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Travaux sur une conjecture de Green-Griffiths quant à la dégénérescence algébrique des courbes holomorphes entières à valeurs dans des hypersurfaces projectives complexes

[362 pages]

Document 1, p. 2: Travail récemment prépublié sur arxiv.org: étude des jets de Green-Griffiths d'ordre $\kappa \to \infty$ pour des courbes holomorphes à valeurs dans des hypersurfaces algébriques lisses $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ de dimension $n \geqslant 2$ arbitraire et de type général. Grâce à une majoration de la cohomologie des fibrés de Schur, j'établis l'existence de différentielles de jets en degré optimal $\deg X \geqslant \dim X + 3$.

Document 2, p. 94: Travail paru en février 2010 aux *Inventiones Mathematicæ* qui établit la dégénérescence algébrique des courbes holomorphes entières à valeurs dans des hypersurfaces $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ génériques de degré $\deg X \geqslant 2^{n^5}$.

Document 3, p. 154: Travail (publié) sur les jets verticaux de l'hypersurface universelle et dont le théorème principal est appliqué dans le Document 2.

Document 4, p. 179: Second travail (paru récemment) sur les jets de Demailly-Semple; production d'un algorithme d'engendrement général; deux applications complètes: $n = 2$, $\kappa = 5$; et: $n = 4$, $\kappa = 4$. Calculs: 80% à la main; 20% sur machine.

Document 5, p. 288: Premier travail (publié) sur les jets de Demailly-Semple: découverte que les crochets sont insuffisants; présence implicite et volontairement cachée du bon algorithme (Document 4).

[5 documents, travaux effectués entre mars 2007 et avril 2010.]

Algebraic differential equations for entire holomorphic curves in projective hypersurfaces of general type: optimal lower degree bound

Joël Merker

Abstract. Let $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a geometrically smooth projective algebraic complex hypersurface. Using Green-Griffiths jets, we establish the existence of nonzero global algebraic differential equations that must be satisfied by every nonconstant entire holomorphic curve $\mathbb{C} \to X$ if X is of general type, namely if its degree d satisfies the optimal possible lower bound:

$$d \geqslant n + 3.$$

The case $n = 2$ dates back to Green-Griffiths 1979, while according to very recent advances (Invent. Math. **180**, pp. 161–223, February 2010), the best (and only) lower degree bound known previously in arbitrary dimension n was, using instead Demailly-Semple jets:

$$d \geqslant 2^{n^4}\, n^{4n^3}\, 3^{n^3}\, n^{3n^2}\, (n+1)^{n^2+1}\, n^{2n}\, 12,$$

which, visibly, was far from the conjectured $n + 3$.

arxiv.org/abs/1005.0405/

Table of contents

§1. Introduction

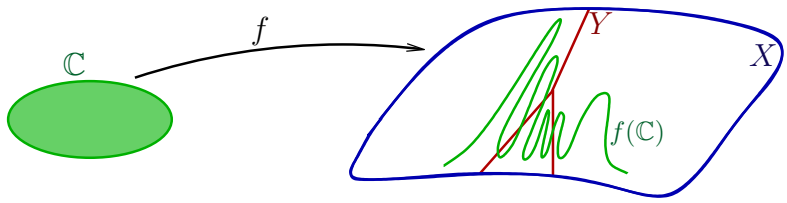
Let X be an n-dimensional ($n \geqslant 1$) compact complex manifold and assume it to be *of general type*, i.e., if as usual $K_X = \Lambda^n T^*_X$ denotes its canonical line bundle, assume that the dimension of the space of global pluricanonical sections:

$$h^0\big(X,\, (K_X)^{\otimes m}\big) \geqslant \mathsf{Constant} \cdot m^{\dim X} \qquad (\mathsf{Constant} > 0)$$

grows the fastest it can, as $m \to \infty$, namely the Kodaira dimension of X is maximal equal to n. According to a theorem due to Kodaira, X can then

be embedded as a geometrically smooth projective algebraic complex manifold in a certain complex projective space $\mathbb{P}^N(\mathbb{C})$. Though it is somewhat delicate to select good embeddings, it is algebraically convenient to view X as being projective *per se*.

In 1979, Green and Griffiths [17] conjectured that there should exist in X a certain *proper* algebraic *sub*variety $Y \subsetneqq X$ (possibly with singularities) inside which all nonconstant entire holomorphic curves $f\colon \mathbb{C} \to X$ must necessarily lie, without any such f being allowed to wander anywhere else in $X \backslash Y$.



According to a strategy of thought going back to Bloch, modernized by Green-Griffiths and viewed in a new light by Siu, the 'first half' of this conjecture — so to say — consists in showing that there exist some nonzero global algebraic jet differentials that must be satisfied by every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$ (*see* [35, 30, 17] for aspects of the 'second half', not at all considered here).

The principal theorem of this memoir is presented specifically in the case where $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ is a (geometrically smooth) n-dimensional *hypersurface*, because the main mathematical difficulty is essentially to reach arbitrary dimensions $n \geqslant 1$, as was shown recently by the complexity of some of the formal computations sketched in [22, 17] for the case of dimension $n = 4$. But because a substantial part of our proof relies upon works of Brückmann ([5, 1, 7]) which hold in fact for complete intersections, it is very likely that our results may be transferred to such a more general context. Also, one could consider entire holomorphic maps $\mathbb{C}^p \to X^n$ having maximal generic rank p with for any fixed $1 \leqslant p \leqslant n$, as did Pacienza and Rousseau ([32]) recently for $p = 2$ in the case of $X^3 \subset \mathbb{P}^4(\mathbb{C})$. Furthermore, we hope more generally that the techniques developed here could in the future enable us to handle any $X^n \subset \mathbb{P}^N(\mathbb{C})$ of general type having arbitrary codimension $N - n$, but probably requiring more than just general type as a workable assumption ([43]). At least in codimension 1, we are able to gain the following optimal result toward the Green-Griffiths conjecture.

Main Theorem. *Let $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a geometrically smooth n-dimensional projective algebraic complex hypersurface. If X is of general type, namely if its degree d satisfies the optimal lower bound:*

$$d \geqslant n + 3,$$

then there exist global algebraic differential equations on X that must be identically satisfied by every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$.

More precisely, if $\mathscr{E}^{GG}_{\kappa,m}T^*_X$ denotes the bundle of Green-Griffiths jet polynomials[1] *of order κ and of weight m over X, then the following holds true.*

Firstly: *for any fixed ample line bundle $\mathscr{A} \to X$ — take* e.g. *simply $\mathscr{A} := \mathscr{O}_X(1)$ —, one has:*

(1)
$$h^0\big(X,\, \mathscr{E}^{GG}_{\kappa,m}T^*_X \otimes \mathscr{A}^{-1}\big) \geqslant \frac{m^{(\kappa+1)n-1}}{(\kappa!)^n\,((\kappa+1)n-1)!}\Big\{\tfrac{(\log\kappa)^n}{n!}d(d-n-2)^n - \mathsf{Constant}_{n,d}\cdot(\log\kappa)^{n-1}\Big\} - \\ - \mathsf{Constant}_{n,d,\kappa}\cdot m^{(\kappa+1)n-2},$$

and the right-hand side minorant visibly tends to ∞, as soon as both $\kappa \geqslant \kappa^0_{n,d}$ and $m \geqslant m^0_{n,d,\kappa}$ are large enough.

Secondly: *If P is any global section of $\mathscr{E}^{GG}_{\kappa,m}T^*_X \otimes \mathscr{A}^{-1}$, hence which vanishes on the ample divisor associated to $\mathscr{A}$, then every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$ must satisfy the corresponding algebraic differential equation $P(j^\kappa f) = 0$.*

Since the late 1990's, after fundamental works of Bloch, Green-Griffiths and Siu, the so-called Ahlfors-Schwarz for entire holomorphic curves was clarified in full generality, and the second statement above is nowadays (well) known to be a consequence of the first (*see* e.g. Section 7 in [4]).

The case $n = 2$ of this theorem dates back to Green-Griffiths 1979 ([17]). In [30], Rousseau was the first to study effective (Demailly-Semple) jet differentials in dimension 3, under the conditions $d \geqslant 97$. In [10], Diverio treated the next dimensions $n = 4$ and $n = 5$ (improving also $n = 3$ with $d \geqslant 74$), under the conditions $d \geqslant 298$ and $d \geqslant 1222$. In [22], the author of the present article improved for $n = 4$ the lower bound to $d \geqslant 259$. In [15], Diverio showed the (noneffective) existence of a lower bound degree d_n such that $d \geqslant d_n$ insures existence of nonzero global jet differentials. An effective d_n was captured in [17] (*see* $\widetilde{d}^1_n$, p. 192):

$$d \geqslant 2^{n^4}\, n^{4n^3}\, 3^{n^3}\, n^{3n^2}\, (n+1)^{n^2+1}\, n^{2n}\, 12,$$

far from the optimal $n + 3$, *see* Section 13 for some explanations. Also, one must mention that except in the original Green-Griffiths article [17] for the case $n = 2$, none of the references cited provides explicit effective minorations of h^0 as (1) just above.

In conference talks given in the CIRM (June 2009), in the Hong-Kong University (August 2009) and also later in some seminars (Paris, Marseille, Lyon), the author announced that he was able to gain the Main Theorem

[1] See Sections 2 and 3 for exact definitions.

under the conjectural assumption that one can majorate the dimensions of the positive cohomology groups of the general Schur bundles[2]:

$$h^q = \dim H^q\big(X,\ \mathscr{S}^{(\ell_1,\dots,\ell_n)}T^*_X\big) \qquad\qquad {\scriptstyle (q=1\cdots n)}$$

over X by a specific formula of the general type:

(2)

$$h^q \leqslant \mathsf{Constant}_n \cdot \big[1+d+\cdots+d^{n+1}\big] \cdot \prod_{1\leqslant i<j\leqslant n} \big(\ell_i-\ell_j\big) \cdot \sum_{\substack{\beta_1+\cdots+\beta_{n-1}+\beta_n=n\\ \beta_n\leqslant n-1}} (\ell_1-\ell_2)^{\beta_1}\cdots(\ell_{n-1}-\ell_n)^{\beta_{n-1}}(\ell_n)^{\beta_n} + \\ + \mathsf{Constant}_{n,d}\,\big(1+|\ell|^{\frac{n(n+1)}{2}-1}\big),$$

where $|\ell| = \ell_1+\cdots+\ell_n$, in which, principally, the exponent β_n of ℓ_n is constrained to be $\leqslant n-1$. In fact, we shall establish that a certain graded bundle $\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X$ introduced by Green and Griffiths themselves which is naturally associated to the jet bundle $\mathscr{E}^{GG}_{\kappa,m}T^*_X$ decomposes as a certain well controlled direct sum of Schur bundles:

$$\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X = \bigoplus_{\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_n\geqslant 0} \Big(\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)}T^*_X\Big)^{M^{\kappa,m}_{\ell_1,\ell_2,\dots,\ell_n}},$$

with multiplicities $M^{\kappa,m}_{\ell_1,\ell_2,\dots,\ell_n} \in \mathbb{N}$ understood in combinatorial terms and which are zero for some obvious reasons when $|\ell|$ is less than $\frac{m}{\kappa}$. As is known (cf. Section 2), the cohomology of $\mathscr{E}^{GG}_{\kappa,m}T^*_X$ is controlled (majorated) by the cohomology of $\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X$. The conjectural majoration (2) above was then the most prudent majoration which would insure that each positive cohomology dimension ($1\leqslant q\leqslant n$):

$$h^q\big(X,\ \mathscr{E}^{GG}_{\kappa,m}T^*_X\big) \leqslant \sum_{\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_n\geqslant 0} M^{\kappa,m}_{\ell_1,\ell_2,\dots,\ell_n}\cdot h^q\big(X,\ \mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)}T^*_X\big) \\ \leqslant \tfrac{m^{(\kappa+1)n-1}}{(\kappa!)^n\,((\kappa+1)n-1)!}\cdot \mathsf{O}_{n,d}\big(\log(\kappa)^{n-1}\big) + \mathsf{O}_{n,d,\kappa}\big(m^{(\kappa+1)n-2}\big)$$

(Sections 7, 8, 9 and 10 are devoted to establishing the second inequality) is majorated[3] by a quantity which becomes asymptotically negligible in comparison to the Euler-Poincaré characteristic computed in 1979 by Green and Griffiths:

$$\chi\big(X,\ \mathscr{E}^{GG}_{\kappa,m}T^*_X\big) = \tfrac{m^{(\kappa+1)n-1}}{(\kappa!)^n\,((\kappa+1)n-1)!}\Big\{d(d-n-2)^n\,\frac{(\log\kappa)^n}{n!} + \mathsf{O}_{n,d}\big(\log(\kappa)^{n-1}\big)\Big\} + \\ + \mathsf{O}_{n,d,\kappa}\big(m^{(\kappa+1)n-2}\big),$$

[2] Definitions and properties may be found in Sections 11 and 4.

[3] Throughout the paper, we shall sometimes write $\mathsf{O}_{n,d,\kappa}\big(m^{(\kappa+1)n-2}\big)$ to denote a quantity which is majorated by $\mathsf{Constant}_{n,d,\kappa}\cdot m^{(\kappa+1)n-2}$.

so that a minoration for h^0 of the sort claimed by the Main Theorem above then immediately follows from $\chi = h^0 - h^1 + \cdots + (-1)^n h^n$ by observing simply:

$$h^0 \geqslant \chi - h^2 - h^4 - h^6 - \cdots .$$

But in fact, we shall obtain a majoration better than (2):

$$h^q \leqslant \mathsf{Constant}_{n,d} \prod_{1 \leqslant i < j \leqslant n} (\ell_i - \ell_j) \Big[\sum_{\beta'_1 + \cdots + \beta'_{n-1} = n} (\ell_1 - \ell_2)^{\beta'_1} \cdots (\ell_{n-1} - \ell_n)^{\beta'_{n-1}} \Big] + \\ + \mathsf{Constant}_{n,d} \big(1 + |\ell|^{\frac{n(n+1)}{2} - 1} \big).$$

Acknowledgments. The author would like to thank Professor Ngaiming Mok, Director, and Professor Yum-Tong Siu, C.V. Starr Visiting Professor, for inviting him at the Hong Kong University's Department of Mathematics during August 4–14, 2009 when parts of the present memoir were finalized and presented orally at a Summer conference entitled "*Workshop on Complex Geometry*". Mainly, he must express his debt to the works [29, 30] by Erwan Rousseau for $n = \kappa = 3$ to whom the present general Schur bundle strategy was borrowed, and also, he would like to mention his debt to the works [5, 1] by Peter Brückmann. Besides, the present memoir has benefited of enlightening discussions with Simone Diverio and with Alain Lascoux. Pascal Dingoyan indicated appropriate references. Jacky Cresson and Michel Petitot provided information about multiple polylogarithms. Lionel Darondeau suggested a few expositional improvements.

§2. Universal combinatorics of Green-Griffiths jets

Algebraic hypersurfaces in a complex projective space. Let $n \geqslant 1$ be a positive integer. On the complex euclidean space $\mathbb{C}^{n+1} \setminus \{0\}$ with the origin deleted, consider so-called *homogeneous coordinates*:

$$[z] := \big[z_0 \colon z_1 \colon \cdots \colon z_n \colon z_{n+1} \big]$$

with the convention that for every nonzero $c \in \mathbb{C} \setminus \{0\}$, any $[cz]$ all of which coordinates are equally c-dilated is equivalent to $[z]$:

$$\big[cz_0 \colon cz_1 \colon \cdots \colon cz_n \colon cz_{n+1} \big] = \big[z_0 \colon z_1 \colon \cdots \colon z_n \colon z_{n+1} \big] .$$

The set of such $\big[z_0 \colon \cdots \colon z_{n+1} \big]$ constitutes the so-called *complex projective space* of dimension $n+1$:

$$\begin{aligned} \mathbb{P}_{n+1}(\mathbb{C}) &:= \big\{ \big[z_0 \colon z_1 \colon \cdots \colon z_n \colon z_{n+1} \big] \big\} \\ &= \big\{ z = (z_0, z_1, \ldots, z_n, z_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\} \big\} \big/ \big(z \sim \mathbb{C}^* z \big), \end{aligned}$$

and is a compact complex manifold. Consider now a complex projective algebraic hypersurface:

$$X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$$

of dimension n given as the zero-set:

$$X = \big\{ \big[z_0 \colon z_1 \colon \ldots \colon z_n \colon z_{n+1} \big] \in \mathbb{P}^{n+1}(\mathbb{C}) \colon \\ P(z_0, z_1, \ldots, z_n, z_{n+1}) = 0 \big\} .$$

of some polynomial which is homogeneous of a certain degree $d \geqslant 1$, say of the general form:

$$P := \sum_{\beta_0+\beta_1+\cdots+\beta_n+\beta_{n+1}=d} \mathsf{coeff} \cdot z_0^{\beta_0} z_1^{\beta_1} \cdots z_n^{\beta_n} z_{n+1}^{\beta_{n+1}}$$

with fixed complex coefficients $P_{\beta_0,\beta_1,\ldots,\beta_n,\beta_{n+1}}$, one of them at least being nonzero. We assume throughout that X is geometrically smooth, namely the differential of P is nonzero at every point of X.

Jets of holomorphic discs in X. In a local chart $(X, x_0) \simeq (\mathbb{C}^n, 0)$ centered at a point $x_0 \in X$ equipped with n complex coordinates $(x_1, \ldots, x_n)$, one looks at holomorphic discs passing through x_0:

$$f : (\mathbb{D}, 0) \to (\mathbb{C}^n, 0) \simeq (X, x_0),$$

namely with $f(0) = x_0$, which possess of course n components:

$$\big(f_1(\zeta), f_2(\zeta), \ldots, f_n(\zeta)\big).$$

For any integer $\kappa \geqslant 1$, the associated κ-jet map of any such a holomorphic disc gathers all its $n\kappa$ derivatives up to order κ with respect to the (single) source variable $\zeta \in \mathbb{D}$:

$$j^\kappa f(\zeta) = \big(f_1', \ldots, f_n', f_1'', \ldots, f_n'', \ldots\ldots, f_1^{(\kappa)}, \ldots, f_n^{(\kappa)}\big)(\zeta).$$

Accordingly, one is led to introduce $n\kappa$ new independent *jet coordinates* that will simply be denoted as:

$$\big(x_1', \ldots, x_n',\ x_1'', \ldots, x_n'',\ \ldots\ldots,\ x_n^{(\kappa)}, \ldots, x_n^{(\kappa)}\big),$$

so that $\big(x, x', x'', \ldots, x^{(\kappa)}\big)$ provide $n + n\kappa$ coordinates on the space of *un*centered κ-jets of maps $\mathbb{D} \to X$.

Weighted homogeneous jet polynomials. Above any fixed point $x_0 \in X$, Green-Griffiths ([17]) introduced a certain "fiber" which consists of all polynomials in these jet variables $x_i^{(\lambda)}$, $1 \leqslant i \leqslant n$, $1 \leqslant \lambda \leqslant \kappa$, that are of the following type:

(3)
$$\mathsf{P}\big(x', x'', \ldots, x^{(\kappa)}\big) = \sum_{\substack{\alpha_1,\alpha_2,\ldots,\alpha_\kappa \in \mathbb{N}^n \\ |\alpha_1|+2|\alpha_2|+\cdots+\kappa|\alpha_\kappa|=m}} \mathsf{coeff}_{\alpha_1,\alpha_2,\ldots,\alpha_\kappa} \cdot \big(x'\big)^{\alpha_1} \big(x''\big)^{\alpha_2} \cdots \big(x^{(\kappa)}\big)^{\alpha_\kappa},$$

where $m \geqslant 1$ is some integer, where $\alpha_\lambda = (\alpha_{\lambda,1}, \ldots, \alpha_{\lambda,n}) \in \mathbb{N}^n$ for $1 \leqslant \lambda \leqslant \kappa$ are multiindices of length $|\alpha_\lambda| = \alpha_{\lambda,1} + \cdots + \alpha_{\lambda,n}$, and where $\mathsf{coeff}_{\alpha_1,\alpha_2,\ldots,\alpha_\kappa}$ are arbitrary complex coefficients, or equivalently if written in greater length:

$$\sum_{\substack{\alpha_1,\alpha_2,\ldots,\alpha_\kappa \in \mathbb{N}^n \\ |\alpha_1|+2|\alpha_2|+\cdots+\kappa|\alpha_\kappa|=m}} \mathsf{coeff}_{\alpha_1,\alpha_2,\ldots,\alpha_\kappa} \cdot \prod_{1\leqslant i\leqslant n} \big(x_i'\big)^{\alpha_{1,i}} \prod_{1\leqslant i\leqslant n} \big(x_i''\big)^{\alpha_{2,i}} \cdots \prod_{1\leqslant i\leqslant n} \big(x_i^{(\kappa)}\big)^{\alpha_{\kappa,i}}.$$

Visibly, such polynomials enjoy weighted homogeneity:

$$\mathsf{P}\big(\delta x',\, \delta^2 x'',\, \ldots,\, \delta^\kappa x^{(\kappa)}\big) = \delta^m\, \mathsf{P}\big(x',\, x'',\, \ldots,\, x^{(\kappa)}\big)$$

of the fixed weight m with respect to the anisotropic complex jet dilation defined by:

$$\delta \cdot \big(x'_{i_1}, x''_{i_2}, \ldots, x^{(\kappa)}_{i_\kappa}\big) := \big(\delta\, x'_{i_1}, \delta^2 x''_{i_2}, \ldots, \delta^\kappa x^{(\kappa)}_{i_\kappa}\big), \qquad\qquad \delta \in \mathbb{C},$$

whence for memory in all what follows one sees that:

$$m = \mathsf{weight} = \text{(fixed) total number of appearing “primes”}.$$

Lemma. *As the point x_0 runs in X, these polynomial fibers organize coherently as a* holomorphic vector *bundle $\mathscr{E}^{GG}_{\kappa,m} T^*_X$ over X of rank equal to the number of arbitrary coefficients* $\mathsf{coeff}_{\alpha_1,\ldots,\alpha_\kappa}$*, namely to:*

$$\mathrm{Card}\,\big\{(\alpha_1, \alpha_2, \ldots, \alpha_\kappa) \in (\mathbb{N}^n)^\kappa \colon |\alpha_1| + 2|\alpha_2| + \cdots + \kappa|\alpha_\kappa| = m\big\}.$$

Proof. Consider an arbitrary change of (local) holomorphic chart on X:

$$(x_1, \ldots, x_n) \longmapsto (y_1, \ldots, y_n) = \big(\Psi_1(x_1, \ldots, x_n),\, \ldots\ldots,\, \Psi_n(x_1, \ldots, x_n)\big),$$

understood as inducing a change of local trivialization for the bundle. One must establish that the new coefficients of the transformed jet polynomials express *linearly* in terms of the coefficients of (3).

To begin with, the knowledge of how the $y^{(\lambda)}_j$ express in terms of the $x^{(\tau)}_i$ is provided by an application of the chain rule for the differentiation of a composed holomorphic disc $\zeta \mapsto \Psi\big(f_1(\zeta), \ldots, f_n(\zeta)\big)$. The closed combinatorial formula writes as follows, for any λ with $1 \leqslant \lambda \leqslant \kappa$ and for any j with $1 \leqslant j \leqslant n$.

Theorem. (see [2, 27]) *The λ-jet of $\Psi_j(f_1, \ldots, f_n)$ is given by the following multivariate Faà di Bruno formula, written without the argument ζ:*

$$\big[\Psi_j(f_1, \ldots, f_n)\big]^{(\lambda)} = \sum_{e=1}^{\lambda} \sum_{1 \leqslant \tau_1 < \cdots < \tau_e \leqslant \lambda} \sum_{\mu_1 \geqslant 1, \ldots, \mu_e \geqslant 1} \sum_{\mu_1 \tau_1 + \cdots + \mu_e \tau_e = \lambda} \frac{\lambda!}{(\tau_1!)^{\mu_1} \mu_1! \cdots (\tau_e!)^{\mu_e} \mu_e!}$$
$$\sum_{j^1_1, \ldots, j^1_{\mu_1} = 1}^{n} \cdots\cdots \sum_{j^e_1, \ldots, j^e_{\mu_e} = 1}^{n} \frac{\partial^{\mu_1 + \cdots + \mu_e} \Psi_j}{\partial x_{j^1_1} \cdots \partial x_{j^1_{\mu_1}} \cdots\cdots \partial x_{j^e_1} \cdots \partial x_{j^e_{\mu_e}}} \cdot$$
$$\cdot\, f^{(\tau_1)}_{j^1_1} \cdots f^{(\tau_1)}_{j^1_{\mu_1}} \cdots\cdots f^{(\tau_e)}_{j^e_1} \cdots f^{(\tau_e)}_{j^e_{\mu_e}}.$$

To read this general formula, we comment it backward, understanding it rather as a (polynomial, invertible) transformation between independent jet

variables:

$$y_j^{(\lambda)} = \sum_{e=1}^{\lambda} \sum_{1\leqslant\tau_1<\cdots<\tau_e\leqslant\lambda} \sum_{\mu_1\geqslant 1,\ldots,\mu_e\geqslant 1} \sum_{\mu_1\tau_1+\cdots+\mu_e\tau_e=\lambda} \frac{\lambda!}{(\tau_1!)^{\mu_1}\mu_1!\cdots(\tau_e!)^{\mu_e}\mu_e!}$$
$$\sum_{j_1^1,\ldots,j_{\mu_1}^1=1}^{n} \cdots\cdots \sum_{j_1^e,\ldots,j_{\mu_e}^e=1}^{n} \frac{\partial^{\mu_1+\cdots+\mu_e}\Psi_j}{\partial x_{j_1^1}\cdots\partial x_{j_{\mu_1}^1}\cdots\cdots\partial x_{j_1^e}\cdots\partial x_{j_{\mu_e}^e}}\cdot$$
$$\cdot\, x_{j_1^1}^{(\tau_1)}\cdots x_{j_{\mu_1}^1}^{(\tau_1)}\cdots\cdots x_{j_1^e}^{(\tau_e)}\cdots x_{j_{\mu_e}^e}^{(\tau_e)}.$$

The general monomial $\prod x_\bullet^{(\tau_1)}\prod x_\bullet^{(\tau_2)}\cdots\prod x_\bullet^{(\tau_2)}$ in the jet variables gathers derivatives of increasing orders $\tau_1<\tau_2<\cdots<\tau_e$ with $\mu_1,\mu_2,\ldots,\mu_e$ counting their respective numbers. Then Ψ_j is subjected to a partial derivative of order $\mu_1+\mu_2+\cdots+\mu_e$, the total number of letters $x_\bullet^\bullet$ in the monomial in question. Because there are $n+1$ variables x_i, the dots in the $x_\bullet^{(\tau_c)}$ should receive indices, and in fact, there appear general sums $\sum_{j_1^c,\ldots,j_{\mu_c}^c=1}^{n}$ over *all possible* such indices.

This precise closed combinatorial formula is not really needed for the proof of our lemma, and instead, it is sufficient to know that each $y_j^{(\lambda)}$ is a certain polynomial in the $x_i^{(\tau_c)}$, with coefficients depending linearly upon the λ-jet of Ψ, the weight $\mu_1\tau_1+\cdots+\mu_e\tau_e$ of each appearing monomial $x_{j_1^1}^{(\tau_1)}\cdots x_{j_{\mu_1}^1}^{(\tau_1)}\cdots\cdots x_{j_1^e}^{(\tau_e)}\cdots x_{j_{\mu_e}^e}^{(\tau_e)}$ being constant equal to λ, and this fact is easily proved by a rough induction argument. We can abbreviate this as:

$$y_j^{(\nu)} = \sum_{i_1=1}^{n}\frac{\partial\Psi_j}{\partial x_{i_1}} + \cdots + \sum_{i_1,\ldots,i_\lambda=1}^{n}\frac{\partial^\lambda\Psi_j}{\partial x_{i_1}\cdots\partial x_{i_\lambda}}$$

Consequently, we must, as said, examine how an y-jet general polynomial of weight m like the x-jet polynomial P in (3):

$$\mathsf{Q}\big(y',\,y'',\,\ldots,\,y^{(\kappa)}\big) = \sum_{\substack{\alpha_1,\,\alpha_2,\,\ldots,\,\alpha_\kappa\in\mathbb{N}^n\\ |\alpha_1|+2|\alpha_2|+\cdots+\kappa|\alpha_\kappa|=m}} \mathsf{Q}_{\alpha_1,\alpha_2,\ldots,\alpha_\kappa}\cdot\big(y'\big)^{\alpha_1}\big(y''\big)^{\alpha_2}\cdots\big(y^{(\kappa)}\big)^{\alpha_\kappa}$$

is transformed. From the theorem stated above (or from the rough induction argument), we deduce that a general monomial of a weight m:

$$\begin{aligned}
&\big(y'\big)^{\alpha_1}\big(y''\big)^{\alpha_2}\cdots\big(y^{(\lambda)}\big)^{\alpha_\lambda}\cdots\big(y^{(\kappa)}\big)^{\alpha_\kappa} = \\
&\quad = \Big(\sum_{i_1}\Psi_{x_{i_1}}\,x'_{i_1}\Big)^{\alpha_1}\Big(\sum_{i_1}\Psi_{x_{i_1}}\,x''_{i_1} + \sum_{i_1,\,i_2}\Psi_{x_{i_1}x_{i_2}}\,x'_{i_1}x'_{i_2}\Big)^{\alpha_2}\cdots \\
&\quad\cdots\Big(\sum_{i_1}\Psi_{x_{i_1}}\,x_{i_1}^{(\lambda)} + \cdots + \sum_{i_1,\ldots,i_\lambda}\Psi_{x_{i_1}\ldots x_{i_\lambda}}\,x'_{i_1}\ldots x'_{i_\lambda}\Big)^{\alpha_\lambda}\cdots \\
&\quad\cdots\Big(\sum_{i_1}\Psi_{x_{i_1}}\,x_{i_1}^{(\kappa)} + \cdots + \sum_{i_1,\ldots,i_\kappa}\Psi_{x_{i_1}\ldots x_{i_\kappa}}\,x'_{i_1}\ldots x'_{i_\kappa}\Big)^{\alpha_\kappa}
\end{aligned} \tag{4}$$

is clearly transformed to a jet polynomial of weight m:

$$\big(y'\big)^{\alpha_1}\cdots\big(y^{(\kappa)}\big)^{\alpha_\kappa}=\sum_{|\beta_1|+\cdots+\kappa|\beta_\kappa|=m}\mathsf{H}^{\alpha_1,\ldots,\alpha_\kappa}_{\beta_1,\ldots,\beta_\kappa}\big(j^\kappa\Psi\big)\cdot(x')^{\beta_1}\cdots(x^{(\kappa)})^{\beta_\kappa}$$

having coefficients that are certain universal polynomials in the κ-jet of Ψ. It therefore follows that $\mathsf{Q}(y',\ldots,y^{(\kappa)})$ is transformed to:

$$\sum_{|\alpha_1|+\cdots+\kappa|\alpha_\kappa|=m}\sum_{|\beta_1|+\cdots+\kappa|\beta_\kappa|=m}\mathsf{Q}_{\alpha_1,\ldots,\alpha_\kappa}\cdot\mathsf{H}^{\alpha_1,\ldots,\alpha_\kappa}_{\beta_1,\ldots,\beta_\kappa}\big(j^\kappa\Psi\big)\cdot(x')^{\beta_1}\cdots(x^{(\kappa)})^{\beta_\kappa}$$
$$=:\sum_{|\beta_1|+\cdots+\kappa|\beta_\kappa|=m}\mathsf{P}_{\beta_1,\ldots,\beta_\kappa}\cdot(x')^{\beta_1}\cdots(x^{(\kappa)})^{\beta_\kappa}$$

with the following *linear* relationship between coefficients:

$$\mathsf{P}_{\beta_1,\ldots,\beta_\kappa}=\sum_{|\alpha_1|+\cdots+\kappa|\alpha_\kappa|=m}\mathsf{H}^{\alpha_1,\ldots,\alpha_\kappa}_{\beta_1,\ldots,\beta_\kappa}\big(j^\kappa\Psi\big)\cdot\mathsf{Q}_{\alpha_1,\ldots,\alpha_\kappa}.$$

This shows that $\mathscr{E}^{GG}_{\kappa,m}T^*_X$ effectively is a vector bundle, because the cocycle relations and the inverse trivializations follow from the transitivity and from the invertibility of change of local coordinates on X. $\square$

Symmetric pluri-tensor decomposition. As is known in the domain ([17, 4, 17]), a certain graded holomorphic vector bundle $\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X$ naturally associated to this Green-Griffiths bundle $\mathscr{E}^{GG}_{\kappa,m}T^*_X$ happens to decompose into the following direct sum of multi-tensored symmetric powers of the cotangent bundle T^*_X of X:

$$\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X=\bigoplus_{\ell_1+2\ell_2+\cdots+\kappa\ell_\kappa=m}\mathrm{Sym}^{\ell_1}T^*_X\otimes\mathrm{Sym}^{\ell_2}T^*_X\otimes\cdots\otimes\mathrm{Sym}^{\ell_\kappa}T^*_X.$$

Informally speaking, such a decomposition just relates to the fact that the general m-weighted polynomial (3) on p. 7 looks like a linear combination of (tensor!) products of the (individually symmetric!) monomials $(x')^{\alpha_1}$, $(x'')^{\alpha_2}$, $\ldots$, $(x^{(\kappa)})^{\alpha_\kappa}$ with, say for a good correspondence:

$$\ell_1\equiv|\alpha_1|,\qquad\ell_2\equiv|\alpha_2|,\qquad\ldots\ldots,\qquad\ell_\kappa\equiv|\alpha_\kappa|.$$

But this view is not rigorous, so let us explain with more details than in [17, 4, 17] how one builds the graded bundle $\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X$.

Consider again the transformation (4). If $\alpha_{\lambda+1}=\cdots=\alpha_\kappa=0$ for some λ, one sees that after expansion, the total number $|\alpha_1|+2|\alpha_2|+\cdots+\lambda|\alpha_\lambda|$ of primes remains unchanged. However, if there exists some μ with $\lambda+1\leqslant\mu\leqslant\kappa$ such that $\alpha_\mu\neq0$, then in general the expansion of the factor $\big(\Psi(x)^{(\mu)}\big)^{\alpha_\mu}$ adds a total of $\mu|\alpha_\mu|$ further primes to the monomials in x', x'', $\ldots$, $x^{(\lambda)}$ that was already obtained by expanding the first λ factors $\big(\Psi(x)'\big)^{\alpha_1}\cdots\big(\Psi(x)^{(\lambda)}\big)^{\alpha_\lambda}$. Thus in all cases, after an arbitrary change of coordinates $x\mapsto\Psi(x)$, the λ-restricted weight $|\alpha_1|+2|\alpha_2|+\cdots+\lambda|\alpha_\lambda|$

can only increase. Following [17], one may hence define for any λ fixed in advance with $1 \leqslant \lambda \leqslant \kappa$ a (decreasing) *filtered sequence*:

$$\mathscr{E}^{GG}_{\kappa,m}T^*_X = \mathscr{F}^0_\lambda \supset \mathscr{F}^1_\lambda \supset \mathscr{F}^2_\lambda \supset \cdots \supset \mathscr{F}^m_\lambda \supset \{0\} = \mathscr{F}^{m+1}_\lambda$$

of subbundles of $\mathscr{E}^{GG}_{\kappa,m}T^*_X$ whose pieces for any $q = 1, 2, \ldots, m$ are naturally defined by:

$$\mathscr{F}^q_\lambda = \mathscr{F}^q_\lambda\big(\mathscr{E}^{GG}_{\kappa,m}T^*_X\big) = \left\{ \begin{array}{l} \mathsf{P}\big(x', \ldots, x^{(\lambda)}, \ldots, x^{(\kappa)}\big) \in \mathscr{E}^{GG}_{\kappa,m}T^*_X \text{ involving only monomials} \\ (x')^{\alpha_1} \cdots (x^{(\lambda)})^{\alpha_\lambda} \cdots (x^{(\kappa)})^{\alpha_\kappa} \text{ with } |\alpha_1| + \cdots + \lambda|\alpha_\lambda| \geqslant q \end{array} \right\}.$$

Notice that $\mathscr{F}^m_\lambda = \mathscr{E}^{GG}_{\lambda,m}T^*_X$. If we now set $\lambda = \kappa - 1$, the graded bundle associated with this filtration:

$$\mathsf{Gr}^\bullet \mathscr{E}^{GG}_{\kappa,m}T^*_X = \big(\mathscr{F}^0_{\kappa-1}\big/\mathscr{F}^1_{\kappa-1}\big) \oplus \cdots \oplus \big(\mathscr{F}^{m-1}_{\kappa-1}\big/\mathscr{F}^m_{\kappa-1}\big) \oplus \mathscr{E}^{GG}_{\kappa-1,m}T^*_X$$

is constituted of quotient factors:

$$\mathscr{G}^q_{\kappa-1} := \mathscr{F}^q_{\kappa-1}\big/\mathscr{F}^{q+1}_{\kappa-1} \qquad\qquad {\scriptstyle (q=0\cdots m-1)}$$

which consist of polynomials P as above for which:

$$|\alpha_1| + \cdots + (\kappa - 1)|\alpha_{\kappa-1}| = q$$

modulo polynomials for which $|\alpha_1| + \cdots + (\kappa-1)|\alpha_{\kappa-1}| \geqslant q+1$. It follows at once that $q + \kappa|\alpha_\kappa| = m$ in such polynomials, that is to say:

$$q = m - \kappa\ell_\kappa$$

for some integer $\ell_\kappa \in \mathbb{N}$ with $\mathsf{Ent}\big[\frac{m}{\kappa}\big] \geqslant \ell_\kappa \geqslant 0$. In particular, this quotient $\mathscr{F}^q_{\kappa-1}\big/\mathscr{F}^{q+1}_{\kappa-1}$ reduces to $\{0\}$ whenever $m - q$ is *not* divisible by κ.

We now claim that:

$$\mathscr{G}^{m-\kappa\ell_\kappa}_{\kappa-1}\big(\mathscr{E}^{GG}_{\kappa,m}T^*_X\big) \simeq \mathscr{E}^{GG}_{\kappa-1,\,m-\kappa\ell_\kappa}T^*_X \otimes \mathrm{Sym}^{\ell_\kappa}T^*_X.$$

Indeed, under an arbitrary change of coordinates $x \mapsto \Psi(x) = y$, the constituents $x^{(\kappa)}$ of the monomials of highest jet $\big(x^{(\kappa)}\big)^{\alpha_\kappa}$ with $|\alpha_\kappa| = \ell_\kappa$ are transformed to:

$$y^{(\kappa)} = \sum_{i_1=1}^n \Psi_{x_{i_1}}\, x^{(\kappa)}_{i_1} \ \text{ modulo } \ \big(x', \ldots, x^{(\kappa-1)}\big),$$

so that they visibly transform in exactly the same covariant way as the covectors in T^*_X, namely:

$$d\Psi = \sum_{i_1=1}^n \Psi_{x_{i_1}}\, dx_{i_1},$$

and so, the $\big(x^{(\kappa)}\big)^{\alpha_\kappa}$ transform as $\mathrm{Sym}^{\ell_\kappa}T^*_X$. The other constituents of $\mathscr{F}^{m-\kappa\ell_\kappa}_{\kappa-1}$ depend only on the $(\kappa-1)$-jet and are of the remaining weight $m - \kappa\ell_\kappa$, whence the claimed isomorphism follows.

Putting together all these isomorphisms, we get:

$$\begin{aligned}
\mathsf{Gr}^\bullet \mathscr{E}^{GG}_{\kappa,m} T^*_X &= \bigoplus_{\mathsf{Ent}[\frac{m}{\kappa}] \geqslant \ell_\kappa \geqslant 1} \mathscr{G}^{m-\kappa\ell_\kappa}_{\kappa-1} \bigoplus \mathscr{E}^{GG}_{\kappa-1,m} T^*_X \\
&= \bigoplus_{\mathsf{Ent}[\frac{m}{\kappa}] \geqslant \ell_\kappa \geqslant 1} \Big(\mathscr{E}^{GG}_{\kappa-1,\, m-\kappa\ell_\kappa} T^*_X \otimes \mathrm{Sym}^{\ell_\kappa} T^*_X \Big) \bigoplus \big(\mathscr{E}^{GG}_{\kappa-1,m} T^*_X \otimes \mathrm{Sym}^0 T^*_X \big) \\
&= \bigoplus_{\mathsf{Ent}[\frac{m}{\kappa}] \geqslant \ell_\kappa \geqslant 0} \mathscr{E}^{GG}_{\kappa-1,\, m-\kappa\ell_\kappa} T^*_X \otimes \mathrm{Sym}^{\ell_\kappa} T^*_X .
\end{aligned}$$

Now an induction of this isomorphism applied to $\mathsf{Gr}^\bullet \big(\mathscr{E}^{GG}_{\kappa-1,\, m-\kappa\ell_\kappa} T^*_X \big)$ yields the announced decomposition for $\mathsf{Gr}^\bullet \mathscr{E}^{GG}_{\kappa,m} T^*_X$.

Theorem. *The holomorphic vector bundle $\mathscr{E}^{GG}_{\kappa,m} T^*_X$ of Green-Griffiths polynomials of weight m in the κ-jet of local complex curves $\mathbb{D} \to X$ admits a natural filtration whose associated graded bundle is isomorphic to the following direct sum of multi-tensored symmetric powers of the cotangent bundle:*

$$\boxed{\mathsf{Gr}^\bullet \mathscr{E}^{GG}_{\kappa,m} T^*_X = \bigoplus_{\ell_1 + 2\ell_2 + \cdots + \kappa\ell_\kappa = m} \mathrm{Sym}^{\ell_1} T^*_X \otimes \mathrm{Sym}^{\ell_2} T^*_X \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T^*_X} .$$

Furthermore, for every $q = 1, 2, \ldots, n$, one has the inequalities:

$$\dim H^q \big(X,\, \mathscr{E}^{GG}_{\kappa,m} T^*_X \big) \leqslant \sum_{\ell_1 + 2\ell_2 + \cdots + \kappa\ell_\kappa = m} \dim H^q \Big(X,\ \mathrm{Sym}^{\ell_1} T^*_X \otimes \mathrm{Sym}^{\ell_2} T^*_X \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T^*_X \Big).$$

To complete the proof, it only remains to check the inequality between the cohomology dimensions. Let us consider instead in greater generality the following situation, which clearly embraces the last claim above.

Lemma. *Suppose a holomorphic vector bundle $E \to X$ admits a filtration:*

$$\{0\} = E_{r+1} \subset E_r \subset E_{r-1} \subset \cdots \subset E_{k+1} \subset E_k \subset E_{k-1} \subset \cdots \subset E_1 \subset E_0 = E$$

by nested holomorphic subbundles E_k, the associated graded bundle being:

$$\mathrm{Gr}^\bullet E = E_r \oplus \big(E_{r-1}/E_r \big) \oplus \cdots \oplus \big(E_k / E_{k+1} \big) \oplus \big(E_{k-1}/E_k \big) \oplus \cdots \oplus \big(E_0 / E_1 \big).$$

where, for a good notational correspondence: $\mathsf{Gr}^k E := E_k / E_{k+1}$. Then for every $q = 0, 1, \ldots, n$, the following inequality between cohomological dimensions holds:

$$\dim H^q \big(X,\, E \big) \leqslant \sum_{k=0}^{r} \dim H^q \big(X,\, E_k / E_{k+1} \big).$$

Proof. To each obviously true short exact sequence:

$$0 \longrightarrow \mathsf{Gr}^k E \longrightarrow E/E_{k+1} \longrightarrow E/E_k \longrightarrow 0 \qquad (k = 0, 1, \ldots, r) \tag{5}$$

is associated the long exact sequence between cohomology groups:
(6)
$$\cdots \longrightarrow H^q\big(X,\, \mathsf{Gr}^k E\big) \longrightarrow H^q\big(X,\, E/E_{k+1}\big) \longrightarrow H^q\big(X,\, E/E_k\big) \longrightarrow \cdots,$$
and the trivial majoration: $\dim B \leqslant \dim A + \dim C$ of the dimension of any member B of any long exact sequence of vector spaces by the sum of the dimensions of its two immediate neighbors gives us here:
$$\dim H^q\big(X,\, E/E_{k+1}\big) \leqslant \dim H^q\big(X,\, \mathsf{Gr}^k E\big) + \dim H^q\big(X,\, E/E_k\big) \qquad (k=0,1,\dots,r).$$
Starting from $k = 0$ for which $\mathsf{Gr}^0 E = E/E_1$ and $E/E_0 = \{0\}$, a plain summation up to $k = r$ of these inequalities cancels out all factors involving E/E_k except only one on the left: $E/E_{r+1} = E$, and we get the desired inequality:
$$\dim H^q\big(X,\, E\big) = \dim H^q\big(X,\, E/E_{r+1}\big) \leqslant \sum_{k=0}^{r} \dim H^q\big(X,\, \mathsf{Gr}^k E\big)$$
which, when applied to the Green-Griffiths bundle, terminates our detailed restitution of the theorem. □

Furthermore, in specific situations where the dimensions of the first cohomology groups of the graded pieces E_k/E_{k+1} do not vanish but happen to become asymptotically (much) smaller than the dimensions of their zeroth cohomology groups, a useful second lemma is as follows.

Lemma. *Under the same assumptions and just for $i = 0$, in addition to the above majoration $h^0(X, E) \leqslant \sum_{k=0}^{r} h^0(X,\, E_k/E_{k+1})$, one has the* minoration:
$$h^0(X,\, E) \geqslant \sum_{k=0}^{r} h^0\big(X,\, E_k/E_{k+1}\big) - \sum_{k=0}^{r} h^1\big(X,\, E_k/E_{k+1}\big).$$

Proof. As a preliminary, we observe that any long exact sequence
$$0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D \xrightarrow{d} \cdots$$
can be stopped at its fourth term by replacing it with the four-terms sequence:
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow \mathrm{Im}(c) \longrightarrow 0$$
which is easily checked to be also exact. Then by considering the basic equality which comes out by alternately summing the dimensions of its members:
$$0 = \dim(A) - \dim(B) + \dim(C) - \dim(\mathrm{Im}(c)),$$
we deduce from the trivial inequality $\dim(\mathrm{Im}(c)) \leqslant \dim(D)$, the useful minoration:
$$\dim(B) \geqslant \dim(A) + \dim(C) - \dim(D).$$

Applying now such an inequality to the first four terms of the long exact sequence associated to the k-th quotient exact sequence (5) above:

$$0 \longrightarrow H^0\big(X,\, \mathsf{Gr}^k E\big) \longrightarrow H^0\big(X,\, E/E_{k+1}\big) \longrightarrow H^0\big(X,\, E/E_k\big) \longrightarrow H^1\big(X,\, \mathsf{Gr}^k E\big) \longrightarrow \cdots,$$

we readily deduce:

$$h^0\big(X,\, E/E_{k+1}\big) \geqslant h^0\big(X,\, \mathsf{Gr}^k E\big) + h^0\big(X,\, E/E_k\big) - h^1\big(X,\, \mathsf{Gr}^k E\big).$$

Starting then from $k = 0$ for which $\mathsf{Gr}^0 E = E/E_1$ and $E/E_0 = \{0\}$, a plain summation of these inequalities up to $k = r$ cancels out all terms involving an E/E_k except one: $E/E_{r+1} = E$, and we get the announced minoration. $\square$

As a corollary, by applying this second elementary lemma to the Green-Griffiths bundle, we gain a possibly useful general minoration:

(7)
$$\begin{aligned} h^0\big(X,\, \mathscr{E}^{GG}_{\kappa,m} T^*_X\big) \geqslant & \sum_{\ell_1+2\ell_2+\cdots+\kappa\ell_\kappa=m} h^0\Big(X,\, \mathrm{Sym}^{\ell_1} T^*_X \otimes \mathrm{Sym}^{\ell_2} T^*_X \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T^*_X\Big) - \\ & - \sum_{\ell_1+2\ell_2+\cdots+\kappa\ell_\kappa=m} h^1\Big(X,\, \mathrm{Sym}^{\ell_1} T^*_X \otimes \mathrm{Sym}^{\ell_2} T^*_X \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T^*_X\Big). \end{aligned}$$

Hence it is now clear that, in order to establish existence nonzero global sections of $\mathscr{E}^{GG}_{\kappa,m} T^*_X$ on hypersurfaces of general type, it would suffice that the considered sum of h^1's grows less substantially than the sum of h^0's, as κ tends to ∞ and as $m \gg \kappa$ tends to ∞ too. We conclude this section by recalling that the Euler-Poincaré characteristic transfers better than cohomology dimensions through exact sequences, namely without inequalities.

Lemma. *The Euler-Poincaré characteristic of the Green-Griffiths bundle is equal to that of its associated graded bundle:*

$$\chi\big(X, \mathscr{E}^{GG}_{\kappa,m} T^*_X\big) = \chi\big(X, \mathsf{Gr}^\bullet \mathscr{E}^{GG}_{\kappa,m} T^*_X\big).$$

Proof. In fact, in the general context of the previous two lemmas, because for any exact sequence of finite-dimensional vector spaces:

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_N \longrightarrow 0$$

one may easily check that the alternating sum of dimensions vanishes:

$$0 = \dim A_1 - \dim A_2 + \cdots + (-1)^N \dim A_N,$$

one deduces from the long exact sequence of cohomology (6) that:

$$\begin{aligned} 0 = {} & h^0(X,\, \mathsf{Gr}^k E) - h^0(X,\, E/E_{k+1}) + h^0(X,\, E/E_k) - \\ & - h^1(X,\, \mathsf{Gr}^k E) - h^1(X,\, E/E_{k+1}) + h^1(X,\, E/E_k) + \\ & + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\ & + (-1)^n h^n(X,\, \mathsf{Gr}^k E) - (-1)^n h^n(X,\, E/E_{k+1}) + (-1)^n h^n(X,\, E/E_k), \end{aligned}$$

or else after gathering terms column by column, that:

$$0 = \chi(X,\, \mathsf{Gr}^k E) - \chi(X,\, E/E_{k+1}) + \chi(X,\, E/E_k).$$

Finally, a plain summation $\sum_{k=0}^{n}$ yields the formula claimed. □

§3. Euler-Poincaré characteristic computations

Theorem. ([17]) *On an arbitrary compact complex projective manifold of dimension $n \geqslant 1$, the Green-Griffiths jet bundle $\mathscr{E}_{\kappa,m}^{GG} T_X^*$ has an Euler-Poincaré characteristic asymptotically given by:*

$$\begin{aligned}\chi\big(X, \mathscr{E}_{\kappa,m}^{GG} T_X^*\big) &= \chi\big(X, \mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*\big)\\ &= \sum_{\ell_1+2\ell_2+\cdots+\kappa\ell_\kappa=m} \chi\Big(X,\, \mathrm{Sym}^{\ell_1} T_X^* \otimes \mathrm{Sym}^{\ell_2} T_X^* \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T_X^*\Big)\\ &= \tfrac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!\,(\kappa!)^n}\Big\{(\mathsf{c}_1^*)^n \frac{(\log\kappa)^n}{n!} + \mathsf{O}_n\big((\log\kappa)^{n-1}\big)\Big\} + \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big),\end{aligned}$$

where $\mathsf{c}_1^* = \mathsf{c}_1(T_X^*) = -\,\mathsf{c}_1(T_X)$ is the first Chern class of T_X^*, a $(1,1)$-cohomology class on X, and where:

(i) *the first remainder is a linear combination of homogeneous*[4] *terms $(\mathsf{c}_1^*)^{\lambda_1}(\mathsf{c}_2^*)^{\lambda_2}\cdots(\mathsf{c}_n^*)^{\lambda_n}$ with $\lambda_1 + 2\,\lambda_2 + \cdots + n\,\lambda_n = n$, with rational coefficients all bounded in absolute value by $\mathsf{Constant}_n\,(\log\kappa)^{n-1}$;*

(ii) *the second remainder is a polynomial in m of submaximal degree $\leqslant (\kappa+1)n - 2$ whose coefficients are linear combinations of the same $(\mathsf{c}_1^*)^{\lambda_1}(\mathsf{c}_2^*)^{\lambda_2}\cdots(\mathsf{c}_n^*)^{\lambda_n}$ with rational coefficients all also bounded in absolute value by $\mathsf{Constant}_{n,\kappa}$.*

In the case where $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ is a hypersurface of degree $d \geqslant 1$, each homogeneous monomial $(\mathsf{c}_1^*)^{\lambda_1}(\mathsf{c}_2^*)^{\lambda_2}\cdots(\mathsf{c}_n^*)^{\lambda_n}$ — implicitly integrated on X, as usually understood in complex algebraic geometry — expresses in terms of n and d by means of some universal formulas as follows. Let $h := \mathsf{c}_1\big(\mathscr{O}_{\mathbb{P}^{n+1}}(1)\big)$ denotes the hyperplane $(1,1)$-cohomology class, which satisfies $h^n = \int_X h^n = d$. Then one may represent ([17], p. 170):

$$\left[\begin{array}{l}\mathsf{c}_1 = -h\big(d-n-2\big)\\ \mathsf{c}_2 = h^2\big(d^2 - \frac{(n+2)!}{(n+1)!\,1!}\,d + \frac{(n+2)!}{n!\,2!}\big)\\ \mathsf{c}_3 = -h^3\big(d^3 - \frac{(n+2)!}{(n+1)!\,1!}\,d^2 + \frac{(n+2)!}{n!\,2!}\,d - \frac{(n+2)!}{(n-1)!\,3!}\big)\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \mathsf{c}_n = (-1)^n\,h^n\big(d^n - \frac{(n+2)!}{(n+1)!\,1!}\,d^{n-1} + \cdots + (-1)^n\,\frac{(n+2)!}{2!\,n!}\big).\end{array}\right. \tag{8}$$

[4] As usual, we understand implicitly that each (n,n)-cohomology class $(\mathsf{c}_1^*)^{\lambda_1}\cdots(\mathsf{c}_n^*)^{\lambda_n}$ is *integrated* over X, hence represents the *numerical* value $\int_X (\mathsf{c}_1^*)^{\lambda_1}\cdots(\mathsf{c}_n^*)^{\lambda_n}$.

Recall that the Chern classes of the tangent T_X and of the cotangent bundle T_X^* of the hypersurface $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ are linked together by the simple relation:

$$\mathsf{c}_k^* := \mathsf{c}_k\big(T_X^*\big) = (-1)^k\, \mathsf{c}_k\big(T_X\big) = (-1)^k\, \mathsf{c}_k,$$

for $k = 1, 2, \dots, n$. It follows notably for instance that:

$$(\mathsf{c}_1^*)^n = (-1)^n (-h)^n \big(d-n-2\big)^n = d\big(d-n-2\big)^n.$$

Generally speaking, one easily convinces oneself that each homogeneous degree n monomial $\mathsf{c}_1^{\tau_1}\mathsf{c}_2^{\tau_2}\cdots\mathsf{c}_n^{\tau_n}$ identifies with a certain polynomial:

$$\mathsf{c}_1^{\tau_1}\mathsf{c}_2^{\tau_2}\cdots\mathsf{c}_n^{\tau_n} = \sum_{k=1}^{n+1} C_k^{\tau_1,\tau_2,\dots,\tau_n} \cdot d^k$$

with respect to $d = \deg X$ having degree $\leqslant n+1$ with integer coefficients $C_k^{\tau_1,\tau_2,\dots,\tau_n} \in \mathbb{Z}$. Furthermore, the constant coefficient $C_0^{\tau_1,\tau_2,\dots,\tau_n} = 0$ is zero, because the factor $h^{\tau_1+2\tau_2+\cdots+n\tau_n} = h^n = d$ is clearly always present in every $\mathsf{c}_1^{\tau_1}\mathsf{c}_2^{\tau_2}\cdots\mathsf{c}_n^{\tau_n}$.

As a result, the Euler-Poincaré characteristic of $\mathscr{E}_{\kappa,m}^{GG}T_X^*$:

$$\tfrac{m^{(\kappa+1)n-1}}{(\kappa!)^n((\kappa+1)n-1)!}\Big\{d\,(d-n-2)^n\frac{(\log\kappa)^n}{n!} + \mathsf{O}_{n,d}\big((\log\kappa)^{n-1}\big)\Big\} + \mathsf{O}_{n,d,\kappa}\big(m^{(\kappa+1)n-2}\big),$$

visibly tends to $+\infty$ as m and κ tend to $+\infty$ as soon as X is of general type, that it is to say, as soon as:

$$\deg X \geqslant \dim X + 3.$$

Proof. With more details, we redo Green-Griffiths' proof; fundamentals may be found in [18] (pp. 50–59 plus Chap. 15), in [3] and in [18].

To begin with, introduce the *formal root decomposition*:

$$\mathsf{c}\big(T_X^*\big) = 1 + \mathsf{c}_1^* + \mathsf{c}_2^* + \cdots + \mathsf{c}_n^* = (1+\mathsf{a}_1^*)(1+\mathsf{a}_2^*)\cdots(1+\mathsf{a}_n^*)$$

of the *total Chern class*, namely of the sum $\mathsf{c}(T_X^*)$ of the c_i^* so that c_i^* is the i-th elementary symmetric function of the *Chern roots* a_j^*:

$$\mathsf{c}_i^* = \sum_{1\leqslant j_1<j_2<\cdots<j_i\leqslant n} \mathsf{a}_{j_1}^*\mathsf{a}_{j_2}^*\cdots\mathsf{a}_{j_i}^* \qquad (i=0,\,1\cdots n).$$

Similarly, let the a_j denote the Chern roots of $\mathsf{c}(T_X) = 1 + \mathsf{c}_1 + \cdots + \mathsf{c}_n$ and let the symbol $[\ \ ]_j$ denote projection to the (j,j)-cohomology class, so that for instance $\big[1 + \mathsf{c}_1^* + \cdots + \mathsf{c}_n^*\big]_j = \mathsf{c}_j^*$. To prove the theorem, we must apply the Riemann-Roch-Hirzebruch theorem [18] which states that the Euler-Poincaré characteristic:

$$\chi\big(X,\ \mathscr{E}_{\kappa,m}^{GG}T_X^*\big) \overset{def}{=} \sum_{0\leqslant q\leqslant n} (-1)^q\ \dim H^q\big(X,\ \mathscr{E}_{\kappa,m}^{GG}T_X^*\big)$$

is equal to the integral over X:

$$\chi = \int_X \big[\mathsf{ch}\big(\mathscr{E}^{GG}_{\kappa,m}T^*_X\big)\cdot \mathsf{td}(T_X)\big]_n = \int_X \sum_{j=0}^{n} \big[\mathsf{ch}\big(\mathscr{E}^{GG}_{\kappa,m}T^*_X\big)\big]_{n-j}\big[\mathsf{td}(T_X)\big]_j$$

of the (n,n)-part of the product between the *Chern character* $\mathsf{ch}\big(\mathscr{E}^{GG}_{\kappa,m}T^*_X\big)$ of $\mathscr{E}^{GG}_{\kappa,m}T^*_X$ (to be computed in a while) and the *Todd class*[5] of T_X:

$$\mathsf{td}(T_X) = \frac{\mathsf{a}_1}{1-e^{-\mathsf{a}_1}}\,\frac{\mathsf{a}_2}{1-e^{-\mathsf{a}_2}}\cdots\frac{\mathsf{a}_n}{1-e^{-\mathsf{a}_n}} = 1+\tfrac{1}{2}\,\mathsf{c}_1+\tfrac{1}{12}\,\big[\mathsf{c}_1^2+\mathsf{c}_2\big]+\cdots.$$

As usual in asymptotic complex algebraic geometry (cf. [12, 24, 25]), for the product: $\mathsf{chern}\cdot\mathsf{todd}$, picking cohomology classes of positive degree $\geqslant 1$ in $\mathsf{td}(T_X)$ forces to pick classes in $\mathsf{ch}\big(\mathscr{E}^{GG}_{\kappa,m}T^*_X\big)$ of bidegree $\leqslant (n-1,\,n-1)$, and then the associated m-contributions are *smaller* than the maximal possible: $m^{(\kappa+1)n-1}$. More precisely:

Lemma. *For every $j=0,1,\dots,n$, one has:*

$$\big[\mathsf{ch}\big(\mathscr{E}^{GG}_{\kappa,m}T^*_X\big)\big]_{n-j} = \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-1-j}\big)$$

and consequently all terms $\sum_{j=1}^{n}$ are negligible for our purposes, whence:

$$\chi\big(X,\ \mathscr{E}^{GG}_{\kappa,m}T^*_X\big) = \int_X \big[\mathsf{ch}\big(\mathscr{E}^{GG}_{\kappa,m}T^*_X\big)\big]_n + \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big).$$

Proof. As is known, the Chern character of the jet bundle equals that of its graded decomposition, and the Chern character is both additive on direct sums and multiplicative on tensor products, so that we can write:

$$\mathsf{ch}\big(\mathscr{E}^{GG}_{\kappa,m}T^*_X\big) = \sum_{\ell_1+2\,\ell_2+\cdots+\kappa\ell_\kappa=m} \mathsf{ch}\big(\mathrm{Sym}^{\ell_1}T^*_X\big)\,\mathsf{ch}\big(\mathrm{Sym}^{\ell_2}T^*_X\big)\cdots\mathsf{ch}\big(\mathrm{Sym}^{\ell_\kappa}T^*_X\big).$$

Furthermore, recall that the Chern character of an arbitrary symmetric power of T^*_X is given, in terms of the a^*_i, by the known formula:

$$\mathsf{ch}\big(\mathrm{Sym}^{\ell}T^*_X\big) = \sum_{\substack{x_1+\cdots+x_n=\ell\\ x_1,\dots,x_n\in\mathbb{N}}} e^{x_1\mathsf{a}^*_1+\cdots+x_n\mathsf{a}^*_n}.$$

We can therefore apply this to $\ell=\ell_\lambda$ for all λ with $1\leqslant\lambda\leqslant\kappa$:

$$\mathsf{ch}\big(\mathrm{Sym}^{\ell_\lambda}T^*_X\big) = \sum_{x_{\lambda 1}+\cdots+x_{\lambda n}=\ell_\lambda} e^{x_{\lambda 1}\,\mathsf{a}^*_1+\cdots+x_{\lambda n}\,\mathsf{a}^*_n},$$

where we have introduced nonnegative integers $x_{\lambda i}\in\mathbb{N}$, $i=1,\dots,n$ parametrized by λ. When we expand the product of all the κ sums involved,

[5] Of course, all terms of degree $\geqslant n+1$ in the a_j are $\equiv 0$, since the associated cohomology classes vanish, as does any form of bidegree (p,q) with $p\geqslant n+1$ or $q\geqslant n+1$.

the exponentiated terms add up and the obtained sum together with the initial sum $\sum_{\ell_1+\cdots+\kappa\ell_\kappa=m}$ unify as a single big sum:

$$\mathsf{ch}\big(\mathscr{E}^{GG}_{\kappa,m}T^*_X\big) = \sum_{\substack{x_{11}+\cdots+x_{1n}\\ +\cdots\cdots\cdots\cdots+\\ +\kappa(x_{\kappa 1}+\cdots+x_{\kappa n})=m}} \exp\big\{(x_{11}+\cdots+x_{\kappa 1})\mathsf{a}^*_1+\cdots+(x_{1n}+\cdots+x_{\kappa n})\mathsf{a}^*_n\big\}$$

in which the ℓ_λ have been naturally removed, with the only constraint that $\sum_{\lambda=1}^{\kappa}\lambda\,(x_{\lambda 1}+\cdots+x_{\lambda n})$ be constant equal to m. Now we observe the general summation rule:

$$\textstyle\sum_{u_1+u_2+\cdots+u_\mu=m} \equiv \sum_{u_2+\cdots+u_\mu\leqslant m},$$

by simply taking $u_1 := m-u_2-\cdots-u_\mu$, where the $u_j\in\mathbb{N}$. Thus, we may eliminate x_{11} in our argument of summation and it follows at once for any $j=0,1,\ldots,n$ that the quantity we want to estimate is equal to:

$$\big[\mathsf{ch}\big(\mathscr{E}^{GG}_{\kappa,m}T^*_X\big)\big]_{n-j} = \sum_{\substack{x_{12}+\cdots+x_{1n}\\ +\cdots\cdots\cdots\cdots+\\ +\kappa(x_{\kappa 1}+\cdots+x_{\kappa n})\leqslant m}} \frac{1}{(n-j)!}\,\big[(\widehat{x_{11}}+x_{21}+\cdots+x_{\kappa 1})\mathsf{a}^*_1+\cdots+(x_{1n}+\cdots+x_{\kappa n})\mathsf{a}^*_n\big]^{n-j},$$

where the symbol $\widehat{x_{11}}$ means that x_{11} is replaced by its value $m-x_{12}-\cdots-\kappa\,x_{\kappa n}$. Classically, by making the change of variables:

$$y_{12} := \frac{x_{12}}{m},\ \ldots,\ y_{1n} := \frac{x_{1n}}{m},\ \ldots\ldots,\ y_{\kappa 1} := \frac{x_{\kappa 1}}{m},\ \ldots,\ y_{\kappa n} := \frac{x_{\kappa n}}{m},$$

the discrete Riemann-like sum just obtained can be approximated by a continuous integral performed on a $(\kappa n-1)$-dimensional simplex against the standard measure of $\mathbb{R}^{\kappa n-1}_+$:

$$\begin{aligned}\big[\mathsf{ch}\big(\mathscr{E}^{GG}_{\kappa,m}T^*_X\big)\big]_{n-j} &= m^{\kappa n-1+n-j}\int_{\substack{y_{12}+\cdots+y_{1n}\\ +\cdots\cdots\cdots\cdots+\\ +\kappa(y_{\kappa 1}+\cdots+y_{\kappa n})\leqslant 1}} dy_{12}\cdots dy_{1n}\cdots\cdots dy_{\kappa 1}\cdots dy_{\kappa n}\cdot\\ &\cdot\frac{1}{(n-j)!}\,\big[(\widehat{y_{11}}+y_{21}+\cdots+y_{\kappa 1})\mathsf{a}^*_1+\cdots+(y_{1n}+\cdots+y_{\kappa n})\mathsf{a}^*_n\big]^{n-j}+\\ &\qquad+\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-j-2}\big),\end{aligned}$$

the remainder being automatically at most of the order of the submaximal power of m. The integral remaining being visibly independent of m, the conclusion is got. □

Consequently, in order to compute asymptotically our Euler-Poincaré characteristic, we only have to estimate the integral above for $j=0$, in which $\widehat{y_{11}}$ is of course an abbreviation for $1-y_{12}-\cdots-\kappa\,y_{\kappa n}$. To this aim, we make the multidilational change of variables: $y_{\lambda i}\mapsto\lambda\,y_{\lambda i}=:z_{\lambda i}$ and

the asymptotic under study becomes an integral over the *standard* $(n\kappa-1)$-dimensional simplex:

$$\begin{aligned}\chi\big(X,\,\mathscr{E}^{GG}_{\kappa,m}T^*_X\big) &= \int_X \big[\mathsf{ch}(\mathscr{E}^{GG}_{\kappa,m}T^*_X)\big]_n + \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big)\\ &\equiv \frac{m^{(\kappa+1)n-1}}{n!\,(\kappa!)^n}\int_{\substack{z_{21}+\cdots+z_{1n}\\ +\cdots\cdots\cdots\cdots+\\ +z_{\kappa1}+\cdots+z_{\kappa n}\leqslant 1}} dz_{12}\cdots dz_{1n}\cdots\cdots dz_{\kappa1}\cdots dz_{\kappa n}\cdot\\ &\quad\cdot\Big[\Big(\widehat{z_{11}}+\frac{z_{21}}{2}+\cdots+\frac{z_{\kappa1}}{\kappa}\Big)\mathsf{a}^*_1+\cdots+\Big(\frac{z_{1n}}{1}+\frac{z_{2n}}{2}+\cdots+\frac{z_{\kappa n}}{\kappa}\Big)\mathsf{a}^*_n\Big]^n,\end{aligned}$$

where now the sign "$\equiv$" means modulo $\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big)$ and where $\widehat{z_{11}}=1-z_{12}-\cdots-z_{\kappa n}$. Applying now Newton's multinomial formula:

$$\big(Z_1+Z_2+\cdots+Z_n\big)^n=\sum_{q_1+q_2+\cdots+q_n=n}\frac{n!}{q_1!\,q_2!\,\cdots\,q_n!}\,(Z_1)^{q_1}(Z_2)^{q_2}\cdots(Z_n)^{q_n},$$

we may expand the n-th power in the second line above, getting:

$$\begin{aligned}\chi\big(X,\,\mathscr{E}^{GG}_{\kappa,m}T^*_X\big) &= \frac{m^{(\kappa+1)n-1}}{\underline{n!}_\circ\,(\kappa!)^n}\sum_{q_1+\cdots+q_n=n}\frac{\underline{n!}_\circ}{q_1!\,\cdots\,q_n!}\,(\mathsf{a}^*_1)^{q_1}\cdots(\mathsf{a}^*_n)^{q_n}\cdot\\ &\quad\cdot\int_{\substack{z_{21}+\cdots+z_{1n}\\ +\cdots\cdots\cdots\cdots+\\ +z_{\kappa1}+\cdots+z_{\kappa n}\leqslant 1}} dz_{21}\cdots dz_{1n}\cdots\cdots dz_{\kappa1}\cdots dz_{\kappa n}\cdot\\ &\quad\cdot\Big(\widehat{z_{11}}+\frac{z_{21}}{2}+\cdots+\frac{z_{\kappa1}}{\kappa}\Big)^{q_1}\cdots\Big(\frac{z_{1n}}{1}+\frac{z_{2n}}{2}+\cdots+\frac{z_{\kappa n}}{\kappa}\Big)^{q_n}.\end{aligned}$$

The $n!$ drops, a fact denoted with the symbol "$\_\circ$". Furthermore, in the integral — call it $\mathsf{I}_{q_1,\ldots,q_n}$ — which appears naturally in the last two lines, we yet expand the q_1-th, ..., the q_n-th powers:

$$\begin{aligned}\mathsf{I}_{q_1,\ldots,q_n} &= \sum_{q_{11}+q_{21}+\cdots+q_{\kappa1}=q_1}\cdots\sum_{q_{1n}+q_{2n}+\cdots+q_{\kappa n}}\frac{q_1!}{q_{11}!\,q_{21}!\,\cdots\,q_{\kappa1}!}\cdots\frac{q_n!}{q_{1n}!\,q_{2n}!\,\cdots\,q_{\kappa n}!}\cdot\\ &\qquad\cdot\frac{1}{(2)^{q_{21}}\,\cdots\,(\kappa)^{q_{\kappa1}}}\cdots\frac{1}{(1)^{q_{1n}}\,(2)^{q_{2n}}\,\cdots\,(\kappa)^{q_{\kappa n}}}\cdot\\ &\quad\cdot\int_{\substack{z_{21}+\cdots+z_{1n}\\ +\cdots\cdots\cdots\cdots+\\ +z_{\kappa1}+\cdots+z_{\kappa n}\leqslant 1}} dz_{21}\cdots dz_{1n}\cdots\cdots dz_{\kappa1}\cdots dz_{\kappa n}\cdot\\ &\qquad\cdot(\widehat{z_{11}})^{q_{11}}\,(z_{21})^{q_{21}}\,\cdots\,(z_{\kappa1})^{q_{\kappa1}}\,\cdots\cdots\,(z_{1n})^{q_{1n}}\,(z_{2n})^{q_{2n}}\,\cdots\,z_{\kappa n}^{q_{\kappa n}}.\end{aligned}$$

Lemma. *For any integer $p\geqslant 2$ and for any nonnegative integer exponents $j_1,j_2,\ldots,j_p\in\mathbb{N}$, one has:*

$$\int_{\substack{u_2+\cdots+u_p\leqslant 1\\ u_2\geqslant 0,\ \ldots,\ u_p\geqslant 0}}[1-u_2-\cdots-u_p]^{j_1}u_2^{j_2}\cdots u_p^{j_p}\,du_2\cdots du_p=\frac{j_1!\,j_2!\,\cdots\,j_p!}{(j_1+j_2+\cdots+j_p+p-1)!}.$$

Proof. By decomposing the integrations, we may write this integral as:

$$\int_0^1 u_2^{j_2}\,du_2\int_0^{1-u_2}u_3^{j_3}\,du_3\,\cdots\cdots\int_0^{1-u_2-\cdots-u_{p-1}}(1-u_2-\cdots-u_{p-1}-u_p)^{j_1}\,u_p^{j_p}\,du_p=:\mathsf{J}^p_{j_1,j_2,j_3,\ldots,j_p}.$$

Taking j_p+1 times the primitive of the first factor in the last integral and integrating successively by parts, this integral in question receives the value:

$$\Big[-\tfrac{(1-u_2-\cdots-u_{p-1}-u_p)^{j_1+j_p+1}}{(j_1+1)\cdots(j_1+j_p)(j_1+j_p+1)}\,j_p!\Big]_0^{1-u_2-\cdots-u_{p-1}} = \tfrac{j_1!}{(j_1+j_p+1)!}\,(1-u_2-\cdots-u_{p-1})^{j_1+j_p+1}\,j_p!.$$

Thus, the case $p=2$ is settled. If $p\geqslant 3$, inserting this value just computed:

$$\mathsf{J}^p_{j_1,j_2,\ldots,j_{p-1},j_p} = \tfrac{j_1!\,j_p!}{(j_1+j_p+1)!}\,\mathsf{J}^{p-1}_{j_1+j_p+1,j_2,\ldots,j_{p-1}} = \tfrac{j_1!\,j_p!}{\underline{(j_1+j_p+1)!}_{\circ}}\,\tfrac{\underline{(j_1+j_p+1)!}_{\circ}\,j_2!\cdots j_{p-1}!}{(j_1+j_p+1+j_2+\cdots+j_{p-1}+p-2)!},$$

we get without effort the general conclusion by induction on p. $\square$

So applying this elementary lemma, we may finish to compute our integral:

$$\begin{aligned}\mathsf{I}_{q_1,\ldots,q_n} = \sum_{q_{11}+q_{21}+\cdots+q_{\kappa 1}=q_1}\cdots\sum_{q_{1n}+q_{2n}+\cdots+q_{\kappa n}} &\tfrac{q_1!}{\underline{q_{11}!\,q_{21}!\cdots q_{\kappa 1}!}_{\circ}}\cdots\tfrac{q_n!}{\underline{q_{1n}!\,q_{2n}!\cdots q_{\kappa n}!}_{\circ}}\cdot\\ &\cdot\tfrac{1}{(2)^{q_{21}}!\cdots(\kappa)^{q_{\kappa 1}}}\cdots\tfrac{1}{(1)^{q_{1n}}\,(2)^{q_{2n}}\cdots(\kappa)^{q_{\kappa n}}}\cdot\\ &\cdot\tfrac{\underline{q_{11}!\,q_{21}\cdots q_{\kappa 1}!}_{\circ}\cdots\cdots\underline{q_{1n}!\,q_{2n}!\cdots q_{\kappa n}!}_{\circ}}{(q_{11}+q_{21}+\cdots+q_{\kappa 1}+\cdots\cdots+q_{1n}+q_{2n}+\cdots+q_{\kappa n}+\kappa n-1)!}.\end{aligned}$$

Remarkably, all the factorials $q_{\lambda i}!$ drop. Furthermore, the big factorial in the denominator visibly simplifies as

$$(q_1+\cdots+q_n+\kappa n-1)! = (n+\kappa n-1)!,$$

and we get a formula for $\mathsf{I}_{q_1,\ldots,q_n}$ in which it will appear soon to be convenient to reconstitute a product of n independent big sums, and to this aim, we add in advance the innocuous factor $\frac{1}{(1)^{q_{11}}}$:

$$\begin{aligned}&\mathsf{I}_{q_1,\ldots,q_n} =\\ &= \tfrac{q_1!\cdots q_n!}{(q_1+\cdots+q_n+\kappa n-1)!}\sum_{q_{11}+\cdots+q_{\kappa 1}=q_1}\cdots\sum_{q_{1n}+\cdots+q_{\kappa n}=q_n}\tfrac{1}{(1)^{q_{11}}\cdots(\kappa)^{q_{\kappa 1}}}\cdots\tfrac{1}{(1)^{q_{1n}}\cdots(\kappa)^{q_{\kappa n}}}\\ &= \tfrac{q_1!\cdots q_n!}{((\kappa+1)n-1)!}\Big(\sum_{q_{11}+\cdots+q_{\kappa 1}=q_1}\tfrac{1}{(1)^{q_{11}}\cdots(\kappa)^{q_{\kappa 1}}}\Big)\cdots\Big(\sum_{q_{1n}+\cdots+q_{\kappa n}=q_n}\tfrac{1}{(1)^{q_{1n}}\cdots(\kappa)^{q_{\kappa n}}}\Big).\end{aligned}$$

Now, when we plug this formula in the computation of $\chi\big(X,\ \mathscr{E}^{GG}_{\kappa,m}T^*_X\big)$ that we interrupted before stating the lemma, all the factorials $q_1!,\ldots,q_n!$ appear once at a numerator place and once at a denominator place, so they drop all and we finally get:

$$\begin{aligned}\chi\big(X,\ \mathscr{E}^{GG}_{\kappa,m}T^*_X\big) = \tfrac{m^{(\kappa+1)n-1}}{(\kappa!)^n\,((\kappa+1)n-1)!}\Big[\sum_{q_1+\cdots+q_n=n}&(\mathsf{a}^*_1)^{q_1}\cdots(\mathsf{a}^*_n)^{q_n}\cdot\\ \cdot\Big(\sum_{q_{11}+\cdots+q_{\kappa 1}=q_1}\tfrac{1}{(1)^{q_{11}}\cdots(\kappa)^{q_{\kappa 1}}}\Big)\cdots\Big(\sum_{q_{1n}+\cdots+q_{\kappa n}=q_n}&\tfrac{1}{(1)^{q_{1n}}\cdots(\kappa)^{q_{\kappa n}}}\Big)\Big]+\\ &+\mathsf{O}\big(m^{(\kappa+1)n-2}\big).\end{aligned}$$

We therefore have to deal with the asymptotic character, as $\kappa \to \infty$, of the poly-logarithmic sums of the type:

$$\Sigma_1^\kappa(q) := \sum_{\substack{q_1+\cdots+q_\kappa=q \\ q_1\geqslant 0,\ \cdots,\ q_\kappa\geqslant 0}} \frac{1}{(1)^{q_1}\cdots(\kappa)^{q_\kappa}},$$

where $q \in \mathbb{N}$ is arbitrary.

Lemma. *As $\kappa \to \infty$, one has:*

$$\Sigma_1^\kappa(q) = \frac{(\log\kappa)^q}{q!} + \mathsf{O}_n\big((\log\kappa)^{q-1}\big).$$

Proof. Easily re-doable, and in fact also known in the literature on polylogarithms ([9]). □

From this last lemma, it follows at once that:

$$\begin{aligned}\chi\big(X,\ \mathscr{E}^{GG}_{\kappa,m}T^*_X\big) &= \frac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!\,(\kappa!)^n}\Bigg[\sum_{q_1+\cdots+q_n=n}(\mathsf{a}_1^*)^{q_1}\cdots(\mathsf{a}_n^*)^{q_n}\cdot \\ &\qquad\cdot\frac{(\log\kappa)^{q_1}}{q_1!}\cdots\frac{(\log\kappa)^{q_n}}{q_n!}+\mathsf{O}_n\big((\log\kappa)^{n-1}\big)\Bigg]+\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big) \\ &= \tfrac{m^{(\kappa+1)n-1}}{(\kappa!)^n\,((\kappa+1)n-1)!}\Big[(\mathsf{a}_1^*+\cdots+\mathsf{a}_n^*)^n\,\tfrac{(\log\kappa)^n}{n!}+\mathsf{O}_n\big((\log\kappa)^{n-1}\big)\Big]+\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big),\end{aligned}$$

so the asymptotic formula exhibited in the theorem is established. To conclude the proof, one easily convinces oneself by inspecting the remainders that they indeed have the form claimed in **(i)** and **(ii)**. □

Open problem. Applying the concepts and the combinatorics partly achieved in [9, 4, 47], find *closed explicit formulas* firstly for the remainder terms $\mathsf{O}_n\big((\log\kappa)^{n-1}\big)$, secondly, for the remainder terms $\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big)$. As an accessible preliminary, study the $\Sigma_q(\kappa)$ completely.

