{2, \alpha}$-smoothness.

Therefore we fix a small $\delta>0$ and approximate the family $\partial Q_{\tau}, 1 \leq$ $\tau<\tau_{1}+\delta$, by a nearby family of $\mathscr{C}^{2, \alpha}$-smooth hypersurfaces $\partial \widetilde{Q}_{\tau}, 1 \leq$ $\tau<\tau_{1}+\delta$. Clearly this can be done so that the $\partial \widetilde{Q}_{\tau}$ are still boundaries of
increasing domains $\widetilde{Q}_{\tau}$ of approximately the same size as $Q_{\tau}$ and so that the points where the $\partial \widetilde{Q}_{\tau}$ are tangent to $L$ are also contained in $M \backslash E_{\text {nr }}$ near the equator $\Upsilon$ of $Q_{\tau}$.

The same reasoning as above shows that there exist a real number $\widetilde{\tau}_{1}>1$, a point $p_{1} \in E_{\mathrm{nr}}$ and a generic hypersurface $M_{0}$ passing through $p_{1}$ (which is a piece of $\partial \widetilde{Q}_{\widetilde{\tau}_{1}}$ ) such that $E_{\text {nr }}$ lies in the left closed side $M_{0}^{-} \cup M_{0}$ in a neighborhood of $p_{1}$ (see Figure 1). We want more: $E_{\text {nr }} \backslash\left\{p_{1}\right\} \subset M_{1}^{-}$. To achieve this last condition, it suffices to choose a $\mathscr{C}^{2, \alpha}$-smooth hypersurface $M_{1}$ passing through $p_{1}$ with $T_{p_{1}} M_{0}=T_{p_{1}} M_{1}$ such that $M_{1} \backslash\left\{p_{1}\right\}$ is contained in $M_{0}^{+}$.

## 3. Analytic discs

Let $p_{1}$ be as in Lemma 2.1. First, we can choose coordinates vanishing at $p_{1}$ and represent $M$ near $p_{1}$ by the vectorial equation

$$
\begin{equation*}
x=h(w, y), \quad w \in \mathbb{C}^{m}, \quad z=x+i y \in \mathbb{C}^{n} \tag{3.1}
\end{equation*}
$$

where $h=\left(h_{1}, \ldots, h_{n}\right)$ is of class $\mathscr{C}^{2, \alpha}$ and satisfies $h_{j}(0)=0$ and $d h_{j}(0)=0$.

Let us recall some generalities (see [Bo] for background). Denote by $\Delta$ the open unit disc in $\mathbb{C}$. An analytic disc attached to $M$ is a holomorphic mapping $A: \Delta \rightarrow \mathbb{C}^{N}$ which extends continuously (or $\mathscr{C}^{k, \alpha}$-smoothly) up to the boundary $\partial \Delta$ and fulfills $A(\partial \Delta) \subset M$.

Discs of small size (for example with respect to the $\mathscr{C}^{2, \alpha}$-norm, $0<\alpha<$ 1) which are attached to $M$ are then obtained as the solutions of the (modified) Bishop equation

$$
\begin{equation*}
Y=T_{1}[h(W(\cdot), Y(\cdot))]+y_{0} \tag{3.2}
\end{equation*}
$$

where $T_{1}$ denotes the harmonic conjugate operator (Hilbert transform on $\partial \Delta$ ) normalized at $\zeta=1$, namely satisfying $T_{1} u(1)=0$ for any $u \in$ $\mathscr{C}^{2, \alpha}\left(b \Delta, \mathbb{R}^{n}\right)$. One verifies that every small $\mathscr{C}^{2, \alpha}$-smooth disc $A(\zeta)=$ $(W(\zeta), Z(\zeta))=(W(\zeta), X(\zeta)+i Y(\zeta))$ attached to $M$ satisfies (3.2). Conversely, for $W(\zeta)$ of small $\mathscr{C}^{2, \alpha}$-norm, equation (3.2) possesses a unique solution $Y(\zeta)$, and one easily checks that $A(\zeta):=(W(\zeta), h(W(\zeta), Y(\zeta))+$ $i Y(\zeta))$ is then the unique disc attached to $M$ with $Y(1)=y_{0}$ and $w$ component equal to $W(\zeta)$. According to an optimal analysis of the regularity of Bishop's equation due to Tumanov [Tu2] (and valid more generally in the classes $\mathscr{C}^{k, \alpha}$ for $k \geq 1$ and $\left.0<\alpha<1\right), Y(\zeta)$ and then $A(\zeta)$ are of class $\mathscr{C}^{2, \alpha}$ over $\bar{\Delta}$.

After a linear transformation we can assume that the tangent space to $M_{1}$ is given by $\left\{x=0, u_{1}=0\right\}$ and that $T_{0} M^{+}$is given by $\left\{u_{1}>0\right\}$ near the origin. Let $\rho_{0}>0$ be small and let $A$ be the analytic disc we obtain by
solving

$$
\begin{equation*}
Y=T_{1} h[(W(\cdot), Y(\cdot))], \quad \text { with } \quad W(\zeta):=\left(\rho_{0}-\rho_{0} \zeta, 0, \ldots, 0\right) \tag{3.3}
\end{equation*}
$$

Notice that the disc $W_{1}(\zeta):=\left(\rho_{0}-\rho_{0} \zeta\right)$ satisfies $W_{1}(1)=0$ and $W_{1}(\bar{\Delta} \backslash\{1\}) \subset\left\{u_{1}+i v_{1} \in \mathbb{C}: u_{1}>0\right\}$. Elementary properties of Bishop's equation yield $A(\partial \Delta) \backslash\{1\} \subset M^{+}$if $\rho_{0}>0$ is sufficiently small (cf. [MP1, Lemma 2.4]). Figure 2 below is devoted to provide a geometric intuition of the relative situation of the boundary of the disc $A$ with respect to $M_{1}$.


At first, we explain how one usually constructs small wedges attached to $M$ at $p_{1}$ by means of deformations of analytic discs and then in Sections 4, 5 and 6 below, we shall explain some of the modifications which are needed in the presence of a singularity $E_{\mathrm{nr}}$ in order to produce wedge extension at $p_{1}$. Following [MP3, pp. 863-864], we shall include (or say, "deform") $A$ in a parametrized family $A_{\rho, s, v}$ with varying radius $\rho$ plus supplementary parameters $s, v$ and with $A_{\rho_{0}, 0,0}=A$. During the construction, we shall sometimes permit ourselves to decrease parameters, related constants, neighborhoods and domains of existence without explicit mentioning. At present, our goal is to explain how we can add some conveninent extra simplifying assumptions to the hypotheses of Theorem 1.1, see especially conditions 1), 2) and 3) before Theorem 3.1 below.

Let $\mathscr{W}_{1}$ be the wedgelike domain attached to $M \backslash E_{\text {nr }}$ constructed in Section 2 and let $f \in \mathscr{O}\left(\mathscr{W}_{1}\right)$. We want to extend $f$ holomorphically to a wedge of edge a small neighborhood of the special point $p_{1} \in E_{\text {nr }}$ picked thanks to Lemma 2.1. Let $\mathscr{W}_{2} \subset \mathscr{W}_{1}$ be a small wedge attached to a neighborhood
of $A(-1)$ in $M^{+}$. As in [Tu1,2], [MP1,3], we can construct analytic discs $A_{\rho, s, v}=\left(W_{\rho, s, v}, Z_{\rho, s, v}\right)$ attached to $M \cup \mathscr{W}_{2}$ with the following properties:
(1) The parameters $s, v$ belong to neighborhoods $U_{s}, U_{v}$ of 0 in $\mathbb{R}^{2 m+n-1}, \mathbb{R}^{n-1}$ respectively and $\rho$ belongs to the interval $\left[0, \rho_{1}\right)$, for some $\rho_{1}>\rho_{0}$.
(2) The mapping $(\rho, s, v) \mapsto A_{\rho, s, v}$ is of class $\mathscr{C}^{2, \beta}$ for all $0<\beta<\alpha$. For $\rho \neq 0$, these maps are embeddings of $\bar{\Delta}$ into $\mathbb{C}^{m+n}$. Finally, we have $A_{\rho_{0}, 0,0}=A$ and the discs $A_{0, s, v}$ are constant.
(3) For every fixed $v_{0} \in U_{v}$, the union $\bigcup_{s \in U_{s}} A_{\rho_{0}, s, v_{0}}\left(\left\{e^{i \theta}:|\theta|<\pi / 4\right\}\right)$ is an open subset of $M$ containing the origin which is $\mathscr{C}^{2, \beta}$-smoothly foliated by the curves $A_{\rho_{0}, s, v_{0}}\left(\left\{e^{i \theta}:|\theta|<\pi / 4\right\}\right)$.
(4) The mapping $U_{v} \ni v \mapsto\left[\frac{d}{d \theta} A_{\rho_{0}, 0, v}\left(e^{i \theta}\right)\right]_{\theta=0} \in T_{0} M / T_{0}^{c} M \simeq \mathbb{R}^{n}$ has rank $n-1$ and its image is transverse to the vector $\left[\frac{d}{d \theta} A\left(e^{i \theta}\right)\right]_{\theta=0} \in$ $T_{0} M / T_{0}^{c} M \simeq \mathbb{R}^{n}$. In geometric terms, this property means that the union of tangent real lines

$$
\mathbb{R}\left[\frac{d}{d r} A_{\rho_{0}, 0, v}\left(r e^{i \theta}\right)\right]_{\zeta=1}=-i \mathbb{R}\left[\frac{d}{d \theta} A_{\rho_{0}, 0, v}\left(e^{i \theta}\right)\right]_{\theta=0}
$$

spans an open cone in the normal bundle to $M$, namely $T_{0} \mathbb{C}^{m+n} / T_{0} M \cong i\left(T_{0} M / T_{0}^{c} M\right)$.
(5) Let $\omega=\{\zeta \in \Delta:|\zeta-1|<\delta\}$ be a neighborhood of 1 in $\Delta$, with some small $\delta>0$. It follows from properties (3) and (4) that the union $\mathscr{W}=\bigcup_{s \in U_{s}, v \in U_{v}} A_{\rho_{0}, s, v}(\omega)$ is an open wedge of edge a neighborhood of the origin in $M$ which is foliated by the discs $A_{\rho_{0}, s, v}(\omega)$.
(6) The sets $D_{s, v}=\bigcup_{0 \leq \rho<\rho_{1},|\zeta|=1} A_{\rho, s, v}(\zeta)$ are real two-dimensional discs of class $\mathscr{C}^{2, \beta}$ embedded in $M$ which are foliated (with a circle degenerating to a point for $\rho=0$ ) by the circles $A_{\rho, s, v}(\partial \Delta)$.
(7) There exists a $(2 m+n-2)$-dimensional submanifold $H$ of $\mathbb{R}^{2 m+n-1}$ passing through the origin such that for every fixed $v_{0} \in U_{v}$, the union $\bigcup_{s \in H} D_{s, v_{0}}$ is a ( $\operatorname{dim} M$ )-dimensional open box foliated by real 2-discs which is contained in $M$ and which contains the origin. Intruitively, it is a stack of plates.

Let us make some commentaries. We stress that the family $A_{\rho, s, v}$ is obtained by solving the Bishop equation for explicitly prescribed data (see [MP3, p. 837] or [MP1, p. 863]; the important Lemma 2.7 in [MP1] which produces the parameter $v$ satisfying (4) above is due to Tumanov [Tu1]). Since Bishop's equation is very flexible, this entails that every geometrical property of the family is stable under slight perturbation of the data. Notice for instance that as $A$ is an embedding of $\bar{\Delta}$ into $\mathbb{C}^{m+n}$, all its small deformations will stay embeddings. In particular we get a likewise family $A_{\rho, s, v}^{d}$
if we replace $M$ by a slightly deformed $\mathscr{C}^{2, \alpha}$-smooth manifold $M^{d}$ (this corresponds to replacing $h$ by a function $h^{d}$ close to $h$ in $\mathscr{C}^{2, \alpha}$-norm in (3.1), (3.2) and (3.3)).