§4. Exact Schur Bundle Decomposition

Schur bundles and Pieri rule. Thanks to the filtration provided by the theorem on p. 12 and to the basic cohomology inequalities reproved in Section 2, the study of the Green-Griffiths jet bundle can in principle be led back to the study of multitensored symmetric powers:

$$\mathrm{Sym}^{\ell_1}T^*_X\otimes\mathrm{Sym}^{\ell_2}T^*_X\otimes\cdots\otimes\mathrm{Sym}^{\ell_\kappa}T^*_X$$

of the cotangent bundle. But it is known since the works of Isai Schur at the turn to the 20th century that these multitensored bundles can even be decomposed in more atomic independent bricks.

Since the complex linear group $\mathsf{GL}_n(\mathbb{C})$ acts naturally on T^*_X and on all of its tensor powers $\big(T^*_X\big)^{\otimes r}$ as well ($r = 1, 2, 3, \dots$), then by fundamental facts of representation theory (Schur's theorems), it follows that the (in fact complicated) direct sum $\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X$ provided by the theorem on p. 12 can

in principle be represented as a certain direct sum of the so-called *Schur bundles*:

$$\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)}T^*_X,$$

in which $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0$; we employ the notation of [30] and the reader is referred to the works of Brückmann [5, 7, 1] and to the monographs [16, 37, 26, 19, 27] for background material, or alternatively to p. 73 below. In order to determine which $\mathscr{S}^{(\ell_1,\dots,\ell_n)}T^*_X$ appear in $\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X$, possibly with some multiplicity $\geqslant 1$, two options present themselves.

The first option would be to apply step by step the so-called *Pieri formula* ([16], p. 455) to the direct sum representation:

$$\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X = \bigoplus_{\ell_1+2\ell_2+\cdots+\kappa\ell_\kappa=m} \mathscr{S}^{(\ell_1,0,\dots,0)}T^*_X \otimes \mathscr{S}^{(\ell_2,0,\dots,0)}T^*_X \otimes \cdots \otimes \mathscr{S}^{(\ell_\kappa,0,\dots,0)}T^*_X.$$

Pieri indeed provides a neat combinatorial rule for representing any tensor product of a Schur bundle with a symmetric power as a certain direct sum of well controlled Schur bundles over X:

$$\mathscr{S}^{(t_1,\dots,t_n)}T^*_X \otimes \mathscr{S}^{(\ell,0,\dots,0)}T^*_X = \sum_{\substack{s_1+\cdots+s_n=\ell+t_1+\cdots+t_n \\ s_1\geqslant t_1\geqslant s_2\geqslant t_2\geqslant\cdots\geqslant s_n\geqslant t_n\geqslant 0}} \mathscr{S}^{(s_1,\dots,s_n)}T^*_X. \tag{9}$$

However, when one tries to induct on such a formula, the complexity increases dramatically as soon the number κ of tensor factors in $\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X$ passes above $\kappa = 5$, even in dimension $n = 2$, and apparently, nothing really effective or exploitable for us exists in the literature.

Invariant theory approach. The second option, more direct and more suited to asymptotic approximations, consists in interpreting the problem directly in terms of classical invariant theory, starting with the original definition (3) on p. 7 of Green-Griffiths jets. Indeed, the general $n \times n$ complex unipotent matrix:

$$\mathsf{u} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \mathsf{u}_{21} & 1 & 0 & \cdots & 0 \\ \mathsf{u}_{31} & \mathsf{u}_{32} & 1 & \cdots & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ \mathsf{u}_{n1} & \mathsf{u}_{n2} & \mathsf{u}_{n3} & \cdots & 1 \end{pmatrix},$$

where the $\mathsf{u}_{ij} \in \mathbb{C}$ are arbitrary complex numbers, acts naturally and linearly on all the jet variables in such a way that for any jet level λ with $1 \leqslant \lambda \leqslant \kappa$, one sets in matrix notation:

$$g^{(\lambda)} := \mathsf{u} \cdot f^{(\lambda)},$$

that is to say in greater length:

$$\begin{cases} g_1^{(\lambda)} := f_1^{(\lambda)} \\ g_2^{(\lambda)} := f_2^{(\lambda)} + \mathsf{u}_{21}\, f_1^{(\lambda)} \\ g_3^{(\lambda)} := f_3^{(\lambda)} + \mathsf{u}_{32}\, f_2^{(\lambda)} + \mathsf{u}_{31}\, f_1^{(\lambda)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ g_n^{(\lambda)} = f_n^{(\lambda)} + \mathsf{u}_{n,n-1}\, f_{n-1}^{(\lambda)} + \cdots + \mathsf{u}_{n1}\, f_1^{(\lambda)}. \end{cases}$$

A general fact from the classical representation theory of $\mathsf{GL}_n(\mathbb{C})$ states that the so-called *vectors of highest weight* identify precisely to those that remain invariant by this unipotent action, namely to jet polynomials $\mathsf{P}\big(j^\kappa f\big)$ which satisfy the invariancy condition:

$$\mathsf{P}\big(j^\kappa g\big) = \mathsf{P}\big(\mathsf{u} \cdot j^\kappa f\big) \equiv \mathsf{P}\big(j^\kappa f\big),$$

for every unipotent matrix $\mathsf{u} \in \mathsf{U}_n(\mathbb{C})$. Furthermore and most importantly, there is a one-to-one correspondence between the vectors of highest weight and the Schur bundles appearing in the decomposition of $\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*$, the rule being as follows. Precisely speaking, the vector space of unipotent-invariant polynomials (vectors of highest weight) is shown to decompose as a direct sum of (linearly independent) one-dimensional spaces generated by vectors $\mathsf{Q} = \mathsf{Q}\big(j^\kappa f\big)$ that are eigenvalues for the action $\mathsf{e} \cdot f_i^{(\lambda)} := \mathsf{e}_i\, f_i^{(\lambda)}$ of all diagonal matrices of the form:

$$\mathsf{e} := \begin{pmatrix} \mathsf{e}_1 & 0 & \cdots & 0 \\ 0 & \mathsf{e}_2 & \cdots & 0 \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ 0 & 0 & \cdots & \mathsf{e}_n \end{pmatrix},$$

where $\mathsf{e}_1, \mathsf{e}_2, \ldots, \mathsf{e}_n$ are arbitrary complex numbers, so that there are certain characteristic exponents ℓ_i with the property that:

$$\mathsf{Q}\big(\mathsf{e} \cdot j^\kappa f\big) = (\mathsf{e}_1)^{\ell_1} (\mathsf{e}_2)^{\ell_2} \cdots (\mathsf{e}_n)^{\ell_n}\, \mathsf{Q}\big(j^\kappa f\big).$$

One shows that $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0$ and that such an eigenvector Q (of highest weight) is precisely linked to the Schur bundle $\mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)} T_X^*$ which corresponds to an irreducible representation on a fiber over a point $x \in X$. Of course, a specific Schur bundle $\mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)} T_X^*$ could well occur several times in the sought decomposition of $\mathsf{Gr}^\bullet \mathscr{E}_{\kappa,m}^{GG} T_X^*$, hence have a certain multiplicity $\geqslant 2$, because some different linearly independent Q's could share the same characteristic exponents ℓ_i. In fact, this will indeed be the case below, and determining such multiplicities, at least asymptotically as $\kappa \to \infty$, will be crucial for us.

Serendipity. The knowledge of the algebra of invariants of the full unipotent group $\mathsf{U}_n(\mathbb{C}) \subset \mathsf{GL}_n(\mathbb{C})$ dates back to the nineteenth century. As a matter of fact, the following four basic statements Theorems A, B, C and D below, which will precede a main starting theorem specially designed for our future purposes, are essentially known and they are established in various sources.

Theorem A. ([37, 19, 24, 27]) *The algebra of jet polynomials invariant under the above action of the full unipotent group* $\mathsf{U}_n(\mathbb{C}) \subset \mathsf{GL}_n(\mathbb{C})$ *is* generated*,* as an algebra*, by the collection of all the determinants (minors):*

$$\left|f_1^{(\lambda_1)}\right| =: \Delta_1^{\lambda_1}, \quad \begin{vmatrix} f_1^{(\lambda_1)} & f_2^{(\lambda_1)} \\ f_1^{(\lambda_2)} & f_2^{(\lambda_2)} \end{vmatrix} =: \Delta_{1,2}^{\lambda_1,\lambda_2}, \quad \begin{vmatrix} f_1^{(\lambda_1)} & f_2^{(\lambda_1)} & f_3^{(\lambda_1)} \\ f_1^{(\lambda_2)} & f_2^{(\lambda_2)} & f_3^{(\lambda_2)} \\ f_1^{(\lambda_3)} & f_2^{(\lambda_3)} & f_3^{(\lambda_3)} \end{vmatrix} =: \Delta_{1,2,3}^{\lambda_1,\lambda_2,\lambda_3},$$

$$\cdots\cdots, \quad \begin{vmatrix} f_1^{(\lambda_1)} & f_2^{(\lambda_1)} & \cdots & f_n^{(\lambda_1)} \\ f_1^{(\lambda_2)} & f_2^{(\lambda_2)} & \cdots & f_n^{(\lambda_2)} \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(\lambda_n)} & f_2^{(\lambda_n)} & \cdots & f_n^{(\lambda_n)} \end{vmatrix} =: \Delta_{1,2,\ldots,n}^{\lambda_1,\lambda_2,\ldots,\lambda_n},$$

in which the jet orders $1 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_n \leqslant \kappa$ *are all arbitrary* [6]*, and these determinants are visibly invariant with respect to the* $\mathsf{U}_n(\mathbb{C})$*-action.*

However, although all the determinants in question happen to be linearly independent, one cannot just pretend that the whole unipotent-invariant algebra identifies with the plain polynomial algebra:

$$\mathbb{C}\big[\Delta_1^{\lambda_1}, \Delta_{1,2}^{\lambda_1,\lambda_2}, \ldots, \Delta_{1,2,\ldots,n}^{\lambda_1,\ldots,\lambda_n}\big],$$

because several elementary *non*linear relations exist between all these determinants; for instance, there exist the basic quadratic Plücker relations[7] of first and of second type:

$$\begin{aligned} 0 &\equiv \Delta_1^{\lambda_1}\, \Delta_{1,2}^{\lambda_2,\lambda_3} + \Delta_1^{\lambda_3}\, \Delta_{1,2}^{\lambda_1,\lambda_2} + \Delta_1^{\lambda_2}\, \Delta_{1,2}^{\lambda_3,\lambda_1} \\ 0 &\equiv \Delta_{1,2}^{\lambda_1,\lambda_2}\, \Delta_{1,2}^{\lambda_3,\lambda_4} + \Delta_{1,2}^{\lambda_1,\lambda_4}\, \Delta_{1,2}^{\lambda_2,\lambda_3} + \Delta_{1,2}^{\lambda_1,\lambda_3}\, \Delta_{1,2}^{\lambda_4,\lambda_2}, \end{aligned}$$

so that the binomial in the first line $\Delta_1^{\lambda_1}\Delta_{1,2}^{\lambda_2,\lambda_3}$, viewed in the plain polynomial algebra $\mathbb{C}\big[\Delta_1^{\lambda_1},\, \Delta_{1,2}^{\lambda_1,\lambda_2}\big]$, would possess *two* distinct representations: itself, and:

$$-\,\Delta_1^{\lambda_3}\, \Delta_{1,2}^{\lambda_1,\lambda_2} - \Delta_1^{\lambda_2}\, \Delta_{1,2}^{\lambda_3,\lambda_1}.$$

---

[6] It is only necessary to consider strictly increasing integers λ_l, since for every i with $1 \leqslant i \leqslant n$, and for every permutation σ of $\{1, 2, \ldots, i\}$ one clearly has:

$$\Delta_{1,2,\ldots,i}^{\lambda_{\sigma(1)},\lambda_{\sigma(2)},\ldots,\lambda_{\sigma(i)}} = (-1)^{\mathrm{sign}(\sigma)}\, \Delta_{1,2,\ldots,i}^{\lambda_1,\lambda_2,\ldots,\lambda_i}.$$

[7] — the knowledge of which surely goes back to the seventeenth century theory, at a time when elimination was the main tool in the search for solving algebraic equations of degrees 2, 3, 4 and 5.

Fortunately, the ideal of all relations between these Δ-determinants is also completely known and understood. However, presenting *explicitly* this ideal of all relations requires a bit of preparation and a few more indices.

Ideal of relations between all Δ jet-determinants. We therefore consider the collection of all determinants $\Delta_{1,2,\dots,i}^{\lambda_1,\lambda_2,\dots,\lambda_i}$ for every $i \in \{1,\dots,n\}$ and for every choice of i jet-line indices $\lambda_1,\lambda_2,\dots,\lambda_i \in \{1,\dots,\kappa\}$. At first, we equip this collection with a *partial* order by declaring that:

$$\Delta_{1,2,\dots,i}^{\lambda_1,\lambda_2,\dots,\lambda_i} <_{one} \Delta_{1,2,\dots,j}^{\mu_1,\mu_2,\dots,\mu_j}$$

if firstly:

$$i \geqslant j$$

and if secondly all the following inequalities hold:

$$\lambda_1 \leqslant \mu_1, \quad \lambda_2 \leqslant \mu_2, \quad \dots\dots, \quad \lambda_j \leqslant \mu_j. \tag{10}$$

Not all determinants are comparable for this order, *e.g.* $\Delta_{1,2}^{1,4}$ and $\Delta_{1,2}^{2,3}$ are *in*comparable, and similarly, $\Delta_{1,2}^{1,4}$ and $\Delta_{1,2,3}^{2,3,4}$ are incomparable too. We will now see that there is a one-to-one correspondence between incomparable Δ-determinants and (generalized) Plücker relations.

Thus, let us pick any two general determinants $\Delta_{1,\dots,i}^{\lambda_1,\dots,\lambda_i}$ and $\Delta_{1,\dots,j}^{\mu_1,\dots,\mu_j}$ that are incomparable and distinct. Permuting the pair if necessary, we may assume that $i \geqslant j$. Furthermore, if $i = j$, we may also assume without loss of generality that $(\lambda_1,\dots,\lambda_i)$ is smaller than $(\mu_1,\dots,\mu_{i=j})$ in the lexicographic ordering, namely there exists an index $s \in \{1,\dots,i = j\}$ such that:

$$\lambda_1 = \mu_1, \quad \dots\dots, \quad \lambda_{s-1} = \mu_{s-1}, \quad \lambda_s < \mu_s.$$

Therefore in both cases $i > j$ and $i = j$, we at least insure by these preliminary choices that:

$$\Delta_{1,\dots,i}^{\lambda_1,\dots,\lambda_i} \not>_{one} \Delta_{1,\dots,j}^{\mu_1,\dots,\mu_j}.$$

Since by assumption, these two determinants are incomparable, the reverse inequality must also fail:

$$\Delta_{1,\dots,i}^{\lambda_1,\dots,\lambda_i} \not<_{one} \Delta_{1,\dots,j}^{\mu_1,\dots,\mu_j},$$

and hence in the two cases $i > j$ and $i = j$, there must exist a smallest index $t \in \{1,\dots,j\}$ such that:

$$\lambda_1 \leqslant \mu_1, \quad \dots\dots, \quad \lambda_{t-1} \leqslant \mu_{t-1}, \quad \lambda_t > \mu_t,$$

because if otherwise all the inequalities (10) would hold, one would have $\Delta_{1,\dots,i}^{\lambda_1,\dots,\lambda_i} <_{one} \Delta_{1,\dots,j}^{\mu_1,\dots,\mu_j}$. In the case $i = j$, it is clear that t can only be $\geqslant s+1$.

Remind that in any circumstance, the jet-line indices of the determinants are strictly increasing:

$$\lambda_1 < \cdots < \lambda_t < \cdots < \lambda_i \qquad \text{and} \qquad \mu_1 < \cdots < \mu_t < \cdots < \mu_j.$$

Diagrammatically, we may then represent a set of inequalities with a pivotal solder, at the index t, between the μ_i and the λ_i:

$$\mu_1 < \cdots < \mu_t \underset{\mathsf{solder}}{<} \lambda_t < \cdots < \lambda_i,$$

by exhibiting, in two adjusted lines, the vertical spot where the join takes place:

$$\begin{aligned} &\mu_1 < \mu_2 < \cdots < \mu_t \qquad < \underline{\mu_{t+1} < \mu_{t+2} < \cdots < \mu_j}_{\mathsf{FIX}} \\ &\qquad \underline{\lambda_1 < \cdots < \lambda_{t-1}}_{\mathsf{FIX}} < \lambda_t < \lambda_{t+1} < \cdots < \lambda_{j-1} < \lambda_j < \cdots < \lambda_i. \end{aligned}$$

Letting now $\pi \in \mathfrak{S}_{i+1}$ be any permutation of the set $\{1, 2, \ldots, i, i+1\}$ with $i+1$ elements, we shall let it act on the $i+1$ elements that are *not* underlined, so that π transforms the $i+1$ integers:

$$\begin{aligned} &\mu_1 < \mu_2 < \cdots < \mu_t < \\ &\qquad\qquad < \lambda_t < \lambda_{t+1} < \cdots < \lambda_{j-1} < \lambda_j < \cdots < \lambda_i \end{aligned}$$

to the $i+1$ permuted integers (not anymore necessarily ordered increasingly):

$$\begin{aligned} &\big(\pi(\mu_1),\, \pi(\mu_2),\, \cdots,\, \pi(\mu_t), \\ &\qquad\qquad \pi(\lambda_t),\, \pi(\lambda_{t+1}),\, \ldots,\, \pi(\lambda_{j-1}),\, \pi(\lambda_j),\, \ldots,\, \pi(\lambda_i)\big). \end{aligned}$$

Since our Δ-determinants are skew-symmetric with respect to any permutation of their lines, it is convenient to restrict attention only to those permutations that respect strict ordering in the two blocks:

$$\begin{aligned} &\pi(\mu_1) < \pi(\mu_2) < \cdots < \pi(\mu_t) \\ \text{and:}\quad &\pi(\lambda_t) < \pi(\lambda_{t+1}) < \cdots < \pi(\lambda_{j-1}) < \pi(\lambda_j) < \cdots < \pi(\lambda_i). \end{aligned}$$

At last, we are in a position to write down the most general quadratic Plücker relations that are fundamental for the subject.

Theorem B. ([37, 19, 24, 27]) *For any two determinants* $\Delta^{\lambda_1,\ldots,\lambda_i}_{1,\ldots,i}$ *and* $\Delta^{\mu_1,\ldots,\mu_j}_{1,\ldots,j}$ *with* $i \geqslant j$ *that are incomparable with respect to the partial ordering "*$<_{one}$*", namely which have the concrete properties that:*

- *when* $i > j$*, there exists an index* $t \in \{1, \ldots, j\}$ *such that:*

$$\lambda_1 \leqslant \mu_1, \quad \ldots\ldots, \quad \lambda_{t-1} \leqslant \mu_{t-1}, \quad \textit{but:} \quad \lambda_t > \mu_t;$$

- *when $i = j$, there exist two indices $s \in \{1, \dots, j\}$ and $t \in \{1, \dots, j\}$ with $t \geqslant s+1$ such that:*

$$\lambda_1 = \mu_1, \ \dots, \ \lambda_{s-1} = \mu_{s-1}, \ \lambda_s < \mu_s,$$
$$\lambda_{s+1} \leqslant \mu_{s+1}, \ \dots, \ \lambda_{t-1} \leqslant \mu_{t-1}, \ \textit{but again:} \ \lambda_t > \mu_t;$$

the following general quadratic (Plücker) relation holds identically in the ground ring $\mathbb{C}\big[f'_{i_1}, f''_{i_2}, \dots, f^{(\kappa)}_{i_\kappa}\big]$:

$$\boxed{0 \equiv \sum_{\pi \in \mathfrak{S}_{i+1}} \sum_{\substack{\pi(\lambda_t) < \cdots < \pi(\lambda_i) \\ \pi(\mu_1) < \cdots < \pi(\mu_t)}} \mathsf{sign}(\pi) \cdot \Delta^{\lambda_1, \dots, \lambda_{t-1}, \pi(\lambda_t), \pi(\lambda_{t+1}), \dots, \pi(\lambda_{j-1}), \pi(\lambda_j), \dots, \pi(\lambda_i)}_{1, \dots, t-1, t, t+1, \dots, j-1, j, \dots, i} \cdot \Delta^{\pi(\mu_1), \pi(\mu_2), \dots, \pi(\mu_t), \mu_{t+1}, \mu_{t+2}, \dots, \mu_j}_{1, 2, \dots, t, t+1, t+2, \dots, j}}.$$

We will not reproduce the proof here, but extract instead from the cited references the further important information that the ideal of relations between all our Δ jet-determinants:

$$\Delta^{\lambda_1, \dots, \lambda_i}_{1, \dots, i} = \begin{vmatrix} f_1^{(\lambda_1)} & \cdots & f_i^{(\lambda_1)} \\ \cdot\cdot & \cdots & \cdot\cdot \\ f_1^{(\lambda_i)} & \cdots & f_i^{(\lambda_i)} \end{vmatrix}$$
$$(i = 1 \cdots n \,;\, 1 \leqslant \lambda_1 < \cdots < \lambda_i \leqslant \kappa)$$

is generated (as an ideal) by all the above quadratic Plücker relations. Moreover, these relations written explicitly above do constitute a *Gröbner basis* for a certain term order, presented as follows.

Introduce first as many independent variables $\nabla^{\lambda^i_1, \dots, \lambda^i_i}$ as there are Δ jet-determinants and consider the ring $\mathbb{C}\big[\nabla^{\lambda^1_1}, \dots, \nabla^{\lambda^n_1, \dots, \lambda^n_n}\big]$. Totally order these variables by declaring that:

$$\nabla^{\lambda^i_1, \dots, \lambda^i_i} <_{two} \nabla^{\mu^j_1, \dots, \mu^j_j}$$

if either $i > j$ or else if $i = j$ *and* $(\lambda^i_1, \dots, \lambda^i_i)$ comes before $(\mu^j_1, \dots, \mu^j_{j=i})$ in the lexicographic ordering, which simply means that there exists an index $s \in \{1, \dots, j\}$ such that:

$$\lambda^i_1 = \mu^j_1, \quad \dots\dots, \quad \lambda^i_{s-1} = \mu^j_{s-1}, \quad \lambda^i_s < \mu^j_s.$$

This total order extend the partial order "$<_{one}$". Finally, let also "$<_{two}$" denote the *reverse lexicographic*[8] term ordering on $\mathbb{C}\big[\nabla^{\lambda^1_1}, \dots, \nabla^{\lambda^n_1, \dots, \lambda^n_n}\big]$ that is induced by this variable ordering $<_{two}$. The set of polynomials

[8] Generally, if $x_1 <_{two} \cdots <_{two} < x_n$, the reverse lexicographic (total) ordering induced on monomials says that $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is *smaller* than $x_1^{\beta_1} \cdots x_n^{\beta_n}$ if, when reading exponents from right to left, equality holds: $\alpha_n = \beta_n, \dots, \alpha_{u+1} = \beta_{u+1}$ until a first difference occurs: $\alpha_u \neq \beta_u$ for which $\alpha_u > \beta_u$ is *bigger* than β_u.

$\mathsf{R}\big(\nabla^{\lambda_1^1}, \dots, \nabla^{\lambda_1^n,\dots,\lambda_n^n}\big)$ which annihilate identically after replacement by the determinants:

$$0 \equiv \mathsf{R}\big(\Delta_1^{\lambda_1^1}, \dots, \Delta_{1,\dots,n}^{\lambda_1^n,\dots,\lambda_n^n}\big)$$

constitutes clearly an *ideal* of $\mathbb{C}\big[\nabla^{\lambda_1^1}, \dots, \nabla^{\lambda_1^n,\dots,\lambda_n^n}\big]$.

Theorem C. ([37, 19, 24, 27]) *The ideal of relations* $\mathsf{Id\text{-}rel}(\Delta)$ *between all* Δ *jet-determinants is generated by all the Plücker relations written above. Moreover, the collection of all these Plücker relations constitutes already* per se *a* Gröbner basis *for* $\mathsf{Id\text{-}rel}(\Delta)$ *under the term ordering "*$<_{two}$*". Finally, the products:*

$$\Delta_{1,\dots,i}^{\lambda_1,\dots,\lambda_i} \cdot \Delta_{1,\dots,j}^{\mu_1,\dots,\mu_j}$$

of all possible incomparable pairs generate the (monomial) ideal of leading monomials of elements of $\mathsf{Id\text{-}rel}(\Delta)$.

Polynomials modulo relations. Thanks to this statement, we will be able to find a basis of the $\mathbb{C}$-vector space:

$$\mathsf{all}\ \Delta\text{-}\mathsf{polynomials} \,\big/\, \mathsf{modulo\ their\ relations}.$$

This will be very useful, for we saw that basis vectors are in one-to-one correspondence with Schur bundles $\mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^*$ (see also below). The general Gröbner basis theory then tells us that this vector space is generated by all Δ-monomials that are *not* multiple of any product of incomparable pairs (leading monomials). In order to describe explicitly this quotient vector space, we need a classical combinatorial object.

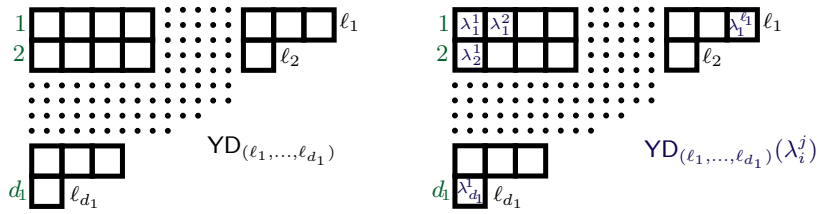
Young diagrams. Let $d_1 \geqslant 1$ be an integer and let $\ell_1, \ell_2, \dots, \ell_{d_1}$ be any collection of d_1 nonnegative integers collected in decreasing order:

$$\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{d_1} \geqslant 1.$$

The *Young diagram* $\mathsf{YD}_{(\ell_1,\dots,\ell_{d_1})}$ associated to such a d_1-tuple $(\ell_1, \dots, \ell_{d_1})$ sits in the right-bottom quadrant $\{x \geqslant 0,\ y \leqslant 0\}$ of the plane $\mathbb{R}^2 = \mathbb{R}^2(x,y)$ and it consists, in the i-th horizontal strip $\{-i \leqslant y \leqslant -i+1\}$ from above, for $i = 1, \dots, d_1$, of the ℓ_i empty unit squares:

$$\square_i^j := \big\{(x,y) \in \mathbb{R}^2 \colon\ -i \leqslant y \leqslant -i+1,\ j-1 \leqslant x \leqslant j\big\}$$

placed, for $j = 1, \dots, \ell_i$, one after the other and starting from the vertical y-axis (left-justification).

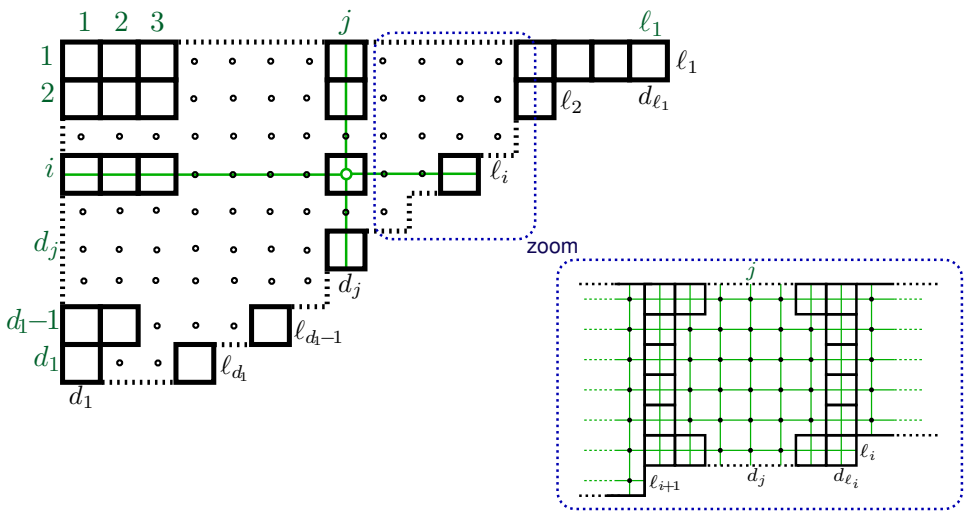


It will be convenient to give names to the column lengths, say d_j will denote that of the j-th, for $j = 1, \dots, \ell_1$. In summary and for memory:

$$\ell_i = \text{length of the } i\text{-th row}; \qquad d_j = \text{length of the } j\text{-th column}.$$

Observe that the longest column lengths are equal to d_1 for all indices j between 1 and ℓ_{d_1}, and more generally at any i-th row (see the zoom below), that the following coincidence of column lengths holds:

$$i = d_{1+\ell_{i+1}} = \cdots = d_{\ell_i} \qquad (1 \leqslant i \leqslant d_1).$$



Semi-standard Young tableaux. If $\lambda_i^j \geqslant 1$ denote as many nonnegative integers as there are empty squares $\square_i^j$, namely with $i = 1, \dots, d_1$ and $j = 1, \dots, \ell_i$, a *filling* $\mathsf{YD}_{(\ell_1,\dots,\ell_{d_1})}(\lambda_i^j)$ of the Young diagram $\mathsf{YD}_{(\ell_1,\dots,\ell_{d_1})}$ by means of the λ_i^j consists in putting each λ_i^j in each square $\square_i^j$. A *semi-standard (Young) tableau* is a filled Young diagram $\mathsf{YD}_{(\ell_1,\dots,\ell_{d_1})}(\lambda_i^j)$ having the property that when reading its full content:

$$\begin{array}{ccccccccccccc}
\lambda_1^1 & \cdots & \lambda_1^{\ell_{d_1}} & \cdots & \lambda_1^{\ell_{d_1-1}} & \cdots\cdots & \lambda_1^{\ell_i} & \cdots\cdots & \lambda_1^{\ell_2} & \cdots & \lambda_1^{\ell_1} \\
\lambda_2^1 & \cdots & \lambda_2^{\ell_{d_1}} & \cdots & \lambda_2^{\ell_{d_1-1}} & \cdots\cdots & \lambda_2^{\ell_i} & \cdots\cdots & \lambda_2^{\ell_2} & & \\
\cdot & \cdots & \cdot & \cdots & \cdot & \cdots\cdots & \cdot & \cdots & & & \\
\lambda_i^1 & \cdots & \lambda_i^{\ell_{d_1}} & \cdots & \lambda_i^{\ell_{d_1-1}} & \cdots\cdots & \lambda_i^{\ell_i} & & & & \\
\cdot & \cdots & \cdot & \cdots & \cdot & \cdots & & & & & \\
\lambda_{d_1-1}^1 & \cdots & \lambda_{d_1-1}^{\ell_{d_1-1}} & \cdots & \lambda_{d_1-1}^{\ell_{d_1-1}} & & & & & & \\
\lambda_{d_1}^1 & \cdots & \lambda_{d_1}^{\ell_{d_1}}, & & & & & & & &
\end{array}$$

the integers λ_i^j increase from top to bottom in each column, and they are nondecreasing[9] in each row from left to right, that is to say and more precisely:

$$\begin{aligned} &\lambda_1^j < \lambda_2^j < \cdots < \lambda_{d_j}^j && (1 \leqslant j \leqslant \ell_1) \\ &\lambda_i^1 \leqslant \lambda_i^2 \leqslant \cdots \leqslant \lambda_i^{\ell_i} && (1 \leqslant i \leqslant d_1). \end{aligned}$$

Vector space basis for the algebra of Δ jet-determinants. Coming back to our algebra of determinants $\Delta_{1,2,\ldots,i}^{\lambda_1,\lambda_2,\ldots,\lambda_i}$, the increasing sequence of their exponents $\lambda_1 < \lambda_2 < \cdots < \lambda_i$ will sit in a column of such a Young diagram. Since the row-size i of any not identically zero minor $\Delta_{1,2,\ldots,i}^{\lambda_1,\lambda_2,\ldots,\lambda_i}$ must be $\leqslant n = \operatorname{rank}(T_X^*)$, we will consider in fact only semi-standard tableaux whose depth d_1 is always $\leqslant n$. Accordingly, when it happens that $d_1 < n$ we shall adopt the natural convention that:

$$\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{d_1-1} \geqslant \ell_{d_1} > 0 = \ell_{d_1+1} = \cdots = \ell_n.$$

We are at last in a position to state the starting point theorem.

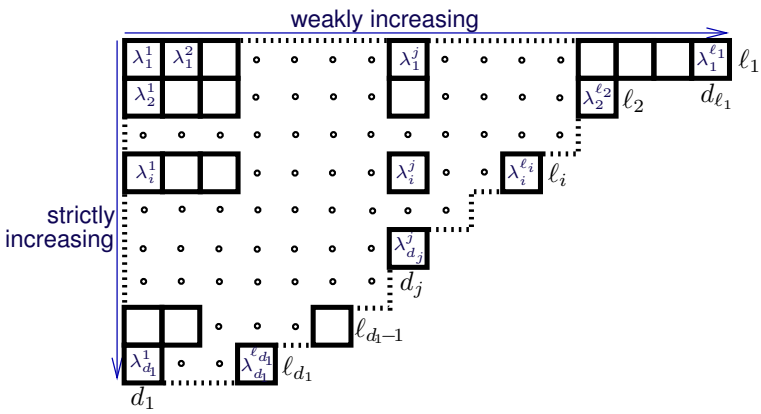
Theorem D. ([37, 19, 24, 27]) *The infinite-dimensional quotient vector space:*

$$\text{all } \Delta\text{-polynomials} \Big/ \text{modulo their relations}$$

possesses a basis over $\mathbb{C}$ consisting of all possible Δ-monomials:

$$\prod_{1 \leqslant j \leqslant \ell_{d_1}} \Delta_{1,\ldots,d_1}^{\lambda_1^j,\ldots,\lambda_{d_1}^j} \prod_{\ell_{d_1}+1 \leqslant j \leqslant \ell_{d_1-1}} \Delta_{1,\ldots,d_1-1}^{\lambda_1^j,\ldots,\lambda_{d_1-1}^j} \cdots \prod_{\ell_2+1 \leqslant j \leqslant \ell_1} \Delta_1^{\lambda_1^j}$$

such that the collection of appearing upper exponents (λ_i^j) constitutes a semi-standard Young tableau:



[9] A so-called *standard tableau* would require that the integers λ_i^j also increase along the rows.

Exact Schur bundle decomposition of $\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X$. In order to apply this combinatorial information to our problem, we may also represent the general Δ-monomial written above more concisely as:

$$\prod_{d_1\geqslant i\geqslant 1}\ \prod_{1+\ell_{i+1}\leqslant j\leqslant \ell_i} \Delta^{\lambda_1^j,\dots,\lambda_i^j}_{1,\dots,i}. \tag{11}$$

First of all, every Δ-determinant read off from such a product happens to be an eigenvector for the action on jets of the diagonal matrices $\mathsf{e} = \mathrm{diag}\big(\mathsf{e}_1,\dots,\mathsf{e}_n\big)$:

$$\mathsf{e}\cdot\Delta^{\lambda_1^j,\lambda_2^j,\dots,\lambda_i^j}_{1,2,\dots,i} = \mathsf{e}_1\,\mathsf{e}_2\cdots\mathsf{e}_i\,\Delta^{\lambda_1^j,\lambda_2^j,\dots,\lambda_i^j}_{1,2,\dots,i},$$

as is clear because the diagonal action just multiplies columns of such a determinant by the quantities $\mathsf{e}_1,\mathsf{e}_2,\dots,\mathsf{e}_j$:

$$\mathsf{e}\cdot\Delta^{\lambda_1^j,\lambda_2^j,\dots,\lambda_i^j}_{1,2,\dots,i} = \begin{vmatrix} \mathsf{e}_1 f_1^{(\lambda_1^j)} & \mathsf{e}_2 f_2^{(\lambda_1^j)} & \cdots & \mathsf{e}_i f_i^{(\lambda_1^j)} \\ \mathsf{e}_1 f_1^{(\lambda_2^j)} & \mathsf{e}_2 f_2^{(\lambda_2^j)} & \cdots & \mathsf{e}_i f_i^{(\lambda_2^j)} \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ \mathsf{e}_1 f_1^{(\lambda_i^j)} & \mathsf{e}_2 f_2^{(\lambda_i^j)} & \cdots & \mathsf{e}_i f_i^{(\lambda_i^j)} \end{vmatrix}.$$

Consequently, every general monomial in the Δ-determinants represented by the above arbitrary semi-standard tableau is also an eigenvector:

$$\begin{aligned}\mathsf{e}\cdot\Bigg(\prod_{d_1\geqslant i\geqslant 1}\ \prod_{1+\ell_{i+1}\leqslant j\leqslant \ell_i} \Delta^{\lambda_1^j,\dots,\lambda_i^j}_{1,\dots,i}\Bigg) &= \mathsf{e}\cdot\big(\text{general }\Delta\text{-monomial}\big)\\ &= \prod_{d_1\geqslant i\geqslant 1}\ \prod_{1+\ell_{i+1}\leqslant j\leqslant \ell_i} \mathsf{e}_1\cdots\mathsf{e}_i\cdot\big(\text{same }\Delta\text{-monomial}\big)\\ &= \prod_{d_1\geqslant i\geqslant 1}\big(\mathsf{e}_1\cdots\mathsf{e}_i\big)^{\ell_i-\ell_{i+1}}\cdot\big(\text{same }\Delta\text{-monomial}\big)\\ &= (\mathsf{e}_1)^{\ell_1}(\mathsf{e}_2)^{\ell_2}\cdots(\mathsf{e}_n)^{\ell_n}\cdot\big(\text{same }\Delta\text{-monomial}\big).\end{aligned}$$

As a result, we deduce generally that:

$$\boxed{\text{Single semi-standard }\Delta\text{-monomial}\longleftrightarrow\text{Unique Schur bundle}},$$

and more precisely, to the general monomial associated with a semi-standard tableau $\mathsf{YD}_{(\ell_1,\dots,\lambda_n)}(\lambda_i^j)$ corresponds bijectively the Schur bundle $\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)}T^*_X$. Thus notably, the related Schur bundle depends only on the diagram, and it *does not depend on its filling by integers* λ_i^j.

Although essentially not new since it follows from Theorem A, B, C and D above, the following basic statement appears nowhere as such in the literature devoted the application of the jet bundle machinery to the conjectures of Green-Griffiths and of Kobayashi, but it will nonetheless constitute our basic starting point.

Theorem. *The graded vector bundle* $\mathsf{Gr}^\bullet \mathscr{E}^{GG}_{\kappa,m} T^*_X$ *associated to the bundle* $\mathscr{E}^{GG}_{\kappa,m} T^*_X$ *of* κ*-th* m*-weighted Green-Griffiths jets identifies to the following* exact direct sum *of Schur bundles:*

$$\mathsf{Gr}^\bullet \mathscr{E}^{GG}_{\kappa,m} T^*_X = \bigoplus_{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0} \Big(\mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)} T^*_X \Big)^{\oplus M^{\kappa,m}_{\ell_1,\ell_2,\ldots,\ell_n}},$$

with multiplicities $M^{\kappa,m}_{\ell_1,\ell_2,\ldots,\ell_n} \in \mathbb{N}$ *equal to the number of times a Young diagram* $\mathsf{YD}_{(\ell_1,\ldots,\ell_n)}$ *with row lengths equal to* $\ell_1, \ell_2, \ldots, \ell_n$ *can be filled in with positive integers* $\lambda^j_i \leqslant \kappa$ *placed at its* i*-th line and* j*-th column so as to constitute a semi-standard tableau, with the further constraint that the sum of all such integers:*

$$\begin{aligned} m = {} & \lambda^1_1 + \cdots + \lambda^{\ell_n}_1 + \cdots + \lambda^{\ell_2}_1 + \cdots + \lambda^{\ell_1}_1 \\ & + \lambda^1_2 + \cdots + \lambda^{\ell_n}_2 + \cdots + \lambda^{\ell_2}_2 \\ & + \cdots\cdots\cdots\cdots\cdots\cdots + \\ & + \lambda^1_n + \cdots + \lambda^{\ell_n}_n \end{aligned}$$

equals the prescribed weighted homogeneity degree m*.*

This apparently complete statement should not hide the fact that the exact computation of the multiplicities $M^{\kappa,m}_{\ell_1,\ell_2,\ldots,\ell_n}$ is not provided in terms of κ, m and $\ell_1, \ell_2, \ldots, \ell_n$. Manual attempts to find a usable, closed and explicit formula for $M^{\kappa,m}_{\ell_1,\ell_2,\ldots,\ell_n}$ showed us that the task could be hard, and we will proceed differently, in an asymptotic manner, so as to avoid several unnecessary computations which would anyway be inaccessible to us.

Corollary. *One has the following inequalities between the cohomology dimensions* h^q *for all* $q = 1, 2, \ldots, n$*:*

$$\begin{aligned} h^q\big(X,\, \mathscr{E}^{GG}_{\kappa,m} T^*_X\big) & \leqslant \sum_{\ell_1 + 2\ell_2 + \cdots + \kappa\ell_\kappa = m} h^q\Big(X,\ \mathrm{Sym}^{\ell_1} T^*_X \otimes \mathrm{Sym}^{\ell_2} T^*_X \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T^*_X\Big) \\ & \leqslant \sum_{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0} M^{\kappa,m}_{\ell_1,\ell_2,\ldots,\ell_n} h^q\big(X,\, \mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)} T^*_X\big). \end{aligned}$$

Proof. The first one was already derived on p. 12. Then the decomposition into Schur bundles of each tensored factor $\mathrm{Sym}^{\ell_1} T^*_X \otimes \cdots \otimes \mathrm{Sym}^{\ell_\kappa} T^*_X$ obtained *e.g.* by an application of Pieri's rule (9) on p. 22 enables one to define a subfiltration to which the same reasoning as on p. 12 applies. □

Thus, as in Rousseau's papers [29, 30] for $n = 3$ and $\kappa = 3$ and as in [22] for $n = 4$ and $\kappa = 4$, the study of the cohomology of the Green-Griffiths bundle $\mathscr{E}^{GG}_{\kappa,m} T^*_X$ is led back to the study of the cohomology of Schur bundles, which might in turn be complicated.

§5. Asymptotic characteristic and asymptotic cohomology

Giambelli determinants of Chern classes. From the lemma on p. 14 and from the theorem on p. 32, we deduce at once from the additivity of Euler-Poincaré characteristic that:

$$\chi\big(X,\ \mathscr{E}^{GG}_{\kappa,m}T^*_X\big) = \sum_{\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_n\geqslant 0} M^{\kappa,m}_{\ell_1,\ell_2,\ldots,\ell_n}\cdot\chi\big(X,\ \mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)}T^*_X\big).$$

But there is a closed asymptotic general formula for:

$$\chi\big(X,\ \mathscr{S}^{(\ell_1,\ldots,\ell_n)}T^*_X\big) = (-1)^n\,\chi\big(X,\ \mathscr{S}^{(\ell_1,\ldots,\ell_n)}T_X\big),$$

where the $(-1)^n$ comes from $\mathsf{c}^*_k = (-1)^k\mathsf{c}_k$. Recall that a *partition* $(\nu_1,\nu_2,\ldots,\nu_n)$ of n is just a collection of nonnegative integers $\nu_1\geqslant\nu_2\geqslant\cdots\geqslant\nu_n\geqslant 0$ whose sum $\nu_1+\nu_2+\cdots+\nu_n$ equals n.

Theorem. ([22][10]) *The terms of highest order with respect to $|\ell| = \ell_1+\cdots+\ell_n$ in the Euler-Poincaré characteristic of the Schur bundle $\mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)}\,T^*_X$ are homogeneous of order $\frac{n(n+1)}{2}$ and they are given by a sum of determinants indexed by all the partitions $(\nu_1,\ldots,\nu_n)$ of n:*

$$(-1)^n\,\chi\Big(X,\ \mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)}\,T^*_X\Big) =$$

$$= \sum_{\nu\text{ partition of }n}\frac{\mathsf{C}_{\nu^c}}{(\nu_1+n-1)!\cdots\nu_n!}\begin{vmatrix} {\ell'_1}^{\nu_1+n-1} & {\ell'_2}^{\nu_1+n-1} & \cdots & {\ell'_n}^{\nu_1+n-1}\\ {\ell'_1}^{\nu_2+n-2} & {\ell'_2}^{\nu_2+n-2} & \cdots & {\ell'_n}^{\nu_2+n-2}\\ \vdots & \vdots & \ddots & \vdots\\ {\ell'_1}^{\nu_n} & {\ell'_2}^{\nu_n} & \cdots & {\ell'_n}^{\nu_n}\end{vmatrix} +$$

$$+\,\mathsf{O}_n\big(|\ell|^{\frac{n(n+1)}{2}-1}\big),$$

where $\ell'_i := \ell_i+n-i$ for notational brevity, with coefficients C_{ν^c} being expressed in terms of the Chern classes $\mathsf{c}_k = \mathsf{c}_k\big(T_X\big)$ of T_X by means of Giambelli's determinantal expression *depending upon the* conjugate *partition ν^c:*

$$\mathsf{C}_{\nu^c} = \mathsf{C}_{(\nu^c_1,\ldots,\nu^c_n)} = \begin{vmatrix} \mathsf{c}_{\nu^c_1} & \mathsf{c}_{\nu^c_1+1} & \mathsf{c}_{\nu^c_1+2} & \cdots & \mathsf{c}_{\nu^c_1+n-1}\\ \mathsf{c}_{\nu^c_2-1} & \mathsf{c}_{\nu^c_2} & \mathsf{c}_{\nu^c_2+1} & \cdots & \mathsf{c}_{\nu^c_2+n-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathsf{c}_{\nu^c_n-n+1} & \mathsf{c}_{\nu^c_n-n+2} & \mathsf{c}_{\nu^c_n-n+3} & \cdots & \mathsf{c}_{\nu^c_n}\end{vmatrix},$$

with the understanding, by convention, that $\mathsf{c}_k := 0$ for $k<0$ or $k>n$, and that $\mathsf{c}_0 := 1$. Furthermore, the remainder $\mathsf{O}_n\big(|\ell|^{\frac{n(n+1)}{2}}\big)$ is a linear combination of homogeneous terms $\mathsf{c}_1^{\tau_1}\mathsf{c}_2^{\tau_2}\cdots\mathsf{c}_n^{\tau_n}$ with $\tau_1+2\tau_2+\cdots+n\tau_n =$

[10] After [22] was posted on arxiv.org, the author was informed by E. Rousseau that Brückmann's Theorem 4 in [1] entails the above statement and moreover, that it shows how to explicit the remainders.

n each multiplied by some polynomial of degree $\leqslant \frac{n(n+1)}{2}-1$ in the ℓ_i whose coefficients are rational and bounded in absolute value by $\mathsf{Constant}_n$.

Because it is elementarily checked that modulo $\mathsf{O}_n\big(|\ell|^{\frac{n(n+1)}{2}-1}\big)$, one has:

$$\begin{vmatrix} {\ell'_1}^{\nu_1+n-1} & {\ell'_2}^{\nu_1+n-1} & \cdots & {\ell'_n}^{\nu_1+n-1} \\ {\ell'_1}^{\nu_2+n-2} & {\ell'_2}^{\nu_2+n-2} & \cdots & {\ell'_n}^{\nu_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ {\ell'_1}^{\nu_n} & {\ell'_2}^{\nu_n} & \cdots & {\ell'_n}^{\nu_n} \end{vmatrix} \equiv \begin{vmatrix} {\ell_1}^{\nu_1+n-1} & {\ell_2}^{\nu_1+n-1} & \cdots & {\ell_n}^{\nu_1+n-1} \\ {\ell_1}^{\nu_2+n-2} & {\ell_2}^{\nu_2+n-2} & \cdots & {\ell_n}^{\nu_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ {\ell_1}^{\nu_n} & {\ell_2}^{\nu_n} & \cdots & {\ell_n}^{\nu_n} \end{vmatrix},$$

we may equivalently replace the ℓ'_i-determinants by the corresponding ℓ_i-determinants in the formula of the theorem. Then for coherence between the above theorem and the computation of the Euler-Poincaré characteristic of $\mathscr{E}^{GG}_{\kappa,m}T^*_X$ conducted independently in Section 3, it should be true that the sum of remainders attached to Schur bundles corresponds to the last remainder of the theorem on p. 15:

$$\sum_{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0} M^{\kappa,m}_{\ell_1,\ell_2,\ldots,\ell_n} \cdot \mathsf{O}_n\big(|\ell|^{\frac{n(n+1)}{2}-1}\big) = \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big).$$

This fact will be established later by the proposition on p. 54 below. Also, using (45) on p. 102, one should consider that all homogeneous products of Chern classes $\mathsf{c}_1^{\tau_1}\cdots\mathsf{c}_n^{\tau_n}$ are implicitly reexpressed in terms of n and d, whence both remainders are in fact of the form $\mathsf{O}_{n,d}\big(|\ell|^{\frac{n(n+1)}{2}-1}\big)$ and $\mathsf{O}_{n,d,\kappa}\big(m^{(\kappa+1)n-2}\big)$.

Dimensions 2, 3 and 4. In greater length, let us for instance write down the expanded sums over partitions, firstly in dimension $n=2$, with two partitions $2=2+0=1+1$:

$$-\chi\big(X,\, \mathscr{S}^{(\ell_1,\ell_2)}T^*_X\big) = \frac{\mathsf{c}_1^2-\mathsf{c}_2}{0!\;3!}\begin{vmatrix} \ell_1^3 & \ell_2^3 \\ 1 & 1 \end{vmatrix} + \frac{\mathsf{c}_2}{1!\,2!}\begin{vmatrix} \ell_1^2 & \ell_2^2 \\ \ell_1 & \ell_2 \end{vmatrix} + \mathsf{O}\big(|\ell|^2\big);$$

next in dimension $n=3$, with three partitions $3=3+0+0=2+1+0=1+1+1$:

$$\begin{aligned} &\chi\big(X,\, \mathscr{S}^{(\ell_1,\ell_2,\ell_3)}\,T^*_X\big) = \\ &= \frac{\mathsf{c}_1^3-2\,\mathsf{c}_1\mathsf{c}_2+\mathsf{c}_3}{0!\;1!\;5!}\begin{vmatrix} \ell_1^5 & \ell_2^5 & \ell_3^5 \\ \ell_1 & \ell_2 & \ell_3 \\ 1 & 1 & 1 \end{vmatrix} + \frac{\mathsf{c}_1\mathsf{c}_2-\mathsf{c}_3}{0!\;2!\;4!}\begin{vmatrix} \ell_1^4 & \ell_2^4 & \ell_3^4 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 \\ 1 & 1 & 1 \end{vmatrix} + \\ &\qquad + \frac{\mathsf{c}_3}{1!\;2!\;3!}\begin{vmatrix} \ell_1^3 & \ell_2^3 & \ell_3^3 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 \\ \ell_1 & \ell_2 & \ell_3 \end{vmatrix} + \mathsf{O}\big(|\ell|^5\big). \end{aligned}$$

and finally in dimension $n = 4$, with 5 partitions $4 = 4+0+0+0 = 3+1+0+0 = 2+2+0+0 = 2+1+1+0 = 1+1+1+1$:

$$\chi\big(X,\, \mathscr{S}^{(\ell_1,\ell_2,\ell_3,\ell_4)}\, T^*_X\big) =$$

$$= \frac{\mathsf{c}_1^4 - 3\,\mathsf{c}_1^2\mathsf{c}_2 + \mathsf{c}_2^2 + 2\,\mathsf{c}_1\mathsf{c}_3 - \mathsf{c}_4}{0!\ 1!\ 2!\ 7!} \begin{vmatrix} \ell_1^7 & \ell_2^7 & \ell_3^7 & \ell_4^7 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^1 & \ell_2^1 & \ell_3^1 & \ell_4^1 \\ 1 & 1 & 1 & 1 \end{vmatrix} +$$

$$+ \frac{\mathsf{c}_1^2\mathsf{c}_2 - \mathsf{c}_2^2 - \mathsf{c}_1\mathsf{c}_3 + \mathsf{c}_4}{0!\ 1!\ 3!\ 6!} \begin{vmatrix} \ell_1^6 & \ell_2^6 & \ell_3^6 & \ell_4^6 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \frac{-\mathsf{c}_1\mathsf{c}_3 + \mathsf{c}_2^2}{0!\ 1!\ 4!\ 5!} \begin{vmatrix} \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \\ \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} +$$

$$+ \frac{\mathsf{c}_1\mathsf{c}_3 - \mathsf{c}_4}{0!\ 2!\ 3!\ 5!} \begin{vmatrix} \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \frac{\mathsf{c}_4}{1!\ 2!\ 3!\ 4!} \begin{vmatrix} \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \end{vmatrix} + \mathsf{O}\big(|\ell|^9\big).$$

Cohomology of Schur bundles. One could be led to presume that the cohomology dimensions:

$$h^q = \dim H^q\big(X,\ \mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)} T^*_X\big) \qquad\qquad (q = 0,\, 1 \cdots n)$$

of any Schur bundle over X might be expressed similarly by means of a general formula of the kind:

$$h^q = \sum_{\tau_1+2\tau_2+\cdots+n\tau_n=n} \mathsf{c}_1^{\tau_1}\mathsf{c}_2^{\tau_2}\cdots\mathsf{c}_n^{\tau_n} \sum_{\alpha_1+\alpha_2+\cdots+\alpha_n \leqslant \frac{n(n+1)}{2}} \sum_{\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_n\geqslant 0} h^{q;\ \ell_1,\ldots,\ell_n}_{\tau_1,\ldots,\tau_n;\ \alpha_1,\ldots,\alpha_n} \cdot (\ell_1)^{\alpha_1}(\ell_2)^{\alpha_2}\cdots(\ell_n)^{\alpha_n}$$

involving the Chern classes c_k, the ℓ_i and certain rational coefficients $h^{q;\ \ell_1,\ldots,\ell_n}_{\tau_1,\ldots,\tau_n\ ;\ \alpha_1,\ldots,\alpha_n} \in \mathbb{Q}$, or alternatively, after making the substitution (45) on p. 102, as follows:

$$h^q = \sum_{k=1}^{n+1} d^k \sum_{\alpha_1+\alpha_2+\cdots+\alpha_n \leqslant \frac{n(n+1)}{2}} \sum_{\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_n\geqslant 0} h^{q;\ \ell_1,\ldots,\ell_n}_{k;\ \alpha_1,\ldots,\alpha_n} \cdot (\ell_1)^{\alpha_1}(\ell_2)^{\alpha_2}\cdots(\ell_n)^{\alpha_n}.$$

However, it turns out to be already known that purely algebraic formulas are certainly impossible, only *semi-algebraic* formulas can be hoped for. Indeed, Brückmann computed in [5] the exact cohomology dimensions:

$$\dim H^q\big(X,\ \Lambda^r T^*_X \otimes \mathscr{O}_X(t)\big)$$

for any $q = 0, 1, \ldots, n$, any $r = 0, 1, \ldots, n$ and any $t \in \mathbb{Z}$, where $\Lambda^r T^*_X$ identifies with $\mathscr{S}^{(1,\ldots,1,0,\ldots,0)} T^*_X$ (r times 1), and it turns out that the obtained formulas are only piecewise polynomial with respect to the data (n, d, q, r, t). In fact, making the convention that $\Lambda^0 T^*_X \equiv \mathscr{O}_X(0)$, it is at

first well known that:

$$\begin{aligned}
\dim H^0\big(X,\, \mathscr{O}_X(t)\big) &= \tbinom{t+n+1}{n+1} - \tbinom{t+n+1-d}{n+1},\\
\dim H^q\big(X,\, \mathscr{O}_X(t)\big) &= 0 \qquad \text{for all } q \text{ with } 1 \leqslant q \leqslant n-1,\\
\dim H^n\big(X,\, \mathscr{O}_X(t)\big) &= \tbinom{d-n-2-t+n+1}{n+1} - \tbinom{d-n-2-t+n+1-d}{n+1}.
\end{aligned}$$

Using then $\Lambda^n T^*_X = K_X = \mathscr{O}_X(d-n-2)$, one deduces:

$$\begin{aligned}
\dim H^0\big(X,\, \Lambda^n T^*_X \otimes \mathscr{O}_X(t)\big) &= \tbinom{d-n-2+t+n+1}{n+1} - \tbinom{d-n-2+t+n+1-d}{n+1},\\
\dim H^q\big(X,\, \Lambda^n T^*_X \otimes \mathscr{O}_X(t)\big) &= 0 \qquad \text{for all } q \text{ with } 1 \leqslant q \leqslant n-1,\\
\dim H^n\big(X,\, \Lambda^n T^*_X \otimes \mathscr{O}_X(t)\big) &= \tbinom{-t+n+1}{n+1} - \tbinom{-t+n+1-d}{n+1}.
\end{aligned}$$

On the other hand, for $1 \leqslant r \leqslant n-1$, Brückmann ([5]) obtained complete dimension formulas:

$$\begin{aligned}
\dim H^0\big(X,\, \Lambda^r T^*_X \otimes \mathscr{O}_X(t)\big) &= \tbinom{t-1}{r}\tbinom{t+n+1-r}{n+1-r},\\
\dim H^q\big(X,\, \Lambda^r T^*_X \otimes \mathscr{O}_X(t)\big) &= \delta_{q,r}\cdot\delta_{t,0} \qquad \text{for all } q \text{ with } 1 \leqslant q \leqslant n-1 \text{ and } q+r \neq n,\\
\dim H^{n-r}\big(X,\, \Lambda^r T^*_X \otimes \mathscr{O}_X(t)\big) &= \sum_{\mu=0}^{n+2} (-1)^\mu \tbinom{n+2}{\mu}\tbinom{-t-rd-(\mu-1)(d-1)}{n+1} + \delta_{n,2r}\cdot\delta_{t,0},\\
\dim H^n\big(X,\, \Lambda^r T^*_X \otimes \mathscr{O}_X(t)\big) &= \tbinom{-t-1}{n-r}\tbinom{-t+n+1-2r}{n+1-2r}.
\end{aligned}$$

Clearly, these formulas are only semi-algebraic. One does not find in the literature complete formulas for cohomology dimensions of Schur bundles having at least three distinct row lengths.

Majorating the cohomology. Rousseau's strategy developed in [30] and in [17] for dimensions 3 and 4 consists in avoiding exact, probably unfeasible cohomology computations and in substituting for that cohomology *inequalities*.

Let as before X be a geometrically smooth projective algebraic complex hypersurface in $\mathbb{P}^{n+1}(\mathbb{C})$. Let $Fl(T^*_X)$ denote the (complete) flag manifold of T^*_X which organizes as a holomorphic vector bundle $\pi\colon Fl(T^*_X) \to X$ of rank $\frac{n(n+1)}{2}$ over X, the fiber of which above an arbitrary point $x \in X$ consists of complete flags:

$$0 = E_{0,x} \subset E_{1,x} \subset \cdots \subset E_{n,x} = T_{X,x},$$

where $\dim E_{i,x} = i$. Let as before $\ell = (\ell_1, \ell_2, \ldots, \ell_n)$ with $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0$. According to Bott ([3]), there is a canonical *line* bundle $\mathscr{B}^\ell(T^*_X)$ over $Fl(T^*_X)$ with the property that the Schur bundle $\mathscr{S}^{(\ell_1,\ldots,\ell_n)}T^*_X \to X$ coincides with the direct image $\pi_*(\mathscr{B}^\ell) \to X$ and whose fiber above an arbitrary flag $E_x \in Fl(T^*_X)$ is $\otimes_{i=1}^n \big(\det(E_{x,i}/E_{x,i-1})\big)^{\otimes \ell_i}$. The fundamental theorem of Bott states that the two bundles $\mathscr{S}^{(\ell_1,\ldots,\ell_n)}T^*_X$ and $\mathscr{B}^\ell(T^*_X)$ have

the same cohomology, and it is therefore somewhat more convenient to deal with $\mathscr{B}^\ell(T^*_X)$, because *line* bundles are better understood and more studied.

In fact, a certain control of the cohomology by means of inequalities is available thanks to the so-called *Holomorphic Morse inequalities* due to Demailly which state as follows in a general version ([12]) suitable for applications devised by Trapani ([38]). Let $\mathscr{E} \to X$ be a completely arbitrary holomorphic vector bundle of rank $r \geqslant 1$ over a compact Kähler manifold of dimension n, and let $\mathscr{L} \to X$ be a holomorphic line bundle subjected to the specific restriction that it can be written as the difference: $\mathscr{L} = \mathscr{F} \otimes \mathscr{G}^{-1}$ between two line bundles that are ample, or more generally, numerically effective. Then about $\mathscr{L}^k \otimes \mathscr{E}$ as $k \to \infty$, we have the following two collections of asymptotic inequalities, firstly for plain cohomology dimensions, secondly for their alternating sums:

• Weak Morse inequalities: For any $q = 0, 1, \dots, n$, one has:

$$h^q\big(X,\ \mathscr{L}^k \otimes \mathscr{E}\big) \leqslant r\, k^n \, \frac{1}{(n-q)!\ q!} \int_X \mathsf{c}_1(\mathscr{F})^{n-q} \cdot \mathsf{c}_1(\mathscr{G})^q + \mathsf{o}(k^n)$$
$$(q = 0,\ 1 \cdots n).$$

• Strong Morse inequalities: For any $q = 0, 1, \dots, n$, one has:

$$\sum_{0 \leqslant q' \leqslant q} (-1)^{q-q'}\, h^{q'}\big(X,\ \mathscr{L}^k \otimes \mathscr{E}\big) \leqslant r\, k^n \sum_{0 \leqslant q' \leqslant q} \frac{(-1)^{q-q'}}{(n-q')!\ q'!} \int_X \mathsf{c}_1(\mathscr{F})^{n-q'} \cdot \mathsf{c}_1(\mathscr{G})^{q'} + \\ + \mathsf{o}(k^n).$$

An algebraic proof of these inequalities (without $\mathscr{E}$ and for X projective) by plain induction on dimension but not using any tools from Analysis was given by Angelini in [1]. We then borrow this scheme of proof, as it was applied by Rousseau ([30]) within the Schur bundle context. Weak type inequalities will suffice for us, and the goal is somehow to represent $\mathscr{B}^\ell(T^*_X)$ as a difference between two line bundles that will be positive, hence ample.

To begin with, since $T^*_X \otimes \mathscr{O}_X(2)$ is generated by its global sections, it is semi-positive. According to a general property ([10]), if a holomorphic vector bundle $\mathscr{E} \to X$ is semi-positive, i.e. if $E \geqslant 0$, then the corresponding line bundle $\mathscr{B}^\ell(\mathscr{E})$ is also semi-positive, i.e. $\mathscr{B}^\ell(\mathscr{E}) \geqslant 0$. Applying this to $\mathscr{E} := T^*_X \otimes \mathscr{O}_X(2)$, we get, thanks to a natural isomorphism, that:

$$\mathscr{B}^\ell\big(T^*_X \otimes \mathscr{O}_X(2)\big) \simeq \mathscr{B}^\ell(T^*_X) \otimes \pi^*\mathscr{O}_X(2|\ell|) \geqslant 0 \tag{12}$$

is semi-positive, where $|\ell| = \ell_1 + \cdots + \ell_n$. Tensoring then by $\pi^*\mathscr{O}_X(|\ell|) > 0$, it thus trivially follows that:

$$\mathscr{B}^\ell(T^*_X) \otimes \pi^*\mathscr{O}_X(3|\ell|) > 0$$

is positive. Hence we can write (somehow artificially) $\mathscr{B}^\ell(T^*_X)$, which we will now write $\mathscr{B}^\ell$ for short, as the following difference:

$$\mathscr{B}^\ell = \big[\mathscr{B}^\ell \otimes \pi^*\mathscr{O}_X(3|\ell|)\big] \otimes \big[\pi^*\mathscr{O}_X(3|\ell|)\big]^{-1}$$

between two positive line bundles over $Fl(T^*_X)$, with plainly:

$$\mathscr{F} := \mathscr{B}^\ell \otimes \pi^*\mathscr{O}_X(3|\ell|) \qquad \text{and} \qquad \mathscr{G} := \pi^*\mathscr{O}_X(3|\ell|),$$

in the above notations for Morse inequalities.

Following Angelini and Rousseau, we need even more in order to force the positive cohomologies $H^q\big(Fl(T^*_X),\, \mathscr{F}\big)$, $q = 1, \dots, n$, to be vanishing. We remind the Kodaira vanishing theorem which stipulates that, on a projective algebraic complex manifold Z, for every ample line bundle $\mathscr{A} \to X$ one has:

$$0 = H^q\big(Z,\, \mathscr{A} \otimes K_Z\big),$$

for all $q = 1, \dots, n$. So on the flag manifold $Z := Fl(T^*_X)$, we not only need that $\mathscr{F}$ be positive (hence ample), but we need also, after decomposing in advance:

$$\mathscr{F} = \big(\mathscr{F} \otimes (K_{Fl(T^*_X)})^{-1}\big) \otimes K_{Fl(T^*_X)},$$

that $\mathscr{A} := \mathscr{F} \otimes (K_{Fl(T^*_X)})^{-1}$ be positive (hence ample).

For this, we recall at first the known isomorphisms ([3, 10]):

$$\begin{aligned} K_{Fl(T^*_X)} &\simeq \big[\mathscr{B}^{2n-1,\dots,3,1}\big]^{-1} \otimes \pi^*(K_X)^{\otimes(n+1)} \\ &\simeq \big[\mathscr{B}^{2n-1,\dots,3,1}\big]^{-1} \otimes \pi^*\mathscr{O}_X\big((n+1)(d-n-2)\big), \end{aligned}$$

from which we hence deduce:

$$\begin{aligned} \mathscr{F} \otimes (K_{Fl(T^*_X)})^{-1} &\simeq \mathscr{B}^\ell \otimes \mathscr{B}^{2n-1,\dots,3,1} \otimes \pi^*\mathscr{O}_X\big(3|\ell| - (n+1)(d-n-2)\big) \\ &\simeq \mathscr{B}^{\ell_1+2n-1,\dots,\ell_{n-1}+3,\ell_n+1} \otimes \pi^*\mathscr{O}_X\big(3|\ell| - (n+1)(d-n-2)\big). \end{aligned}$$

But similarly as in (12) a short while ago, we know that the bundle:

$$\mathscr{B}^{\ell_1+2n-1,\dots,\ell_{n-1}+3,\ell_n+1} \otimes \pi^*\mathscr{O}_X\big(2[\ell_1 + 2n - 1 + \cdots + \ell_{n-1} + 3 + \ell_n + 1]\big)$$

is semi-positive, whence it is surely positive after it is tensored only by $\pi^*\mathscr{O}_X(1)$. Consequently, observing $2n - 1 + \cdots + 3 + 1 = n^2$, our bundle $\mathscr{F} \otimes (K_{Fl(T^*_X)})^{-1}$ will be positive when:

$$3|\ell| - (n+1)(d-n-2) \geqslant 1 + 2(|\ell| + n^2),$$

that is to say when:

$$|\ell| \geqslant 1 + 2n^2 + (n+1)(d-n-2),$$

or with less effective information, when $|\ell| \geqslant \mathsf{Constant}_{n,d}$. Under this restriction concerning $|\ell|$ which insures the applicability of Kodaira's vanishing theorem, Rousseau's scheme of proof works in arbitrary dimension n

(cf. [39] and also [17] for the case $n = 4$), and it yields the following majorations:

$$\begin{aligned} h^q\big(X,\, \mathscr{S}^{(\ell_1,\dots,\ell_n)}T^*_X\big) \leqslant\ & \chi\big(X,\, \mathscr{S}^{(\ell_1,\dots,\ell_n)}T^*_X \otimes \mathscr{O}_X(3(q+1)|\ell|)\big) - \\ & - \tbinom{q}{1}\, \chi\big(X,\, \mathscr{S}^{(\ell_1,\dots,\ell_n)}T^*_X \otimes \mathscr{O}_X(3q|\ell|)\big) + \\ & + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\ & + (-1)^{q-1}\, \tbinom{q}{q-1}\, \chi\big(X,\, \mathscr{S}^{(\ell_1,\dots,\ell_n)}T^*_X \otimes \mathscr{O}_X(6|\ell|)\big) + \\ & + (-1)^{q}\, \tbinom{q}{q}\, \chi\big(X,\, \mathscr{S}^{(\ell_1,\dots,\ell_n)}T^*_X \otimes \mathscr{O}_X(3|\ell|)\big), \end{aligned}$$

in terms of alternating sums of Euler-Poincaré characteristics. Applying Brückmann's formula for the explicit computation of the appearing Euler-Poincaré characteristics (Theorem 4 in [1]), we then get the following result.

Theorem. *Let $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a geometrically smooth projective algebraic complex hypersurface of general type, i.e. of degree $d \geqslant n+3$, and let $\ell = (\ell_1, \dots, \ell_{n-1}, \ell_n)$ with $\ell_1 \geqslant \cdots \geqslant \ell_{n-1} \geqslant \ell_n \geqslant 0$. If:*

$$|\ell| = \ell_1 + \cdots + \ell_{n-1} + \ell_n \geqslant \mathsf{Constant}_{n,d},$$

then for every $q = 1, 2, \dots, n$, the dimensions of the positive cohomology groups of the Schur bundle $\mathscr{S}^{(\ell_1,\dots,\ell_{n-1},\ell_n)}T^*_X$ over X satisfy a general majoration of the form:

$$\begin{aligned} h^q\big(X,\ \mathscr{S}^{(\ell_1,\dots,\ell_{n-1},\ell_n)}T^*_X\big) \leqslant \mathsf{Constant}_{n,d} \prod_{1\leqslant i<j\leqslant n} (\ell_i - \ell_j) \Bigg[\sum_{\beta_1+\cdots+\beta_{n-1}+\beta_n=n} \ell_1^{\beta_1} \cdots \ell_{n-1}^{\beta_{n-1}} \ell_n^{\beta_n} \Bigg] + \\ + \mathsf{Constant}_{n,d} \Bigg[\sum_{\alpha_1+\cdots+\alpha_n \leqslant \frac{n(n+1)}{2}-1} \ell_1^{\alpha_1} \cdots \ell_n^{\alpha_n} \Bigg], \end{aligned}$$

with leading terms being homogeneous of degree $\frac{n(n+1)}{2}$ with respect to the ℓ_i and divisible by all the differences $(\ell_i - \ell_j)$, where $1 \leqslant i < j \leqslant n$.

For the estimates that we will conduct in the next sections, we need none of the three $\mathsf{Constant}_{n,d}$ above to be effective. Admitting this, raising if necessary the two $\mathsf{Constants}_{n,d}$ appearing in the right-hand side, it follows that the majoration is in fact valid for every ℓ, since the restriction that $|\ell|$ be large enough can obviously be absorbed by the $\mathsf{Constants}_{n,d}$. Also, one must observe that the Euler-Poincaré characteristic provided by the theorem on p. 33 satisfies the same kind of majoration, hence *all* cohomology dimensions $h^0, h^1, \dots, h^n$ do the same. However, we want to underline that, even with an effective control on $\mathsf{Constant}_{n,d}$ similar as in [30, 17] for $n = 3$ and $n = 4$, the above kind of majoration cannot at all conduct to the optimal degree bound $d \geqslant n+3$ of the Main Theorem, because we will see that the presence of the monomial ℓ_n^n in $\sum_\beta \ell^\beta$ forces to lose a nonzero portion of the $(\log \kappa)^n$ entering in the Euler-Poincaré characteristic when summing $\sum M_\ell^{\kappa,m}\, \mathscr{S}^{(\ell)}$ over Schur bundles.

§6. Emergence of basic numerical sums

Expanding and rewriting. At least, the explicit formula for the Euler characteristic and the cohomology bounds for Schur bundles firmly motivates to consider basic numerical sums of the form:

$$\sum_{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0} M^{\kappa,m}_{\ell_1,\ell_2,\ldots,\ell_n} \, \ell_1^{\alpha_1} \ell_2^{\alpha_2} \cdots \ell_n^{\alpha_n},$$

for any n-tuple of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ of length $\leqslant \frac{n(n+1)}{2} - 1$ if remainders are to be taken into consideration, or else of length $|\alpha| = \alpha_1 + \cdots + \alpha_n$ constant equal to $\frac{n(n+1)}{2}$, if major terms are to be studied. After some reflections based on manuscript explorations and on intense thought, it appears *a posteriori* convenient, if not adequate, to express all quantities in terms of the successive differences between horizontal lengths in the Young diagram:

$$\ell_1 - \ell_2, \qquad \ell_2 - \ell_3, \qquad \cdots\cdots, \qquad \ell_{n-1} - \ell_n, \qquad \ell_n,$$

that is to say, in terms of the horizontal lengths of the appearing successive blocks of constant depths. Thus accordingly, we may rewrite any appearing monomial $\ell_1^{\alpha_1} \cdots \ell_n^{\alpha_n}$ by inserting differences as follows:

$$\begin{aligned} \ell_1^{\alpha_1} \ell_2^{\alpha_2} \cdots \ell_{n-1}^{\alpha_{n-1}} \ell_n^{\alpha_n} = \big(\ell_1 - \ell_2 + \ell_2 - \ell_3 + \cdots + \ell_{n-1} - \ell_n + \ell_n\big)^{\alpha_1} \cdot \\ \cdot \big(\ell_2 - \ell_3 + \cdots + \ell_{n-1} - \ell_n + \ell_n\big)^{\alpha_2} \cdot \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdot \big(\ell_{n-1} - \ell_n + \ell_n\big)^{\alpha_{n-1}} \cdot \\ \cdot \big(\ell_n\big)^{\alpha_n}, \end{aligned}$$

and then we may simply expand all the appearing powers to obtain a certain sum, with integer integer coefficients, of interesting monomials of the specific form:

$$\big(\ell_1 - \ell_2\big)^{\alpha'_1} \big(\ell_2 - \ell_3\big)^{\alpha'_2} \cdots\cdots \big(\ell_{n-1} - \ell_n\big)^{\alpha'_{n-1}} \big(\ell_n\big)^{\alpha'_n},$$

the total degree in the ℓ_i being evidently preserved:

$$\alpha'_1 + \alpha'_2 + \cdots + \alpha'_{n-1} + \alpha'_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Only multinomial coefficients being involved in the expansion, we have a simple inequality of the form:

(13)
$$\ell_1^{\alpha_1} \cdots \ell_{n-1}^{\alpha_{n-1}} \ell_n^{\alpha_n} \leqslant \mathsf{Constant}_n \cdot \sum_{\alpha'_1 + \cdots + \alpha'_{n-1} + \alpha'_n = \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n} \big(\ell_1 - \ell_2\big)^{\alpha'_1} \cdots \big(\ell_{n-1} - \ell_n\big)^{\alpha'_{n-1}} \big(\ell_n\big)^{\alpha'_n}.$$

Basic numerical sums. As a consequence, in order to verify that the contribution in $\sum M_\ell^{\kappa,m} \cdot \ell^\alpha$ of any monomial $\ell_1^{\alpha_1} \cdots \ell_{n-1}^{\alpha_{n-1}} \ell_n^{\alpha_n}$ of total degree:

$$\alpha_1 + \cdots + \alpha_{n-1} + \alpha_n \leqslant \tfrac{n(n+1)}{2} - 1$$

which possibly appears in a general remainder of the form $\mathsf{O}_{n,d}\big(|\ell|^{\frac{n(n+1)}{2}-1}\big)$ still falls into the corresponding m-remainder $\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big)$, we are led back to studying the asymptotic behavior, as $m \to \infty$, of basic numerical sums of the general form:

$$\sum_{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{n-1} \geqslant \ell_n \geqslant 0} M_{\ell_1,\ell_2,\ldots,\ell_{n-1},\ell_n}^{\kappa,m} \cdot \big(\ell_1 - \ell_2\big)^{\alpha'_1} \cdots\cdots \big(\ell_{n-1} - \ell_n\big)^{\alpha'_{n-1}} \big(\ell_n\big)^{\alpha'_n},$$

with again $\alpha'_1 + \cdots + \alpha'_{n-1} + \alpha'_n \leqslant \frac{n(n+1)}{2} - 1$, and now, everything has become purely combinatorial, that is to say, complex geometry concepts have entirely disappeared.

On the other hand, after expanding any $\prod_{i<j}(\ell_i - \ell_j)\, \ell^\beta$ with $|\beta| = n$ appearing both as principal terms in the Euler-Poincaré characteristic of $\mathscr{S}^{(\ell)} T_X^*$ and in the cohomology majorations provided by the theorem on p. 39, and after performing the rewriting in terms of $\ell_1 - \ell_2, \ldots, \ell_{n-1} - \ell_n$, ℓ_n as above, we are led back to estimating the same kind of basic numerical sums, but this time with $\alpha'_1 + \cdots + \alpha'_{n-1} + \alpha'_n = \frac{n(n+1)}{2}$.

Since we will not attempt to compute, even asymptotically, the multiplicities $M_{\ell_1,\ldots,\ell_n}^{\kappa,m}$ for which only semi-algebraic formulas could exist as an examination for small values of n and κ shows, we will rewrite such basic numerical sums under the following more archetypal form[11]:

(14)

$$\boxed{\begin{aligned} &\sum_{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{n-1} \geqslant \ell_n \geqslant 0} M_{\ell_1,\ell_2,\ldots,\ell_{n-1},\ell_n}^{\kappa,m} \cdot \big(\ell_1 - \ell_2\big)^{\alpha'_1} \cdots\cdots \big(\ell_{n-1} - \ell_n\big)^{\alpha'_{n-1}} \big(\ell_n\big)^{\alpha'_n} = \\ &= \sum_{\substack{\mathsf{YT}\ \mathsf{semi-standard} \\ \mathsf{weight}(\mathsf{YT}) = m}} \big(\ell_1(\mathsf{YT}) - \ell_2(\mathsf{YT})\big)^{\alpha'_1} \cdots \big(\ell_{n-1}(\mathsf{YT}) - \ell_n(\mathsf{YT})\big)^{\alpha'_{n-1}} \big(\ell_n(\mathsf{YT})\big)^{\alpha'_n} \end{aligned}},$$

where in the second line, YT runs over all the possible semi-standard Young tableaux, where $\ell_i(\mathsf{YT})$ denote the length of the i-th line of YT, and where as before $\mathsf{weight}(\mathsf{YT})$ denotes the total number of primes appearing in the associated Δ-monomial, that is to say, the sum of all the λ_i^j occupying the

[11] The equality written follows immediately from the definitions: the passage from the second line to the first line just consists in counting the semi-standard Young Tableaux of weight m which have the same underlying Young diagram $\mathsf{YD}_{(\ell_1,\ell_2,\ldots,\ell_{n-1},\ell_n)}$, and their number is just what we denoted by the multiplicity $M_{\ell_1,\ell_2,\ldots,\ell_{n-1},\ell_n}^{\kappa,m}$.

squares of YT:

$$\begin{aligned}\mathsf{weight}\big(\mathsf{YT}\big) &= \mathsf{weight}\big(\mathsf{YD}_{(\ell_1,\dots,\ell_n)}(\lambda_i^j)\big)\\ &= \sum_{1\leqslant j_1\leqslant \ell_1}\lambda_1^{j_1}+\sum_{1\leqslant j_2\leqslant \ell_2}\lambda_2^{j_2}+\cdots+\sum_{1\leqslant j_n\leqslant \ell_n}\lambda_n^{j_n}.\end{aligned}$$

Then more tractable computations and partially explicit formulas will come up as being somewhat available in the next sections.

Thus, assuming from now on that $\kappa\geqslant n$ is at least equal to the dimension, our first main goal will be to establish (corollary on p. 56 below) that for every $(\alpha'_1,\dots,\alpha'_{n-1},\alpha'_n)\in\mathbb{N}^n$, the following precise logarithmic-like majoration holds:

$$\sum_{\substack{\mathsf{YT}\ \mathsf{semi-standard}\\ \mathsf{weight}(\mathsf{YT})=m}}\big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha'_1}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha'_{n-1}}\big(\ell_n(\mathsf{YT})\big)^{\alpha'_n}\leqslant$$
$$\leqslant \mathsf{Constant}_{n,\kappa}\cdot m^{\alpha'_1+\cdots+\alpha'_{n-1}+\alpha'_n}\cdot m^{n\kappa-\frac{n(n-1)}{2}}.$$

Applying these majorations when $|\alpha'|\leqslant\frac{n(n+1)}{2}-1$, it will then follow in particular that the right-hand side majorant is an $\mathsf{O}_{n,\kappa}\big(m^{(n+1)\kappa-2}\big)$, whence remainders match through summation in Euler-Poincaré characteristics, as was announced a bit after the theorem on p. 33.

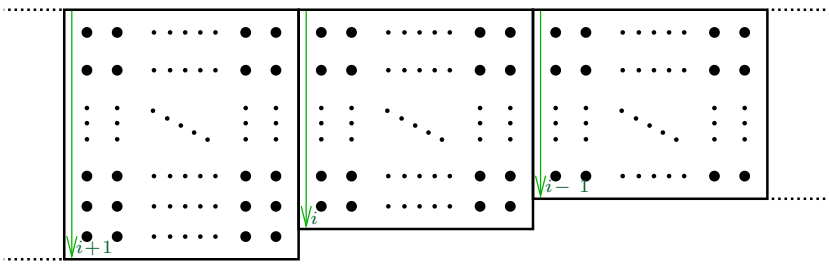
Afterward, we will study what arises when $\alpha'_1+\cdots+\alpha'_{n-1}+\alpha'_n=\frac{n(n+1)}{2}$. In any case, we need to analyze more deeply what comes out from the semi-standard Young tableaux of weight m.

§7. Asymptotic combinatorics of semi-standard Young tableaux

Repetitions in the Δ-monomials. In the general Δ-monomial modulo the Plücker relations given by (11) on p. 31:

$$\prod_{d_1\geqslant i\geqslant 1}\ \prod_{1+\ell_{i+1}\leqslant j\leqslant \ell_i}\Delta_{1,\dots,i}^{\lambda_1^j,\dots,\lambda_i^j},$$

there may exist (several) repetitions of a given determinant $\Delta_{1,\dots,i}^{\lambda_1^j,\dots,\lambda_i^j}$, since in the semi-standard Young tableau, the increasing property enjoyed by the λ_i^j is only weak along its rows. So in $\mathsf{YD}_{(\ell_1,\dots,\ell_n)}(\lambda_i^j)$, we should describe with more explicit information the typical block of depth i:



which is naturally located between a block of depth $i+1$ on its left, and a block of depth $i-1$ on its right. To this aim, let us rewrite such a block as follows:

$$\boxed{\begin{matrix} \lambda_1^{1+\ell_{i+1}} & \cdots & \lambda_1^{\ell_i} \\ \lambda_2^{1+\ell_{i+1}} & \cdots & \lambda_2^{\ell_i} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \lambda_i^{1+\ell_{i+1}} & \cdots & \lambda_i^{\ell_i} \end{matrix}} = \boxed{\begin{bmatrix} \mu_1^j \\ \mu_2^j \\ \cdot\cdot \\ \mu_i^j \end{bmatrix}^{a_{\mu_1^j,\mu_2^j,\ldots,\mu_i^j}} \cdots \begin{bmatrix} \lambda_1^j \\ \lambda_2^j \\ \cdot\cdot \\ \lambda_i^j \end{bmatrix}^{a_{\lambda_1^j,\lambda_2^j,\ldots,\lambda_i^j}} \cdots \begin{bmatrix} \nu_1^j \\ \nu_2^j \\ \cdot\cdot \\ \nu_i^j \end{bmatrix}^{a_{\nu_1^j,\nu_2^j,\ldots,\nu_i^j}}}.$$

Here firstly, looking at the two extreme (right and left) columns, we changed the notation for later purposes, denoting $\mu_l^j := \lambda_l^{1+\ell_{i+1}}$ and $\nu_l^j := \lambda_l^j$ for any row index $l = 1, 2, \ldots, i$; secondly, the appearing exponents $a_{*,*,\ldots,*}$ are meant to denote repetitions of (bracketed) columns, so that naturally their sum equals the horizontal length of the initially considered i-th block:

$$\ell_i - \ell_{i+1} = a_{\mu_1^j,\mu_2^j,\ldots,\mu_i^j} + \cdots + a_{\lambda_1^j,\lambda_2^j,\ldots,\lambda_i^j} + \cdots + a_{\nu_1^j,\nu_2^j,\ldots,\nu_i^j};$$

thirdly and lastly, the succession of columns now increases strictly when one disregards the repetitions:

$$\begin{bmatrix} \mu_1^j \\ \mu_2^j \\ \cdot\cdot \\ \mu_i^j \end{bmatrix} < \cdots < \begin{bmatrix} \lambda_1^j \\ \lambda_2^j \\ \cdot\cdot \\ \lambda_i^j \end{bmatrix} < \cdots < \begin{bmatrix} \nu_1^j \\ \nu_2^j \\ \cdot\cdot \\ \nu_i^j \end{bmatrix},$$

where by definition we declare that a column $(\lambda'_l)_{1 \leqslant l \leqslant i}$ is smaller (strictly) than another column $(\lambda''_l)_{1 \leqslant l \leqslant i}$ if all its row elements are smaller (weakly): $\lambda'_l \leqslant \lambda''_l$ for $l = 1, \ldots, i$, *and if* there exists at least one row index l_0 for which $\lambda'_{l_0} < \lambda''_{l_0}$. As a result, we have represented our typical semi-standard Young tableau of depth d_1 as follows by emphasizing precisely the column

repetitions, all the appearing columns being now pairwise distinct and ordered increasingly:

$$\boxed{\begin{bmatrix} \mu_1^{d_1} \\ \mu_2^{d_1} \\ \mu_3^{d_1} \\ \cdot \\ \cdot \\ \mu_{d_1-1}^{d_1} \\ \mu_{d_1}^{d_1} \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^{d_1} \\ \nu_2^{d_1} \\ \nu_3^{d_1} \\ \cdot \\ \cdot \\ \nu_{d_1-1}^{d_1} \\ \nu_{d_1}^{d_1} \end{bmatrix}^*} \boxed{\begin{bmatrix} \mu_1^{d_1-1} \\ \mu_2^{d_1-1} \\ \mu_3^{d_1-1} \\ \cdot \\ \cdot \\ \mu_{d_1-1}^{d_1-1} \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^{d_1-1} \\ \nu_2^{d_1-1} \\ \nu_3^{d_1-1} \\ \cdot \\ \cdot \\ \nu_{d_1-1}^{d_1-1} \end{bmatrix}^*} \cdots \boxed{\begin{bmatrix} \mu_1^3 \\ \mu_2^3 \\ \mu_3^3 \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^3 \\ \nu_2^3 \\ \nu_3^3 \end{bmatrix}^*} \boxed{\begin{bmatrix} \mu_1^2 \\ \mu_2^2 \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^2 \\ \nu_2^2 \end{bmatrix}^*} \boxed{\begin{bmatrix} \mu_1^1 \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^1 \end{bmatrix}^*} .$$

Here, for reasons of space, the multiplicities $*$ are not written in length, but as above, they should be read for a typical column as an integer $a_{\lambda_1^j,\lambda_2^j,\ldots,\lambda_i^j}$ depending on the column which is $\geqslant 1$; so we understand that the multiplicities of appearing columns are always positive, but it may well happen that some blocks of given depths are completely missing[12], so that at some places, there are contacts between a block of depth, say $i+c$ on the left for some $c \geqslant 2$, and a block of depth i on the right. Furthermore, inside any block, semi-standard inequalities must hold, and between the two contacting columns of two neighboring blocks, say of depth $i+1$ and of depth i:

$$\cdots \boxed{\begin{bmatrix} \mu_1^{i+1} \\ \mu_2^{i+1} \\ \cdot \\ \cdot \\ \mu_i^{i+1} \\ \mu_{i+1}^{i+1} \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^{i+1} \\ \nu_2^{i+1} \\ \cdot \\ \cdot \\ \nu_i^{i+1} \\ \nu_{i+1}^{i+1} \end{bmatrix}^*} \boxed{\begin{bmatrix} \mu_1^i \\ \mu_2^i \\ \cdot \\ \cdot \\ \mu_i^i \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^i \\ \nu_2^i \\ \cdot \\ \cdot \\ \nu_i^i \end{bmatrix}^*} \cdots ,$$

there must of course in addition exist the semi-standard-like truncated inequalities:

$$\begin{array}{ccc} \nu_1^{i+1} & \leqslant & \mu_1^i \\ \nu_2^{i+1} & \leqslant & \mu_2^i \\ \cdot & \leqslant & \cdot \\ \nu_i^{i+1} & \leqslant & \mu_i^i \\ \nu_{i+1}^{i+1}, & & \end{array} \tag{15}$$

with nothing about the last element of the longest column; if more generally, the contact holds between a nonvoid block of depth $i+c$ on the left and a nonvoid block of depth i on the right, in the case where blocks of the intermediate depths $i+c-1, \ldots, i+1$ are inextant, then the last c elements $\nu_{i+1}^{i+c}, \ldots, \nu_{i+c}^{i+c}$ of the rightmost column of the longest block located on the left are subjected to no constraint at all.

[12] However, in what we will be interested in later for applications to the Green-Griffiths bundle of jets, we will have to study only Young diagrams $\mathsf{YD}_{(\ell_1,\ldots,\ell_n)}$ for which $\ell_1-\ell_2, \ldots, \ell_{n-1}-\ell_n$ and ℓ_n are all positive, and even in fact large, so letting all blocks appear in diagrams is harmless.

A notable example of such a semi-standard Young tableau representing a Δ-monomial written in such a way more informative than before is the following:

$$\underbrace{\boxed{\begin{bmatrix}1\\2\\3\\\cdot\\\cdot\\n-1\\n\end{bmatrix}^{*}\cdots\begin{bmatrix}1\\2\\3\\\cdot\\\cdot\\n-1\\\kappa\end{bmatrix}^{*}}}_{\kappa-n+1}\underbrace{\boxed{\begin{bmatrix}1\\2\\3\\\cdot\\\cdot\\n-1\end{bmatrix}^{*}\cdots\begin{bmatrix}1\\2\\3\\\cdot\\\cdot\\\kappa\end{bmatrix}^{*}}}_{\kappa-n+2}\cdots\underbrace{\boxed{\begin{bmatrix}1\\2\\3\end{bmatrix}^{*}\cdots\begin{bmatrix}1\\2\\\kappa\end{bmatrix}^{*}}}_{\kappa-2}\underbrace{\boxed{\begin{bmatrix}1\\2\end{bmatrix}^{*}\cdots\begin{bmatrix}1\\\kappa\end{bmatrix}^{*}}}_{\kappa-1}\underbrace{\boxed{\begin{bmatrix}1\end{bmatrix}^{*}\cdots\begin{bmatrix}\kappa\end{bmatrix}^{*}}}_{\kappa}.$$

It has depth $d_1 = n$ and its first column on the left corresponds naturally to the n-dimensional Wronskian:

$$\Delta^{1,2,3,\ldots,n-1,n}_{1,2,3,\ldots,n-1,n} = \begin{vmatrix} f'_1 & f'_2 & f'_3 & \cdots & f'_{n-1} & f'_n \\ f''_1 & f''_2 & f''_3 & \cdots & f''_{n-1} & f''_n \\ f'''_1 & f'''_2 & f'''_3 & \cdots & f'''_{n-1} & f'''_n \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ f^{(n-1)}_1 & f^{(n-1)}_2 & f^{(n-1)}_3 & \cdots & f^{(n-1)}_{n-1} & f^{(n-1)}_n \\ f^{(n)}_1 & f^{(n)}_2 & f^{(n)}_3 & \cdots & f^{(n)}_{n-1} & f^{(n)}_n \end{vmatrix}$$

raised to a certain power $* = a_{1,2,3,\ldots,n-1,n} \geqslant 1$; in its first block, the bottom indices of extant columns are $n, n+1, \ldots, \kappa-1, \kappa$ while all other indices above are constant horizontally; in its second block, the bottom indices of extant columns are $n-1, n, n+1, \ldots, \kappa+1, \kappa$; and so on. Therefore, the number of pairwise distinct columns is equal to[13]:

$$(\kappa-n+1)+(\kappa-n+2)+\cdots+(\kappa-2)+(\kappa-1)+\kappa = n\kappa - \tfrac{n(n-1)}{2}.$$

Observe that this example of semi-standard Young tableau constitutes not just one Δ-monomial, but in fact it represents a whole specific *family* of Δ-monomials which are labelled by all possible column exponents $* \geqslant 1$. Of course, the row lengths, or equivalently and what is clearer, the differences of row lengths, may, as already seen, be expressed in terms of these

[13] Notice *passim* that this number minus 1 plus the (constant) degree of any homogeneous monomial $\big(\ell_1-\ell_2\big)^{\alpha_1}+\cdots+\big(\ell_{n-1}-\ell_n\big)^{\alpha_{n-1}}\big(\ell_n\big)^{\alpha_n}$:

$$n\kappa - \tfrac{n(n-1)}{2} - 1 + \tfrac{n(n+1)}{2} = (\kappa+1)n - 1$$

equals the exponent of m in the formula of the theorem on p. 15 about the asymptotic behavior of the Euler-Poincaré characteristic.

exponents:

$$\begin{cases} \ell_n = a_{1,2,3,\dots,n-1,n} + \cdots + a_{1,2,3,\dots,n-1,\kappa} \\ \ell_{n-1} - \ell_n = a_{1,2,3,\dots,n-1} + \cdots + a_{1,2,3,\dots,\kappa} \\ \cdot\cdot \quad \cdot\cdot = \cdots\cdots\cdots\cdots\cdots\cdots \\ \ell_3 - \ell_4 = a_{1,2,3} + \cdots + a_{1,2,\kappa} \\ \ell_2 - \ell_3 = a_{1,2} + \cdots + a_{1,\kappa} \\ \ell_1 - \ell_2 = a_1 + \cdots + a_\kappa. \end{cases}$$

One should also remember from the theorem on p. 32 that the total number of primes in any considered Δ-monomial should be constant equal to m, a condition that can now be read here as:

$$\begin{aligned} m = {} & \big[1+2+3+\cdots+n-1+n\big]a_{1,2,3,\dots,n-1,n} + \cdots + \big[1+2+3+\cdots+n-1+\kappa\big]a_{1,2,3,\dots,n-1,\kappa} + \\ & + \big[1+2+3+\cdots+n-1\big]a_{1,2,3,\dots,n-1} + \cdots + \big[1+2+3+\cdots+\kappa\big]a_{1,2,3,\dots,\kappa} + \\ & + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\ & + \big[1+2+3\big]a_{1,2,3} + \cdots + \big[1+2+\kappa\big]a_{1,2,\kappa} + \\ & + \big[1+2\big]a_{1,2} + \cdots + \big[1+\kappa\big]a_{1,\kappa} + \\ & + \big[1\big]a_1 + \cdots + \big[\kappa\big]a_\kappa; \end{aligned}$$

in this equation, each exponent $a_{1,2,3,\dots,i-1,\lambda}$, $i \leqslant \lambda \leqslant \kappa$, is multiplied by the sum $1+2+3+\cdots+i-1+\lambda$ of its lower indices, an integer which equals the total number of primes in the determinant $\Delta^{1,2,3,\dots,i-1,\lambda}_{1,2,3,\dots,i-1,i}$, so that the total number of primes in the power $\big[\Delta^{1,2,3,\dots,i-1,\lambda}_{1,2,3,\dots,i-1,i}\big]^{a_{1,2,3,\dots,i-1,\lambda}}$ is indeed equal to the product:

$$\big[1+2+3+\cdots+i-1+\lambda\big]a_{1,2,3,\dots,i-1,\lambda}.$$

Maximal number of pairwise distinct columns in $\mathsf{YD}_{(\ell_1,\dots,\ell_n)}(\lambda^j_i)$. We now come back to a general semi-standard Young tableau of depth d_1 with extreme columns in each block that can be arbitrary:

(16)

$$\underbrace{\boxed{\begin{bmatrix} \mu_1^{d_1} \\ \mu_2^{d_1} \\ \mu_3^{d_1} \\ \cdot \\ \cdot \\ \mu_{d_1-1}^{d_1} \\ \mu_{d_1}^{d_1} \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^{d_1} \\ \nu_2^{d_1} \\ \nu_3^{d_1} \\ \cdot \\ \cdot \\ \nu_{d_1-1}^{d_1} \\ \nu_{d_1}^{d_1} \end{bmatrix}^*}}_{\#\leqslant D_{d_1}} \underbrace{\boxed{\begin{bmatrix} \mu_1^{d_1-1} \\ \mu_2^{d_1-1} \\ \mu_3^{d_1-1} \\ \cdot \\ \cdot \\ \mu_{d_1-1}^{d_1-1} \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^{d_1-1} \\ \nu_2^{d_1-1} \\ \nu_3^{d_1-1} \\ \cdot \\ \cdot \\ \nu_{d_1-1}^{d_1-1} \end{bmatrix}^*}}_{\#\leqslant D_{d_1-1}} \cdots \underbrace{\boxed{\begin{bmatrix} \mu_1^3 \\ \mu_2^3 \\ \mu_3^3 \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^3 \\ \nu_2^3 \\ \nu_3^3 \end{bmatrix}^*}}_{\#\leqslant D_3} \underbrace{\boxed{\begin{bmatrix} \mu_1^2 \\ \mu_2^2 \end{bmatrix}^* \cdots \begin{bmatrix} \nu_1^2 \\ \nu_2^2 \end{bmatrix}^*}}_{\#\leqslant D_2} \underbrace{\boxed{\big[\mu_1^1\big]^* \cdots \big[\nu_1^1\big]^*}}_{\#\leqslant D_1}.$$

What is the maximal possible number of pairwise distinct $*$-ed columns? In the rightmost block, the number of entries in the single row is clearly less than or equal to:

$$D_1 := 1 + \nu_1^1 - \mu_1^1.$$

In full generality, there may well be several gaps from μ_1^1 to ν_1^1 in the 'cdots', so D_1 is an upper bound. If a general $*$-ed column of depth i with $1 \leqslant i \leqslant d_1$ has two immediate neighbors, with no possible intermediate neighbors:

$$\begin{bmatrix} \lambda'_1 \\ \lambda'_2 \\ \cdot\cdot \\ \lambda'_i \end{bmatrix}^* < \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \cdot\cdot \\ \lambda_i \end{bmatrix}^* < \begin{bmatrix} \lambda''_1 \\ \lambda''_2 \\ \cdot\cdot \\ \lambda''_i \end{bmatrix}^*,$$

then necessarily the two sums of horizontal differences:

$$\lambda_1 - \lambda'_1 + \lambda_2 - \lambda'_2 + \cdots + \lambda_i - \lambda'_i = 1$$
$$\lambda''_1 - \lambda_1 + \lambda''_2 - \lambda_2 + \cdots + \lambda''_i - \lambda_i = 1$$

are both smallest possible, equal to 1. It follows easily from this observation and from the constraint of semi-standarcy that in the penultimate block, in the antepenultimate block and in the general i-th block, the numbers of pairwise distinct $*$-ed columns are at most equal to, respectively:

$$D_2 := 1 + (\nu_1^2 - \mu_1^2) + (\nu_2^2 - \mu_2^2),$$
$$D_3 := 1 + (\nu_1^3 - \mu_1^3) + (\nu_2^3 - \mu_2^3) + (\nu_3^3 - \mu_3^3),$$

and to:

$$D_i := 1 + \sum_{l=1}^{i} (\nu_l^i - \mu_l^i). \tag{17}$$

Consequently, the total number of pairwise distinct columns in an arbitrary semi-standard Young tableau is at most equal to $D_1 + D_2 + D_3 + \cdots + D_{d_1-1} + D_{d_1}$, that is to say to:

$$\begin{array}{ccccccccccccc}
 & 1 & + & 1 & + & \cdots & + & 1 & + & 1 & + & 1 & + \\
+ & \nu_1^{d_1} - \mu_1^{d_1} & + & \nu_1^{d_1-1} - \mu_1^{d_1-1} & + & \cdots & + & \nu_1^3 - \mu_1^3 & + & \nu_1^2 - \mu_1^2 & + & \nu_1^1 - \mu_1^1 & + \\
+ & \nu_2^{d_1} - \mu_2^{d_1} & + & \nu_2^{d_1-1} - \mu_2^{d_1-1} & + & \cdots & + & \nu_2^3 - \mu_2^3 & + & \nu_2^2 - \mu_2^2 & + & & \\
+ & \nu_3^{d_1} - \mu_3^{d_1} & + & \nu_3^{d_1-1} - \mu_3^{d_1-1} & + & \cdots & + & \nu_3^3 - \mu_3^3 & + & & & & \\
+ & \cdots\cdots\cdots & + & \cdots\cdots\cdots & + & \cdots & + & & & & & & \\
+ & \nu_{d_1-1}^{d_1} - \mu_{d_1-1}^{d_1} & + & \nu_{d_1-1}^{d_1-1} - \mu_{d_1-1}^{d_1-1} & + & & & & & & & & \\
+ & \nu_{d_1}^{d_1} - \mu_{d_1}^{d_1}. & & & & & & & & & & &
\end{array}$$

But by permuting the order of appearance of ν and μ in each subtraction of every line, this sum becomes:

$$\begin{array}{ccccccccccc}
 & & 1 & + & \cdots & + & 1 & + & 1 & + & 1 & - \\
-\mu_1^{d_1} & + & \underline{\nu_1^{d_1} - \mu_1^{d_1-1}}_{<} & + & \cdots & + & \underline{\nu_1^3 - \mu_1^2}_{<} & + & \underline{\nu_1^2 - \mu_1^1}_{<} & + & \nu_1^1 & \\
-\mu_2^{d_1} & + & \underline{\nu_2^{d_1} - \mu_2^{d_1-1}}_{<} & + & \cdots & + & \underline{\nu_2^3 - \mu_2^2}_{<} & + & \nu_2^2 & & & \\
\cdots & + & \cdots\cdots\cdots & + & \cdots & & & & & & & \\
-\mu_{d_1-1}^{d_1} & + & \underline{\nu_{d_1-1}^{d_1} - \mu_{d_1-1}^{d_1-1}}_{<} & + & & & & & & & & \\
-\mu_{d_1}^{d_1} & + & \nu_{d_1}^{d_1}, & & & & & & & & &
\end{array}$$

and then by taking account just of the semi-standard-like inequalities (15) on p. 44 (about the columns of contact between two neighboring blocks), we see that all the pairs that we have underlined above are $\leqslant 0$, whence:

$$D_1 + D_2 + D_3 + \cdots + D_{d_1-1} + D_{d_1} \leqslant d_1 \cdot 1 + (-\mu_1^{d_1} + \nu_1^1) + (-\mu_2^{d_1} + \nu_2^2) + \cdots + \\ + (-\mu_{d_1-1}^{d_1} + \nu_{d_1-1}^{d_1-1}) + (-\mu_{d_1}^{d_1} + \nu_{d_1}^{d_1}).$$

Finally, the strict inequalities $0 < \mu_1^{d_1} < \mu_2^{d_1} < \cdots < \mu_{d_1-1}^{d_1} < \mu_{d_1}^{d_1}$ between the entries of the first column yield trivial majorants:

$$-\mu_1^{d_1} \leqslant -1, \quad -\mu_2^{d_1} \leqslant -2, \ldots\ldots, -\mu_{d_1-1}^{d-1} \leqslant -(d_1-1), \quad -\mu_{d_1} \leqslant -d_1,$$

and since all the ν_i^j are $\leqslant \kappa$ anyway, we deduce about any semi-standard Young tableau the following majoration:

$$\begin{aligned} \text{total number of distinct } *\text{-ed columns} &\leqslant d_1 + (-1+\kappa) + (-2+\kappa) + \cdots + \\ &\quad + (-(d_1-1)+\kappa) + (-d_1+\kappa) \\ &\leqslant n + (-1+\kappa) + (-2+\kappa) + \cdots + \\ &\quad + (-(n-1)+\kappa) + (-n+\kappa) \\ &= n\kappa - \tfrac{n(n-1)}{2}. \end{aligned}$$

Lemma. *The total number of pairwise distinct columns in a semi-standard Young tableau of depth $\leqslant n$ filled in with integers $\lambda_i^j \leqslant \kappa$ is in any case $\leqslant n\kappa - \frac{n(n-1)}{2}$.* □

We now introduce several families of Δ-monomials parametrized by a fixed collection of pairs of columns (having various multiplicities $* \geqslant 1$) that should occupy the left and right extreme positions in blocks of decreasing depths.

Main definition. Let d_1 be a positive integer $\leqslant n$ and let μ_l^i and ν_l^i be integers defined for $i = 1, 2, 3, \ldots, d_1 - 1, d_1$ and $1 \leqslant l \leqslant i$ with $\mu_l^i \leqslant \kappa$ and $\nu_l^i \leqslant \kappa$ which satisfy all the following inequalities:

- vertical downward increasing:

$$0 < \mu_1^i < \cdots < \mu_i^i \quad \text{and} \quad 0 < \nu_1^i < \cdots < \nu_i^i \qquad (i=1,\ldots,d_1);$$

- weak increasing inside blocks:

$$\mu_l^i \leqslant \nu_l^i \qquad (i=1,\ldots,d_1;\ 1 \leqslant l \leqslant i);$$

- weak increasing for the contacts between neighboring blocks[14]:

$$\nu_l^{i+1} \leqslant \mu_l^i \qquad\qquad (i=1,\dots,d_1-1;\ 1\leqslant l\leqslant i)$$

Then with such data, the family of semi-standard tableaux:

$$\boxed{\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i\big)}$$

is defined to consist of all possible concatenations:

$$\mathsf{block}^{d_1}\big(\mu^{d_1},\nu^{d_1}\big)\cdots\mathsf{block}^{i}\big(\mu^{i},\nu^{i}\big)\cdots\mathsf{block}^{1}\big(\mu^{1},\nu^{1}\big)$$

of *semi-standard* blocks[15] of the form:

$$\mathsf{block}^i\big(\mu^i,\nu^i\big)=\boxed{\begin{bmatrix}\mu_1^i\\ \mu_2^i\\ \cdot\\ \cdot\\ \mu_i^i\end{bmatrix}^{a_{\mu_1^i,\dots,\mu_i^i}}\quad\cdots\quad\begin{bmatrix}\lambda_1^i\\ \lambda_2^i\\ \cdot\\ \cdot\\ \lambda_i^i\end{bmatrix}^{a_{\lambda_1^i,\dots,\lambda_i^i}}\quad\cdots\quad\begin{bmatrix}\nu_1^i\\ \nu_2^i\\ \cdot\\ \cdot\\ \nu_i^i\end{bmatrix}^{a_{\nu_1^i,\dots,\nu_i^i}}},$$

all $*$-ed columns being pairwise distinct and ordered increasingly from left to right, with multiplicities:

$$a_{\mu_1^i,\dots,\mu_i^i}\geqslant 1,\qquad\dots\qquad a_{\lambda_1^i,\dots,\lambda_i^i}\geqslant 1,\qquad\dots\qquad a_{\nu_1^i,\dots,\nu_i^i}\geqslant 1$$

which are *all* positive and with the further important constraint that:

$$\begin{aligned}m=\mathrm{weight}\big(\mathsf{block}^{d_1}(\mu^{d_1},\nu^{d_1})\big)+\cdots+\\ +\mathrm{weight}\big(\mathsf{block}^{i}(\mu^{i},\nu^{i})\big)+\cdots+\mathrm{weight}\big(\mathsf{block}^{1}(\mu^{1},\nu^{1})\big),\end{aligned}$$

where according to an expectable, natural definition:

$$\begin{aligned}\mathrm{weight}\big(\mathsf{block}^{i}(\mu^{i},\nu^{i})\big)\overset{\text{def}}{=}\big(\mu_1^i+\cdots+\mu_i^i\big)a_{\mu_1^i,\dots,\mu_i^i}+\cdots+\\ +\big(\lambda_1^i+\cdots+\lambda_i^i\big)a_{\lambda_1^i,\dots,\lambda_i^i}+\cdots+\big(\nu_1^i+\cdots+\nu_i^i\big)a_{\nu_1^i,\dots,\nu_i^i}\end{aligned}$$

simply denotes the total number of primes (remember the theorem on p. 24) in the associated Δ-monomial:

$$\big(\Delta_{1,\dots,i}^{\mu_1^i,\dots,\mu_i^i}\big)^{a_{\mu_1^i,\dots,\mu_i^i}}\cdots\big(\Delta_{1,\dots,i}^{\lambda_1^i,\dots,\lambda_i^i}\big)^{a_{\lambda_1^i,\dots,\lambda_i^i}}\cdots\big(\Delta_{1,\dots,i}^{\nu_1^i,\dots,\nu_i^i}\big)^{a_{\nu_1^i,\dots,\nu_i^i}}.$$

In a specific family $\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i\big)$, the freedom of variation lies: 1) in the choice of some intermediate columns; 2) in the choice of the number of such intermediate columns; 3) in the choice of the positive multiplicities of all the columns.

---

[14] The most general case where certain block lengths i are missing, so that block lengths sometimes jump for more than one unit, is implicitly also embraced by such a definition, for it suffices, about the indices i of blocks that are thought to be missing, to just prescribe somewhat arbitrarily some integers μ_l^i and ν_l^i that violate the second condition ; the third condition is then supposed to hold for the direct contacts between the really extant neighboring blocks. Since in our later principal considerations, we will not be dealing with semi-standard tableaux having block gaps, it is not necessary to introduce further specific notations here.

[15] By a *semi-standard block* is of course meant a block in which strict increase holds downward along columns, while weak increase holds from left to right along rows.

Lemma. *The collection of all semi-standard Young tableaux* YT *of depth* $\leqslant n$ *filled in with positive integers* $\lambda_i^j \leqslant \kappa$ *whose weight equals* m *is identical to the disjoint union:*

$$\bigcup_{\mu_l^i, \nu_l^i} \mathsf{YT}_{\kappa,m}\big(\mu_l^i, \nu_l^i\big)$$

of the so-defined families of semi-standard tableaux.

Proof. According to the preceding considerations, an arbitrary semi-standard Young tableau looks like (16) on p. 46, hence belongs to $\mathsf{YT}_{\kappa,m}\big(\mu_l^i, \nu_l^i\big)$ for some μ_l^i, ν_l^i. Disjointness follows from the fact that the extreme column data $\big(\mu_l^i, \nu_l^i\big)$ are obviously pairwise distinct. $\square$

By what has already been seen, the number of pairwise distinct columns in any $\mathsf{block}^i(\mu^i, \nu^i)$ may well be equal to zero[16] and is always smaller than or equal to:

$$D_i := 1 + \textstyle\sum_{l=1}^i (\nu_l^i - \mu_l^i).$$

In order to fix ideas about the exact number of distinct columns, we shall in addition split each (big) family $\mathsf{YT}_{\kappa,m}(\mu_l^i, \nu_l^i)$ in distinct, finer (sub)families as follows.

For every $i = 1, \ldots, n$ and for every nonnegative integer:

$$\tau_i \leqslant D_i - 1 = \textstyle\sum_{l=1}^i (\nu_l^i - \mu_l^i)$$

less than the maximal possible number of distinct columns inside $\mathsf{block}^i(\mu^i, \nu^i)$ *minus* 1, let us choose a discrete increasing path[17]:

$$\gamma^i\colon\ \big\{0, 1, 2, \ldots, \tau^i\big\} \longrightarrow \text{downward increasing columns} \in \{1, \ldots, \kappa\}^i$$

from the μ^i-column to the ν^i-column, namely:

$$\begin{bmatrix} \mu_1^i = \gamma_1^i(0) \\ \mu_2^i = \gamma_2^i(0) \\ .. \\ \mu_i^i = \gamma_i^i(0) \end{bmatrix}^* < \begin{bmatrix} \gamma_1^i(1) \\ \gamma_2^i(1) \\ . \\ \gamma_i^i(1) \end{bmatrix}^* < \cdots < \begin{bmatrix} \gamma_1^i(s^i) \\ \gamma_2^i(s^i) \\ . \\ \gamma_i^i(s^i) \end{bmatrix}^* < \cdots < \begin{bmatrix} \gamma_1^i(\tau^i) = \nu_1^i \\ \gamma_2^i(\tau^i) = \nu_2^i \\ .. \\ \gamma_i^i(\tau^i) = \nu_i^i \end{bmatrix}^*,$$

with $s^i = 0, 1, 2, \ldots, \tau^i$ denoting the current (discrete) time variable, such that the associated block:

$$\mathsf{block}^i(\gamma^i) := \boxed{\begin{bmatrix} \gamma_1^i(0) \\ \gamma_2^i(0) \\ . \\ \gamma_i^i(0) \end{bmatrix}^* \begin{bmatrix} \gamma_1^i(1) \\ \gamma_2^i(1) \\ . \\ \gamma_i^i(1) \end{bmatrix}^* \cdots \begin{bmatrix} \gamma_1^i(s^i) \\ \gamma_2^i(s^i) \\ . \\ \gamma_i^i(s^i) \end{bmatrix}^* \cdots \begin{bmatrix} \gamma_1^i(\tau^i) \\ \gamma_2^i(\tau^i) \\ . \\ \gamma_i^i(\tau^i) \end{bmatrix}^*}$$

[16] This would correspond to the empty block case, *cf.* a preceding footnote.

[17] When a block of depth i is inextant, we set $\tau^i := -1$ so that the length $1 + \tau^i$ of any associated path γ^i equals 0: possible paths γ^i are thus necessarily empty in this case.

is semi-standard. Then with such data, the (sub)family of semi-standard tableaux:

$$\boxed{\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)}$$

is defined to consist of all possible concatenations:

$$\mathsf{block}^n\big(\gamma^n\big)\cdots\mathsf{block}^i\big(\gamma^i\big)\cdots\mathsf{block}^1\big(\gamma^1\big)$$

of the above specific blocks, with $*$-multiplicities:

$$a_{\gamma_1^i(0),\ldots,\gamma_i^i(0)}\geqslant 1,\quad\ldots\quad a_{\gamma_1^i(s^i),\ldots,\gamma_i^i(s^i)}\geqslant 1,\quad\ldots\quad a_{\gamma_1^i(\tau^i),\ldots,\gamma_i^i(\tau^i)}\geqslant 1$$

which are *all* positive, and with the further constraint, similar as before, that:

$$m=\mathrm{weight}\big(\mathsf{block}^n(\gamma^n)\big)+\cdots+\mathrm{weight}\big(\mathsf{block}^i(\gamma^i)\big)+\cdots+\mathrm{weight}\big(\mathsf{block}^1(\gamma^1)\big).$$

Here of course, the weight of a general single column, namely having with multiplicity 1, equals:

$$\gamma_1^i(s^i)+\gamma_2^i(s^i)+\cdots+\gamma_i^i(s^i),$$

hence the $*$-ed column has weight:

$$\big[\gamma_1^i(s^i)+\gamma_2^i(s^i)+\cdots+\gamma_i^i(s^i)\big]\,a_{\gamma_1^i(s^i),\ldots,\gamma_i^i(s^i)}.$$

From now on, we shall denote the multiplicity of a general $*$-ed column shortly by $a_{s^i}^i$, instead of $a_{\gamma_1^i(s^i),\ldots,\gamma_i^i(s^i)}$. The weight homogeneity condition therefore reads:

$$m=\sum_{i=1}^n\sum_{0\leqslant s^i\leqslant\tau^i}\big[\gamma_1^i(s^i)+\cdots+\gamma_i^i(s^i)\big]a_{s^i}^i. \tag{18}$$

In a specific family $\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)$, the freedom of variation now lies only in the multiplicities, since all the pairwise distinct $*$-ed columns are fully prescribed in it. Notice that as m is supposed to be quite large[18] in comparison to n and κ, then for any choice of pairwise distinct column data $\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)$, the column weights $\big[\gamma_1^i(s^i)+\cdots+\gamma_i^i(s^i)\big]$ being fixed and finite, there is still much freedom for the multiplicities to fulfill the homogeneity condition in question. We will in fact realize in a while that the number of semi-standard Young tableaux of weight m in any family $\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)$ is an $\mathsf{O}_{n,\kappa}\big(m^{D-1}\big)$, where $D=\sum_{i=1}^n(1+\tau^i)$ as before is the total number of pairwise distinct columns.

By construction, it is clear that the union of the (sub)families $\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)$ fills the previously introduced larger family:

$$\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i\big)=\bigcup_{\tau^i,\gamma^i(s^i)}\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big).$$

[18] We will eventually let $m\to\infty$, similarly as in the Euler-Poincaré characteristic of $\mathscr{E}_{\kappa,m}^{GG}T_X^*$.

Lemma. *The collection of all semi-standard Young tableaux* YT *of depth* $\leqslant n$ *filled in with positive integers* $\lambda_i^j \leqslant \kappa$ *whose weight equals* m *is identical to the* disjoint *union:*

$$\bigcup_{\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)} \mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)$$

of the so-defined families of semi-standard tableaux. Furthermore, the number of possible such families $\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)$ *is smaller than or equal to the (nonoptimal) constant:*

$$\prod_{i=1}^{n} \big(1+\tfrac{\kappa!}{(\kappa-i)!\,i!}\big)^{1+i(\kappa-i)} = \mathsf{Constant}_{n,\kappa},$$

independently of m.

Proof. Disjointness (not yet argued) of subfamilies inside a family $\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i\big)$ comes from the fact that any collection of paths $(\gamma^1,\gamma^2,\dots,\gamma^n)$ prescribes all the mutually distinct $*$-ed columns that are extant, their multiplicities being all $\geqslant 1$.

In a block of depth i, a single $*$-ed column is either empty, or it consists of i numbers $\lambda_1,\dots,\lambda_i$ chosen without repetition in $\{1,2,\dots,\kappa\}$ and ordered afterward increasingly. So the number of possible such columns (including the empty one) is equal to $1+\frac{\kappa!}{(\kappa-i)!\,i!}$. Since all $*$-ed columns are pairwise distinct, the maximal number of $*$-ed columns that one may put in a semi-standard block of depth i will be attained for the blocks having the following two extreme columns, which are the farthest ones for the ordering between columns of depth i:

$$\boxed{\begin{bmatrix}1\\2\\\cdot\\i-1\\i\end{bmatrix}^* \cdots \begin{bmatrix}\kappa-i+1\\\kappa-i+2\\\cdot\\\kappa-1\\\kappa\end{bmatrix}^*}.$$

It follows from (17) on p. 47 that one may put at most:

$$1+(\kappa-i+1-1)+(\kappa-i+2-2)+\cdots+\kappa-i = 1+i(\kappa-i) \tag{19}$$

pairwise distinct $*$-ed columns in between so as to constitute a semi-standard block. Without optimality, we then majorate the number of possible semi-standard blocks of depth i (including the empty one) simply by the number $1+\frac{\kappa!}{(\kappa-i)!\,i!}$ of possible $*$-ed columns raised to a power equal to the maximal number $1+i(\kappa-i)$ of pairwise distinct such $*$-ed columns. What matters for the sequel is only that the obtained majorant is independent of m. $\square$

In summary, here is how we constitute our refined view of an arbitrary semi-standard Young tableau: the data $(\mu_l^i, \nu_l^i)_{1\leqslant l\leqslant i}$, subjected to the natural inequalities of the Main definition on p. 48, prescribe the left and right extreme $*$-ed column in all blocks of depth $i = 1, 2, \dots, n$ (with possible block gaps); $\tau^i + 1$ is the number of pairwise distinct $*$-ed columns in the i-th block, and these columns are $\gamma^i(0), \dots, \gamma^i(\tau^i)$; all $*$-multiplicities of these columns are $\geqslant 1$.

Asymptotically negligible families of Δ-monomials. By definition, for each semi-standard Young tableau YT belonging to a fixed family:

$$\mathsf{YT}_{\kappa,m}\big(\mu_l^i, \nu_l^i, \tau^i, \gamma^i(s^i)\big),$$

the number D of pairwise distinct $*$-ed columns is equal to the sum of the lengths of the paths between two extreme $*$-ed columns in every block[19]:

$$D = (1+\tau^1) + \cdots + (1+\tau^i) + \cdots + (1+\tau^n).$$

However, the common horizontal length of each of the i rows in the block:

$$\mathsf{block}^i(\gamma^i) = \boxed{\begin{bmatrix}\gamma_1^i(0)\\ \gamma_2^i(0)\\ \cdot\\ \gamma_i^i(0)\end{bmatrix}^{a_0^i} \begin{bmatrix}\gamma_1^i(1)\\ \gamma_2^i(1)\\ \cdot\\ \gamma_i^i(1)\end{bmatrix}^{a_1^i} \cdots \begin{bmatrix}\gamma_1^i(s^i)\\ \gamma_2^i(s^i)\\ \cdot\\ \gamma_i^i(s^i)\end{bmatrix}^{a_{s^i}^i} \cdots \begin{bmatrix}\gamma_1^i(\tau^i)\\ \gamma_2^i(\tau^i)\\ \cdot\\ \gamma_i^i(\tau^i)\end{bmatrix}^{a_{\tau^i}^i}}$$

depends visibly on the multiplicities, and is equal to:

$$a_0^i + a_1^i + \cdots + a_{s^i}^i + \cdots + a_{\tau^i}^i = \sum_{0\leqslant s^i\leqslant \tau^i} a_{s^i}^i.$$

It follows that the lengths $\ell_1, \ell_2, \dots, \ell_{n-1}, \ell_n$ of the first, second, $\dots$, $(n-1)$-th and n-th horizontal lines in the semi-standard Young tableau YT are equal to[20]:

$$\begin{aligned}
\ell_1 &= \sum_{0\leqslant s^n\leqslant\tau^n} a_{s^n}^n + \sum_{0\leqslant s^{n-1}\leqslant\tau^{n-1}} a_{s^{n-1}}^{n-1} + \cdots + \sum_{0\leqslant s^2\leqslant\tau^2} a_{s^2}^2 + \sum_{0\leqslant s^1\leqslant\tau^1} a_{s^1}^1\\
\ell_2 &= \sum_{0\leqslant s^n\leqslant\tau^n} a_{s^n}^n + \sum_{0\leqslant s^{n-1}\leqslant\tau^{n-1}} a_{s^{n-1}}^{n-1} + \cdots + \sum_{0\leqslant s^2\leqslant\tau^2} a_{s^2}^2\\
\cdots &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\ell_{n-1} &= \sum_{0\leqslant s^n\leqslant\tau^n} a_{s^n}^n + \sum_{0\leqslant s^{n-1}\leqslant\tau^{n-1}} a_{s^{n-1}}^{n-1}\\
\ell_n &= \sum_{0\leqslant s^n\leqslant\tau^n} a_{s^n}^n.
\end{aligned}$$

[19] By the preceding convention, inextant blocks contribute with 0 to this sum, *e.g.* all blocks of depths $d_1+1, \dots, n$ when the depth d_1 of the tableau is $< n$.

[20] Sums $\sum_{0\leqslant s^i\leqslant\tau^i} a_{s^i}^i$ for which $\tau^i = -1$ (inextant blocks) are naturally thought to be inextant.

As we already said in Section 6 above, it appears *a posteriori* more adequate to write everything in terms of the differences:

$$\ell_1 - \ell_2 = \sum_{0 \leqslant s^1 \leqslant \tau^1} a^1_{s^1}, \ \ldots, \ \ell_{n-1} - \ell_n = \sum_{0 \leqslant s^{n-1} \leqslant \tau^{n-1}} a^{n-1}_{s^{n-1}}, \qquad \ell_n = \sum_{0 \leqslant s^n \leqslant \tau^n} a^n_{s^n}$$

of lengths of lines, which are nothing but row lengths of blocks.

Proposition. *Fix μ^i_l, ν^i_l, τ^i and $\gamma^i(s^i)$. If $\alpha'_1, \ldots, \alpha'_{n-1}, \alpha'_n$ are arbitrary nonnegative integers satisfying:*

$$\alpha'_1 + \cdots + \alpha'_{n-1} + \alpha'_n \leqslant \tfrac{n(n+1)}{2},$$

then there exists a positive quantity $\mathsf{Constant}_{n,\kappa} > 0$ depending on n and on κ which is independent of m such that:

$$\sum_{\mathsf{YT} \in \mathsf{YT}_{\kappa,m}\left(\mu^i_l, \nu^i_l, \tau^i, \gamma^i(s^i)\right)} \big(\ell_1(\mathsf{YT}) - \ell_2(\mathsf{YT})\big)^{\alpha'_1} \cdots \big(\ell_{n-1}(\mathsf{YT}) - \ell_n(\mathsf{YT})\big)^{\alpha'_{n-1}} \big(\ell_n(\mathsf{YT})\big)^{\alpha'_n} \leqslant$$
$$\leqslant \mathsf{Constant}_{n,\kappa} \cdot m^{\alpha'_1 + \cdots + \alpha'_{n-1} + \alpha'_n} \cdot m^{D-1},$$

where $D = \sum_{i=1}^n (1 + \tau^i)$ is the common number of pairwise distinct $*$-ed columns shared by all Young tableaux $\mathsf{YT} \in \mathsf{YT}_{\kappa,m}\big(\mu^i_l, \nu^i_l, \tau^i, \gamma^i(s^i)\big)$.

Proof. Substituting the values $\ell_i - \ell_{i+1} = \sum_{0 \leqslant s^i \leqslant \tau^i} a^i_{s^i}$ in the monomial:

$$\big(\ell_1 - \ell_2\big)^{\alpha'_1} \cdots \big(\ell_{n-1} - \ell_n\big)^{\alpha'_{n-1}} \big(\ell_n\big)^{\alpha'_n},$$

and expanding the result, we may majorate:

$$\big(\ell_1 - \ell_2\big)^{\alpha'_1} \cdots \big(\ell_{n-1} - \ell_n\big)^{\alpha'_{n-1}} \big(\ell_n\big)^{\alpha'_n} \leqslant$$
$$\leqslant \mathsf{Constant}_{\tau^1, \ldots, \tau^n} \cdot \sum_{\sum \beta^1_{s^1} + \cdots + \sum \beta^n_{s^n} = \alpha'_1 + \cdots + \alpha'_{n-1} + \alpha'_n} \left(\prod_{i=1}^n \prod_{0 \leqslant s^i \leqslant \tau^i} \big(a^i_{s^i}\big)^{\beta^i_{s^i}} \right).$$

Since according to (19) on p. 52 above, the $\tau^i \leqslant i(\kappa - i) \leqslant n\kappa$ are majorated in terms of n and κ, we have:

$$\mathsf{Constant}_{\tau^1, \ldots, \tau^n} \leqslant \mathsf{Constant}_{n,\kappa}.$$

Moreover, since $\alpha'_1 + \cdots + \alpha'_{n-1} + \alpha'_n \leqslant \frac{n(n+1)}{2}$, the number of terms in the sum:

$$\sum_{\sum \beta^1_{s^1} + \cdots + \sum \beta^n_{s^n} = \alpha'_1 + \cdots + \alpha'_{n-1} + \alpha'_n} (\bullet)$$

is also $\leqslant \mathsf{Constant}_{n,\kappa}$. Consequently, in order to prove the proposition, it suffices to majorate by the same claimed majorant:

$$\mathsf{Constant}_{n,\kappa} \cdot m^{\alpha'_1 + \cdots + \alpha'_{n-1} + \alpha'_n} \cdot m^{D-1}$$

every single sum of the form:

$$\sum_{\mathsf{YT}\in\mathsf{YT}_{\kappa,m}\left(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\right)} \left(\prod_{0\leqslant s^1\leqslant\tau^1} \left(a_{s^1}^1\right)^{\beta_{s^1}^1} \cdots \prod_{0\leqslant s^n\leqslant\tau^n} \left(a_{s^n}^n\right)^{\beta_{s^n}^n} \right),$$

where the exponents $\beta_{s^i}^i$, $i = 1, \ldots, n$, $0 \leqslant s^i \leqslant \tau^i$, are now fixed and subjected to the same property that their sum equals:

$$\sum_{i=1}^n \sum_{0\leqslant s^i\leqslant\tau^i} \beta_{s^i}^i = \alpha_1' + \cdots + \alpha_{n-1}' + \alpha_n'.$$

Recall that Young tableaux in the family $\mathsf{YT}_{\kappa,m}\left(\mu_l^i, \nu_l^i, \tau^i, \gamma^i(s^i)\right)$ have fixed set of pairwise distinct columns, and that the freedom only lies in the multiplicities $a_{s^i}^i \geqslant 1$, $i = 1, \ldots, n$, $0 \leqslant s^i \leqslant \tau^i$, of the columns. The considered sum:

$$\sum_{\mathsf{YT}\in\mathsf{YT}_{\kappa,m}\left(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\right)} (\bullet)$$

coincides therefore with the sum:

$$\sum_{\sum_{i=1}^n \sum_{0\leqslant s^i\leqslant\tau^i} \left[\gamma_1^i(s^i)+\cdots+\gamma_i^i(s^i)\right] a_{s^i}^i = m} (\bullet),$$

which takes precisely account of the homogeneity constraint (18) on p. 51. Let us now set:

$$b_{s^i}^i := \left[\gamma_1^i(s^i) + \cdots + \gamma_i^i(s^i)\right] \, a_{s^i}^i,$$

whence $a_{s^i}^i \leqslant b_{s^i}^i$ always[21], so that the sum in question now writes:

$$\sum_{\sum b_{s^1}^1+\cdots+\sum b_{s^n}^n=m} \left(\prod_{i=1}^n \prod_{0\leqslant s^i\leqslant\tau^i} \left(a_{s^i}^i\right)^{\beta_{s^i}^i} \right)$$
$$\leqslant \sum_{\sum b_{s^1}^1+\cdots+\sum b_{s^n}^n=m} \left(\prod_{i=1}^n \prod_{0\leqslant s^i\leqslant\tau^i} \left(b_{s^i}^i\right)^{\beta_{s^i}^i} \right).$$

The number of nonzero variables $b_{s^i}^i \in \mathbb{N}$ here is the same, equal to D, as the number of nonzero exponents $a_{s^i}^i$. The conclusion now follows from the elementary general inequality:

$$\sum_{\substack{b_1+\cdots+b_D=m \\ b_1\geqslant 1,\ldots,b_D\geqslant 1}} b_1^{\beta_1} \cdots b_D^{\beta_D} \leqslant \mathsf{Constant}_D \cdot m^{\beta_1+\cdots+\beta_D} \cdot m^{D-1},$$

that can be established by approximating the sum by a Riemann integral; of course, $\mathsf{Constant}_D \leqslant \mathsf{Constant}_{n,\kappa}$. $\square$

[21] Even in the case where the block of depth i is inextant.

From this proposition, we will deduce a few corollaries. Firstly, as announced earlier on at the end of Section 6, we have:

Corollary. *Let $\alpha'_1, \ldots, \alpha'_{n-1}, \alpha'_n$ be nonnegative integers satisfying:*

$$\alpha'_1 + \cdots + \alpha'_{n-1} + \alpha'_n \leqslant \tfrac{n(n+1)}{2} - 1.$$

Then the following majoration holds for the summation over all *semi-standard Young tableaux of weight m:*

$$\sum_{\substack{\mathsf{YT\ semi-standard} \\ \mathsf{weight}(\mathsf{YT})=m}} \big(\ell_1(\mathsf{YT}) - \ell_2(\mathsf{YT})\big)^{\alpha'_1} \cdots \big(\ell_{n-1}(\mathsf{YT}) - \ell_n(\mathsf{YT})\big)^{\alpha'_{n-1}} \big(\ell_n(\mathsf{YT})\big)^{\alpha'_n} \leqslant$$

$$\leqslant \mathsf{Constant}_{n,\kappa} \cdot m^{\alpha'_1 + \cdots + \alpha'_{n-1} + \alpha'_n} \cdot m^{n\kappa - \frac{n(n-1)}{2}}$$

$$\leqslant \mathsf{Constant}_{n,\kappa} \cdot m^{(\kappa+1)n-2}.$$

Proof. According to the lemma on p. 52:

$$\sum_{\substack{\mathsf{YT\ semi-standard} \\ \mathsf{weight}(\mathsf{YT})=m}} (\bullet) = \sum_{\mu^i_l,\, \nu^i_l} \sum_{\tau_i} \sum_{\gamma^i(s^i)} \sum_{\mathsf{YT} \in \mathsf{YT}(\mu^i_l, \nu^i_l, \tau^i, \gamma^i(s^i))} (\bullet),$$

and furthermore, the number of terms in the three first sums of the right-hand side is $\leqslant \mathsf{Constant}_{n,\kappa}$. It suffices then to apply the proposition which controls each fourth sum, reminding from the lemma on p. 48 that each $D = \sum_{i=1}^n (1 + \tau^i)$ is in any case $\leqslant n\kappa - \frac{n(n-1)}{2}$. $\square$

Secondly, from the simple inequality (13) on p. 40, we immediately deduce:

Corollary. *If $\alpha_1, \ldots, \alpha_n$ are any nonnegative integers satisfying $\alpha_1 + \cdots + \alpha_n \leqslant \frac{n(n+1)}{2} - 1$, then:*

$$\sum_{\substack{\mathsf{YT\ semi-standard} \\ \mathsf{weight}(\mathsf{YT})=m}} \big(\ell_1(\mathsf{YT})\big)^{\alpha_1} \cdots \big(\ell_n(\mathsf{YT})\big)^{\alpha_n} \leqslant \mathsf{Constant}_{n,\kappa} \cdot m^{\alpha_1 + \cdots + \alpha_n} \cdot m^{n\kappa - \frac{n(n-1)}{2}}$$

$$\leqslant \mathsf{Constant}_{n,\kappa} \cdot m^{(\kappa+1)n-2}.$$

Lastly, we now introduce a certain collection of semi-standard Young tableaux the contribution of which appears to also fall in the remainder $\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big)$: we gather all the ones for which the number of pairwise distinct columns is (strictly) less than the maximal possible number $n\kappa - \frac{n(n-1)}{2}$:

$$\boxed{\mathsf{NGYT}_{\kappa,m} := \bigcup_{\substack{\mu^i_l, \nu^i_l, \tau^i, \gamma^i(s^i) \\ \sum_{i=1}^n (1+\tau^i) \leqslant n\kappa - \frac{n(n-1)}{2} - 1}} \mathsf{YT}_{\kappa,m}\big(\mu^i_l, \nu^i_l, \tau^i, \gamma^i(s^i)\big)}.$$

The number of appearing such families $\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)$ is of course less than the majorant:

$$\prod_{i=1}^{n}\big(1+\tfrac{\kappa!}{(\kappa-i)!\,i!}\big)^{1+i(\kappa-i)}\equiv\mathsf{Constant}_{n,\kappa}$$

provided by the lemma on p. 52 for the total number of all families.

Lemma. *For any integers $\alpha'_1,\dots,\alpha'_{n-1},\alpha'_n$ whose sum equals $\frac{n(n+1)}{2}$, the contribution of:*

$$\sum_{\mathsf{YT}\in\mathsf{NGYT}_{\kappa,m}}\big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha'_1}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha'_{n-1}}\big(\ell_n(\mathsf{YT})\big)^{\alpha'_n}\leqslant$$
$$\leqslant\mathsf{Constant}_{n,\kappa}\cdot m^{(\kappa+1)n-2}$$

is asymptotically negligible in comparison to the dominant power $m^{(\kappa+1)n-1}$.

Proof. By what has been just seen, it suffices to verify that such a majoration holds for a sum $\sum_{\mathsf{YT}\in(\bullet)}$ running over a single family $\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)$ contained in $\mathsf{NGYT}_{\kappa,m}$, and this was already achieved by the proposition on p. 54 above, since $\frac{n(n+1)}{2}+D-1\leqslant(\kappa+1)n-2$ always when $D\leqslant n\kappa-\frac{n(n-1)}{2}-1$. $\square$

§8. Maximal length families of semi-standard Young tableaux

Relevant families of Δ-monomials. In the remainder of the paper, we shall now consider only exponents α'_i satisfying $\alpha'_1+\cdots+\alpha'_{n-1}+\alpha'_n=\frac{n(n+1)}{2}$. From the proposition just proved, we deduce that the complete sum:

$$\sum_{\substack{\mathsf{YT}\ \mathsf{semi-standard}\\ \mathsf{weight}(\mathsf{YT})=m}}\big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha'_1}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha'_{n-1}}\big(\ell_n(\mathsf{YT})\big)^{\alpha'_n}\tag{20}$$

splits up as a negligible sum plus a relevant sum that we should now study:

$$\sum_{\mathsf{YT}\in\mathsf{NGYT}_{\kappa,m}}+\sum_{\mathsf{YT}\notin\mathsf{NGYT}_{\kappa,m}}.$$

Hence, what remains to be studied is the collection of all families of semi-standard Young tableaux:

$$\boxed{\mathsf{YT}^{\max}_{\kappa,m}:=\bigcup_{\substack{\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\\ \sum_{i=1}^n(1+\tau^i)=n\kappa-\frac{n(n-1)}{2}}}\mathsf{YT}_{\kappa,m}\big(\mu_l^i,\nu_l^i,\tau^i,\gamma^i(s^i)\big)}$$

for which the number of pairwise distinct columns is maximal, equal to $n\kappa-\frac{n(n-1)}{2}$. The following statement describes them in great details.

Proposition. *The number D of pairwise distinct columns in any semi-standard Young tableau written as in* (16) *on p. 46 is in any case $\leqslant n\kappa - \frac{n(n-1)}{2}$. Furthermore, a given semi-standard Young tableau reaches the maximal number:*

$$D = n\kappa - \tfrac{n(n-1)}{2}$$

of pairwise distinct columns if and only if all the following conditions are fulfilled:

- *the depth of the tableau is maximal: $d_1 = n$;*
- *nonvoid blocks of any depth $i = 1, 2, 3, \dots, n-1, n$ are all extant, so that the number of nonvoid blocks is maximal, equal to n;*
- *the leftmost $*$-ed column of the tableau corresponds to the n-dimensional Wronskian $\Delta^{1,2,3,\dots,n-1,n}_{1,2,3,\dots,n-1,n}$, reproduced a certain number $* \geqslant 1$ of times;*
- *the bottom-right entry of every block is maximal:*

$$\nu^1_1 = \nu^2_2 = \nu^3_3 = \cdots = \nu^{n-1}_{n-1} = \nu^n_n = \kappa;$$

$$\underbrace{\boxed{\begin{bmatrix} 1 \\ 2 \\ 3 \\ \cdot \\ \cdot \\ n-1 \\ n \end{bmatrix}^* \cdots \begin{bmatrix} \mu^{n-1}_1 \\ \mu^{n-1}_2 \\ \mu^{n-1}_3 \\ \cdot \\ \cdot \\ \mu^{n-1}_{n-1} \\ \kappa \end{bmatrix}^*}}_{1+\tau^n} \underbrace{\boxed{\begin{bmatrix} \mu^{n-1}_1 \\ \mu^{n-1}_2 \\ \mu^{n-1}_3 \\ \cdot \\ \cdot \\ \mu^{n-1}_{n-1} \end{bmatrix}^* \cdots \begin{bmatrix} \mu^{n-2}_1 \\ \mu^{n-2}_2 \\ \mu^{n-2}_3 \\ \cdot \\ \cdot \\ \kappa \end{bmatrix}^*}}_{1+\tau^{n-1}} \cdots \underbrace{\boxed{\begin{bmatrix} \mu^3_1 \\ \mu^3_2 \\ \mu^3_3 \end{bmatrix}^* \cdots \begin{bmatrix} \mu^2_1 \\ \mu^2_2 \\ \kappa \end{bmatrix}^*}}_{1+\tau^3} \underbrace{\boxed{\begin{bmatrix} \mu^2_1 \\ \mu^2_2 \end{bmatrix}^* \cdots \begin{bmatrix} \mu^1_1 \\ \kappa \end{bmatrix}^*}}_{1+\tau^2} \underbrace{\boxed{\begin{bmatrix} \mu^1_1 \end{bmatrix}^* \cdots \begin{bmatrix} \kappa \end{bmatrix}^*}}_{1+\tau^1} .$$

- *the border column entries (excepting the last one, equal to κ, of the longest column) of any pair of neighboring blocks are equal:*

$$\cdots \boxed{\begin{bmatrix} \mu^{i+1}_1 \\ \mu^{i+1}_2 \\ \cdot \\ \cdot \\ \mu^{i+1}_{i-1} \\ \mu^{i+1}_i \\ \mu^{i+1}_{i+1} \end{bmatrix}^* \cdots \begin{bmatrix} \nu^{i+1}_1 = \mu^i_1 \\ \nu^{i+1}_2 = \mu^i_2 \\ \cdot \\ \cdot \\ \nu^{i+1}_{i-1} = \mu^i_{i-1} \\ \nu^{i+1}_i = \mu^i_i \\ \kappa \end{bmatrix}^*} \boxed{\begin{bmatrix} \mu^i_1 \\ \mu^i_2 \\ \cdot \\ \cdot \\ \mu^i_{i-1} \\ \mu^i_i \end{bmatrix}^* \cdots \begin{bmatrix} \nu^i_1 = \mu^{i-1}_1 \\ \nu^i_2 = \mu^{i-1}_2 \\ \cdot \\ \cdot \\ \nu^i_{i-1} = \mu^{i-1}_{i-1} \\ \kappa \end{bmatrix}^*} \boxed{\begin{bmatrix} \mu^{i-1}_1 \\ \mu^{i-1}_2 \\ \cdot \\ \cdot \\ \mu^{i-1}_{i-1} \end{bmatrix}^* \cdots \begin{bmatrix} \nu^{i-1}_1 = \mu^{i-2}_1 \\ \nu^{i-1}_2 = \mu^{i-2}_2 \\ \cdot \\ \cdot \\ \kappa \end{bmatrix}^*} \cdots ,$$

- *the number of pairwise distinct columns in each block of depth i, for $i = 1, 2, 3, \dots, n-1, n$ is maximal*[22]*, equal to:*

$$\begin{aligned} 1 + \tau^i &:= 1 + (\mu^{i-1}_1 - \mu^i_1) + (\mu^{i-1}_2 - \mu^i_2) + \cdots + (\mu^{i-1}_{i-1} - \mu^i_{i-1}) + (\kappa - \mu^i_i) \\ &= 1 + \kappa + \sum_{l=1}^{i-1} \mu^{i-1}_l - \sum_{l=1}^{i} \mu^i_l, \end{aligned}$$

[22] By convention, we shall also call $\mu^n_1, \mu^n_2, \mu^n_3, \dots, \mu^n_{n-1}, \mu^n_n$ the entries $1, 2, 3, \dots, n-1, n$ of the leftmost $*$-ed column.

so that the total number of pairwise distinct columns is accordingly indeed equal to:

$$(1+\tau^1)+(1+\tau^2)+(1+\tau^3)+\cdots+(1+\tau^{n-1})+(1+\tau^n) = n+n\kappa-\sum_{l=1}^{n}\mu_l^n$$
$$= n\kappa-\tfrac{n(n-1)}{2}.$$

Proof. The majorant $n\kappa-\frac{n(n-1)}{2}$ has already been obtained above, before the introduction of families of Young tableaux. The remaining statements follow by thinking once again about what has already been seen above. □

So the families of semi-standard Young tableaux having maximal number $n\kappa-\frac{n(n-1)}{2}$ of pairwise distinct columns is parameterized by all the collections of integers μ_i^j satisfying the inequalities:

$$\begin{array}{l} 1\leqslant \mu_1^1<\kappa \\ 1\leqslant \mu_1^2<\mu_2^2<\kappa \\ 1\leqslant \mu_1^3<\mu_2^3<\mu_3^3<\kappa \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 1\leqslant \mu_1^{n-1}<\mu_2^{n-1}<\mu_3^{n-1}<\cdots<\mu_{n-1}^{n-1}<\kappa \\ 1\leqslant \mu_1^n<\mu_2^n<\mu_3^n<\cdots<\mu_{n-1}^n<\mu_n^n, \end{array} \tag{21}$$

together with the further semi-standard-like inequalities[23]:

$$\mu_l^l\geqslant\mu_l^{l+1}\geqslant\cdots\geqslant\mu_l^{n-1}(\geqslant l). \tag{22}$$

For brevity, let us write as:

$$\boxed{\mu_l^i\in\nabla_{n,\kappa}}$$

the condition that the μ_l^i satisfy the two sets of inequalities (21) and (22). For any such choice of $\mu_l^i\in\nabla_{n,\kappa}$, we shall denote by:

$$\boxed{\mathsf{YT}_{\kappa,m}^{\max}\big(\mu_l^i\big)}$$

the family of semi-standard Young tableaux which consist of all possible concatenations:

$$\mathsf{block}^n\big(\gamma^n\big)\cdots\mathsf{block}^i\big(\gamma^i\big)\cdots\mathsf{block}^1\big(\gamma^1\big)$$

[23] Diagramatically, this second set of inequalities reads as saying that *vertically* in each column of the first array (21) of inequalities, the integers μ_l^i are weakly decreasing. In particular, $\kappa\geqslant\mu_n^n=n$, as was assumed throughout earlier on.

of pathed blocks of the form:

$$\boxed{\mathsf{block}^i(\gamma^i) := \begin{bmatrix} \mu_1^i = \gamma_1^i(0) \\ \mu_2^i = \gamma_2^i(0) \\ .. \\ \mu_{i-1}^i = \gamma_{i-1}^i(0) \\ \mu_i^i = \gamma_i^i(0) \end{bmatrix}^{a_0^i} \begin{bmatrix} \gamma_1^i(1) \\ \gamma_2^i(1) \\ . \\ \gamma_{i-1}^i(1) \\ \gamma_i^i(1) \end{bmatrix}^{a_1^i} \cdots \begin{bmatrix} \gamma_1^i(s^i) \\ \gamma_2^i(s^i) \\ . \\ \gamma_{i-1}^i(s^i) \\ \gamma_i^i(s^i) \end{bmatrix}^{a_{s^i}^i} \cdots \begin{bmatrix} \gamma_1^i(\tau^i) = \mu_1^{i-1} \\ \gamma_2^i(\tau^i) = \mu_2^{i-1} \\ .. \\ \gamma_{i-1}^i(\tau^i) = \mu_{i-1}^{i-1} \\ \gamma_i^i(\tau^i) = \kappa \end{bmatrix}^{a_{\tau^i}^i}},$$

where the lengths $1+\tau^i$ of paths are maximal equal to:

$$1+\tau^i = 1+\kappa+\sum_{l=1}^{i-1}\mu_l^{i-1} - \sum_{l=1}^{i}\mu_l^i,$$

so that between two successive $*$-ed columns:

$$\begin{bmatrix} \gamma_1^i(s^i) \\ \gamma_2^i(s^i) \\ . \\ \gamma_i^i(s^i) \end{bmatrix}^* < \begin{bmatrix} \gamma_1^i(s^i+1) \\ \gamma_2^i(s^i+1) \\ .. \\ \gamma_i^i(s^i+1) \end{bmatrix}^*$$

one has the semi-standard inequalities $\gamma_l^i(s^i) \leqslant \gamma_l^i(s^i+1)$ for $l=1,2,\ldots,i$ but the jump is smallest possible, namely $\gamma_l^i(s^i+1) = \gamma_l^i(s^i)$ for all l except only one l_0 for which:

$$\gamma_{l_0}^i(s^i+1) = 1+\gamma_{l_0}^i(s^i).$$

Such paths all of which jumps are unit will be called *tight paths*. With all these notations, the initial complete sum (20) to be studied now writes:

$$\sum_{\substack{\mathsf{YT\ semi-standard} \\ \mathsf{weight}(\mathsf{YT})=m}} (\bullet) = \sum_{\mathsf{YT}\in\mathsf{NGYT}_{\kappa,m}} (\bullet) + \sum_{\mu_l^i\in\nabla_{n,\kappa}} \sum_{\mathsf{YT}\in\mathsf{YT}_{\kappa,m}^{\max}(\mu_l^i)} (\bullet),$$

the first sum being negligible, in the sense that:

(23)

$$\boxed{\begin{aligned} &\sum_{\substack{\mathsf{YT\ semi-standard} \\ \mathsf{weight}(\mathsf{YT})=m}} \big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha_1'}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha_{n-1}'}\big(\ell_n(\mathsf{YT})\big)^{\alpha_n'} = \\ &= \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big)+ \\ &+ \sum_{\mu_l^i\in\nabla_{n,\kappa}} \sum_{\mathsf{YT}\in\mathsf{YT}_{\kappa,m}^{\max}(\mu_l^i)} \big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha_1'}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha_{n-1}'}\big(\ell_n(\mathsf{YT})\big)^{\alpha_n'} \end{aligned}}.$$

Grouping sums. Let us therefore fix $\mu_l^i \in \nabla_{n,\kappa}$, let us keep aside (and in mind) the first summation $\sum_{\mu_l^i\in\nabla_{n,\kappa}}(\bullet)$, and let us consider only the second summation:

$$\sum_{\mathsf{YT}\in\mathsf{YT}_{\kappa,m}^{\max}(\mu_l^i)} \big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha_1'}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha_{n-1}'}\big(\ell_n(\mathsf{YT})\big)^{\alpha_n'}.$$

With $\tau^i := \kappa + \sum_{l=1}^{i-1} \mu_l^{i-1} - \sum_{l=1}^{i} \mu_l^i$, this apparently compact sum identifies in fact with the multiple sums:

$$\sum_{\gamma^1(s^1)} \cdots \sum_{\gamma^n(s^n)} \sum_{a^1_{s^1}} \cdots \sum_{a^n_{s^n}} (\bullet),$$

the paths $\gamma^i(s^i)$ being all tight and the multiplicities $a^i_{s^i}$ being constrained only by the weight condition:

$$\sum_{i=1}^{n} \sum_{0 \leqslant s^i \leqslant \tau^i} \big[\gamma^i_1(s^i) + \cdots + \gamma^i_i(s^i)\big]\, a^i_{s^i} = m.$$

Since the paths are tight, the sums (equal to the weights of columns):

$$\gamma^i_1(s^i) + \cdots + \gamma^i_i(s^i)$$

jump of a single unit as $s^i = 0, 1, 2, \ldots, \tau^i$, and to be precise, they take the following exact values:

$$\gamma^i_1(s^i) + \cdots + \gamma^i_i(s^i) = \begin{cases} \sum_{l=1}^{i} \mu_l^i & \text{for } s^i = 0 \\ 1 + \sum_{l=1}^{i} \mu_l^i & \text{for } s^i = 1 \\ 2 + \sum_{l=1}^{i} \mu_l^i & \text{for } s^i = 2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \kappa + \sum_{l=1}^{i-1} \mu_l^i & \text{for } s^i = \tau^i, \end{cases}$$

independently of the paths. In other words:

$$\gamma^i_1(s^i) + \cdots + \gamma^i_i(s^i) = s^i + \mu^i_1 + \cdots + \mu^i_i,$$

with of course at the end (as already written):

$$\begin{aligned} \gamma^i_1(\tau^i) + \cdots + \gamma^i_i(\tau^i) &= \kappa + \sum_{l=1}^{i-1} \mu_l^{i-1} - \sum_{l=1}^{i} \mu_l^i + \mu^i_1 + \cdots + \mu^i_i \\ &= \kappa + \mu^{i-1}_1 + \cdots + \mu^{i-1}_{i-1}. \end{aligned}$$

Recalling that[24]:

$$\big(\ell_i(\mathsf{YT}) - \ell_{i+1}(\mathsf{YT})\big)^{\alpha'_i} = \big(a^i_0 + \cdots + a^i_{\tau^i}\big)^{\alpha'_i} \qquad (1 \leqslant i \leqslant n),$$

we may therefore write:

$$\sum_{\mathsf{YT} \in \mathsf{YT}^{\max}_{\kappa,m}(\mu^i_l)} \big(\ell_1(\mathsf{YT}) - \ell_2(\mathsf{YT})\big)^{\alpha'_1} \cdots \big(\ell_{n-1}(\mathsf{YT}) - \ell_n(\mathsf{YT})\big)^{\alpha'_{n-1}} \big(\ell_n(\mathsf{YT})\big)^{\alpha'_n} =$$

$$= \sum_{\gamma^1(s^1)} \cdots \sum_{\gamma^n(s^n)} \sum_{\sum_{i=1}^n \sum_{0 \leqslant s^i \leqslant \tau^i} [s^i + \mu^i_1 + \cdots + \mu^i_i] a^i_{s^i} = m} \prod_{i=1}^{n} \big(a^i_0 + \cdots + a^i_{\tau^i}\big)^{\alpha'_i}.$$

[24] By convention, $\ell_{n+1} = \ell_{n+1}\big(\mathsf{YT}\big) = 0$ always.

Observing that the last written sum is independent of the paths, each one of the first n sums $\sum_{\gamma^i(s^i)}$ then collapses as just multiplication by the number of considered paths $\gamma^i(s^i)$. Thus, let $N^{\kappa}_{\mu^1_1}$ denote the number[25] of tight paths $\gamma^1(s^1)$ from μ^1_1 to κ; let $N^{\mu^1_1,\kappa}_{\mu^2_1,\mu^2_2}$ denote the number of tight paths $\gamma^2(s^2)$ from the column ${}^t(\mu^2_1,\mu^2_2)$, where ${}^t(\bullet)$ denotes transposition, to the column ${}^t(\mu^1_1,\kappa)$; and generally, let:

$$\boxed{N^{\mu^{i-1}_1,\mu^{i-1}_2,\ldots,\mu^{i-1}_{i-1},\kappa}_{\mu^i_1,\mu^i_2,\ldots,\mu^i_{i-1},\mu^i_i}}$$

denote the number of tight paths $\gamma^i(s^i)$ from the column ${}^t\big(\mu^i_1,\mu^i_2,\ldots,\mu^i_{i-1},\mu^i_i\big)$ to the column ${}^t\big(\mu^{i-1}_1,\mu^{i-1}_2,\ldots,\mu^{i-1}_{i-1},\kappa\big)$, with the natural convention that, for $i=n$, one has the notational equivalence:

$${}^t\big(\mu^n_1,\mu^n_2,\ldots,\mu^n_{n-1},\mu^n_n\big)\equiv{}^t\big(1,2,\ldots,n-1,n\big).$$

Then with such notations, we may represent our sum as:

$$\sum_{\gamma^1(s^1)}\cdots\sum_{\gamma^n(s^n)}\ \sum_{\sum_{i=1}^n\sum_{0\leqslant s^i\leqslant\tau^i}\left[s^i+\mu^i_1+\cdots+\mu^i_i\right]a^i_{s^i}=m}\ \prod_{i=1}^n\big(a^i_0+\cdots+a^i_{\tau^i}\big)^{\alpha'_i}=$$
$$=N^{\kappa}_{\mu^1_1}N^{\mu^1_1,\kappa}_{\mu^2_1,\mu^2_2}N^{\mu^2_1,\mu^2_2,\kappa}_{\mu^3_1,\mu^3_2,\mu^3_3}\cdots N^{\mu^{n-1}_1,\ldots,\mu^{n-1}_{n-1},\kappa}_{1,\ldots,n-1,n}\cdot$$
$$\cdot\sum_{\sum_{i=1}^n\sum_{0\leqslant s^i\leqslant\tau^i}\left[s^i+\mu^i_1+\cdots+\mu^i_i\right]a^i_{s^i}=m}\ \prod_{i=1}^n\big(a^i_0+\cdots+a^i_{\tau^i}\big)^{\alpha'_i}.$$

In conclusion, remembering the dropped $\sum_{\mu^i_l\in\nabla_{n,\kappa}}$, we have established that:

$$\sum_{\mu^i_l\in\nabla_{n,\kappa}}\ \sum_{\mathsf{YT}\in\mathsf{YT}^{\max}_{\kappa,m}(\mu^i_l)}\big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha'_1}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha'_2}\big(\ell_n(\mathsf{YT})\big)^{\alpha'_n}=$$

$$=\sum_{\mu^i_l\in\nabla_{n,\kappa}}N^{\kappa}_{\mu^1_1}N^{\mu^1_1,\kappa}_{\mu^2_1,\mu^2_2}N^{\mu^2_1,\mu^2_2,\kappa}_{\mu^3_1,\mu^3_2,\mu^3_3}\cdots N^{\mu^{n-1}_1,\ldots,\mu^{n-1}_{n-1},\kappa}_{1,\ldots,n-1,n}\cdot$$
$$\cdot\sum_{\sum_{i=1}^n\sum_{0\leqslant s^i\leqslant\tau^i}\left[s^i+\mu^i_1+\cdots+\mu^i_i\right]a^i_{s^i}=m}\ \prod_{i=1}^n\big(a^i_0+\cdots+a^i_{\tau^i}\big)^{\alpha'_i}.$$

Approximating sums by integrals. If we now set, similarly as in §3:

$$b^i_{s^i}:=\frac{1}{m}\,a^i_{s^i},$$

[25] In fact trivially, $N^{\kappa}_{\mu^1_1}=1$.

the sum of the last line of the preceding, boxed equation:

$$\mathsf{S}^{\alpha'_1,\dots,\alpha'_n}_{n,\kappa,m}\big(\mu^i_l\big) := \sum_{\sum_{i=1}^n \sum_{0\leqslant s^i\leqslant \tau^i}\left[s^i+\mu^i_1+\cdots+\mu^i_i\right]a^i_{s^i}=m} \prod_{i=1}^n \big(a^i_0+\cdots+a^i_{\tau^i}\big)^{\alpha'_i}$$

becomes:

$$\begin{aligned}\mathsf{S}^{\alpha'_1,\dots,\alpha'_n}_{n,\kappa,m}\big(\mu^i_l\big) &= \sum_{\sum_{i=1}^n \sum_{0\leqslant s^i\leqslant \tau^i}\left[s^i+\mu^i_1+\cdots+\mu^i_i\right]\frac{a^i_{s^i}}{m}=1} m^{\alpha'_1+\cdots+\alpha'_n}\prod_{i=1}^n\Big(\sum_{0\leqslant s^i\leqslant\tau^i}\frac{a^i_{s^i}}{m}\Big)^{\alpha'_i}\\ &= m^{\frac{n(n+1)}{2}}\sum_{\sum_{i=1}^n \sum_{0\leqslant s^i\leqslant \tau^i}\left[s^i+\mu^i_1+\cdots+\mu^i_i\right]\frac{a^i_{s^i}}{m}=1}\prod_{i=1}^n\Big(\sum_{0\leqslant s^i\leqslant\tau^i}\frac{a^i_{s^i}}{m}\Big)^{\alpha'_i}.\end{aligned}$$

Approximating the so obtained sum by a Riemann integral[26], we get up to a negligible power of m:

$$\begin{aligned}\mathsf{S}^{\alpha'_1,\dots,\alpha'_n}_{n,\kappa,m}\big(\mu^i_l\big) &= \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big)+m^{\frac{n(n+1)}{2}}\cdot m^{n\kappa-\frac{n(n-1)}{2}-1}\cdot\\ &\cdot\int_{\sum_{i=1}^n \sum_{0\leqslant s^i\leqslant \tau^i}\left[s^i+\mu^i_1+\cdots+\mu^i_i\right]b^i_{s^i}=1}\prod_{i=1}^n\Big(\sum_{0\leqslant s^i\leqslant\tau^i}b^i_{s^i}\Big)^{\alpha'_i}\end{aligned}$$

Performing next the changes of variables:

$$c^i_{s^i} := \left[s^i+\mu^i_1+\cdots+\mu^i_i\right]b^i_{s^i},$$

whence:

$$b^i_{s^i} = \frac{c^i_{s^i}}{s^i+\mu^i_1+\cdots+\mu^i_i},$$

we transform the integral as:

$$\begin{aligned}\mathsf{S}^{\alpha'_1,\dots,\alpha'_n}_{n,\kappa,m}\big(\mu^i_l\big) &= m^{(\kappa+1)n-1}\cdot\prod_{0\leqslant s^1\leqslant\tau^1}\frac{1}{s^1+\mu^1_1}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\frac{1}{s^n+\mu^n_1+\cdots+\mu^n_n}\cdot\\ &\cdot\int_{c^1_0+\cdots+c^1_{\tau^1}+\cdots\cdots+c^n_0+\cdots+c^n_{\tau^n}=1}\prod_{i=1}^n\Big(\sum_{0\leqslant s^i\leqslant\tau^i}\frac{c^i_{s^i}}{s^i+\mu^i_1+\cdots+\mu^i_i}\Big)^{\alpha'_i}dc'+\\ &+\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big),\end{aligned}$$

where:

$$dc' := dc^1_0\cdots dc^1_{\tau^1}\cdots\cdots dc^n_0\cdots dc^n_{\tau^n}.$$

Using the multinomial formula, we now expand the product of all the α'_i-th powers under the sum in the second line above:

$$\begin{aligned}&\prod_{i=1}^n\Big[\sum_{0\leqslant s^i\leqslant\tau^i}\frac{c^i_{s^i}}{s^i+\mu^i_1+\cdots+\mu^i_i}\Big]^{\alpha'_i}=\\ &=\prod_{i=1}^n\Big[\sum_{q^i_0+\cdots+q^i_{\tau^i}=\alpha'_i}\frac{\alpha'_i!}{q^i_0!\cdots q^i_{\tau^i}!}\prod_{0\leqslant s^i\leqslant\tau^i}\Big(\frac{c^i_{s^i}}{s^i+\mu^i_1+\cdots+\mu^i_i}\Big)^{q^i_{s^i}}\Big]\end{aligned}$$

[26] A different view may be found in [2].

$$= \sum_{q_0^1+\cdots+q_{\tau^1}^1=\alpha_1'} \cdots \sum_{q_0^n+\cdots+q_{\tau^n}^n=\alpha_n'} \frac{\alpha_1'!}{q_0^1!\cdots q_{\tau^1}^1!} \cdots \frac{\alpha_n'!}{q_0^n!\cdots q_{\tau^n}^n!} \cdot$$
$$\cdot \prod_{0\leqslant s^1\leqslant \tau^1} \Big(\frac{c_{s^1}^1}{s^1+\mu_1^1}\Big)^{q_{s^1}^1} \cdots \prod_{0\leqslant s^n\leqslant \tau^n} \Big(\frac{c_{s^n}^n}{s^n+\mu_1^n+\cdots+\mu_n^n}\Big)^{q_{s^n}^n}$$

$$= \sum_{q_0^1+\cdots+q_{\tau^1}^1=\alpha_1'} \cdots \sum_{q_0^n+\cdots+q_{\tau^n}^n=\alpha_n'} \frac{\alpha_1'!}{q_0^1!\cdots q_{\tau^1}^1!} \cdots \frac{\alpha_n'!}{q_0^n!\cdots q_{\tau^n}^n!} \cdot$$
$$\cdot \prod_{0\leqslant s^1\leqslant \tau^1} \frac{1}{\big(s^1+\mu_1^1\big)^{q_{s^1}^1}} \cdots \prod_{0\leqslant s^n\leqslant \tau^n} \frac{1}{\big(s^n+\mu_1^n+\cdots+\mu_n^n\big)^{q_{s^n}^n}} \cdot$$
$$\cdot \prod_{0\leqslant s^1\leqslant \tau^1} \big(c_{s^1}^1\big)^{q_{s^1}^1} \cdots \prod_{0\leqslant s^n\leqslant \tau^n} \big(c_{s^n}^n\big)^{q_{s^n}^n}.$$

After these expansions are done, in order to complete the computation of $\mathsf{S}_{n,\kappa,m}^{\alpha_1',\ldots,\alpha_n'}\big(\mu_l^i\big)$, we are left with the task of computing the integrals:

$$\int_{c_0^1+\cdots+c_{\tau^1}^1+\cdots\cdots+c_0^n+\cdots+c_{\tau^n}^n=1} \prod_{0\leqslant s^1\leqslant \tau^1} \big(c_{s^1}^1\big)^{q_{s^1}^1} \cdots \prod_{0\leqslant s^n\leqslant \tau^n} \big(c_{s^n}^n\big)^{q_{s^n}^n}\, dc'.$$

To this aim, we simply apply the elementary lemma on page 19, and this then yields to us the desired value:

$$\int_{c_0^1+\cdots+c_{\tau^1}^1+\cdots\cdots+c_0^n+\cdots+c_{\tau^n}^n=1} \prod_{0\leqslant s^1\leqslant \tau^1} \big(c_{s^1}^1\big)^{q_{s^1}^1} \cdots \prod_{0\leqslant s^n\leqslant \tau^n} \big(c_{s^n}^n\big)^{q_{s^n}^n} =$$
$$= \frac{q_0^1!\cdots q_{\tau^1}^1!\cdots\cdots q_0^n!\cdots q_{\tau^n}^n!}{(q_0^1+\cdots+q_{\tau^1}^1+\cdots\cdots+q_0^n+\cdots+q_{\tau^n}^n+(1+\tau^1)+\cdots+(1+\tau^n)-1)!}$$

$$= \frac{q_0^1!\cdots q_{\tau^1}^1!\cdots\cdots q_0^n!\cdots q_{\tau^n}^n!}{(\alpha_1'+\cdots\cdots+\alpha_n'+n\kappa-\frac{n(n-1)}{2}-1)!}$$
$$= \frac{q_0^1!\cdots q_{\tau^1}^1!\cdots\cdots q_0^n!\cdots q_{\tau^n}^n!}{\big((\kappa+1)n-1\big)!},$$

since $q_0^i+\cdots+q_{\tau^i}^i=\alpha_i'$ and since $\alpha_1'+\cdots+\alpha_n'=\frac{n(n+1)}{2}$.