Further remark. If $A^{\prime}$ is an arbitrary disc which is sufficiently close to $A$ in $\mathscr{C}^{1, \beta}$-norm, for some $0<\beta<\alpha$, we can also include $A^{\prime}$ in a similar $\mathscr{C}^{1, \gamma_{-}}$ smooth $(0<\gamma<\beta)$ family $A_{\rho, s}^{\prime}$, without the parameter $v$, which satisfies the geometric properties (3), (6) and (7) above. This remark will be useful in the end of Section 4 below.

Using such a nice family $A_{\rho, s, v}$ which gently deforms as a family $A_{\rho, s, v}^{d}$ under perturbations, let us begin to remind from [MP1] how we can add three simplifying geometric assumptions to Theorem 1.1, without loss of generality.

First of all, using a partition of unity, we can perform arbitrarily small $\mathscr{C}^{2, \alpha}$-smooth deformations $M^{d}$ of $M$ leaving $E_{\mathrm{nr}}$ fixed and moving $M \backslash E_{\mathrm{nr}}$ inside the wedgelike domain $\mathscr{W}_{1}$. Further, we can make $M^{d}$ to depend on a single small real parameter $d \geq 0$ with $M^{0}=M$ and $M^{d} \backslash E_{\mathrm{nr}} \subset \mathscr{W}_{1}$ for all $d>0$. Now, the wedgelike domain $\mathscr{W}_{1}$ becomes a neighborhood of $M^{d}$ in $\mathbb{C}^{m+n}$. In the sequel, we shall denote this neighborhood by $\Omega$. By stability of Bishop's equation, we obtain a deformed disc $A^{d}$ attached to $M^{d}$ by solving (3.3) with $h^{d}$ in place of $h$. In the sequel, we will also consider a small neighborhood $\Omega_{1}$ of $A^{d}(-1)$ in $\mathbb{C}^{m+n}$ which contains the intersection of the above wedge $\mathscr{W}_{2}$ with a neighborhood of $A(-1)$ in $\mathbb{C}^{m+n}$.

Again by stability of Bishop's equation, we also obtain deformed families $A_{\rho, s, v}^{d}$ attached to $M^{d} \cup \Omega_{1}$, satisfying properties (1)-(7) above. Recall that according to [Tu2], the mapping $(\rho, s, v, d) \mapsto A_{\rho, s, v}^{d}$ is $\mathscr{C}^{2, \beta}$-smooth for all $0<\beta<\alpha$. In the core of the proof of our main Theorem 1.1 (Sections 4,5 and 6 below), we will show that, for each sufficiently small fixed $d>0$, we get holomorphic extension to the wedgelike set $\mathscr{W}^{d}=\bigcup_{s \in U_{s}, v \in U_{v}} A_{\rho_{0}, s, v}^{d}(\omega)$ attached to a neighborhood of 0 in $M^{d}$. But this implies Theorem 2.4: In the limit $d \rightarrow 0$, the wedges $\mathscr{W}^{d}$ tend smoothly to the wedge $\mathscr{W}:=\mathscr{W}^{0}$ attached to a neighborhood of 0 in $M^{0}=M$. As the construction depends smoothly on the deformations $d$, we derive univalent holomorphic extension to $\mathscr{W}$ thereby arriving at a contradiction to the definition of $E_{\mathrm{nr}}$.

As a summary of the above discussion, we formulate below the local statement that remains to prove. Essentially, we have shown that it suffices to prove Theorem 1.1 with the following three extra simplifying assumptions:

1) Instead of functions which are holomorphic in a wedgelike open set attached to $M \backslash E_{\mathrm{nr}}$, we consider functions which are holomorphic in a neighborhood of $M \backslash E_{\mathrm{nr}}$ in $\mathbb{C}^{m+n}$.
2) Proceeding by contradiction, we have argued that it suffices to remove at least one point of $E_{\mathrm{nr}}$.
3) Moreover, we can assume that the point $p_{1} \in E_{\mathrm{nr}}$ we want to remove is behind a generic "wall" $M_{1}$ as depicted in Figure 2.
Consequently, from now on, we shall denote the set $E_{\text {nr }}$ simply by $E$. We also denote $M^{d}$ simply by $M$. We take again the disc $A$ defined by (3.3) and its deformation $A_{\rho, s, v}$. The goal is now to show that holomorphic functions in a neighborhood of $M \backslash E$ in $\mathbb{C}^{m+n}$ extend holomorphically to a wedge at $p_{1}$, assuming the "nice" geometric situation of Figure 2. To be precise, we have argued that Theorem 1.1 is reduced to the following precise and geometrically more concrete statement.
Theorem 2.4. Let $M$ be a $\mathscr{C}^{2, \alpha}$-smooth generic CR manifold in $\mathbb{C}^{m+n}$ of codimension n. Let $M_{1} \subset M$ be a $\mathscr{C}^{2, \alpha}$-smooth generic $C R$ manifold of dimension $2 m+n-1$ and let $p_{1} \in M_{1}$. Let $M^{+}$and $M^{-}$denote the two local open sets in which $M$ is divided by $M_{1}$, in a neighborhood of $p_{1}$. Suppose that $E \subset M$ is a nonempty closed subset with $p_{1} \in E$ satisfying the Hausdorff condition $H^{2 m+n-2}(E)=0$ and suppose that $E \subset M^{-} \cup\left\{p_{1}\right\}$ (Figure 2). Let $\Omega$ be a neighborhood of $M \backslash E$ in $\mathbb{C}^{m+n}$, let $A$ be the disc defined by (3.3), let $\Omega_{1}$ be a neighborhood of $A(-1)$ in $\mathbb{C}^{m+n}$ which is contained in $\Omega$ and let $A_{\rho, s, v}$ be a family of discs attached to $M \cup \Omega_{1}$ with the properties (1)-(7) explained above. Then every function $f$ which is holomorphic in $\Omega$ extends holomorphically to the wedge $\mathscr{W}=\bigcup_{s \in U_{s}, v \in U_{v}} A_{\rho_{0}, s, v}(\omega)$.

Of course, Theorem 3.1 would be obvious if $E$ would be empty, but we have to take account of $E$.

## 4. Proof of Theorem 2.4 , part I

This section contains the part of the proof of Theorem 2.4 above which relies on constructions with the small discs $A_{\rho, s, v}$ attached to $M \cup \Omega$. Since we want the boundaries of our discs to avoid $E$, we shall employ the following elementary lemma several times, which is simply a convenient particularization of a general property of Hausdorff measures [C, Appendix A6].

Lemma 5.3. Let $N$ be a real d-dimensional manifold and let $E \subset N$ be a closed subset. Let $U$ be a small neighborhood of the origin in $\mathbb{R}^{d-1}$ and let $\Phi: \partial \Delta \times U \rightarrow N($ resp. $\Psi:(0,1) \times U \rightarrow M)$ be an embedding.
(i) If $H^{d-2}(E)=0$, then the set of $x \in U$ for which $\Phi(\partial \Delta \times\{x\}) \cap E$ is nonempty (resp. $\Psi((0,1) \times\{x\}) \cap E \neq \emptyset$ ) is of zero $(d-2)$ dimensional Hausdorff measure.
(ii) If $H^{d-1}(E)=0$, then for almost every $x \in U$ in the sense of Lebesgue measure, we have $\Phi(\partial \Delta \times\{x\}) \cap E=\emptyset$ (resp. $\Psi((0,1) \times\{x\}) \cap E=\emptyset)$.

Proof of Theorem 2.4: We divide the proof in five steps.

Step 1: Holomorphic extension to a dense subset of $\mathscr{W}$. We shall start by constructing a holomorphic extension to an everywhere dense open subdomain of the wedge $\mathscr{W}=\bigcup_{s \in U_{s}, v \in U_{v}} A_{\rho_{0}, s, v}(\omega)$ by means of the disc technique (continuity principle).

For each fixed $v_{0} \in U_{v}$, the first dimensional count of Lemma 4.1 (which applies by the foliation property (3) of the discs) yields a closed subset $\mathscr{S}_{v_{0}} \subset U_{s}$ depending on $v_{0}$ and satisfying $H^{2 m+n-2}\left(\mathscr{S}_{v_{0}}\right)=0$ such that for every $s \notin \mathscr{S}_{v_{0}}$ we have $A_{\rho_{0}, s, v_{0}}(\partial \Delta) \cap E=\emptyset$. Notice also that $\mathscr{S}_{v_{0}}$ does not locally disconnect $U_{s}$, for dimensional reasons ([C, Appendix A6]).

By property (7) of Section 3, the real two-dimensional discs $D_{s, v_{0}}$ foliate an open subset of $M$, for $s$ running in a manifold $H$ of dimension $2 m+n-2$. Consequently, for almost every $s \in H$, (in the sense of Lebesgue measure), we have $D_{s, v_{0}} \cap E=\emptyset$.


Since $E$ is closed, we claim that for every $s \notin \mathscr{S}_{v_{0}}$, it follows that we can contract every boundary $A_{\rho_{0}, s, v_{0}}(\partial \Delta)$ which does not meet $E$, to a point in $M$ without meeting $E$ by an analytic isotopy ( $c f$. [MP3, p. 864]). Indeed, by shifting $s$ to some nearby $s^{\prime}$, we first move $A_{\rho_{0}, s, v_{0}}$ into a disc $A_{\rho_{0}, s^{\prime}, v_{0}}$ which also satisfies $A_{\rho_{0}, s^{\prime}, v_{0}}(\partial \Delta) \cap E=\emptyset$. Choosing well $s^{\prime}$, this boundary belongs to a real disc $D_{s^{\prime}, v_{0}}$ satisfying $D_{s^{\prime}, v_{0}} \cap E=\emptyset$. This can be achieved with $s^{\prime}$ arbitrarily close to $s$, since $\mathscr{S}_{v_{0}}$ does not disconnect $U_{s}$. Then we contract in the obvious manner the disc $A_{\rho_{0}, s^{\prime}, v_{0}}$ to the point $A_{0, s^{\prime}, v_{0}}(\bar{\Delta})$ by isotoping its boundary inside $D_{s^{\prime}, v_{0}}$ (recall that $D_{s^{\prime}, v_{0}}$ is a union of boundary of discs). Applying the continuity principle to this analytic isotopy of discs, we see that we can extend every function $f \in \mathscr{O}(\Omega)$ holomorphically to a neighborhood of $A_{\rho_{0}, s, v_{0}}(\bar{\Delta})$ in $\mathbb{C}^{m+n}$, for every $s \notin \mathscr{S}_{v_{0}}$ and for every $v_{0} \in U_{v}$.

From the nice geometry (5) of the family $A_{\rho, s, v}$ one easily derives that the various local extensions near $A_{\rho_{0}, s, v_{0}}(\omega)$ for $s \notin \mathscr{S}_{v_{0}}$ fit in a univalent function $F \in \mathscr{O}\left(\mathscr{W} \backslash E_{\mathscr{W}}\right)$, where $E_{\mathscr{W}}:=\bigcup_{s \in \mathscr{S}_{v_{0}}, v_{0} \in U_{v}} A_{\rho_{0}, s, v_{0}}(\omega)$. Furthermore we observe that $E_{\mathscr{W}}$ is laminated by holomorphic discs and satisfies $H^{2 m+2 n-1}\left(E_{\mathscr{W}}\right)=0$. This metrical property implies that $\mathscr{W} \backslash E_{\mathscr{W}}$ is locally connected. The remainder of the proof is devoted to show how to extend $F$ through $E_{\mathscr{W}}$. This occupies the paper up to its end. The difficulty and the length of the proof comes from the fact that the disc method necessarily increases by a factor 1 the dimension of the singularity: it transforms a singularity set $E \subset M$ of codimension $2^{+0}$ into a bigger singularity set $E_{\mathscr{W}} \subset \mathscr{W}$ which is of codimension $1^{+0}$.