Resuming what has been done, we therefore get:

$$\begin{aligned}
\mathsf{S}_{n,\kappa,m}^{\alpha'_1,\dots,\alpha'_n}\big(\mu_i^l\big) = m^{(\kappa+1)n-1} \cdot \prod_{0\leqslant s^1\leqslant \tau^1} \frac{1}{s^1+\mu_1^1} \cdots \prod_{0\leqslant s^n\leqslant \tau^n} \frac{1}{s^n+\mu_1^n+\cdots+\mu_n^n} \cdot \\
\cdot \sum_{q_0^1+\cdots+q_{\tau^1}^1=\alpha'_1} \cdots \sum_{q_0^n+\cdots+q_{\tau^n}^n=\alpha'_n} \frac{\alpha'_1!}{\underline{q_0^1!\cdots q_{\tau^1}^1!}_{\circ}} \cdots \frac{\alpha'_n!}{\underline{q_0^n!\cdots q_{\tau^n}^n!}_{\circ}} \cdot \\
\cdot \prod_{0\leqslant s^1\leqslant \tau^1} \frac{1}{\big(s^1+\mu_1^1\big)^{q_{s^1}^1}} \cdots \prod_{0\leqslant s^n\leqslant \tau^n} \frac{1}{\big(s^n+\mu_1^n+\cdots+\mu_n^n\big)^{q_{s^n}^n}} \cdot \\
\cdot \frac{\underline{q_0^1!\cdots q_{\tau^1}^1!}_{\circ} \cdots\cdots \underline{q_0^n!\cdots q_{\tau^n}^n!}_{\circ}}{\big((\kappa+1)n-1\big)!} + \\
+ \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big).
\end{aligned}$$

The products of the factorials of the $q_{s^i}^i$ disappear and a reorganization gives:

$$\begin{aligned}
\mathsf{S}_{n,\kappa,m}^{\alpha'_1,\dots,\alpha'_n}\big(\mu_i^l\big) = \frac{m^{(\kappa+1)n-1}}{\big((\kappa+1)n-1\big)!} \cdot \\
\cdot \prod_{0\leqslant s^1\leqslant \tau^1} \frac{1}{s^1+\mu_1^1} \cdots \prod_{0\leqslant s^n\leqslant \tau^n} \frac{1}{s^n+\mu_1^n+\cdots+\mu_n^n} \cdot \\
\cdot \alpha'_1!\cdots\alpha'_n! \cdot \sum_{q_0^1+\cdots+q_{\tau^1}^1=\alpha'_1} \cdots \sum_{q_0^n+\cdots+q_{\tau^n}^n=\alpha'_n} \Bigg(\\
\Bigg(\prod_{0\leqslant s^1\leqslant \tau^1} \frac{1}{\big(s^1+\mu_1^1\big)^{q_{s^1}^1}} \cdots \prod_{0\leqslant s^n\leqslant \tau^n} \frac{1}{\big(s^n+\mu_1^n+\cdots+\mu_n^n\big)^{q_{s^n}^n}} \Bigg) + \\
+ \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big).
\end{aligned}$$

Symbolically, instead of:

$$\prod_{0\leqslant s^1\leqslant \tau^1} \frac{1}{s^1+\mu_1^1} \prod_{0\leqslant s^2\leqslant \tau^2} \frac{1}{s^2+\mu_1^2+\mu_2^2} \cdots \prod_{0\leqslant s^n\leqslant \tau^n} \frac{1}{s^n+\mu_1^n+\cdots+\mu_n^n},$$

we shall write without any risk of ambiguity:

$$\frac{1}{\kappa\cdots\mu_1^1} \frac{1}{(\kappa+\mu_1^1)\cdots(\mu_2^2+\mu_1^2)} \cdots \frac{1}{(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})\cdots(n+(n-1)+\cdots+1)},$$

the dots in the denominators meaning that one takes the product of all integers, decreasingly, that are extant between the two written extremal integers.

In conclusion, we have established that:

$$
\begin{aligned}
&\sum_{\substack{\text{YT semi}-\text{standard}\\ \text{weight}(\mathsf{YT})=m}} \big(\ell_1(\mathsf{YT})-\ell_2(\mathsf{YT})\big)^{\alpha'_1}\cdots\big(\ell_{n-1}(\mathsf{YT})-\ell_n(\mathsf{YT})\big)^{\alpha'_{n-1}}\big(\ell_n(\mathsf{YT})\big)^{\alpha'_n}=\\
&=\frac{m^{(\kappa+1)n-1}}{\big((\kappa+1)n-1\big)!}\sum_{\mu^i_l\in\nabla_{n,\kappa}}\frac{N^{\kappa}_{\mu^1_1}}{\kappa\cdots\mu^1_1}\,\frac{N^{\mu^1_1,\kappa}_{\mu^2_1,\mu^2_2}}{(\kappa+\mu^1_1)\cdots(\mu^2_2+\mu^2_1)}\cdots\\
&\qquad\cdots\frac{N^{\mu^{n-1}_1,\dots,\mu^{n-1}_{n-1},\kappa}_{\mu^n_1,\dots,\mu^n_{n-1},\mu^n_n}}{(\kappa+\mu^{n-1}_{n-1}+\cdots+\mu^{n-1}_1)\cdots(\mu^n_n+\mu^n_{n-1}+\cdots+\mu^n_1)}\cdot\\
&\alpha'_1!\cdots\alpha'_n!\cdot\sum_{q^1_0+\cdots+q^1_{\tau^1}=\alpha'_1}\cdots\sum_{q^n_0+\cdots+q^n_{\tau^n}=\alpha'_n}\\
&\Big(\prod_{0\leqslant s^1\leqslant\tau^1}\frac{1}{\big(s^1+\mu^1_1\big)^{q^1_{s^1}}}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\frac{1}{\big(s^n+\mu^n_1+\cdots+\mu^n_{n-1}+\mu^n_n\big)^{q^n_{s^n}}}\Big)+\\
&\qquad\qquad+\mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big),
\end{aligned}
\tag{24}
$$

where we recall for completeness that $\tau^i=\kappa+\sum_{l=1}^{i-1}\mu^{i-1}_l-\sum_{l=1}^{i}\mu^i_l$ for convenient abbreviation.

§9. Number of tight paths in semi-standard Young tableaux

Summary. Thus, we are left with the task of computing or of majorating, for any $\alpha'_1,\dots,\alpha'_n$ with $\alpha'_1+\cdots+\alpha'_n=\frac{n(n+1)}{2}$, sums:

$$
\begin{aligned}
\square^{\alpha'_1,\dots,\alpha'_n}_{n,\kappa}:=&\sum_{\mu^i_l\in\nabla_{n,\kappa}}(\kappa!)^n\cdot\frac{N^{\kappa}_{\mu^1_1}}{\kappa\cdots\mu^1_1}\,\frac{N^{\mu^1_1,\kappa}_{\mu^2_1,\mu^2_2}}{(\kappa+\mu^1_1)\cdots(\mu^2_2+\mu^2_1)}\cdots\\
&\cdots\frac{N^{\mu^{n-1}_1,\dots,\mu^{n-1}_{n-1},\kappa}_{\mu^n_1,\dots,\mu^n_{n-1},\mu^n_n}}{(\kappa+\mu^{n-1}_{n-1}+\cdots+\mu^{n-1}_1)\cdots(\mu^n_n+\mu^n_{n-1}+\cdots+\mu^n_1)}\cdot\\
&\alpha'_1!\cdots\alpha'_n!\cdot\sum_{q^1_0+\cdots+q^1_{\tau^1}=\alpha'_1}\cdots\sum_{q^n_0+\cdots+q^n_{\tau^n}=\alpha'_n}\\
&\Big(\prod_{0\leqslant s^1\leqslant\tau^1}\frac{1}{\big(s^1+\mu^1_1\big)^{q^1_{s^1}}}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\frac{1}{\big(s^n+\mu^n_1+\cdots+\mu^n_{n-1}+\mu^n_n\big)^{q^n_{s^n}}}\Big)
\end{aligned}
$$

in which the weight m has completely disappeared, while only the dimension n and the jet order κ remain present.

At first, we would like to remind from the two theorems on p. 33 and on p. 39 that the basic numerical sums $\sum M\cdot\ell^{\alpha}$ we must compute were in fact born when $\alpha_1+\cdots+\alpha_n$ is maximal equal to $\frac{n(n+1)}{2}$, after rewriting in terms of $\ell_1-\ell_2,\dots,\ell_{n-1}-\ell_n$ and ℓ_n expressions of the form:

$$
\Big[\prod_{1\leqslant i<j\leqslant n}(\ell_i-\ell_j)\Big]\cdot\ell_1^{\beta_1}\cdots\ell_n^{\beta_n}
$$

that are multiple of the product of the $\ell_i - \ell_j$ with $\beta_1 + \cdots + \beta_n = n$. Each $\ell_i - \ell_j$ then writes as $\ell_i - \ell_{i+1} + \cdots + \ell_{j-1} - \ell_j$ with no ℓ_n at all, and it follows that after the rewriting, the exponent α'_n of ℓ_n is at most equal to n:

(25)
$$\Big[\prod_{1\leqslant i<j\leqslant n}(\ell_i-\ell_j)\Big]\cdot \ell_1^{\beta_1}\cdots\ell_{n-1}^{\beta_{n-1}}\ell_n^{\beta_n}$$
$$\leqslant \mathsf{Constant}_n\cdot \sum_{\substack{\alpha'_1+\cdots+\alpha'_{n-1}+\alpha'_n=\frac{n(n+1)}{2}\\ \alpha'_n\leqslant n}}(\ell_1-\ell_2)^{\alpha'_1}\cdots(\ell_{n-1}-\ell_n)^{\alpha'_{n-1}}\ell_n^{\alpha'_n}.$$

Thus, the sums $\square_{n,\kappa}^{\alpha'_1,\ldots,\alpha'_n}$ we consider are such that $\alpha'_1+\cdots+\alpha'_n=\frac{n(n+1)}{2}$ and $\alpha'_n\leqslant n$.

Logarithmic equivalents. Next, we observe that for any integer $\alpha'\geqslant 1$, as soon as $\tau\geqslant\alpha'$, one has:

$$\sum_{q_0+\cdots+q_\tau=\alpha'}\frac{1}{(k)^{q_0}\cdots(k+\tau)^{q_\tau}}=\frac{1}{\alpha'!}\big[\log(k+\tau)-\log(k)\big]^{\alpha'}+\mathsf{O}_{\alpha'}\big(\big[\log(k+\tau)-\log(k)\big]^{\alpha'-1}\big).$$

If $\tau\leqslant\alpha'-1$, this sum is smaller than the written power of a difference between two logarithms. Since our goal now will be to establish only an inequality of the form:

$$\boxed{\square_{n,\kappa}^{\alpha'_1,\ldots,\alpha'_n}\leqslant\mathsf{Constant}_n\cdot(\log\kappa)^{\alpha'_n}},\tag{26}$$

in which no particular knowledge about the $\mathsf{Constant}_n$ will be required, again with $\alpha'_1+\cdots+\alpha'_n=\frac{n(n+1)}{2}$ and with $\alpha'_n\leqslant n$, it will even suffice to observe that the last two lines in the definition of $\square_{n,\kappa}^{\alpha'_1,\ldots,\alpha'_n}$ enjoy a majoration of the sort:

$$\begin{aligned}&\alpha'_1!\,\alpha'_2!\cdots\alpha'_n!\cdot\sum_{q_0^1+\cdots+q_\tau^1=\alpha'_1}\ \sum_{q_0^2+\cdots+q_\tau^2=\alpha'_2}\cdots\sum_{q_0^n+\cdots+q_\tau^n=\alpha'_n}\ \prod_{0\leqslant s^1\leqslant\tau^1}\frac{1}{\big(s^1+\mu_1^1\big)^{q_{s^1}^1}}\\&\prod_{0\leqslant s^2\leqslant\tau^2}\frac{1}{\big(s^2+\mu_1^2+\mu_2^2\big)^{q_{s^2}^2}}\cdots\prod_{0\leqslant s^n\leqslant\tau^n}\frac{1}{\big(s^n+\mu_1^n+\cdots+\mu_{n-1}^n+\mu_n^n\big)^{q_{s^n}^n}}\leqslant\\&\leqslant\mathsf{Constant}_n\cdot\big[\log(\kappa)-\log(\mu_1^1)\big]^{\alpha'_1}\big[\log(\kappa+\mu_1^1)-\log(\mu_2^2+\mu_1^2)\big]^{\alpha'_2}\cdots\\&\quad\cdots\big[\log(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})-\log(n+\cdots+2+1)\big]^{\alpha'_n}.\end{aligned}$$

Consequently, we are left with establishing the following proposition.

Proposition. *Let $\alpha'_1,\ldots,\alpha'_n\in\mathbb{N}$ with $\alpha'_1+\cdots+\alpha'_n=\frac{n(n+1)}{2}$ and with $\alpha'_n\leqslant n$. Then for $\kappa\geqslant n$, one has the majoration:*

$$\widetilde{\square}_{n,\kappa}^{\alpha'_1,\ldots,\alpha'_n} := \sum_{\mu_l^i \in \nabla_{n,\kappa}} (\kappa!)^n \cdot \frac{N_{\mu_1^1}^{\kappa}}{\kappa \cdots \mu_1^1} \frac{N_{\mu_1^2,\mu_2^2}^{\mu_1^1,\kappa}}{(\kappa+\mu_1^1)\cdots(\mu_2^2+\mu_1^2)} \cdots$$
$$\cdots \frac{N_{\mu_1^n,\ldots,\mu_{n-1}^n,\mu_n^n}^{\mu_1^{n-1},\ldots,\mu_{n-1}^{n-1},\kappa}}{(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})\cdots(\mu_n^n+\mu_{n-1}^n+\cdots+\mu_1^n)} \cdot$$
$$\big[\log(\kappa)-\log(\mu_1^1)\big]^{\alpha'_1}\big[\log(\kappa+\mu_1^1)-\log(\mu_2^2+\mu_1^2)\big]^{\alpha'_2}\cdots$$
$$\cdots\big[\log(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})-\log(n+\cdots+2+1)\big]^{\alpha'_n} \leqslant$$
$$\leqslant \mathsf{Constant}_n \cdot (\log\kappa)^{\alpha'_n}.$$

Tight paths. According to the proposition on p. 58 and to the definition made on p. 62, the integer $N_{\mu_1^1}^{\kappa}$ denotes the number of tight paths from the column μ_1^1 to the column κ, hence it is equal to 1. When the dimension n is equal to 2, one may show that:

$$N_{1,2}^{\mu_1^1,\kappa} = \tfrac{(\kappa+\mu_1^1)!}{(\kappa-3)!\,(\mu_1^1-1)!} - \tfrac{(\kappa+\mu_1^1-4)!}{(\kappa-1)!\,(\mu_1^1-3)!}.$$

In higher dimensions, the exact computation of the numbers $N_{\mu_1^2,\mu_2^2}^{\mu_1^1,\kappa}$, $N_{\mu_1^3,\mu_2^3,\mu_3^3}^{\mu_1^2,\mu_2^2,\kappa}$, ...may certainly be done and it involves only differences of multinomial coefficients, but very many cases are to be considered according to certain inequalities between the μ_i^j. However, after some explorations, it appears that in order to get the majoration claimed by the proposition, it suffices to *majorate* these numbers uniformly as follows.

Majoration of the tight path numbers $N_{\mu_1^i,\ldots,\mu_{i-1}^i,\mu_i^i}^{\mu_1^{i-1},\ldots,\mu_{i-1}^{i-1},\kappa}$. By definition, $N_{\mu_1^i,\ldots,\mu_{i-1}^i,\mu_i^i}^{\mu_1^{i-1},\ldots,\mu_{i-1}^{i-1},\kappa}$ counts the number of strictly increasing tight paths from the column $\big[\mu_1^i \ \cdots \ \mu_{i-1}^i \ \mu_i^i\big]^{\text{transposed}}$ to the column $\big[\mu_1^{i-1} \ \cdots \ \mu_{i-1}^{i-1} \ \kappa\big]^{\text{transposed}}$ in the i-dimensional lattice $\mathbb{N}^i$ with the supplementary constraint that at each point $\big[\gamma_1^i(s^i) \ \cdots \ \gamma_{i-1}^i(s^i) \ \gamma_i(s^i)\big]^{\text{transposed}}$ of the path, the inequalities $\gamma_1^i(s^i) < \cdots < \gamma_{i-1}^i(s^i) < \gamma_i(s^i)$ (strict increase inside columns, downward) must be satisfied.

If we relax this last constraint, there are clearly more paths. But the number of strictly increasing paths in a complete lattice is elementarily computed. Thus we deduce that:

$$N_{\mu_1^i,\ldots,\mu_{i-1}^i,\mu_i^i}^{\mu_1^{i-1},\ldots,\mu_{i-1}^{i-1},\kappa} \leqslant \tfrac{(\mu_1^{i-1}-\mu_1^i+\cdots+\mu_{i-1}^{i-1}-\mu_{i-1}^i+\kappa-\mu_i^i)!}{(\mu_1^{i-1}-\mu_1^i)!\cdots(\mu_{i-1}^{i-1}-\mu_{i-1}^i)!\,(\kappa-\mu_i^i)!}.$$

Removal of α'_n. On the other hand, for any choice of $\mu_{n-1}^{n-1},\ldots,\mu_1^{n-1}$ as in the proposition on p. 58, the last difference between logarithms:

$$\big[\log(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})-\log(n+\cdots+2+1)\big]^{\alpha'_n}$$

enjoys, when $\kappa \gg n$, a doubly controlling inequality of the form:

$$\frac{1}{\mathsf{C}_n}\cdot[\log(\kappa)]^{\alpha'_n} \leqslant \big[\log(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})-\log(n+\cdots+2+1)\big]^{\alpha'_n} \leqslant \mathsf{C}_n\cdot[\log(\kappa)]^{\alpha'_n}$$

where the constant $\mathsf{C}_n > 1$ can be chosen arbitrarily close to 1 provided that $\kappa \geqslant \kappa_{\mathsf{C}_n} \gg n$ is large enough. Consequently, in order to establish the inequality boxed on p. 67, it suffices now to establish the following concrete proposition, in which $\alpha'_n = 0$.

Proposition. *Let $\alpha'_1, \dots, \alpha'_{n-1} \in \mathbb{N}$ with $\alpha'_1 + \cdots + \alpha'_{n-1} = \frac{n(n+1)}{2}$ and assume $\kappa \geqslant n$. Then the following sum is bounded independently of κ:*

$$\begin{aligned}
\Delta_{n,\kappa}^{\alpha'_1,\dots,\alpha'_{n-1},0} := \sum_{\mu_l^i \in \nabla_{n,\kappa}} (\kappa!)^n \cdot 1\, \frac{(\mu_1^1-1)!}{\kappa!} \cdot \frac{(\mu_1^1-\mu_1^2+\kappa-\mu_2^2)!}{(\mu_1^1-\mu_1^2)!\,(\kappa-\mu_2^2)!}\, \frac{(\mu_2^2+\mu_1^2-1)!}{(\kappa+\mu_1^1)!}\cdot \\
\cdot \frac{(\mu_1^2-\mu_1^3+\mu_2^2-\mu_2^3+\kappa-\mu_3^3)!}{(\mu_1^2-\mu_1^3)!\,(\mu_2^2-\mu_2^3)!\,(\kappa-\mu_3^3)!}\, \frac{(\mu_3^3+\mu_2^3+\mu_1^3-1)!}{(\kappa+\mu_2^2+\mu_1^2)!} \cdots \\
\cdots \frac{(\mu_1^{n-2}-\mu_1^{n-1}+\cdots+\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1}+\kappa-\mu_{n-1}^{n-1})!}{(\mu_1^{n-2}-\mu_1^{n-1})!\cdots(\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1})!\,(\kappa-\mu_{n-1}^{n-1})!}\, \frac{(\mu_{n-1}^{n-1}+\mu_{n-2}^{n-1}+\cdots+\mu_1^{n-1}-1)!}{(\kappa+\mu_{n-2}^{n-2}+\cdots+\mu_1^{n-2})!}\cdot \\
\cdot \frac{(\mu_1^{n-1}-\mu_1^n+\cdots+\mu_{n-1}^{n-1}-\mu_{n-1}^n+\kappa-\mu_n^n)!}{(\mu_1^{n-1}-\mu_1^n)!\cdots(\mu_{n-1}^{n-1}-\mu_{n-1}^n)!\,(\kappa-\mu_n^n)!}\, \frac{(\mu_n^n+\mu_{n-1}^n+\cdots+\mu_1^n-1)!}{(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})!}\cdot \\
\cdot \big[\log(\kappa)-\log(\mu_1^1)\big]^{\alpha'_1} \cdot \big[\log(\kappa+\mu_1^1)-\log(\mu_2^2+\mu_1^2)\big]^{\alpha'_2}\cdot \\
\cdot \big[\log(\kappa+\mu_2^2+\mu_1^2)-\log(\mu_3^3+\mu_2^3+\mu_1^3)\big]^{\alpha'_3}\cdots \\
\cdots \big[\log(\kappa+\mu_{n-2}^{n-2}+\cdots+\mu_1^{n-2})-\log(\mu_{n-1}^{n-1}+\mu_{n-2}^{n-1}+\cdots+\mu_1^{n-1})\big]^{\alpha'_{n-1}} \leqslant \\
\leqslant \mathsf{Constant}_n.
\end{aligned}$$

§10. Bounded behavior of plurilogarithmic sums

Simplifying the kernel. To begin with, disregarding the logarithmic factors, or equivalently, considering that $\alpha'_1 = \alpha'_2 = \alpha'_3 = \cdots = \alpha'_{n-1} = 0$, we observe that the rational factor simplifies a bit (a factorial $\kappa!$ disappears) and can be majorated as follows:

$$\begin{aligned}
\underline{\kappa!}_{\circ}\,(\kappa!)^{n-2}\,\underline{\kappa!}_{ⓐ}\cdot 1\, \frac{(\mu_1^1-1)!}{\underline{\kappa!}_{\circ}} \cdot \frac{(\mu_1^1-\mu_1^2+\kappa-\mu_2^2)!}{(\mu_1^1-\mu_1^2)!\,(\kappa-\mu_2^2)!}\, \frac{(\mu_2^2+\mu_1^2-1)!}{(\kappa+\mu_1^1)!}\cdots \\
\cdot \frac{(\mu_1^2-\mu_1^3+\mu_2^2-\mu_2^3+\kappa-\mu_3^3)!}{(\mu_1^2-\mu_1^3)!\,(\mu_2^2-\mu_2^3)!\,(\kappa-\mu_3^3)!}\, \frac{(\mu_3^3+\mu_2^3+\mu_1^3-1)!}{(\kappa+\mu_2^2+\mu_1^2)!}\cdots \\
\cdots \frac{(\mu_1^{n-2}-\mu_1^{n-1}+\cdots+\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1}+\kappa-\mu_{n-1}^{n-1})!}{(\mu_1^{n-2}-\mu_1^{n-1})!\cdots(\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1})!\,(\kappa-\mu_{n-1}^{n-1})!}\, \frac{(\mu_{n-1}^{n-1}+\mu_{n-2}^{n-1}+\cdots+\mu_1^{n-1}-1)!}{(\kappa+\mu_{n-2}^{n-2}+\cdots+\mu_1^{n-2})!}\cdot \\
\cdot \frac{(\mu_1^{n-1}-1+\cdots+\mu_{n-1}^{n-1}-(n-1)+\kappa-n)!_{ⓒ}}{(\mu_1^{n-1}-1)!\cdots(\mu_{n-1}^{n-1}-(n-1))!\,\underline{(\kappa-n)!}_{ⓐ}}\, \frac{(n+(n-1)+\cdots+1-1)!_{ⓑ}}{\underline{(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})!}_{ⓒ}} \leqslant \\
\leqslant \mathsf{Constant}_n\cdot(\kappa!)^{n-2}\cdot(\mu_1^1-1)!\cdot \frac{(\mu_1^1-\mu_1^2+\kappa-\mu_2^2)!}{(\mu_1^1-\mu_1^2)!\,(\kappa-\mu_2^2)!}\, \frac{(\mu_2^2+\mu_1^2-1)!}{(\kappa+\mu_1^1)!}\cdots \\
\cdot \frac{(\mu_1^2-\mu_1^3+\mu_2^2-\mu_2^3+\kappa-\mu_3^3)!}{(\mu_1^2-\mu_1^3)!\,(\mu_2^2-\mu_2^3)!\,(\kappa-\mu_3^3)!}\, \frac{(\mu_3^3+\mu_2^3+\mu_1^3-1)!}{(\kappa+\mu_2^2+\mu_1^2)!}\cdots
\end{aligned}$$

$$\cdots \frac{(\mu_1^{n-2}-\mu_1^{n-1}+\cdots+\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1}+\kappa-\mu_{n-1}^{n-1})!}{(\mu_1^{n-2}-\mu_1^{n-1})!\cdots(\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1})!\,(\kappa-\mu_{n-1}^{n-1})!}\,\frac{(\mu_{n-1}^{n-1}+\mu_{n-2}^{n-1}+\cdots+\mu_1^{n-1}-1)!}{(\kappa+\mu_{n-2}^{n-2}+\cdots+\mu_1^{n-2})!}\cdot$$
$$\cdot\,\frac{1}{(\mu_1^{n-1}-1)!\cdots(\mu_{n-1}^{n-1}-(n-1))!}\cdot$$
$$\cdot\,\frac{\kappa(\kappa-1)\cdots(\kappa-n+1)}{(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})\cdots(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1}-\frac{n(n+1)}{2}+1)},$$

since the two pairs of terms underlined with ⓐ and ⓒ appended can be put at the end and simplified, while the pair of terms with ⓑ appended, equal to the factorial $(\frac{n(n+1)}{2}-1)!$, may be considered as just a $\mathsf{Constant}_n$. But now, the last line is controlled as follows:

$$\mathsf{C}_n^{-1}\,\kappa^{-\frac{n(n-1)}{2}} \leqslant \frac{\kappa(\kappa-1)\cdots(\kappa-n+1)}{(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1})\cdots(\kappa+\mu_{n-1}^{n-1}+\cdots+\mu_1^{n-1}-\frac{n(n+1)}{2}+1)} \leqslant \mathsf{C}_n\,\kappa^{-\frac{n(n-1)}{2}},$$

for some constant $\mathsf{C}_n > 1$. Consequently, we are reduced to the following proposition.

Proposition. *Let $\alpha'_1,\ldots,\alpha'_{n-1}\in\mathbb{N}$ with $\alpha'_1+\cdots+\alpha'_{n-1}=\frac{n(n+1)}{2}$ and assume $\kappa\geqslant n$. Then the following sum is bounded independently of κ:*

$$\begin{aligned}
\mathsf{K}^n_{\alpha'_1,\ldots,\alpha'_{n-1}}(\kappa) := \sum_{\mu_l^i\in\nabla_{n,\kappa}} & \frac{1}{\kappa^{\frac{n(n-1)}{2}}}\cdot(\kappa!)^{n-2}\cdot(\mu_1^1-1)!\cdot \\
& \cdot\,\frac{(\mu_1^1-\mu_1^2+\kappa-\mu_2^2)!}{(\mu_1^1-\mu_1^2)!\,(\kappa-\mu_2^2)!}\,\frac{(\mu_2^2+\mu_1^2-1)!}{(\kappa+\mu_1^1)!}\cdot\frac{(\mu_1^2-\mu_1^3+\mu_2^2-\mu_2^3+\kappa-\mu_3^3)!}{(\mu_1^2-\mu_1^3)!\,(\mu_2^2-\mu_2^3)!\,(\kappa-\mu_3^3)!}\,\frac{(\mu_3^3+\mu_2^3+\mu_1^3-1)!}{(\kappa+\mu_2^2+\mu_1^2)!}\cdots \\
& \cdots\,\frac{(\mu_1^{n-2}-\mu_1^{n-1}+\cdots+\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1}+\kappa-\mu_{n-1}^{n-1})!}{(\mu_1^{n-2}-\mu_1^{n-1})!\cdots(\mu_{n-2}^{n-2}-\mu_{n-2}^{n-1})!\,(\kappa-\mu_{n-1}^{n-1})!}\,\frac{(\mu_{n-1}^{n-1}+\mu_{n-2}^{n-1}+\cdots+\mu_1^{n-1}-1)!}{(\kappa+\mu_{n-2}^{n-2}+\cdots+\mu_1^{n-2})!}\cdot \\
& \cdot\,\frac{1}{(\mu_1^{n-1}-1)!\cdots(\mu_{n-1}^{n-1}-(n-1))!}\cdot \\
& \cdot\,\big[\log(\kappa)-\log(\mu_1^1)\big]^{\alpha'_1}\cdot\big[\log(\kappa+\mu_1^1)-\log(\mu_2^2+\mu_1^2)\big]^{\alpha'_2}\cdot \\
& \cdot\,\big[\log(\kappa+\mu_2^2+\mu_1^2)-\log(\mu_3^3+\mu_2^3+\mu_1^3)\big]^{\alpha'_3}\cdots \\
& \cdots\big[\log(\kappa+\mu_{n-2}^{n-2}+\cdots+\mu_1^{n-2})-\log(\mu_{n-1}^{n-1}+\mu_{n-2}^{n-1}+\cdots+\mu_1^{n-1})\big]^{\alpha'_{n-1}} \leqslant \\
\leqslant\ & \mathsf{Constant}_n.
\end{aligned}$$

Here, the summation $\sum_{\mu_l^i\in\nabla_{n,\kappa}}$ holds for μ_l^i satisfying the two collections of inequalities (21) and (22), and we may expand it symbolically using two symbols Σ in order to notify well these two conditions:

$$\sum_{\mu_l^i\in\nabla_{n,\kappa}} \equiv \sum_{\substack{1\leqslant\mu_1^1<\kappa \\ 1\leqslant\mu_1^2<\mu_2^2<\kappa \\ 1\leqslant\mu_1^3<\mu_2^3<\mu_3^3<\kappa \\ \cdots\cdots\cdots\cdots \\ 1\leqslant\mu_1^{n-2}<\mu_2^{n-2}<\mu_3^{n-2}<\cdots<\mu_{n-2}^{n-2}<\kappa \\ 1\leqslant\mu_1^{n-1}<\mu_2^{n-1}<\mu_3^{n-1}<\cdots<\mu_{n-2}^{n-1}<\mu_{n-1}^{n-1}<\kappa}} \sum_{\substack{\mu_1^1\geqslant\mu_1^2\geqslant\mu_1^3\geqslant\cdots\geqslant\mu_1^{n-2}\geqslant\mu_1^{n-1} \\ \mu_2^2\geqslant\mu_2^3\geqslant\cdots\geqslant\mu_2^{n-2}\geqslant\mu_2^{n-1} \\ \mu_3^3\geqslant\cdots\geqslant\mu_3^{n-2}\geqslant\mu_3^{n-1} \\ \cdots\cdots\cdots \\ \mu_{n-2}^{n-2}\geqslant\mu_{n-2}^{n-1}}}$$

In dimension $n=2$, the sum writes:

$$\mathsf{K}^2_{\alpha'_1}(\kappa)=\sum_{1\leqslant\mu^1_1<\kappa}\frac{1}{\kappa}\cdot\underline{(\mu^1_1-1)!}_{\circ}\cdot\frac{1}{\underline{(\mu^1_1-1)!}_{\circ}}\cdot\big[\log(\kappa)-\log(\mu^1_1)\big]^{\alpha'_1},$$

and is seen to be an approximation of the Riemann integral:

$$\int_0^1\big(-\log(x)\big)^{\alpha'_1}=\alpha'_1!,$$

which is finite. In dimensions $n=3$ and $n=4$, the sum writes:

$$\mathsf{K}^3_{\alpha'_1,\alpha'_2}(\kappa)=\sum_{\substack{1\leqslant\mu^1_1<\kappa\\1\leqslant\mu^2_1<\mu^2_2<\kappa}}\sum_{\mu^1_1\geqslant\mu^2_1}\frac{1}{\kappa^3}\cdot\kappa!\cdot(\mu^1_1-1)!\cdot\frac{(\mu^1_1-\mu^2_1+\kappa-\mu^2_2)!}{(\mu^1_1-\mu^2_1)!\,(\kappa-\mu^2_2)!}\,\frac{(\mu^2_2+\mu^2_1-1)!}{(\kappa+\mu^1_1)!}\cdot$$
$$\cdot\frac{1}{(\mu^2_1-1)!\,(\mu^2_2-2)!}\cdot\big[\log\kappa-\log\mu^1_1\big]^{\alpha'_1}\big[\log(\kappa+\mu^1_1)-\log(\mu^2_2+\mu^2_1)\big]^{\alpha'_2},$$

and:

$$\mathsf{K}^4_{\alpha'_1,\alpha'_2,\alpha'_3}(\kappa)=\sum_{\substack{1\leqslant\mu^1_1<\kappa\\1\leqslant\mu^2_1<\mu^2_2<\kappa\\1\leqslant\mu^3_1<\mu^3_2<\mu^3_3<\kappa}}\sum_{\substack{\mu^1_1\geqslant\mu^2_1\geqslant\mu^3_1\\\mu^2_2\geqslant\mu^3_2}}\frac{1}{\kappa^6}\cdot(\kappa!)^2\cdot(\mu^1_1-1)!\cdot\frac{(\mu^1_1-\mu^2_1+\kappa-\mu^2_2)!}{(\mu^1_1-\mu^2_1)!\,(\kappa-\mu^2_2)!}\,\frac{(\mu^2_2+\mu^2_1-1)!}{(\kappa+\mu^1_1)!}\cdot$$
$$\cdot\frac{(\mu^2_1-\mu^3_1+\mu^2_2-\mu^3_2+\kappa-\mu^3_3)!}{(\mu^2_1-\mu^3_1)!\,(\mu^2_2-\mu^3_2)!\,(\kappa-\mu^3_3)!}\,\frac{(\mu^3_3+\mu^3_2+\mu^3_1-1)!}{(\kappa+\mu^2_2+\mu^2_1)!}\cdot$$
$$\cdot\frac{1}{(\mu^3_1-1)!\,(\mu^3_2-2)!\,(\mu^3_3-3)!}\cdot$$
$$\cdot\big[\log\kappa-\log\mu^1_1\big]^{\alpha'_1}\big[\log(\kappa+\mu^1_1)-\log(\mu^2_2+\mu^2_1)\big]^{\alpha'_2}$$
$$\cdot\big[\log(\kappa+\mu^2_2+\mu^2_1)-\log(\mu^3_3+\mu^3_2+\mu^3_1)\big]^{\alpha'_3}.$$

When $n=3$, we have:

$$\sum_{\substack{1\leqslant\mu^1_1<\kappa\\1\leqslant\mu^2_1<\mu^2_2<\kappa}}\sum_{\mu^1_1\geqslant\mu^2_1}\equiv\sum_{\mu^2_1=1}^{\kappa-1}\Bigg(\sum_{\mu^2_2=\mu^2_1+1}^{\kappa-1}\Bigg(\sum_{\mu^1_1=\mu^2_1}^{\kappa-1}\bullet\Bigg)\Bigg).$$

When $\alpha'_1=\alpha'_2=0$, the first summation $\sum_{\mu^1_1=\mu^2_1}^{\kappa-1}$ disregarding the $\frac{1}{\kappa^3}$ gives:

$$\sum_{\mu^1_1=\mu^2_1}^{\kappa-1}\kappa!\cdot(\mu^1_1-1)!\cdot\frac{(\mu^1_1-\mu^2_1+\kappa-\mu^2_2)!}{(\mu^1_1-\mu^2_1)!\,(\kappa-\mu^2_2)!}\,\frac{(\mu^2_2+\mu^2_1-1)!}{(\kappa+\mu^1_1)!}\cdot\frac{1}{(\mu^2_1-1)!\,(\mu^2_2-2)!}=$$
$$=(c-1)-\frac{(\kappa-1)!\,(2\kappa-\mu^2_1-\mu^2_2)!\,(\mu^2_1+\mu^2_2-1)!\,\kappa!}{(2\kappa)!\,(\kappa-\mu^2_2)!\,(\kappa-\mu^2_1)!\,(\mu^2_1-1)!\,(\mu^2_2-2)!}\sum_{l=0}^{\infty}\frac{(k)_l\,(2k+1-\mu^2_1-\mu^2_2)_l}{(2\kappa+1)_l\,(\kappa-\mu^2_1+1)},$$

where $(j)_l:=j(j+1)\cdots(j+l-1)=\frac{(j+l-1)!}{j!}$. The sum being positive, the absolute value of the second negative term is necessarily $<(c-1)$. But then:

$$\frac{1}{\kappa^3}\sum_{\mu^2_1=1}^{\kappa-1}\Bigg(\sum_{\mu^2_2=\mu^2_1+1}^{\kappa-1}(c-1)\Bigg)=\frac{1}{k^3}\big[\tfrac{1}{3}\,k^3-\tfrac{3}{2}\,k^2+\tfrac{13}{6}\,k-1\big]$$

is clearly bounded independently of κ. So $\mathsf{K}^3_{0,0}(\kappa) \leqslant \mathsf{Constant}_3$ in any case, and so on.

Indirect majorations. Looking back at the Euler-Poincaré characteristic, in the summation formula:

$$\chi\big(X,\, \mathsf{Gr}^\bullet \mathscr{E}^{GG}_{\kappa,m} T^*_X\big) = \sum_{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0} M_{\ell_1,\ell_2,\cdots,\ell_n} \cdot \chi\big(\big(X,\, \mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)} T^*_X\big),$$

the coefficients of each Chern monomial $\mathsf{c}_1^{\lambda_1} c_2^{\lambda_2} \cdots \mathsf{c}_n^{\lambda_n}$ must identify. In the Euler-Poincaré characteristic of the Schur bundle, the coefficient of c_1^n is, up to a rational factor:

$$\prod_{1 \leqslant i < j \leqslant n} (\ell_i - \ell_j) \sum_{\beta_1 + \cdots + \beta_{n-1} + \beta_n = n} \ell_1^{\beta_1} \cdots \ell_{n-1}^{\beta_{n-1}} \ell_n^{\beta_n}.$$

We then rewrite:

$$\sum_{\beta_1 + \cdots + \beta_{n-1} + \beta_n = n} \ell_1^{\beta_1} \cdots \ell_{n-1}^{\beta_{n-1}} \ell_n^{\beta_n} = \sum_{\beta'_1 + \cdots + \beta'_{n-1} + \beta'_n} C_{\beta'_1,\ldots,\beta'_{n-1},\beta'_n} (\ell_1 - \ell_2)^{\beta'_1} \cdots (\ell_{n-1} - \ell_n)^{\beta'_{n-1}} \ell_n^{\beta'_n},$$

with coefficients $C_{\beta'_1,\ldots,\beta'_{n-1},\beta'_n} \in \mathbb{N}$. Notice that $C_{0,\ldots,0,n} = \binom{n+n-1}{n}$. Identifying then the coefficients of c_1^n:

$$\boxed{\begin{aligned} \sum_{\substack{\text{YT semi-standard} \\ \text{weight}(\text{YT}) = m}} \prod_{1 \leqslant i < j \leqslant n} (\ell_i - \ell_j) \cdot (\ell_n)^n &= \tfrac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!}\, 1!\, 2! \cdots (n-1)!\, (\log \kappa)^n + \\ &\quad + \mathsf{O}_n\big(m^{(\kappa+1)n-1} \cdot (\log \kappa)^{n-1}\big) + \mathsf{O}_{n,\kappa}\big(m^{(\kappa+1)n-2}\big). \end{aligned}}$$

The power $(\ell_n)^n$ of ℓ_n corresponds to $(\log \kappa)^n$.

Similarly, in dimension $n = 2$, looking at the coefficient of c_2 and making identification, one gets:

$$\sum_{1 \leqslant \lambda \leqslant \kappa} \frac{1}{\lambda^2} = \sum_{1 \leqslant \mu_1^1 < \kappa} (\kappa!)^2 \, \frac{N^\kappa_{\mu_1^1}}{\kappa \cdots \mu_1^1} \, \frac{N^{\mu_1^1,\kappa}_{1,2}}{(\kappa + \mu_1^1) \cdots (2+1)} \left(\sum_{q_0^1 + \cdots + q_{\tau^1}^1 = 3} \frac{1}{(\mu_1^1)^{q_0^1} \cdots (\kappa)^{q_{\tau^1}^1}} \right),$$

so one deduces without computation that the sum:

$$\sum_{1 \leqslant \mu_1^1 < \kappa} (\kappa!)^2 \, \frac{N^\kappa_{\mu_1^1}}{\kappa \cdots \mu_1^1} \, \frac{N^{\mu_1^1,\kappa}_{1,2}}{(\kappa + \mu_1^1) \cdots (2+1)} \, \big[\log \kappa - \log \mu_1^1\big]^3$$

is finite and bounded independently of κ. In dimensions $n = 3$ and higher, looking at the coefficient of c_n, one sees indirectly, without computations and without majorations that all the sums $\square^{\alpha'_1,\ldots,\alpha'_{n-1},0}_{n,\kappa}$ which appear after expressing:

$$\prod_{1 \leqslant i < j \leqslant n} \sum_{\beta'_1 + \cdots + \beta'_{n-1} = n} \ell_1^{\beta'_1} \cdots \ell_{n-1}^{\beta'_{n-1}}$$

in terms of $(\ell_1 - \ell_2)$, $\ldots$, $(\ell_{n-1} - \ell_n)$, ℓ_n are finite. This suffices for our purposes.

Summary. In conclusion, either directly or indirectly by identification without computations and without majorations, we have seen that for any $\alpha'_1, \dots, \alpha'_{n-1}, \alpha'_n \in \mathbb{N}$ with $\alpha'_1 + \cdots + \alpha'_{n-1} + \alpha'_n \leqslant \frac{n(n+1)}{2}$, the quantity $\Delta_{n,\kappa}^{\alpha'_1,\dots,\alpha'_{n-1},0}$ is $\leqslant \mathsf{Constant}_n$, whence $\square_{n,\kappa}^{\alpha'_1,\dots,\alpha'_n}$ is $\leqslant \mathsf{Constant}_n \cdot (\log \kappa)^{\alpha'_n}$, and from (24), it follows at the end that:

$$\begin{aligned} &\sum_{\substack{\mathsf{YT\ semi-standard} \\ \mathsf{weight}(\mathsf{YT})=m}} \big(\ell_1(\mathsf{YT}) - \ell_2(\mathsf{YT})\big)^{\alpha'_1} \cdots \big(\ell_{n-1}(\mathsf{YT}) - \ell_n(\mathsf{YT})\big)^{\alpha'_{n-1}} \leqslant \\ &\qquad \leqslant \frac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!\,(\kappa!)^n} \cdot \mathsf{Constant}_n + \mathsf{Constant}_{n,\kappa} \cdot m^{(\kappa+1)n-2}. \end{aligned} \tag{27}$$

§11. Algebraic sheaf theory and Schur bundles

Complex projective hypersurface and line bundles $\mathscr{O}_X(k)$. Let $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a geometrically smooth complex projective hypersurface of degree $d \geqslant 1$, defined in homogeneous coordinates $z = [z_0 \colon z_1 \colon \cdots \colon z_n \colon z_{n+1}]$ as the zero-set:

$$X = \big\{[z_0 \colon z_1 \colon \cdots \colon z_n \colon z_{n+1}] \in \mathbb{P}^{n+1}(\mathbb{C}) \colon P(z_0, z_1, \dots, z_n, z_{n+1}) = 0\big\}$$

of a certain holomorphic polynomial $P = P(z) \in \mathbb{C}[z_0, z_1, \dots, z_n, z_{n+1}]$ which is homogeneous of a certain degree $d \geqslant 1$ and whose differential $P_{z_0} dz_0 + \cdots + P_{z_{n+1}} dz_{n+1}$ does not vanish at any point of X, so that X has no singularities. We will sometimes use the letter N to denote $n+1$:

$$N \overset{\mathsf{notation}}{\equiv} n+1.$$

The *tautological line bundle* over $\mathbb{P}^N$ will be denoted by $\mathscr{O}_{\mathbb{P}^N}(-1)$ and its dual by $\mathscr{O}_{\mathbb{P}^N}(1) := \mathscr{O}_{\mathbb{P}^N}(-1)^*$. For various values of the integer $k \in \mathbb{Z}$, the standard line bundles:

$$\mathscr{O}_{\mathbb{P}^N}(k) := \mathscr{O}_{\mathbb{P}^N}(\pm 1)^{\otimes |k|},$$

where $\pm = \mathrm{sign}(k)$, will play a very decisive rôle in what follows, as well as their restrictions to X, namely the bundles:

$$\mathscr{O}_X(k) := \mathscr{O}_{\mathbb{P}^N}(k)\big|_X.$$

Canonical line bundles. For any $\mathbb{P}^N$, the (line) bundle of holomorphic differential forms of maximal degree N on $\mathbb{P}^N$:

$$K_{\mathbb{P}^N} = \Lambda^N T^*_{\mathbb{P}^N} \simeq \mathscr{O}_{\mathbb{P}^N}(-N-1),$$

is known, thanks to the *adjunction formula*, to be isomorphic to $\mathscr{O}_{\mathbb{P}^N}(-N-1)$. Similarly, the (line) bundle of holomorphic differential forms of maximal degree n on X:

$$K_X \overset{\mathsf{notation}}{\equiv} \Lambda^n T^*_X \simeq \mathscr{O}_X(d-n-2)$$

called the *canonical bundle* of X and central in complex algebraic geometry, is known, again thanks to the adjunction formula, to be isomorphic to $\mathscr{O}_X(d-n-2)$.

Normal exact sequence. To begin with, one has the so-called *normal exact sequence*:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(-d) \xrightarrow{\textit{incl}} \mathscr{O}_{\mathbb{P}^{n+1}}(0) \xrightarrow{\textit{rest}} \mathscr{O}_X(0) \longrightarrow 0. \tag{28}$$

Here, the inclusion *incl* is defined by multiplication with the defining polynomial $P(z_0, \dots, z_{n+1})$ for X, and the restriction *rest*, of course from $\mathbb{P}^{n+1}$ to X, concerns functions, differential forms, bundles and sheaves.

General sheaves of differential forms. Let r be an integer with $0 \leqslant r \leqslant n+1$ and consider the bundle $\Lambda^r T^*_X$ of differential forms of degree r on X, with the convention that:

$$\Lambda^0 T^*_X \overset{\text{collapse}}{\equiv} \mathscr{O}_X(0). \tag{29}$$

The functor $\mathscr{F} \longmapsto \mathscr{F} \otimes \mathscr{G}$ is right exact, for any sheaf $\mathscr{G}$, and is furthermore also left exact when $\mathscr{G}$ is locally free (in what follows, only such sheaves will be considered). Here at any point $z \in X$, the bundle $\Lambda^k T^*_X$ is, for any k with $0 \leqslant k \leqslant n$, a free $\mathscr{O}_{X,z}$-module of rank $\binom{n}{k}$, hence by tensoring the above normal exact sequence, we obtain the following exact sequence:

$$\begin{aligned} 0 \longrightarrow \Lambda^k T^*_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(-d) &\longrightarrow \Lambda^k T^*_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(0) \longrightarrow \\ &\longrightarrow \Lambda^k T^*_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(0) \longrightarrow 0. \end{aligned}$$

Hook lengths of Young diagrams. More generally, let $r \geqslant 1$ be any non-negative integer and let:

$$(\ell) = (\ell_1, \ell_2, \dots, \ell_n)$$

be an arbitrary *partition* of r in at most n parts, namely the sum:

$$\ell_1 + \ell_2 + \cdots + \ell_n = r$$

equals r, and the *parts* ℓ_i are ordered decreasingly:

$$\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0.$$

Let $d_1, d_2, \dots, d_{\ell_1}$ denote the column lengths of the diagram consisting of ℓ_1 blank squares above ℓ_2 blank squares, ..., above ℓ_n blank squares.
With a bit more precisions, we hence can denote our arbitrary partition as:

$$\left[\begin{array}{l} (\ell) = \big(\ell_1, \ell_2, \dots, \ell_{d_1}, 0, \dots, 0\big) \\ \ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{d_1} \geqslant 1, \end{array} \right.$$

and as in Section 4, we will denote by:

$$\mathsf{YD}_{(\ell)} = \mathsf{YD}_{(\ell_1, \ell_2, \dots, \ell_{d_1}, 0, \dots, 0)}$$

the associated Young diagram. The *hook-length* $h_{i,j}$ of the diagram at the square of coordinates (i,j) is equal to:

$$h_{i,j} := \ell_i - j + d_j - i + 1.$$

A preliminary combinatorial fact, useful soon, is as follows.

Theorem. ([16]) *The number of ways to fill in the r blank cases of the diagram $\mathsf{YD}_{(\ell_1,\dots,\ell_n)}$ just with the first r nonnegative integers $1,2,3,\dots,r$ in such a way that the appearing integers do increase (strictly) along each row and do also increase (strictly) along each column is equal to the* integer:

$$\nu_{(\ell)} := \frac{r!}{\prod_{i,j} h_{i,j}}.$$

Schur bundles. On every fiber $\big(T^*_{X,x}\big)^{\otimes r}$ of the r-th tensor bundle $\big(T^*_X\big)^{\otimes r}$ over a point $x \in X$, the full linear group $\mathsf{GL}_n(\mathbb{C}) \ni \mathsf{w}$ acts in a natural way:

$$\mathsf{w} \cdot v^*_{i_1} \otimes v^*_{i_2} \otimes \cdots \otimes v^*_{i_r} := \mathsf{w}(v^*_{i_1}) \otimes \mathsf{w}(v^*_{i_2}) \otimes \cdots \otimes \mathsf{w}(v^*_{i_r}),$$

if by $(v^*_1, v^*_2, \dots, v^*_n)$ one denotes any fixed basis of $T^*_{X,x}$. Since the works of Schur?? at the end of the 19th, it is known (*see* [16]) how one may decompose this action into irreducible (nondecomposable) representations which generate the Schur bundles $\mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)}T^*_X$ that were already considered in Section 4. Let us provide more information here.

A *Young tableau* $\mathsf{YT}_{1,2,\dots,r}$ is a filling of a given Young diagram $\mathsf{YD}_{(\ell_1,\dots,\ell_n)}$ having $r = \ell_1 + \cdots + \ell_n$ blank boxes precisely by means of the first r positive integers $1,2,\dots,r$. Notice *passim* that only a special kind of Young tableaux was considered in the theorem above, namely those which enjoy strict increase both along lines and columns, and such combinatorial objects are usually called *standard Young tableaux*.

Idempotents in the group algebra of permutations. Introduce also the *group algebra* $\mathbb{Q} \cdot \mathfrak{S}_r$ over the permutation group:

$$\mathfrak{S}_r = \mathsf{Perm}\big(\{1,2,\dots,r\}\big),$$

whose general element is a typical sum $\sum_{\sigma \in \mathfrak{S}_r} c_\sigma \cdot \sigma$ having arbitrary rational coefficients $c_\sigma \in \mathbb{Q}$, the addition:

$$\textstyle\sum_{\sigma \in \mathfrak{S}_r} c_\sigma \cdot \sigma + \sum_{\sigma \in \mathfrak{S}_r} d_\sigma \cdot \sigma = \displaystyle\sum_{\sigma \in \mathfrak{S}_r} \big(c_\sigma + d_\sigma\big) \cdot \sigma$$

being obvious and the "multiplication":

$$\Big(\textstyle\sum_{\sigma' \in \mathfrak{S}_r} c_{\sigma'} \cdot \sigma'\Big) \circ \Big(\textstyle\sum_{\sigma'' \in \mathfrak{S}_r} c_{\sigma''} \cdot \sigma''\Big) = \displaystyle\sum_{\sigma' \in \mathfrak{S}_r} \sum_{\sigma'' \in \mathfrak{S}_r} c_{\sigma'}\, c_{\sigma''} \cdot \sigma' \circ \sigma''$$

corresponding naturally to the composition $\sigma' \circ \sigma''$ of permutations. For a given Young tableau $\mathsf{YT}_{1,\ldots,r}$ which shall also be denoted shortly by T, one introduces the following element:

$$e_{\mathsf{T}} := \frac{\nu_{(\ell)}}{r!} \cdot \Big(\sum_{q \in Q_{\mathsf{T}}} \mathsf{sgn}(q) \cdot q \Big) \circ \Big(\sum_{p \in P_{\mathsf{T}}} p \Big) \tag{30}$$

of the group algebra $\mathbb{Q} \cdot \mathfrak{S}_r$, where Q_{T} denotes the set of permutations that preserve the numbers present in each column of T, and where similarly P_{T} denotes the set of permutations that preserve the numbers present in each row of T.

Theorem. ([16]) *This element e_{T} is an idempotent:*

$$e_{\mathsf{T}} \circ e_{\mathsf{T}} = e_{\mathsf{T}},$$

and the identity permutation $\mathsf{Id} \in \mathbb{Q} \cdot \mathfrak{S}_r$ decomposes as the sum of all such idempotents:

$$\mathsf{Id} = \sum_{\substack{\mathsf{T}=\text{Young tableau} \\ \mathsf{Card}(\mathsf{T})=r}} e_{\mathsf{T}}.$$

Canonical decomposition of tensor powers of the cotangent bundle. The symmetric group $\mathfrak{S}_r$ and therefore also the group algebra $\mathbb{Q} \cdot \mathfrak{S}_r$, act on $\big(T^*_X\big)^{\otimes r}$ just by permuting the spots inside the tensor product:

$$\sigma \cdot v_1 \otimes v_2 \otimes \cdots \otimes v_r := v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(r)}.$$

The identity decomposition (30) then yields at any point $x \in X$ the direct sum decomposition of the r-th tensor power of the cotangent space:

$$\big(T^*_{X,x}\big)^{\otimes r} = \bigoplus_{\substack{\mathsf{T}=\text{Young tableau} \\ \mathsf{Card}(\mathsf{T})=r}} = \mathscr{S}^{\mathsf{T}} T^*_{X,x} \quad \text{with} \quad \mathscr{S}^{\mathsf{T}} T^*_{X,x} := e_{\mathsf{T}} \cdot \big(T^*_{X,x}\big)^{\otimes r}.$$

This generalizes the simple well known case $r = 2$:

$$\big(T_{X,x}\big)^{\otimes 2} = \Lambda^2 T^*_{X,x} \oplus \mathrm{Sym}^2 T^*_{X,x}.$$

Theorem. *For any Young Tableau T, a basis of $\mathscr{S}^{\mathsf{T}} T^*_{X,x}$ as a $\mathbb{C}$-vector space is constituted of all vectors of the form:*

$$e_{\mathsf{T}}\big(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}\big),$$

for any choice of integers $i_1, i_2, \ldots, i_r \in \{1, \ldots, n\}$ having the property that the filling of the blank boxes of the underlying Young diagram with the integers $i_1, \ldots, i_{\ell_1}$ in the first line, then with the integers $i_{\ell_1+1}, \ldots, i_{\ell_1+\ell_2}$ in the second line, and so on, provides at the end a semi-standard Young tableau*, in the sense that integers are always nondecreasing when read in each row from left to right, and are always increasing (strictly) when read in each column from top to bottom.*

It turns out ([16, 1, 18, 4, 29]) that, if two arbitrary Young tableaux T and $\widetilde{\mathsf{T}}$ correspond to the same Young diagram, *i.e.* to the same partition, then $\mathscr{S}^{\mathsf{T}}T^*_{X,x}$ and $\mathscr{S}^{\widetilde{\mathsf{T}}}T^*_{X,x}$ are *isomorphic*. Moreover, for any T, the linear action of $\mathsf{GL}_n(\mathbb{C})$ being compatible with the changes of chart on X and on T^*_X, one may show that the various fibers $\mathscr{S}^{\mathsf{T}}T^*_{X,x}$ organize coherently as a holomorphic bundle over X. In conclusion, a fundamental Schur bundle decomposition theorem holds which gives the complete generalization of, say:

$$\begin{aligned}\big(T^*_X\big)^{\otimes 2} &= \mathscr{S}^{(2,0,\ldots,0)}T^*_X \oplus \mathscr{S}^{(1,1,0,\ldots,0)}T^*_X,\\ \big(T^*_X\big)^{\otimes 3} &= \mathscr{S}^{(3,0,\ldots,0)}T^*_X \oplus \big[\mathscr{S}^{(2,1,0,\ldots,0)}T^*_X\big]^{\oplus 2} \oplus \mathscr{S}^{(1,1,1,0,\ldots,0)}T^*_X,\end{aligned}$$

provided X is of dimension $\geqslant 3$; the last factor is dropped when $\dim X = 2$.

Theorem. *For any integer $r \geqslant 1$, the r-th tensor power of the cotangent bundle T^*_X of an arbitrary n-dimensional complex manifold X splits up as a direct sum of so-called Schur bundles:*

$$\big(T^*_X\big)^{\otimes r} = \bigoplus_{\substack{\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n \geqslant 0\\ \ell_1+\ell_2+\cdots+\ell_n = r}} \big(\mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)}T^*_X\big)^{\oplus \nu(\ell)}$$

indexed by all the partitions (ℓ) of r. The rank of $\mathscr{S}^{(\ell_1,\ldots,\ell_n)}T^*_X$ as a complex vector bundle equals:

$$\operatorname{rank}\big(\mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)}T^*_X\big) = \prod_{1 \leqslant i < j \leqslant n} \left(\frac{\ell_i - \ell_j}{i-j} + 1\right),$$

and the integer multiplicities:

$$\nu_{(\ell)} = \frac{r!}{\prod_{i,j} h_{i,j}}$$

appearing in the decomposition are expressible in terms of the hook lengths $h_{i,j}$ of the concerned Young diagram $\mathsf{YD}_{(\ell_1,\ell_2,\ldots,\ell_n)}$.

Dividing by K_X. Our main goal will now be to control the cohomology of the $\mathscr{S}^{(\ell_1,\ldots,\ell_n)}T^*_X$ by a formula which will complement the inequality of the theorem on p. 39, in the case where ℓ_n is large (whence all the ℓ_i are so too). It is then natural to use the known formula:

$$\mathscr{S}^{(\ell_1,\ldots,\ell_n)}T^*_X \otimes K_X = \mathscr{S}^{(\ell_1,\ldots,\ell_n)}T^*_X \otimes \mathscr{S}^{(1,\ldots,1)}T^*_X = \mathscr{S}^{(\ell_1+1,\ldots,\ell_n+1)}T^*_X$$

under the subtraction form:

$$\begin{aligned}\mathscr{S}^{(\ell_1,\ldots,\ell_{n-1},\ell_n)}T^*_X &= \mathscr{S}^{(\ell_1-\ell_n,\ldots,\ell_{n-1}-\ell_n,0)}T^*_X \otimes (K_X)^{\otimes \ell_n}\\ &= \mathscr{S}^{(\ell_1-\ell_n,\ldots,\ell_{n-1}-\ell_n,0)}T^*_X \otimes \mathscr{O}_X\big(\ell_n(d-n-2)\big),\end{aligned}$$

which underlines twisting by a certain $\mathscr{O}_X(t)$. On the occasion, it is known thanks to analytical tools (cf. Section 6 in [12]) that if $\mathscr{E}$ is *any* holomorphic

vector bundle on the hypersurface $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ and if $\mathscr{L}$ is an ample (or even nef) *line* bundle on X, then:

$$\dim H^q\big(X,\ \mathscr{E} \otimes \mathscr{L}^{\otimes k}\big) = \mathsf{O}(k^{n-q}),$$

for any $q = 0, 1, 2, \dots, n$. Using purely algebraic tools, what we will do now is to make this estimate much more effective in the case we are interested in, namely when $\mathscr{E} = \mathscr{S}^{(\ell_1-\ell_n,\dots,\ell_{n-1}-\ell_n,0)}T^*_X$ and when $\mathscr{L} = \mathscr{O}_X(1)$ on a general type hypersurface $X \subset \mathbb{P}^{n+1}$; in this case, X is of degree $d \geqslant n+3$, whence:

$$K_X = \mathscr{O}_X(d-n-2) = \mathscr{O}_X(1)^{\otimes(d-n-2)}$$

is ample of course, so that the exponent $k := \ell_n(d-n-2)$ in:

$$(K_X)^{\ell_n} = \big(\mathscr{O}_X(1)\big)^{\otimes(\ell_n(d-n-2))} = \mathscr{L}^{\otimes(\ell_n(d-n-2))}$$

is positive and in fact large. However, the Landau-type estimate "O" above provided by analytic techniques is not precise enough and we need instead explicit *inequalities*. To achieve more effective estimates, three fundamental exact sequences of holomorphic vector bundles due to Lascoux ([23]) and to Brückmann ([1]) will be very helpful. Thus our goal is to study the cohomology of the twisted Schur bundles:

$$\mathscr{S}^{(\ell'_1,\dots,\ell'_{n-1},0)}T^*_X \otimes \mathscr{O}_X(t),$$

when t is large.

First fundamental (long) exact sequence. Dualizing the Euler exact sequence:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(0) \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(1)^{\oplus(n+2)} \longrightarrow T_{\mathbb{P}^{n+1}} \longrightarrow 0,$$

we get as a starter the exact sequence:

$$0 \longrightarrow T^*_{\mathbb{P}^{n+1}} \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(-1)^{\oplus(n+2)} \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(0) \longrightarrow 0.$$

The procedure explained by Brückmann in [1] of taking the r-th tensor power of the extracted complex composed of the last two bundles:

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(-1)^{\oplus(n+2)} \longrightarrow \mathscr{O}_{\mathbb{P}^{n+1}}(0) \longrightarrow 0$$

and more generally, of taking any of its Schur powers, provides a useful long exact sequence of holomorphic vector bundles which gives a free resolution of $\mathscr{S}^{(\ell_1,\dots,\ell_n,\ell_{n+1})}T^*_{\mathbb{P}^{n+1}}$ on $\mathbb{P}^{n+1}$. Instead of using the same letter $\mathscr{S}$ for Schur bundles over $\mathbb{P}^{n+1}$ and over X, we shall, in order to underline a clearly visible distinction between $\mathbb{P}^{n+1}$ and X, write:

$$\mathscr{S}^{(\ell_1,\dots,\ell_n,\ell_{n+1})}T^*_{\mathbb{P}^{n+1}} \overset{\text{notation}}{\equiv} \Omega^{(\ell_1,\dots,\ell_n,\ell_{n+1})}_{\mathbb{P}^{n+1}},$$

using the Greek letter[27] Ω with '$\mathbb{P}^{n+1}$' placed at the lower index place.

[27] Justification: in several articles, the letter Ω is employed to denote the bundles Ω^k or $\Omega^k T^*_X$, $0 \leqslant k \leqslant n$, that we denoted by $\Lambda^k T^*_X$ above.

Let now T be a Young tableau with r boxes and with row lengths $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_{n+1} \geqslant 0$, hence of depth $\leqslant n+1$. For convenient abbreviation, we introduce the general notation:

$$\Delta\big(\theta_1, \theta_2, \ldots, \theta_K\big) := \prod_{1 \leqslant i < j \leqslant K} \big(\theta_i - \theta_j\big)$$

which is, up to sign, the value:

$$\begin{vmatrix} 1 & \theta_1 & \theta_1^2 & \cdots & \theta_1^{K-1} \\ 1 & \theta_2 & \theta_2^2 & \cdots & \theta_2^{K-1} \\ .. & .. & .. & \cdots & .. \\ 1 & \theta_K & \theta_K^2 & \cdots & \theta_K^{K-1} \end{vmatrix} = (-1)^{\frac{K(K-1)}{2}} \Delta\big(\theta_1, \theta_2, \ldots, \theta_K\big)$$

of a corresponding Van der Monde determinant.

Theorem. ([1]) *Let $d_1 = \mathrm{depth}(\mathsf{T})$ be the depth of T, which is $\leqslant n+1$, let $r = \ell_1 + \cdots + \ell_{d_1}$ be the number of boxes of T, set:*

$$t_i := r + \ell_i - i$$

for all $i = 1, 2, \ldots, n+1, n+2$ with of course:

$$t_{d_1+1} = r - d_1 - 1, \ \ldots\ldots, \ t_{n+1} = r - n - 1, \ \ t_{n+2} = r - n - 2,$$

and define the rational number:

$$b_0 := \tfrac{1}{1!\,2!\cdots n!\,(n+1)!} \cdot \Delta\big(t_1, \ldots, t_{n+1}, t_{n+2}\big),$$

together with, for any $s = 1, 2, \ldots, d_1$, the rational numbers:

$$b_s := \tfrac{1}{1!\,2!\cdots n!\,(n+1)!} \cdot \sum_{1 \leqslant i_1 < \cdots < i_s \leqslant d_1} \Delta\big(t_1, t_2, \ldots, t_{i_1} - 1, \ldots, t_{i_s} - 1, \ldots, t_{n+1}, t_{n+2}\big).$$

Then there is a long exact sequence of holomorphic vector bundles over $\mathbb{P}^{n+1}$ of the form:

$$0 \longrightarrow \Omega_{\mathbb{P}^{n+1}}^{(\ell_1, \ldots, \ell_{d_1}, 0, \ldots, 0)} \longrightarrow \bigoplus_{b_0} \mathscr{O}_{\mathbb{P}^{n+1}}(-r) \longrightarrow \bigoplus_{b_1} \mathscr{O}_{\mathbb{P}^{n+1}}(-r+1) \longrightarrow \cdots$$
$$\cdots \longrightarrow \bigoplus_{b_{d_1}} \mathscr{O}_{\mathbb{P}^{n+1}}(-r+d_1) \longrightarrow 0.$$

Then tensoring by $\mathscr{O}_{\mathbb{P}^{n+1}}(t)$ with an arbitrary $t \in \mathbb{Z}$, we get the useful:

(31)
$$0 \longrightarrow \Omega_{\mathbb{P}^{n+1}}^{(\ell_1, \ldots, \ell_{d_1}, 0, \ldots, 0)} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t) \longrightarrow \bigoplus_{b_0} \mathscr{O}_{\mathbb{P}^{n+1}}(t-r) \longrightarrow \bigoplus_{b_1} \mathscr{O}_{\mathbb{P}^{n+1}}(t-r+1) \longrightarrow \cdots$$
$$\cdots \longrightarrow \bigoplus_{b_{d_1}} \mathscr{O}_{\mathbb{P}^{n+1}}(t-r+d_1) \longrightarrow 0.$$

Second fundamental (short) exact sequence. Because any Schur bundle $\Omega^{(\ell_1,\dots,\ell_n,\ell_{n+1})}_{\mathbb{P}^{n+1}}$ over $\mathbb{P}^{n+1}$ is, according to what precedes, a locally free sheaf of $\mathscr{O}_{\mathbb{P}^{n+1}}$-modules of finite rank $\prod_{1\leqslant i<j\leqslant n+1}\big(\frac{\ell_i-\ell_j}{j-i}+1\big)$, a tensorisation of the normal exact sequence (28) on p. 74 yields the general short exact sequence ([5, 29]):

$$\begin{aligned}0 \longrightarrow \Omega^{(\ell_1,\dots,\ell_n,\ell_{n+1})}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(-d) \longrightarrow \Omega^{(\ell_1,\dots,\ell_n,\ell_{n+1})}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(0) \longrightarrow \\ \longrightarrow \Omega^{(\ell_1,\dots,\ell_n,\ell_{n+1})}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(0) \longrightarrow 0.\end{aligned}$$

Tensoring in addition again by $\mathscr{O}_{\mathbb{P}^{n+1}}(t)$ where $t\in\mathbb{Z}$ is arbitrary, knowing $\mathscr{O}_X(0)\otimes\mathscr{O}_{\mathbb{P}^{n+1}}(t)=\mathscr{O}_X(t)$, we deduce the general form of this (second, short) exact sequence that will be useful below:

(32)
$$\begin{aligned}0 \longrightarrow \Omega^{(\ell_1,\dots,\ell_n,\ell_{n+1})}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t-d) \longrightarrow \Omega^{(\ell_1,\dots,\ell_n,\ell_{n+1})}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t) \longrightarrow \\ \longrightarrow \Omega^{(\ell_1,\dots,\ell_n,\ell_{n+1})}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(t) \longrightarrow 0.\end{aligned}$$

Here, we make the convention similar to (29) on p. 74 that when all the ℓ_i are zero, $\Omega^{(0,\dots,0,0)}_{\mathbb{P}^{n+1}}$ identifies to $\mathscr{O}_{\mathbb{P}^{n+1}}(0)$, whence in this case the written exact sequence reduces just to (28) on p. 74, tensored of course by $\mathscr{O}_{\mathbb{P}^{n+1}}(t)$.

Third fundamental exact sequence. Lastly, starting from the cotangential normal exact sequence:

$$0 \longrightarrow \mathscr{O}_X(-d) \longrightarrow T^*_{\mathbb{P}^{n+1}}|_X \longrightarrow T^*_X \longrightarrow 0, \tag{33}$$

(recall that $T^*_{\mathbb{P}^{n+1}}|_X = T^*_{\mathbb{P}^{n+1}}\otimes\mathscr{O}_X(0)$), Brückmann established that the Schur power of the extracted complex:

$$0 \longrightarrow \mathscr{O}_X(-d) \longrightarrow T^*_{\mathbb{P}^{n+1}}\otimes\mathscr{O}_X(0) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

provides a free resolution of $\mathscr{S}^{(\ell_1,\dots,\ell_n)}T^*_X$ (Theorem 3 in [1]) which may be written in great details as follows when $\ell_n\geqslant 1$:

$$\begin{aligned}0 \longrightarrow \bigoplus_{\substack{\delta_1+\cdots+\delta_n=n\\ \delta_i=0 \text{ or } 1}} \Omega^{(\ell_1,\dots,\ell_n,0)-(\delta_1,\dots,\delta_n,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(-nd) \longrightarrow \cdots \\ \cdots \longrightarrow \bigoplus_{\substack{\delta_1+\cdots+\delta_n=k\\ \delta_i=0 \text{ or } 1}} \Omega^{(\ell_1,\dots,\ell_n,0)-(\delta_1,\dots,\delta_n,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(-kd) \longrightarrow \cdots \\ \cdots \longrightarrow \Omega^{(\ell_1,\dots,\ell_n,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(0) \longrightarrow \mathscr{S}^{(\ell_1,\dots,\ell_n)}T^*_X \longrightarrow 0.\end{aligned}$$

Notice that the last upper entry ℓ_{n+1} of each Ω is zero. Of course, the direct sum for the first entry reduces just to the single term:

$$\Omega^{(\ell_1-1,\dots,\ell_n-1,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(-nd).$$

In full generality, if d_1 denotes the depth of the considered Young diagram, hence if one has $\ell_1 \geqslant \cdots \geqslant \ell_{d_1} \geqslant 1$ but $0 = \ell_{d_1+1} = \cdots = \ell_n = \ell_{n+1}$, the locally free resolution of $\mathscr{S}^{(\ell_1,\ldots,\ell_{d_1},0,\ldots,0)}T^*_X$ reads ([1]):

$$0 \longrightarrow \bigoplus_{\substack{\delta_1+\cdots+\delta_{d_1}=d_1\\ \delta_i=0 \text{ or } 1}} \Omega^{(\ell_1,\ldots,\ell_{d_1},0,\ldots,0,0)-(\delta_1,\ldots,\delta_{d_1},0,\ldots,0,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(-d_1 d) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{\substack{\delta_1+\cdots+\delta_{d_1}=k\\ \delta_i=0 \text{ or } 1}} \Omega^{(\ell_1,\ldots,\ell_{d_1},0,\ldots,0,0)-(\delta_1,\ldots,\delta_{d_1},0,\ldots,0,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(-kd) \longrightarrow \cdots$$

$$\cdots \longrightarrow \Omega^{(\ell_1,\ldots,\ell_{d_1},0,\ldots,0,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(0) \longrightarrow \mathscr{S}^{(\ell_1,\ldots,\ell_{d_1},0,\ldots,0)}T^*_X \longrightarrow 0,$$

hence it just looks like a truncation of the preceding resolution valid when $d_1 = n$. Tensoring this by $\mathscr{O}_X(t)$, we finally get what will be useful below:

(34)

$$0 \longrightarrow \bigoplus_{\substack{\delta_1+\cdots+\delta_{d_1}=d_1\\ \delta_i=0 \text{ or } 1}} \Omega^{(\ell_1,\ldots,\ell_{d_1},0,\ldots,0,0)-(\delta_1,\ldots,\delta_{d_1},0,\ldots,0,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(t-d_1 d) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{\substack{\delta_1+\cdots+\delta_{d_1}=k\\ \delta_i=0 \text{ or } 1}} \Omega^{(\ell_1,\ldots,\ell_{d_1},0,\ldots,0,0)-(\delta_1,\ldots,\delta_{d_1},0,\ldots,0,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(t-kd) \longrightarrow \cdots$$

$$\cdots \longrightarrow \Omega^{(\ell_1,\ldots,\ell_{d_1},0,\ldots,0,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_X(t) \longrightarrow \mathscr{S}^{(\ell_1,\ldots,\ell_{d_1},0,\ldots,0)}T^*_X \otimes \mathscr{O}_X(t) \longrightarrow 0.$$

Cohomology of Schur bundles over $\mathbb{P}^{n+1}$. In [1] too, using the first exact sequence above plus further arguments, Brückmann established the following theorem which computes completely the dimensions of all the cohomology groups of twisted Schur bundles over $\mathbb{P}^{n+1}$. As above, for fixed $n+1 \geqslant 2$ and for fixed $\ell_1 \geqslant \cdots \geqslant \ell_n \geqslant \ell_{n+1} \geqslant 0$, we introduce the integers:

$$t_i := \ell_i - i + \textstyle\sum_{i=1}^{n+1} \ell_i \qquad (i = 1 \cdots n,\ n+1),$$

which, visibly, are ordered decreasingly:

$$t_1 > t_2 > \cdots > t_n > t_{n+1}.$$

Theorem. ([1]) *For any $t \in \mathbb{Z}$, the Euler-Poincaré characteristic of $\Omega^{(\ell_1,\ldots,\ell_n,\ell_{n+1})}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t)$ is equal to:*

$$\chi(t) := \chi\big(\mathbb{P}^{n+1},\, \Omega^{(\ell_1,\ldots,\ell_n,\ell_{n+1})}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t)\big) = \tfrac{1}{1!\,2!\cdots n!\,(n+1)!} \prod_{1\leqslant i<j\leqslant n+1} (t_i - t_j) \prod_{1\leqslant i\leqslant n} (t - t_i),$$

whence it vanishes for t equal to each one of the t_i. Furthermore, as t varies in $\mathbb{Z}$, at most one of the cohomology dimensions:

$$h^q(t) := \dim H^q\big(\mathbb{P}^{n+1},\, \Omega^{(\ell_1,\ldots,\ell_n,\ell_{n+1})}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t)\big)$$

may be nonzero, and more precisely, $h^q(t)$ is nonzero and equal to $(-1)^q\chi(t)$ if and only if $t_{q+1}+1 \leqslant t \leqslant t_q - 1$, while the other $h^{q'}(t)$ do vanish for all t in the same range. In particular, for all:

$$t \geqslant \ell_1 + \textstyle\sum_{i=1}^{n+1} \ell_i, \tag{35}$$

all the positive cohomology dimensions vanish:

$$0 = \dim H^q\big(\mathbb{P}^{n+1},\, \Omega_{\mathbb{P}^{n+1}}^{(\ell_1,\dots,\ell_n,\ell_{n+1})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t)\big) \qquad {\scriptstyle (q=1,\,2\cdots n)}.$$

Applications. For an application to the study of the cohomology of Schur bundles over $X^n \subset \mathbb{P}^{n+1}$, we shall apply the above theorems specifically to the Young diagrams $\mathsf{YD}_{(\ell_1,\dots,\ell_n,0)}$ of depth $d_1 \leqslant n = \dim X$, with $\ell_{n+1} = 0$ in order to gain the following complement to the theorem on p. 39.

Theorem. *Let $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a geometrically smooth projective algebraic complex hypersurface of general type, i.e. of degree $d \geqslant n+3$, and let $\ell = (\ell_1, \dots, \ell_{n-1}, \ell_n)$ with $\ell_1 \geqslant \cdots \geqslant \ell_{n-1} \geqslant \ell_n \geqslant 1$. If:*

$$\ell_n \geqslant \tfrac{1}{d-n-2}\big\{n(d-1) + \ell_1 - \ell_n + \textstyle\sum_{i=1}^{n-1}(\ell_i - \ell_n)\big\},$$

then all the positive cohomologies vanish:

$$0 = H^q\big(X,\, \mathscr{S}^{(\ell_1,\dots,\ell_{n-1},\ell_n)}T_X^*\big) \qquad {\scriptstyle (q=1,\,2\cdots n)}.$$

Proof. As anticipated above, after division by $(K_X)^{\ell_n}$, we are lead back to examining the cohomology of:

$$\mathscr{S}^{(\ell_1-\ell_n,\dots,\ell_{n-1}-\ell_n,0)}T_X^* \otimes \mathscr{O}_X\big(\ell_n(d-n-2)\big).$$

A bit more generally, using the second and the third exact sequences (32) and (34), we shall examine when the positive cohomologies of:

$$\mathscr{S}^{(\ell'_1,\dots,\ell'_{n-1},0)}T_X^* \otimes \mathscr{O}_X(t')$$

do all vanish, and afterward, we shall set:

$$\ell'_1 := \ell_1 - \ell_n,\ \dots,\ \ell'_{n-1} := \ell_{n-1} - \ell_n \quad \text{and} \quad t' := \ell_n(d-n-2).$$

We assume first that $\ell'_{n-1} \geqslant 1$ and we shall discuss the quite similar case $\ell'_{n-1} = 0$ afterward. The consideration of the third exact sequence (34) with $d_1 = n-1$ and $(\ell'_1, \dots, \ell'_{n-1}, 0)$ instead of $(\ell_1, \dots, \ell_{n-1}, 0)$ then necessarily conducts us to the study of $\mathscr{O}_X$-twisted Schur bundles over $\mathbb{P}^{n+1}$:

$$\Omega_{\mathbb{P}^{n+1}}^{(\ell''_1,\dots,\ell''_{n-1},0)} \otimes \mathscr{O}_X(t'')$$

whose Young diagram exponents ℓ''_i have values:

$$(\ell''_1, \dots, \ell''_{n-1}, 0) = (\ell'_1, \dots, \ell'_{n-1}, 0) - (\delta'_1, \dots, \delta'_{n-1}, 0)$$

shifted a bit from the values of the ℓ'_i, where $\delta'_1 + \cdots + \delta'_{n-1} = k$ for $k = 0, 1, \dots, n-1$, with of course $\delta'_i = 0$ or 1. So to begin with, it is advisable to study the cohomology of these $\mathscr{O}_X$-twisted Schur bundles over $\mathbb{P}^{n+1}$.

To this aim, we look at the second (short) exact sequence (32) written with:

$$(\ell_1, \dots, \ell_{n-1}, \ell_n, \ell_{n+1}) := (\ell''_1, \dots, \ell''_{n-1}, 0, 0),$$

for some arbitrary $\ell''_1 \geqslant \cdots \geqslant \ell''_{n-1} \geqslant 0$ and we abbreviate this exact sequence as:

$$0 \longrightarrow \mathscr{P} \longrightarrow \mathscr{Q} \longrightarrow \mathscr{R} \longrightarrow 0,$$

where $\mathscr{P} \to \mathbb{P}^{n+1}$, $\mathscr{Q} \to \mathbb{P}^{n+1}$ and $\mathscr{R} \to X$ are the bundles:

$$\begin{aligned}
\mathscr{P} &:= \Omega^{(\ell''_1,\dots,\ell''_{n-1},0,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t'' - d),\\
\mathscr{Q} &:= \Omega^{(\ell''_1,\dots,\ell''_{n-1},0,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t''),\\
\mathscr{R} &:= \Omega^{(\ell''_1,\dots,\ell''_{n-1},0,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{X}(t''),
\end{aligned}$$

so that all the cohomology dimensions of $\mathscr{P}$ and of $\mathscr{Q}$ are known thanks to Brückmann's theorem on p. 81. Then in the long exact cohomology sequence associated to the short exact sequence:

$$\begin{aligned}
0 &\longrightarrow H^0(\mathbb{P}^{n+1},\, \mathscr{P}) \longrightarrow H^0(\mathbb{P}^{n+1},\, \mathscr{Q}) \longrightarrow H^0(X,\, \mathscr{R}) \longrightarrow\\
&\longrightarrow \underline{H^1(\mathbb{P}^{n+1},\, \mathscr{P})}_{\circ} \longrightarrow \underline{H^1(\mathbb{P}^{n+1},\, \mathscr{Q})}_{\circ} \longrightarrow H^1(X,\, \mathscr{R}) \longrightarrow\\
&\longrightarrow \underline{H^2(\mathbb{P}^{n+1},\, \mathscr{P})}_{\circ} \longrightarrow \underline{H^2(\mathbb{P}^{n+1},\, \mathscr{Q})}_{\circ} \longrightarrow H^2(X,\, \mathscr{R}) \longrightarrow \cdots\\
\cdots &\longrightarrow \underline{H^n(\mathbb{P}^{n+1},\, \mathscr{P})}_{\circ} \longrightarrow \underline{H^n(\mathbb{P}^{n+1},\, \mathscr{Q})}_{\circ} \longrightarrow H^n(X,\, \mathscr{R}) \longrightarrow\\
\cdots &\longrightarrow \underline{H^{n+1}(\mathbb{P}^{n+1},\, \mathscr{P})}_{\circ} \longrightarrow \underline{H^{n+1}(\mathbb{P}^{n+1},\, \mathscr{Q})}_{\circ} \longrightarrow 0
\end{aligned}$$

(the last 0 because $\mathscr{R} \to X$ is a bundle over an n-dimensional basis), all the underlined terms will vanish, namely one will have:

$$\begin{aligned}
0 &= H^q\big(\mathbb{P}^{n+1},\ \Omega^{(\ell''_1,\dots,\ell''_{n-1},0,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t'' - d)\big) \qquad {\scriptstyle (q=1,\, 2\cdots n,\, n+1)},\\
0 &= H^q\big(\mathbb{P}^{n+1},\ \Omega^{(\ell''_1,\dots,\ell''_{n-1},0,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(t'')\big) \qquad {\scriptstyle (q=1,\, 2\cdots n,\, n+1)},
\end{aligned}$$

as soon as the following two inequalities are satisfied by t'':

$$\begin{aligned}
t'' - d &\geqslant \ell''_1 + \textstyle\sum_{i=1}^{n-1} \ell''_i,\\
t'' &\geqslant \ell''_1 + \textstyle\sum_{i=1}^{n-1} \ell''_i,
\end{aligned}$$

as is guaranteed by the inequality (35) of the theorem on p. 81. But the first inequality obviously entails the second one, hence we deduce that all positive cohomologies:

$$0 = H^q\big(X,\ \Omega^{(\ell''_1,\dots,\ell''_{n-1},0,0)}_{\mathbb{P}^{n+1}} \otimes \mathscr{O}_{X}(t'')\big) \qquad {\scriptstyle (q=1,\, 2\cdots n)}$$

of $\mathscr{R}$ vanish as soon as:

$$t'' \geqslant d + \ell''_1 + \textstyle\sum_{i=1}^{n-1} \ell''_i. \tag{36}$$

We observe that this fact is valid also when $\ell''_{n_1+1} = \cdots = \ell''_{n-1}$ for some largest integer $n_1 \geqslant 0$ with $\ell''_{n_1} \geqslant 1$, because the second exact sequence (32) we used is subjected to no restriction.

We now come to dealing with the third exact sequence (34). Cutting a long exact sequence in short exact sequences, one may establish the following standard lemma, used e.g. in [30].

Lemma. *Consider a holomorphic vector bundle $\mathscr{S} \to X$ equipped with a free resolution of length $\leqslant n$ provided by a long exact sequence of holomorphic vector bundles $\mathscr{A}^0, \mathscr{A}^1, \ldots, \mathscr{A}^n$ over X:*

$$0 \longrightarrow \mathscr{A}^n \longrightarrow \mathscr{A}^{n-1} \longrightarrow \cdots \longrightarrow \mathscr{A}^1 \longrightarrow \mathscr{A}^0 \longrightarrow \mathscr{S} \longrightarrow 0.$$

Then in order that all the positive cohomology groups vanish:

$$0 = H^1(X, \mathscr{S}) = \cdots = H^n(X, \mathscr{S}),$$

it suffices that:

$$\begin{aligned} 0 = H^1(X, \mathscr{A}^0) = H^2(X, \mathscr{A}^1) = H^3(X, \mathscr{A}^2) &= \cdots = H^n(X, \mathscr{A}^{n-1}) \\ 0 = H^2(X, \mathscr{A}^0) = H^3(X, \mathscr{A}^1) &= \cdots = H^n(X, \mathscr{A}^{n-2}) \\ 0 = H^3(X, \mathscr{A}^0) &= \cdots = H^n(X, \mathscr{A}^{n-3}) \\ &\cdots\cdots\cdots\cdots \\ 0 &= H^n(X, \mathscr{A}^0). \end{aligned}$$

So as said a short while ago, we aim to apply this lemma when looking at the third exact sequence (34) which, for the case we are interested in, writes precisely under the form:

$$0 \longrightarrow \bigoplus_{\substack{\delta'_1+\cdots+\delta'_{n-1}=n-1 \\ \delta'_i=0 \text{ or } 1}} \Omega_{\mathbb{P}^{n+1}}^{(\ell'_1,\ldots,\ell'_{n-1},0,0)-(\delta'_1,\ldots,\delta'_{n-1},0,0)} \otimes \mathscr{O}_X\big(t' - (n-1)d\big) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{\substack{\delta'_1+\cdots+\delta'_{n-1}=k \\ \delta'_i=0 \text{ or } 1}} \Omega_{\mathbb{P}^{n+1}}^{(\ell'_1,\ldots,\ell'_{n-1},0,0)-(\delta'_1,\ldots,\delta'_{n-1},0,0)} \otimes \mathscr{O}_X\big(t' - kd\big) \longrightarrow \cdots$$

$$\cdots \longrightarrow \Omega_{\mathbb{P}^{n+1}}^{(\ell'_1,\ldots,\ell'_{n-1},0,0)} \otimes \mathscr{O}_X(t') \longrightarrow \mathscr{S}^{(\ell'_1,\ldots,\ell'_{n-1},0)}\mathscr{O}_X(t') \longrightarrow 0.$$

In the notations of the lemma, the resolution of:

$$\mathscr{S} := \mathscr{S}^{(\ell'_1,\ldots,\ell'_{n-1},0)} \otimes \mathscr{O}_X(t')$$

is hence of length $n-1$ when we set:

$$\mathscr{A}^k := \bigoplus_{\substack{\delta'_1+\cdots+\delta'_{n-1}=k \\ \delta'_i=0 \text{ or } 1}} \Omega_{\mathbb{P}^{n+1}}^{(\ell'_1,\ldots,\ell'_{n-1},0,0)-(\delta'_1,\ldots,\delta'_{n-1},0,0)} \otimes \mathscr{O}_X\big(t' - kd\big)$$

$$(k = 0,\, 1 \cdots n-1).$$

Then for the lemma to yield the vanishing of all the positive cohomologies of $\mathscr{S} = \mathscr{S}^{(\ell'_1,\dots,\ell'_{n-1},0)} \otimes \mathscr{O}_X(t')$, it is evidently sufficient that plainly all positive cohomologies of the $\mathscr{A}^k$ vanish:

$$0 = H^q(X,\, \mathscr{A}^k) \qquad\qquad {\scriptstyle (q=1,\,2\cdots n\,;\ k=0,\,1\cdots n-1)},$$

which is more than what is required in fact. But since each $\mathscr{A}^k$ is a direct sum, it even suffices that:

$$0 = H^q\big(X,\ \Omega_{\mathbb{P}^{n+1}}^{(\ell'_1,\dots,\ell'_{n-1},0,0)-(\delta'_1,\dots,\delta'_{n-1},0,0)} \otimes \mathscr{O}_X(t'-kd)\big)$$
$$\scriptstyle (q=1,\,2\cdots n\,;\ \delta'_1+\cdots+\delta'_{n-1}=k\,;\ k=0,\,1\cdots n-1).$$

According to (36), this holds true provided all the following inequalities are satisfied:

$$t'-kd \geqslant d+\ell'_1-\delta'_1+\textstyle\sum_{i=1}^{n-1}(\ell'_i-\delta'_i),$$

for every $k=0,1,\dots,n-1$ and every $\delta'_1,\dots,\delta'_{n-1}\in\{0,1\}$ with $\delta'_1+\cdots+\delta'_{n-1}=k$. But since $-\delta'_i\leqslant 1$ and since $\sum(-\delta'_i)=-k$, it suffices that, firstly for $k=0,1,\dots,n-2$:

$$t'-kd \geqslant d+\ell'_1+\textstyle\sum_{i=1}^{n-1}\ell'_i-k,$$

and secondly for $k=n-1$, whence $-\delta'_1=-1$ necessarily:

$$t'-(n-1)d \geqslant d+\ell'_1-1+\textstyle\sum_{i=1}^{n-1}\ell'_i-(n-1). \tag{37}$$

But this last inequality, rewritten under the form:

$$t' \geqslant n(d-1)+\ell'_1+\textstyle\sum_{i=1}^{n-1}\ell'_i$$

visibly entails all the inequalities for $k=0,1,\dots,n-2$. Lastly, replacing $t'=\ell_n(d-n-2)$ and the $\ell'_i=\ell_i-\ell_n$ by their values, we finally come to the numerical condition claimed by the theorem for positive cohomologies of the $\mathscr{S}^{(\ell_1,\dots,\ell_{n-1},\ell_n)}T^*_X$ to vanish.

To conclude the argument, it only remains to examine what happens with the case, left aside, when $\ell'_{n-1}=0$. In this case, there is a nonnegative integer $n_1\leqslant n-2$ with $\ell'_1\geqslant\cdots\geqslant\ell'_{n_1}\geqslant 1$ while $0=\ell'_{n_1+1}=\cdots=\ell'_{n-1}$. At first, if $n_1=0$, i.e. if all the ℓ_i are equal to ℓ_n, then $\mathscr{S}^{(\ell_n,\dots,\ell_n)}T^*_X = \mathscr{O}_X\big(\ell_n(d-n-2)\big)$ reduces to a standard line bundle $\mathscr{O}_X(t')$, and it is well known that:

$$0 = H^q\big(X,\ \mathscr{O}_X(t')\big) \qquad\qquad {\scriptstyle (q=1,\,2\cdots n)}$$

whenever $t'\geqslant 0$.

Therefore, we may assume that n_1 satisfies $1\leqslant n_1\leqslant n-2$. As before, the subtraction of $(K_X)^{\ell_n}$ yields:

$$\ell'_1=\ell_1-\ell_n,\ \dots,\ \ell_{n'_1}=\ell_{n_1}-\ell_n \quad\text{and}\quad 0=\ell'_{n_1+1}=\cdots=\ell'_{n-1}=\ell'_n,$$

and again as always $t' = \ell_n(d-n-2)$. In the third exact sequence, the factors then are:

$$\mathscr{A}^k = \bigoplus_{\substack{\delta'_1+\cdots+\delta'_{n_1}=k \\ \delta'_i=0 \text{ or } 1}} \Omega_{\mathbb{P}^{n+1}}^{(\ell'_1,\ldots,\ell'_{n_1},0,\ldots,0,0)-(\delta'_1,\ldots,\delta'_{n_1},0,\ldots,0,0)} \otimes \mathscr{O}_X(t'-kd)$$

$$(k=0,\,1\cdots n_1),$$

so the positive cohomologies vanish all provided that:

$$t'-kd \geqslant d+\ell'_1-\delta'_1+\textstyle\sum_{i=1}^{n_1}(\ell'_i-\delta'_i) \qquad (k=0,\,1\cdots n_1\,;\ \delta'_1+\cdots+\delta'_{n_1}=n_1),$$

and because $k \leqslant n_1 \leqslant n-2$, these inequalities are all less stringent than the one (37) we found previously in the case when $\ell'_{n-1} \geqslant 1$ (or equivalently, when $n_1 = n-1$). This completes the proof of the theorem. □

§12. Asymptotic cohomology vanishing

Synthesis: uniform majoration for the cohomology of Schur bundles. Two cohomology controls have been achieved. Firstly, according to the theorem stated above on p. 82 and just proved, when:

$$\ell_n \geqslant \tfrac{1}{d-n-2}\big\{n(d-1)+\ell_1-\ell_n+\textstyle\sum_{i=1}^{n-1}(\ell_i-\ell_n)\big\},$$

the positive cohomologies of Schur bundles vanish:

$$0 = h^q\big(X,\,\mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)}T_X^*\big) \qquad (q=1,2\cdots n).$$

Secondly, according to the theorem stated on p. 39, when:

$$|\ell| \geqslant 1+2n^2+(n+1)(d-n-2),$$

the positive cohomologies enjoy a majoration of the shape:

$$\begin{aligned} h^q\big(X,\,\mathscr{S}^{(\ell_1,\ell_2,\ldots,\ell_n)}T_X^*\big) \leqslant \mathsf{Constant}_{n,d}\cdot \prod_{1\leqslant i<j\leqslant n}(\ell_i-\ell_j)\Big\{ \\ \Big\{\sum_{\beta_1+\cdots+\beta_{n-1}+\beta_n=n}(\ell_1-\ell_2)^{\beta_1}\cdots(\ell_{n-1}-\ell_n)^{\beta_{n-1}}\ell_n^{\beta_n}\Big\}+ \\ +\mathsf{Constant}_{n,d}\big(1+|\ell|^{\frac{n(n+1)}{2}-1}\big) \qquad (q=1,2\cdots n). \end{aligned}$$

But then we may assume here that:

$$\ell_n < \tfrac{1}{d-n-2}\big\{n(d-1)+\ell_1-\ell_n+\textstyle\sum_{i=1}^{n-1}(\ell_i-\ell_n)\big\},$$

since otherwise the right-hand side majorant can be replaced by 0, and consequently, because it follows by exponentiation from such a restriction on ℓ_n that:

$$\ell_n^{\beta_n} \leqslant \mathsf{Constant}_{n,d}\cdot \sum_{\beta'_1+\cdots+\beta'_{n-1}\leqslant\beta_n}(\ell_1-\ell_2)^{\beta'_1}\cdots(\ell_{n-1}-\ell_n)^{\beta'_{n-1}},$$

we conclude that whenever $|\ell| \geqslant 1 + 2n^2 + (n+1)(d-n-2)$, one has:

$$\boxed{\begin{aligned} h^q\big(X,\, \mathscr{S}^{(\ell_1,\dots,\ell_n)}T^*_X\big) \leqslant \mathsf{Constant}_{n,d} \cdot \prod_{1\leqslant i<j\leqslant n} (\ell_i - \ell_j) \Bigg[\sum_{\beta'_1+\cdots+\beta'_{n-1}=n} (\ell_1-\ell_2)^{\beta'_1} \cdots (\ell_{n-1}-\ell_n)^{\beta'_{n-1}} \Bigg] + \\ + \mathsf{Constant}_{n,d}\big(1 + |\ell|^{\frac{n(n+1)}{2}-1}\big) \qquad\qquad (q=1,2\cdots n). \end{aligned}}$$

Application: cohomology control for $\mathscr{E}^{GG}_{\kappa,m}T^*_X$. Now, we make the following observation: no Schur bundle $\mathscr{S}^{(\ell_1,\dots,\ell_n)}T^*_X$ for which $|\ell| < \frac{m}{\kappa}$ can appear in the decomposition of $\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X$ provided by the theorem on p. 32, just because all the integers λ^j_i filling the Young diagram $\mathsf{YD}_{(\ell_1,\dots,\ell_n)}$ satisfy all $1 \leqslant \lambda^j_i \leqslant \kappa$, whence:

$$|\ell| \leqslant m \leqslant \kappa\,|\ell|$$

always. Thus, if we assume only that $\frac{m}{\kappa}$ is larger than the above constant $1 + 2n^2 + (n+1)(d-n-2)$, and we certainly can assume this since both $m \gg \kappa$ and $\kappa \gg n$ are supposed to tend to infinity, then the cohomology majoration boxed above can be applied to *all* Schur bundles entering the decomposition of $\mathsf{Gr}^\bullet\mathscr{E}^{GG}_{\kappa,m}T^*_X$.

We are thus now in a position to accomplish the final series of inequalities. For any $q = 1, 2, \dots, n$, reminding Sections 8, 9 and 10, we have:

$$\begin{aligned} h^q\big(X,\, \mathscr{E}^{GG}_{\kappa,m}T^*_X\big) &\leqslant \sum_{\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_n\geqslant 0} M^{\kappa,m}_{\ell_1,\ell_2,\dots,\ell_n} \cdot h^q\big(X,\, \mathscr{S}^{(\ell_1,\ell_2,\dots,\ell_n)}T^*_X\big) \\ &\leqslant \mathsf{Constant}_{n,d} \sum_{\substack{\mathsf{YT\ semi-standard} \\ \mathsf{weight}(\mathsf{YT})=m}} \prod_{1\leqslant i<j\leqslant n} \big(\ell_i(\mathsf{YT}) - \ell_j(\mathsf{YT})\big) \Bigg\{ \\ &\qquad \Bigg\{ \sum_{\beta'_1+\cdots+\beta'_{n-1}=n} \big(\ell_1(\mathsf{YT}) - \ell_2(\mathsf{YT})^{\beta'_1} \cdots \big(\ell_{n-1}(\mathsf{YT}) - \ell_n(\mathsf{YT})\big)^{\beta'_{n-1}} \Bigg\} + \\ &\quad + \mathsf{Constant}_{n,d} \sum_{\substack{\mathsf{YT\ semi-standard} \\ \mathsf{weight}(\mathsf{YT})=m}} \sum_{\alpha_1+\cdots+\alpha_n\leqslant\frac{n(n+1)}{2}-1} \ell_1(\mathsf{YT})^{\alpha_1} \cdots \ell_n(\mathsf{YT})^{\alpha_n} \\ &\leqslant \mathsf{Constant}_{n,d} \sum_{\mathsf{YT}\in\mathsf{YT}^{\max}_{\kappa,m}} \sum_{\alpha'_1+\cdots+\alpha'_{n-1}=\frac{n(n+1)}{2}} \big(\ell_1(\mathsf{YT}) - \ell_2(\mathsf{YT})\big)^{\alpha'_1} \cdots \big(\ell_{n-1}(\mathsf{YT}) - \ell_n(\mathsf{YT})\big)^{\alpha'_{n-1}} + \\ &\quad + \mathsf{Constant}_{n,d,\kappa} \cdot m^{(\kappa+1)n-2} \end{aligned}$$

$$\leqslant \mathsf{Constant}_{n,d}\,\frac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!\,(\kappa!)^n}\sum_{\alpha'_1+\cdots+\alpha'_{n-1}=\frac{n(n+1)}{2}}\Bigg\{$$

$$\Bigg\{\sum_{\mu^i_l\in\nabla_{n,\kappa}}(\kappa!)^n\cdot\frac{N^{\kappa}_{\mu^1_1}}{\kappa\cdots\mu^1_1}\cdot\frac{N^{\mu^1_1,\kappa}_{\mu^2_1,\mu^2_2}}{(\kappa+\mu^1_1)\cdots(\mu^2_2+\mu^2_1)}\cdots$$

$$\cdots\frac{N^{\mu^{n-2}_1,\ldots,\mu^{n-2}_{n-2},\kappa}_{\mu^{n-1}_1,\ldots,\mu^{n-1}_{n-2},\mu^{n-1}_{n-1}}}{(\kappa+\mu^{n-2}_{n-2}+\cdots+\mu^{n-2}_1)\cdots(\mu^{n-1}_{n-1}+\mu^{n-1}_{n-2}+\cdots+\mu^{n-1}_1)}\cdot\frac{N^{\mu^{n-1}_1,\ldots,\mu^{n-1}_{n-1},\kappa}_{\mu^{n}_1,\ldots,\mu^{n}_{n-1},\mu^{n}_{n-1}}}{(\kappa+\mu^{n-2}_{n-2}+\cdots+\mu^{n-2}_1)\cdots(\mu^{n-1}_{n-1}+\mu^{n-1}_{n-2}+\cdots+\mu^{n-1}_1)}\cdot$$

$$\cdot\big[\log(\kappa)-\log(\mu^1_1)\big]^{\alpha'_1}\big[\log(\kappa+\mu^1_1)-\log(\mu^2_2+\mu^2_1)\big]^{\alpha'_2}\cdots$$

$$\cdots\big[\log(\kappa+\mu^{n-2}_{n-2}+\cdots+\mu^{n-2}_1)-\log(\mu^{n-1}_{n-1}+\mu^{n-1}_{n-2}+\cdots+\mu^{n-1}_1)\big]^{\alpha'_{n-1}}\Bigg\}+$$

$$+\,\mathsf{Constant}_{n,d,\kappa}\cdot m^{(\kappa+1)n-2}$$

$$\leqslant \mathsf{Constant}_{n,d}\,\frac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!\,(\kappa!)^n}\sum_{\alpha'_1+\cdots+\alpha'_{n-1}=\frac{n(n+1)}{2}}\Delta^{\alpha'_1,\ldots,\alpha'_{n-1},0}_{n,\kappa}+$$

$$+\,\mathsf{Constant}_{n,d,\kappa}\cdot m^{(\kappa+1)n-2}$$

$$\leqslant \mathsf{Constant}_{n,d}\,\frac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!\,(\kappa!)^n}+\mathsf{Constant}_{n,d,\kappa}\cdot m^{(\kappa+1)n-2}.$$

Lastly, in the trivial minoration:

$$h^0\geqslant\chi-h^2-h^4-h^6-\cdots$$

for $\mathscr{E}^{GG}_{\kappa,m}T^*_X$, we may apply the majorations just obtained with q even and deduce that:

$$h^0\big(X,\,\mathscr{E}^{GG}_{\kappa,m}T^*_X\big)\geqslant\chi\big(X,\,\mathscr{E}^{GG}_{\kappa,m}T^*_X\big)-\mathsf{Constant}_{n,d}\,\frac{m^{(\kappa+1)n-1}}{((\kappa+1)n-1)!\,(\kappa!)^n}(\log\kappa)^0-$$
$$-\,\mathsf{Constant}_{n,d,\kappa}\cdot m^{(\kappa+1)n-2},$$

so that we even get a better minoration of h^0 than the one stated in the Main Theorem. □

§13. Speculations about invariant jet differentials

Demailly-Semple invariant jet differentials. The group G_κ of κ-jets at the origin of local reparametrizations $\phi(\zeta)=\zeta+\phi''(0)\,\frac{\zeta^2}{2!}+\cdots+\phi^{(\kappa)}(0)\,\frac{\zeta^\kappa}{\kappa!}+\cdots$ of $(\mathbb{C},0)$ that are tangent to the identity, namely which satisfy $\phi'(0)=1$, may be seen to act linearly on the $n\kappa$-tuples of jet variables $\big(f'_{j_1},f''_{j_2},\ldots,f^{(\kappa)}_{j_\kappa}\big)$ by plain matrix multiplication, *i.e.* when we set $g^{(\lambda)}_i:=\big(f_i\circ\phi\big)^{(\lambda)}$, a computation applying the chain rule gives for each index i:

$$\begin{pmatrix} g'_i\\ g''_i\\ g'''_i\\ g''''_i\\ \vdots\\ g^{(\kappa)}_i\end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0\\ \phi'' & 1 & 0 & 0 & \cdots & 0\\ \phi''' & 3\phi'' & 1 & 0 & \cdots & 0\\ \phi'''' & 4\phi'''+3\phi''^2 & 6\phi'' & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \phi^{(\kappa)} & \cdots & \cdots & \cdots & \cdots & 1\end{pmatrix}\begin{pmatrix} f'_i\circ\phi\\ f''_i\circ\phi\\ f'''_i\circ\phi\\ f''''_i\circ\phi\\ \vdots\\ f^{(\kappa)}_i\circ\phi\end{pmatrix}\qquad{\scriptstyle(i=1\cdots n).}$$

By definition (*see* [4, 29, 22, 17]), Demailly-Semple invariant jet polynomials $\mathsf{P}\big(j^\kappa f\big)$ satisfy, for some integer m:

$$\mathsf{P}\big(j^\kappa g\big) = \mathsf{P}\big(j^\kappa(f\circ\phi)\big) = \phi'(0)^m \cdot \mathsf{P}\big((j^\kappa f)\circ\phi\big) = \mathsf{P}\big((j^\kappa f)\circ\phi\big),$$

for any ϕ.

Then obviously when $\phi'(0) = 1$, the algebra E^n_κ just coincides with the algebra of invariants for the linear group action represented by the group of matrices just written:

$$\mathsf{P}\big(j^\kappa g\big) = \mathsf{P}\big(\mathsf{M}_{\phi'',\phi''',\ldots,\phi^{(\kappa)}} \cdot j^\kappa f\big) = \mathsf{P}\big(j^\kappa f\big),$$

with $\phi'', \phi''', \ldots, \phi^{(\kappa)}$ interpreted as arbitrary complex constants. Such a group clearly has dimension $\kappa - 1$.

This group of matrices is a subgroup of the full unipotent group, hence it is *non-reductive*, and for this reason, it not immediate to deduce finite generation, valid in the so developed invariant theory of reductive actions, from Hilbert's averaging Reynold operator trick. Moreover, though the invariants of the full unipotent group are well understood (cf. Section 4), as soon as one looks at a *proper* subgroup of it, formal harmonies happen to be rapidly destroyed.

Three challenging questions about effectiveness that are, though, only preliminary. If one prefers to work with Demailly-Semple jets (instead of working with plain Green-Griffiths jets), then in order to reach the first stage which would correspond to knowing the exact Schur bundle decomposition for $\mathscr{E}^{DS}_{\kappa,m}T^*_X$ (instead of the one for $\mathscr{E}^{GG}_{\kappa,m}T^*_X$ provided by the theorem stated at the end of Section 4), one would have to answer *in an effective way* the following three challenging questions, for which, step by step, we explain why one should not be naive as a platonist-structuralist about what it really means to answer a mathematical question.

Question 1: *Is the Demailly-Semple algebra finitely generated?*

More precisely: is the algebra of *bi-invariant* ([22]) jet polynomials finitely generated? However, contrary to what is sometimes believed, knowing that something is "finite in cardinal" is closer to ignorance than to real knowledge, mathematically-ontologically speaking, and as an instance of this philosophical claim, it would be absolutely useless to know that the algebra of Demailly-Semple jet polynomials is generated, as an algebra, by a certain huge number, say $\leqslant 2^{2^{n\kappa}}$, of basic jet polynomials. For *effective* applications to the Green-Griffiths conjecture, one would in fact need to know not only the *exact minimal* number for such a system of generators, but also the *weight* of each a generator, and even *all the generators themselves*. However and most importantly, this would even not at all be enough, as the second obstacle comes immediately.

Question 2: *Is the ideal of relations between a set of basic generating Demailly-Semple invariants finitely generated?*

Again, in order to be able to describe the *exact* Schur bundle decomposition as was done in Section 4, it is absolutely necessary to describe effectively and for arbitrary n, κ the ideal of relations. For jets of order $\kappa = 4$ in dimension $n = 4$, we were unable to describe the full ideal of relations between the 2835 basic generating invariants listed in [22], not to mention that we ignore what is the minimal number of generators. We were saved in [22] by the fact that there are "only" 16 basic bi-invariants (minimal number) and "only" 41 relations between them[28] (in a Gröbner basis for a certain lexicographic order).

All these speculative considerations lead us *in fine* to the main metaphysical question: *Are there observable, simple mathematical harmonies in a certain set of generators and for all the relations between them?* Without harmonies, there is absolutely no hope to treat the case where n and κ are arbitrary. For $n = 2$, $\kappa = 5$ and for $n = 4$, $\kappa = 4$, we were unable, in [22], to discover any combinatorially convincing global formal harmonies. Nevertheless, there could yet be some slight hope as follows.

Question 3: *Is the algebra of bi-invariants Cohen-Macaulay?*

Well, this would be nice. For reductive group actions, this is known to be true, but however, *almost never in a neat effective way*. At least, one could dream that the Demailly-Semple algebra is Cohen-Macaulay and that a basis of so-called *primary invariants* presents some understandable harmonies. It is known, then, that the effective calculations about Euler-Poincaré characteristic and cohomologies with an adapted reduced Schur bundle decomposition become much more tractable when one looks only at primary invariants. But for $n = 4$, $\kappa = 4$ and for $n = 2$, $\kappa = 5$, going through the mutually independent bi-invariant we exhibited in [22] and trying to change the generators, we were unable to see or to devise a neat basis of primary invariants, though for $n = 2$, $\kappa = 4$, one easily discovers such a basis at first glance. Again, a non-effective theorem claiming "the algebra of Demailly-Semple is Cohen-Macaulay" would be useless toward the Green-Griffiths conjecture because rather, one would really need to know the exact description of a basis of primary invariants with all their weights in order to start continuing working toward the Green-Griffiths conjecture. But if the algebra is not even Cohen-Macaulay, well, the next tasks could be even more extremely challenging because, as we already saw, the end of Section 4 opens several doors to other fields of hard effective computations when one just deals with the much simpler Green-Griffiths jets.

[28] We believe that one could attack the seemingly accessible case $n = \mathbf{5}$, $\kappa = \mathbf{5}$.

Last but not least, we would like to insist on the fact that in the state of affairs which is current since the 19th Century, even for the most studied reductive action of $\mathsf{SL}_2(\mathbb{C})$ on binary forms of degree d in (only) two variables, the *effective answers* to Questions 1, 2 and 3 is unknown for arbitrary d, and is rather extremely challenging in fact. Cayley, Sylvester, Gordan, Noether, Popov, Grosshans, Springer, Dixmier, Lazard, Bedratyuk and others did not find any complete closed global tamed combinatorial harmonies.

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Effective algebraic degeneracy

Joël Merker (with S. Diverio et E. Rousseau)

Abstract. We show that for every smooth projective hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d and of arbitrary dimension $n \geqslant 2$, if X is generic, then there exists a proper algebraic subvariety $Y \subsetneqq X$ such that every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$ has image $f(\mathbb{C})$ which lies in Y, as soon as its degree satisfies the effective lower bound $d \geqslant 2^{n^5}$.

Table of contents

1. Introduction

In 1979, Green and Griffiths [17] conjectured that every projective algebraic variety X of general type contains a certain *proper* algebraic *sub*variety $Y \subsetneqq X$ inside which all nonconstant entire holomorphic curves $f\colon \mathbb{C} \to X$ must necessarily lie.

A positive answer to this conjecture has been given for surfaces by McQuillan [11] under the assumption that the second Segre number $c_1^2 - c_2$ is positive. In the survey article [35] (*cf.* also [20]), Siu provided a beautiful strategy to establish algebraic degeneracy of entire holomorphic curves in generic hypersurfaces $X \subset \mathbb{P}^{n+1}$ of high degree larger than a certain $d_n \gg 1$, and also *Kobayashi-hyperbolicity* of such X's if d_n is even much higher.

Siu's strategy is based on two key steps: 1) the explicit construction, in projective coordinates, of global holomorphic jet differentials; 2) the deformation of such jet differentials by means of slanted vector fields having low pole order. The explicit construction of jet differentials can be seen as a replacement of the argument using Riemann-Roch which is known to be difficult to realize since it involves a control of the cohomology. The reason to perform explicit constructions is also a better access to the base-point set, in order to provide hyperbolicity instead of just algebraic degeneracy. Complete up-to-date survey considerations may further be found in [36, 4, 12, 5, 10, 25].

In this paper, we overcome the difficulty of the Riemann-Roch argument thanks to an alternative approach for Siu's first key step based on Demailly's

bundle of invariant jets [4]. The advantage of this method is also that it usually yields better bounds on the degree. Indeed, after performing in Sections 4 and 5 below some explicit, delicate elimination computations, we finally obtain a lower bound on the degree $d_n = d(n)$ as an explicit function of n, for generic projective hypersurfaces of arbitrary dimension $n \geqslant 2$.

Theorem 1.1. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree d and of arbitrary dimension $n \geqslant 2$. If X is generic and if its degree satisfies the* effective *lower bound:*

$$d \geqslant 2^{n^5},$$

then there exists a proper *algebraic subvariety $Y \subsetneqq X$ such that every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$ has image $f(\mathbb{C})$ contained in Y.*

As in [20, 35], we thereby confirm, for generic projective hypersurfaces of high degree, the Green-Griffiths-Lang conjecture. Even if our lower bound is far from the one $\deg X \geqslant n+3$ insuring general type, to our knowledge, Theorem 1.1 is, in this direction, the first n-dimensional result with, moreover, an explicit degree lower bound. In addition, as a byproduct of our constructions, the subvarieties absorbing the images of nonconstant entire curves vary as a holomorphic family with the generic projective hypersurface.

Two main ingredients enter our proof: 1) the existence of invariant jet differentials vanishing on an ample divisor in projective hypersurfaces of high degree, following [4, 6]; and Siu's second key step: 2) the global generation of a sufficiently high twisting of the tangent bundle to the so-called *manifold of vertical n-jets*, which is canonically associated to the universal family of projective hypersurfaces, following [35, 21].

The first ingredient dates back to the seminal work of Bloch [1], revisited by Green-Griffiths in [17], by Siu in [33, 36, 35] and by Demailly in [4]. Bloch's main philosophical idea is that global jet differentials vanishing on an ample divisor provide some algebraic differential equations that every entire holomorphic curve $f\colon \mathbb{C} \to X$ must satisfy. Five decades later, Green and Griffiths [17] modernized Bloch's concepts and established several results — still fundamental nowadays — about the geometry of entire curves.

Later on, Demailly [4] refined and enlarged the whole theory by defining jet differentials that are invariant under reparametrization of the source $\mathbb{C}$. Through this geometrically adequate, new point of view, one looks only at the conformal class of all entire curves. In [6, 7], the first-named author combined Demailly's approach with Trapani's [19] algebraic version of the holomorphic Morse inequalities, so as to construct global invariant jet differentials in *any* dimension $n \geqslant 2$. The first effective aspect of our proof is to make somewhat explicit such a construction.

Indeed, by following [6, 7], we consider a certain intersection product (*see* (47) and (50) below), the positivity of which yields — thanks to a suitable application of the holomorphic Morse inequalities — a lower bound for the (asymptotic) dimension of the space of global sections of a certain *weighted subbundle* of Demailly's full bundle $E_{n,m}T_X^*$ of invariant n-jet differentials. This intersection product lives in the cohomology algebra of the n-th projectivized jet bundle over X, a polynomial algebra in n indeterminates $u_1, u_2, \ldots, u_n$ equipped with canonical, geometrically significant relations ([4, 6]). The u_i here are the first Chern classes of the successive (anti)tautological line bundles which arise during the projectivization process. The task of reducing the mentioned intersection product in terms of the Chern classes of T_X — after eliminating *all* the Chern classes living at each level of Demailly's tower — happens to be of high algebraic complexity, because four combinatorics are intertwined there: 1) presence of several relations shared by all the Chern classes of the lifted horizontal distributions; 2) Newton expansion of large n^2-powers; 3) differences of various binomial coefficients; 4) emergence of many Jacobi-Trudy determinants.

The second ingredient, *viz.* the *vertical jets*, comes from ideas developed for 1-jets by Voisin [24] in order to generalize works of Clemens [1] and Ein on the positivity of the canonical bundles of subvarieties of generic projective hypersurfaces of high degree. In [35], Siu showed how the corresponding *global generation property* for 1-jets devised by Voisin generalizes to the bundle of tangents to the space of vertical n-jets. Siu then established that one may use the available tangential generators, which are meromorphic vector fields with a certain *pole order* $c_n \geqslant 1$, so as to produce, by plain differentiation, many new algebraically independent invariant jet differentials when starting from just a single *nonzero* jet differential. At the end, one obtains in this way sufficiently many independent jet differentials, and this then forces entire curves to lie in a positive-codimensional subvariety $Y \subsetneqq X$.

This strategy was realized in details for 2-jets in dimension 2 by Păun [26] with pole order $c_2 = 7$, and similarly, for 3-jets in dimension 3 by the third-named author in [18] with $c_3 = 12$. In both works, global generation holds outside a certain exceptional set. The general case of n-jets in dimension n was performed recently by the second-named author in [21] with $c_n = \frac{n^2+5n}{2}$ and with a quite similar exceptional set. It then became clear, when [21] appeared, that Demailly's invariant jets combined with Siu's second key step could yield *weak* algebraic degeneracy (nonexistence of Zariski-dense entire curves) in *any* dimension $n \geqslant 2$. But to reach effectivity, it yet remained to perform what the present article is aimed at: taming somehow the complicated combinatorics of Demailly's tower. Furthermore, at the cost of increasing the pole order up to $c'_n = n^2 + 2n$, the exceptional set is shrunk

to be just the set of singular jets ([21]), and then *strong effective* algebraic degeneracy is gained. This is Theorem 1.1.

These brief words summarize how we combine *several ideas*, both of *conceptual* and of *technical* nature which stem from Algebra, from Analysis and from Geometry; deep conjectures always confirm the unity of mathematics.

As the effective lower bound $\deg X \geqslant 2^{n^5}$ of the main theorem above is not optimal, Sections 6 and 7 of the paper are intended to provide numerically better estimates in small dimensions. For surfaces, the best known effective lower bound for the degree is $d \geqslant 18$ ([26]), after $d \geqslant 21$ ([5]) and $d \geqslant 36$ ([12]). In [18], the third-named author obtained the first effective result for weak algebraic degeneracy of entire curves inside threefolds X of $\mathbb{P}^4$, whenever $\deg X \geqslant 593$.

Theorem 1.2. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree d. If X is generic, then there exists a proper closed subvariety $Y \subsetneqq X$ such that every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$ has image $f(\mathbb{C})$ contained in Y*

- *for* $\dim X = 3$, *whenever* $\deg X \geqslant 593$;
- *for* $\dim X = 4$, *whenever* $\deg X \geqslant 3203$;
- *for* $\dim X = 5$, *whenever* $\deg X \geqslant 35355$;
- *for* $\dim X = 6$, *whenever* $\deg X \geqslant 172925$.

The last three effective lower bounds in dimensions 4, 5 and 6 are entirely new. In dimension 3, our bound 593 is the same as in [18]. Indeed, an inspection of the exceptional set in [18] shows that the part of the degeneracy locus which may depend on f is in fact of codimension 2 (*cf.* [21]), and therefore is empty, thanks to Clemens' result [1] which excludes elliptic and rational curves. Using $c_4 = 18$ and $c_5 = 25$ instead of $c'_4 = 24$ and $c'_5 = 35$, we would have obtained the two lower bounds $\deg X \geqslant 2432$ and $\deg X \geqslant 25586$ which were announced in our first `arxiv.org` preprint and which insured only *weak* algebraic degeneracy (*cf.* [21]; using $c_6 = 33$ instead of $c'_6 = 48$, the bound would be $\deg X \geqslant 120176$).

For dimensions 5 and 6, our strategy of proof is the same as for Theorem 1.1, except that we choose a numerically better weighted subbundle of Demailly's bundle of invariant jet differentials, exactly as in [6].

Quite differently, for dimensions 3 and 4, the construction of nonzero jet differentials is based on a *complete* algebraic description of the full Demailly bundles $E_{n,m}T^*_X$, $n = 3, 4$, due respectively to the third-named author ([29]) and to the second-named author ([22]), after Demailly [4] and Demailly-El Goul [5] for $n = 2$. The invariant theory approach requires finding the composition series of the $E_{n,m}T^*_X$, but this is understood only in dimensions 2, 3 and 4, because of the proliferation of secondary invariants — a well known phenomenon, *cf.* [22] and the references therein. Then by appropriately

summing the Euler characteristics of the composing Schur bundles [29], taking account of the numerous syzygies shared by a collection of fundamental bi-invariants [22], one establishes the positivity of the Euler characteristics $\chi\big(E_{n,m}T_X^*\big)$ for $n = 3, 4$, at least asymptotically as m goes to infinity. Furthermore, realizing also in dimension 4 the strategy finalized in dimension 3 by the third-named author [30], we estimate from above the contribution of the even cohomology dimensions $h^{2i}\big(X, E_{n,m}T_X^*\big)$, thereby gaining a suitable lower bound for the dimension of the space $h^0\big(X, E_{n,m}T_X^*\big)$ of global sections. Such estimates are done by means of Demailly's [4] generalization of a vanishing theorem due to Bogomolov for the top cohomology, and also by means of the algebraic version of the weak holomorphic Morse inequalities for the intermediate cohomologies [30].

Even if the numerical bounds obtained in this way in dimensions 3 and 4 are better than the ones we obtained in all dimensions, the extreme intricacy of the algebras of invariants by reparametrization (*cf.* [22]) is the main obstacle to run the process in the higher dimensions $n \geqslant 5$. This was our central motivation to follow the strategy of [6, 7].

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2. Preliminaries

2.1. **Jet differentials.** We briefly present here useful geometric concepts selected from the theory of Green-Griffiths' and Demailly's jets [17, 4] (*cf.* also [29, 6]). Let (X, V) be a *directed manifold*, *i.e.* a pair consisting of a complex manifold X together with a (not necessarily integrable) holomorphic subbundle $V \subset T_X$ of the tangent bundle to X. This category will be very useful later on, when we will consider the situation where X is the universal family of projective hypersurfaces of fixed degree and V the relative tangent bundle to the family. The bundle J_kV is the bundle of k-jets of germs of holomorphic curves $f\colon (\mathbb{C}, 0) \to X$ which are tangent to V, *i.e.*, such that $f'(t) \in V_{f(t)}$ for all t near 0, together with the projection map $f \mapsto f(0)$ onto X.

Let $\mathbb{G}_k$ be the group of germs of k-jets of biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \mapsto \varphi(t) = a_1\, t + a_2\, t^2 + \cdots + a_k\, t^k, \quad a_1 \in \mathbb{C}^*,\ a_j \in \mathbb{C},\ j \geqslant 2$$

of $(\mathbb{C}, 0)$, the composition law being taken modulo terms t^j of degree $j > k$. Then $\mathbb{G}_k$ admits a natural fiberwise right action on J_kV which consists in reparametrizing k-jets of curves by such changes φ of parameters. In [21], one finds the multivariate FaÃă di Bruno formulae yielding explicit reparametrization for the so-called *absolute case* $V = T_X$. Moreover the subgroup $\mathbb{H} \simeq \mathbb{C}^*$ of homotheties $\varphi(t) = \lambda\, t$ is a (non-normal) subgroup of

$\mathbb{G}_k$ and we have a semidirect decomposition $\mathbb{G}_k = \mathbb{G}'_k \ltimes \mathbb{H}$, where $\mathbb{G}'_k$ is the group of k-jets of biholomorphisms tangent to the identity, *i.e.* with $a_1 = 1$. The corresponding action on k-jets is described in coordinates by

$$\lambda \cdot \big(f', f'', \dots, f^{(k)}\big) = \big(\lambda f', \lambda^2 f'', \dots, \lambda^k f^{(k)}\big). \tag{38}$$

As in [17], we introduce the *Green-Griffiths vector bundle* $E^{GG}_{k,m}V^* \to X$, the fibers of which are complex-valued polynomials $Q(f', f'', \dots, f^{(k)})$ in the fibers of J_kV having weighted degree m with respect to the $\mathbb{C}^*$ action, namely such that:

$$Q\big(\lambda f', \lambda^2 f'', \dots, \lambda^k f^{(k)}\big) = \lambda^m Q\big(f', f'', \dots, f^{(k)}\big),$$

for all $\lambda \in \mathbb{C}^*$ and all $\big(f', f'', \dots, f^{(k)}\big) \in J_kV$. Demailly refined this concept.

Definition 2.1 ([4]). The *bundle of invariant jet differentials of order k and weighted degree m* is the subbundle $E_{k,m}V^* \subset E^{GG}_{k,m}V^*$ of polynomial differential operators $Q(f', f'', \dots, f^{(k)})$ which are invariant under *arbitrary* changes of parametrization, *i.e.* which, for every $\varphi \in \mathbb{G}_k$, satisfy:

$$Q\big((f\circ\varphi)', (f\circ\varphi)'', \dots, (f\circ\varphi)^{(k)}\big) = \varphi'(0)^m\, Q\big(f', f'', \dots, f^{(k)}\big).$$

Alternatively, $E_{k,m}V^* = \big(E^{GG}_{k,m}V^*\big)^{\mathbb{G}'_k}$ is the set of invariants of $E^{GG}_{k,m}V^*$ under the action of $\mathbb{G}'_k$.

We now define a filtration on $E^{GG}_{k,m}V^*$. A coordinate change $f \mapsto \Psi\circ f$ transforms every monomial $(f^{(\bullet)})^\ell = (f')^{\ell_1}(f'')^{\ell_2}\cdots(f^{(k)})^{\ell_k}$ having, for any s with $1 \leqslant s \leqslant k$, the partial weighted degrees $|\ell|_s := |\ell_1| + 2|\ell_2| + \cdots + s|\ell_s|$, into a new polynomial $\big((\Psi\circ f)^{(\bullet)}\big)^\ell$ in $(f', f'', \dots, f^{(k)})$, which has the same partial weighted degree of order s when $\ell_{s+1} = \cdots = \ell_k = 0$, and a larger or equal partial degree of order s otherwise (use the chain rule). Hence, for each $s = 1, \dots, k$, we get a well defined decreasing filtration $F^\bullet_s$ on $E^{GG}_{k,m}V^*$ as follows:

$$F^p_s\big(E^{GG}_{k,m}V^*\big) = \left\{\begin{matrix} Q(f', f'', \dots, f^{(k)}) \in E^{GG}_{k,m}V^* \text{ involving} \\ \text{only monomials } (f^{(\bullet)})^\ell \text{ with } |\ell|_s \geqslant p \end{matrix}\right\}, \quad \forall\, p \in \mathbb{N}.$$

The graded terms $\mathrm{Gr}^p_{k-1}\big(E^{GG}_{k,m}V^*\big)$ associated with the $(k-1)$-filtration $F^p_{k-1}(E^{GG}_{k,m}V^*)$ are the homogeneous polynomials $Q(f', f'', \dots, f^{(k)})$ all the monomials $(f^{(\bullet)})^\ell$ of which have partial weighted degree $|\ell|_{k-1} = p$; hence, their degree ℓ_k in $f^{(k)}$ is such that $m - p = k\ell_k$ and $\mathrm{Gr}^p_{k-1}(E^{GG}_{k,m}V^*) = 0$ unless $k|m-p$. Looking at the transition automorphisms of the graded bundle induced by the coordinate change $f \mapsto \Psi\circ f$, it turns out that $f^{(k)}$ transforms as an element of $V \subset T_X$ and, by means of a simple computation, one finds

$$\mathrm{Gr}^{m-k\ell_k}_{k-1}\big(E^{GG}_{k,m}V^*\big) = E^{GG}_{k-1,m-k\ell_k}V^* \otimes S^{\ell_k}V^*.$$

Combining all filtrations $F_s^\bullet$ together, we find inductively a filtration $F^\bullet$ on $E_{k,m}^{GG}V^*$ the graded terms of which are

$$\mathrm{Gr}^\ell\left(E_{k,m}^{GG}V^*\right) = S^{\ell_1}V^* \otimes S^{\ell_2}V^* \otimes \cdots \otimes S^{\ell_k}V^*, \quad \ell \in \mathbb{N}^k, \ |\ell|_k = m.$$

Moreover ([4]), invariant jet differentials enjoy the natural induced filtration:

$$F_s^p(E_{k,m}V^*) = E_{k,m}V^* \cap F_s^p\big(E_{k,m}^{GG}V^*\big),$$

the associated graded bundle being, if we employ $(\bullet)^{\mathbb{G}'_k}$ to denote $\mathbb{G}'_k$-invariance:

$$\mathrm{Gr}^\bullet(E_{k,m}V^*) = \Bigg(\bigoplus_{|\ell|_k=m} S^{\ell_1}V^* \otimes S^{\ell_2}V^* \otimes \cdots \otimes S^{\ell_k}V^*\Bigg)^{\mathbb{G}'_k}.$$

2.2. **Projectivized k-jet bundles.** Next, we recall briefly Demailly's construction [4] of the tower of projectivized bundles providing a (relative) smooth compactification of $J_k^{\mathrm{reg}}V/\mathbb{G}_k$, where $J_k^{\mathrm{reg}}V$ is the bundle of *regular k-jets tangent to* V, that is, k-jets such that $f'(0) \neq 0$.

Let (X, V) be a directed manifold, with $\dim X = n$ and $\operatorname{rank} V = r$. With (X, V), we associate another directed manifold $(\widetilde{X}, \widetilde{V})$ where $\widetilde{X} = P(V)$ is the projectivized bundle of lines of V, $\pi\colon \widetilde{X} \to X$ is the natural projection and $\widetilde{V}$ is the subbundle of $T_{\widetilde{X}}$ defined fiberwise as

$$\widetilde{V}_{(x_0,[v_0])} \stackrel{\text{def}}{=} \big\{\xi \in T_{\widetilde{X},(x_0,[v_0])} \mid \pi_*\xi \in \mathbb{C}\cdot v_0\big\},$$

for any $x_0 \in X$ and $v_0 \in T_{X,x_0} \setminus \{0\}$. We also have a "lifting" operator which assigns to a germ of holomorphic curve $f\colon (\mathbb{C}, 0) \to X$ tangent to V a germ of holomorphic curve $\widetilde{f}\colon (\mathbb{C}, 0) \to \widetilde{X}$ tangent to $\widetilde{V}$ in such a way that $\widetilde{f}(t) = (f(t), [f'(t)])$.

To construct the projectivized k-jet bundle we simply set inductively $(X_0, V_0) = (X, V)$ and $(X_k, V_k) = (\widetilde{X}_{k-1}, \widetilde{V}_{k-1})$. Clearly $\operatorname{rank} V_k = r$ and $\dim X_k = n + k(r-1)$. Of course, we have for each $k > 0$ a tautological line bundle $\mathscr{O}_{X_k}(-1) \to X_k$ and a natural projection $\pi_k\colon X_k \to X_{k-1}$. We call $\pi_{j,k}$ the composition of the projections $\pi_{j+1} \circ \cdots \circ \pi_k$, so that the total projection is given by $\pi_{0,k}\colon X_k \to X$. We have, for each $k > 0$, two short exact sequences

$$0 \to T_{X_k/X_{k-1}} \to V_k \to \mathscr{O}_{X_k}(-1) \to 0, \tag{39}$$

$$0 \to \mathscr{O}_{X_k} \to \pi_k^* V_{k-1} \otimes \mathscr{O}_{X_k}(1) \to T_{X_k/X_{k-1}} \to 0. \tag{40}$$

Here, we also have an inductively defined k-lifting for germs of holomorphic curves such that $f_{[k]}\colon (\mathbb{C}, 0) \to X_k$ is obtained as $f_{[k]} = \widetilde{f}_{[k-1]}$.

Theorem 2.1 ([4]). *Suppose that* $\operatorname{rank} V \geqslant 2$. *The quotient* $J_k^{\mathrm{reg}} V \big/ \mathbb{G}_k$ *has the structure of a locally trivial bundle over* X, *and there is a holomorphic embedding* $J_k^{\mathrm{reg}} V \big/ \mathbb{G}_k \hookrightarrow X_k$ *over* X, *which identifies* $J_k^{\mathrm{reg}} V \big/ \mathbb{G}_k$ *with* X_k^{reg}, *that is the set of points in* X_k *of the form* $f_{[k]}(0)$ *for some non singular* k*-jet* f. *In other words* X_k *is a relative compactification of* $J_k^{\mathrm{reg}} V / \mathbb{G}_k$ *over* X. *Moreover, one has the direct image formula:*

$$(\pi_{0,k})_* \mathscr{O}_{X_k}(m) = \mathscr{O}\big(E_{k,m} V^*\big).$$

Next, we are in position to recall the fundamental application of jet differentials to Kobayashi-hyperbolicity and to Green-Griffiths algebraic degeneracy.

Theorem 2.2 ([17, 36, 4]). *Assume that there exist integers* $k, m > 0$ *and an ample line bundle* $A \to X$ *such that*

$$H^0\big(X_k, \mathscr{O}_{X_k}(m) \otimes \pi_{0,k}^* A^{-1}\big) \simeq H^0\big(X, E_{k,m} V^* \otimes A^{-1}\big)$$

has non zero sections $\sigma_1, \ldots, \sigma_N$ *and let* $Z \subset X_k$ *be the base locus of these sections. Then every entire holomorphic curve* $f \colon \mathbb{C} \to X$ *tangent to* V *necessarily satisfies* $f_{[k]}(\mathbb{C}) \subset Z$. *In other words, for every global* $\mathbb{G}_k$*-invariant differential equation* P *vanishing on an ample divisor, every entire holomorphic curve* f *must satisfy the algebraic differential equation* $P\big(j^k f(t)\big) \equiv 0$. *Furthermore, the same result also holds true for the bundle* $E_{k,m}^{GG} T_X^*$.

2.3. **Existence of invariant jet differentials.** Now, we recall some results obtained by the first-named author in [7], concerning the existence of invariant jet differentials on projective hypersurfaces which generalized to all dimensions n previous works by Demailly [4] and of the third-named author [30].

Denote by $c_\bullet(E)$ the total Chern class of a vector bundle E. The two short exact sequences (39) and (40) give, for each $k > 0$, the following two formulae:

$$\begin{aligned} c_\bullet(V_k) &= c_\bullet\big(T_{X_k/X_{k-1}}\big)\, c_\bullet\big(\mathscr{O}_{X_k}(-1)\big) \\ c_\bullet\big(\pi_k^* V_{k-1} \otimes \mathscr{O}_{X_k}(1)\big) &= c_\bullet\big(T_{X_k/X_{k-1}}\big), \end{aligned}$$

so that by a plain substitution:

$$c_\bullet(V_k) = c_\bullet\big(\mathscr{O}_{X_k}(-1)\big)\, c_\bullet\big(\pi_k^* V_{k-1} \otimes \mathscr{O}_{X_k}(1)\big). \tag{41}$$

Let us call $u_j = c_1\big(\mathscr{O}_{X_j}(1)\big)$ and $c_l^{[j]} = c_l(V_j)$. With these notations, (41) becomes:

$$c_l^{[k]} = \sum_{s=0}^{l} \Big[\tbinom{n-s}{l-s} - \tbinom{n-s}{l-s-1}\Big] u_k^{l-s} \cdot \pi_k^* c_s^{[k-1]}, \quad 1 \leqslant l \leqslant r. \tag{42}$$

Since X_j is the projectivized bundle of line of V_{j-1}, we also have the polynomial relations

$$u_j^r + \pi_j^* c_1^{[j-1]} \cdot u_j^{r-1} + \cdots + \pi_j^* c_{r-1}^{[j-1]} \cdot u_j + \pi_j^* c_r^{[j-1]} = 0, \quad 1 \leqslant j \leqslant k. \tag{43}$$

After all, the cohomology ring of X_k is defined in terms of generators and relations as the polynomial algebra $H^\bullet(X)[u_1, \dots, u_k]$ with the relations (43) in which, using inductively (42), one may express in advance all the $c_l^{[j]}$ as certain polynomials with integral coefficients in the variables $u_1, \dots, u_j$ and $c_1(V), \dots, c_l(V)$. In particular, for the first Chern class of V_k, a simple explicit formula is available:

$$c_1^{[k]} = \pi_{0,k}^* c_1(V) + (r-1) \sum_{s=1}^{k} \pi_{s,k}^* \, u_s. \tag{44}$$

Also, it is classically known that the Chern classes $c_j(X)$ of a smooth projective hypersurface $X \subset \mathbb{P}^{n+1}$ are polynomials in $d := \deg X$ and the hyperplane class $h := c_1\big(\mathscr{O}_{\mathbb{P}^{n+1}}(1)\big)$, *viz.* for $1 \leqslant j \leqslant n$:

$$c_j(X) = c_j(T_X) = (-1)^j \, h^j \sum_{i=0}^{j} (-1)^i \, \tbinom{n+2}{i} \, d^{j-i}. \tag{45}$$

Now, let $X \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree $\deg X = d$ and consider, for all what follows in the sequel, the absolute case $V = T_X$ with jet order $k = n$ equal to the dimension. Given any $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we define (*cf.* [4, 6]) the following line bundle $\mathscr{O}_{X_n}(\mathbf{a})$ on X_n:

$$\mathscr{O}_{X_n}(\mathbf{a}) = \pi_{1,n}^* \mathscr{O}_{X_1}(a_1) \otimes \pi_{2,n}^* \mathscr{O}_{X_2}(a_2) \otimes \cdots \otimes \mathscr{O}_{X_n}(a_n).$$

Using the algebraic version — first appeared in Trapani's article [19] — of Demailly's holomorphic Morse inequalities, the first-named author showed in [7] that, in order to check the *bigness* of $\mathscr{O}_{X_n}(1)$, it suffices to show the *positivity*, for some $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ lying arbitrarily in the cone defined by:

$$a_1 \geqslant 3a_2, \dots, a_{n-2} \geqslant 3a_{n-1} \quad \text{and} \quad a_{n-1} \geqslant 2a_n \geqslant 1, \tag{46}$$

of the following intersection product:

$$F^N - N \, F^{N-1} \cdot G,$$

where $N = \dim X_n = n^2$, and where the two bundles $F := \mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|)$ and $G := \pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|)$ are both globally nef on X_n ([7], Proposition 2); here, $\mathscr{O}_X(1)$ is the hyperplane bundle over X and we abbreviate $|\mathbf{a}| := a_1 + \cdots + a_n$. In other words, we express $\mathscr{O}_{X_n}(\mathbf{a})$ as a "difference" $F \otimes G^{-1}$ between two nef line bundles over X_n:

$$\mathscr{O}_{X_n}(\mathbf{a}) = \big(\mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|)\big) \otimes \big(\pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|)\big)^{-1}.$$

Thus in sum, we have to find some $\mathbf{a} \in \mathbb{Z}^n$ lying in the cone (46) for which the concerned intersection product written in length:

$$\begin{aligned}(47)\qquad &\big(\mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|)\big)^{n^2} - \\ &\qquad - n^2 \big(\mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|)\big)^{n^2-1} \cdot \pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|)\end{aligned}$$

is positive. This was done by the first-named author, and an application of the mentioned Morse inequalities yielded the following.

Theorem 2.3 ([7]). *Let $X \subset \mathbb{P}^{n+1}$ by a smooth complex hypersurface of degree $\deg X = d$ and fix any ample line bundle $A \to X$. Then, for jet order $k = n$ equal to the dimension, there exists a positive integer d_n such that the two isomorphic spaces of sections:*

$$H^0\big(X_n, \mathscr{O}_{X_n}(m) \otimes \pi_{0,n}^* A^{-1}\big) \simeq H^0\big(X, E_{n,m} T_X^* \otimes A^{-1}\big) \neq 0,$$

are nonzero*, whenever $d \geqslant d_n$ provided that m is large enough.*

It is also proved in [6] that for any jet order $k < n$ *smaller* than the dimension, no nonzero sections, though, are available: $H^0\big(X_k, \mathscr{O}_{X_k}(m) \otimes \pi_{0,k}^* A^{-1}\big) = 0$; in fact, this vanishing property is used as a technical tool in the proof of Theorem 2.3.

In our applications, it will be crucial to be able to control in a more precise way the order of vanishing of these differential operators along the ample divisor. Thus, we shall need here a slightly different theorem, inspired from [35, 26, 18]. Recall at first that for X a smooth projective hypersurface of degree d in $\mathbb{P}^{n+1}$, the canonical bundle has the following expression in terms of the hyperplane bundle:

$$K_X \simeq \mathscr{O}_X(d - n - 2),$$

whence it is ample as soon as $d \geqslant n + 3$.

Theorem 2.4. *Let $X \subset \mathbb{P}^{n+1}$ by a smooth complex hypersurface of degree $\deg X = d$. Then, for all positive rational numbers δ small enough, there exists a positive integer d_n such that the space of twisted jet differentials:*

$$H^0\big(X_n, \mathscr{O}_{X_n}(m) \otimes \pi_{0,n}^* K_X^{-\delta m}\big) \simeq H^0\big(X, E_{n,m} T_X^* \otimes K_X^{-\delta m}\big) \neq 0,$$

is nonzero, whenever $d \geqslant d_{n,\delta}$ provided again that m is large enough and that δm is an integer.

Observe that all nonzero sections $\sigma \in H^0\big(X, E_{n,m} T_X^* \otimes K_X^{-\delta m}\big)$ then have vanishing order at least equal to $\delta m(d - n - 2)$, when viewed as sections of $E_{n,m} T_X^*$.

Proof of Theorem 2.4. For each weight $\mathbf{a} \in \mathbb{N}^n$ satisfying (46), we first of all express $\mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* K_X^{-\delta|\mathbf{a}|}$ as the following difference of two nef line bundles:

$$\big(\mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|)\big) \otimes \big(\pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|) \otimes \pi_{0,n}^* K_X^{\delta|\mathbf{a}|}\big)^{-1}.$$

In order to apply the algebraic holomorphic Morse inequalities to obtain the existence of sections for high powers, we are thus led to compute the following intersection product:

(48)
$$\begin{aligned}&\big(\mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|)\big)^{n^2} - \\ &\quad - n^2 \big(\mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|)\big)^{n^2-1} \cdot \big(\pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|) \otimes \pi_{0,n}^* K_X^{\delta|\mathbf{a}|}\big),\end{aligned}$$

and to decide whether it is positive. After reducing it in terms of the Chern classes of X, and then in terms of $d = \deg X$ using (45), this intersection product becomes a polynomial — difficult to compute explicitly, but effective aspects will start in Section 4 — in d of degree less than or equal to $n+1$, having coefficients which are polynomials in $(\mathbf{a}, \delta)$ of bidegree $(n^2, 1)$, homogeneous in $\mathbf{a}$ or identically zero. Notice that for $\delta = 0$, the intersection product identifies with (47); we claim that there exists a weight $\mathbf{a}'$ such that (47) is positive. Thus by continuity, with the same choice of weight, for all $\delta > 0$ small enough, the leading coefficient still remains positive. So the polynomial in question again takes only positive values when $d \geqslant d_n$, for some (noneffective) d_n. Holomorphic Morse inequalities then insure the existence of nonzero sections.

Coming back to our claim, the argument is as follow. First of all, the three intersection products: (47), $\mathscr{O}_{X_n}(\mathbf{a})^{n^2}$ and $\big(\mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|)\big)^{n^2}$, once evaluated with respect to the degree d of the hypersurface, are all polynomials in the variable d with coefficients in $\mathbb{Z}[a_1, \ldots, a_n]$ of degree at most $n+1$ and the coefficients of d^{n+1} of the three expressions are the same (*cf.* Proposition 3 in [7]). Next, by Proposition 2 in [7], $\mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|)$ is nef if $\mathbf{a}$ satisfies (46); therefore the coefficient of d^{n+1} of its top self-intersection must be non-negative. Thus, by Lemma 1 in [7], in order to find a weight $\mathbf{a}'$ in the cone defined by (46) as in the claim, it suffices to show that this coefficient is not an identically zero polynomial in $\mathbb{Z}[a_1, \ldots, a_n]$. So, we have to prove that it contains at least one non-zero monomial: but by Lemma 3 in [7], the coefficient of its monomial $a_1^n \cdot a_2^n \cdots a_n^n$ is $(n^2)!/(n!)^n$ and we are done (*cf.* also Subsection 4.4). $\square$

2.4. **Global generation of the tangent bundle to the variety of vertical jets.** We now briefly present the second ingredient, as said in the Introduction. Let $\mathscr{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d^n}$ be the universal family of projective n-dimensional hypersurfaces of degree d in $\mathbb{P}^{n+1}$; its parameter space is the

projectivization $\mathbb{P}\big(H^0(\mathbb{P}^{n+1}, \mathscr{O}(d))\big) = \mathbb{P}^{N_d^n}$, where $N_d^n = \binom{n+d+1}{d} - 1$. We have two canonical projections:

$$\begin{tikzcd} & \mathscr{X} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathbb{P}^{n+1} & & \mathbb{P}^{N_d^n}. \end{tikzcd}$$

Consider the relative tangent bundle $\mathscr{V} \subset T_{\mathscr{X}}$ with respect to the second projection $\mathscr{V} := \ker(\mathrm{pr}_2)_*$, and form the corresponding directed manifold $(\mathscr{X}, \mathscr{V})$. It is clear that $\mathscr{V}$ is integrable and that any entire holomorphic curve from $\mathbb{C}$ to $\mathscr{X}$ tangent to $\mathscr{V}$ has its image entirely contained in some fiber $\mathrm{pr}_2^{-1}(s) = X_s$, $s \in \mathbb{P}^{N_d^n}$.

Now, let $p\colon J_n\mathscr{V} \to \mathscr{X}$ be the bundle of n-jets of germs of holomorphic curves in $\mathscr{X}$ tangent to $\mathscr{V}$, the so-called *vertical jets*, and consider the subbundle $J_n^{\mathrm{reg}}\mathscr{V}$ of *regular n-jets* of maps $f\colon (\mathbb{C}, 0) \to \mathscr{X}$ tangent to $\mathscr{V}$ such that $f'(0) \neq 0$.

Theorem 2.5 ([21]). *The twisted tangent bundle to vertical n-jets:*

$$T_{J_n\mathscr{V}} \otimes p^*\mathrm{pr}_1^*\,\mathscr{O}_{\mathbb{P}^{n+1}}(n^2+2n) \otimes p^*\mathrm{pr}_2^*\,\mathscr{O}_{\mathbb{P}^{N_d^n}}(1)$$

is generated over $J_n^{\mathrm{reg}}\mathscr{V}$ by its global holomorphic sections. Moreover, one may choose such global generating vector fields to be invariant with respect to the reparametrization action of $\mathbb{G}_n$ on $J_n\mathscr{V}$.

This means that we have enough independent, global, invariant vector fields having *meromorphic* coefficients over $J_n\mathscr{V}$ in order to linearly generate the tangent space $T_{J_n\mathscr{V},j^n}$ at every arbitrary fixed regular jet $j^n \in J_n^{\mathrm{reg}}\mathscr{V}$. The poles of these vector fields occur only in the base variables of $\mathscr{X}$, but not in the vertical jet variables of positive differentiation order. *Most importantly*, the maximal pole order here is $\leqslant n^2 + 2n$, hence it is compensated by the first twisting $(\bullet) \otimes p^*\mathrm{pr}_1^*\,\mathscr{O}_{\mathbb{P}^{n+1}}(n^2+2n)$.

3. Algebraic degeneracy of entire curves

Now, we are fully in position to establish the *noneffective* version of Theorem 1.1. The proof (*cf.* the Introduction) incorporates two main ingredients: 1) the existence, already established by Theorem 2.4, of at least *one* nonzero global invariant jet differential vanishing on an ample divisor; 2) Theorem 2.5 just above to produce sufficiently many *new algebraically independent* jet differentials.

Theorem 3.1. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of arbitrary dimension $n \geqslant 2$. Then there exists a positive integer d_n such that whenever $\deg X \geqslant d_n$ and X is generic, there exists a* proper *algebraic*

subvariety $Y \subsetneqq X$ *such that every nonconstant entire holomorphic curve* $f\colon \mathbb{C} \to X$ *has image* $f(\mathbb{C})$ *contained in* Y.

Proof. As above, consider the universal projective hypersurface $\mathbb{P}^{n+1} \xleftarrow{\mathrm{pr}_1} \mathscr{X} \xrightarrow{\mathrm{pr}_2} \mathbb{P}^{N_d^n}$ of degree d in $\mathbb{P}^{n+1}$. Observe that $X_s = \mathrm{pr}_2^{-1}(s)$ is a smooth projective hypersurface of $\mathbb{P}^{n+1}$ for generic $s \in \mathbb{P}^{N_d^n}$ and that $\mathscr{V} = \ker(\mathrm{pr}_2)_*$ restricted to X_s coincides with the tangent bundle to X_s. We infer therefore that:

$$H^0\big(X_s,\, E_{n,m}\mathscr{V}^* \otimes \mathrm{pr}_1^*\mathscr{O}_{\mathbb{P}^{n+1}}\big(-\delta m(d-n-2)\big)\big|_{X_s}\big) \simeq H^0\big(X_s,\, E_{n,m}T^*_{X_s} \otimes K_{X_s}^{-\delta m}\big).$$

Thanks to Theorem 2.4, the latter space of sections is nonzero, for small rational $\delta > 0$, for $d \geqslant d_{n,\delta}$ and for m large enough, independently of s. Fix any $s_0 \in \mathbb{P}^{N_d^n}$ and pick a nonzero jet differential $P_0 \in H^0\big(X_{s_0},\, E_{n,m}T^*_{X_{s_0}} \otimes K_{X_{s_0}}^{-\delta m}\big)$. In order to employ the vector fields of Theorem 2.5, we must at first extend P_0 as a *holomorphic family* of nonzero jet differentials. Thus, we invoke the following classical extension result.

Theorem 3.2 ([9], p. 288)**.** *Let* $\tau\colon \mathscr{Y} \to S$ *be a flat holomorphic family of compact complex spaces and let* $\mathscr{L} \to \mathscr{Y}$ *be a holomorphic vector bundle. Then there exists a proper subvariety* $Z \subset S$ *such that for each* $s_0 \in S \setminus Z$, *the restriction map* $H^0\big(\tau^{-1}(U_{s_0}), \mathscr{L}\big) \to H^0\big(\tau^{-1}(s_0), \mathscr{L}|_{\tau^{-1}(s_0)}\big)$ *is onto, for some Zariski-dense open set* $U_{s_0} \subset S$ *containing* s_0.

We remark that this theorem implies that the weighted degree of the jet differential constructed above may be chosen to be independent of the hypersurface X_s of degree d. Now, we apply this statement $\tau = \mathrm{pr}_2$, to $\mathscr{Y} = \mathscr{X}$, to $S = \mathbb{P}^{N_d^n}$, to $\mathscr{L} = E_{n,m}\mathscr{V}^* \otimes \mathrm{pr}_1^*\mathscr{O}_{\mathbb{P}^{n+1}}\big(-\delta m(d-n-2)\big)$ and we similarly denote by $Z \subset \mathbb{P}^{N_d^n}$ the embarrassing proper algebraic subvariety. The genericity of X assumed in the two theorems 1.1 and 3.1 will just consist in requiring that $s_0 \notin Z$ (notice *passim* that we do not have a constructive access to Z) and of course also, that s does not belong to the set for which X_s is singular.

We therefore obtain a holomorphic family of jet differentials:

$$P = \big\{P|_s \in H^0\big(X_s,\, E_{n,m}T^*_{X_s} \otimes K_{X_s}^{-\delta m}\big)\big\}$$

parametrized by s with $P|_{s_0} = P_0 \not\equiv 0$ and vanishing on $K_{X_s}^{\delta m}$; for our purposes, it will suffice that s varies in some neighborhood of s_0.

Now, take a *nonconstant* entire holomorphic curve $f\colon \mathbb{C} \to \mathscr{X}$ tangent to $\mathscr{V}$. Since the distribution $\mathscr{V}$ has integral manifolds $\mathrm{pr}_2^{-1}(s) = X_s$, f maps $\mathbb{C}$ into some X_{s_0}, for some $s_0 \in \mathbb{P}^{N_d^n}$. Of course, we assume that $s_0 \notin Z$ and that X_{s_0} is non-singular. Consider now the zero-set locus

$$Y_{s_0} := \big\{x \in X_{s_0}\colon P|_{s_0}(x) = 0\big\},$$

where $P|_{s_0} \not\equiv 0$ vanishes as a section of the vector bundle $E_{n,m}T^*_{X_{s_0}} \otimes K^{-\delta m}_{X_{s_0}}$. Then Y_{s_0} is a *proper algebraic subvariety* of X_{s_0}. We then claim that

$$f(\mathbb{C}) \subset Y_{s_0},$$

which will complete the proof of the theorem. (It will even come out that we obtain strong algebraic degeneracy of entire curves $f : \mathbb{C} \to X_s$ inside a $Y_s \subsetneqq X_s$ defined by $Y_s = \big\{x \in X : \ P|_s(x) = 0\big\}$ and parametrized by s near s_0.)

Reasoning by contradiction, suppose that there exists $t_0 \in \mathbb{C}$ with $f(t_0) \notin Y_{s_0}$. Consider the n-jet map $j^n f \colon \mathbb{C} \to J_n\mathscr{V}$ induced by f. If $j^n f(\mathbb{C})$ would be entirely contained in $J_n^{\text{sing}}\mathscr{V} \overset{\text{def}}{=} J_n\mathscr{V} \setminus J_n^{\text{reg}}\mathscr{V}$, then f would be *constant*, since singular n-jets satisfy $f'(t) = 0$. So necessarily $j^n f(\mathbb{C}) \not\subset J_n\mathscr{V}^{\text{sing}}$, namely $f' \not\equiv 0$. Then by shifting a bit t_0 if necessary, we can assume that we in addition have $f'(t_0) \neq 0$, *viz.* $j^n f(t_0) \in J_n^{\text{reg}}\mathscr{V}$.

Theorem 2.2 ensures that $P|_{s_0}\big(j^n\, f(t)\big) \equiv 0$. Denote $U := \mathbb{P}^{N_d^n} \setminus Z$.

We may now view the family $P = \{P|_s\}$ as being a holomorphic map

$$P \colon J_n\mathscr{V}\big|_{\text{pr}_2^{-1}(U)} \longrightarrow p^*\text{pr}_1^*\mathscr{O}_{\mathbb{P}^{n+1}}\big(-\delta m(d-n-2)\big)\big|_{\text{pr}_2^{-1}(U)}$$

which is polynomial of weighted degree m in the jet variables. Let V be any of the global invariant holomorphic vector fields on $J_n\mathscr{V}$ with values in $p^*\text{pr}_1^*\mathscr{O}_{\mathbb{P}^{n+1}}(n^2+2n)$ that were provided by Theorem 2.5. Then we observe that the Lie derivative $L_V P$ together with the natural duality pairing

$$\mathscr{O}_{\mathbb{P}^{n+1}}(p) \times \mathscr{O}_{\mathbb{P}^{n+1}}(-q) \to \mathscr{O}_{\mathbb{P}^{n+1}}(p-q) \qquad\qquad (p,q \geqslant 1)$$

provides a new holomorphic map (notice the shift by n^2+2n):

$$L_V P \colon J_n\mathscr{V}\big|_{\text{pr}_2^{-1}(U)} \longrightarrow p^*\text{pr}_1^*\mathscr{O}_{\mathbb{P}^{n+1}}\big(-\delta m(d-n-2)+n^2+2n\big)\big|_{\text{pr}_2^{-1}(U)},$$

again polynomial of weighted degree m in the jet variables, thus a new parameterized family of invariant jet differentials. In particular, the restriction $L_V P|_{s_0}$ of $L_V P$ to $\{s = s_0\}$ yields a *nonzero* global holomorphic section in

$$\begin{aligned} &H^0\big(X_{s_0},\, E_{n,m}T^*_{X_{s_0}} \otimes K^{-\delta m}_{X_{s_0}} \otimes \mathscr{O}_{X_{s_0}}(n^2+2n)\big) = \\ &\qquad = H^0\big(X_{s_0},\, E_{n,m}T^*_{X_{s_0}} \otimes \mathscr{O}_{X_{s_0}}(-\delta m(d-n-2)+n^2+2n)\big), \end{aligned}$$

which is a global invariant jet differential on X_{s_0} vanishing on an ample divisor provided that $-\delta m(d-n-2)+n^2+2n$ *still remains negative*; therefore, if we ensure such a negativity (*see* below), Theorem 2.2 shows that $[L_V P|_{s_0}]\big(j^n f(t)\big) \equiv 0$. As a result, the n-jet of f now satisfies *two* global algebraic differential equations:

$$P_{s_0}\big(j^n f(t)\big) \equiv \big[L_V P|_{s_0}\big]\big(j^n f(t)\big) \equiv 0.$$

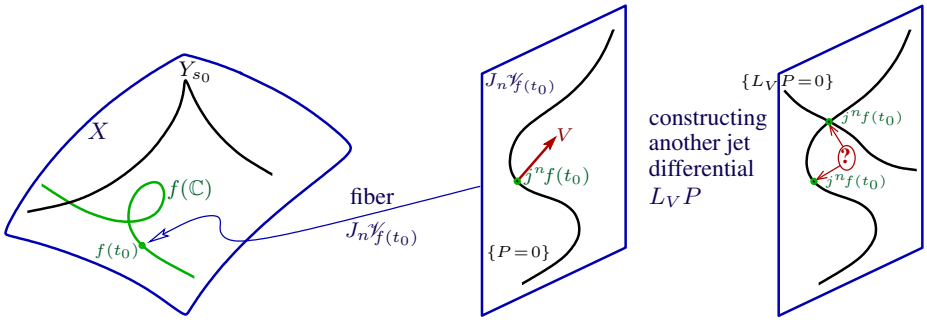

Fig. 1: Producing from P a new jet differential L_VP having distinct zero locus in $J_n\mathscr{V}$

Heuristically (*cf.* the figure), if the fiber $J_n\mathscr{V}_{f(t_0)}$ would be, say, 2-dimensional, and if the intersection of $\{P_{s_0} = 0\}$ with $\{L_VP|_{s_0} = 0\}$, viewed in the fiber $J_n\mathscr{V}_{f(t_0)}$, would be a point *distinct from the original* $j^nf(t_0)$, we would get the sought contradiction. Now we realize this idea (*cf.* [35, 26, 18]) by producing enough new jet differential divisors whose intersection becomes *empty*.