Step 2: Plan for the removal of $E_{\mathscr{W}}$. Let us remember that our goal is to show that $p_{1}$ is $\mathscr{W}$-removable in order to achieve the final step in our reasoning by contradiction which begins in Section 2. To show that $p_{1}$ is removable, it suffices to extend $F$ through $E_{\mathscr{W}}$. At first, we notice that because $H^{2 m+2 n-1}\left(E_{\mathscr{W}}\right)=0$, it follows that $\mathscr{W} \backslash E_{\mathscr{W}}$ is locally connected, so the part of the envelope of holomorphy of $\mathscr{W} \backslash E_{\mathscr{W}}$ which is contained in $\mathscr{W}$ is not multisheeted: it is necessarily a subdomain of $\mathscr{W}$. In analogy with the beginning of Section 2, let us therefore denote by $E_{\mathscr{W}}^{\mathrm{nr}}$ the set of points of $E_{\mathscr{W}}$ through which our holomorphic function $F \in \mathscr{O}\left(\mathscr{W} \backslash E_{\mathscr{W}}\right)$ does not extend holomorphically. If $E_{\mathscr{W}}^{\mathrm{nr}}$ is empty, we are done, gratuitously. As it might certainly be nonempty, we shall suppose therefore that $E_{\mathscr{W}}^{\mathrm{nr}} \neq \emptyset$ and we shall construct a contradiction in the remainder of the paper. Let $q \in E_{\mathscr{W}}^{\mathrm{nr}} \neq \emptyset$. To derive a contradiction, it suffices to show that $F$ extends holomorphically through $q$. Philosophically again, it will suffice to remove one single point, which will simplify the presentation and the geometric reasonings. Finally, as $E_{\mathscr{W}}^{\mathrm{nr}} \neq \emptyset$ is contained in $E_{\mathscr{W}}$, there exist a point $\zeta_{0} \in \partial \Delta$ and parameters ( $\rho_{0}, s_{0}, v_{0}$ ) such that $q=A_{\rho_{0}, s_{0}, v_{0}}\left(\zeta_{0}\right)$. In the sequel, we shall simply denote the disc $A_{\rho_{0}, s_{0}, v_{0}}$ by $A_{\mathrm{nr}}$. Obviously also, $H^{2 m+2 n-1}\left(E_{\mathscr{W}}^{\mathrm{nr}}\right)=0$.

Step 3: Smoothing the boundary of the singular disc $A_{\mathrm{nr}}$ near $\zeta=-1$. In step 4 below, our goal will be to deform $A_{\text {nr }}$ to extend $F$ through $q$. As we shall need to glue a maximally real submanifold $R_{1}$ of $M$ along $A_{\text {nr }}(\partial \Delta \backslash\{|\zeta+1|<\varepsilon\})$ to some collection of maximally real planes along $A_{\text {nr }}(\zeta)$ for $\zeta \in \partial \Delta$ near -1 , and because $\mathscr{C}^{2, \beta}$-smoothness of $A_{\text {nr }}$ will not be sufficient to keep the $\mathscr{C}^{2, \beta}$-smoothness of the glued object, it is convenient to smooth out first $A_{\text {nr }}$ near $\zeta=-1$ (see especially Step 2 of Section 6 below). Fortunately, we can use the freedom $\Omega_{1}$ (the small neighborhood of $A(-1)$ in Theorem 3.1) to modify the boundary of $A_{\text {nr }}$. Thus, for technical reasons only, we need the following preliminary lemma, which is simply
obtained by reparametrizing an almost full subdisc of $A_{\mathrm{nr}}$. This preparatory reparametrization is indispensible to state our Main Lemma 4.3 below correctly.
Lemma 5.3. For every $\varepsilon>0$, there exists an analytic disc $A^{\prime}$ satisfying
(a) $A^{\prime}$ is a $\mathscr{C}^{2, \beta}$-smooth subdisc of $A_{\mathrm{nr}}$, namely $A^{\prime}(\bar{\Delta}) \subset A_{\mathrm{nr}}(\bar{\Delta})$, such that moreover $A^{\prime}(\bar{\Delta}) \supset A_{\mathrm{nr}}(\bar{\Delta} \backslash\{|\zeta+1|<2 \varepsilon\})$.
(b) $A^{\prime}$ is real analytic over $\{\zeta \in \partial \Delta:|\zeta+1|<\varepsilon\}$.
(c) $\left\|A^{\prime}-A_{\text {nr }}\right\|_{\mathscr{C}^{2}, \beta} \leq \varepsilon$.
(d) $A^{\prime}(\partial \Delta) \subset M \cup \Omega_{1}$.

Proof. Of course, (d) follows immediately from (a) and (c) if $\varepsilon$ is sufficiently small. To construct $A^{\prime}$, we consider a $\mathscr{C}^{\infty}$-smooth cut-off function $\mu_{\varepsilon}$ : $\partial \Delta \rightarrow[0,1]$ with $\mu_{\varepsilon}(\zeta)=1$ for $|\zeta+1|>2 \varepsilon$ and $\mu_{\varepsilon}(\zeta)$ equal to a constant $c_{\varepsilon}<1$ with $c_{\varepsilon}>1-\varepsilon$ for $|\zeta+1|<\varepsilon$. Let $\Delta_{\mu_{\varepsilon}}$ be the (almost full) subdisc of $\Delta$ defined by $\left\{\zeta \in \Delta:|\zeta|<\mu_{\varepsilon}(\zeta /|\zeta|)\right\}$. Let $\psi_{\varepsilon}$ be the Riemann conformal map $\Delta \rightarrow \Delta_{\mu_{\varepsilon}}$. We can assume that $\psi_{\varepsilon}(-1)=-c_{\varepsilon} \in \partial \Delta_{\mu_{\varepsilon}} \cap \mathbb{R}$. By Caratheodory's theorem and by the Schwarz symmetry principle, $\psi_{\varepsilon}$ is $\mathscr{C}^{\infty}$ smooth up to the boundary and real analytic near $\zeta=-1$. If $\varepsilon$ is sufficiently small and $c_{\varepsilon}$ sufficiently close to 1 , the stability of Riemann's uniformization theorem under small $\mathscr{C}^{\infty}$-smooth perturbations shows that the disc

$$
A^{\prime}(\zeta):=A_{\mathrm{nr}}\left(\psi_{\varepsilon}(\zeta)\right)
$$

satisfies the desired properties, possibly with a slightly different small $\varepsilon$.
Step 4: Variation of the singular disc. In the sequel, we shall constantly denote the disc of Lemma 4.2 by $A^{\prime}$. We set $\zeta_{q}:=\psi_{\varepsilon}^{-1}\left(\zeta_{0}\right)$, so that $A^{\prime}\left(\zeta_{q}\right)=$ $q$. Of course, after a reparametrization by a Blaschke transformation, we can (and we will) assume that $\zeta_{q}=0$. By construction, $\left.A^{\prime}\right|_{\partial \Delta}$ is real analytic near -1 and the point $q=A^{\prime}(0)$ is contained $E_{\mathscr{W}}^{\mathrm{nr}}$, the set through which our partial extension $F$ does not extend a priori. To derive a contradiction, our next purpose is to produce a disc $A^{\prime \prime}$ close to $A^{\prime}$ and passing through the fixed point $q$ such that $q$ can be encircled by a small closed curve in $A^{\prime \prime}(\Delta) \backslash E_{\mathscr{W}}^{\mathrm{nr}}$, because in such a situation, we will be able to apply the continuity principle as in the typical local situation of Hartog's theorem (see (4) of Lemma 4.3 and Step 5 below).

At first glance it seems that we can produce $A^{\prime \prime}$ simply by turning $A^{\prime}$ a little around $q$ : indeed, Lemma 4.1 applies, since $H^{2 m+2 n-1}\left(E_{\mathscr{Y}}^{\mathrm{nr}}\right)=0$. However, the difficult point is to guarantee that $A^{\prime \prime}$ is still attached to the union of $M$ with the small neighborhood $\Omega_{1}$ of $A(-1)$ in $\mathbb{C}^{m+n}$. The following key lemma asserts that these additional requirements can be fulfilled.
Lemma 5.3. Let $A^{\prime}$ be the disc of Lemma 4.1, let $q=A^{\prime}(0) \in E_{\mathscr{W}}^{\mathrm{nr}}$ and let $0<\beta<\alpha$ be arbitrarily close to $\alpha$. Then there exists a parameterized family $A_{t^{\prime}}^{\prime}$ of analytic discs with the following properties:
(1) The parameter $t^{\prime}$ ranges in a neighborhood $U_{t^{\prime}}$ of 0 in $\mathbb{R}^{2 m+2 n-1}$ and $A_{0}^{\prime}=A^{\prime}$.
(2) The mapping $U_{t^{\prime}} \times \bar{\Delta} \ni\left(t^{\prime}, \zeta\right) \mapsto A_{t^{\prime}}^{\prime}(\zeta) \in \mathbb{C}^{m+n}$ is of class $\mathscr{C}^{1, \beta}$ and each $A_{t^{\prime}}^{\prime}$ is an embedding of $\bar{\Delta}$ into $\mathbb{C}^{m+n}$.
(3) For all $t^{\prime} \in U_{t^{\prime}}$, the point $q=A_{t^{\prime}}^{\prime}(0)$ is fixed and $A_{t^{\prime}}^{\prime}(\partial \Delta) \subset M \cup \Omega_{1}$. Furthermore, there exists a small $\delta>0$ such that the large boundary part $A_{t^{\prime}}^{\prime}(\partial \Delta \backslash\{|\zeta+1|<\delta\})$ is attached to a fixed maximally real $(m+n)$-dimensional $\mathscr{C}^{2, \alpha}$-smooth submanifold $R_{1}$ of $M$.
(4) For every fixed $\rho_{\varepsilon}>0$ which is sufficiently small and for $t^{\prime}$ ranging in a sufficiently small neighborhood of the origin, the union of circles

$$
\bigcup_{t^{\prime}}\left\{A_{t^{\prime}}^{\prime}\left(\rho_{\varepsilon} e^{i \theta}\right): \theta \in \mathbb{R}\right\}
$$

foliates a neighborhood in $\mathbb{C}^{m+n}$ of the small fixed circle $\left\{A^{\prime}\left(\rho_{\varepsilon} e^{i \theta}\right)\right.$ : $\theta \in \mathbb{R}\}$ which encircles the point $q$ inside $A^{\prime}(\Delta)$. Consequently, by Lemma 4.1, for almost all $t^{\prime} \in U_{t^{\prime}}$, the circle $\left\{A_{t^{\prime}}^{\prime}\left(\rho_{\varepsilon} e^{i \theta}\right): \theta \in \mathbb{R}\right\}$ does not meet $E_{\mathscr{W}}^{\mathrm{nr}}$.