Indeed, with t_0 such that $f(t_0) \notin Y_{s_0}$ and $j^nf(t_0) \in J_n^{\mathrm{reg}}\mathscr{V}$, and with W_i, V_j denoting some global meromorphic vector fields in

$$H^0\big(J_n\mathscr{V},\, T_{J_n\mathscr{V}} \otimes p^*\mathrm{pr}_1^*\mathscr{O}_{\mathbb{P}^{n+1}}(n^2+2n) \otimes p^*\mathrm{pr}_2^*\mathscr{O}_{\mathbb{P}^{N_d^n}}(1)\big),$$

that are supplied by Theorem 2.5, we claim that the following two *evidently contradictory* conditions can be satisfied, and this will achieve the proof.

(i) For every $p \leqslant m$ and for arbitrary such fields $W_1, \dots, W_p$, the restriction $L_{W_p} \cdots L_{W_1}P\big|_{s_0}$ yields a nonzero global holomorphic section in

$$H^0\big(X_{s_0},\, E_{n,m}T^*_{X_{s_0}} \otimes \mathscr{O}_{X_{s_0}}(-\delta m(d-n-2) + p(n^2+2n))\big)$$

with the property that $\big[L_{W_p} \cdots L_{W_1}P\big]\big(s_0,\, j^nf(t)\big) \equiv 0$.

(ii) there exist some $p \leqslant m$ and some invariant fields $V_1, \dots, V_p$ such that $\big[L_{V_p} \cdots L_{V_1}P\big]\big(s_0,\, j^nf(t_0)\big) \neq 0$.

The first condition **(i)** will automatically be ensured by Theorem 2.2 provided the resulting jet differential still vanishes on an ample divisor, *i.e.* provided that

$$-\delta m(d-n-2) + p(n^2+2n) < 0$$

is still negative. But since p will be $\leqslant m$, it suffices that $-\delta m(d-n-2) + m(n^2+2n) < 0$, and then after erasing m, that:

$$d > \tfrac{n^2+2n}{\delta} + n + 2. \tag{49}$$

To get **(i)**, we first fix a rational $\delta > 0$ so that Theorem 2.4 gives a *nonzero* jet differential for any $d \geqslant d_{n,\delta}$, we increase (if necessary) this lower bound

by taking account of (49), we construct the holomorphic family $P|_s$, and **(i)** holds.

To establish **(ii)**, we choose local coordinates:

$$\big(s, z, z', \dots, z^{(n)}\big) \in \mathbb{C}^{N_d^n} \times \mathbb{C}^n \times \mathbb{C}^n \times \cdots \times \mathbb{C}^n$$

on $J_n\mathscr{V}$ near $\big(s_0, j^n f(t_0)\big)$, where $z \in \mathbb{C}^n$ provides some local coordinates on X_s for any fixed s near s_0, and where $\big(z', \dots, z^{(n)}\big)$ are the jet coordinates associated with z. We also choose a local trivialization of the line bundle $K_{X_s}^{-\delta m}$. Then our holomorphic family of jet differentials $P|_s \in H^0\big(X_s, E_{n,m}T^*_{X_s} \otimes K_{X_s}^{-\delta m}\big)$ writes locally as a weighted m-homogeneous jet-polynomial:

$$P = \sum_{|i_1|+\cdots+n|i_n|=m} q_{i_1,\dots,i_n}(s,z)\,(z')^{i_1}\cdots(z^{(n)})^{i_n},$$

where $i_1, \dots, i_n \in \mathbb{N}^n$ and where the $q_{i_1,\dots,i_n}(s,z)$ are holomorphic near $(s_0, f(t_0))$. Locally, the proper subvariety $Y_{s_0} \subset X$ is represented as the common zero-locus:

$$Y_{s_0} = \big\{z \in X_{s_0} \colon q_{i_1,\dots,i_n}(s_0, z) = 0, \ \ \forall\, i_1, \dots, i_n\big\}.$$

By our assumption that $f(t_0) \notin Y_{s_0}$, there exist $i_1^0, \dots, i_n^0 \in \mathbb{N}^n$ such that $q_{i_1^0,\dots,i_n^0}\big(s_0, f(t_0)\big) \neq 0$. If we make the translational change of jet coordinates $\overline{z}' := z' - f'(t_0)$, $\dots$, $\overline{z}^{(n)} := z^{(n)} - f^{(n)}(t_0)$, our jet-polynomial transfers to:

$$\overline{P} = \sum_{|i_1|+\cdots+n|i_n|\leqslant m} \overline{q}_{i_1,\dots,i_n}(s,z)\,(\overline{z}')^{i_1}\cdots(\overline{z}^{(n)})^{i_n},$$

(notice "$\leqslant m$") with new coefficients $\overline{q}_{i_1,\dots,i_n}(s,z)$ that depend linearly upon the old ones and polynomially upon $\big(f'(t_0), \dots, f^{(n)}(t_0)\big)$. Again, there exist $\overline{i}_1^0, \dots, \overline{i}_n^0 \in \mathbb{N}^n$ such that $\overline{q}_{\overline{i}_1^0,\dots,\overline{i}_n^0}\big(s_0, f(t_0)\big) \neq 0$, because otherwise the two jet-polynomials $P\big|_{s_0,f(t_0)}$ and $\overline{P}\big|_{s_0,f(t_0)}$ would be both identically zero.

Since $j^n f(t_0) \in J_n^{\mathrm{reg}}\mathscr{V}$, by the property 2.5 of generation by global sections, we get that for every k with $1 \leqslant k \leqslant n$ and for every i with $1 \leqslant i \leqslant n$, there exists an invariant vector field V_i^k with

$$V_i^k\big|_{(s_0,\overline{j}^n f(t_0))} = \frac{\partial}{\partial \overline{z}_i^{(k)}}\bigg|_{(s_0,\overline{j}^n f(t_0))},$$

where we have denoted the translated central jet by $\overline{j}^n f(t_0) := \big(f(t_0), 0, \dots, 0\big)$.

To achieve the proof of **(ii)**, we may suppose that for every integer p with $p < |\overline{i}_1^0| + |\overline{i}_2^0| + \cdots + |\overline{i}_n^0|$, whence $p < |\overline{i}_1^0| + 2\,|\overline{i}_2^0| + \cdots + n\,|\overline{i}_n^0| = m$, and for every p invariant vector fields $W_1, \dots, W_p$, one has $\big[W_1 \cdots W_p\, \overline{P}\,\big]\big(s_0, \overline{j}^n f(t_0)\big) = 0$, since if any such an expression is already

$\neq 0$, **(ii)** would be got gratuitously. Thanks to the global generation Theorem 2.5, this vanishing property then holds for any vector fields W_i involving all the possible differentiations $\frac{\partial}{\partial s}$, $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \overline{z}'}$, $\ldots$, $\frac{\partial}{\partial \overline{z}^{(n)}}$. Then under this assumption, the contribution of the remainder differentiations present in V_i^k after $\partial/\partial \overline{z}_i^{(k)}\big|_{(s_0, \overline{j}^n f(t_0))}$ will vanish at the point $\big(s_0, \overline{j}^n f(t_0)\big)$ when performing any multi-derivation of length equal to $|\overline{i}_1^0| + \cdots + |\overline{i}_n^0|$, hence if we write in length $\overline{i}_k^0 = \big(\overline{i}_{k,1}^0, \ldots, \overline{i}_{k,n}^0\big) \in \mathbb{N}^n$ all the multiindices present in the specific coefficient $\overline{q}_{\overline{i}_1^0, \ldots, \overline{i}_n^0}$, it follows that:

$$\begin{aligned}
&\big[V^n_{\overline{i}^0_{n,n}} \cdots V^n_{\overline{i}^0_{n,1}} \cdots\cdots V^1_{\overline{i}^0_{1,n}} \cdots V^1_{\overline{i}^0_{1,n}} \overline{P}\,\big]\big(s_0,\, \overline{j}^n f(t_0)\big) = \\
&\qquad = \Big[\frac{\partial}{\partial \overline{z}^{(n)}_{\overline{i}^0_{n,n}}} \cdots \frac{\partial}{\partial \overline{z}^{(n)}_{\overline{i}^0_{n,1}}} \cdots\cdots \frac{\partial}{\partial \overline{z}^{(1)}_{\overline{i}^0_{1,n}}} \cdots \frac{\partial}{\partial \overline{z}^{(1)}_{\overline{i}^0_{1,1}}} \overline{P}\Big]\big(s_0,\, f(t_0), 0, \ldots, 0\big) \\
&\qquad = \overline{i}^0_{n,n}! \cdots \overline{i}^0_{n,1}! \cdots\cdots \overline{i}^0_{1,n}! \cdots \overline{i}^0_{1,1}!\; \overline{q}_{\overline{i}^0_1, \ldots, \overline{i}^0_n}\big(s_0,\, f(t_0)\big) \neq 0,
\end{aligned}$$

which is nonzero. Thus **(ii)** holds and the proof of Theorem 3.1 is complete. Theorem 3.1 being *not* effective regarding the condition $d \geqslant d_n$, the next two Sections 4 and 5 are devoted to the proof of the effective main Theorem 1.1. $\square$

4. Effectiveness of the degree lower bound

It is known (*cf.* [33, 4, 25, 35, 29, 6, 22]) that reaching an explicit lower bound degree $\deg X \geqslant d_n$ both for Green-Griffiths algebraic degeneracy and for Kobayashi hyperbolicity (in nonoptimal degree) still remained an open question in arbitrary dimension n, due to the existence of *substantial algebraic obstacles*. In order to render somewhat explicit the lower bound d_n of Theorem 3.1, one has to expand the n^2-powered intersection product (48) and then to reduce it as an explicit polynomial $\mathsf{P}_{\mathbf{a},\delta}(d)$, as was foreseen in the proof of Theorem 2.4. To this aim, one should descend Demailly's tower *step by step*, each time using the two relations (42) and (43). As a matter of fact, one must perform some numerous, explicit eliminations and substitutions and thereby tame the exponential growth of computations. At several places, we shall leave aside optimality of majorations in order to reach the neat announced lower bound 2^{n^5}.

4.1. **Reduction of the basic intersection product.**

We remind from Theorem 2.4 that, in order to produce a global invariant jet differential with controlled vanishing order on hypersurfaces X whose degree $d \geqslant d_n$ would be bounded from below by an effectively known function $d_n = d(n)$ of n, we

should ensure *in an effective way* the positivity of the intersection product:

$$\big(\mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|)\big)^{n^2} - \\ - n^2 \big(\mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|)\big)^{n^2-1} \cdot \big(\pi_{0,n}^* \mathscr{O}_X(2|\mathbf{a}|) \otimes \pi_{0,n}^* K_X^{\delta|\mathbf{a}|}\big),$$

for a certain n-tuple of integers $\mathbf{a} = \mathbf{a}(n) \in \mathbb{N}^n$ belonging to the cone (46) (with $k = n$) which would depend *effectively* upon n, and for a certain rational number $\delta = \delta(n) > 0$ which would also depend *effectively* upon n.

As in [7], denote $u_\ell = c_1\big(\mathscr{O}_{X_\ell}(1)\big)$ for $\ell = 1, \dots, n$, denote $c_k = c_k(T_X)$ for $k = 1, \dots, n$, and $h = c_1\big(\mathscr{O}_X(1)\big)$. With these standard notations, the intersection product we have to evaluate becomes:

$$\Pi_\delta := \big(a_1 u_1 + \cdots + a_n u_n + 2|\mathbf{a}|h\big)^{n^2} - \\ - n^2 \big(a_1 u_1 + \cdots + a_n u_n + 2|\mathbf{a}|h\big)^{n^2-1} \cdot \big(2|\mathbf{a}|h - \delta|\mathbf{a}|c_1\big); \tag{50}$$

here and from now on, admitting a slight abuse of notation which will greatly facilitate the reading of formal computations, *we systematically omit every pull-back symbol* $\pi_{j,k}^*(\bullet)$. After elimination and reduction using the relations (42) and (43) (*see* below), our intersection product gives in principle a polynomial (difficult to compute, *see* the end of the paper) of degree $\leqslant n+1$ with respect to $d = \deg X$, which is affine in δ, and all of which coefficients are homogeneous polynomials in $\mathbf{a}$ of degree n^2. Thus, let us call it:

$$\mathsf{P}_{\mathbf{a},\delta}(d) = \mathsf{P}_{\mathbf{a}}(d) + \delta\, \mathsf{P}'_{\mathbf{a}}(d) = \sum_{k=0}^{n+1} \mathsf{p}_{k,\mathbf{a}}\, d^k + \delta \sum_{k=0}^{n+1} \mathsf{p}'_{k,\mathbf{a}}\, d^k.$$

Now, suppose in advance that we have an effective control, through explicit inequalities, of all the coefficients $\mathsf{p}_{k,\mathbf{a}} \in \mathbb{Z}$ and $\mathsf{p}'_{k,\mathbf{a}} \in \mathbb{Z}$ of both $\mathsf{P}_{\mathbf{a}}$ and $\mathsf{P}'_{\mathbf{a}}$, and more precisely, that we already know inequalities of the type:

$$|\mathsf{p}_{k,\mathbf{a}}| \leqslant \mathsf{E}_k \quad {\scriptstyle (k=0,\dots,n)}, \qquad \mathsf{p}_{n+1,\mathbf{a}} \geqslant \mathsf{G}_{n+1}, \qquad |\mathsf{p}'_{k,\mathbf{a}}| \leqslant \mathsf{E}'_k \quad {\scriptstyle (k=0,\dots,n,n+1)},$$

with the $\mathsf{E}_k \in \mathbb{N}$, with $\mathsf{G}_{n+1} \in \mathbb{N} \setminus \{0\}$ and with the $\mathsf{E}'_k \in \mathbb{N}$ all depending upon n only. According to the proof of Theorem 2.4, a good choice of weight $\mathbf{a}$ indeed makes $\mathsf{p}_{n+1,\mathbf{a}}$ positive; we will see below that $\mathsf{p}'_{n+1,\mathbf{a}}$ is then necessarily negative.

If we now set $\delta := \frac{1}{2}\frac{\mathsf{G}_{n+1}}{\mathsf{E}'_{n+1}}$ so that δ also depends *a posteriori* explicitly upon n, the leading d^{n+1}-coefficient of $\mathsf{P}_{\mathbf{a},\delta}$ becomes positive and bounded from below:

$$\mathsf{p}_{n+1,\mathbf{a}} + \delta\, \mathsf{p}'_{n+1,\mathbf{a}} = \mathsf{p}_{n+1,\mathbf{a}} - \delta\, \big|\mathsf{p}'_{n+1,\mathbf{a}}\big| \geqslant \mathsf{G}_{n+1} - \tfrac{1}{2}\tfrac{\mathsf{G}_{n+1}}{\mathsf{E}'_{n+1}}\, \mathsf{E}'_{n+1} = \tfrac{1}{2}\, \mathsf{G}_{n+1}.$$

The largest real root of a polynomial $\mathsf{a}_{n+1}\, d^{n+1} + \mathsf{a}_n\, d^n + \cdots + \mathsf{a}_0$ having integer coefficients and positive leading coefficient $\mathsf{a}_{n+1} \geqslant 1$ may be checked to be less than $1 + (\mathsf{a}_n + \cdots + \mathsf{a}_0)\big/\mathsf{a}_{n+1}$; instead of the finer bound

$2\max_{0\leqslant j\leqslant n}\left(\frac{|\mathsf{a}_j|}{|\mathsf{a}_{n+1}|}\right)^{1/n+1-j}$, we use this easier-to-write-down majoration because at the end of Section 4, this will make no difference in reaching the bound $\deg X \geqslant 2^{n^5}$ of Theorem 1.1. Applied to our situation:

Lemma 4.1. *If one chooses* $\delta := \frac{1}{2}\frac{\mathsf{G}_{n+1}}{\mathsf{E}'_{n+1}}$, *then the intersection product* $\sum_{k=0}^{n+1}\left(\mathsf{p}_{k,\mathbf{a}}+\delta\,\mathsf{p}'_{k,\mathbf{a}}\right)d^k$ *has positive leading coefficient* $\mathsf{p}_{n+1,\mathbf{a}}+\delta\,\mathsf{p}'_{n+1,\mathbf{a}} \geqslant \frac{1}{2}\mathsf{G}_{n+1}$ *and has other coefficients enjoying the majorations:*

$$\left|\mathsf{p}_{k,\mathbf{a}}+\delta\,\mathsf{p}'_{k,\mathbf{a}}\right| \leqslant \mathsf{E}_k + \frac{1}{2}\frac{\mathsf{G}_{n+1}}{\mathsf{E}'_{n+1}}\mathsf{E}'_k \qquad (k=0,\dots,n),$$

and therefore it takes only positive values for all degrees

$$d \geqslant 1 + \left(\mathsf{E}_n+\cdots+\mathsf{E}_0+\frac{1}{2}\frac{\mathsf{G}_{n+1}}{\mathsf{E}'_{n+1}}\left\{\mathsf{E}'_n+\cdots+\mathsf{E}'_0\right\}\right)\Big/\frac{1}{2}\mathsf{G}_{n+1} =: d_n^1. \quad \square$$

Thus, this d_n^1 will be effectively known in terms of n when E_k, G_{n+1}, E'_k will be so. In order to have not only the existence of global invariant jet differentials with controlled vanishing order, but also algebraic degeneracy, we have also to take account of condition (49), and this condition now reads:

$$d \geqslant 1+n+2+2\left(n^2+2n\right)\frac{\mathsf{E}'_{n+1}}{\mathsf{G}_{n+1}} =: d_n^2.$$

In conclusion, we would obtain the *effective* estimate of Theorem 1.1 provided we compute the bounds E_k, G_{n+1}, E'_k in terms of n and provided we establish that:

$$2^{n^5} \geqslant \max\left\{d_n^1, d_n^2\right\} =: d_n. \tag{51}$$

4.2. **Expanding the intersection product.** By expanding the n^2- and the (n^2-1)-powers, the intersection product Π_δ in (50) writes as a certain sum, with coefficients being polynomials in $\mathbb{Z}\left[a_1,\dots,a_n,\delta\right]$, of monomials in the present Chern classes that are of the general form:

$$h^l u_1^{i_1}\cdots u_n^{i_n} \quad \text{or} \quad h^l c_1 u_1^{j_1}\cdots u_n^{j_n},$$

where $l+i_1+\cdots+i_n=n^2$ or $l+1+j_1+\cdots+j_n=n^2$.

Lemma 4.2 ([4, 6])**.** *After several elimination computations which take account of the relations* (42) *and* (43)*, any such monomial reduces to a certain polynomial in* $\mathbb{Z}\left[h,c_1,\dots,c_n\right]$ *which is homogeneous of degree* $n=\dim X$*, if* h *is assigned the weight* 1 *and each* c_k *receives the weight* k*. Furthermore, after a last substitution by means of* (45) *which uses* $h^n \equiv \int_X h^n = d = \deg X$*, the polynomial in question becomes a plain polynomial in* $\mathbb{Z}[d]$ *of degree* $\leqslant n+1$. $\square$

We illustrate with $h^l u_1^{i_1}\cdots u_{n-1}^{i_{n-1}}u_n^{i_n}$ three fundamental processes of reduction that will be intensively used. Recall that any *sub*monomial $h^l u_1^{i_1}\cdots u_\ell^{i_\ell} = \pi_{0,\ell}^*(h^l)\pi_{1,\ell}^*(u_1^{i_1})\cdots u_\ell^{i_\ell}$ denotes a differential form living X_ℓ and that $\dim X_\ell = n+\ell(n-1)$. Such a form is of bidegree (p,p) where

$p = l + i_1 + \cdots + i_\ell$. We shall allow the (slight) abuse of language to say that p itself is the *degree* of a (p,p)-form.

At first, if $i_n \leqslant n-2$, then $l+i_1+\cdots+i_{n-1} \geqslant n^2-n+2 = 1+\dim_{\mathbb{C}} X_{n-1}$, whence the (sub)form $h^l u_1^{i_1} \cdots u_{n-1}^{i_{n-1}}$ which lives on X_{n-1} annihilates, as then does $h^l u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} u_n^{i_n}$ too. We call this (straightforward) first kind of reduction process:

"vanishing for degree-form reasons",

and we symbolically point out the annihilating subform by underlining it with a small circle appended, *viz.*:

$$\underline{h^l u_1^{i_1} \cdots u_{n-1}^{i_{n-1}}}_{\circ} u_n^{i_n} = 0 \qquad \text{when } \ i_n \leqslant n-2.$$

This will greatly improve readability of elimination computations below.

Secondly, in the case where $i_n = n-1$, using an appropriate version of the Fubini theorem and taking account of the fact that $\int_{\text{fiber}} u_n^{n-1} = \int_{\mathbb{P}^{n-1}} u_n^{n-1} = 1$, where all the fibers of $\pi_{n-1,n} : X_n \to X_{n-1}$ are $\simeq \mathbb{P}^{n-1}(\mathbb{C})$ ([4, 18, 6, 7]), we may simplify as follows our monomial:

$$h^l u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} \underline{u_n^{n-1}}_{\int} = h^l u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} \cdot 1 = h^l u_1^{i_1} \cdots u_{n-1}^{i_{n-1}}.$$

We shall call this second kind of reduction process:

"fiber-integration".

The third process of course consists in substituting the two relations (42) and (43) as many times as necessary. With $r = n$ and without any $\pi_{j,k}^*(\bullet)$, they now read:

$$c_j^{[\ell]} = \sum_{k=0}^{j} \lambda_{j,j-k} \cdot c_k^{[\ell-1]} \big(u_\ell\big)^{j-k}, \tag{52}$$

where $1 \leqslant j,\, \ell \leqslant n$, with the conventions $c_0^{[\ell]} = 1$ and $c_j^{[0]} = c_j$, where we set

$$\lambda_{j,j-k} := \tbinom{n-k}{j-k} - \tbinom{n-k}{j-k-1} = \tfrac{(n-k)!}{(j-k)!\,(n-j)!} - \tfrac{(n-k)!}{(j-k-1)!(n-j+1)!},$$

and also, with upper indices of u_ℓ denoting exponents:

$$u_\ell^n = -c_1^{[\ell-1]}\, u_\ell^{n-1} - c_2^{[\ell-1]}\, u_\ell^{n-2} - \cdots - c_{n-1}^{[\ell-1]}\, u_\ell - c_n^{[\ell-1]}. \tag{53}$$

Estimating the coefficient of d^{n+1}. Our first main task is to reach a lower bound $\mathsf{G}_{n+1} - \delta\, \mathsf{E}'_{n+1}$ for the coefficient of d^{n+1} in Π_δ, and this cannot be straightforward, because there are *very numerous* monomials in the expansion of Π_δ. In a first reading, one might jump directly to Subsection 4.4 just after Corollary 4.1. Here is an initial observation.

Lemma 4.3 ([7]). *Assume $l+i_1+\cdots+i_n=n^2$ or $l+1+j_1+\cdots+j_n=n^2$. Then as soon as $l\geqslant 1$, one has:*

$$0=\mathsf{coeff}_{d^{n+1}}\big[h^l u_1^{i_1}\cdots u_n^{i_n}\big]\quad\text{and}\quad 0=\mathsf{coeff}_{d^{n+1}}\big[h^l c_1 u_1^{j_1}\cdots u_n^{j_n}\big].$$

Proof. Indeed, after reduction of either u-monomial in terms of the Chern classes c_k of the base, one obtains a sum with integer coefficients of terms of the form:

$$h^l c_1^{\lambda_1} c_2^{\lambda_2}\cdots c_n^{\lambda_n}$$

with $l+\lambda_1+2\lambda_2+\cdots+n\lambda_n=n$. But then if we replace the Chern classes by their expressions (45) in terms of h and of the degree, we get:

$$\begin{aligned}\mathsf{coeff}_{d^{n+1}}\big[h^l c_1^{\lambda_1} c_2^{\lambda_2}\cdots c_n^{\lambda_n}\big]&=\mathsf{coeff}_{d^{n+1}}\big[(-1)^{\lambda_1+\cdots+\lambda_n}\,h^n\cdot d^{\lambda_1+\lambda_2+\cdots+\lambda_n}+\mathsf{l.o.t}\big]\\&=\mathsf{coeff}_{d^{n+1}}\big[(-1)^{\lambda_1+\cdots+\lambda_n}\,d\cdot d^{\lambda_1+\lambda_2+\cdots+\lambda_n}+\mathsf{l.o.t}\big]\\&=0,\end{aligned}$$

since $1+\lambda_1+\lambda_2+\cdots+\lambda_n\leqslant l+\lambda_1+2\lambda_2+\cdots+n\lambda_n=n$. □

As a result, a glance at (50) immediately shows that:

$$\mathrm{coeff}_{d^{n+1}}\big[\Pi_\delta\big]=\mathrm{coeff}_{d^{n+1}}\Big[\big(a_1u_1+\cdots+a_nu_n\big)^{n^2}+\delta|\mathbf{a}|c_1\big(a_1u_1+\cdots+a_nu_n\big)^{n^2-1}\Big].$$

4.3. **Reverse lexicographic ordering for the u-monomials.** We order the collection of all homogeneous monomials $u_1^{i_1}\cdots u_n^{i_n}$ with $i_1+\cdots+i_n=n^2$ appearing in the expansion of $\big(a_1u_1+\cdots+a_nu_n\big)^{n^2}$ above by declaring that the monomial $u_1^{i_1}\cdots u_n^{i_n}$ is *smaller*, for the *reverse lexicographic ordering*, than another monomial $u_1^{j_1}\cdots u_n^{j_n}$, again of course with $j_1+\cdots+j_n=n^2$, if:

$$\begin{cases} \quad i_n>j_n\\ \text{or if } \ i_n=j_n \ \text{ but } \ i_{n-1}>j_{n-1}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \text{or if } \ i_n=j_n,\ \ldots,\ i_3=j_3 \ \text{ but } \ i_2>j_2.\end{cases}$$

Observe that $i_n=j_n,\ldots,i_2=j_2$ implies $i_1=j_1$. An equivalent language says that the multiindices themselves are ordered in this way:

$$(i_1,\ldots,i_n)<_{\mathsf{revlex}}(j_1,\ldots,j_n).$$

Proposition 4.1. *The coefficient of d^{n+1} in any monomial $u_1^{i_1}\cdots u_n^{i_n}$ which is* larger *than $u_1^n\cdots u_n^n$ is zero:*

$$\mathsf{coeff}_{d^{n+1}}\big[u_1^{i_1}\cdots u_n^{i_n}\big]=0\qquad\text{for any}\qquad(i_1,\ldots,i_n)>_{\mathsf{revlex}}(n,\ldots,n).$$

Proof. Thus, assume $(i_1,\ldots,i_n)>_{\mathsf{revlex}}(n,\ldots,n)$. Firstly, if $i_n=n$, the claimed vanishing property is in all concerned subcases yielded by **(iii)** of the lemma just below. Secondly, if $i_n=n-1$, an integration on the fiber of $\pi_{n-1,n}:X_n\to X_{n-1}$ replaces u_n^{n-1} by the constant $+1$, hence we are

left with $u_1^{i_1}\cdots u_{n-1}^{i_{n-1}}$ and **(i)** of the same lemma then yields the conclusion. Thirdly and lastly, if $i_n \leqslant n-2$, then the form $u_1^{i_1}\cdots u_{n-1}^{i_{n-1}}$ vanishes identically for degree-form reasons. Thus, granted the lemma, the proposition is proved. $\square$

Lemma 4.4. *The coefficient of d^{n+1} in all the following four sorts of u-monomials is equal to zero:*

- **(i)** $u_1^{i_1}\cdots u_k^{i_k}$ *for any* $k \leqslant n-1$ *and any* $i_1,\dots,i_k$ *with* $i_1+\cdots+i_k = n+k(n-1)$;
- **(ii)** $(c_1)^{n-k}\,u_1^{i_1}\cdots u_k^{i_k}$ *for any* $k \leqslant n-1$, *and any* $i_1,\dots,i_k$ *with* $i_k \leqslant n-1$ *and* $i_1+\cdots+i_k = kn$;
- **(iii)** $u_1^{i_1}\cdots u_l^{i_l}\,u_{l+1}^n\cdots u_n^n$ *for any* $l \leqslant n$, *any* $i_1,\dots,i_l$ *with* $i_l \leqslant n-1$ *and* $i_1+\cdots+i_l = ln$;
- **(iv)** $c_1 u_1^{i_1}\cdots u_l^{i_l} u_{l+1}^n\cdots u_{n-1}^n$ *for any* $l \leqslant n-1$, *any* $i_l \leqslant n-1$, *any* $i_1,\dots,i_l$ *with* $i_1+\cdots+i_l = ln$.

Proof. Property **(i)** is established in Section 3 of [7]. So **(i)** holds.

Applying (52) written for $j=1$, namely $c_1^{[\ell]} = c_1^{[\ell-1]} + (n-1)\,u_\ell$, we get:

$$c_1^{[\ell]} = c_1 + (n-1)\,u_1 + \cdots + (n-1)\,u_\ell. \tag{54}$$

To begin with, we start from **(i)** for $k=n-1$, $i_{n-1}=n$ and $i_1+\cdots+i_{n-2} = n+(n-1)(n-1)-i_{n-1} = n^2-2n+1$ arbitrary, namely:

$$0 = \mathsf{coeff}_{d^{n+1}}\big[u_1^{i_1}\cdots u_{n-2}^{i_{n-2}} u_{n-1}^n\big].$$

Next, thanks to (53), we may replace in this equality u_{n-1}^n by $-c_1^{[n-2]}u_{n-1}^{n-1} - c_2^{[n-2]}u_{n-1}^{n-2} - \cdots - c_n^{[n-2]}$:

$$\begin{aligned}
0 &= \mathsf{coeff}_{d^{n+1}}\Big[u_1^{i_1}\cdots u_{n-2}^{i_{n-2}}\big(-c_1^{[n-2]}u_{n-1}^{n-1} - \underline{c_2^{[n-2]}u_{n-1}^{n-2} - \cdots - c_n^{[n-2]}}_{\circ}\big)\Big]\\
&= \mathsf{coeff}_{d^{n+1}}\big[u_1^{i_1}\cdots u_{n-2}^{i_{n-2}}\big(-c_1^{[n-2]}\,u_{n-1}^{n-1}\big)\big] \qquad \text{[degree-form reasons]} \quad \text{[use (54)]}\\
&= \mathsf{coeff}_{d^{n+1}}\big[u_1^{i_1}\cdots u_{n-2}^{i_{n-2}}\big(-c_1 - \underline{(n-1)u_1 - \cdots - (n-1)u_{n-2}}_{\circ}\big)\,u_{n-1}^{n-1}\big]\\
&= \mathsf{coeff}_{d^{n+1}}\big[-c_1 u_1^{i_1}\cdots u_{n-2}^{i_{n-2}} u_{n-1}^{n-1}\big] \qquad \text{[apply (i) again]},
\end{aligned}$$

and we therefore get **(ii)** for $k=n-1$ when $i_{n-1}=n-1$. But in all the other remaining cases when $i_{n-1} \leqslant n-2$, then by the assumption that the sum of the indices i_l is equal to $(n-1)n$:

$$i_1+\cdots+i_{n-2} \geqslant (n-1)n-(n-2) = n^2-2n+2 = \dim X_{n-2},$$

and consequently, the degree of the form $c_1 u_1^{i_1}\cdots u_{n-2}^{i_{n-2}}$ is $\geqslant 1+\dim X_{n-2}$, whence this form vanishes identically. Thus **(ii)** is proved completely for $k=n-1$.

Next, consider **(iii)** for $l=n$. If $i_n \leqslant n-2$, then by degree-form reasons $0 \equiv u_1^{i_1}\cdots u_{n-1}^{i_{n-1}}$, whence $\mathsf{coeff}_{d^{n+1}}\big[u_1^{i_1}\cdots u_{n-1}^{i_{n-1}} u_n^{i_n}\big] = 0$ gratuitously. So

we assume $i_n = n-1$. But then $i_1 + \cdots + i_{n-1} = n^2 - n + 1$, hence **(i)** applies to give:

$$\begin{aligned} 0 &= \mathsf{coeff}_{d^{n+1}}\big[u_1^{i_1} \cdots u_{n-1}^{i_{n-1}}\big] \qquad \text{[reconstitute hidden integration of } u_n^{n-1}\text{]} \\ &= \mathsf{coeff}_{d^{n+1}}\big[u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} u_n^{n-1}\big], \end{aligned}$$

and therefore this proves **(iii)** completely for $l = n$. But we also get at the same time the property **(iii)** for $l = n-1$. Indeed, with $i_1 + \cdots + i_{n-1} = (n-1)n$ and with $i_{n-1} \leqslant n-1$, we may reduce, using (53):

$$\begin{aligned} u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} u_n^n &= u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} \big[- c_1^{[n-1]} u_n^{n-1} - \underline{c_2^{[n-1]} u_n^{n-2} - \cdots - c_n^{[n-1]}}_{\circ} \big] \\ &= u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} \big[- c_1^{[n-1]} u_n^{n-1} \big] \qquad \text{[degree-form reasons]} \quad \text{[use (54)]} \\ &= u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} \big[- c_1 - (n-1)u_1 - \cdots - (n-1)u_{n-1} \big] \end{aligned}$$

Thanks to **(i)**, after expansion, the pure u-monomials give no contribution to d^{n+1}, and consequently:

$$\mathsf{coeff}_{d^{n+1}}\big[u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} u_n^n\big] = \mathsf{coeff}_{d^{n+1}}\big[- c_1 u_1^{i_1} \cdots u_{n-1}^{i_{n-1}}\big] = 0,$$

where the last equality holds true thanks to the property **(ii)** already proved for $k = n-1$. Thus **(iii)** is completely proved for $l = n$ and for $l = n-1$.

Lastly, we just observe that **(iv)** for $l = n-1$ coincides with **(ii)** for $k = n-1$. In summary, we have completed a first loop of proofs.

Consider now the second loop. We start from **(ii)** for $k = n-1$ (already got) with $i_{n-1} = n-1$ and with $i_{n-2} = n$, so that $i_1 + \cdots + i_{n-3} = (n-1)n - i_{n-2} - i_{n-1} = n^2 - 3n + 1$, and then we compute:

$$\begin{aligned} 0 &= \mathsf{coeff}_{d^{n+1}}\big[c_1 u_1^{i_1} \cdots u_{n-3}^{i_{n-3}} u_{n-2}^n \underline{u_{n-1}^{n-1}}_{\int}\big] \qquad \text{[fiber-integration]} \\ &= \mathsf{coeff}_{d^{n+1}}\big[c_1 u_1^{i_1} \cdots u_{n-3}^{i_{n-3}} \big(- c_1^{[n-3]} u_{n-2}^{n-1} - \underline{c_2^{[n-3]} u_{n-2}^{n-2} - \cdots - c_n^{[n-3]}}_{\circ} \big)\big] \qquad \text{[use (53)]} \\ &= \mathsf{coeff}_{d^{n+1}}\big[c_1 u_1^{i_1} \cdots u_{n-3}^{i_{n-3}} \big(- c_1^{[n-3]} \big) u_{n-2}^{n-1}\big] \qquad \text{[degree-form reasons]} \quad \text{[use (54)]} \\ &= \mathsf{coeff}_{d^{n+1}}\big[c_1 u_1^{i_1} \cdots u_{n-3}^{i_{n-3}} \big(- c_1 - \underline{(n-1)u_1 - \cdots - (n-1)u_{n-3}}_{\circ} \big) u_{n-2}^{n-1} \underline{u_{n-1}^{n-1}}_{\int}\big] \\ &= \mathsf{coeff}_{d^{n+1}}\big[- c_1 c_1 u_1^{i_1} \cdots u_{n-3}^{i_{n-3}} u_{n-2}^{n-1} \underline{u_{n-1}^{n-1}}_{\int}\big] \qquad \text{[apply (ii) for } k = n-1 \text{ again]} \\ &= \mathsf{coeff}_{d^{n+1}}\big[- c_1 c_1 u_1^{i_1} \cdots u_{n-3}^{i_{n-3}} u_{n-2}^{n-1}\big] \qquad \text{[fiber-integration]}, \end{aligned}$$

where we have reintroduced u_{n-1}^{n-1} (artificially) in the fourth line, so as to apply **(ii)** for $k = n-1$ (got). As a result of the last obtained equation, we have gained **(ii)** for $k = n-2$ when $i_{n-2} = n-1$, but since when $i_{n-2} \leqslant n-2$, the form $c_1 c_1 u_1^{i_1} \cdots u_{n-3}^{i_{n-3}}$ vanishes identically for degree reasons, we finally have fully established **(ii)** for $k = n-2$.

Next, we look at **(iii)** for $l = n-2$. Then $i_1 + \cdots + i_{n-2} = (n-2)n$ with $i_{n-2} \leqslant n-1$. So we ask whether the following coefficient vanishes:

$$\begin{aligned}
&\mathsf{coeff}_{d^{n+1}}\big[u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} u_{n-1}^n u_n^n\big] = \\
&= \mathsf{coeff}_{d^{n+1}}\big[u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} u_{n-1}^n \big(c_1 - \underline{(n-1)u_1 - \cdots - (n-1)u_{n-1}}_{\circ}\big)\big] \\
&= \mathsf{coeff}_{d^{n+1}}\big[-c_1 u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} u_{n-1}^n\big] \\
&= \mathsf{coeff}_{d^{n+1}}\big[-c_1 u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \big(-c_1 - \underline{(n-1)u_1 - \cdots - (n-1)u_{n-2}}\big) u_{n-1}^{n-1}\big] \\
&= \mathsf{coeff}_{d^{n+1}}\big[c_1 c_1 u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \underline{u_{n-1}^{n-1}}_{\int}\big] \\
&= 0,
\end{aligned}$$

and in fact, this coefficient vanishes actually, thanks to **(ii)** for $k = n-2$ seen a moment ago. This therefore proves **(iii)** for $l = n-2$ completely.

Finally, consider **(iv)** for $l = n-2$. Then $i_1 + \cdots + i_{n-2} = (n-2)n$ and $i_{n-2} \leqslant n-1$. But coming back to the third line of the equations just above, where $i_{n-2} \leqslant n-1$ too, we have in fact already implicitly proved that:

$$0 = \mathsf{coeff}_{d^{n+1}}\big[c_1 u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} u_{n-1}^n\big],$$

and this is **(iv)** for $l = n-2$. Thus, the second loop is completed, and the general induction, similar, is now intuitively clear. □

Corollary 4.1. *The coefficient of d^{n+1} in any monomial $c_1 u_1^{j_1} \cdots u_{n-1}^{j_{n-1}} u_n^{j_n}$ with $1 + j_1 + \cdots + j_{n-1} + j_n = n^2$ which is larger than $c_1 u_1^n \cdots u_{n-1}^n u_n^{n-1}$ is zero:*

$$\mathsf{coeff}_{d^{n+1}}\big[c_1 u_1^{j_1} \cdots u_{n-1}^{j_{n-1}} u_n^{j_n}\big] = 0, \qquad \text{for any} \quad (j_1, \ldots, j_{n-1}, j_n) >_{\mathsf{revlex}} (n, \ldots, n, n-1).$$

Furthermore:

$$\begin{aligned}
\mathsf{coeff}_{d^{n+1}}\big[u_1^n \cdots u_{n-1}^n u_n^n\big] &= \mathsf{coeff}_{d^{n+1}}\big[(-1)^n (c_1)^n\big] = +1. \\
\mathsf{coeff}_{d^{n+1}}\big[c_1 u_1^n \cdots u_{n-1}^n u_n^{n-1}\big] &= \mathsf{coeff}_{d^{n+1}}\big[(-1)^{n-1} (c_1)^n\big] = -1.
\end{aligned}$$

Proof. The first claim is just a rephrasing of the property **(iv)** of the lemma, after one notices that $c_1 u_1^{j_1} \cdots u_{n-1}^{j_{n-1}} u_n^{j_n}$ vanishes identically for degree reasons when $j_n \leqslant n-2$, while the term $u_n^{n-1} = u_n^{j_n}$ disappears after fiber integration when $j_n = n-1$. The identities stated just after now have obvious proofs. □

4.4. **Minorating** $\mathsf{coeff}_{d^{n+1}}\big[\Pi\big]$**.** Let us decompose the intersection product Π_δ defined by (50) as $\Pi + \delta\Pi'$, where:

$$\begin{aligned}
\Pi &:= \big(a_1 u_1 + \cdots + a_n u_n + 2|\mathbf{a}|h\big)^{n^2} - n^2 h \big(a_1 u_1 + \cdots + a_n u_n + 2|\mathbf{a}|h\big)^{n^2-1} 2|\mathbf{a}|, \\
\Pi' &:= n^2 c_1 \big(a_1 u_1 + \cdots + a_n u_n + 2|\mathbf{a}|h\big)^{n^2-1} |\mathbf{a}|.
\end{aligned}$$

The (ineffective) Lemma 4.2 insures that the reduction of Π in terms of $d = \deg X$ is a certain polynomial:

$$\mathsf{P}_{\mathbf{a}}(d) = \sum_{k=0}^{n+1} \mathsf{p}_{k,\mathbf{a}}\, d^k,$$

having certain coefficients $\mathsf{p}_{k,\mathbf{a}} \in \mathbb{Z}\big[a_1, \dots, a_n\big]$. Moreover, Lemma 4.3 showed that positive powers of h do not contribute to the leading coefficient, whence:

$$\begin{aligned}\mathsf{p}_{n+1,\mathbf{a}} = \mathsf{coeff}_{d^{n+1}}\big[\Pi\big] &= \mathsf{coeff}_{d^{n+1}}\big[\big(a_1u_1 + \cdots + a_nu_n\big)^{n^2}\big] \\ &= \mathsf{coeff}_{d^{n+1}}\big[\big(a_1u_1 + \cdots + a_nu_n + 2|\mathbf{a}|h\big)^{n^2}\big].\end{aligned}$$

By Proposition 2 in [7], the bundle:

$$\mathscr{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^*\mathscr{O}_X(2|\mathbf{a}|)$$

is nef whenever $\mathbf{a}$ belongs to the cone defined by (46), therefore its top self-intersection must be non-negative. Thus, once this top self-intersection is evaluated in term of the degree d of the hypersurface, its dominating coefficient must be non-negative, too. In other words we must have:

$$\mathsf{p}_{n+1,\mathbf{a}} \geqslant 0.$$

But from the corollary just above, we know that $\mathsf{p}_{n+1,\mathbf{a}} \in \mathbb{Z}[\mathbf{a}]$ is not identically zero, for it incorporates at least the nonzero (central) monomial:

$$\mathsf{coeff}_{d^{n+1}}\big[\tfrac{n^2!}{n!\cdots n!}\, a_1^n \cdots a_n^n\, u_1^n \cdots u_n^n\big] = \tfrac{n^2!}{n!\cdots n!}\, a_1^n \cdots a_n^n.$$

Then, in order to capture a weight $\mathbf{a}$ for which $\mathsf{p}_{n+1,\mathbf{a}} > 0$, we at first observe that the cube of $\mathbb{N}^n$ having edges of length n^2 which consists of all integers $(a_1, \dots, a_n)$ satisfying the inequalities:

$$\begin{aligned}1 \leqslant a_n \leqslant 1 + n^2, \quad 3n^2 \leqslant a_{n-1} \leqslant (3+1)n^2, \quad (3^2+3)n^2 \leqslant a_{n-2} \leqslant (3^2+3+1)n^2 \\ \dots, \quad (3^{n-1} + \cdots + 3)n^2 \leqslant a_1 \leqslant (3^{n-1} + \cdots + 3 + 1)n^2\end{aligned}$$

is visibly contained in the cone in question:

$$a_n \geqslant 1, \quad a_{n-1} \geqslant 2a_n, \quad a_{n-2} \geqslant 3a_{n-1}, \dots, a_1 \geqslant 3a_2.$$

We now claim that there exists at least one n-tuple of integers $\mathbf{a}^* = (a_1^*, \dots, a_n^*)$ belonging to this cube with the property that $\mathsf{p}_{n+1,\mathbf{a}^*}$ is nonzero, and hence:

$$\mathsf{p}_{n+1,\mathbf{a}^*} \geqslant 1 =: \mathsf{G}_{n+1},$$

so that we can take 1 as the minorant introduced at the beginning. Indeed, $\mathsf{p}_{n+1,\mathbf{a}}$ is a homogeneous polynomial of degree n^2 to which an elementary lemma applies.

Lemma 4.5. *Let* $\mathsf{q} = \mathsf{q}(b_1, \dots, b_\nu) \in \mathbb{Z}\big[b_1, \dots, b_\nu\big]$ *be a polynomial of degree* $c \geqslant 1$. *Then* q *can vanish at all points of a cube of integers having edges of length equal to its degree* c *only when it is identically zero.*

Proof. Expand $\mathsf{q} = \sum_{k_1=0}^{c} b_1^{k_1}\, \mathsf{q}_{k_1}(b_2, \dots, b_\nu)$, recognize a $(c+1) \times (c+1)$ Van der Monde determinant, deduce that each $\mathsf{q}_{k_1}(b_2, \dots, b_\nu)$ vanishes at all points of a similar cube in a space of dimension $\nu - 1$, and terminate by induction. $\square$

4.5. **Majorating the other coefficients** $\mathsf{coeff}_{d^k}\big[\Pi\big]$**.** Now, for such an $\mathbf{a}^*$ which is not very precisely located in the cube, we nevertheless have the effective control, which is useful below:

$$\max_{1 \leqslant i \leqslant n} a_i^* = a_1^* = \tfrac{3^n - 1}{2}\, n^2 \leqslant \tfrac{3^n}{2}\, n^2.$$

From now on, we shall simply denote $\mathbf{a}^*$ by $\mathbf{a}$. At present, for any integer k with $0 \leqslant k \leqslant n$, let us denote by $\mathsf{D}_k(n)$ any available bound (*see* in advance Theorem 5.1) in terms of n only for the maximal absolute value of the coefficient of d^k in all monomials $h^l u_1^{i_1} \cdots u_n^{i_n}$ with $l + i_1 + \cdots + i_n = n^2$, namely:

$$\max_{l + i_1 + \cdots + i_n = n^2} \big|\mathsf{coeff}_{d^k}\big[h^l u_1^{i_1} \cdots u_n^{i_n}\big]\big| \leqslant \mathsf{D}_k(n).$$

Then for any k with $0 \leqslant k \leqslant n$, we now aim at estimating from above the coefficient of d^k in our intersection product Π, using two new lemmas and starting from its expansion, all terms of which we shall have to control:

$$\begin{aligned}
&\big|\mathsf{coeff}_{d^k}\big[\Pi\big]\big| \leqslant \\
&\leqslant \sum_{l + i_1 + \cdots + i_n = n^2} \tfrac{n^2!}{l!\, i_1! \cdots i_n!} \cdot (2|\mathbf{a}|)^l a_1^{i_1} \cdots a_n^{i_n} \cdot \big|\mathsf{coeff}_{d^k}\big[h^l u_1^{i_1} \cdots u_n^{i_n}\big]\big| + \\
&+ \sum_{l + j_1 + \cdots + j_n = n^2 - 1} n^2\, \tfrac{(n^2-1)!}{l!\, j_1! \cdots j_n!} \cdot 2|\mathbf{a}|(2|\mathbf{a}|)^l a_1^{j_1} \cdots a_n^{j_n} \cdot \big|\mathsf{coeff}_{d^k}\big[h h^l u_1^{j_1} \cdots u_n^{j_n}\big]\big|.
\end{aligned}$$

Lemma 4.6. *Let* $l, i_1, \dots, i_n \in \mathbb{N}$ *satisfying* $l + i_1 + \cdots + i_n = n^2$ *and let* $l, j_1, \dots, j_n \in \mathbb{N}$ *satisfying* $l + j_1 + \cdots + j_n = n^2 - 1$. *Then:*

$$\tfrac{n^2!}{l!\, i_1! \cdots i_n!} \leqslant (n+1)^{n^2} \quad \textit{and:} \quad n^2\, \tfrac{(n^2-1)!}{l!\, j_1! \cdots j_n!} \leqslant (n+1)^{n^2+1}.$$

Furthermore, the number of summands in $\sum_{l + i_1 + \cdots + i_n = n^2}$ *and the number of summands in* $\sum_{l + j_1 + \cdots + j_n = n^2 - 1}$, *which are both plain binomial coefficients, enjoy the following two elementary majorations:*

$$\tfrac{(n^2+n)!}{n^2!\, n!} \leqslant 4\, n^{2n-1} \quad \textit{and:} \quad \tfrac{(n^2-1+n)!}{(n^2-1)!\, n!} \leqslant 2\, n^{2n-1}.$$

Proof. Indeed, any multinomial coefficient $\frac{n^2!}{l!\, i_1! \cdots i_n!}$ is less than or equal to the sum of all multinomial coefficients $(1 + 1 + \cdots + 1)^{n^2} = (n+1)^{n^2}$. At the same time, we deduce: $n^2\, \frac{(n^2-1)!}{l! j_1! \cdots j_n!} = n^2 (n+1)^{n^2-1} \leqslant (n+1)^{n^2+1}$.

For the second claim, we as a preliminary have:

$$\tfrac{(n^2+n-1)!}{n^2!\,(n-1)!} = \tfrac{(n^2+1)\cdots(n^2+n-1)}{1\,\cdots\,(n-1)} \leqslant \tfrac{(n^2+n^2)\cdots(n^2+n^2)}{(n-1)!} = \tfrac{2^{n-1}\,n^{2n-2}}{(n-1)!} \leqslant 2\,n^{2n-2},$$

since $2^{n-1} \leqslant 2\,(n-1)!$ for any $n \geqslant 1$. Consequently, we deduce:

$$\tfrac{(n^2+n)!}{n^2!\,n!} = \tfrac{(n^2+n-1)!}{n^2!\,(n-1)!} \cdot \tfrac{(n^2+n)}{n} \leqslant 2\,n^{2n-2} \cdot (n+\tfrac{1}{n}) \leqslant 4\,n^{2n-1},$$

and similarly: $\tfrac{(n^2-1+n)!}{(n^2-1)!\,n!} \leqslant \tfrac{(n^2+n-1)!}{n^2!\,(n-1)!} \cdot \tfrac{n^2}{n} \leqslant 2\,n^{2n-2} \cdot n = 2\,n^{2n-1}$. □

Lemma 4.7. *For any $l, i_1, \ldots, i_n \in \mathbb{N}$ satisfying $l + i_1 + \cdots + i_n = n^2$, one has:*

$$(2|\mathbf{a}|)^l a_1^{i_1} \cdots a_n^{i_n} \leqslant n^{3n^2}\, 3^{n^3}.$$

Proof. Indeed, we majorate each a_i by $|\mathbf{a}|$ and $|\mathbf{a}| = a_1 + \cdots + a_n$ by na_1, and also l by n^2, so that $(2|\mathbf{a}|)^l a_1^{i_1} \cdots a_n^{i_n} \leqslant 2^{n^2}\big(na_1\big)^{n^2}$ and we apply $a_1 \leqslant \frac{3^n}{2}\, n^2$. □

Thanks to these two lemmas, we may perform majorations:

$$\begin{aligned}\big|\mathsf{coeff}_{d^k}\big[\Pi\big]\big| &\leqslant 4\,n^{2n-1} \cdot (n+1)^{n^2} \cdot n^{3n^2}\, 3^{n^3} \cdot \mathsf{D}_k(n) + \\ &\quad + 2\,n^{2n-1} \cdot (n+1)^{n^2+1} \cdot n^{3n^2}\, 3^{n^3} \cdot \mathsf{D}_k(n) \\ &\leqslant 6\,n^{2n-1} \cdot (n+1)^{n^2+1} \cdot n^{3n^2}\, 3^{n^3} \cdot \mathsf{D}_k(n) \qquad {\scriptstyle (k=0,\ldots,n)}.\end{aligned}$$

Lemma 4.8. *For any exponent k with $0 \leqslant k \leqslant n$, one has:*

$$\big|\mathsf{coeff}_{d^k}\big[\Pi\big]\big| \leqslant 6\,n^{2n-1} \cdot (n+1)^{n^2} \cdot n^{3n^2}\, 3^{n^3} \cdot \mathsf{D}_k(n). \quad \square$$

To conclude these estimates, for any integer $k = 0, 1, \ldots, n, n+1$, let us denote by $\mathsf{D}'_k(n)$ any available majorant for all the monomials appearing in Π':

$$\max_{1+l+j_1+\cdots+j_n=n^2} \big|\mathsf{coeff}_{d^k}\big[c_1 h^l u_1^{j_1} \cdots u_n^{j_n}\big]\big| \leqslant \mathsf{D}'_k(n).$$

Lemma 4.9. *For any exponent k with $0 \leqslant k \leqslant n+1$, one has:*

$$\big|\mathsf{coeff}_{d^k}\big[\Pi'\big]\big| \leqslant n^{2n-1} \cdot (n+1)^{n^2+1} \cdot n^{3n^2}\, 3^{n^3} \cdot \mathsf{D}'_k(n).$$

Proof. Indeed, one performs the similar majorations:

$$\begin{aligned}&\big|\mathsf{coeff}_{d^k}\big[\Pi'\big]\big| \leqslant \\ &\leqslant \sum_{l+j_1+\cdots+j_n=n^2-1} n^2\, \tfrac{(n^2-1)!}{l!\,j_1!\cdots j_n!} \cdot |\mathbf{a}|(2|\mathbf{a}|)^l a_1^{j_1} \cdots a_n^{j_n} \cdot \big|\mathsf{coeff}_{d^k}\big[c_1 h^l u_1^{j_1} \cdots u_n^{j_n}\big]\big| \\ &\leqslant 2\,n^{2n-1} \cdot (n+1)^{n^2+1} \cdot \tfrac{1}{2}\, n^{3n^2}\, 3^{n^3} \cdot \mathsf{D}'_k(n) \\ &\leqslant n^{2n-1} \cdot (n+1)^{n^2+1} \cdot n^{3n^2}\, 3^{n^3} \cdot \mathsf{D}'_k(n),\end{aligned}$$

hence the bound we obtain is exactly the same, up to the factor 6. □

4.6. **Final effective estimations.** We can now explain how to achieve the proof of Theorem 1.1. At first, we shall realize in Section 5 that both constant coefficients $\mathsf{coeff}_{d^0}\big[\Pi\big] = \mathsf{coeff}_{d^0}\big[\Pi'\big] = 0$ vanish, hence $\mathsf{D}_0(n) = \mathsf{D}'_0(n) = 0$ works. Most importantly, we shall establish in Section 5 that one may choose:

$$\mathsf{D}_1(n) = \cdots = \mathsf{D}_n(n) = \mathsf{D}'_1(n) = \cdots = \mathsf{D}'_n(n) = \mathsf{D}'_{n+1}(n) = n^{4n^3}2^{n^4}.$$

Taking $n^{4n^3}2^{n^4}$ for granted, remind that with the above choice of weight $\mathbf{a}^*$ (now denoted $\mathbf{a}$), we ensure that:

$$\mathsf{coeff}_{d^{n+1}}\big[\Pi\big] = \mathsf{p}_{n+1,\mathbf{a}} \geqslant 1 =: \mathsf{G}_{n+1}.$$

From the preceding two lemmas, we therefore deduce that:

$$\big|\mathsf{coeff}_{d^k}\big[\Pi\big]\big| \leqslant 6\, n^{2n-1} \cdot (n+1)^{n^2+1} \cdot n^{3n^2}\, 3^{n^3} \cdot n^{4n^3}2^{n^4} =: 6\,\mathsf{H}(n) \qquad {\scriptstyle (k=1\cdots n)}$$
$$\big|\mathsf{coeff}_{d^k}\big[\Pi'\big]\big| \leqslant n^{2n-1} \cdot (n+1)^{n^2+1} \cdot n^{3n^2}\, 3^{n^3} \cdot n^{4n^3}2^{n^4} =: \mathsf{H}(n) \qquad {\scriptstyle (k=1\cdots n+1)}.$$

so that, coming back to the beginning of Section 4, we may choose $\mathsf{E}_0 = \mathsf{E}'_0 = 0$ (since $\mathsf{D}_0(n) = \mathsf{D}'_0(n) = 0$) and also explicitly in terms of n:

$$\mathsf{E}_1 = \cdots = \mathsf{E}_n = 6\,\mathsf{H}(n)$$
$$\mathsf{E}'_1 = \cdots = \mathsf{E}'_n = \mathsf{E}'_{n+1} = \mathsf{H}(n).$$

Coming back to the definition of d^1_n, d^2_n given at the end of Lemma 4.1 and just after, we may now majorate:

$$d^1_n \leqslant 1 + \big(n\,6\,\mathsf{H}(n) + \tfrac{n+1}{2}\big)/\tfrac{1}{2} =: \widetilde{d}^1_n,$$
$$d^2_n \leqslant 1 + n + 2 + 2\,(n^2+2n)\,\mathsf{H}(n) =: \widetilde{d}^2_n.$$

Notice that $\widetilde{d}^2_n \geqslant \widetilde{d}^1_n$ as soon as $n \geqslant 3$. Finally, by comparing the growth of all terms in $\mathsf{H}(n)$ as $n \to \infty$, one sees that 2^{n^4} dominates and hence that the following inequality:

$$\widetilde{d}^2_n = 1 + n + 2 + 2\,(n^2+2n) \cdot n^{2n-1} \cdot (n+1)^{n^2+1} \cdot n^{3n^2}\, 3^{n^3} \cdot n^{4n^3}2^{n^4} \leqslant 2^{n^5},$$

holds for all large n. However, any symbolic computer shows that for $n = 2, 3, 4$, one in fact has $\widetilde{d}^2_2 > 2^{2^5}$, $\widetilde{d}^2_3 > 2^{3^5}$, $\widetilde{d}^2_4 > 2^{4^5}$, while $\widetilde{d}^2_5 < 2^{n^5}$ and $\widetilde{d}^2_n \ll 2^{n^5}$ for $n = 6, 7, 8, 9$ so that $\widetilde{d}^2_n < 2^{n^5}$ holds for any $n \geqslant 5$ by an elementary inspection of the function $n \mapsto \widetilde{d}^2_n$. Fortunately, the three left cases $n = 2$, $n = 3$ and $n = 4$ of Theorem 1.1 are covered, firstly for the classical surface case $n = 2$ by, say [5] in which $\deg X \geqslant 21$ with $21 \ll 2^{2^5}$, and secondly for $n = 3$ and $n = 4$ by our second Theorem 1.2, because $2^{3^5} \gg 593$ and $2^{4^5} \gg 3203$. So we conclude that if we take for granted: 1) that one may take all the $\mathsf{D}_k(n)$ and all the $\mathsf{D}'_k(n)$ equal to $n^{4n^3}2^{n^4}$, a technical and crucial statement to which Section 5 below is entirely devoted; and 2) that Theorem 1.2 is got, an effective statement to which the two Sections 6 and 7

below are devoted, then the proof of our main Theorem 1.1 is to be considered as complete, and finally, the neat uniform degree bound $\deg X \geqslant 2^{n^5}$ works in all dimensions $n \geqslant 2$. $\square$

5. Estimations of the quantities $\mathsf{D}_k(n)$ and $\mathsf{D}'_k(n)$

To complete our program, it now remains only to capture somewhat effective upper bounds $\mathsf{D}_k(n)$, $0 \leqslant k \leqslant n$ and $\mathsf{D}'_k(n)$, $0 \leqslant k \leqslant n+1$.

Theorem 5.1. *With $n \geqslant 2$, for any $l, i_1, \dots, i_n \in \mathbb{N}$ with $l+i_1+\cdots+i_n = n^2$ and any $l, j_1, \dots, j_n \in \mathbb{N}$ with $1+l+j_1+\cdots+j_n = n^2$, one has:*

$$0 = \mathsf{coeff}_{d^0}\big[h^l u_1^{i_1} \cdots u_n^{i_n}\big] = \mathsf{coeff}_{d^0}\big[c_1 h^l u_1^{j_1} \cdots u_n^{j_n}\big].$$

Moreover and above all, for every $k = 1, \dots, n+1$, the following uniform effective upper bound holds:

$$\big|\mathsf{coeff}_{d^k}\big[h^l u_1^{i_1} \cdots u_n^{i_n}\big]\big| \leqslant n^{4n^3} 2^{n^4},$$
$$\big|\mathsf{coeff}_{d^k}\big[c_1 h^l u_1^{j_1} \cdots u_n^{j_n}\big]\big| \leqslant n^{4n^3} 2^{n^4}.$$

In other words, in the above notations, one may choose $\mathsf{D}_0(n) = \mathsf{D}'_0(n) = 0$ and $\mathsf{D}_k(n) = \mathsf{D}'_k(n) = n^{4n^3} 2^{n^4}$ for $k = 1, \dots, n+1$.

5.1. Jacobi-Trudy determinants.

One key observation towards these estimations is that the reduction process from one level to the lower level in Demailly's tower involves Jacobi-Trudy determinants in the Chern classes of the lower level in question.

Definition 5.1. At any level ℓ with $0 \leqslant \ell \leqslant n-1$ and for any J with $0 \leqslant J \leqslant n+\ell(n-1) = \dim X_\ell$, we define the corresponding *Jacobi-Trudy determinant*:

$$\mathscr{C}_J^\ell := \begin{vmatrix} c_1^{[\ell]} & c_2^{[\ell]} & c_3^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & c_1^{[\ell]} & c_2^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ 0 & 1 & c_1^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix},$$

where, again by convention, we set any $c_k^{[\ell]} := 0$ as soon as $k \geqslant n+1$; by convention also, $\mathscr{C}_J^\ell := 0$ is set to zero when $J > \dim X_\ell$ and when $J < 0$; lastly, we set $\mathscr{C}_0^\ell := 1$.

Expanding the determinant $\mathscr{C}_J^\ell$ along its first line, and expanding again the obtained block-determinants, one easily convinces oneself of the induction formulae:

$$\mathscr{C}_J^\ell = c_1^{[\ell]} \mathscr{C}_{J-1}^\ell - c_2^{[\ell]} \mathscr{C}_{J-2}^\ell + c_3^{[\ell]} \mathscr{C}_{J-3}^\ell - \cdots, \tag{55}$$

the last term in this expansion being either $(-1)^{n-1}\, c_n^{[\ell]}\, \mathscr{C}^{\ell}_{J-n}$ when $J \geqslant n$ or else $(-1)^{J-1}\, c_J^{[\ell]}\, \mathscr{C}^{\ell}_0$ when $J < n$.

In the proof of Theorem 5.1, the study of the monomials $u_1^{i_1} \cdots u_n^{i_n}$ will appear *a posteriori* to be exactly the same as the study of the monomials $h^l u_1^{i_1} \cdots u_n^{i_n}$ and $c_1 h^l u_1^{j_1} \cdots u_n^{j_n}$.

Generally speaking, fixing ℓ with $1 \leqslant \ell \leqslant n$ and exponents $i_1, \ldots, i_\ell \in \mathbb{N}$ satisfying $i_1 + \cdots + i_\ell = n + \ell(n-1) = \dim X_\ell$, let us therefore study the reduction, in term of the degree d of X, of the specific monomial $u_1^{i_1} \cdots u_{\ell-1}^{i_{\ell-1}} u_\ell^{i_\ell}$. We write it as $\Omega_K^{\ell-1} u_\ell^{i_\ell}$, where $\Omega_K^{\ell-1} := u_1^{i_1} \cdots u_{\ell-1}^{i_{\ell-1}}$ is a (K,K)-form living on $X_{\ell-1}$ with $K + i_\ell = n + \ell(n-1)$.

If $i_\ell \leqslant n-2$, then $\Omega_K^{\ell-1}$ vanishes form degree-form reasons. If $i_\ell = n-1$, then a fiber-integration gives $\Omega_K^{\ell-1} \underline{u_\ell^{n-1}}_{\int} = \Omega_K^{\ell-1} \cdot 1 = \Omega_K^{\ell-1} \mathscr{C}_0^{\ell-1}$.

Lemma 5.1. *For any ℓ with $1 \leqslant \ell \leqslant n$, given any (K,K)-form $\Omega_K^{\ell-1}$ at level $\ell - 1$ and any integer i_ℓ with $i_\ell \geqslant n-1$ and $i_\ell + K = \dim X_\ell$, the reduction of $\Omega_K^{\ell-1}\, u_\ell^{i_\ell}$ down to level $\ell - 1$ precisely reads:*

$$\begin{aligned}
\Omega_K^{\ell-1} u_\ell^{i_\ell} &= (-1)^{i_\ell - n + 1}\, \Omega_K^{\ell-1} \begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell-1]} & \cdots & c_{i_\ell-n+1}^{[\ell-1]} \\ 1 & c_1^{[\ell-1]} & \cdots & c_{i_\ell-n}^{[\ell-1]} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & c_1^{[\ell-1]} \end{vmatrix} \\
&= (-1)^{i_\ell - n + 1}\, \Omega_K^{\ell-1}\, \mathscr{C}^{\ell-1}_{i_\ell - n + 1}.
\end{aligned}$$

Proof. Assume first that $i_\ell = n$ and use (53) to get:

$$\begin{aligned}
\Omega_K^{\ell-1}\, u_\ell^n &= -\Omega_K^{\ell-1}\, c_1^{[\ell-1]}\, \underline{u_\ell^{n-1}}_{\int} - \underline{\Omega_K^{\ell-1} c_2^{[\ell-1]}}_{\circ}\, u_\ell^{n-2} - \cdots - \underline{\Omega_K^{\ell-1} c_n^{[\ell-1]}}_{\circ} \\
&= -\Omega_K^{\ell-1}\, \mathscr{C}_1^{\ell-1}.
\end{aligned}$$

Reasoning by induction, assume now that the lemma holds for all i'_ℓ with $n \leqslant i'_\ell \leqslant i_\ell$ for some $i_\ell \geqslant n$. Take an arbitrary (L,L)-form $\Omega_L^{\ell-1}$ on $X_{\ell-1}$ with $L + i_\ell + 1 = \dim X_\ell$, multiply (53) by $\Omega_L^{\ell-1}\, u_\ell^{i_\ell+1-n}$ to get:

$$\begin{aligned}
\Omega_L^{\ell-1}\, u_\ell^{i_\ell+1} &= -\Omega_L^{\ell-1} \Big(c_1^{[\ell-1]}\, u_\ell^{i_\ell} + c_2^{[\ell-1]}\, u_\ell^{i_\ell-1} + c_3^{[\ell-1]}\, u_\ell^{i_\ell-2} + \cdots \Big) \\
&= (-1)^{1+i_\ell-n+1}\, \Omega_L^{\ell-1} \Big(c_1^{[\ell-1]}\, \mathscr{C}^{\ell-1}_{i_\ell-n+1} - c_2^{[\ell-1]}\, \mathscr{C}^{\ell-1}_{i_\ell-n} + c_3^{[\ell-1]}\, \mathscr{C}^{\ell-1}_{i_\ell-n-1} - \cdots \Big) \\
&= (-1)^{i_\ell+1-n+1}\, \Omega_L^{\ell-1}\, \mathscr{C}^{\ell-1}_{i_\ell+1-n+1},
\end{aligned}$$

thanks to (55), which gives the claimed reduction for the exponent $i_\ell + 1$. $\square$

Applying this lemma to the monomial $u_1^{i_1} \cdots u_\ell^{i_\ell} u_{\ell+1}^{i_{\ell+1}}$, we thus reduce it to

$$u_1^{i_1} \cdots u_\ell^{i_\ell} u_{\ell+1}^{i_{\ell+1}} = (-1)^{i_{\ell+1}-n+1}\, u_1^{i_1} \cdots u_\ell^{i_\ell}\, \mathscr{C}^{\ell}_{i_{\ell+1}-n+1}\,.$$

To obtain effective estimations, we will need to further reduce such a Jacobi-Trudy determinant $\mathscr{C}^{\ell}_{i_{\ell+1}-n+1}$ from level ℓ down to level $\ell - 1$. A whole

program begins. In the application we have in mind, one should think that $\Omega_K^\ell = (-1)^{i_{\ell+1}-n+1}\, u_1^{i_1} \cdots u_\ell^{i_\ell}$ and that $J = i_{\ell+1} - n + 1$.

Lemma 5.2. *At an arbitrary level ℓ with $1 \leqslant \ell \leqslant n-1$, consider the Jacobi-Trudy determinant $\mathscr{C}_J^\ell$ of an arbitrary size $J \times J$ with $1 \leqslant J \leqslant \dim X_\ell$ and furthermore, let Ω_K^ℓ be any (K,K)-form on X_ℓ whose degree K satisfies $K + J = \dim X_\ell = n + \ell(n-1)$. Then the reduction of $\Omega_K^\ell \mathscr{C}_J^\ell$ down to level $\ell - 1$ relies upon the following formulae:*

$$\Omega_K^\ell \mathscr{C}_J^\ell = \Omega_K^\ell \big[\mathscr{C}_J^{\ell-1} + \mathscr{C}_0^\ell \mathsf{A}_J^\ell + \mathscr{C}_1^\ell \mathsf{A}_{J-1}^\ell + \cdots + \mathscr{C}_{J-1}^\ell \mathsf{A}_1^\ell\big],$$

in which, for any k with $1 \leqslant k \leqslant J$, one has set:

$$\mathsf{A}_k^\ell := \mathsf{X}_1^\ell \mathscr{C}_{k-1}^{\ell-1} - \mathsf{X}_2^\ell \mathscr{C}_{k-2}^{\ell-1} + \cdots + (-1)^{k-1} \mathsf{X}_k^\ell \mathscr{C}_0^{\ell-1},$$

where the X-terms here gather all the terms after $c_j^{[\ell-1]}$ in a convenient rewriting of (52) *under the following form:*

$$c_j^{[\ell]} = c_j^{[\ell-1]} + \underbrace{\lambda_{j,1}\, c_{j-1}^{[\ell-1]} u_\ell + \lambda_{j,2}\, c_{j-2}^{[\ell-1]} u_\ell^2 + \cdots + \lambda_{j,j}\, u_\ell^j}_{\stackrel{\text{def}}{=} \mathsf{X}_j^\ell},$$

with the convention that $\mathsf{X}_j^\ell = 0$ for any $j \geqslant n+1$.

Proof. Naturally, we should expand the Jacobi-Trudy determinant in question after inserting in it the relation (52). This is based on linear algebra considerations and we shall drop Ω_K^ℓ in the computations.

More precisely, let us write down the determinant $\mathscr{C}_J^\ell$ we have to expand:

$$\mathscr{C}_J^\ell = \begin{vmatrix} c_1^{[\ell]} & c_2^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & c_1^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix} = \begin{vmatrix} \mathsf{X}_1^\ell + c_1^{[\ell-1]} & c_2^{[\ell]} & \cdots & c_J^{[\ell]} \\ 0+1 & c_1^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix}$$

by emphasizing the induction on ℓ which represents its first column naturally as the sum of two columns. As already devised, we expand it by linearity, getting:

$$\mathscr{C}_J^\ell = \begin{vmatrix} \mathsf{X}_1^\ell & c_2^{[\ell]} & \cdots & c_J^{[\ell]} \\ 0 & c_1^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix} + \begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & c_1^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix},$$

and just afterwards immediately, we expand the first determinant along its first column, while at the same time, in the second column of the second

determinant, we again emphasize the induction on ℓ:

$$\mathscr{C}_J^\ell = \mathsf{X}_1^\ell \cdot \mathscr{C}_{J-1}^\ell + \begin{vmatrix} c_1^{[\ell-1]} & \mathsf{X}_2^\ell + c_2^{[\ell-1]} & c_3^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & \mathsf{X}_1^\ell + c_1^{[\ell-1]} & c_2^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ 0 & 0+1 & c_1^{[\ell]} & \cdots & c_{J-2}^{[\ell]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix}.$$

Next, we similarly expand by linearity the obtained determinant, realizing again that its second column is a sum of two columns:

$$\mathscr{C}_J^\ell = \mathsf{X}_1^\ell \cdot \mathscr{C}_{J-1}^\ell + \begin{vmatrix} c_1^{[\ell-1]} & \mathsf{X}_2^\ell & c_3^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & \mathsf{X}_1^\ell & c_2^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ 0 & 0 & c_1^{[\ell]} & \cdots & c_{J-2}^{[\ell]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix} + \begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell-1]} & c_3^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & c_1^{[\ell-1]} & c_2^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ 0 & 1 & c_1^{[\ell]} & \cdots & c_{J-2}^{[\ell]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix},$$

and evidently again, we must expand the first obtained determinant along its second column, getting:

$$\mathscr{C}_J^\ell = \mathsf{X}_1^\ell \cdot \mathscr{C}_{J-1}^\ell - \mathsf{X}_2^\ell \cdot \begin{vmatrix} 1 & c_2^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ 0 & c_1^{[\ell]} & \cdots & c_{J-2}^{[\ell]} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix} + \mathsf{X}_1^\ell \cdot \begin{vmatrix} c_1^{[\ell-1]} & c_3^{[\ell]} & \cdots & c_J^{[\ell]} \\ 0 & c_1^{[\ell]} & \cdots & c_{J-2}^{[\ell]} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix}$$
$$+ \begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell-1]} & \mathsf{X}_3^\ell + c_3^{[\ell-1]} & c_4^{[\ell]} & \cdots & c_J^{[\ell]} \\ 1 & c_1^{[\ell-1]} & \mathsf{X}_2^\ell + c_2^{[\ell-1]} & c_3^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\ 0 & 1 & \mathsf{X}_1^\ell + c_1^{[\ell-1]} & c_2^{[\ell]} & \cdots & c_{J-2}^{[\ell]} \\ 0 & 0 & 0+1 & c_1^{[\ell]} & \cdots & c_{J-3}^{[\ell]} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_1^{[\ell]} \end{vmatrix},$$

and we are supposed to iterate once again the same two processes:

$$
\begin{aligned}
\mathscr{C}_J^\ell = {} & \mathsf{X}_1^\ell \cdot \mathscr{C}_{J-1}^\ell - \mathsf{X}_2^\ell \cdot 1 \cdot \mathscr{C}_{J-2}^\ell + \mathsf{X}_1^\ell \cdot \mathscr{C}_1^{\ell-1} \cdot \mathscr{C}_{J-2}^\ell \\
& + \mathsf{X}_3^\ell \cdot \begin{vmatrix} 1 & c_1^{[\ell-1]} \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} c_1^{[\ell]} & \cdots & c_{J-3}^{[\ell]} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_1^{[\ell]} \end{vmatrix} \\
& - \mathsf{X}_2^\ell \cdot \begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell-1]} \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} c_1^{[\ell]} & \cdots & c_{J-3}^{[\ell]} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_1^{[\ell]} \end{vmatrix} \\
& + \mathsf{X}_1^\ell \cdot \begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell-1]} \\ 1 & c_1^{[\ell-1]} \end{vmatrix} \cdot \begin{vmatrix} c_1^{[\ell]} & \cdots & c_{J-3}^{[\ell]} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_1^{[\ell]} \end{vmatrix} \\
& + \begin{vmatrix}
c_1^{[\ell-1]} & c_2^{[\ell-1]} & c_3^{[\ell-1]} & \mathsf{X}_4^\ell + c_4^{[\ell-1]} & c_5^{[\ell]} & \cdots & c_J^{[\ell]} \\
1 & c_1^{[\ell-1]} & c_2^{[\ell-1]} & \mathsf{X}_3^\ell + c_3^{[\ell-1]} & c_4^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\
0 & 1 & c_1^{[\ell-1]} & \mathsf{X}_2^\ell + c_2^{[\ell-1]} & c_3^{[\ell]} & \cdots & c_{J-2}^{[\ell]} \\
0 & 0 & 1 & \mathsf{X}_1^\ell + c_1^{[\ell-1]} & c_2^{[\ell]} & \cdots & c_{J-3}^{[\ell]} \\
0 & 0 & 0 & 0+1 & c_1^{[\ell]} & \cdots & c_{J-4}^{[\ell]} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & c_1^{[\ell]}
\end{vmatrix}.
\end{aligned}
$$

At this point where things start to become clearer, we make the following general observation. Consider the determinant that one obtains after a finite number of steps:

$$
\begin{vmatrix}
c_1^{[\ell-1]} & c_2^{[\ell-1]} & \cdots & c_{k-1}^{[\ell-1]} & \mathsf{X}_k^\ell + c_k^{[\ell-1]} & c_{k+1}^{[\ell]} & \cdots & c_J^{[\ell]} \\
1 & c_1^{[\ell-1]} & \cdots & c_{k-2}^{[\ell-1]} & \mathsf{X}_{k-1}^\ell + c_{k-1}^{[\ell-1]} & c_k^{[\ell]} & \cdots & c_{J-1}^{[\ell]} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & c_1^{[\ell-1]} & \mathsf{X}_2^\ell + c_2^{[\ell-1]} & c_3^{[\ell]} & \cdots & c_{J-k+2}^{[\ell]} \\
0 & 0 & \cdots & 1 & \mathsf{X}_1^\ell + c_1^{[\ell-1]} & c_2^{[\ell]} & \cdots & c_{J-k+1}^{[\ell]} \\
0 & 0 & \cdots & 0 & 0+1 & c_1^{[\ell]} & \cdots & c_{J-k}^{[\ell]} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & c_1^{[\ell]}
\end{vmatrix},
$$

where the central-looking column is the k-th one, for some k with $1 \leqslant k \leqslant J$. Write this determinant as a sum of two determinants by linearity, and expand the first obtained determinant, let us call it Δ_k, along its k-th column in which are present all the X_k^ℓ's. We thus get that the first determinant is

equal to:

$$\begin{aligned}
\Delta_k := (-1)^{k+1}\,\mathsf{X}_k^\ell \cdot \begin{vmatrix} 1 & \cdots & c_{k-2}^{[\ell-1]} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{vmatrix} \cdot \mathscr{C}_{J-k}^\ell \\
+ (-1)^{k+2}\,\mathsf{X}_{k-1}^\ell \cdot \begin{vmatrix} c_1^{[\ell-1]} & * & \cdots & * \\ 0 & 1 & \cdots & c_{k-3}^{[\ell-1]} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} \cdot \mathscr{C}_{J-k}^\ell \\
+ (-1)^{k+3}\,\mathsf{X}_{k-2}^\ell \cdot \begin{vmatrix} c_1^{[\ell-1]} & c_2^{[\ell-1]} & * & \cdots & * \\ 1 & c_1^{[\ell-1]} & * & \cdots & * \\ 0 & 0 & 1 & \cdots & c_{k-4}^{[\ell-1]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} \cdot \mathscr{C}_{J-k}^\ell \\
+ \cdots + (-1)^{k+k}\,\mathsf{X}_1^\ell \cdot \begin{vmatrix} c_1^{[\ell-1]} & \cdots & c_{k-1}^{[\ell-1]} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_1^{[\ell-1]} \end{vmatrix} \cdot \mathscr{C}_{J-k}^\ell,
\end{aligned}$$

while the second determinant is of the same kind as the one we started with, except that the X's are now located in the $(k+1)$-th column. Thus after mild simplifications, what we called the first determinant equals:

$$\begin{aligned}
\Delta_k &= (-1)^{k+1}\,\mathsf{X}_k^\ell \cdot 1 \cdot \mathscr{C}_{J-k}^\ell + (-1)^{k+2}\,\mathsf{X}_{k-1}^\ell \cdot \mathscr{C}_1^{\ell-1} \cdot \mathscr{C}_{J-k}^\ell + \\
&\quad + (-1)^{k+3}\,\mathsf{X}_{k-2}^\ell \cdot \mathscr{C}_2^{\ell-1} \cdot \mathscr{C}_{J-k}^\ell + \cdots + \mathsf{X}_1^\ell \cdot \mathscr{C}_{k-1}^{\ell-1} \cdot \mathscr{C}_{J-k}^\ell \\
&= \mathsf{A}_k^\ell \mathscr{C}_{J-k}^\ell.
\end{aligned}$$

In conclusion, the initial Jacobi-Trudy determinant $\mathscr{C}_J^\ell$ we started with now equals:

$$\mathscr{C}_J^\ell = \Delta_1 + \cdots + \Delta_k + \cdots + \Delta_J + \begin{vmatrix} c_1^{[\ell-1]} & \cdots & c_J^{[\ell-1]} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_1^{[\ell-1]} \end{vmatrix},$$

where the last written determinant, equal to $\mathscr{C}_J^{\ell-1}$ and living at the $(\ell-1)$-th level, is the remainder determinant after all X-terms are removed by expansion. Summing the $\Delta_k = \mathsf{A}_k^\ell \mathscr{C}_{J-k}^\ell$, we obtain the formula announced in the lemma. $\square$

As J varies, the formulae given by this lemma:

$$\mathscr{C}_J^\ell = \mathscr{C}_J^{\ell-1} + \mathscr{C}_0^\ell \mathsf{A}_J^\ell + \mathscr{C}_1^\ell \mathsf{A}_{J-1}^\ell + \cdots + \mathscr{C}_{J-1}^\ell \mathsf{A}_1^\ell,$$

are still imperfect, for their right-hand sides still involve Jacobi-Trudy determinants at the level ℓ. So necessarily, we must perform further reductions.