Let us make some explanatory commentaries. Notice that the discs are only $\mathscr{C}^{1, \beta}$-smooth, because the underlying method of Sections 5 and 6 (implicit function theorem in Banach spaces, $c f$. [G1]) imposes a real loss of smoothness. If we could have produce a $\mathscr{C}^{2, \beta}$-smooth family (assuming for instance that $M$ was $\mathscr{C}^{3, \alpha}$-smooth from the beginning, or asking whether the regularity methods of [ Tu 2 ] are applicable to the global Bishop equation), we would have constructed a slightly different family and stated instead of (4) the following conic-like differential geometric property:
(4') The parameter $t^{\prime}$ ranges over a neighborhood $U_{t^{\prime}}$ of the origin in $\mathbb{R}^{2 m+2 n-2}$ with $A_{t^{\prime}}^{\prime}(0)=q$ for all $t^{\prime}$ and the mapping

$$
U_{t^{\prime}} \ni t^{\prime} \mapsto\left[\partial A_{t^{\prime}}^{\prime} / \partial \zeta\right](0) \in T_{q} \mathbb{C}^{m+n}
$$

has maximal rank at $t^{\prime}=0$ with its image being transverse to the tangent space of $A^{\prime}(\Delta)$ at $q$.

 discs passing through $q$ which sweeps out an open cone with vertex in $q$. Using some basic differential geometric computations, the reader can easily check that the geometric property ( $\mathbf{4}^{\prime}$ ) implies (4) after adding one supplementary real parameter $t_{2 m+2 n-1}^{\prime}$ corresponding to the radius $\rho=|\zeta|$ of the disc. Fortunately, for the needs of Step 5 below, the essential foliation property stated in (4) will be valuable with an only $\mathscr{C}^{1, \beta}$-smooth family and, as stated in the end of (4), this family yields an appropriate disc $A_{t^{\prime}}^{\prime}$ with empty intersection with the singularity, namely $A_{t^{\prime}}^{\prime}\left(\left\{\rho_{\varepsilon} e^{i \theta}: \theta \in \mathbb{R}\right\}\right) \cap E_{\mathscr{W}}^{\mathrm{nr}}=\emptyset$. Using this Main Lemma 4.3, we can now accomplish the last step of the proof of Theorem 3.1.
Step 5: Removal of the point $q \in E_{\mathscr{Y}}^{\mathrm{nr}}$. Let $A_{t^{\prime}}^{\prime}$ the family that we obtain by applying Main Lemma 5.3 to $A^{\prime}$. According to the last sentence of Main Lemma 4.3, we may choose $t^{\prime}$ arbitrarily small and a positive radius $\rho_{\varepsilon}>0$ sufficiently small so that the boundary of analytic subdisc $A_{t^{\prime}}^{\prime}\left(\left\{\rho_{\varepsilon} e^{i \theta}: \theta \in\right.\right.$ $\mathbb{R}\}$ ) does not intersect $E_{\mathscr{\mathscr { W }}}^{\mathrm{nr}}$. Let us denote such a disc $A_{t^{\prime}}^{\prime}$ simply by $A^{\prime \prime}$ in the sequel. Furthermore, we can assume that $A^{\prime \prime}\left(\left\{\rho_{\varepsilon} e^{i \theta}: \theta \in \mathbb{R}\right\}\right)$ is contained in the small ball $B_{\varepsilon}:=\{|z-q| \leq \varepsilon\}$ in which we shall localize an application of the continuity principle (see Figure 5). Thus, it remains essentially to check that $F$ extends analytically to a neighborhood of $q$ in $\mathbb{C}^{m+n}$ by constructing an analytic isotopy of $A^{\prime \prime}$ in $\left(\mathscr{W} \backslash E_{\mathscr{W}}^{\mathrm{nr}}\right) \cup \Omega$ and by applying the continuity principle.

One idea would be to translate a little bit in $\mathbb{C}^{m+n}$ the small disc $A^{\prime \prime}\left(\left\{\rho e^{i \theta}: \rho \leq \rho_{\varepsilon}, \theta \in \mathbb{R}\right\}\right)$. However, there is a priori no reason for which
such a small translated disc (which is of real dimension two) would avoid the singularity $E_{\mathscr{W}}^{\mathrm{nr}}$. Indeed, since we only know that $H^{2 m+2 n-1}\left(E_{\mathscr{W}}^{\mathrm{nr}}\right)=0$, it is impossible in general that a two-dimensional manifold avoids such a "big" set of Hausdorff codimension $1^{+0}$.

Of course, there is no surprise here: it is clear that functions which are holomorphic in the domain $\mathscr{W} \backslash E_{\mathscr{W}}^{\mathrm{nr}}$ do not extend automatically through a set with $H^{2 m+2 n-1}\left(E_{\mathscr{W}}^{\mathrm{nr}}\right)=0$, since for instance, such a set $E_{\mathscr{W}}^{\mathrm{nr}}$ might contain infinitely many complex hypersurfaces, which are certainly not removable. So we really need to consider the whole disc $A^{\prime \prime}$ and to include it into another family of discs attached to $M \cup \Omega_{1}$ in order to produce an appropriate analytic isotopy.

The good idea is to include $A^{\prime \prime}$ in a family $A_{\rho, s}^{\prime \prime}$ similar to the one in Section 2 (with of course $A_{\rho_{0}, 0}=A^{\prime \prime}$, but without the unnecessary parameter $v$ ), since we already know that for almost all $s \in U_{s}$, we can show as in Step 1 above that $f$ (hence $F$ too) extends holomorphically to a neighborhood of $A_{\rho_{0}, s}(\bar{\Delta})$ in $\mathbb{C}^{m+n}$.

To construct this family, we observe that $A^{\prime \prime}$ is not attached to $M$, but as $A^{\prime \prime}$ can be chosen arbitrarily close in $\mathscr{C}^{1, \beta}$-norm to the original disc $A$ attached to $M$, it follows that $A^{\prime \prime}$ is certainly attached to some $\mathscr{C}^{1, \beta}$-smooth manifold $M^{\prime \prime}$ close to $M$ which coincides with $M$ except in a neighborhood of $A^{\prime \prime}(-1)$. Finally, the family $A_{\rho, s}^{\prime \prime}$ is constructed as in Section 2 (but without the parameter $v$, because in order to add the parameter $v$ satisfying the second order condition (4) of Section 3, one would need $\mathscr{C}^{2, \beta}$-smoothness of the disc). By Tumanov's regularity theorem [Tu2], this family is again of class $\mathscr{C}^{1, \beta}$ for all $0<\beta<\alpha$. Using properties (3) and (6) and reasoning as in Step 1 of this Section 4 (continuity principle), we deduce that the function $f$ of Theorem 3.1 extends holomorphically to a neighborhood of $A_{\rho_{0}, s}(\bar{\Delta})$ in $\mathbb{C}^{m+n}$ for all $s \in U_{s}$, except those belonging to some closed thin set $\mathscr{S}$ with $H^{2 m+n-2}(\mathscr{S})=0$. Since $\mathscr{S}$ does not locally disconnect $U_{s}$, such an extension necessarily coincides with the extension $F$ in the intersection of their domains.

In summary, by using the family $A_{\rho, s}^{\prime \prime}$, we have shown that for almost all $s$, the function $F$ extends holomorphically to a neighborhood of $A_{\rho_{0}, s}^{\prime \prime}(\bar{\Delta})$. We can therefore apply the continuity principle to remove the point $q$.

Indeed, we remind that $A^{\prime \prime}=A_{\rho_{0}, 0}^{\prime \prime}$ and that by construction the small boundary $A_{\rho_{0}, 0}^{\prime \prime}\left(\left\{\rho_{\varepsilon} e^{i \theta}: \theta \in \mathbb{R}\right\}\right)$ which encircles $q$ does not intersect $E_{\mathscr{W}}^{\mathrm{nr}}$. It is now clear that the usual continuity principle along the family of small $\operatorname{discs} A_{\rho_{0}, s}^{\prime \prime}\left(\left\{\rho e^{i \theta}: \rho<\rho_{\varepsilon}, \theta \in \mathbb{R}\right\}\right)$ yields holomorphic extension of $F$ at $q$ (see again Figure 5).


Finally, the proof of Theorem 3.1 is complete modulo the proof of Main Lemma 4.3, to which the remainder of the paper is devoted.

## 5. ANALYTIC DISCS ATTACHED TO MAXIMALLY REAL MANIFOLDS

A crucial ingredient of the proof of Theorem 2.4 is the description of a family of analytic discs which are close to the given disc $A^{\prime}$ of Main Lemma 4.3 and which are attached to a maximally real submanifold $R \subset$ $M \cup \Omega_{1}$ (we shall construct such an $R$ with $A^{\prime}(\zeta) \in R$ for each $\zeta \in \partial \Delta$ in Section 6 below). This topic was developed by E. Bedford-B. Gaveau, F. Forstnerič in complex dimension two and generalized by J. Globevnik to higher dimensions. In this introductory section, we shall closely follow [G1,2].

We need the solution of the following more general distribution problem. Instead of a fixed maximally real submanifold $R$, we consider a smooth family $R(\zeta), \zeta \in \partial \Delta$, of maximally real submanifolds of $\mathbb{C}^{N}, N \geq 2$, and we study the discs attached to this family which are close to an attached disc $A^{\prime}$ of reference, i.e. fulfilling $A^{\prime}(\zeta) \in R(\zeta), \forall \zeta \in \partial \Delta$. Let $\alpha>0$ be as in Theorem 1.1 and let $0<\beta<\alpha$ be arbitrarily close to $\alpha$, as in Main Lemma 4.3.

Concretely, the manifolds $R(\zeta)$ are given by defining functions $r_{j} \in$ $\mathscr{C}^{2, \beta}(\partial \Delta \times B, \mathbb{R}), j=1, \ldots, N$, where $B \subset \mathbb{C}^{N}$ is a small open ball containing the origin, so that $r_{j}(\zeta, 0)=0$ and $\partial r_{1}(\zeta, p) \wedge \cdots \wedge \partial r_{N}(\zeta, p)$ never vanishes for $\zeta \in \partial \Delta$ and $p \in B$. We would like to mention that in [G1, p. 289], the author considers the more general regularity $r_{j} \in \mathscr{C}^{\beta}\left(\partial \Delta, \mathscr{C}^{2}(B)\right)$, but that for us, the simpler smoothness category $\mathscr{C}^{2, \beta}(\partial \Delta \times B, \mathbb{R})$ will be enough. Then we represent

$$
R(\zeta):=\left\{p \in A(\zeta)+B: r_{j}\left(\zeta, p-A^{\prime}(\zeta)\right)=0, j=1, \ldots, N\right\},
$$

which is a $\mathscr{C}^{2, \beta}$-smooth maximally real manifold by the condition on $\partial r_{j}$. We suppose the given reference disc $A^{\prime}$ to be of class $\mathscr{C}^{2, \beta}$ up to the boundary. Following [G1], we describe the family of nearby attached disc as a $\mathscr{C}^{1, \beta}$-smooth submanifold of the space $\mathscr{C}^{2, \beta}\left(\partial \Delta, \mathbb{C}^{N}\right)$, with a loss of smoothness.
Remark. At first glance the transition from a fixed manifold to the family $R(\zeta)$ may appear purely technical. Nevertheless it gives in our application a decisive additional degree of freedom: If we had to construct a fixed manifold $R$ containing the boundary of our given disc $A^{\prime}$, the boundary of $A^{\prime}$ would prescribe one direction of $T R$. It will prove very convenient to avoid this constraint by the transition to distributions $R(\zeta)$ and this freedom will be used in an essential way in Section 6 below.

It turns out that the problem is governed by an $N$-tuple $\kappa_{1}, \ldots, \kappa_{N} \in \mathbb{Z}$ of coordinate independent partial indices which are defined as follows. As in [G1], we shall always assume that the pull-back bundle $\left(\left.A^{\prime}\right|_{\partial \Delta}\right)^{*}(T R(\zeta))$ is topologically trivial (this condition is dispensible, see [O]). For each $\zeta \in$ $\partial \Delta$, let us denote by $L(\zeta)$ the tangent space to $R(\zeta)$ at $A^{\prime}(\zeta)$. Then there is a $\mathscr{C}^{1, \beta}$-smooth map $G: \partial \Delta \rightarrow G L(N, \mathbb{C})$ such that for each $\zeta \in \partial \Delta$ the columns of $G$ are a (real) basis of $L(\zeta)$. By results of Plemelj and Vekua, we can decompose the matrix function $B(\zeta)=G(\zeta) \overline{G(\zeta)^{-1}}, \zeta \in \partial \Delta$, as

$$
B(\zeta)=F^{+}(\zeta) \Lambda(\zeta) F^{-}(\zeta)
$$

with matrix functions

$$
\begin{array}{r}
F^{+} \in \mathscr{O}(\Delta, G L(N, \mathbb{C})) \cap \mathscr{C}^{1, \beta}(\bar{\Delta}, G L(N, \mathbb{C})), \\
F^{-} \in \mathscr{O}(\mathbb{C} \backslash \bar{\Delta}, G L(N, \mathbb{C})) \cap \mathscr{C}^{1, \beta}(\mathbb{C} \backslash \Delta, G L(N, \mathbb{C})),
\end{array}
$$

and where $\Lambda(\zeta)$ is the matrix with powers $\zeta^{\kappa_{j}}$ on the diagonal and zero elsewhere. In [G1] it is shown that the matrix $B(\zeta)$ depends only on the family of maximally real linear space $L(\zeta)$ and that the $\kappa^{j}$ are unique up to permutation. They are called the partial indices of $R$ along $A^{\prime}(\partial \Delta)$ and their sum $\kappa=\kappa_{1}+\cdots+\kappa_{N}$ the total index. We stress that only $\kappa$ is a topological invariant, in fact twice the winding number of $\operatorname{det}(G(\zeta))$ around the origin. In the literature on symplectic topology, $\kappa$ is called Maslov index of the loop $\zeta \mapsto L(\zeta)$.