Lemma 5.3. *For any J with $0 \leqslant J \leqslant \dim X_\ell$ and any ℓ with $1 \leqslant \ell \leqslant n$, one has:*

$$\mathscr{C}_J^\ell = \sum_{j=0}^{J} \mathscr{C}_{J-j}^{\ell-1} \Bigg(\sum_{\nu=1}^{j} \sum_{\substack{k_1+\cdots+k_\nu=j \\ k_1,\ldots,k_\nu \geqslant 1}} \mathsf{A}_{k_1}^\ell \cdots \mathsf{A}_{k_\nu}^\ell \Bigg),$$

with the convention that for $j = 0$, the empty sum in parentheses equals 1.

Proof. First, for $J = 0$, recall that by convention $\mathscr{C}_0^\ell = \mathscr{C}_0^{\ell-1} = 1$. Next, for $J = 1$, we start from the formula of the preceding lemma and we perform an evident computation:

$$\mathscr{C}_1^\ell = \mathscr{C}_1^{\ell-1} + \mathscr{C}_0^\ell \mathsf{A}_1^\ell = \mathscr{C}_1^{\ell-1} \Sigma_0^\ell(\mathsf{A}) + \mathscr{C}_0^{\ell-1} \Sigma_1^\ell(\mathsf{A}),$$

if, generally speaking, we denote for convenient abbreviation:

$$\Sigma_j^\ell(\mathsf{A}) := \sum_{\nu=1}^{j} \sum_{\substack{k_1+\cdots+k_\nu=j \\ k_1,\ldots,k_\nu \geqslant 1}} \mathsf{A}_{k_1}^\ell \cdots \mathsf{A}_{k_\nu}^\ell, \tag{56}$$

with of course $\Sigma_0^\ell(\mathsf{A}) = 1$. These $\Sigma_j^\ell(\mathsf{A})$ satisfy useful induction formulae:

(57)

$$\begin{aligned}
\Sigma_j^\ell(\mathsf{A}) &= \mathsf{A}_j^\ell + \sum_{\nu=2}^{j} \sum_{\substack{k_1+k_2+\cdots+k_\nu=j \\ k_1,k_2,\ldots,k_\nu \geqslant 1}} \mathsf{A}_{k_1}^\ell \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell \\
&= \mathsf{A}_j^\ell + \sum_{\nu=2}^{j} \Bigg(\mathsf{A}_1^\ell \sum_{\substack{k_2+\cdots+k_\nu=j-1 \\ k_2,\ldots,k_\nu \geqslant 1}} \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell + \mathsf{A}_2^\ell \sum_{\substack{k_2,\ldots,k_\nu=j-2 \\ k_2,\ldots,k_\nu \geqslant 1}} \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell + \\
&\qquad + \cdots + \mathsf{A}_{j-1}^\ell \sum_{\substack{k_2+\cdots+k_\nu=1 \\ k_1,\ldots,k_\nu \geqslant 1}} \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell \Bigg) \\
&= \mathsf{A}_j^\ell + \mathsf{A}_1^\ell \sum_{\nu=2}^{j-1} \sum_{\substack{k_2+\cdots+k_\nu=j-1 \\ k_2,\ldots,k_\nu \geqslant 1}} \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell + \mathsf{A}_2^\ell \sum_{\nu=2}^{j-2} \sum_{\substack{k_2+\cdots+k_\nu=j-2 \\ k_2,\ldots,k_\nu \geqslant 1}} \mathsf{A}_{k_2}^\ell \cdots \mathsf{A}_{k_\nu}^\ell + \\
&\qquad + \cdots + \mathsf{A}_{j-1}^\ell \sum_{\nu=2}^{2} \sum_{\substack{k_2=1 \\ k_2 \geqslant 1}} \mathsf{A}_{k_2}^\ell \\
&= \mathsf{A}_j^\ell \Sigma_0^\ell(\mathsf{A}) + \mathsf{A}_1^\ell \, \Sigma_{j-1}^\ell(\mathsf{A}) + \mathsf{A}_2^\ell \Sigma_{j-2}^\ell(\mathsf{A}) + \cdots + \mathsf{A}_{j-1}^\ell \Sigma_1^\ell(\mathsf{A}).
\end{aligned}$$

Next, for $J = 2$, starting again from the known (imperfect) formula and using what has just been seen:

$$\begin{aligned}
\mathscr{C}_2^\ell &= \mathscr{C}_2^{\ell-1} + \mathscr{C}_0^\ell \mathsf{A}_2^\ell + \mathscr{C}_1^\ell \mathsf{A}_1^\ell \\
&= \mathscr{C}_2^{\ell-1} + \mathscr{C}_0^{\ell-1} \mathsf{A}_2^\ell + \big[\mathscr{C}_1^{\ell-1} \Sigma_0^\ell(\mathsf{A}) + \mathscr{C}_0^{\ell-1} \Sigma_1^\ell(\mathsf{A}) \big] \mathsf{A}_1^\ell \\
&= \mathscr{C}_2^{\ell-1} \Sigma_0^\ell(\mathsf{A}) + \mathscr{C}_1^{\ell-1} \big[\Sigma_0^\ell(\mathsf{A}) \mathsf{A}_1^\ell \big] + \mathscr{C}_0^{\ell-1} \big[\Sigma_1^\ell(\mathsf{A}) \mathsf{A}_1^\ell + \mathsf{A}_2^\ell \big] \\
&= \mathscr{C}_2^{\ell-1} \Sigma_0^\ell(\mathsf{A}) + \mathscr{C}_1^{\ell-1} \Sigma_1^\ell(\mathsf{A}) + \mathscr{C}_0^{\ell-1} \Sigma_2^\ell(\mathsf{A}).
\end{aligned}$$

Suppose now by induction that we have already proved that:

$$\mathscr{C}^{\ell}_{J'} = \mathscr{C}^{\ell-1}_{J'}\Sigma^{\ell}_{0}(\mathsf{A}) + \mathscr{C}^{\ell-1}_{J'-1}\Sigma^{\ell}_{1}(\mathsf{A}) + \mathscr{C}^{\ell-1}_{J'-2}\Sigma^{\ell}_{2}(\mathsf{A}) + \cdots + \mathscr{C}^{\ell-1}_{0}\Sigma^{\ell}_{J}(\mathsf{A}),$$

for all J' with $0 \leqslant J' \leqslant J$, for some $J \geqslant 2$. Then we apply the known general (imperfect) formula with J replaced by $J+1$ in it, and afterwards, we use the induction hypothesis, which gives:

$$\begin{aligned}
\mathscr{C}^{\ell}_{J+1} &= \mathscr{C}^{\ell-1}_{J+1} + \mathscr{C}^{\ell}_{0}\mathsf{A}^{\ell}_{J+1} + \mathscr{C}^{\ell}_{1}\mathsf{A}^{\ell}_{J} + \cdots + \mathscr{C}^{\ell}_{J-1}\mathsf{A}^{\ell}_{2} + \mathscr{C}^{\ell}_{J}\mathsf{A}^{\ell}_{1} \\
&= \mathscr{C}^{\ell-1}_{J+1}\Sigma^{\ell}_{0}(\mathsf{A}) + \\
&\quad + \big[\mathscr{C}^{\ell-1}_{0}\Sigma^{\ell}_{0}(\mathsf{A})\big]\mathsf{A}^{\ell}_{J+1} + \\
&\quad + \big[\mathscr{C}^{\ell-1}_{1}\Sigma^{\ell}_{0}(\mathsf{A}) + \mathscr{C}^{\ell-1}_{0}\Sigma^{\ell}_{1}(\mathsf{A})\big]\mathsf{A}^{\ell}_{J} + \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\
&\quad + \big[\mathscr{C}^{\ell-1}_{J-1}\Sigma^{\ell}_{0}(\mathsf{A}) + \mathscr{C}^{\ell-1}_{J-2}\Sigma^{\ell}_{1}(\mathsf{A}) + \mathscr{C}^{\ell-1}_{J-3}\Sigma^{\ell}_{2}(\mathsf{A}) + \cdots + \mathscr{C}^{\ell-1}_{0}\Sigma^{\ell}_{J-1}(\mathsf{A})\big]\mathsf{A}^{\ell}_{2} + \\
&\quad + \big[\mathscr{C}^{\ell-1}_{J}\Sigma^{\ell}_{0}(\mathsf{A}) + \mathscr{C}^{\ell-1}_{J-1}\Sigma^{\ell}_{1}(\mathsf{A}) + \mathscr{C}^{\ell-1}_{J-2}\Sigma^{\ell}_{2}(\mathsf{A}) + \cdots + \mathscr{C}^{\ell-1}_{1}\Sigma^{\ell}_{J-1}(\mathsf{A}) + \mathscr{C}^{\ell-1}_{0}\Sigma^{\ell}_{J}(\mathsf{A})\big]\mathsf{A}^{\ell}_{1}.
\end{aligned}$$

A necessary and natural reorganization then gives:

$$\begin{aligned}
\mathscr{C}^{\ell}_{J+1} &= \mathscr{C}^{\ell-1}_{J+1}\big[\Sigma_{0}(\mathsf{A})\big] + \\
&\quad + \mathscr{C}^{\ell-1}_{J}\big[\Sigma^{\ell}_{0}(\mathsf{A})\mathsf{A}^{\ell}_{1}\big] + \\
&\quad + \mathscr{C}^{\ell-1}_{J-1}\big[\Sigma^{\ell}_{1}(\mathsf{A})\mathsf{A}^{\ell}_{1} + \Sigma^{\ell}_{0}(\mathsf{A})\mathsf{A}^{\ell}_{2}\big] + \\
&\quad + \mathscr{C}^{\ell-1}_{J-2}\big[\Sigma^{\ell}_{2}(\mathsf{A})\mathsf{A}^{\ell}_{1} + \Sigma^{\ell}_{1}(\mathsf{A})\mathsf{A}^{\ell}_{2} + \Sigma^{\ell}_{0}(\mathsf{A})\mathsf{A}^{\ell}_{3}\big] + \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\
&\quad + \mathscr{C}^{\ell-1}_{0}\big[\Sigma^{\ell}_{J}(\mathsf{A})\mathsf{A}^{\ell}_{1} + \Sigma^{\ell}_{J-1}(\mathsf{A})\mathsf{A}^{\ell}_{2} + \Sigma^{\ell}_{J-2}(\mathsf{A})\mathsf{A}^{\ell}_{3} + \cdots + \Sigma^{\ell}_{0}(\mathsf{A})\mathsf{A}^{\ell}_{J+1}\big] \\
&= \mathscr{C}^{\ell-1}_{J+1}\Sigma^{\ell}_{0}(\mathsf{A}) + \mathscr{C}^{\ell-1}_{J}\Sigma^{\ell}_{1}(\mathsf{A}) + \mathscr{C}^{\ell-1}_{J-1}\Sigma^{\ell}_{2}(\mathsf{A}) + \mathscr{C}^{\ell-1}_{J-2}\Sigma^{\ell}_{3}(\mathsf{A}) + \cdots + \mathscr{C}^{\ell-1}_{0}\Sigma^{\ell}_{J+1}(\mathsf{A}),
\end{aligned}$$

where at the end, one applies the formulae (57) just seen. Notice *passim* that the number of terms in $\Sigma^{\ell}_{j}(\mathsf{A})$ is equal to 2^{j-1} for all $j \geqslant 1$. $\square$

5.2. **Upper reduction operator.** The reduction process, after several elimination computations involving (52) and (53) and at the end (45), transforms a general monomial of the form $h^{l}u_{1}^{i_1}\cdots u_{n}^{i_n}$ with $l + i_1 + \cdots + i_n = n^2$ into a polynomial $\mathscr{R}\big(h^{l}u_{1}^{i_1}\cdots u_{n}^{i_n}\big)$ of degree $\leqslant n+1$ in d, where the symbol "$\mathscr{R}$" stands for "*reduction*".

From now on, complete explicit algebraic computations will not be conducted anymore, and instead, to tame their complexity, *inequalities* will be dealt with.

For our majoration purposes, we now introduce an important *upper reduction operator* $\mathscr{R}^{+}$ which by definition, at each computational step of the reduction process, while going down in the Demailly's tower, always replaces any incoming sign "$-$" by a sign "$+$". Accordingly, for any two monomials $h^{l}u_{1}^{i_1}\cdots u_{n}^{i_n}$ and $h^{l'}u_{1}^{i'_1}\cdots u_{n}^{i'_n}$, we shall say that:

$$\mathscr{R}^{+}\big(h^{l}u_{1}^{i_1}\cdots u_{n}^{i_n}\big) \leqslant \mathscr{R}^{+}\big(h^{l'}u_{1}^{i'_1}\cdots u_{n}^{i'_n}\big),$$

and write more briefly:

$$h^l u_1^{i_1} \cdots u_n^{i_n} \leqslant_{\mathscr{R}^+} h^{l'} u_1^{i'_1} \cdots u_n^{i'_n},$$

if the corresponding two (upper) reduced polynomials $\sum_{k=0}^{n+1} \mathsf{p}_k \cdot d^k$ and $\sum_{k=0}^{n+1} \mathsf{p}'_k \cdot d^k$ have all their coefficients satisfying:

$$\big(0 \leqslant\big)\, \mathsf{p}_k \leqslant \mathsf{p}'_k \quad \text{for every } \ k = 0, 1, \ldots, n+1.$$

Then obviously the absolute values of the coefficients of the reduction are smaller than the (nonnegative) coefficients of the upper reduction:

$$\big|\mathsf{coeff}_{d^k}\big[h^l u_1^{i_1} \cdots u_n^{i_n}\big]\big| \leqslant \mathsf{coeff}_{d^k}\big[\mathscr{R}^+\big(h^l u_1^{i_1} \cdots u_n^{i_n}\big)\big]\,.$$

To obtain the desired bound $n^{4n^3} 2^{n^4}$ we need to handle the Jacobi-Trudy determinants seen above. The following lemma will be useful.

Lemma 5.4. *For any $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $n = \lambda_1 + 2\lambda_2 + \cdots + n\lambda_n$, one has:*

$$c_1^{\lambda_1} \big(\mathscr{C}_2^0\big)^{\lambda_2} \cdots \big(\mathscr{C}_n^0\big)^{\lambda_n} \leqslant_{\mathscr{R}^+} \mathscr{C}_n^0\,.$$

Proof. An inspection of the determinant $\mathscr{C}_n^0$ shows that one may view all the pure monomials $c_1^{\lambda_1}$, $\big(\mathscr{C}_2^0\big)^{\lambda_2}$, $\ldots$, $\big(\mathscr{C}_k^0\big)^{\lambda_k}$ as diagonal subblocks of the corresponding sizes lying inside $\mathscr{C}_n^0$. Since the operator $\mathscr{R}^+$ expands the determinants and replaces all the minus signs by plus signs, it is then clear that there are more terms in the right-hand side than there are in the left-hand side, which completes the proof. $\square$

The same arguments yield determinantal inequalities at any level.

Lemma 5.5. *For any two J_1, J_2 with $0 \leqslant J_1, J_2 \leqslant \dim X_\ell$ satisfying in addition $J_1 + J_2 \leqslant \dim X_\ell$, and for any j_1 with $0 \leqslant j_1 \leqslant n$ satisfying in addition $j_1 + J_2 \leqslant \dim X_\ell$, one has the two majorations:*

$$\mathscr{R}^+\big(\Omega_K^\ell \cdot \mathscr{C}_{J_1}^\ell \cdot \mathscr{C}_{J_2}^\ell\big) \leqslant \mathscr{R}^+\big(\Omega_K^\ell \cdot \mathscr{C}_{J_1+J_2}^\ell\big) \quad \textit{and} \quad \mathscr{R}^+\big(\Omega_K^\ell \cdot c_{j_1}^{[\ell]} \cdot \mathscr{C}_{J_2}^\ell\big) \leqslant \mathscr{R}^+\big(\Omega_K^\ell \cdot \mathscr{C}_{j_1+J_2}^\ell\big),$$

where Ω_K^ℓ is any (K, K)-form living on X_ℓ completing to $\dim X_\ell$ the degree, namely with $K + J_1 + J_2$ and with $K + j_1 + J_2$ both equal to $\dim X_\ell$.

If $J_1 + J_2 < 0$ or if $J_1 + J_2 > \dim X_\ell$, and if $j_1 + J_2 < 0$ or if $j_1 + J_2 > \dim X_\ell$, the two sides vanish in both inequalities, which hence hold without restriction.

Lemma 5.6. *These coefficients $\lambda_{j,j-k} = \frac{(n-k)!}{(j-k)!\,(n-j)!} - \frac{(n-k)!}{(j-k-1)!(n-j+1)!}$ appearing in* (52) *satisfy the uniform majoration:*

$$\big|\lambda_{j,j-k}\big| \leqslant 2^n =: \lambda$$

expressed in terms of the dimension n only.

Proof. Indeed, the absolute value of the difference $\lambda_{j,j-k} = \lambda'_{j,j-k} - \lambda''_{j,j-k}$ of two nonnegative integers is less than the largest one, and we majorate any appearing binomial coefficient $\frac{n'!}{i'!\,(n'-i')!}$ or $\frac{n''!}{i''!\,(n''-i'')!}$ with $n' \leqslant n$ and $n'' \leqslant n$ plainly by 2^n. $\square$

In the subsequent majorations, while applying the upper majoration operator $\mathscr{R}^+$, we shall also replace any incoming $\lambda_{j,j-k}$ by this majorant $\lambda = 2^n$. As a result, we define a generalized upper majoration operator "$\mathscr{R}^+_\lambda$" which both replaces any minus sign by a plus sign and any $\lambda_{j,j-k}$ by $\lambda = 2^n$.

Also, when executing inequalities, we shall sometimes not write the left differential form Ω^ℓ_K which completes to $\dim X_\ell$ the total degree of the considered forms, for one knows well now that forms to be reduced always have degree equal to the dimension of the level on which they sit, unless they vanish identically for degree-form reasons.

Lemma 5.7. *For all $k = 1, 2, \dots, n$, one has the $\mathscr{R}^+_\lambda$ majorations:*

$$\mathsf{A}^\ell_k \ \leqslant_{\mathscr{R}^+_\lambda} \ k\lambda\big(\mathscr{C}^{\ell-1}_{k-1} u_\ell + \mathscr{C}^{\ell-1}_{k-2} u^2_\ell + \cdots + u^k_\ell\big).$$

Proof. Starting from the evident majoration of the X^ℓ_j that were defined at the end of Lemma 5.2:

$$\mathsf{X}^\ell_j \ \leqslant_{\mathscr{R}^+_\lambda} \ \lambda\big(c^{[\ell-1]}_{j-1} u_\ell + c^{[\ell-1]}_{j-2} u^2_\ell + \cdots + u^j_\ell\big),$$

we may perform majorations of an arbitrary A^ℓ_k also defined there:

$$\begin{aligned}
\mathsf{A}^\ell_k &= \mathsf{X}^\ell_1 \mathscr{C}^{\ell-1}_{k-1} - \mathsf{X}^\ell_2 \mathscr{C}^{\ell-1}_{k-2} + \mathsf{X}^\ell_3 \mathscr{C}^{\ell-1}_{k-3} - \cdots + (-1)^{k-1} \mathsf{X}^\ell_k \mathscr{C}^{\ell-1}_0 \\
&\leqslant_{\mathscr{R}^+_\lambda} \ \big[\lambda u_\ell\big] \mathscr{C}^{\ell-1}_{k-1} + \big[\lambda\big(c^{[\ell-1]}_1 u_\ell + u^2_\ell\big)\big] \mathscr{C}^{\ell-1}_{k-2} + \big[\lambda\big(c^{[\ell-1]}_2 u_\ell + c^{[\ell-1]}_1 u^2_\ell + u^3_\ell\big)\big] \mathscr{C}^{\ell-1}_{k-3} + \\
&\qquad + \cdots + \big[\lambda\big(c^{[\ell-1]}_{k-1} u_\ell + \cdots + c^{[\ell-1]}_1 u^{k-1}_\ell + u^k_\ell\big)\big] \mathscr{C}^{\ell-1}_0 \\
&= \lambda\Big(u_\ell \big[\mathscr{C}^{\ell-1}_{k-1} + c^{[\ell-1]}_1 \mathscr{C}^{\ell-1}_{k-2} + c^{[\ell-1]}_2 \mathscr{C}^{\ell-1}_{k-3} + \cdots + c^{[\ell-1]}_{k-1} \mathscr{C}^{\ell-1}_0\big] + \\
&\quad + u^2_\ell \big[\qquad\qquad \mathscr{C}^{\ell-1}_{k-2} + c^{[\ell-1]}_1 \mathscr{C}^{\ell-1}_{k-3} + \cdots + c^{[\ell-1]}_{k-2} \mathscr{C}^{\ell-1}_0\big] + \\
&\quad + u^3_\ell \big[\qquad\qquad\qquad\qquad \mathscr{C}^{\ell-1}_{k-3} + \cdots + c^{[\ell-1]}_{k-3} \mathscr{C}^{\ell-1}_0\big] + \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\
&\quad + u^k_\ell \big[\qquad\qquad\qquad\qquad\qquad\qquad \mathscr{C}^{\ell-1}_0\big]\Big).
\end{aligned}$$

Now, we use the majoration of an arbitrary product of a Jacobi-Trudy determinant by a Chern class that was provided in advance by Lemma 5.5 to obtain:

$$\begin{aligned}
\mathsf{A}^\ell_k \ &\leqslant_{\mathscr{R}^+_\lambda} \ \lambda\Big(u_\ell \big[k \cdot \mathscr{C}^{\ell-1}_{k-1}\big] + u^2_\ell \big[(k-1) \cdot \mathscr{C}^{\ell-1}_{k-2}\big] + \cdots + u^k_\ell \big[\mathscr{C}^{\ell-1}_0\big]\Big) \\
&\leqslant_{\mathscr{R}^+_\lambda} \ k\lambda\big(\mathscr{C}^{\ell-1}_{k-1} u_\ell + \mathscr{C}^{\ell-1}_{k-2} u^2_\ell + \cdots + u^k_\ell\big),
\end{aligned}$$

as was to be proved. $\square$

We now have to majorate conveniently the A-polynomials $\Sigma_j^\ell(\mathsf{A})$ defined by (56) in terms of Jacobi-Trudy determinants living at the inferior level $\ell-1$, and in terms of u_ℓ, too. For this purpose, let us define what will play the role of a convenient majorant:

$$\Theta_k^\ell := \mathscr{C}_{k-1}^{\ell-1} u_\ell + \mathscr{C}_{k-2}^{\ell-1} u_\ell^2 + \cdots + \mathscr{C}_1^{\ell-1} u_\ell^{k-1} + u_\ell^k,$$

and let us keep in mind that the lemma just proved provided the majorations $\mathsf{A}_k^\ell \leqslant_{\mathscr{R}_\lambda^+} k\lambda\,\Theta_k^\ell$. To majorate products of A_k^ℓ's, we majorate products of Θ_k^ℓ's.

Lemma 5.8. *For any $k_1, k_2, \ldots, k_\nu$ with $k_1, k_2, \ldots, k_\nu \geqslant 1$ whose sum $k_1 + k_2 + \cdots + k_\nu = j$ equals j, one has the majoration:*

$$\Theta_{k_1}^\ell \Theta_{k_2}^\ell \cdots \Theta_{k_\nu}^\ell \leqslant_{\mathscr{R}_\lambda^+} k_1 k_2 \cdots k_\nu\, \Theta_{k_1+k_2+\cdots+k_\nu}^\ell.$$

Proof. In greater length, the considered product writes:

$$\big(\mathscr{C}_{k_1-1}^{\ell-1} u_\ell + \cdots + u_\ell^{k_1}\big)\big(\mathscr{C}_{k_2-1}^{\ell-1} u_\ell + \cdots + u_\ell^{k_2}\big) \cdots \big(\mathscr{C}_{k_\nu-1}^{\ell-1} u_\ell + \cdots + u_\ell^{k_\nu}\big),$$

and the total number of terms, after expansion, is hence clearly $\leqslant k_1 k_2 \cdots k_\nu$. Using the already known inequality $\mathscr{C}_{J_1}^{\ell-1} \cdot \mathscr{C}_{J_2}^{\ell-1} \leqslant_{\mathscr{R}_\lambda^+} \mathscr{C}_{J_1+J_2}^{\ell-1}$, we may majorate as follows any monomial appearing after expansion:

$$\mathscr{C}_{k_1'}^{\ell-1} \mathscr{C}_{k_2'}^{\ell-1} \cdots \mathscr{C}_{k_\nu'}^{\ell-1}\, u_\ell^{k''} \leqslant_{\mathscr{R}_\lambda^+} \mathscr{C}_{k_1'+\cdots+k_\nu'}^{\ell-1}\, u_\ell^{k''},$$

where $k_1' + k_2' + \cdots + k_\nu' + k'' = k_1 + k_2 + \cdots + k_\nu = j$ of course, which completes the proof. $\square$

At last, we can state and prove the main useful majoration proposition which will enable us to achieve the proof of Theorem 5.1, *cf.* the program launched just before Lemma 5.2.

Proposition 5.1. *At any level ℓ with $1 \leqslant \ell \leqslant n-1$, consider the Jacobi-Trudy determinant $\mathscr{C}_J^\ell$ of an arbitrary size $J \times J$ with $1 \leqslant J \leqslant \dim X_\ell$ and furthermore, let Ω_K^ℓ be any (K,K)-form on X_ℓ the degree K of which satisfies $K + J = \dim X_\ell = n + \ell(n-1)$. Then the upper reduction $\mathscr{R}_\lambda^+(\bullet)$ of $\Omega_K^\ell \mathscr{C}_J^\ell$ in which any incoming $\lambda_{j,j-k}$ is replaced by $\lambda = 2^n \geqslant \big|\lambda_{j,j-k}\big|$ enjoys the following majoration in the right-hand side of which, notably, all the appearing Jacobi-Trudy determinants live at level $\ell-1$:*

$$\Omega_K^\ell \mathscr{C}_J^\ell \leqslant_{\mathscr{R}_\lambda^+} J \cdot 2^J \cdot J^{2J} \cdot 2^{nJ} \cdot \Omega_K^\ell \left[\mathscr{C}_J^{\ell-1} + \mathscr{C}_{J-1}^{\ell-1} u_\ell + \cdots + \mathscr{C}_1^{\ell-1} u_\ell^{J-1} + u_\ell^J\right].$$

Proof. Recall that

$$\mathscr{C}_J^\ell = \sum_{j=1}^J \mathscr{C}_{J-j}^\ell\, \Sigma_j^\ell(\mathsf{A}) = \sum_{j=0}^J \mathscr{C}_{J-j}^{\ell-1} \Bigg(\sum_{\nu=1}^j \sum_{\substack{k_1+\cdots+k_\nu=j \\ k_1,\ldots,k_\nu \geqslant 1}} \mathsf{A}_{k_1}^\ell \cdots \mathsf{A}_{k_\nu}^\ell \Bigg).$$

Using the last two lemmas, we deduce that for any $k_1, \dots, k_\nu \geqslant 1$ with $k_1 + \cdots + k_\nu$ the sum of which $k_1 + \cdots + k_\nu$ equals j, we have the majoration:

$$\begin{aligned}
\mathsf{A}^\ell_{k_1} \cdots \mathsf{A}^\ell_{k_\nu} &\leqslant_{\mathscr{R}^+_\lambda} k_1 \cdots k_\nu \, \lambda^\nu \, \Theta^\ell_{k_1} \cdots \Theta^\ell_{k_\nu} && \text{[Lemma 5.7]} \\
&\leqslant_{\mathscr{R}^+_\lambda} \big(k_1 \cdots k_\nu\big)^2 \, \lambda^\nu \, \Theta^\ell_{k_1 + \cdots + k_\nu} && \text{[Lemma 5.8]} \\
&\leqslant_{\mathscr{R}^+_\lambda} j^{2j} \, \lambda^j \, \Theta^\ell_j.
\end{aligned}$$

Since there are $2^{j-1} \leqslant 2^j$ terms in the sum $\sum_{\nu=1}^j \sum_{\substack{k_1+\cdots+k_\nu=j \\ k_1,\dots,k_\nu \geqslant 1}}$, we receive the useful majoration:

$$\begin{aligned}
\Sigma^\ell_j(\mathsf{A}) &= \sum_{\nu=1}^j \sum_{\substack{k_1+\cdots+k_\nu=j \\ k_1,\dots,k_\nu \geqslant 1}} \mathsf{A}^\ell_{k_1} \cdots \mathsf{A}^\ell_{k_\nu} \\
&\leqslant_{\mathscr{R}^+_\lambda} 2^j \, j^{2j} \, \lambda^j \, \Theta^\ell_j.
\end{aligned}$$

In conclusion, starting from Lemma 5.3 and using Lemma 5.5, we may lastly perform the following (not optimal) majoration:

$$\begin{aligned}
\mathscr{C}^\ell_J &= \mathscr{C}^{\ell-1}_J + \mathscr{C}^{\ell-1}_{J-1} \Sigma^\ell_1(\mathsf{A}) + \mathscr{C}^{\ell-1}_{J-2} \Sigma^\ell_2(\mathsf{A}) + \cdots + \mathscr{C}^{\ell-1}_{J-j} \Sigma^\ell_j(\mathsf{A}) + \cdots + \mathscr{C}^{\ell-1}_0 \Sigma^\ell_J(\mathsf{A}) \\
&\leqslant_{\mathscr{R}^+_\lambda} \mathscr{C}^{\ell-1}_J + \mathscr{C}^{\ell-1}_{J-1} \, 2^1 1^2 \lambda^1 \big[u_\ell\big] + \mathscr{C}^{\ell-1}_{J-2} 2^2 2^4 \lambda^2 \big[\mathscr{C}^{\ell-1}_1 u_\ell + u^2_\ell\big] \\
&\qquad + \cdots + \mathscr{C}^{\ell-1}_{J-j} \, 2^j j^{2j} \lambda^j \big[\mathscr{C}^{\ell-1}_{j-1} u_\ell + \cdots + u^j_\ell\big] \\
&\qquad + \cdots + \mathscr{C}^{\ell-1}_0 \, 2^J J^{2J} \lambda^J \big[\mathscr{C}^{\ell-1}_{J-1} u_\ell + \cdots + u^J_\ell\big] \\
&\leqslant_{\mathscr{R}^+_\lambda} 2^1 1^2 \lambda^1 \big[\mathscr{C}^{\ell-1}_J + \mathscr{C}^{\ell-1}_{J-1} u_\ell\big] + 2^2 2^4 \lambda^2 \big[\mathscr{C}^{\ell-1}_{J-1} u_\ell + \mathscr{C}^{\ell-1}_{J-2} u^2_\ell\big] \\
&\qquad + \cdots + 2^j j^{2j} \lambda^j \big[\mathscr{C}^{\ell-1}_{J-1} u_\ell + \cdots + \mathscr{C}^{\ell-1}_{J-j} u^j_\ell\big] \\
&\qquad + \cdots + 2^J J^{2J} \lambda^J \big[\mathscr{C}^{\ell-1}_{J-1} u_\ell + \cdots + u^J_\ell\big] \\
&\leqslant_{\mathscr{R}^+_\lambda} J \cdot 2^J \cdot J^{2J} \cdot \lambda^J \Big[\mathscr{C}^{\ell-1}_J + \mathscr{C}^{\ell-1}_{J-1} u_\ell + \mathscr{C}^{\ell-1}_{J-2} u^2_\ell + \cdots + \mathscr{C}^{\ell-1}_1 u^{J-1}_\ell + u^J_\ell\Big],
\end{aligned}$$

where the introduction of supplementary terms in the brackets aims at producing a uniform right-hand side. □

5.3. **Proof of Theorem 5.1.** The vanishing of the d^0-coefficient comes from the fact that after reduction to the ground level $\ell = 0$, one gets a sum of homogeneous monomials of the form $h^l c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n}$ with $l + \lambda_1 + 2\lambda_2 + \cdots + n\lambda_n = n$, and then after expressing each c_k in terms of d through (45), one always has the power $h^n = d$ of h in factor.

Notice that the integer J of the Proposition 5.1 will always be less than or equal to $\dim X_{n-1} = n^2 - n + 1$. To simplify the computations and to receive at the end as simple majorants as possible, we shall apply the

following elementary majoration, using $J \leqslant n^2 - n + 1$:

$$\begin{aligned} J \cdot 2^J \cdot J^{2J} \cdot 2^{nJ} &= 2^{(n+1)J} \cdot J^{2J+1} \\ &\leqslant 2^{n^3+1} \left(n^2 - n + 1\right)^{2n^2-2n+3} \\ &\leqslant 2^{n^3} \left(n^2\right)^{2n^2}, \end{aligned}$$

because $2\,(n^2 - n + 1)^{2n^2-2n+3} \leqslant 2\,(n^2)^{2n^2-2n+3} \leqslant (n^2)^{2n^2}$ for any $n \geqslant 2$ (an assumption of Theorem 5.1). Let us temporarily denote this bound by:

$$\mathsf{N} := 2^{n^3}\, n^{4n^2}.$$

As expected, we can now perform a uniform upper majoration of an arbitrary monomial $u_1^{i_1} \cdots u_n^{i_n}$ with $i_1 + \cdots + i_n = n^2$ down to level $\ell = 0$ as follows:

$$\begin{aligned} u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} u_n^{i_n} &= u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} \mathscr{C}_{i_n-n+1}^{n-1} \\ &\leqslant_{\mathscr{R}_\lambda^+} \mathsf{N} \cdot u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} u_{n-1}^{i_{n-1}} \big[\mathscr{C}_{i_n-n+1}^{n-2} + \mathscr{C}_{i_n-n}^{n-2} u_{n-1} \\ &\qquad + \cdots + \mathscr{C}_1^{n-2} u_{n-1}^{i_n-n} + u_{n-1}^{i_n-n+1}\big] \qquad \text{[Proposition 5.1]} \\ &\leqslant_{\mathscr{R}_\lambda^+} \mathsf{N} \cdot u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \big[\underline{\mathscr{C}_{i_n-n+1}^{n-2} u_{n-1}^{i_{n-1}} + \cdots}_{\circ} \\ &\qquad + \mathscr{C}_{i_{n-1}+i_n-2n+2}^{n-2} \underline{u_{n-1}^{n-1}}_{\int} + \cdots + u_{n-1}^{i_{n-1}+i_n-n+1}\big] \\ &\leqslant_{\mathscr{R}_\lambda^+} \mathsf{N} \cdot u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \big[\mathscr{C}_{i_{n-1}+i_n-2n+2}^{n-2} + \mathscr{C}_{i_{n-1}+i_n-2n+1}^{n-2} u_{n-1}^{n} \\ &\qquad + \cdots + u_{n-1}^{i_{n-1}+i_n-n+1}\big] \\ &\leqslant_{\mathscr{R}_\lambda^+} \mathsf{N} \cdot u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \big[\mathscr{C}_{i_{n-1}+i_n-2n+2}^{n-2} + \mathscr{C}_{i_{n-1}+i_n-2n+1}^{n-2} \mathscr{C}_1^{n-2} \\ &\qquad + \cdots + \mathscr{C}_{i_{n-1}+i_n-2n+2}^{n-2}\big] \qquad \text{[Lemma 5.1]} \\ &\leqslant_{\mathscr{R}_\lambda^+} \mathsf{N}\, n^2 \cdot u_1^{i_1} \cdots u_{n-2}^{i_{n-2}} \mathscr{C}_{i_{n-1}+i_n-2n+2}^{n-2} \qquad \text{[Lemma 5.5]} \\ &\leqslant_{\mathscr{R}_\lambda^+} \left(\mathsf{N}\, n^2\right)^2 \cdot u_1^{i_1} \cdots u_{n-3}^{i_{n-3}} \mathscr{C}_{i_{n-2}+i_{n-1}+i_n-3n+3}^{n-3} \qquad \text{[induction]} \\ &\leqslant_{\mathscr{R}_\lambda^+} \left(\mathsf{N}\, n^2\right)^3 \cdot u_1^{i_1} \cdots u_{n-4}^{i_{n-4}} \mathscr{C}_{i_{n-3}+i_{n-2}+i_{n-1}+i_n-4n+4}^{n-4} \qquad \text{[induction].} \end{aligned}$$

In the third line, we exhibit the general case where i_{n-1} can be $< n-1$, we underline the terms vanishing for degree-form reasons and we point out the fiber-integration of u_{n-1}^{n-1}; when $i_{n-1} \geqslant n-1$, the underlined terms are absent. In the sixth line, we majorate plainly by n^2 the number of terms inside the brackets. (Recall that here by convention again, $\mathscr{C}_J^\ell = 0$ if either $J < 0$ or $J > \dim X_\ell$, so that some of the written $\mathscr{C}_J^\ell$ might well vanish, depending on $i_1, \ldots, i_n$.) A now clear induction down to level $\ell = 1$ therefore yields:

$$\begin{aligned} u_1^{i_1} \cdots u_{n-1}^{i_{n-1}} u_n^{i_n} &\leqslant_{\mathscr{R}_\lambda^+} \left(\mathsf{N}\, n^2\right)^{n-2} \cdot u_1^{i_1} \mathscr{C}_{i_2+\cdots+i_n-(n-1)n+n-1}^{1} \\ &\leqslant_{\mathscr{R}_\lambda^+} \left(\mathsf{N}\, n^2\right)^{n-2} \cdot \mathsf{N} \cdot \big[\underline{\mathscr{C}_{2n-1-i_1}^{0} + \cdots}_{\circ} + \\ &\qquad + \mathscr{C}_n^0 \underline{u_1^{n-1}}_{\int} + \cdots + u_1^{2n-1}\big] \\ &\leqslant_{\mathscr{R}_\lambda^+} \left(\mathsf{N}\, n^2\right)^{n-1} \mathscr{C}_n^0. \end{aligned}$$

It only remains to majorate $\mathscr{C}_n^0$. This last reduction using only (45) without any $\lambda_{j,j-k}$, let us denote by $\mathscr{R}_d^+$ the upper reduction operator restricted to level $\ell = 0$.

Lemma 5.9. *The $n \times n$ Jacobi-Trudy determinant $\mathscr{C}_0^n$ enjoys the majoration:*

$$\mathscr{C}_n^0 \leqslant_{\mathscr{R}_d^+} 2^{n^2+2n}\, n!\, n^n \big[d^{n+1} + d^n + \cdots + d\big].$$

Proof. The number of monomials in the universal $n \times n$ determinant $|a_i^j|$ is $\leqslant n!$ (and is $< n!$ when some a_i^j are zero). Hence:

$$\mathscr{C}_n^0 \leqslant_{\mathscr{R}_d^+} n! \max_{\lambda_1+2\lambda_2+\cdots+n\lambda_n=n} c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n}.$$

The general binomial coefficient $\binom{n+2}{k}$ which appears in (45) is less than or equal to 2^{n+2}, so that:

$$c_j \leqslant_{\mathscr{R}_d^+} 2^{n+2}\, h^j \big[d^j + \cdots + d + 1\big].$$

We majorate as follows the products of these basic polynomials in d:

$$\big[d^{j_1} + \cdots + d + 1\big]\big[d^{j_2} + \cdots + d + 1\big] \leqslant_{\mathscr{R}_d^+} j_1 j_2 \big[d^{j_1+j_2} + \cdots + d + 1\big],$$

and we therefore deduce a majorant for the general homogeneous degree n monomial in the ground Chern classes:

$$\begin{aligned} c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} &\leqslant_{\mathscr{R}_d^+} \big(2^{n+2}\big)^{\lambda_1+\lambda_2+\cdots+\lambda_n}\, 1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n}\, h^{\lambda_1+2\lambda_2+\cdots+n\lambda_n} \\ &\qquad \cdot \big[d^{\lambda_1+2\lambda_2+\cdots+n\lambda_n} + \cdots + d + 1\big] \\ &\leqslant_{\mathscr{R}_d^+} \big(2^{n+2}\big)^n\, n^{\lambda_1+\lambda_2+\cdots+\lambda_n}\, h^n \big[d^n + \cdots + d + 1\big] \\ &\leqslant_{\mathscr{R}_d^+} 2^{n^2+2n}\, n^n\, d \big[d^n + \cdots + d + 1\big] \end{aligned}$$

which completes the proof. □

Applying this lemma to the last obtained inequality:

$$u_1^{i_1} \cdots u_n^{i_n} \leqslant_{\mathscr{R}_\lambda^+} (Nn^2)^{n-1}\, 2^{n^2+2n}\, n!\, n^n \cdot \big[d^{n+1} + d^n + \cdots + 1\big],$$

we then obtain the announced bound $n^{4n^3} 2^{n^4}$ as follows:

$$\begin{aligned} \big|\mathsf{coeff}_{d^k}\big[u_1^{i_1} \cdots u_n^{i_n}\big]\big| &\leqslant \big(2^{n^3}\, n^{4n^2}\, n^2\big)^{n-1}\, 2^{n^2+2n}\, n!\, n^n \\ &\leqslant 2^{n^4-n^3+n^2+2n}\, n^{4n^3-4n^2+2n-2}\, n^n\, n^n \\ &\leqslant n^{4n^3} 2^{n^4}. \end{aligned}$$

By an inspection of the final inequalities which enabled us to descend from the top of Demailly's tower to its ground level, one easily convinces oneself

that the monomials $h^l u_1^{i_1} \cdots u_n^{i_n}$ and $c_1 h^l u_1^{j_1} \cdots u_n^{j_n}$ satisfy exactly the same upper bound reduction:

$$h^l u_1^{i_1} \cdots u_n^{i_n} \leqslant_{\mathscr{R}_\lambda^+} \big(N\, n^2\big)^{n-1} \mathscr{C}_n^0 \quad \text{and}$$
$$c_1 h^l u_1^{j_1} \cdots u_n^{j_n} \leqslant_{\mathscr{R}_\lambda^+} \big(N\, n^2\big)^{n-1} \mathscr{C}_n^0,$$

since the forms h^l and $c_1 h^l$ do intervene only at the very end of the process. This completes the proof of Theorem 5.1. At the same time, the proof of Theorem 1.1 can be considered as complete, as soon as we take for granted Theorem 1.2, as was already explained at the end of Section 4. □

6. Effective bounds in dimensions 2, 3 and 4 through the invariant theory approach

The goal of this section is to obtain the effective bound $\deg X^4 \geqslant 3203$ of Theorem 1.2 in dimension $n = 4$ which insures strong algebraic degeneracy of entire curves inside a generic projective four-fold $X^4 \subset \mathbb{P}^5$. As was said in the Introduction, our reasonings will be based on a complete knowledge ([22]) of the full algebra $\bigoplus_{m \geqslant 0} E_{4,m} T^*_{X^4, x_0}$ of germs of invariant 4-jet differentials at a point $x_0 \in X^4$, which, unfortunately, is still unavailable at present times for jets of order $k \geqslant n$ in the higher dimensions $n = 5, 6, 7, \ldots$ (remind that by Theorem 1.1 in [30] and by Theorem 1 in [6], $H^0\big(X^n,\, E_{k,m} T^*_{X^n}\big) = 0$ whenever $k \leqslant n-1$). The so obtained bound $\deg X^4 \geqslant 3203$ happens to be sharper than the one $\deg X^4 \geqslant 6527$ that one would obtain using the intersection product (48). For completeness and in parallel, we also recall what happens in the lower dimensions 2 ([4, 5]) and 3 ([29, 30]).

6.1. **Algebras of bi-invariant k-jet differentials.** Let $(x_1, \ldots, x_n)$ be local coordinates centered at some point $x_0 \in X$ and let $f = (f_1, \ldots, f_n) \colon (\mathbb{C}, 0) \to (X, x_0)$ be a germ of holomorphic curve. For each fixed l-th jet level ($1 \leqslant l \leqslant k$) over x_0, the constant matrices $v = (v_i^j)_{1 \leqslant i \leqslant n}^{1 \leqslant j \leqslant n}$ in $\mathrm{GL}_n(\mathbb{C})$ act in a natural way on the n jet coordinates $(f_1^{(l)}, \ldots, f_n^{(l)})$ simply by:

$$v \cdot f^{(l)} := \Big(\textstyle\sum_j v_i^j f_j^{(l)}\Big)_{1 \leqslant i \leqslant n}.$$

In order to know what is the precise decomposition of $\mathrm{Gr}^\bullet(E_{k,m} T^*_X)$ as a direct sum of Schur bundles $\Gamma^{(\ell_1, \ldots, \ell_n)} T^*_X$, the classical representation theory of $\mathrm{GL}_n(\mathbb{C})$ tells us that one should look at jet polynomials $Q(f', f'', \ldots, f^{(k)})$ that are not only invariant under reparametrization in the sense of Definition 2.1, but also invariant under the action of the full unipotent subroup $\mathrm{U}_n(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C})$ consisting of matrices with 1 on the diagonal, 0 above the diagonal, and arbitrary complex number below the diagonal; background

information may be found in [4, 29, 30, 22]. Accordingly, one may define the algebra of *bi-invariant* k-jet polynomials in dimension n:

$$\mathsf{UE}_k^n := \Big(\bigoplus_{m\geqslant 0} E_{k,m}T^*_{X^n,x_0}\Big)^{\mathrm{U}_n(\mathbb{C})}.$$

This algebra does not depend on the base point $x_0 \in X$. We shall employ the abbreviations $\Delta_{i_1,i_2}^{(\alpha),(\beta)} := f_{i_1}^{(\alpha)} f_{i_2}^{(\beta)} - f_{i_2}^{(\alpha)} f_{i_1}^{(\beta)}$ for 2×2 determinants, and similarly $\Delta_{i_1,i_2,i_3}^{(\alpha),(\beta),(\gamma)}$ for the analogous 3×3 determinants. The upper indices of all the appearing 16 bi-invariants f_1', Λ^3, Λ^5, Λ^7, D^6, D^8, N^{10}, W^{10}, M^8, E^{10}, L^{12}, Q^{14}, R^{15}, U^{17}, V^{19} and X^{21} below just denote their weighted degree m.

Theorem 6.1. *The following three algebraic descriptions hold.*

- [4] *In dimension* 2*, one has:* $\mathsf{UE}_2^2 = \mathbb{C}\big[f_1',\, \Lambda^3\big]$*, where* $\Lambda^3 := \Delta_{1,2}^{',''} = f_1'f_2'' - f_2'f_1''$ *is the two-dimensional Wronskian.*
- [29] *In dimension* 3*, one has:*
$$\mathsf{UE}_3^3 = \mathbb{C}\big[f_1',\, \Lambda^3,\, \Lambda^5,\, D^6\big],$$
where $\Lambda^5 := \Delta_{1,2}^{','''} f_1' - 3\,\Delta_{1,2}^{',''} f_1''$ *and where* $D^6 := \Delta_{1,2,3}^{',''',''''}$ *is the three-dimensional Wronskian.*
- [22] *In dimension* 4*, one has:*
$$\begin{aligned}\mathsf{UE}_4^4 =&\mathbb{C}\big[f_1',\, \Lambda^3,\, \Lambda^5,\, \Lambda^7,\, D^6,\, D^8,\, N^{10},\, W^{10},\, M^8,\, E^{10},\, L^{12},\\ &Q^{14},\, R^{15},\, U^{17},\, V^{19},\, X^{21}\big]\Big/\textit{a certain ideal of } 41 \textit{ relations},\end{aligned}$$
where:
$$\begin{aligned}\Lambda^7 &= \Delta_{1,2}^{',''''} f_1'f_1' + \Delta_{1,2}^{'','''} f_1'f_1' - 10\,\Delta_{1,2}^{','''} f_1'f_1'' + 15\,\Delta_{1,2}^{',''} f_1''f_1'',\\ D^8 &= \Delta_{1,2,3}^{',''',''''} f_1' - 3\,\Delta_{1,2,3}^{','',''''} f_1'',\\ N^{10} &= \Delta_{1,2,3}^{',''',''''} f_1'f_1' - 3\,\Delta_{1,2,3}^{','',''''} f_1'f_1'' + 4\,\Delta_{1,2,3}^{','','''} f_1'f_1''' + 3\,\Delta_{1,2,3}^{','','''} f_1''f_1'',\end{aligned}$$
where $W^{10} = \Delta_{1,2,3,4}^{','',''',''''}$ *is the four-dimensional Wronskian and where the eight remaining bi-invariants defined by:*
$$\begin{aligned}M^8 &:= \tfrac{-5\,\Lambda^5\Lambda^5+3\,\Lambda^3\Lambda^7}{f_1'f_1'}, & E^{10} &:= \tfrac{-6\,\Lambda^5\,D^6+3\,\Lambda^3\,D^8}{f_1'}, & L^{12} &:= \tfrac{-\Lambda^7D^6+5\,\Lambda^3N^{10}}{f_1'},\\ Q^{14} &:= \tfrac{\Lambda^7D^8-10\,\Lambda^5N^{10}}{f_1'}, & R^{15} &:= \tfrac{D^8D^8-12\,D^6N^{10}}{f_1'}, & U^{17} &:= \tfrac{4\,D^8E^{10}+3\,\Lambda^3R^{15}}{f_1'},\\ V^{19} &:= \tfrac{8\,N^{10}E^{10}+\Lambda^5R^{15}}{f_1'}, & X^{21} &:= \tfrac{4\,D^8Q^{14}-5\,\Lambda^7R^{15}}{f_1'}\end{aligned}$$
happen all to be true polynomials *in* $\mathbb{C}\big[f_1',\ldots,f_4''''\big]$*, and where an explicit Gröbner basis, with respect to the pure lexicographic term-order* $f_1' < \Lambda^3 < \cdots < X^{21}$*, for the ideal of relations that they share, is provided in §11 of* [22].

For instance, the first three relations among the 41 written just before the theorem of §11 in [22] are:

$$\begin{aligned}
0 &\overset{1}{\equiv} -5\,\Lambda^5\Lambda^5 + 3\,\Lambda^3\Lambda^7 - f_1'f_1'M^8,\\
0 &\overset{2}{\equiv} -2\,\Lambda^5 D^6 + \Lambda^3 D^8 - \tfrac{1}{3}f_1'\,E^{10},\\
0 &\overset{3}{\equiv} -\Lambda^7 D^6 + 5\,\Lambda^3 N^{10} - f_1' L^{12}.
\end{aligned}$$

Although the complexity of the algebra of bi-invariants increases dramatically as soon as $n \geqslant 4$, one finds in [22] a complete algorithm which generates all bi-invariants together with all the relations that they share, this in arbitrary dimension $n \geqslant 1$ and for arbitrary jet order $k \geqslant 1$.

6.2. **Schur bundle decompositions.** In dimension 3, there are no relations between the four basic bi-invariants f_1', Λ^3, Λ^5 and D^6 and we hence clearly have:

$$\big(E_{k,m}T^*_{X^n,x_0}\big)^{\mathrm{U}_3(\mathbb{C})} = \mathrm{Span}_{\mathbb{C}}\Big\{(f_1')^a(\Lambda^3)^b(\Lambda^5)^c(D^6)^d\colon \\ a,b,c,d\in\mathbb{N},\ a+3b+5c+6d=m\Big\}.$$

Then to any such general monomial $(f_1')^a(\Lambda^3)^b(\Lambda^5)^c(D^6)^d$ having weighted degree $m = a+3b+5c+6d$, the representation theory of $\mathrm{GL}_n(\mathbb{C})$ tells us that there corresponds the Schur bundle:

$$\Gamma^{(a+b+2c+d,\ b+c+d,\ d)}T^*_X,$$

just because the diagonal 3×3 matrices $t = \mathrm{diag}(t_1,t_2,t_3)$ act as: $t\cdot f_i^{(\lambda)} := t_i\, f_i^{(\lambda)}$, whence:

$$t\cdot f_1' = t_1 f_1', \qquad t\cdot\Lambda^3 = t_1t_2\,\Lambda^3, \qquad t\cdot\Lambda^5 = t_1t_1t_2\,\Lambda^5, \qquad t\cdot D^6 = t_1t_2t_3\,D^6,$$

so that indeed the three exponents of the t_i in:

$$t\cdot(f_1')^a(\Lambda^3)^b(\Lambda^5)^c(D^6)^d = t_1^{a+b+2c+d}\ t_2^{b+c+d}\ t_3^d\ (f_1')^a(\Lambda^3)^b(\Lambda^5)^c(D^6)^d$$

indicate the three corresponding integers in $\Gamma^{(\lambda_1,\lambda_2,\lambda_3)}T^*_X$. The same elementary process enables one, in dimensions 2 and 4, to immediately deduce from the preceding statement the following important decomposition theorem for the graded bundle $\mathrm{Gr}^\bullet E_{k,m}T^*_{X^n}$ associated to $E_{k,m}T^*_{X^n}$, which is valuable without assuming that X is projective.

Theorem 6.2. *Let X be a compact complex manifold and let $m\in\mathbb{N}$.*

- [4] *If* $\dim X = 2$ *then*

$$\mathrm{Gr}^\bullet\, E_{2,m}T^*_X = \bigoplus_{a+3b=m} \Gamma^{(a+b,\ b)}T^*_X.$$

- [29] *If* $\dim X = 3$ *then*

$$\mathrm{Gr}^\bullet E_{3,m}T_X^* = \bigoplus_{a+3b+5c+6d=m} \Gamma^{(a+b+2c+d,\ b+c+d,\ d)}T_X^*.$$

- [22] *If* $\dim X = 4$ *then*

$$\mathrm{Gr}^\bullet E_{4,m}T_X^* = \bigoplus_{\substack{(a,b,c,d,e,f,g,h,i,j,k,l,m',n)\in\mathbb{N}^{14}\setminus(\square_1\cup\cdots\cup\square_{41})\\ o+3a+5b+7c+6d+8e+10f+8g+10h+12i+14j+15k+17l+19m'+21n+10p=m}}$$

$$\Gamma\begin{pmatrix} o+a+2b+3c+d+2e+3f+2g+2h+3i+4j+3k+3l+4m'+5n+p \\ a+b+c+d+e+f+2g+2h+2i+2j+2k+3l+3m'+3n+p \\ d+e+f+h+i+j+2k+2l+2m'+2n+p \\ p \end{pmatrix} T_X^*,$$

where the 41 subsets $\square_i$, $i = 1, 2, \ldots, 41$, *of* $\mathbb{N}^{14} \ni (a, b, \ldots, l, m', n)$ *are explicitly defined in §12 of* [22].

6.3. **Euler-Poincaré characteristic of Schur bundles.** With $X = X^n \subset \mathbb{P}^{n+1}$ projective as before and with $c_j = c_j(T_X)$ for $j = 1, \ldots, n$ being the Chern classes of T_X as in (45), a general asymptotic formula for the Euler-Poincaré characteristic of a Schur bundle is given in §13 of [22] (*see* also Theorem 4 in [1]), and for $n = 4$, this formula expands as:

$$\chi\big(X,\, \Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)}\, T_X^*\big) = \frac{c_1^4 - 3\,c_1^2c_2 + c_2^2 + 2\,c_1c_3 - c_4}{0!\ 1!\ 2!\ 7!} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^7 & \ell_2^7 & \ell_3^7 & \ell_4^7 \end{vmatrix} +$$

(58)

$$+\frac{c_1^2c_2 - c_2^2 - c_1c_3 + c_4}{0!\ 1!\ 3!\ 6!} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^6 & \ell_2^6 & \ell_3^6 & \ell_4^6 \end{vmatrix} + \frac{-c_1c_3 + c_2^2}{0!\ 1!\ 4!\ 5!} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \\ \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \end{vmatrix} +$$

$$+\frac{c_1c_3 - c_4}{0!\ 2!\ 3!\ 5!} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \end{vmatrix} + \frac{c_4}{1!\ 2!\ 3!\ 4!} \begin{vmatrix} \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \end{vmatrix} + \mathrm{O}\big(|\ell|^9\big).$$

Of course, similar expanded — though shorter — formulae exist also in dimensions 2 and 3, *cf.* again §13 of [22].

6.4. **Riemann-Roch computations.** Recalling that the n-th power $h^n = d$ of the hyperplane class $h = c_1\big(\mathscr{O}_{\mathbb{P}^{n+1}}(1)\big)$ equals $\deg X$, the formulae (45) entail that any monomial $c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n}$ whose weighted homogeneous degree $\lambda_1 + 2\lambda_2 + \cdots + n\lambda_n$ equals n is a polynomial in $\mathbb{Z}[d]$, as are c_1^4, $c_1^2c_2$,

c_2c_2, c_1c_3 and c_4 above when $n=4$. Basic additivity, *e.g.* in dimension 3:

$$\chi\big(X,\, E_{3,m}T_X^*\big) = \chi\big(X,\, \mathrm{Gr}^\bullet E_{3,m}T_X^*\big) = \sum_{a+3b+5c+6d=m} \chi\big(X,\, \Gamma^{(a+b+2c+d,\, b+c+d,\, d)}T_X^*\big)$$

enables one to deduce, by plain numerical summation and with some electronic assistance, the following three Euler-Poincaré characteristics, depending upon m and d only. We notice that the summation of the three attached remainders, *e.g.* of $\mathrm{O}\big(|\ell|^9\big)$ in dimension 4, only contributes up to a lower power of m, *e.g.* up to an $\mathrm{O}\big(m^{15}\big)$ in dimension 4.

Theorem 6.3. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d.*

- [4] *For $n=2$:*

$$\chi\big(X, E_{2,m}T_X^*\big) = \frac{m^4}{648}\, d\,\big(4d^2 - 68d + 154\big) + \mathrm{O}(m^3).$$

- [29] *For $n=3$:*

$$\chi\big(X,\, E_{3,m}T_X^*\big) = \frac{m^9}{81648 \times 10^6}\, d\,\big(389d^3 - 20739d^2 + 185559d - 358873\big) + \mathrm{O}(m^8).$$

- [22] *For $n=4$:*

$$\begin{aligned}\chi(X,\, E_{4,m}T_X^*) = {} & \frac{m^{16}}{1313317832303894333210335641600000000000000} \cdot d \cdot \\ & \cdot \big(5004851113579703425623 5\, d^4 - \\ & - 617060662250595525598878 6\, d^3 - \\ & - 928886901354141153880624704\, d + \\ & + 1411704752502476621473639 41\, d^2 + \\ & + 1624908955061039283976041114\big) \\ & + \mathrm{O}\big(m^{15}\big).\end{aligned}$$

6.5. **The strategy of controlling the even cohomology dimensions.** Remember from Theorem 2.2 that the first step towards the algebraic degeneracy of entire curves $f\colon \mathbb{C} \to X$ consists in proving the existence of nonzero global sections in $H^0\big(X,\, E_{k,m}T_X^* \otimes A^{-1}\big)$, for some ample line bundle $A \to X$, *e.g.* $A = \mathscr{O}_X(1)$, and when A does not depend on m, the asymptotic cohomologies, as $m \to \infty$, of the two bundles $E_{k,m}T_X^*$ and $E_{k,m}T_X^* \otimes A^{-1}$ coincide. So a quite natural strategy, followed by the third-named author in [30], consists to rewrite the characteristic, say in dimension four: $\chi = h^0 - h^1 + h^2 - h^4$ under the form:

$$\begin{aligned} h^0 &= \chi + h^1 - h^2 + h^3 - h^4 \\ &\geqslant \chi \qquad\quad - h^2 \qquad\quad - h^4, \end{aligned}$$

and to control asymptotically the dimensions $h^{2i}_{k,m}$ of all the even cohomology groups $H^{2i}(X,\, E_{k,m}T_X^* \otimes A^{-1})$ by some vanishing theorem or by some

appropriate inequalities which would then show that these $h^{2i}_{k,m}$ grow less rapidly than the characteristic $\chi_{k,m}$ as m tends to ∞. In dimensions 2 and 4, the controls of the top even cohomology dimensions h^2 and h^4 are obtained thanks to a vanishing theorem due to Demailly which generalized a theorem of Bogomolov.

Theorem 6.4 ([4]). *Let X be a projective algebraic manifold of dimension $n \geqslant 2$ and let L be a holomorphic line bundle over X. Assume that K_X is big and nef and let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ be a weight with $\mu_1 \geqslant \cdots \geqslant \mu_n$. If either L is pseudo-effective and $|\mu| = \mu_1 + \cdots + \mu_n > 0$, or L is big and $|\mu| \geqslant 0$, then*:

$$H^0\big(X,\, \Gamma^{(\mu_1,\dots,\mu_n)}T_X \otimes L^*\big) = 0.$$

Recall that if some μ_i is negative, we may use the identity:

$$\Gamma^{(\mu_1,\dots,\mu_n)}T^*_X = \Gamma^{(\mu_1+l,\dots,\mu_n+l)}T^*_X \otimes K_X^{-l}.$$

For instance in dimension 4, we observe that the above vanishing theorem implies that

$$h^4\big(X,\, E_{4,m}T^*_X \otimes A^{-1}\big) = 0,$$

for all m sufficiently large; indeed, Serre duality and a division by a tensor power of K_X gives:

$$\begin{aligned} &h^4\big(X,\, \Gamma^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}T^*_X \otimes A^{-1}\big) = h^0\big(X,\, \Gamma^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}T_X \otimes A \otimes K_X\big) \\ &= h^0\Big(X,\, \Gamma^{(\lambda_1-\nu,\lambda_2-\nu,\lambda_3-\nu,\lambda_4-\nu)}T_X \otimes {K_X}^{1-\nu} \otimes \mathscr{O}(A)\Big). \end{aligned}$$

But ${K_X}^{\nu-1} \otimes A^{-1}$ is big for ν large enough and then the above theorem applies to provide the vanishing of h^4 as soon as:

$$|\lambda| - 4\nu \geqslant 0,$$

which is satisfied for m large enough since one easily convinces oneself that $|\lambda| \geqslant \frac{4m}{10}$ in the dimension 4 case of Theorem 6.2.

However, it has been discovered by the third-named author [30] that already in dimension three, $H^2\big(X,\, E_{3,m}T^*_X\big) \neq 0$ does not vanish in general. Fortunately, a suitable majoration holds.

Theorem 6.5 ([30]). *Let X be a smooth hypersurface of degree d in $\mathbb{P}^4$. Then for $|\lambda|$ large enough:*

$$\begin{aligned} &h^2\big(X, \Gamma^{(\lambda_1,\lambda_2,\lambda_3)}T^*_X\big) \\ &\leqslant d(d+13)\frac{3(\lambda_1+\lambda_2+\lambda_3)^3}{2}(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3) + \mathrm{O}(|\lambda|^5). \end{aligned}$$

In dimension 4 the same proof provides the new estimate:

Theorem 6.6. *Let X be a smooth hypersurface of degree d in $\mathbb{P}^5$. Then for $|\lambda|$ large enough, we have:*

$$\begin{aligned}
&h^2\big(X, \Gamma^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}T^*_X\big)\\
&\qquad\leqslant \frac{1}{80}\, d\,(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)\\
&\qquad\cdot\big(\lambda_1+\lambda_2+\lambda_3+\lambda_4\big)^2\big[5\lambda_2\lambda_1 d^2+132\lambda_2\lambda_1 d+132\lambda_1\lambda_3 d+5\lambda_2\lambda_3 d^2\\
&\qquad+132\lambda_2\lambda_4 d+5\lambda_2 d^2\lambda_4+132\lambda_1\lambda_4 d+5\lambda_3\lambda_4 d^2+5\lambda_1\lambda_3 d^2\\
&\qquad+132\lambda_3\lambda_4 d+132\lambda_2\lambda_3 d+1308\lambda_2\lambda_1+648\lambda_2^2+648\lambda_3^2\\
&\qquad+72\lambda_3^2 d+648\lambda_1^2+72\lambda_1^2 d+1308\lambda_1\lambda_4+5\lambda_1 d^2\lambda_4+1308\lambda_2\lambda_4\\
&\qquad+1308\lambda_2\lambda_3+648\lambda_4^2+72\lambda_2^2 d+1308\lambda_1\lambda_3+72\lambda_4^2 d+1308\lambda_3\lambda_4\big]\\
&\qquad+\mathrm{O}\big(|\lambda|^9\big).
\end{aligned}$$

Proof. We follow [30] pp. 335-36, summarizing the main arguments for the convenience of the reader. The proof is essentially, again, an application of holomorphic Morse inequalities and the reader will notice strong similarities with the arguments presented in section 2.

Let $Y := Fl(T^*_X)$ be the flag manifold of T^*_X and let $\pi\colon Fl(T^*_X) \to X$ the natural projection. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be a weight and $\mathscr{L}^\lambda$ the line bundle on Y associated to $\Gamma^\lambda T^*_X$ such that $\Gamma^\lambda T^*_X = \pi_*(\mathscr{L}^\lambda)$. By a theorem of Bott, these bundles have the same cohomology (*cf.* [30] p. 327) and therefore we are reduced to control the cohomology of a line bundle. To this aim, we write:

$$\mathscr{L}^\lambda = \big(\mathscr{L}^\lambda \otimes \pi^*\mathscr{O}_X(3\,|\lambda|)\big) \otimes \big(\pi^*\mathscr{O}_X(3\,|\lambda|)\big)^{-1} = F \otimes G^{-1},$$

with $F := \mathscr{L}^\lambda \otimes \pi^*\mathscr{O}_X(3\,|\lambda|)$ and $G := \pi^*\mathscr{O}_X(3\,|\lambda|)$. We observe that $\mathscr{L}^\lambda \otimes \pi^*\mathscr{O}_X(3\,|\lambda|)$ is positive because $T^*_X \otimes \mathscr{O}_X(2)$ is semi-positive ([30]).

Recall also ([30]) that we can write $K_Y = (\mathscr{L}^\sigma)^{-1} \otimes \pi^* K_X^5$ where $\sigma = (7, 5, 3, 1)$, so:

$$F \otimes K_Y^{-1} = \mathscr{L}^{\lambda+\sigma} \otimes \pi^*\mathscr{O}_X(3\,|\lambda|) \otimes \pi^* K_X^{-5}.$$

Then we still have the positivity $F \otimes K_Y^{-1} > 0$ for $|\lambda|$ large enough, because similarly as above, the line bundle:

$$\mathscr{L}^{\lambda+\sigma} \otimes \pi^*\mathscr{O}_X\big(2\,|\lambda+\sigma|\,\big)$$

is semi-positive as soon as $|\lambda| > 5(d-6)+32$.

Now, we take a smooth irreducible divisor D_1 in the linear series $|G|$ of the form $\pi^*(E_1)$ for some divisor in X. On Y, we have the exact sequence:

$$0 \longrightarrow \mathscr{O}_Y\big(F \otimes G^{-1}\big) \longrightarrow \mathscr{O}_Y(F) \longrightarrow \mathscr{O}_{D_1}(F) \longrightarrow 0,$$

and therefore in the associated long exact cohomology sequence:

$$\begin{aligned}
0 = H^i\big(Y, \mathscr{O}_Y(F)\big) \longrightarrow H^i\big(D_1, \mathscr{O}_{D_1}(F)\big) &\longrightarrow H^{i+1}\big(Y, \mathscr{O}_Y(F \otimes G^{-1})\big)\\
&\longrightarrow H^{i+1}\big(Y, \mathscr{O}_Y(F)\big) = 0,
\end{aligned}$$

both the first and last terms vanish for any $i > 0$ by an application of the Kodaira vanishing theorem. We at once deduce:

$$h^i\big(D_1,\, \mathscr{O}_{D_1}(F)\big) = h^{i+1}\big(Y, \mathscr{O}_Y(F \otimes G^{-1})\big).$$

Next, we take a second divisor $D_2 \in |G|$ intersecting properly D_1 and of the form $\pi^*(E_2)$ too. Using the adjunction formula and applying a similar restriction to $D_3 := D_1 \cap D_2$ (word by word, the arguments are exactly the same as in [30], pp. 335–336, so we do not repeat the complete proof), we obtain:

$$h^2\big(Y, \mathscr{O}_Y(F{\otimes}G^{-1})\big) = h^1\big(D_1, \mathscr{O}_{D_1}(F)\big) \leqslant h^0\big(D_3,\, \mathscr{O}_{D_3}(F{\otimes}G^2)\big) = \chi\big(D_3,\, \mathscr{O}_{D_3}(F{\otimes}G^2)\big).$$

Letting $E_3 := E_1 \cap E_2$, one then shows ([30], p. 336) that the latter Euler-Poincaré characteristic equals the following linear combination of characteristics *on the base* X:

$$\begin{aligned}\chi\big(D_3,\, \mathscr{O}_{D_3}(F \otimes G^2)\big) &= \chi\big(E_3,\, \Gamma^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}T^*_X|_{E_3} \otimes \mathscr{O}_{E_3}(9|\lambda|)\big)\\ &= \chi\big(X,\, \Gamma^\lambda T^*_X \otimes \mathscr{O}_X(9|\lambda|)\big) - 2\,\chi\big(X,\, \Gamma^\lambda T^*_X \otimes \mathscr{O}_X(6|\lambda|)\big) + \chi\big(X,\, \Gamma^\lambda T^*_X \otimes \mathscr{O}_X(3|\lambda|)\big).\end{aligned}$$

So the h^2 we want to majorate is less than or equal to this last line. But then by applying a general complete combinatorial formula due to Brückmann (Theorem 4 in [1]) for the characteristic $\chi\big(X,\, \Gamma^\lambda T^*_X{\otimes}\mathscr{O}_X(t)\big)$ of any twisted Schur bundle over X, we may terminate the proof either by hand or with the help of a computer. $\square$

From such controls of higher cohomology groups, we deduce the existence of global algebraic differential equations canalizing all entire holomorphic maps: to obtain minorations of $h^0 \geqslant \chi - h^2$, it suffices indeed as already explained to perform summations, according to the representations of Theorem 6.2, of the asymptotic Euler characteristics (58), subtracting at the same time the majorant of h^2 just obtained. At first, we recall here what is known in dimensions 2 and 3. The twisting $(\bullet) \otimes A^{-1}$ by the negative of a fixed ample line bundle $A \to X$ is erased in the asymptotics.

Theorem 6.7. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d and let A be any ample line bundle over X.*

- [4] *For $n = 2$:*

$$h^0\big(X,\, E_{2,m}T^*_X \otimes A^{-1}\big) \geqslant \frac{m^4}{648}\, d\,\big(4d^2 - 68d + 154\big) + \mathrm{O}(m^3);$$

- [30] *For $n = 3$:*

$$\begin{aligned}h^0\big(X,\, E_{3,m}T^*_X \otimes A^{-1}\big) \geqslant \frac{m^9}{408240000000} \cdot d \cdot \big(1945\, d^3 - 103695\, d^2\\ - 7075491\, d - 105837083\big) + \mathrm{O}(m^8).\end{aligned}$$

In particular, if $d \geqslant 15$ (resp. $d \geqslant 97$) then $E_{2,m}T^*_X \otimes A^{-1}$ (resp. $E_{3,m}T^*_X \otimes A^{-1}$) admits non trivial sections for m large, and every entire curve $f\colon \mathbb{C} \to X$ must satisfy the corresponding algebraic differential equations.

In dimension 4, we may therefore present the following new result.

Theorem 6.8. *Let X be a smooth hypersurface of degree d in $\mathbb{P}^5$ and let A be any ample line bundle over X. Then:*

$$\begin{aligned}
&h^0(X, E_{4,m}T^*_X \otimes A^{-1}) \\
&\qquad \geqslant \frac{m^{16}}{1313317832303894333210335641600000000000000} \cdot d \\
&\qquad\quad \cdot \big[-867659678949860838548185438614 \\
&\qquad\quad - 9348806936076078509405937921 6\, d \\
&\qquad\quad - 1369327265177339103292331439\, d^2 \\
&\qquad\quad - 617060662250595525598878 6\, d^3 \\
&\qquad\quad + 5004851113579703425623 5\, d^4 \big] \\
&\qquad\qquad + \mathrm{O}\big(m^{15}\big).
\end{aligned}$$

In particular, if $d \geqslant 259$ then $E_{4,m}T^*_X \otimes A^{-1}$ admits non trivial sections for m large, and every entire curve $f\colon \mathbb{C} \to X$ must satisfy the corresponding algebraic differential equations.

6.6. **Algebraic degeneracy.** Similarly as in Theorem 2.4 but say in dimension 4 to fix ideas (for the dimension 3, *see* [18], pp. 381–383), one tensors the invariant jet bundle $E_{k,m}T^*_X$ by $A^{-1} := K_X^{-\delta m}$, one uses the standard formula:

$$\Gamma^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}T^*_X \otimes K_X^{-\delta m} = \Gamma^{(\lambda_1-\delta m,\lambda_2-\delta m,\lambda_3-\delta m,\lambda_4-\delta m)}T^*_X$$

in order to reapply the Schur bundle decomposition of Theorem 6.2, one redoes all the computations of Theorem 6.6 and of Theorem 6.8, and one gets in this way a new minorant:

$$h^0\big(X,\, E_{4,m}T^*_X \otimes K_X^{-\delta m}\big) \geqslant \alpha(d,\delta) \cdot m^{16} + \mathrm{O}\big(m^{15}\big),$$

for a certain complicated polynomial $\alpha(d,\delta) \in \mathbb{Q}[d,\delta]$ which we find now superfluous to write down explicitly, and which of course regives for $\delta = 0$ the minorant of Theorem 6.8. Remind now that according to Theorem 2.5, in dimension 4, the maximal pole order of a meromorphic frame on the space of vertical 4-jets of the universal hypersurface parametrizing all degree d hypersurfaces of $\mathbb{P}^5$ is equal to $4^2 + 2 \cdot 4 = 24$. Then following line by line the arguments of the proof of Theorem 3.1, in order to be able to apply sufficiently many meromorphic derivations $L_{W_1} \cdots L_{W_p}$ with $p \leqslant m$ to a given nonzero jet differential so as to deduce — reasoning again by contradiction as in Section 3 — algebraic degeneracy of entire curves, one has to insure: that $d > \frac{24}{\delta} + 6$, as is required by (49) for the general dimension n; and

simultaneously also: that $\alpha(d,\delta) > 0$ for all $d \geqslant d_4$ larger than a certain effective $d_4 \in \mathbb{N}$. But quite similarly as in the dimension 3 case, these two constraints happen to be *compatible*, and thanks to effective computations executed independently on two digital computers by the second and by the third named author using different codes, one verifies in dimension 4 that $d_4 = 3203$ works (with $\delta = \frac{3197}{24}$), and this is how, after so many rational calculations, one gains the new effective lower bound $\deg X \geqslant 3203$ of Theorem 1.2. $\square$

7. Effective algebraic degeneracy in dimensions 5 and 6

Finally, for dimensions 5 and 6, we simply carry out the same strategy as in the general case, but with a choice of weight different from a^* introduced in Subsection 4.4. Our choice specific for these two dimensions are $\mathbf{a} = (54, 18, 6, 2, 1)$ and $\mathbf{a} = (162, 54, 18, 6, 2, 1)$, that is to say: the minimal choice in order to have relative nefness of the weighted (anti)tautological line bundle $\mathscr{O}_{X_n}(\mathbf{a})$, $n = 5, 6$ (*cf.* [4, 6]); also, we choose $\delta = \frac{5^2+2\cdot 5}{d-5-2}$ and $\delta = \frac{6^2+2\cdot 6}{d-6-2}$. The bound is then obtained thanks to computer calculations with GP/PARI, (*cf.* [6] for the code). The same method, in dimension 4 (resp. 3), would have produced $\deg X \geqslant 6527$ (resp. $\geqslant 1019$), less sharp than $\deg X \geqslant 3203$ (resp. $\geqslant 593$).

In dimension $n = 5$, here are the corresponding two polynomials $\mathsf{P}_{\mathbf{a}}(d)$ and $\mathsf{P}'_{\mathbf{a}}(d)$ the length of which confirms the incompressible complexity of the reduction process:

$$\begin{aligned}\mathsf{P}_{54,18,6,2,1}(d) = {} & 829705552526846689513237554474 24\, d^6 - \\ & - 69092357692382960198316008279615424\, d^5 - \\ & - 3759195731318462969721810883195592774 4\, d^4 - \\ & - 2161144497516080476955607837671278699584\, d^3 - \\ & - 2076793172317374111754855583724316380614 4\, d^2 - \\ & - 2373646177903816624611595830455187105638 4\, d.\end{aligned} \tag{59}$$

and:

$$\begin{aligned}\mathsf{P}'_{54,18,6,2,1}(d) = {} & -810649364923821805499061816503472 00\, d^6 - \\ & - 2561926552944387465736285101371322720 0\, d^5 - \\ & - 1138360224016877254137407566642735778400\, d^4 - \\ & - 2649407942988198539201176162753240634400\, d^3 + \\ & + 7039955826593328320294994211810158028080 0\, d^2 + \\ & + 9035595310649985453016931098557894500880 0\, d.\end{aligned} \tag{60}$$

We believe that the sequence of weights $\mathbf{a} = (2 \cdot 3^{n-2}, \ldots, 6, 2, 1)$ instead of a^* should work in any dimension, and that it should provide better effective estimates in all dimensions, though we suspect the bound should remain

exponential. To conclude, we collect our three effective estimates in a comparative table

dim X	Theorem 1.2	Theorem 1.1
3	593	2^{3^5}
4	3203	2^{4^5}
5	35355	2^{5^5}
6	172925	2^{6^5}

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Responses to referees:
Effective Algebraic Degeneracy

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Monsieur,

Nous souhaitons resoumettre aux *Inventiones Mathematicæ* le projet révisé de publication:

« Effective Algebraic Degeneracy »

que vous avez eu l'obligeance de bien vouloir faire relire et vérifier par deux rapporteurs indépendants. Grâce à la vigilance de l'un des deux rapporteurs, nous nous sommes rendu compte que notre manière de résumer les résultats effectifs pour la dégénéréscence algébrique des courbes entières en dimension 4, dans la Section 6, était beaucoup trop elliptique, et donc essentiellement peu lisible par des non spécialistes, le style choisi à cet endroit-là tranchant trop visiblement avec le reste du projet d'article. Ainsi avons-nous complété un grand nombre de détails intermédiaires, en ne nous autorisant que des références à des théorèmes ou propositions importantes qui sont déjà parus dans la littérature.