Building on work of Forstnerič [F], Globevnik [G1, Theorem 7.1] showed that the family of all analytic discs attached to $R(\zeta)$ which are $\mathscr{C}^{1, \beta}$-close to $A^{\prime}$ is a $\mathscr{C}^{1, \beta}$-smooth submanifold of $\mathscr{O}\left(\Delta, \mathbb{C}^{N}\right) \cap \mathscr{C}^{2, \beta}\left(\bar{\Delta}, \mathbb{C}^{N}\right)$ of dimension $\kappa+N$, if all $\kappa_{j}$ are non-negative (by a result due to Oh [O], this is even true if $\kappa_{j} \geq-1$ for all $j$ ). Furthermore the result is stable with respect to small $\mathscr{C}^{2, \beta}$-smooth deformations of $M$.

We shall need some specific ingredients of Globevnik's construction. Since all our later arguments will exclude the appearance of odd partial
indices and since the expression of the square root matrix $\sqrt{\Lambda}$ below is less complicated for even ones, we shall suppose from now on that $\kappa_{j}=$ $2 m_{j}, j=1, \ldots, N$.

Firstly one has to replace $G(\zeta)$ by another basis of $L(\zeta)$ which extends holomorphically to $\Delta$. By [G1, Lemma 5.1], there is a finer decomposition

$$
B(\zeta)=\Theta(\zeta) \Lambda(\zeta) \overline{\Theta(\zeta)^{-1}}
$$

where $\Theta \in \mathscr{O}(\Delta, G L(N, \mathbb{C})) \cap \mathscr{C}^{1, \beta}(\bar{\Delta}, G L(N, \mathbb{C}))$. The substitute for $G(\zeta)$ is

$$
\Theta(\zeta) \sqrt{\Lambda}(\zeta)
$$

where $\sqrt{\Lambda}(\zeta)$ denotes the matrix with $\zeta^{m_{j}}$ on the diagonal. We denote by $X_{j}\left(Y_{j}\right)$ the columns of $\Theta(\zeta) \sqrt{\Lambda}(\zeta)(\sqrt{\Lambda}(\zeta))$ respectively. One can verify that the $X_{j}(\zeta)$ span $L(\zeta)$ ([G1, Theorem 5.1]). Observe $\Theta(\zeta) \sqrt{\Lambda}(\zeta) \in$ $\mathscr{O}\left(\Delta, \mathbb{C}^{N}\right) \cap \mathscr{C}^{1, \beta}\left(\bar{\Delta}, \mathbb{C}^{N}\right)$.

Secondly one studies variations of $\left.A^{\prime}\right|_{\partial \Delta}$ as a function from $\partial \Delta$ to $\mathbb{C}^{N}$. Every nearby $\mathscr{C}^{1, \beta}$-smooth (not necessarily holomorphic) variation is a disc close to $A^{\prime}$ which can be written in the form ([F, p. 20])

$$
G(u, f)(\zeta)=\sum_{j=1}^{N} u_{j}(\zeta) X_{j}(\zeta)+i \sum_{j=1}^{N}\left\{f_{j}(\zeta)+i\left(T_{0} f_{j}\right)(\zeta)\right\} X_{j}(\zeta),
$$

where $u_{j}, f_{j} \in \mathscr{C}^{1, \beta}(\partial \Delta, \mathbb{R})$, are uniquely determined by the variation. Here $T_{0}$ denotes the harmonic conjugation operator normalized at $\zeta=0$. The condition $G(u, f)(\zeta) \in R(\zeta), \forall \zeta \in \partial \Delta$ is equivalent to the validity of the system $r_{j}(\zeta)(G(u, f)(\zeta))=0,1 \leq j \leq N$. The implicit function theorem implies that this system can be solved for $f=\phi(u)$ for $\mathscr{C}^{1, \beta_{-}}$ small $u$ with a $\mathscr{C}^{1, \beta}$-smooth mapping $\phi$ of Banach spaces $\mathscr{C}^{1, \beta}\left(\partial \Delta, \mathbb{R}^{N}\right) \rightarrow$ $\mathscr{C}^{1, \beta}\left(\partial \Delta, \mathbb{R}^{N}\right)$. This follows from [G1, Theorem 6.1] by an application of the implicit function theorem in Banach spaces, except concerning the $\mathscr{C}^{1, \beta_{-}}$ smoothness, which, in our situation, is more direct and elementary than in [G1], since we have supposed that $r_{j} \in \mathscr{C}^{2, \beta}(\partial \Delta \times B, \mathbb{R})$.

Finally one has to determine for which choices of $u$ the function $G(u, \phi(u))$ extends holomorphically to $\Delta$. Writing

$$
G(u, f)(\zeta)=\Theta(\zeta) \sum_{j=1}^{N}\left\{u_{j}(\zeta)+i\left[f_{j}(\zeta)+i\left(T_{0} f_{j}\right)(\zeta)\right]\right\} Y_{j}
$$

we see that $G(u, \phi(u))$ extends holomorphically, if and only if

$$
\begin{equation*}
\Theta^{-1}(\zeta) G(u, \phi(u))(\zeta)=\sum_{j=1}^{N}\left\{u_{j}(\zeta)+i\left[\phi(u)_{j}(\zeta)+i\left(T_{0} \phi(u)_{j}\right)(\zeta)\right]\right\} Y_{j} \tag{5.1}
\end{equation*}
$$

extends, i.e. if and only if the function $\zeta \mapsto \sum_{j=1}^{N} u_{j}(\zeta) Y_{j}(\zeta)$ extends. One can compute ([G1, p. 301]) that this is precisely the case, if $h_{j}(\zeta)=$ $Y^{-1}(\zeta) u_{j}(\zeta)$ has polynomial components of the form

$$
\begin{align*}
h_{j}(\zeta)= & t_{1}^{j}+i t_{2}^{j}+\left(t_{3}^{j}+i t_{4}^{j}\right) \zeta+\cdots+\left(t_{\kappa_{j-1}}^{j}+i t_{\kappa_{j}}^{j}\right) \zeta^{m_{j}-1}+t_{\kappa_{j+1}}^{j} \zeta^{m_{j}}  \tag{5.2}\\
& +\left(t_{\kappa_{j-1}}^{j}-i t_{\kappa_{j}}^{j}\right) \zeta^{m_{j}+1}+\cdots+\left(t_{3}^{j}-i t_{4}^{j}\right) \zeta^{\kappa_{j}-1}+\left(t_{1}^{j}-i t_{2}^{j}\right) \zeta^{\kappa_{j}}
\end{align*}
$$

where all $t_{k}^{j}$ are real. In total we get $\kappa_{j}+1$ real parameters for the choice of $h_{j}$ and hence $\kappa+N$ parameters for our local family of discs attached to $R(\zeta)$.

## 6. Proof of Theorem 3.1, part II

In this section we provide the final part of the proof of Theorem 2.4, namely Main Lemma 4.3, which relies essentially on global properties of analytic discs. The disc $A^{\prime}$ of Main Lemma 4.3 need not be attached to $M$ but since it is close to $A$ in $\mathscr{C}^{2, \beta}$-norm, it is certainly attached to a nearby manifold $M^{\prime}$ of class $\mathscr{C}^{2, \beta}$ which coincides with $M$ except in $\Omega_{1}$. The idea is now to first embed $A^{\prime}(\partial \Delta)$ into a maximal real submanifold of $M \cup \Omega_{1}$ whose partial indices are easy to determine. Then we shall explain how to increase the partial indices separately by twisting $R$ around $A^{\prime}(\partial \Delta)$ inside $\Omega_{1}$. The families of attached discs get richer with increasing indices and will eventually contain the required discs $A_{t^{\prime}}^{\prime}$ as a subfamily. We divide the proof in four essential steps.
Step 1: Construction of a first maximally real manifold $R_{1}$. Let $h^{\prime}$ be a defining function of $M^{\prime}$ as in (3.1). Then $A^{\prime}$ is the solution of a Bishop equation

$$
Y^{\prime}=T_{1}\left(h^{\prime}\left(W^{\prime}, Y^{\prime}\right)\right)+y_{0}
$$

where $W^{\prime} \in \mathscr{C}^{2, \beta}$ is the $w$-component of $A^{\prime}$ and $y_{0} \in \mathbb{R}^{n}$ is close to 0 . Recall that by construction, $W^{\prime}(\zeta)$ is close to the $w$-component $\left(\rho_{0}-\rho_{0} \zeta, 0, \ldots, 0\right)$ of the disc $A$ defined in (3.3). Let $A_{u_{*}, y}^{\prime}$ be the discs defined by the perturbed equation

$$
Y_{u_{*}, y}^{\prime}=T_{1}\left(h^{\prime}\left(W_{u_{*}, y}^{\prime}+\left(0, u_{*}\right), Y_{u_{*}, y}^{\prime}\right)\right)+y_{0}+y
$$

where $u_{*}:=\left(u_{2}, \ldots, u_{m}\right)$ is close to 0 and $y \in \mathbb{R}^{n}$ is close to 0 . We have $A_{0,0}^{\prime}=A^{\prime}$. Since $A$ defined by (3.3) and hence also $A^{\prime}$ are by construction almost parallel to the $w_{1}$-axis, the union

$$
R_{1}:=\bigcup_{u_{*}, y} A_{u_{*}, y}^{\prime}(\partial \Delta)
$$

is a maximally real manifold of class $\mathscr{C}^{2, \beta}$ contained in $M^{\prime}$ and containing $A^{\prime}(\partial \Delta)$. The explicit construction of $R_{1}$ allows an easy determination of the partial indices.

Lemma 5.3. The partial indices of $R_{1}$ with respect to $A^{\prime}(\partial \Delta)$ are $2,0, \ldots, 0$.

Proof. We begin by constructing $N=m+n$ holomorphic vector fields along $A^{\prime}(\partial \Delta)$ which generate (over $\mathbb{R}$ ) the tangent bundle of $R_{1}$. We denote $\zeta=e^{i \theta} \in \partial \Delta$ and define first $G_{1}(\zeta):=\left[\partial A^{\prime}\left(e^{i \theta}\right) / \partial \theta\right]$ as the push-forward of $\partial / \partial \theta$. Next, we put

$$
\begin{cases}G_{k}(\zeta):=\left.\left[\partial A_{u_{*}, 0}^{\prime}(\zeta) / \partial u_{k}\right]\right|_{u_{*}=0}, & \text { for } k=2, \ldots, m, \\ G_{k}(\zeta):=\left.\left[\partial A_{0, y}^{\prime}(\zeta) / \partial y_{k-m}\right]\right|_{y=0}, & \text { for } k=m+1, \ldots, N .\end{cases}
$$

For $k=2, \ldots, N, G_{k}$ is the uniform limit of pointwise holomorphic difference quotients and therefore holomorphic itself. As $A_{u_{*}, y}^{\prime}$ depends $\mathscr{C}^{2, \beta_{-}}$ smoothly on parameters, we obtain $G_{k} \in \mathscr{C}^{1, \beta}\left(\bar{\Delta}, \mathbb{C}^{N}\right), k=2, \ldots, N$.

By [G1, Proposition 10.2], the maximal number of linearly independent holomorphically extendable sections equals the number of non-negative partial indices. Hence we deduce that all $\kappa_{j}$ are non-negative.

Furthermore it is easy to see that the total index $\kappa$, which is twice the winding number of $\left.\operatorname{det} G\right|_{\partial \Delta}$ around 0 , equals 2 . Indeed, $A^{\prime}$ is almost parallel to the $w_{1}$ axis, the direction in which $G_{1}$ has winding number 1 , and the vector fields $G_{2}, \ldots, G_{N}$ have a topologically trivial behaviour in the remaining directions. This heuristic argument can be made precise in the following way. One easily can smoothly deform the complex coordinates $z_{j}, w_{k}$ to (non-holomorphic coordinates) in which the matrix $G(\zeta)$ gets diagonal with diagonal entries $\zeta, 1, \ldots, 1$. In the deformed coordinates the winding number of the determinant is obviously 1 , and this remains unchanged when deforming back to the standard coordinates.

In summary the only possible constellations for the partial indices are $2,0, \ldots, 0$ and $1,1,0, \ldots, 0$. But [G1, Proposition 10.1] excludes the second case as $\partial A^{\prime} / \partial \theta$ does not vanish on $\bar{\Delta}$, which completes the proof.

Step 2: Gluing $R_{1}$ with a family of maximally real planes. Our goal is to twist the manifold $R_{1}$ many times around the boundary of $A^{\prime}$ in the small neighborhood $\Omega_{1}$ of $A^{\prime}(-1)$ in order to increase its partial indices. Since it is rather easy to increase partial indices when a disc is attached to a family of linear maximally real subspaces of $\mathbb{C}^{N}$ (using Lemma 6.3 below, see the reasonings just after the proof), we aim to glue $R_{1}$ with its family of tangent planes $T_{A^{\prime}(\zeta)} R_{1}$ for $\zeta$ near -1 . Before proceeding, we have to take care of a regularity question: the family $\zeta \mapsto T_{A^{\prime}(\zeta)} R_{1}$ being only of class $\mathscr{C}^{1, \beta}$, some preliminary regularizations are necessary. We remind that by Lemma 4.2 (b), the disc $A^{\prime}$ is real analytic near $\zeta=-1$. This choice of smoothness is very adapted to our purpose. Indeed, using cut-off functions and the Weierstrass approximation theorem, we can construct a $\mathscr{C}^{2, \beta}$-smooth maximally real manifold $R_{2}$ to which $A^{\prime}$ is still attached and which is also
real analytic in a neighborhood of $\left\{A^{\prime}(\zeta):|\zeta+1|<\varepsilon / 2\right\}$. Of course, this can be done with $\left\|R_{2}-R_{1}\right\|_{\mathscr{C}_{2, \beta}}$ being arbitrarily small, so the partial indices of $A^{\prime}$ with respect to $R_{2}$ are still equal to $(2,0, \ldots, 0)$.

Using real analyticity, we can now glue $R_{2}$ with its family of maximally real tangent planes $T_{A^{\prime}(\zeta)} R_{2}$ for $|\zeta+1|<\varepsilon / 4$ in a smooth way as follows. After localization near $A^{\prime}(-1)$ using a cut-off function, the gluing problem is reduced to the following statement.

Lemma 5.3. Let $R$ be small real analytic maximally real submanifold of $\mathbb{C}^{N}$, let $p \in R$ and let $\gamma(s), s \in(-\varepsilon, \varepsilon)$, be a real analytic curve in $R$ passing through $p$. Then there exist smooth functions $r_{j}(s, z) \in \mathscr{C}^{\infty}((-\varepsilon, \varepsilon) \times B, \mathbb{R})$, for $j=1, \ldots, N$, where $B$ is a small open ball centered at the origin in $\mathbb{C}^{N}$, such that
(1) $r_{j}(s, \gamma(s)) \equiv 0$.
(2) $r_{j}(s, z) \equiv r_{j}(z) \equiv$ the defining functions of $R$ for $|s| \geq \varepsilon / 2$.
(3) For all $s$ with $|s| \leq \varepsilon / 4$, the set $\left\{z \in \mathbb{C}^{N}: r_{j}(s, z)=0, j=\right.$ $1, \ldots, N\}$ coincides with the tangent space of $R$ at $\gamma(s)$.

Proof. Choosing coordinates $\left(z_{1}, \ldots, z_{N}\right)$ vanishing at $p$, we can asssume that $R$ is given by $r_{j}(z):=y_{j}-\varphi_{j}(x)=0$ with $\varphi_{j}(0)=0$ and $d \varphi_{j}(0)=0$, and that $\gamma_{j}(s)=x_{j}(s)+i y_{j}(s)$, where $y_{j}(s):=\varphi_{j}(x(s))$. Let $\chi(s)$ be a $\mathscr{C}^{\infty}$-smooth cut-off function satisfying $\chi(s) \equiv 0$ for $|s| \leq \varepsilon / 4$ and $\chi(s) \equiv 1$ for $|s| \geq \varepsilon / 2$. We choose for $r_{j}(s, z)$ the following functions:
$y_{j}-y_{j}(s)-\sum_{k=1}^{N} \frac{\partial \varphi_{j}}{\partial x_{k}}(x(s))\left[x_{k}-x_{k}(s)\right]-\chi(s)\left[\sum_{K \in \mathbb{N}^{N},|K| \geq 2} \frac{\partial_{x}^{K} \varphi_{j}(x(s))}{K!}[x-x(s)]^{K}\right]$.
Clearly, the $r_{j}$ are $\mathscr{C}^{\infty}$-smooth and (3) holds. As $\varphi_{j}$ is real analytic in a neighborhood of $\gamma$, property (2) holds by Taylor's formula.

In summary, we have shown that we can attach $A^{\prime}$ to some $\mathscr{C}^{2, \beta}$-smooth family $\left(R_{3}(\zeta)\right)_{\zeta \in \partial \Delta}$ of maximally real submanifolds such that $R_{3}(\zeta)$ coincides with $R_{2}$ for $|\zeta+1| \geq \varepsilon / 2$ and such that $R_{3}(\zeta)$ coincides with the maximally real plane $T_{A^{\prime}(\zeta)} R_{2}$, for $|\zeta+1| \leq \varepsilon / 4$. Clearly, the partial indices of $A^{\prime}$ with respect to the family $R_{3}(\zeta)$ are still equal to $(2,0, \ldots, 0)$.
Step 3: Increasing partial indices. This step is the crucial one in our argumentation. Recall that the partial indices are defined in terms of vector fields along $A^{\prime}(\partial \Delta)$. In the previous section we have described how to select distinguished vector fields $X_{k}$ as the columns of $\Theta \sqrt{\Lambda}$, where $\Theta, \Lambda$ were associated to a decomposition of $G_{3}(\zeta){\overline{G_{3}(\zeta)}}^{-1}$, where the columns of the matrix $G_{3}(\zeta)$ span $T_{A^{\prime}(\zeta)} R_{3}(\zeta)$. Our method is to modify the vector fields
$X_{k}$ by replacing them by products $g_{k} X_{k}$ with the boundary values of certain holomorphic functions $g_{k}$. It turns out that the indices can be read from properties of the $g_{k}$. Here is how the $g_{k}$ are constructed.

For convenience in the following lemma, we shall represent $\partial \Delta$ by the real closed interval $[-\pi, \pi]$ where $\pi$ is identified with $-\pi$.
Lemma 5.3. For every small $\varepsilon>0$, every integer $\ell \in \mathbb{N}$, there exists a holomorphic function $h \in \mathscr{O}(\Delta) \cap \mathscr{C}^{\infty}(\bar{\Delta})$ such that
(1) $h(\zeta) \neq 0$ for all $\zeta \in \bar{\Delta}$.
(2) The function $g(\zeta):=\zeta^{\ell} h(\zeta)$ is real-valued over $\left\{e^{i \theta}:|\theta| \leq \pi-\right.$ $\varepsilon / 8\}$.
It follows that the winding number of $\left.g\right|_{\partial \Delta}$ around $0 \in \Delta$ is equal to $\ell$.
Proof. Let $v(\zeta)$ be an arbitrary $\mathscr{C}^{\infty}$-smooth $2 \pi$-periodic extension to $\mathbb{R}$ of the linear function $-\ell \theta$ defined on $[-\pi+\varepsilon / 8, \pi-\varepsilon / 8]$. Let $T_{0}$ be the harmonic conjugate operator satisfying $\left(T_{0} u\right)(0)=0$ for every $u \in L^{2}(\partial \Delta)$. Since $T_{0}$ is a bounded operator of the $\mathscr{C}^{k, \alpha}$ spaces of norm equal to 1 , the function $T_{0} v$ is $\mathscr{C}{ }^{\infty}$-smooth over $\partial \Delta$. It suffices to set $h:=\exp \left(-T_{0} v+i v\right)$. Indeed,

$$
\zeta^{\ell} h(\zeta)=e^{i \ell \theta} e^{-T_{0} v+i v}
$$

is real for $\theta \in[-\pi+\varepsilon / 8, \pi-\varepsilon / 8]$, as desired.
Let $L(\zeta)$ denote the tangent space $T_{A^{\prime}(\zeta)} R_{3}(\zeta)$ and let $X_{k}(\zeta)$ be $\mathscr{C}^{1, \beta_{-}}$ smooth vector fields as the columns of the matrix $\Theta \sqrt{\Lambda}$ constructed in Section 5 above. We remind that the $X_{k}(\zeta)$ span $L(\zeta)$. Further, as the partial indices of $A^{\prime}(\zeta)$ with respect to $R_{3}(\zeta)$ are $(2,0, \ldots, 0)$, we have

$$
\Theta(\zeta) \sqrt{\Lambda}(\zeta)=\left(\zeta \Theta_{1}(\zeta), \Theta_{2}(\zeta), \ldots, \Theta_{N}(\zeta)\right)
$$

Now, let us choose an arbitrary collection of nonnegative integers $\ell_{1}, \ell_{2}, \ldots, \ell_{N}$ and associated functions $g_{\ell_{1}}(\zeta)=\zeta^{\ell_{1}} h_{\ell_{1}}(\zeta), \ldots$, $g_{\ell_{N}}(\zeta)=\zeta^{\ell_{N}} h_{\ell_{N}}(\zeta)$ satisfying (1) and (2) of Lemma 6.3 above. With these functions, we define a new family of maximally real manifolds to which $A^{\prime}(\zeta)$ is still attached as follows:
(a) For $|\theta| \leq \pi-\varepsilon / 8, R_{4}(\zeta) \equiv R_{3}(\zeta)$.
(b) For $|\theta| \geq \pi-\varepsilon / 8$, namely for $\zeta$ close to -1 ,

$$
R_{4}(\zeta):=\operatorname{span}_{\mathbb{R}}\left(\zeta g_{\ell_{1}}(\zeta) \Theta_{1}(\zeta), g_{\ell_{2}}(\zeta) \Theta_{2}(\zeta), \ldots, g_{\ell_{N}}(\zeta) \Theta_{N}(\zeta)\right)
$$

It is important to notice that this definition yields a true $\mathscr{C}^{2, \beta}$-smooth family of maximally real manifolds, thanks to the fact that the family $R_{3}(\zeta)$ is already a family of real linear spaces for $|\zeta+1| \leq \varepsilon / 4$, by construction. Interestingly, the partial indices have increased:
Lemma 5.3. The partial indices of $R_{4}(\zeta)$ along $\partial A$ are equal to $2+$ $2 \ell_{1}, 2 \ell_{2}, \ldots, 2 \ell_{N}$.

Proof. By construction, since the functions $g_{\ell_{j}}(\zeta)$ are real-valued over $\left\{e^{i \theta}\right.$ : $|\theta| \leq \pi-\varepsilon / 8\}$, the tangent space $T_{A^{\prime}(\zeta)} R_{4}(\zeta)$ is spanned for all $\zeta \in \partial \Delta$ by the $N$ vectors

$$
\zeta g_{\ell_{1}}(\zeta) \Theta_{1}(\zeta), g_{\ell_{2}}(\zeta) \Theta_{2}(\zeta), \ldots, g_{\ell_{N}}(\zeta) \Theta_{N}(\zeta)
$$

which form together a $N \times N$ matrix which we will denote by $G_{4}(\zeta)$. By Section 5 , we can read directly from the matrix identity

$$
\begin{aligned}
G_{4}(\zeta) & {\overline{G_{4}}(\zeta)}^{-1}=\left(h_{\ell_{1}}(\zeta) \Theta_{1}(\zeta), \ldots, h_{\ell_{N}}(\zeta) \Theta_{N}(\zeta)\right) \times \\
& \times \operatorname{diag}\left(\zeta^{2+2 \ell_{1}}, \zeta^{2 \ell_{2}}, \ldots, \zeta^{2 \ell_{N}}\right) \times{\overline{\left(h_{\ell_{1}}(\zeta) \Theta_{1}(\zeta), \ldots, h_{\ell_{N}}(\zeta) \Theta_{N}(\zeta)\right)}}^{-1}
\end{aligned}
$$

that the partial indices of $A^{\prime}(\zeta)$ with respect to $R_{4}(\zeta)$ are equal to $(2+$ $2 \ell_{1}, 2 \ell_{2}, \ldots, 2 \ell_{N}$ ), as stated.

Step 4: Construction of the family $A_{t^{\prime}}^{\prime}$. Now, as we need not very large partial indices, we choose $\ell_{1}=1, \ell_{2}=2, \ldots, \ell_{N}=2$, so the partial indices are simply $(4,4, \ldots, 4)$. Moreover, the matrix $Y(\zeta)$ is equal to the diagonal matrix $\operatorname{diag}\left(\zeta^{2}, \zeta^{2}, \ldots, \zeta^{2}\right)$. Concerning the $5 N$ parameters $\left(t_{1}^{j}, t_{2}^{j}, t_{3}^{j}, t_{4}^{j}, t_{5}^{j}\right)$ appearing in equation (5.2) above, we even choose $t_{1}^{j}=t_{2}^{j}=t_{5}^{j}=0$. Then by the result of Globevnik, we thus obtain a family of discs depending on the $2 N$-dimensional real parameter $t:=\left(t_{3}^{j}+i t_{4}^{j}\right)_{1 \leq j \leq N}$. The functions $h_{j}$ and $u_{j}$ defined in Section 5 are thus equal to

$$
\left\{\begin{array}{l}
h_{j}(t, \zeta):=\left(t_{3}^{j}+i t_{4}^{j}\right) \zeta+\left(t_{3}^{j}-i t_{4}^{j}\right) \zeta^{3}, \\
u_{j}(t, \zeta):=\left(t_{3}^{j}+i t_{4}^{j}\right) \bar{\zeta}+\left(t_{3}^{j}-i t_{4}^{j}\right) \zeta .
\end{array}\right.
$$

It remains to explain how we can extract the desired family $A_{t^{\prime}}^{\prime}$ by reducing this $(2 m+2 n)$-dimensional parameter space to some of dimension $(2 m+$ $2 n-1$ ) such that property (4) of Main Lemma 4.3 is satisfied. Let us denote by $h_{t}$ and $u_{t}$ the maps $\zeta \mapsto h(t, \zeta)$ and $\zeta \mapsto u(t, \zeta)$. By equation (5.1), we have
$G\left(u_{t}, \phi\left(u_{t}\right)\right)(\zeta)=\Theta(\zeta) \sum_{j=1}^{N}\left\{u_{j}(t, \zeta)+i\left[\phi\left(u_{t}\right)_{j}(\zeta)+i T_{0} \phi\left(u_{t}\right)_{j}(\zeta)\right]\right\} Y_{j}(\zeta)$.
and by Section 5 , the $\mathscr{C}^{1, \beta}$-smooth discs

$$
A_{t}^{\prime}(\zeta):=A^{\prime}(\zeta)+G\left(u_{t}, \phi\left(u_{t}\right)\right)(\zeta)
$$

are attached to $R_{4}(\zeta)$. By [G1, p. 299 top], the differential of $\phi$ at 0 is null: $D_{u} \phi(0)=0$. It follows that

$$
\frac{\partial}{\partial t}\left[\Theta(\zeta) \sum_{j=1}^{N} i\left[\phi\left(u_{t}\right)_{j}(\zeta)+i T_{0} \phi\left(u_{t}\right)_{j}(\zeta)\right] Y_{j}(\zeta)\right]_{t=0} \equiv 0
$$

So on the one hand, we can compute for $j=1, \ldots, N$

$$
\left\{\begin{array}{l}
{\left[\frac{\partial A_{t}^{\prime}}{\partial t_{3}^{j}}\right]_{t=0}=\rho e^{i \theta} \Theta(0)(0, \ldots, 0,1,0, \ldots, 0)+\mathrm{O}\left(\rho^{2}\right)}  \tag{6.1}\\
{\left[\frac{\partial A_{t}^{\prime}}{\partial t_{4}^{j}}\right]_{t=0}=\rho e^{i \theta} \Theta(0)(0, \ldots, 0, i, 0, \ldots, 0)+\mathrm{O}\left(\rho^{2}\right)}
\end{array}\right.
$$

where $\mathrm{O}\left(\rho^{2}\right)$ denotes a holomorphic disc in $\mathscr{O}\left(\Delta, \mathbb{C}^{N}\right) \cap \mathscr{C}^{0, \beta}\left(\bar{\Delta}, \mathbb{C}^{N}\right)$ vanishing up to order one at 0 . For $\rho>0$ small enough and $\theta$ arbitrary, it follows that these $2 m+2 n$ vectors span $\mathbb{C}^{m+n}$. On the other hand, we compute

$$
\begin{equation*}
\left[\frac{\partial A_{t}^{\prime}}{\partial \theta}\right]_{t=0}=\rho e^{i \theta} \Theta(0)\left(i a_{1}, \ldots, i a_{N}\right)+\mathrm{O}\left(\rho^{2}\right) \tag{6.2}
\end{equation*}
$$

where the constants $a_{j}$ are defined by $A^{\prime}(\zeta)=\left(a_{1} \zeta, \ldots, a_{N} \zeta\right)+\mathrm{O}\left(\zeta^{2}\right)$ and do not all vanish (since $A^{\prime}$ is an embedding).

Let us choose a $(2 m+2 n-1)$-dimensional real plane $H$ which is supplementary to $\mathbb{R} \Theta(0)\left(i a_{1}, \ldots, i a_{N}\right)$ in $\mathbb{C}^{m+n}$. Using (6.1), we can choose a $(2 m+2 n-1)$-dimensional real linear subspace $T^{\prime} \subset \mathbb{R}^{2 m+2 n}$ and $\rho_{\varepsilon}$ small enough such that, after restricting the family $A_{t}^{\prime}$ with $t^{\prime} \in T^{\prime}$, the $(2 m+2 n-1)$ vectors $\left[\partial A_{t^{\prime}}^{\prime} / \partial t_{j}^{\prime}\right]_{t^{\prime}=0}, j=1, \ldots, 2 m+2 n-1$, are linearly independent with the vector (6.2) for all $\zeta \in \Delta$ of the form $\zeta=\rho_{\varepsilon} e^{i \theta}$. It follows that the mapping

$$
\left(e^{i \theta}, t^{\prime}\right) \mapsto A_{t^{\prime}}^{\prime}\left(\rho_{\varepsilon} e^{i \theta}\right)
$$

is a local embedding of the circle $\partial \Delta$ times a small neighborhood of the origin in $\mathbb{R}^{2 m+2 n-1}$, from which we see that the foliation property (4) of Main Lemma 4.3 holds.

This completes the proof of Step 4, the proof of Main Lemma 4.3, the proof of Theorem 3.1 and the proof of Theorem 1.1.

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[^0]:    ${ }^{1}$ Background about Hölder classes appears in Section 1(IV).

[^1]:    ${ }^{2}$ One may set $e_{1}:=m$ and $e_{2}:=m$ in any case.

[^2]:    ${ }^{3}$ Other rigidity phenomena are: parametrization of CR automorphism groups by a jet of finite order, finiteness of their dimension, genericity of nonalgebraizable CR submanifolds, genericity of CR submanifolds having no infinitesimal CR automorphisms, etc.

[^3]:    ${ }^{4}$ We are grateful to Stormark for pointing out this observation

[^4]:    ${ }^{5}$ In certain references, local $\mathbb{L}$-orbits are considered as germs. Knowing by experience that the language of germs becomes misleading when several quantifiers are involved in complex statements, we will always prefer to speak of local submanifolds of a certain small size.

[^5]:    ${ }^{6}$ We are grateful to Burglind Jöricke who provided the reference [Al1975].

[^6]:    ${ }^{7}$ The origin of this equation may be found in the seminal article [Bi1965] of Bishop. Since then, it has been further exploited in [Pi1974a, Pi1974b, HiTa1978, BeFo1978, We1982, BG1983, BPo1982, KW1982, R1983, HiTa1984, KW1984, BPi1985, FR1985, Trp1986, Fo1986, Tu1988, Ai1989, Tu1990, Trp1990, Bo1991, DH1992, Tu1994a, Tu1994b, BRT1994, CR1994, Gl1994, Me1994, HuKr1995, Jö1995, Trp1996, Tu1996, Jö1996, Jö1997, Me1997, Po1997, MP1998, Tu1998, Hu1998, CR1998, Jö1999a, Jö1999b, MP1999, BER1999, Po2000, MP2000, Tu2001, Da2001, DS2001, Me2002, MP2002, 16, Po2003, JS2004, Me2004c].

[^7]:    ${ }^{8}$ In Lemma 3.15 above, for $t \notin \mathfrak{N}$, the map $e^{i \theta} \mapsto u\left(e^{i \theta}, t\right)$ was shown to be $\mathscr{C}^{0, \alpha / 2}$ on $\partial \Delta$ in order to insure that $\mathrm{T} u(\cdot, t)$ and $\phi(\cdot, t)$ are both bounded on $\partial \Delta$, so that Theorem 2.24 may be applied in the next phrase. In [Tu1996], $u(\cdot, t)$ is only shown to be in $L^{\infty}(\partial \Delta)$, but then $\mathrm{T} u(\cdot, t)$ and $\phi$ are not necessarily bounded.

[^8]:    ${ }^{9}$ Further historical information may be found in [29, Str1988, Fi1991, 30, 16].

[^9]:    ${ }^{10}$ Often, some authors consider instead a compact $K \subset \Omega$ with $\Omega \backslash K$ connected and state that $\mathscr{O}(\Omega \backslash K)=\left.\mathscr{O}(\Omega)\right|_{\Omega \backslash K}$; a technical check shows that the two statements are equivalent.

[^10]:    ${ }^{11}$ Here, $\mathscr{D}^{p, q}$ is the space of $\mathscr{C}^{\infty}$ forms of bidegree $(p, q)$ having compact support; fundamental notions about currents may be found in [Ch1989].

[^11]:    ${ }^{12}$ Using propagation techniques of Section 3, the theorem holds assuming that $M$ is globally minimal and considering continuous CR functions on $M$.

[^12]:    ${ }^{13}$ Although the shape of polydiscs is not invariant by local biholomorphisms, their topology is. To avoid dealing implicitly with possibly wild open sets, we prefer to speak of neighborhoods $U_{p}, V_{p}, W_{p}$ of $p$ that are polydiscs.

[^13]:    ${ }^{14}$ This theorem also holds true with $M \in \mathscr{C}^{2}$ and even with $M \in \mathscr{C}^{1, \alpha}(0<\alpha<1)$, provided one redefines the notion of CR orbit in terms of boundaries of small attached analytic discs.

[^14]:    ${ }^{15}$ This is the so-called Method of analytic discs ; $\bar{\partial}$ techniques are also powerful.

[^15]:    ${ }^{16}$ A more general property holds true (see [Trp1990, Tu1994a, MP2006b]): every small attached disc is necessarily attached to a single (local or global) CR orbit; here, $\Sigma$ is a local orbit, whence $A(\partial \Delta) \subset \Sigma$.

[^16]:    ${ }^{17}$ We recommend mostly the two elegant presentations [Trp1996] and [Tu1998]; other references are: [BRT1994, BER1999]. Excepting a conceptual abstraction involving the implicit function theorem in Banach spaces and the conormal bundle to $M$, the major arguments: differentiation of Bishop's equation and a crucial correspondence between an exit vector mapping and an evaluation mapping defined on the space of discs attached to $M$, the geometric structure of the proof is exactly the same in the original article [Tu1988] as in the restitutions.

[^17]:    ${ }^{18}$ Classical microlocal analysis was devised to measure the analytic wave front set of a distribution in terms of the exponential decay ot the Fourier transform restricted to open, conic submanifolds of the cotangent bundle. We suspect that there might exist higher generalizations of microlocal analysis in which one takes account of the good decay of the Fourier transform on submanifolds of positive codimension in the cotangent bundle.

[^18]:    ${ }^{19}$ Since first order partial derivatives $W_{t_{k}^{\prime}}\left(\zeta, t^{\prime}\right), k=1, \ldots, e$, will appear in a while, we do not write the parameter $t^{\prime}$ as a lower index in $U\left(\zeta, t^{\prime}\right)+i V\left(\zeta, t^{\prime}\right)$.

[^19]:    ${ }^{20}$ We can also check that $\mathscr{J}\left(U_{t_{k}^{\prime}}\right)=-\mathscr{J}\left(\mathrm{T}_{1} V_{t_{k}^{\prime}}\right)=\partial V_{t_{k}^{\prime}}(1,0) / \partial \theta=0$. Indeed, it suffices to differentiate (3.18) with respect to $\theta$ at $\theta=0$, noticing that $\Phi_{u}(0,0,0)=$ $\varphi_{u}(0,0)=0$, that $U_{t_{k}^{\prime}}(1,0)=0$ and that $\Phi_{t_{k}^{\prime}}(z, u, 0)=0$ for $(z, u)$ near $(0,0)$.

[^20]:    ${ }^{21}$ If $\|f\|_{L^{\infty}} \leqslant 1$, setting $g(z):=[f(\infty)-f(z)] /[1-\overline{f(\infty)} f(z)]$, we have $g(\infty)=0$, $\|g\|_{L^{\infty}} \leqslant 1$ and $\left|g^{\prime}(\infty)\right|=\left|f^{\prime}(\infty)\right| /\left(1-|f(\infty)|^{2}\right) \geqslant\left|f^{\prime}(\infty)\right|$, so that in the definition of analytic capacity, we may restrict to take the supremum over functions $g \in H^{\infty}(\mathbb{C} \backslash K)$ with $\|g\|_{L^{\infty}} \leqslant 1$ and $g(\infty)=0$.

[^21]:    ${ }^{22}$ Some refinements of the statements may be formulated, for instance assuming in (rm3) and (rm4) that $N$ is $\mathscr{C}^{1}$ and has some metrically thin singularities ([6]).

[^22]:    ${ }^{23}$ Minimalization at a point takes strong advantage of the rigidity of complex hypersurfaces in this argument.

[^23]:    ${ }^{24} \mathrm{CR}$ distributions may also be considered, but in the sequel, we shall restrict considerations to continuous and integrable CR functions.

[^24]:    ${ }^{25}$ To be rigorous: for every holomorphic function $f \in \mathscr{O}\left(\mathscr{V}^{\prime}(\partial \Omega \backslash K)\right)$, there exists a holomorphic function $F \in \mathscr{O}(\mathscr{V}(\partial \Omega \backslash K))$ that coincides with $f$ in a possibly smaller one-sided neighborhood $\mathscr{V}^{\prime \prime}(\partial \Omega \backslash K) \subset \mathscr{V}^{\prime}(\partial \Omega \backslash K)$. Details of the proof (involving a deformation argument) will not be provided here (see [Me1997, Jö1999a]).

[^25]:    ${ }^{26}$ Indeed, consider for instance the Hartogs figure $\Omega:=\left\{\left|z_{1}\right|<1,\left|z_{2}\right|<2\right\} \cup\{1 \leqslant$ $\left.\left|z_{1}\right|<2,1<\left|z_{2}\right|<2\right\}$ in $\mathbb{C}^{2}$. Then the annulus $\left.K=\left\{z_{1}=1,1 \leqslant\left|z_{2}\right| \leqslant 2\right)\right\} \subset \partial \Omega$ is $\mathscr{O}(\bar{\Omega})$-convex. We claim that the function $g:=1 / z_{2}$, holomorphic in a neighborhood of $K$, cannot be approximated on $K$ by functions $f \in \mathscr{O}(\bar{\Omega})$. Indeed, by Hartogs extension $\mathscr{O}(\bar{\Omega})=\mathscr{O}\left(\left\{\left|z_{1}\right| \leqslant 2,\left|z_{2}\right| \leqslant 2\right\}\right)$, which implies that every $f \in \mathscr{O}(\bar{\Omega})$ has to satisfy the maximum principle on the disc $\left.\left\{z_{1}=1,\left|z_{2}\right| \leqslant 2\right)\right\} \supset K$. Rounding off the corners, we get an example with $\partial \Omega \in \mathscr{C}^{\infty}$.
    ${ }^{27}$ Said differently, the envelope of holomorphy of an arbitrarily thin (interior) one-sided neighborhood of $\partial \Omega \backslash K$ is one-sheeted and identifies with $\Omega$.

[^26]:    ${ }^{28}$ In the general case $f \in \mathscr{C}^{0}$, one shrinks slightly $\Omega_{g}$ inside $\Omega$, rounds off its corners and passes to the limit.

[^27]:    ${ }^{29}$ Appropriate background, further survey of Lupacciolu's results and additional material may be found in [6].

[^28]:    ${ }^{30}$ We believe that $\mathscr{C}^{2, \alpha}$-smoothness of $M^{1}$ is required in the proof built there, since the map $w \mapsto \widehat{h}(w)$ appearing in equation (3.12) of [Jö1999a] (that corresponds essentially to the singular integral $\mathscr{J}(v)$ defined in $(3.20)(\mathrm{V}))$ already requires $M^{1}$ to be $\mathscr{C}^{1, \alpha}$ with

[^29]:     be at least $\mathscr{C}^{2, \alpha}$.

[^30]:    ${ }^{31}$ Observe that since $D$ is totally real, the last step can be done without changing $S^{\prime}$ along $D^{\prime}$.

[^31]:    ${ }^{32}$ The classical notion of $L^{\mathrm{p}}$-removability ([HP1970, Jö1988, Jö1999b, MP1999]) means that $L_{l o c}^{\mathrm{p}}(M) \cap L_{l o c, C R}^{\mathrm{p}}(M \backslash K)=L_{l o c, C R}^{\mathrm{p}}(M)$; CR-removability means that $\mathscr{C}_{C R}^{0}(M \backslash K)$ extends holomorphically to some global one-sided neighborhood $\omega_{M}$ of $M$; $\mathscr{W}$-removability means essentially that the same extension property holds for $\mathscr{O}\left(\omega_{M \backslash K}\right)$; the reader is referred to [MP2002] for rigorous and precise definitions, valuable in arbitrary codimension, and to [6, Jö1999a] for similar concepts, presented from the standard hypersurface perspective.

[^32]:    ${ }^{33}$ Needless to say, there is no direct direct magical translation from characteristic segments to holomorphic curves.

[^33]:    ${ }^{34}$ However, most points $p \in C$ are in fact not locally removable. Indeed, the simplest example of a local CR singularity being the intersection of $M$ with some local holomorphic curve $\Sigma$, which yields a local real curve $\mu:=\Sigma \cap M$, it may well happen that such a curve $\mu$ is fully contained in the closed set $C$, since $C$ which might have nonempty interior in $S$ (as in the figures).

[^34]:    ${ }^{35}$ We remind from [Me1997, MP1999, MP2002] that in codimension $\geqslant 2$, a wedge of edge a deformation $M^{d}$ of $M$ does not in general contain a wedge of edge $M$. This is why stability arguments are needed.

[^35]:    ${ }^{36}$ Complete, self-contained background is provided in [29].

[^36]:    ${ }^{37}$ This property will be crucial to insure uniqueness of the holomorphic extension, when we apply the continuity principle in Lemma 9.16.

[^37]:    ${ }^{38}$ To define the $\mathscr{O}(\bar{\Omega})$-hull, simply replace polynomials by functions holomorphic in some (unspecified) neighborhood of $\bar{\Omega}$.

[^38]:    ${ }^{39}$ We thank an anonymous referee for pointing historical incorrections in the preliminary version of this paper and for providing us with exact informations.

[^39]:    ${ }^{40} \mathrm{~A}$ certain number of other simpler cases will also happen, where the components $\widetilde{\Omega}_{>r}^{c}$ do grow regularly with respect to holomorphic extension, possibly changing topology.

[^40]:    ${ }^{41}$ In fact, one can just take the translated radius $r(z)-r\left(\widehat{p}_{\lambda}\right)$ as the coordinate $v=v(z)$.

[^41]:    ${ }^{42}$ Because in the sequel, the union $\mathscr{U}_{1} \cup \mathscr{U}_{2}$ would sometimes be a rather long, complicated expression (see e.g. (3.9)), hence uneasy to read, we will also say that $\mathscr{O}\left(\mathscr{U}_{1}\right)$ extends holomorphically and uniquely to $\mathscr{U}_{2}$.

[^42]:    ${ }^{43}$ We let the letter $c$ (resp. $C$ ) denote a positive constant $<1$ (resp. $>1$ ), absolute or depending only on $n$, which is allowed to vary with the context.

[^43]:    ${ }^{44}$ If the spheres $\mathrm{S}_{r}^{2 n-1}$ for $r$ close to $r^{\prime}$ would be hyperplanes - they almost are in comparison to $\mathbb{B}^{n}\left(q_{t}, \delta\right)$ - we would have exactly $r\left(p_{t}^{\prime}\right)=r\left(p^{\prime}\right)+t$, whence $\frac{d r\left(p_{t}^{\prime}\right)}{d t}=1$.

[^44]:    ${ }^{45}$ Sets written " $\{\cdot\}$ " here are understood to be subsets of $C_{\eta}$.

[^45]:    ${ }^{46}$ The reader in referred to point $\mathbf{2}$ ) of the proof of Prosition 4.1 below and to Figure 3 in Section 4 for an illustration of the concerned univalent extension argument.

[^46]:    ${ }^{47}$ The previous definitions are known to be independent of the choices - covering, embeddings $\varphi_{j}$, dimensions $N_{j}$, extensions $\widetilde{(\bullet)}$, see $[8,10,13]$.

    48 The first statement yields immediately that every point $z \in X$ has a neighborhood basis $\left(\mathscr{V}_{k}\right)_{k \in \mathbb{N}}$ such that $X_{\text {reg }} \cap \mathscr{V}_{k}$ is connected; also, $X_{\text {reg }}$ itself is connected as soon as $X$ is so. The second statement is known as Levi's extension theorem ([11], p. 185).

[^47]:    49 i.e. by definition, a function which is locally the sum of a psh function and of a $\mathscr{C}^{\infty}$ function, or equivalently, a function $v$ whose complex Hessian $i \partial \bar{\partial} v$ has bounded negative part.
    ${ }^{50}$ In addition, a regularized maximum function ([4]) is used to smoothly glue these different definitions on all finite intersections $A_{j_{1}} \cap \cdots \cap A_{j_{m}}$ and the formula given here is exact on a sub-ball $\widetilde{\mathrm{C}}_{j} \subset \widetilde{\mathrm{~B}}_{j}$.

[^48]:    ${ }^{51}$ The previous four properties being preserved, especially (d) on $\{\rho=C\}$.
    ${ }^{52}$ The perturbed $M$ being still contained in $\{\rho<C\}$ and in the original connected corona $\Omega \backslash K$.

[^49]:    ${ }^{53}$ Since the configuration is always local and biholomorphic to $\mathbb{C}^{n}\left(n=\operatorname{dim} X_{\text {reg }}\right)$ and since holomorphic envelopes coincide with meromorphic envelopes in $\mathbb{C}^{n}$, meromorphic functions enjoy exactly the same extension properties. Thus, in [27], results stated for holomorphic functions are immediately true for meromorphic functions too.
    ${ }^{54}$ We emphasize that, from the point of view of holomorphic extension, jumping across critical points of $\mu$ on $X_{\text {reg }}$ is much simpler than jumping across critical points of $\left.\mu\right|_{M}, c f$. the $\mathbb{C}^{n}$ case [27].
    ${ }^{55}$ In $\mathbb{R}^{3}$ already, this is true for the "exterior" $x^{2}+y^{2}-z^{2}>\delta$ of the standard cone.