Ci-dessous, et aussi dans le fichier latex que nous joignons à cette lettre pdf pour plus de flexibilité, on trouvera des réponses brèves et précises — qui s'accompagneront parfois de remerciements — à chaque point soulevé par les deux rapporteurs, afin d'expliciter les modifications que nous apportons. C'est en anglais, plutôt qu'en français, que ces réponses seront rédigées.

En vous remerciant très sincèrement d'avoir su faire évaluer et relire le projet en neuf mois seulement,

Joël Merker,

Answers to Report A:

• Page 1: We replace the concerned phrase:

*In the survey article [21] (*cf. *also [20]), Siu established that there exists a high integer* d_n *such that generic hypersurfaces* $X \subset \mathbb{P}^{n+1}$ *of degree* $\geqslant d_n$ *are moreover Kobayashi-hyperbolic, namely all entire curves* $f\colon \mathbb{C} \to X$ *must be* constant*, not only algebraically degenerate*.

by the following phrase:

*In the survey article [21] (*cf. *also [20]), Siu provided a beautiful strategy to establish algebraic degeneracy of entire holomorphic curves in generic hypersurfaces* $X \subset \mathbb{P}^{n+1}$ *of high degree larger than a certain* $d_n \gg 1$*, and also* Kobayashi-hyperbolicity *of such* X*'s if* d_n *is even much higher.*

• Page 6: Misprint corrected.

• Page 7: Nice suggestion to improve readability, we add it, and we reorganize carefully all the neighboorhing phrases.

• Page 8: Formulation changed accordingly.

• Page 14: It is true that the estimate: $\max(|\mathrm{roots}|) \leqslant 2 \max\big(|a_j|/|a_0|\big)^{1/j}$ for the maximum of the modulus of roots of a polynomial $a_0X^d + a_1X^{d-1} + \cdots + a_d$ is better than the coarser estimate:

$$\max(|\mathrm{roots}|) \leqslant 1 + |a_1|/|a_0| + \cdots + |a_d|/|a_0|,$$

and we in fact hesitated to use it instead. In our case, both estimates would yield at the end exactly the same degree estimate $\deg X \geqslant 2^{n^5}$ in the Main Theorem 1.1, because all our $|a_j|/|a_0|$ will be equal to each other at the end of Section 4, because for $j = 1$, both the finer and the coarser estimate show $|a_1|/|a_0|$, and because the linear comparison factor "n" between the finer and the coarser estimates will be rather negligible in comparison to 2^{n^5}, or more precisely to the long and last inequality we use:

$$\begin{aligned}\widetilde{d}_n^2 &= 1 + n + 2 + 2\,(n^2 + 2n) \cdot n^{2n-1} \cdot (n+1)^{n^2+1} \cdot n^{3n^2}\, 3^{n^3} \cdot n^{4n^3} 2^{n^4} \\ &\leqslant 2^{n^5}.\end{aligned}$$

In addition, the finer estimate would be rather less convenient to write down throughout all Sections 4 and 5. To explain our choice to the reader [but

we could, if required, use the stronger estimate], we add a phrase before Lemma 4.1 (our polynomial writes $\mathsf{a}_{n+1}\, d^{n+1} + \mathsf{a}_n\, d^n + \cdots + \mathsf{a}_0$ there):

Instead of the finer bound $2\max_{0\leqslant j\leqslant n}\big(\frac{|\mathsf{a}_j|}{|\mathsf{a}_{n+1}|}\big)^{1/n+1-j}$*, we use this easier-to-write-down majoration because at the end of Section 4, this will make no difference in reaching the bound* $\deg X \geqslant 2^{n^5}$ *of Theorem 1.1.*

Answers to Report B:

There are eleven points, that we number.

$\bullet_1$: To clarify that genericity holds inside a family of hypersurfaces of fixed degree d, we reformulate slightly the statement.

Theorem. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree d and of arbitrary dimension $n \geqslant 2$. If X is generic and if its degree satisfies the* effective *lower bound:*

$$\deg(X) \geqslant 2^{n^5} \ldots\ldots$$

In this way, since the degree d of X is specified at first in the assumptions, we believe that the reader will understand correctly that genericity holds within the family of degree d hypersurfaces. Indeed, clarifying this point improves the formulation.

$\bullet_2$: The two misprints are corrected.

$\bullet_3$: We carefully revise the writing of the proof according to this suggestion, being sure that justifications of articles are numerous and complex.

$\bullet_4$: In the arxiv.org version, it was uncorrectly written $N_d^n = \frac{(n+1+d)!}{(n+1)!\,d!}$, while a projectivization forces to subtract "1", and the correct N_d^n equal to $\frac{(n+1+d)!}{(n+1)!\,d!} - 1$ appears in the version published by the second author at Ann. Inst. Fourier (Grenoble).

$\bullet_5$: We do not know. In the edition of Hartshorne's book we used, the theorem we apply appears on page 288. Apparently, there is no "second edition", only reprints of the original edition.

$\bullet_6$: Bars should, we believe, *not* be deleted: we just picked one fixed set of n multi-indices $\bar{i}_1^0, \ldots, \bar{i}_n^0 \in \mathbb{N}^n$ such that $\overline{q}_{\bar{i}_1^0, \ldots, \bar{i}_n^0}\big(s_0, f(t_0)\big) \neq 0$. Then with these multiindices, we want to argue seven lines below with all integers p with $p < |\bar{i}_1^0| + |\bar{i}_2^0| + \cdots + |\bar{i}_n^0|$ and we just mention that one then trivially has: $|\bar{i}_1^0| + |\bar{i}_1^0| + \cdots + |\bar{i}_n^0| \leqslant |\bar{i}_1^0| + 2\,|\bar{i}_1^0| + \cdots + n\,|\bar{i}_n^0| = m$, so that $p < m$: what we wanted to observe. True: this passage can be micro-improved, we cut the inequalities and we explain better what comes in within one supplementary line.

$\bullet_7$: Of course we are disposed to modify according to referee's suggestion. For us, the mentioned statement was a *Proposition*: semantically speaking, it expresses here the principal information. The proof of a major proposition often relies upon lemmas. Here, the Proposition is easier to read than the technical Lemma written just below upon which it relies.

$\bullet_8$: Thanks to the referee. We revise the writing of the proof following this renewed appropriate suggestion.

$\bullet_9$: Precious watchfulness of our proofreader! This passage was re-written (too) rapidly after we devised a strategy of how to improve the double exponential degree lower bound $\deg X \geqslant n^{n+1^{n+5}}$ to the better polynomial exponential bound $\deg X \geqslant 2^{n^5}$. So we correct the whole passage, and of course, for the dimension 2 that we essentially do not touch in our contribution, we just refer to the appropriate references.

Furthermore and most importantly, our (loose here) rhetoric is improved. As the referee points out, there was a risk of confusion about the true logic of our arguments.

Hence we rigorously write that the proof of Theorem 1.1 with its neat and uniform degree lower bound $\deg X \geqslant 2^{n^5}$ will be really complete only after: 1) the crucial estimates of the quantities $\mathsf{D}_k(n) = n^{4n^3}2^{n^4}$ and $\mathsf{D}'_k(n) = n^{4n^3}2^{n^4}$ to which Section 5 is entirely devoted are effectively got; and: 2) the proof of our second Theorem 1.2 in the computer-nowadays-accessible dimensions 3, 4, 5 and 6, to which Sections 6 and 7 are entirely devoted, are also got. Thanks!

$\bullet_{10}$: Page 35, we also modify accordingly.

$\bullet_{11}$: Section 6 was poorly and elliptically written, just because it is clear to insiders of the field that the complete treatment of the dimension $n = 4$ by fully following the strategy achieved before by the third-named author for $n = 3$ (in *three* articles!) and by providing a complete algebraic description of the algebra of 4-jet Demailly-Semple invariants would well have deserved a whole article in a high-level journal. We therefore add many more details, filling thoroughly all the transitions between the main cornerstones, and referring precisely to the works of the second-named author and of the third-named author whenever a technical statement already exists in the literature. In sum, we add a few pages to the project of publication, but the whole Section 6 becomes much more readable than before, and lastly, we cut the ultimate subsection of the previous submission devoted to the dimensions 5 and 6, putting it without any change in a new final Section 7, because the techniques used there: precisely those of the first sections of the article, are

totally different and much more economic than those we employ just before for the dimension 4 in Section 6.

Low pole order frames on vertical jets of the universal hypersurface

Joël Merker

Abstract. For low order jets, it is known how to construct meromorphic frames on the space of the so-called *vertical k-jets* $J^k_{\mathsf{vert}}(\mathscr{X})$ of the universal hypersurface $\mathscr{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1}$ parametrizing all projective hypersurfaces $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ of degree d. In 2004, for $k = n$, Siu announced that there exist two constants $c_n \geqslant 1$ and $c'_n \geqslant 1$ such that the twisted tangent bundle:

$$T_{J^n_{\mathrm{vert}}(\mathscr{X})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}\big(c_n\big) \otimes \mathscr{O}_{\mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1}}(c'_n)$$

is generated at every point by its global sections. In the present article, we establish this property outside a certain exceptional algebraic subset $\Sigma \subset J^n_{\mathsf{vert}}(\mathscr{X})$ defined by the vanishing of certain Wronskians, with the *effective* pole order $c_n = \frac{n^2+5n}{2}$, thus recovering $c_2 = 7$ (Paŭn), $c_3 = 12$ (Rousseau), and with $c'_n = 1$.

Moreover, at the cost of raising c_n up to $c_n = n^2 + 2n$, the same generation property holds outside the smaller set $\widetilde{\Sigma} \subset \Sigma \subset J^n_{\mathsf{vert}}(\mathscr{X})$ which is defined by the vanishing of all first order jets. Applications to *weak* (with Σ) and to *strong* (with $\widetilde{\Sigma}$) algebraic degeneracy of entire holomorphic curves $\mathbb{C} \to X$ are upcoming.

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Table of contents

§1. Introduction

The Kobayashi hyperbolicity conjecture (1970), in optimal degree and taking account of Brody's theorem (1978), expects that all entire holomorphic curves $f : \mathbb{C} \to X$ into a complex projective (algebraic, smooth) hypersurface $X \subset \mathbb{P}^{n+1}$ must be constant if $\deg X \geqslant 2n + 1$, provided X is generic.

In 2004, Siu [35] announced a strategy of proof, valid in arbitrary dimensions for (extremely) high (noneffective) degrees $d \gg n$. Two major techniques are used.

Inspired by Bloch's ideas, one looks firstly for global sections of the Green-Griffiths bundle $E^{GG}_{k,m}T^*_X$ of jet differentials of order k and weighted degree m (*cf.* [17]), which vanish on some ample divisor; an Ahlfors-Schwarz-type theorem then forces every entire curve $f : \mathbb{C} \to X$ to satisfy

the corresponding differential equation ([4]), a first step toward algebraic degeneracy. In 1997, Demailly introduced a refined subbundle $E_{k,m}T_X^*$ having better positivity properties which consists of jet differentials that are invariant under (local) reparametrizations of the source $\mathbb{C}$. In dimension $n = 3$ for jets of order $k = 3$, Rousseau ([29]) completely described the algebraic structure of $E_{k,m}T_X^*$ in its fibers, decomposed it in direct sums of Schur bundles $\Gamma^{(\lambda_1,\lambda_2,\lambda_3)}T_X^*$, computed its Euler characteristic $\chi\big(X,\, E_{k,m}T_X^*\big)$, majorated from above $h^2\big(X,\, E_{k,m}T_X^*\big)$ (*see* [30]), and established existence of global algebraic differential equations in degree $d \geqslant 97$.

In [21, 22], one finds a *complete algorithm* to generate all Demailly-Semple invariants in arbitrary dimension $n \geqslant 1$ and for jets of any order $k \geqslant 1$. In particular, for $n = k = 4$, there are 16 fundamental, mutually independent bi-invariant polynomials generating the Demailly-Semple (unipotent-invariant) algebra sharing 41 (gröbnerized) syzygies, and one deduces by polarization that the algebra of all invariants for $n = k = 4$ is generated by 2835 polynomials. Nonconstant entire holomorphic curves valued in an algebraic 3-fold (resp. 4-fold) $X^3 \subset \mathbb{P}^4(\mathbb{C})$ (resp. $X^4 \subset \mathbb{P}^5(\mathbb{C})$) of degree d satisfy ([22]) global differential equations as soon as $d \geqslant 72$ (resp. $d \geqslant 259$).

In [9], for dimensions $n = 2, 3, 4, 5$ and for jet orders $k = 3, 4, 5, \underline{5}$, resp., it is shown that asymptotically as $m \to \infty$:

$$H^0\big(X,\, E_{k,m}T_X^* \otimes A^{-1}\big) \neq 0,$$

in degrees $d \geqslant 16, 74, 298, 1222$ resp., where $A \to X$ is any auxiliary ample line bundle. But it is also shown that $H^0\big(X,\, E_{k,m}T_X^*\big) = 0$, for all jet orders $k \leqslant \dim X - 1$, generalizing a theorem of Rousseau ([30]) in dimension 3. Furthermore, for jet order k equal to the dimension n, with n arbitrary, Diverio shows in [10] that there exists an integer $\delta_n \gg n$ (up to now not effective) insuring existence of global sections of $E_{n,m}T_X^* \otimes A^{-1}$ in degree $d \geqslant \delta_n$.

The second technique, initiated by Clemens [1], Ein [8], Voisin [20] and pushed further by Siu [35], Paŭn [26], Rousseau [31], consists in constructing meromorphic frames on the space of the so-called *vertical k-jets* $J^k_{\mathsf{vert}}(\mathscr{X})$ in the universal hypersurface $\mathscr{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1}$ parametrizing all $X \subset \mathbb{P}^{n+1}$ of degree d, so as to produce, by frame differentiations, enough *independent* algebraic differential equations from just one global section of $E_{k,m}T_X^* \otimes A^{-1}$.

In [35], p. 557, Siu announced that, for $k = n$, there exist two constants $c_n \geqslant 1$ and $c'_n \geqslant 1$ such that the twisted tangent bundle:

$$T_{J^n_{\mathrm{vert}}(\mathscr{X})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}\big(c_n\big) \otimes \mathscr{O}_{\mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1}}\big(c'_n\big)$$

is generated at every point by its global sections (frame property). In the present article, we establish this property outside a certain exceptional algebraic subset $\Sigma \subset J^n_{\mathsf{vert}}(\mathscr{X})$ defined by the vanishing of certain Wronskians, with the effecive pole order $c_n = \frac{n^2+5n}{2}$, recovering $c_2 = 7$ (Paŭn [26]), $c_3 = 12$ (Rousseau [31]), and with $c'_n = 1$.

Moreover, at the cost of raising c_n up to $c_n = n^2+2n$, the same generation property holds outside the smaller set $\widetilde{\Sigma} \subset \Sigma$ defined by the vanishing of all first order jets. Applications to *weak* (with Σ) and to *strong* (with $\widetilde{\Sigma}$) algebraic degeneracy of entire holomorphic curves are given in [11], following Rousseau's Schur bundle decomposition strategy in dimension $n = 4$, and also in higher dimensions, thanks to Diverio's use ([9, 10]) of the algebraic version of Demailly's Morse inequalities due to Trapani ([19]).

Acknowledgments. During the author's stay at the Mittag-Leffler Institute (1–21 April 2008), Yum-Tong Siu and Mihai Paŭn have provided helpful oral explanations of [35, 26]. The stronger property of generation at every point of $J^n_{\mathsf{vert}}(\mathscr{X})\backslash\widetilde{\Sigma}$ was obtained thanks to fruitful exchanges joint with Simone Diverio and Erwan Rousseau during the *Workshop Complex Hyperbolic Geometry and Related Topics* at the Fields Institute, Toronto, Canada, 17-21 November 2008.

§2. Universal hypersurface and vertical jets

Representation in coordinates. Consider the *universal hypersurface* $\mathscr{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1}$ parametrizing all complex n-dimensional algebraic hypersurfaces of fixed degree $d \geqslant 1$ in $\mathbb{P}^{n+1}$ which is defined, in two collections of homogeneous coordinates:

$$[Z] = [Z_0 : Z_1 : \cdots : Z_n : Z_{n+1}] \in \mathbb{P}^{n+1}$$
$$[A] = \big[(A_\alpha)_{\alpha\in\mathbb{N}^{n+2},\,|\alpha|=d}\big] \in \mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1},$$

as the zero-set locus:

$$\mathscr{X} \ : \qquad 0 = \sum_{\substack{\alpha\in\mathbb{N}^{n+2}\\ |\alpha|=d}} A_\alpha\, Z^\alpha$$

of the general homogeneous degree d polynomial. Here of course, a multiindex $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n+1}) \in \mathbb{N}^{n+2}$ has *length* defined by $|\alpha| := \alpha_0 + \alpha_1 + \cdots + \alpha_{n+1}$ and we abbreviate $Z^\alpha = Z_0^{\alpha_0} Z_1^{\alpha_1} \cdots Z_{n+1}^{\alpha_{n+1}}$.

Our goal is to perform, for jets of order κ equal to the dimension n of hypersurfaces $\mathscr{X}(A) \subset \mathbb{P}^{n+1}$, a construction of meromorphic vector fields on the space of jets of holomorphic discs (or entire maps) valued in $\mathscr{X}$ which was initiated by Clemens [1], Ein [8], Voisin [20] for $\kappa = 1$, $n \geqslant 1$, then announced for higher κ's by Siu [35] and recently detailed by Paŭn [26]

for $n = \kappa = 2$ and by Rousseau [31] for $n = \kappa = 3$. For general $\kappa = n$, a concise book-keeping of indices appears to be available here.

As in [26, 31], we shall mainly work in *in*homogeneous coordinates on the chart $\{Z_0 \neq 0\} \times \{A_{0d0\cdots 0} \neq 0\}$, a copy of $\mathbb{C}^{n+1} \times \mathbb{C}^{\frac{(n+1+d)!}{(n+1)!\,d!}}$. Dividing by $(Z_0)^d$ and by $A_{0d0\cdots 0}$, and setting $z_i := Z_i/Z_0$, the equation of $\mathscr{X}$ then transfers to:

$$\mathscr{X}_0 \ : \qquad 0 = z_1^d + \sum_{\substack{\alpha \in \mathbb{N}^{n+1} \\ |\alpha| \leqslant d,\, \alpha_1 < d}} a_\alpha \, z^\alpha,$$

with new coefficients $a_{\alpha_1 \cdots \alpha_{n+1}} := \frac{A_{\alpha_0 \alpha_1 \cdots \alpha_{n+1}}}{A_{0d0\cdots 0}}$ in which $\alpha_0 := d - \alpha_1 - \cdots - \alpha_{n+1}$. By convention, we shall set $a_{d0\cdots 0} = 1$.

In view of applications to the Green-Griffiths algebraic degeneracy conjecture ($d \geqslant n+3$) or to the Kobayashi hyperbolicity conjecture ($d \geqslant 2n+1$), it will, without loss of generality, be assumed that $d > n$ throughout.

Defining equations for the space of vertical jets. To settle Kobayashi hyperbolicity or Green-Griffiths algebraic degeneracy, the strategy initiated by Bloch and pursued by Green-Griffiths [17], Siu [35], Demailly [4] consists in producing enough (global, algebraic) differential equations that every entire map $\mathbb{C} \ni \zeta \longmapsto \big(z_1(\zeta), \ldots, z_{n+1}(\zeta)\big)$ valued in an algebraic variety $\mathscr{X}(A)$ for (very) generic fixed coefficients A_α should satisfy. Accordingly, if one introduces independent coordinates corresponding to derivatives with respect to ζ:

$$\Big(z_i,\, a_\alpha,\, z'_{j_1},\, z''_{j_2}, \ldots, z^{(\kappa)}_{j_\kappa}\Big) \in \mathbb{C}^{n+1} \times \mathbb{C}^{\frac{(n+1+d)!}{(n+1)!\,d!}} \times \underbrace{\mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \cdots \times \mathbb{C}^{n+1}}_{\kappa \text{ times}},$$

the manifold of κ-jets of such entire maps has equations obtained by just formally differentiating the monomials z^α with respect to the variable $\zeta \in \mathbb{C}$, the a_α being constant. The basic chain rule yields the first five equations, up to $\kappa = 4$:

$$\begin{aligned}
0 &= \sum_{\substack{\alpha \in \mathbb{N}^{n+1} \\ |\alpha| \leqslant d,\, a_{d0\cdots 0} = 1}} a_\alpha \, z^\alpha \\
0 &= \sum_\alpha a_\alpha \Big(\sum_{j_1} \frac{\partial (z^\alpha)}{\partial z_{j_1}} \, z'_{j_1} \Big) \\
0 &= \sum_\alpha a_\alpha \Big(\sum_{j_1} \frac{\partial (z^\alpha)}{\partial z_{j_1}} \, z''_{j_1} + \sum_{j_1, j_2} \frac{\partial^2 (z^\alpha)}{\partial z_{j_1} \partial z_{j_2}} \, z'_{j_1} z'_{j_2} \Big) \\
0 &= \sum_\alpha a_\alpha \Big(\sum_{j_1} \frac{\partial (z^\alpha)}{\partial z_{j_1}} \, z'''_{j_1} + \sum_{j_1, j_2} \frac{\partial^2 (z^\alpha)}{\partial z_{j_1} \partial z_{j_2}} \, 3\, z'_{j_1} z''_{j_2} + \sum_{j_1, j_2, j_3} \frac{\partial^3 (z^\alpha)}{\partial z_{j_1} \partial z_{j_2} \partial z_{j_3}} \, z'_{j_1} z'_{j_2} z'_{j_3} \Big) \\
0 &= \sum_\alpha a_\alpha \Big(\sum_{j_1} \frac{\partial (z^\alpha)}{\partial z_{j_1}} \, z''''_{j_1} + \sum_{j_1, j_2} \frac{\partial^2 (z^\alpha)}{\partial z_{j_1} \partial z_{j_2}} \big(4\, z'_{j_1} z'''_{j_2} + 3\, z''_{j_1} z''_{j_2}\big) + \\
&\quad + \sum_{j_1, j_2, j_3} \frac{\partial^3 (z^\alpha)}{\partial z_{j_1} \partial z_{j_2} \partial z_{j_3}} \, 6\, z'_{j_1} z'_{j_2} z''_{j_3} + \sum_{j_1, j_2, j_3, j_4} \frac{\partial^4 (z^\alpha)}{\partial z_{j_1} \partial z_{j_2} \partial z_{j_3} \partial z_{j_4}} \, z'_{j_1} z'_{j_2} z'_{j_3} z'_{j_4} \Big),
\end{aligned}$$

on understanding that $a_{d0\cdots 0} = 1$ and that all summations $\sum_{j_1}$, $\sum_{j_1,\,j_2}$ *etc.* are performed for the indices j_i running from 1 to $n+1$. Equivalently, this submanifold of $\mathbb{C}^{n+1} \times \mathbb{C}^{\frac{(n+1+d)!}{(n+1)!\;d!}} \times \mathbb{C}^{\kappa(n+1)}$ may be be defined as the submanifold of the full κ-jet manifold $J^\kappa(\mathbb{C},\ \mathscr{X}_0)$ consisting of only the jets tangent to the fibers of the projection $\mathscr{X}_0 \to \mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\;d!}-1}$ onto the second factor. They are called *vertical jets* in [35, 26, 31] and will be denoted by $J^\kappa_{\mathsf{vert}}(\mathscr{X}_0)$.

Formally differentiating any polynomial in the jet variables amounts to applying the *total differentiation operator*:

$$\mathsf{D}(\bullet) := \sum_{\lambda\in\mathbb{N}} \sum_{k=1}^{n+1} \frac{\partial(\bullet)}{\partial z_k^{(\lambda)}} \cdot z_k^{(\lambda+1)},$$

and above, it is clear that each next equation is obtained from the previous one by applying D to it so that, for jets of arbitrary order κ up to κ equal to the dimension n, the $(n+1)$ defining equations of $J^n_{\mathsf{vert}}(\mathscr{X}_0)$ happen to be:

$$0 = \sum_\alpha a_\alpha z^\alpha = \mathsf{D}\Big(\sum_\alpha a_\alpha z^\alpha\Big) = \cdots = \mathsf{D}^n\Big(\sum_\alpha a_\alpha z^\alpha\Big).$$

Then a suitable multivariate version of the classical Faà di Bruno formula provides a *closed, explicit formula* for all such equations.

Lemma 1. ([2, 20]) *The* $(n+1)$ *defining equations of* $J^n_{\mathsf{vert}}(\mathscr{X}_0)$ *write as follows, where* $\kappa = 0, 1, 2, \ldots, n$:

$$0 = \sum_{\substack{\alpha\in\mathbb{N}^{n+1}\\ |\alpha|\leqslant d,\ a_{d0\cdots 0}=1}} a_\alpha \sum_{e=1}^{\kappa} \sum_{1\leqslant\lambda_1<\cdots<\lambda_e\leqslant\kappa} \sum_{\mu_1\geqslant 1,\ldots,\mu_e\geqslant 1} \sum_{\mu_1\lambda_1+\cdots+\mu_e\lambda_e=\kappa} \frac{\kappa!}{(\lambda_1!)^{\mu_1}\mu_1!\cdots(\lambda_e!)^{\mu_e}\mu_e!}$$
$$\sum_{j_1^1,\ldots,j_{\mu_1}^1=1}^{n+1} \cdots \sum_{j_1^e,\ldots,j_{\mu_e}^e=1}^{n+1} \frac{\partial^{\mu_1+\cdots+\mu_e}\big(z^\alpha\big)}{\partial z_{j_1^1}\cdots\partial z_{j_{\mu_1}^1}\cdots\partial z_{j_1^e}\cdots\partial z_{j_{\mu_e}^e}}\, z_{j_1^1}^{(\lambda_1)}\cdots z_{j_{\mu_1}^1}^{(\lambda_1)}\cdots z_{j_1^e}^{(\lambda_e)}\cdots z_{j_{\mu_e}^e}^{(\lambda_e)}.$$

To read this general formula with the help of the formulas specialized above, we comment it backwards from its end.

The general monomial $\prod z_\bullet^{(\lambda_1)} \prod z_\bullet^{(\lambda_2)} \cdots \prod z_\bullet^{(\lambda_e)}$ in the jet variables gathers derivatives of increasing orders $\lambda_1 < \lambda_2 < \cdots < \lambda_e$, with $\mu_1, \mu_2, \ldots, \mu_e$ counting their respective numbers. Then each monomial z^α is subjected to a partial derivative of order $\mu_1 + \mu_2 + \cdots + \mu_e$, the total number of $z_j^{(\lambda_i)}$ in the monomial in question. Since there are $n+1$ variables z_i, the dots in the $z_\bullet^{(\lambda_i)}$ should receive indices, and in fact, there appear general sums $\sum_{j_1^i,\ldots,j_{\mu_i}^i=1}^{n+1}$ over *all possible* such indices. Notice that these observations are confirmed by the formulas developed above up to $\kappa = 4$.

In the sequel, we will in fact not need all the information of such a precise, explicit formula, but it will suffice to know that, among the $(n+1)$ defining equations, the equation numbered κ is a certain finite sum with certain

integer coefficients of terms of the form:

$$\boxed{\sum_{\substack{\beta\in\mathbb{N}^{n+1}\\ |\beta|\leqslant d,\, a_{d0\cdots 0}=1}} a_\beta \bigg(\sum_{j_1,\dots,j_e=1}^{n+1} \frac{\partial^e(z^\beta)}{\partial z_{j_1}\cdots\partial z_{j_e}} \cdot z_{j_1}^{(\nu_1)}\cdots z_{j_e}^{(\nu_e)} \bigg)},$$

where the derivative orders $\nu_i \geqslant 1$ of the jet monomial $z_{j_1}^{(\nu_1)}\cdots z_{j_e}^{(\nu_e)}$ are nondecreasing and where $\nu_1+\cdots+\nu_e=\kappa$. The reader unacquainted with the Faà di Bruno combinatorics could readily prove this less informative representation by reasoning inductively on κ.

Frames and generation by global sections. Now, a globally defined vector field on the ambient space $\mathbb{C}^{n+1}\times\mathbb{C}^{\frac{(n+1+d)!}{(n+1)!\,d!}}\times\mathbb{C}^{n(n+1)}$ writes under the general form:

$$\mathsf{T}=\sum_{i=1}^{n+1}\mathsf{Z}_i\,\frac{\partial}{\partial z_i}+\sum_{\substack{\alpha\in\mathbb{N}^{n+1}\\ |\alpha|\leqslant d,\,\alpha_1<d}}\mathsf{A}_\alpha\,\frac{\partial}{\partial a_\alpha}+\sum_{k=1}^{n+1}\mathsf{Z}'_k\,\frac{\partial}{\partial z'_k}+\sum_{k=1}^{n+1}\mathsf{Z}''_k\,\frac{\partial}{\partial z''_k}+\cdots+\sum_{k=1}^{n+1}\mathsf{Z}_k^{(n)}\,\frac{\partial}{\partial z_k^{(n)}}.$$

We shall seek vector fields of this form which should extend meromorphically to the full space of vertical jets and which should make a spanning frame of vectors tangent to $J^n_{\mathsf{vert}}(\mathscr{X})$ at almost every point, say outside a certain exceptional set. After twisting by $(\bullet)\otimes\mathscr{O}_{\mathbb{P}^{n+1}}(c_n)\otimes\mathscr{O}_{\mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1}}(c'_n)$ for some two suitable constants $c_n\geqslant 1$ and $c'_n\geqslant 1$, one may in fact erase the appearing poles of the meromorphic coefficients, so that one may speak of global *holomorphic* sections instead of meromorphic sections.

Theorem. *Let $\widetilde{\Sigma}$ be the closure, in $J^n_{\mathsf{vert}}(\mathscr{X})$, of the Zariski closed subset of the space $J^n_{\mathsf{vert}}(\mathscr{X}_0)$ of vertical* affine *jets defined by requiring that all* first order *jet vanish:*

$$\widetilde{\Sigma}_0:=\Big\{\big(z_i,a_\alpha,z'_{j_1},\dots,z_{j_n}^{(n)}\big):\ z'_1=z'_2=\cdots=z'_{n+1}=0\Big\},$$

so that in any other standard affine chart $(t_0,\dots,t_{v-1},t_{v+1},\dots,t_{n+1})\in\mathbb{C}^{n+1}$ on $\mathbb{P}^{n+1}(\mathbb{C})$, the representation of $\widetilde{\Sigma}$ is yielded by exactly the same equations $0=t'_0=\cdots=t'_{v-1}=t'_{v+1}=\cdots=t'_{n+1}$. Then the following two properties hold true.

- *$J^n_{\mathsf{vert}}(\mathscr{X})\backslash\Sigma$ is smooth of pure codimension equal to $n+1$ at every point, namely, it is of pure dimension equal to:*

$$\begin{aligned} j_n^d &:= n+1+\tfrac{(n+1+d)!}{(n+1)!\,d!}+n(n+1)-(n+1)\\ &=\tfrac{(n+1+d)!}{(n+1)!\,d!}+n(n+1).\end{aligned}$$

- *The twisted tangent bundle:*

$$T_{J^n_{\mathrm{vert}}(\mathscr{X})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}\big(n^2+2n\big) \otimes \mathscr{O}_{\mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1}}(1)$$

is generated by its global sections on $J^n_{\mathsf{vert}}(\mathscr{X})\big\backslash\widetilde{\Sigma}$, that is to say: at every point $p^{[n]} \in J^n_{\mathsf{vert}}(\mathscr{X})\big\backslash\widetilde{\Sigma}$ not lying in $\widetilde{\Sigma}$, one may find j^d_n global *sections $\mathsf{T}_1, \dots, \mathsf{T}_{j^d_n}$ over X of this twisted tangent bundle such that:*

$$\mathbb{C}\mathsf{T}_1(p^{[n]}) \oplus \cdots \oplus \mathbb{C}\mathsf{T}_{j^d_n}(p^{[n]}) = T_{J^n_{\mathsf{vert}}(\mathscr{X}),\,p^{[n]}}.$$

Comments about applications. The simplicity of the defining equations of the avoided exceptional set $\widetilde{\Sigma} = \{z'_i = 0\}$ has considerable advantages in the study of Green-Griffiths algebraic degeneracy of entire holomorphic curves $f : \mathbb{C} \to X$.

Indeed, by employing jet differentials, one shows in a first moment that the n-jet $j^n f$ of any such an f must satisfy[29] at least one nontrivial global algebraic differential equation $P(j^n f) = 0$. Then in a second moment, following Siu's strategy (*see* [35, 26, 31, 11]) which consists in applying some well chosen multi-derivations $(\mathsf{T}_1)^{\nu_1} \cdots (\mathsf{T}_{j^d_n})^{\nu_{j^d_n}}$ to $P(j^n f) = 0$ so as to get sufficiently many *supplementary* differential equations, one comes down to distinguishing two cases:

- ☐ either $j^n f(\mathbb{C}) \not\subset \widetilde{\Sigma}$; in this first case, one is then able to show ([31, 11]) that $f(\mathbb{C})$ is contained in a certain proper algebraic subvariety $Y \subsetneqq X$ which is independent of f, and this yields *strong* algebraic degeneracy;
- ☐ or else $j^n f(\mathbb{C}) \subset \widetilde{\Sigma}$ fully; in this second case, one cannot apply any derivation $\mathsf{T}_1, \dots, \mathsf{T}_{j^d_n}$, but then the condition $j^n f(\mathbb{C}) \subset \Sigma$ simply reads $0 \equiv f'_1(\zeta) \equiv f'_2(\zeta) \equiv \cdots \equiv f'_n(\zeta)$, hence f is *constant* and strong degeneracy again holds *gratuitously*.

Quite differently, in [26, 31] and in a preliminary version of the present article as well, the exceptional set Σ that one had to avoid was substantially larger than $\widetilde{\Sigma}$. Then as a consequence in these references, the condition $j^n f \subset \Sigma$ in the second case above only meant that $j^n f$ was contained in the intersection of X with some one-codimensional linear subspace H of $\mathbb{P}^{n+1}(\mathbb{C})$ which in general depended upon f, so that only *weak* algebraic degeneracy of $f(\mathbb{C})$ could be deduced[30]. Here is the weaker statement which we generalize in arbitrary dimension $n \geqslant 2$.

Theorem'. *Let Σ be the closure, in $J^n_{\mathsf{vert}}(\mathscr{X})$, of the Zariski closed subset of the space $J^n_{\mathsf{vert}}(\mathscr{X}_0)$ of vertical* affine *jets defined by requiring that all $n \times n$*

[29] *see* [17, 4, 35, 29, 30, 9]; we only summarize very briefly the ideas here.

[30] A more careful inspection shows that in fact, H is two-codimensional (Simone Diverio).

Wronskians vanish:

$$\Sigma_0 := \Big\{ \big(z_i, a_\alpha, z'_{j_1}, \dots, z^{(n)}_{j_n}\big) : \ 0 = \det \ \big(z_i^{(\lambda_j)}\big)_{1\leqslant i\leqslant n+1}^{1\leqslant j\leqslant n} \\ \textit{for all } \lambda_1, \dots, \lambda_n \textit{ with } 1 \leqslant \lambda_j \leqslant n \Big\}.$$

Then the twisted tangent bundle:

$$T_{J^n_{\mathrm{vert}}(\mathscr{X})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}\big(\tfrac{n^2+5n}{2}\big) \otimes \mathscr{O}_{\mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\ d!}-1}}(1)$$

is generated by its global sections at every point of $J^n_{\mathsf{vert}}(\mathscr{X})\big\backslash\Sigma$.

Notice that the twisting order $\frac{n^2+5n}{2}$ along the z-direction is smaller than the one n^2+2n of the preceding theorem: a certain price has to be "paid" in order to shrink the exceptional set, and to thereby gain strong degeneracy.

As said, one may verify that the vanishing of all $n \times n$ minors of the $n \times (n+1)$ Wronskian-like matrix $\big(f_j^{(\lambda)}(\zeta)\big)_{1\leqslant j\leqslant n+1}^{1\leqslant \lambda\leqslant n}$ implies that the components $f_1(\zeta), \dots, f_{n+1}(\zeta)$ satisfy at least two linearly independent linear relations:

$$0 \equiv \textstyle\sum_{i=1}^{n+1} a_i\, f_i(\zeta) \equiv \sum_{i=1}^{n+1} b_i\, f_i(\zeta),$$

for $\zeta \in \mathbb{C}$, with no universal control on the coefficients a_i, b_i.

Before proceeding to establishing the two theorems, let us check that the set Σ is represented by the same kind of equations $0 = t'_1 = \cdots = t'_{\upsilon-1} = t'_{\upsilon+1} = \cdots = t'_{n+1}$ in any other standard chart $\{Z_\upsilon \neq 0\}$ on $\mathbb{P}^{n+1}(\mathbb{C})$ in which the affine coordinates are defined just by:

$$t_0 = \tfrac{Z_0}{Z_\upsilon}, \dots, t_{\upsilon-1} = \tfrac{Z_{\upsilon-1}}{Z_\upsilon},\ t_{\upsilon+1} = \tfrac{Z_{\upsilon+1}}{Z_\upsilon}, \dots, t_{n+1} = \tfrac{Z_{n+1}}{Z_\upsilon}.$$

Indeed, coming back to the definition $z_i = \frac{Z_i}{Z_0}$ of the z_i, $i = 1, \dots, n+1$, the change of chart $\{Z_0 \neq 0\} \to \{Z_\upsilon \neq 0\}$ is given by the well known basic formulas:

$$t_0 = \tfrac{1}{z_\upsilon}, \dots, t_{\upsilon-1} = \tfrac{z_{\upsilon-1}}{z_\upsilon},\ t_{\upsilon+1} = \tfrac{z_{\upsilon+1}}{z_\upsilon}, \dots, t_{n+1} = \tfrac{z_{n+1}}{z_\upsilon},$$

whence by differentiating the right-hand sides as if they virtually depended upon a variable $\zeta \in \mathbb{C}$, we get the transformation rules for the first order jets:

$$t'_0 = -\tfrac{z'_\upsilon}{z_\upsilon^2}, \dots, t'_{\upsilon-1} = \tfrac{z'_{\upsilon-1}}{z_\upsilon} - \tfrac{z_{\upsilon-1}z'_\upsilon}{z_\upsilon^2},\ t'_{\upsilon+1} = \tfrac{z'_{\upsilon+1}}{z_\upsilon} - \tfrac{z_{\upsilon+1}z'_\upsilon}{z_\upsilon^2}, \dots, t'_{n+1} = \tfrac{z'_{n+1}}{z_\upsilon} - \tfrac{z_{n+1}z'_\upsilon}{z_\upsilon^2}.$$

Then visibly, the two representations $\{0 = z'_1 = \cdots = z'_{n+1}\}$ and $\{0 = t'_0 = \cdots = t'_{\upsilon-1} = t'_{\upsilon+1} = \cdots = t'_{n+1}\}$ of the set Σ coincide coherently on the intersection $\{Z_0 \neq 0\} \cap \{Z_\upsilon \neq 0\}$ of the two affine charts. One may verify that $\widetilde{\Sigma}$ also enjoys a similar invariance property.

Organization. The remainder of the paper is entirely devoted to the proof of the first theorem. When necessary, we shall briefly indicate which mild modifications suffice to gain the second theorem at the same time.

§3. First package of coefficient vector fields

First family of global sections. We begin by seeking tangent vector fields globally defined over $\mathbb{C}^{n+1} \times \mathbb{C}^{\frac{(n+1+d)!}{(n+1)!\,d!}} \times \mathbb{C}^{n(n+1)}$ of the specific, short form:

$$\mathsf{T} = \sum_{|\alpha| \leqslant n} \mathsf{A}_\alpha \, \frac{\partial}{\partial a_\alpha}$$

in the space of only the coefficient variables a_α, up to length n. Afterwards, we shall deal with $\sum_{\substack{n \leqslant |\alpha| \leqslant d \\ \alpha_1 < d}} \mathsf{A}_\alpha \, \frac{\partial}{\partial a_\alpha}$, and in Section 4, the remaining directions $\partial / \partial z_i$ and $\partial / \partial z_j^{(\lambda)}$ will complete the sought generating tangent vector fields.

Any arbitrary point $p^{[n]} \in J^n_{\mathsf{vert}}(\mathscr{X}_0)$ not in $\widetilde{\Sigma}$ lies in at least one of the open sets $\{z_i' \neq 0\}$. Fixing such an index i with $1 \leqslant i \leqslant n+1$, we shall construct a collection of vector fields of the above form that are defined in $\{z_i' \neq 0\}$ and that extend meromorphically to $J^n_{\mathsf{vert}}(\mathscr{X})$. To this aim, let us rewrite the defining equations of $J^n_{\mathsf{vert}}(\mathscr{X}_0)$ under the following convenient form, in which we denote by $\epsilon_i = (0, \dots, 1, \dots, 0)$ the i-th basic multiindex having 1 at the i-th place and 0 elsewhere, whence $n\epsilon_i = (0, \dots, n, \dots, 0)$:

$$\begin{cases} 0 = a_0 + a_{\epsilon_i} z_i + \cdots\cdots\cdots\cdots + a_{n\epsilon_i} z_i^n + \displaystyle\sum_{\substack{\beta \neq \epsilon_i, \dots, n\epsilon_i \\ 1 \leqslant |\beta| \leqslant d}} a_\beta \, z^\beta \\ 0 = \qquad a_{\epsilon_i} \mathsf{D}(z_i) + \cdots\cdots + a_{n\epsilon_i} \mathsf{D}(z_i^n) + \displaystyle\sum_{\substack{\beta \neq \epsilon_i, \dots, n\epsilon_i \\ 1 \leqslant |\beta| \leqslant d}} a_\beta \, \mathsf{D}(z^\beta) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = \qquad a_{\epsilon_i} \mathsf{D}^n(z_i) + \cdots + a_{n\epsilon_i} \mathsf{D}^n(z_i^n) + \displaystyle\sum_{\substack{\beta \neq \epsilon_i, \dots, n\epsilon_i \\ 1 \leqslant |\beta| \leqslant d}} a_\beta \, \mathsf{D}^n(z^\beta). \end{cases}$$

Here in the last n lines, we emphasize a generalized $n \times n$ Wronskian-like matrix, about which the next lemma states that its determinant is nonzero *if and only if* $z_i' \neq 0$, an assumption we made. For short in the sequel, we shall write $(z_i^k)^{(\kappa)}$ instead of $D^\kappa(z_i^k)$, since it is now clear and unambiguous that primes denote abstract jet variables.

Lemma. *For every $i = 1, 2, \dots, n+1$, one has:*

$$\begin{vmatrix} z_i' & (z_i^2)' & \cdots & (z_i^n)' \\ z_i'' & (z_i^2)'' & \cdots & (z_i^n)'' \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ z_i^{(n)} & (z_i^2)^{(n)} & \cdots & (z_i^n)^{(n)} \end{vmatrix} = 1!2!\cdots n! \cdot z_i' \, (z_i')^2 \cdots (z_i')^n$$

$$= 1!2!\cdots n! \cdot (z_i')^{\frac{n(n+1)}{2}}.$$

Our appendix is devoted to the proof of this elementary, but not straightforward, determinantal identity. As a result, we immediately deduce that $J^n_{\mathsf{vert}}(\mathscr{X}_0)$ is smooth of pure codimension $(n+1)$ at each one of its points which lies in $\{z_i' \neq 0\}$, for the last n defining equations written above can at first be solved with respect to $a_{\epsilon_i}, \dots, a_{n\epsilon_i}$ thanks to Cramer's rule, while the first defining equation is trivially solvable with respect to a_0. In other words, in our open set $\{z_i' \neq 0\}$, the vertical affine jet manifold may be represented as a plain semi-global *graph*:

$$a_0, a_{\epsilon_i}, \dots, a_{n\epsilon_i} = \text{certain functions of } \big(z, z', \dots, z^{(n)}, \widetilde{a}_i\big),$$

where $\widetilde{a}_i := \big(a_\beta\big)^{1 \leqslant |\beta| \leqslant d}_{\beta \neq 0, \epsilon_i, \dots, n\epsilon_i}$ gathers all the other coefficients of the universal hypersurface.

Now, we seek vector fields of the form $\mathsf{T} = \sum_{|\alpha| \leqslant n} \mathsf{A}_\alpha \frac{\partial}{\partial a_\alpha}$ which would be tangent to $J^n_{\mathsf{vert}}(\mathscr{X}_0)$ with the length of the appearing multiindices being bounded by n. For this reason, and because the equations of $J^n_{\mathsf{vert}}(\mathscr{X}_0)$ are *linear* with respect to the coefficients a_β, when one applies such a derivation T to the equations in question, every monomial z^β with $n+1 \leqslant |\beta| \leqslant d$ disappears automatically, hence we come down to solving the following linear system:

$$\begin{cases} 0 = \mathsf{A}_0 + \mathsf{A}_{\epsilon_i} z_i + \cdots\cdots + \mathsf{A}_{n\epsilon_i} z_i^n + \displaystyle\sum_{\substack{\alpha \neq \epsilon_i, \dots, n\epsilon_i \\ 1 \leqslant |\alpha| \leqslant n}} \mathsf{A}_\alpha \, z^\alpha \\ 0 = \qquad \mathsf{A}_{\epsilon_i} z_i' + \cdots + \mathsf{A}_{n\epsilon_i} (z_i^n)' + \displaystyle\sum_{\substack{\alpha \neq \epsilon_i, \dots, n\epsilon_i \\ 1 \leqslant |\alpha| \leqslant n}} \mathsf{A}_\alpha \, (z^\alpha)' \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = \qquad \mathsf{A}_{\epsilon_i} z_i^{(n)} + \cdots + \mathsf{A}_{n\epsilon_i} (z_i^n)^{(n)} + \displaystyle\sum_{\substack{\alpha \neq \epsilon_i, \dots, n\epsilon_i \\ 1 \leqslant |\alpha| \leqslant n}} \mathsf{A}_\alpha (z^\alpha)^{(n)}, \end{cases}$$

having the A_α as unknowns, where notably, $|\alpha| \leqslant n$ everywhere.

Noticing that the number of directions $\frac{\partial}{\partial a_\alpha}$ equals $\frac{(n+1+n)!}{(n+1)!\,n!}$ while the number of equations above equals $(n+1)$, we may now claim that for every

$\alpha \neq 0, \epsilon_i, \ldots, n\epsilon_i$ with $|\alpha| \leqslant n$, there are $\frac{(n+1+n)!}{(n+1)!\,n!} - (n+1)$ linearly independent vector fields of the specific form:

$$\frac{\partial}{\partial a_\alpha} - \mathsf{B}^i_{\alpha,0}\,\frac{\partial}{\partial a_0} - \mathsf{B}^i_{\alpha,1}\,\frac{\partial}{\partial a_{\epsilon_i}} - \cdots - \mathsf{B}^i_{\alpha,n}\,\frac{\partial}{\partial a_{n\epsilon_i}}$$

that are tangent to the (semi-global) graph $J^n_{\mathsf{vert}}(\mathscr{X}_0) \cap \{z'_i \neq 0\}$, that is to say, the coefficients of which satisfy the written linear system. In order to insure meromorphic prolongation to projective spaces (*see* below), it is convenient to multiply in advance the basic vector field $\frac{\partial}{\partial a_\alpha}$ of such a kind of sought vector field by the Wronskian-like determinant $\Delta(z'_i) := 1!2!\cdots n! \cdot (z'_i)^{\frac{n(n+1)}{2}}$ that the lemma computed. In sum, for any α with $1 \leqslant |\alpha| \leqslant n$ which is different from $0, \epsilon_i, \ldots, n\epsilon_i$, the vector field:

$$\mathsf{T}_\alpha := \Delta(z'_i)\,\frac{\partial}{\partial a_\alpha} - \mathsf{B}^i_{\alpha,0}\,\frac{\partial}{\partial a_0} - \mathsf{B}^i_{\alpha,1}\,\frac{\partial}{\partial a_{\epsilon_i}} - \cdots - \mathsf{B}^i_{\alpha,n}\,\frac{\partial}{\partial a_{n\epsilon_i}}$$

is tangent to $J^n_{\mathsf{vert}}(\mathscr{X}_0) \cap \{z'_i \neq 0\}$ if and only if its unknown coefficients $\mathsf{B}^i_{\alpha,k}$ satisfy the following linear system:

$$\begin{cases} 0 = -\mathsf{B}^i_{\alpha,0} - \mathsf{B}^i_{\alpha,1}\,z_i - \cdots\cdots - \mathsf{B}^i_{\alpha,n}\,z_i^n + \Delta(z'_i)\cdot z^\alpha \\ 0 = \qquad -\mathsf{B}^i_{\alpha,1}\,z'_i - \cdots\cdots - \mathsf{B}^i_{\alpha,n}\,(z_i^n)' + \Delta(z'_i)\cdot (z^\alpha)' \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = \qquad -\mathsf{B}^i_{\alpha,1}\,z_i^{(n)} - \cdots - \mathsf{B}^i_{\alpha,n}\,(z_i^n)^{(n)} + \Delta(z'_i)\cdot (z^\alpha)^{(n)}. \end{cases}$$

A basic application of Cramer's rule now enable us to solve the last n equations, and afterwards, we may then substitute the obtained solutions in the first equation:

$$\left[\begin{array}{l} \mathsf{B}^i_{\alpha,k} := \begin{vmatrix} z'_i & \cdots & (z^\alpha)' & \cdots & (z_i^n)' \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ z_i^{(n)} & \cdots & (z^\alpha)^{(n)} & \cdots & (z_i^n)^{(n)} \end{vmatrix} \qquad (\kappa\text{-th column},\ 1 \leqslant k \leqslant n) \\ \mathsf{B}^i_{\alpha,0} := -\mathsf{B}^i_{\alpha,1}\,z_i - \cdots - \mathsf{B}^i_{\alpha,n}\,z_i^n + \Delta(z'_i)\cdot z^\alpha. \end{array}\right.$$

Clearly, the so obtained vector fields T_α with $\alpha \neq 0, \epsilon_i, \ldots, n\epsilon_i$ are linearly independent at every point of $J^n_{\mathsf{vert}}(\mathscr{X}_0) \cap \{z'_i = 0\}$.

Meromorphic prolongation and computation of pole orders. Recall that any polynomial $P(t_0, \ldots, t_{\upsilon-1}, t_{\upsilon+1}, \ldots, t_{n+1})$ of degree $e \geqslant 1$ on an affine $\mathbb{C}^{n+1} \subset \mathbb{P}^{n+1}$, when viewed as a meromorphic map $\mathbb{P}^{n+1} \to \mathbb{P}^1$, has pole order equal to e, for a change of standard affine chart:

$$t_0 = \tfrac{1}{z_\upsilon}, \ldots, t_{\upsilon-1} = \tfrac{z_{\upsilon-1}}{z_\upsilon},\ t_{\upsilon+1} = \tfrac{z_{\upsilon+1}}{z_\upsilon}, \ldots, t_{n+1} = \tfrac{z_{n+1}}{z_\upsilon},$$

transfers P to $P\big(\frac{1}{z_v}, \dots, \frac{z_{v-1}}{z_v}, \frac{z_{v+1}}{z_v}, \dots, \frac{z_{n+1}}{z_v}\big)$. Through such an inversion map, the first-order jets, second-order jets, *etc*, are transferred to:

$$\frac{z_i'}{z_v} - \frac{z_i z_v'}{z_v^2}, \qquad \frac{z_i''}{z_v} - 2\,\frac{z_i' z_v'}{z_v^2} - \frac{z_i z_v''}{z_v^2} + 2\,\frac{z_i z_v' z_v'}{z_v^3}, \qquad \textit{etc.},$$

hence by just looking at the maximal power of z_v at the denominator, one easily observes by induction that:

$$\textsf{Pole-order}\left[z^\alpha\,(z')^{\alpha^1}\cdots(z^{(n)})^{\alpha^n}\right] = |\alpha| + |\alpha^1| + \cdots + |\alpha^n| + n,$$

and furthermore, one differentiation of such a monomial increases its pole order by just one unit. Now, we claim that:

$$\left[\begin{array}{l} \textsf{Pole-order}\big[\Delta(z_i')\big] = n^2 + n, \\ \quad \textsf{Pole-order}\big[\mathsf{B}^i_{\alpha,k}\big] = |\alpha| + n^2 + n - k \\ \quad \textsf{Pole-order}\big[\mathsf{B}^i_{\alpha,0}\big] = |\alpha| + n^2 + n, \end{array}\right.$$

so that the highest pole order occurs to be the coefficient $\mathsf{B}^i_{\alpha,0}$ of $\frac{\partial}{\partial a_0}$ in each T_α.

Indeed, replacing the entries of the determinant $\Delta(z_i')$ plainly by the non-negative integers which indicate the pole orders, we may write symbolically:

$$\textsf{Pole-order}\big[\Delta(z_i')\big] = \textsf{Pole-order of} \left|\begin{array}{ccccc} 2 & 3 & 4 & \cdots & n+1 \\ 3 & 4 & 5 & \cdots & n+2 \\ 4 & 5 & 6 & \cdots & n+3 \\ .. & .. & .. & \cdots & .. \\ n+1 & n+2 & n+3 & \cdots & 2n \end{array}\right|.$$

When one expands the determinant as a sum of monomials with $\pm$ signs, pole orders are just added, symbolically speaking. Then one easily convinces oneself that *each one* of the obtained monomials has the *same* pole order, hence it suffices to compute the pole order of the monomial of the main diagonal, which is equal to: $2 + 4 + 6 + \cdots + 2n = n(n+1)$.

Next, $\mathsf{B}^i_{\alpha,k}$ is obtained from $\Delta(z_i')$ by replacing the k-th column of $\Delta(z_i')$ by the new column of pole order entries $|\alpha| + 1$, $|\alpha| + 2$, $\dots$, $|\alpha| + n$. The pole order $|\alpha|$ being "factorizable", we get:

$$\textsf{Pole-order}\big[\mathsf{B}^i_{\alpha,k}\big] = |\alpha| + \textsf{Pole-order of} \left|\begin{array}{cccccc} 2 & 3 & \cdots & 1 & \cdots & n+1 \\ 3 & 4 & \cdots & 2 & \cdots & n+2 \\ .. & .. & \cdots & .. & \cdots & .. \\ n+1 & n+2 & \cdots & n & \cdots & 2n \end{array}\right|,$$

where the central-looking column is the k-th, the only which differs from $\Delta(z_i')$. Again, one easily convinces oneself that the pole order of *each one*

of the monomials obtained after expansion is the *same*, so that by looking again at the main diagonal:

$$\begin{aligned}\mathsf{Pole\text{-}order}\big[\mathsf{B}^i_{\alpha,k}\big] &= |\alpha| + 2 + \cdots + 2(k-1) + k + 2(k+1) + \cdots + 2n \\ &= |\alpha| + n(n+1) - k.\end{aligned}$$

Finally, coming back to the definition of $\mathsf{B}^i_{\alpha,0}$, one then immediately sees that each term in $\mathsf{B}^i_{\alpha,0}$ has the same pole order, equal to $|\alpha| + n^2 + n$.

In conclusion, the maximal pole order is reached by any $\mathsf{B}^i_{\alpha,0}$ with $|\alpha| = n$, and is equal to n^2+2n, as it appears in the statement of the main theorem. Clearly, the poles are compensated by the twisting $(\bullet) \otimes \mathscr{O}_{\mathbb{P}^{n+1}}(n^2+2n)$. Notice that the coefficients of the constructed T_α's depend only on the jet variables $(z, z', \ldots, z^{(n)})$, absolutely not on the coefficients a_β.

The other vector fields that we will construct in the remainder of the article, so as to complete a true framing, will all have pole order in the z-direction smaller than $n^2 + 2n$, and will all have pole order at most 1 in the a-direction.

Modifications needed for the second theorem. By assumption, at least one (classical) Wronskian $\det\big(z_i^{(\lambda_j)}\big)_{1\leqslant i\leqslant n+1}^{1\leqslant j\leqslant n}$ does not vanish, where, without loss of generality, we may assume that $1 \leqslant \lambda_1 < \cdots < \lambda_n \leqslant n+1$. To fix ideas, we shall work in the open set $\big\{\det(z_i^{(j)})_{1\leqslant i\leqslant n}^{1\leqslant j\leqslant n} \neq 0\big\}$ and we shall denote: $\mathsf{W} := \det(z_i^{(j)})_{1\leqslant i\leqslant n}^{1\leqslant j\leqslant n}$. Then again, the variety of vertical jets is checked to be, in the open set $\big\{\det(z_i^{(j)})_{1\leqslant i\leqslant n}^{1\leqslant j\leqslant n} \neq 0\big\}$, a semi-global graph of equations:

$$\begin{cases} 0 = a_0 + a_{\epsilon_1} z_1 + \cdots\cdots + a_{\epsilon_n} z_n + \displaystyle\sum_{\substack{\beta\neq\epsilon_1,\ldots,\epsilon_n\\ 1\leqslant|\beta|\leqslant d}} a_\beta\, z^\beta \\ 0 = \qquad a_{\epsilon_1} z_1' + \cdots\cdots + a_{\epsilon_n} z_n' + \displaystyle\sum_{\substack{\beta\neq\epsilon_1,\ldots,\epsilon_n\\ 1\leqslant|\beta|\leqslant d}} a_\beta\, (z^\beta)' \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = \qquad a_{\epsilon_1} z_1^{(n)} + \cdots + a_{\epsilon_n} z_n^{(n)} + \displaystyle\sum_{\substack{\beta\neq\epsilon_1,\ldots,\epsilon_n\\ 1\leqslant|\beta|\leqslant d}} a_\beta\, (z^\beta)^{(n)}, \end{cases}$$

having transversal coordinates $(a_0, a_{\epsilon_1}, \ldots, a_{\epsilon_n})$, the ones that are then clearly solvable here. Thus if, similarly as in the previous paragraphs, one seeks tangent vector fields of the specific form:

$$\mathsf{T}_\alpha := \mathsf{W}\,\frac{\partial}{\partial a_\alpha} - \mathsf{B}_{\alpha,0}\,\frac{\partial}{\partial a_0} - \mathsf{B}_{\alpha,1}\,\frac{\partial}{\partial a_{\epsilon_1}} - \cdots - \mathsf{B}_{\alpha,n}\,\frac{\partial}{\partial a_{\epsilon_n}},$$

for any $\alpha \in \mathbb{N}^{n+1}$ with $|\alpha| \leqslant n$ and with $\alpha \neq 0, \epsilon_1, \dots, \epsilon_n$, then the system we now have to solve becomes:

$$\begin{cases} 0 = -\mathsf{B}_{\alpha,0} - \mathsf{B}_{\alpha,1}\, z_1 - \cdots\cdots - \mathsf{B}_{\alpha,n}\, z_n + \mathsf{W} \cdot z^\alpha \\ 0 = \qquad -\mathsf{B}_{\alpha,1}\, z_1' - \cdots\cdots - \mathsf{B}_{\alpha,n}\, z_n' + \mathsf{W} \cdot (z^\alpha)' \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = \qquad -\mathsf{B}_{\alpha,1}\, z_1^{(n)} - \cdots\cdot - \mathsf{B}_{\alpha,n}\, z_n^{(n)} + \mathsf{W} \cdot (z^\alpha)^{(n)}. \end{cases}$$

The unique solution is then again yielded by Cramer's rule:

$$\left[\begin{array}{l} \mathsf{B}_{\alpha,k} := \begin{vmatrix} z_1' & \cdots & (z^\alpha)' & \cdots & z_n' \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ z_1^{(n)} & \cdots & (z^\alpha)^{(n)} & \cdots & z_n^{(n)} \end{vmatrix} \qquad (k\text{-th column},\ 1 \leqslant k \leqslant n) \\ \mathsf{B}_{\alpha,0} := -\mathsf{B}_{\alpha,1}\, z_1 - \cdots - \mathsf{B}_{\alpha,n}\, z_n + \mathsf{W} \cdot z^\alpha. \end{array}\right.$$

One may now verify that:

$$\begin{aligned} \text{Pole order}\big[\mathsf{W}\big] &= \tfrac{(n+1)(n+2)}{2} \\ \text{Pole order}\big[\mathsf{B}_{\alpha,k}\big] &= 2 + \cdots + (n+1) - (k+1) + |\alpha| + k \\ &= \tfrac{n^2+3n-2}{2} + |\alpha| \\ \text{Pole order}\big[\mathsf{B}_{\alpha,0}\big] &= \tfrac{n^2+3n}{2} + |\alpha|, \end{aligned}$$

so that the maximal pole order is reached by $\mathsf{B}_{\alpha,0}$ for any multiindex α with $|\alpha| = n$, and is equal to $\frac{n^2+5n}{2}$, as this appears in the second theorem.

The other vector fields that we will construct in the sequel will complete a generating set both for the first and for the second theorem and will have lower pole order in the z-direction.

Higher lengths. At present, we construct globally defined tangent vector fields which span the remaining directions $\bigoplus_{\substack{n+1 \leqslant |\alpha| \leqslant d \\ \alpha_1 < d}} \mathbb{C} \cdot \frac{\partial}{\partial a_\alpha}$ in the space of coefficients a_α. For an arbitrary multiindex $\ell = (\ell_1, \ell_2, \dots, \ell_{n+1}) \in \mathbb{N}^{n+1}$ of length:

$$n + 1 = \ell_1 + \ell_2 + \cdots + \ell_{n+1},$$

we introduce the following family of vector fields living only in the space of a-variables:

$$\mathsf{T}_\alpha^{\ell_1,\ell_2,\dots,\ell_{n+1}} = \mathsf{T}_\alpha^\ell = \sum_{\substack{\ell'+\ell''=\ell \\ \ell',\,\ell'' \in \mathbb{N}^{n+1}}} (-1)^{|\ell''|}\, \frac{\ell!}{\ell'!\ \ell''!}\, z^{\ell''}\, \frac{\partial}{\partial a_{\alpha-\ell''}},$$

where the indices α are all possible indices satisfying $\alpha_1 \geqslant \ell_1, \dots, \alpha_{n+1} \geqslant \ell_{n+1}$, $|\alpha| \leqslant d$ and $\alpha_1 < d$, and where the sum abbreviates $\sum_{\ell_1'+\ell_1''=\ell_1} \cdots \sum_{\ell_{n+1}'+\ell_{n+1}''=\ell_{n+1}}$. For instance, for $n+1 = 4$ and with the special choice $\ell_1 = \ell_2 = 2$ (whence necessarily $\ell_3 = \ell_4 = 0$), we get the

following family of vector fields defined for all α with $\alpha_1 \geqslant 2$, $\alpha_2 \geqslant 2$, $\alpha_3 \geqslant 0$, $\alpha_4 \geqslant 0$ and $|\alpha| \leqslant d$, $\alpha_1 < d$ (compare [31], p. 373):

$$\begin{aligned}\mathsf{T}_\alpha^{2,2,0,0} = \frac{\partial}{\partial a_\alpha} - 2z_1 \frac{\partial}{\partial a_{\alpha-\epsilon_1}} - 2z_2 \frac{\partial}{\partial a_{\alpha-\epsilon_2}} + \\ + z_1^2 \frac{\partial}{\partial a_{\alpha-2\epsilon_1}} + 4z_1 z_2 \frac{\partial}{\partial a_{\alpha-\epsilon_1-\epsilon_2}} + z_2^2 \frac{\partial}{\partial a_{\alpha-2\epsilon_2}} - \\ - 2z_1^2 z_2 \frac{\partial}{\partial a_{\alpha-2\epsilon_1-\epsilon_2}} - 2z_1 z_2^2 \frac{\partial}{\partial a_{\alpha-\epsilon_1-2\epsilon_2}} + z_1^2 z_2^2 \frac{\partial}{\partial a_{\alpha-2\epsilon_1-2\epsilon_2}}.\end{aligned}$$

After a moment's reflection, one may convince oneself that as ℓ with $|\ell| = n+1$ runs and as α with $\alpha_i \geqslant \ell_i$ runs, the T_α^ℓ together with the vector fields of the previous paragraph do span $\bigoplus_{\substack{n+1 \leqslant |\alpha| \leqslant d \\ \alpha_1 < d}} \mathbb{C} \cdot \frac{\partial}{\partial a_\alpha}$; there are in fact redundancies among the *triangular* system defined by the T_α^ℓ, whenever one has $\alpha \geqslant \ell_1$ and $\alpha \geqslant \ell_2$ for two distinct ℓ^1, ℓ^2 with $|\ell^1| = |\ell^2| = n+1$.

Lemma. *For every nonnegative integer $e \leqslant n$ and for arbitrary indices $j_1, \dots, j_e$ with $1 \leqslant j_i \leqslant n+1$, one has:*

$$0 \equiv \mathsf{T}_\alpha^\ell \Bigg(\sum_{\substack{\beta \in \mathbb{N}^{n+1} \\ |\beta| \leqslant d,\, a_{d0\cdots 0}=1}} a_\beta \, \frac{\partial^e (z^\beta)}{\partial z_{j_1} \cdots \partial z_{j_e}} \Bigg),$$

and as a result, T_α^ℓ identically annihilates all the defining equations of $J_{\mathsf{vert}}^n(\mathscr{X}_0)$, hence is tangent to $J_{\mathsf{vert}}^n(\mathscr{X}_0)$.

Proof. Let $w_1, w_2, \dots, w_{n+1}$ be auxiliary complex variables. For every derivation order $e \leqslant n$ strictly less than the vanishing order $\sum \ell_i = n+1$, we trivially have:

$$0 \equiv \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_e}} \Big([w_1 - z_1]^{\ell_1} \, [w_2 - z_2]^{\ell_2} \cdots [w_{n+1} - z_{n+1}]^{\ell_{n+1}} \Big) \Big|_{w=z}.$$

In other words, by expanding $[w-z]^\ell = \sum_{\ell'+\ell''=\ell} (-1)^{|\ell''|} \frac{\ell!}{\ell'!\,\ell''!} w^{\ell'} z^{\ell''}$ thanks to the multinomial formula, by letting the derivation $\partial^e(\bullet)/\partial z_{j_1} \cdots \partial z_{j_e}$ act on this expansion, by setting $w = z$, and finally, by multiplying the result obtained by $z^{\alpha-\ell}$, we get the useful identities:

$$\boxed{0 \equiv \sum_{\substack{\ell'+\ell''=\ell \\ \ell',\, \ell' \in \mathbb{N}^{n+1}}} (-1)^{|\ell''|} \frac{\ell!}{\ell'!\,\ell''!} z^{\alpha-\ell''} \cdot \frac{\partial^e \big(z^{\ell''}\big)}{\partial z_{j_1} \cdots \partial z_{j_e}}}.$$

On the other hand, by letting the derivation $\mathsf{T}^{\ell}_{\alpha}$ act as it should, the identities of the lemma that we have to check may be written:

$$\begin{aligned} 0 &\overset{?}{\equiv} \sum_{\substack{\ell'+\ell''=\ell \\ \ell',\,\ell'\in\mathbb{N}^{n+1}}} (-1)^{|\ell''|}\,\frac{\ell!}{\ell'!\;\ell''!}\,\frac{\partial}{\partial a_{\alpha-\ell''}}\Bigg(\sum_{\substack{\beta\in\mathbb{N}^{n+1} \\ |\beta|\leqslant d,\,a_{d0\cdots0}=1}} a_\beta\,\frac{\partial^e(z^\beta)}{\partial z_{j_1}\cdots\partial z_{j_e}}\Bigg)\cdot z^{\ell''} \\ &= \sum_{\substack{\ell'+\ell''=\ell \\ \ell',\,\ell'\in\mathbb{N}^{n+1}}} (-1)^{|\ell''|}\,\frac{\ell!}{\ell'!\;\ell''!}\,\frac{\partial^e\big(z^{\alpha-\ell''}\big)}{\partial z_{j_1}\cdots\partial z_{j_e}}\cdot z^{\ell''}. \end{aligned}$$

Compared to the boxed, known identities, the derivation is now switched to the other monomial. Generally, we claim that for every $e=0,1,\dots,n$ and for every decomposition $e=e_1+(e-e_1)$ with $0\leqslant e_1\leqslant e$, the expression:

$$\big(j_1,\dots,j_{e_1}\big|j_{e_1+1},\dots,j_e\big) := \sum_{\substack{\ell'+\ell''=\ell \\ \ell',\,\ell'\in\mathbb{N}^{n+1}}} (-1)^{|\ell''|}\,\frac{\ell!}{\ell'!\;\ell''!}\,\frac{\partial^{e_1}\big(z^{\alpha-\ell''}\big)}{\partial z_{j_1}\cdots\partial z_{j_{e_1}}}\cdot\frac{\partial^{e-e_1}\big(z^{\ell''}\big)}{\partial z_{j_{e_1+1}}\cdots\partial z_{j_e}}$$

vanishes identically, for all indices $j_1,\dots,j_e=1,2,\dots n+1$. We know that this assertion is true when $e_1=0$ for all $e=0,1,\dots,n$ and the lemma corresponds to $e-e_1=0$ for all $e_1=0,1,\dots,n$.

For $e=0$, the assertion is thus known. Suppose it to be true at level e. Reasoning by induction, we then assume that:

$$0\equiv\big(j_1,\dots,j_{e_1}\big|j_{e_1+1},\dots,j_e\big),$$

for all $e_1=0,1,\dots,e$ and all possible j_i, If $e+1$ is still $\leqslant n$, we differentiate all these identities with respect to z_k using Leibniz' rule and we organize the resulting equations as a convenient array:

$$\begin{aligned} 0 &\equiv \big(j_1,\dots,j_e,k\big|\emptyset\big)+\big(j_1,\dots,j_e\big|k\big) \\ 0 &\equiv \big(j_1,\dots,j_{e-1},k\big|j_e\big)+\big(j_1,\dots,j_{e-1}\big|j_e,k\big) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 &\equiv \big(j_1,k\big|j_2,\dots,j_e\big)+\big(j_1\big|j_2,\dots,j_e,k\big) \\ 0 &\equiv \big(k\big|j_1,\dots,j_e\big)+\underline{\big(\emptyset\big|j_1,\dots,j_e,k\big)}_{\circ}. \end{aligned}$$

We have underlined the last term, known to vanish. Then the first term of the last line vanishes, for all indices $k,j_1,\dots,j_e=1,2,\dots,n+1$. So the second term of the penultimate vanishes, *etc.*, and hence the very first term $\big(j_1,\dots,j_e,k\big|\emptyset\big)$ does vanish identically, as desired. □

§4. Second package of jet, coordinate vector fields

Spanning the $\frac{\partial}{\partial z_i}$-directions. To complete the framing, let us at first span all the $\frac{\partial}{\partial z_i}$ directions. By convention, $a_{d0\cdots0}=1$.

Lemma. *For $i = 1, 2, \dots, n+1$, the vector fields:*

$$\mathsf{T}_i := \frac{\partial}{\partial z_i} - \sum_{|\alpha| \leqslant d-1} a_{\alpha+\epsilon_i} (\alpha_i + 1) \frac{\partial}{\partial a_\alpha}$$

are all tangent to $J^n_{\mathsf{vert}}(\mathscr{X}_0)$.

Proof. Appying the derivation T_i to the first equation $0 = \sum_\alpha a_\alpha z^\alpha$ of $J^n_{\mathsf{vert}}(\mathscr{X}_0)$, we indeed get an identically vanishing result:

$$\sum_{|\alpha| \leqslant d} a_\alpha \frac{\partial(z^\alpha)}{\partial z_i} - \sum_{|\alpha| \leqslant d-1} a_{\alpha+\epsilon_i} (\alpha_i + 1) z^\alpha \equiv 0.$$

Since the T_i commute with the total differentiation operator D, it then follows immediately that T_i annihilates all the other defining equations:

$$0 \equiv \mathsf{T}_i \Big(\mathsf{D} \sum_\alpha a_\alpha z^\alpha \Big) \equiv \cdots \equiv \mathsf{T}_i \Big(\mathsf{D}^n \sum_\alpha a_\alpha z^\alpha \Big),$$

and this yields the tangency property claimed. □

Spanning the $\partial / \partial z_j^{(\lambda)}$ directions. For the last family of vector fields, we transfer to general $\kappa = n \geqslant 2$ the approach of [26] known for $\kappa = n = 2$ and also for $\kappa = n = 3$ [31], with few differences.

Let $\Lambda = \big(\Lambda^l_k\big)^{1 \leqslant l \leqslant n+1}_{1 \leqslant k \leqslant n+1}$ be a matrix in $\mathsf{GL}(n+1, \mathbb{C})$. To span the only remaining directions $\partial / \partial z_j^{(\lambda)}$, one seeks meromorphic vector fields tangent to $J^n_{\mathsf{vert}}(\mathscr{X}_0) \big\backslash \Sigma_0$ that are of the special form:

$$\begin{aligned} \mathsf{T}_\Lambda := & \sum_{k=1}^{n+1} \Big(\textstyle\sum_{l=1}^{n+1} \Lambda^l_k \, z'_l \Big) \frac{\partial}{\partial z'_k} + \cdots + \sum_{k=1}^{n+1} \Big(\textstyle\sum_{l=1}^{n+1} \Lambda^l_k \, z^{(n)}_l \Big) \frac{\partial}{\partial z^{(n)}_k} + \\ & + \sum_{\substack{|\alpha| \leqslant d \\ \alpha_1 < d}} \mathsf{A}_\alpha \big(z, a, \Lambda\big) \frac{\partial}{\partial a_\alpha}, \end{aligned}$$

where, for various jet orders λ's, the coefficients $\mathsf{Z}^{(\lambda)}_k$ of the $\frac{\partial}{\partial z^{(\lambda)}_k}$, $k = 1, \dots, n+1$, are defined a priori to be obtained by multiplying the jet matrix $\big(z^{(\lambda)}_j\big)^{1 \leqslant \lambda \leqslant n}_{1 \leqslant j \leqslant n+1}$ by such a matrix Λ:

$$\begin{pmatrix} \Lambda^1_1 & \cdots & \Lambda^n_1 & \Lambda^{n+1}_1 \\ \Lambda^1_2 & \cdots & \Lambda^n_2 & \Lambda^{n+1}_2 \\ \vdots & \ddots & \vdots & \vdots \\ \Lambda^1_{n+1} & \cdots & \Lambda^n_{n+1} & \Lambda^{n+1}_{n+1} \end{pmatrix} \begin{pmatrix} z'_1 & \cdots & z^{(n)}_1 \\ z'_2 & \cdots & z^{(n)}_2 \\ \vdots & \ddots & \vdots \\ z'_{n+1} & \cdots & z^{(n)}_{n+1} \end{pmatrix} = \begin{pmatrix} \mathsf{Z}'_1 & \cdots & \mathsf{Z}^{(n)}_1 \\ \mathsf{Z}'_2 & \cdots & \mathsf{Z}^{(n)}_2 \\ \vdots & \ddots & \vdots \\ \mathsf{Z}'_{n+1} & \cdots & \mathsf{Z}^{(n)}_{n+1} \end{pmatrix},$$

and where the coefficients $\mathsf{A}_\alpha\big(z, a, \Lambda\big)$, to be computed shortly, should insure that T_Λ is effectively tangent to $J^n_{\mathsf{vert}}(\mathscr{X}_0) \big\backslash \Sigma_0$.

In fact, by plainly inspecting ranks of the matrix multiplication above, one easily sees that, at every point of our basic open set where at least one

$n \times n$ (sub)Wronskian of the jet matrix $\big(z_j^{(\lambda)}\big)$ does not vanish, one has for Λ varying without restriction in $\mathsf{GL}(n+1,\mathbb{C})$:

$$\mathrm{Span}_\Lambda\bigg(\Lambda\, z'\,\frac{\partial}{\partial z'}+\cdots+\Lambda\, z^{(n)}\,\frac{\partial}{\partial z^{(n)}}\bigg) = \bigoplus_{1\leqslant k\leqslant n+1}\mathbb{C}\,\frac{\partial}{\partial z'_k}\ \cdots\ \bigoplus_{1\leqslant k\leqslant n+1}\mathbb{C}\,\frac{\partial}{\partial z_k^{(n)}}.$$

The following proposition will therefore complete the proof of the theorem.

Proposition. *There exist coefficients A_α for $\partial/\partial a_\alpha$ with $|\alpha|\leqslant d$, $\alpha_1<d$, which are polynomials in z of degree at most n:*

$$\mathsf{A}_\alpha\big(z,a,\Lambda\big) = \sum_{|\beta|\leqslant n}\mathscr{L}_\alpha^\beta\big(a,\Lambda\big)\,z^\beta$$

with coefficients $\mathscr{L}_\alpha^\beta\big(a,\Lambda\big)$ being bilinear in the variables $\big(a_\gamma,\,\Lambda_k^l\big)$ such that T_Λ is tangent to $J^n_{\mathsf{vert}}(\mathscr{X}_0)\big\backslash\Sigma_0$.

Proof. While writing down, say, the first two tangency equations, namely when applying the derivative T_Λ to the first two of the five big equations written at the beginning, one gets equations:

$$\textbf{(0)}\qquad 0=\sum_{\substack{|\alpha|\leqslant d\\ \alpha_1<d}}\mathsf{A}_\alpha\cdot z^\alpha$$

$$\textbf{(1}_{j_1}\textbf{)}\qquad 0=\sum_{\substack{|\alpha|\leqslant d\\ \alpha_1<d}}\mathsf{A}_\alpha\cdot\frac{\partial(z^\alpha)}{\partial z_{j_1}}+\sum_{\substack{|\alpha|\leqslant d\\ a_{d0\cdots0}=1}}a_\alpha\sum_{l=1}^{n+1}\frac{\partial(z^\alpha)}{\partial z_l}\,\Lambda_l^{j_1},$$

for which one is allowed to equate to zero the coefficient of each z'_{j_1}, because the sought A_α should be independent of $z'_{j_1}, z''_{j_2},\dots,z^{(n)}_{j_n}$.

Next, when applying T_Λ to the third defining equation of $J^n_{\mathsf{vert}}(\mathscr{X}_0)$, one sees thanks to $(\mathbf{1})_{j_1}$ that the coefficient of each z''_{j_1} then automatically vanishes[31], hence we are left with just equating to zero the coefficients of the monomials $z'_{j_1}z'_{j_2}$, namely:

$\textbf{(2}_{j_1j_2}\textbf{)}$

$$0=\sum_{\substack{|\alpha|\leqslant d\\ \alpha_1<d}}\mathsf{A}_\alpha\cdot\frac{\partial^2(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}}+\sum_{\substack{|\alpha|\leqslant d\\ a_{d0\cdots0}=1}}a_\alpha\sum_{l=1}^{n+1}\bigg(\frac{\partial^2(z^\alpha)}{\partial z_l\partial z_{j_2}}\,\Lambda_l^{j_1}+\frac{\partial^2(z^\alpha)}{\partial z_{j_1}\partial z_l}\,\Lambda_l^{j_2}\bigg).$$

By induction, such a simplification is easily seen to generalize and thus, the $(e+1)$-th condition of tangency, after taking account of the successive

[31] This simplification trick justifies a posteriori, *cf.* [26], the *ad hoc*-looking assumption that the same matrix Λ appears in each jet vector field coefficient $\mathsf{Z}^{(\lambda)}=\Lambda\cdot z^{(\lambda)}$.

cancellations, is obtained by just looking at how T_Λ acts on the jet monomial $z'_{j_1}\cdots z'_{j_e}$, and the result then consists in the family of equations:

$(\mathbf{e}_{j_1\cdots j_e})$

$$0=\sum_{\substack{|\alpha|\leqslant d\\ \alpha_1<d}}\mathsf{A}_\alpha\cdot\frac{\partial^e(z^\alpha)}{\partial z_{j_1}\cdots\partial z_{j_e}}+\sum_{\substack{|\alpha|\leqslant d\\ a_{d0\cdots 0}=1}}a_\alpha\sum_{l=1}^{n+1}\Bigg($$
$$\Bigg(\frac{\partial^e(z^\alpha)}{\partial z_l\partial z_{j_2}\cdots\partial z_{j_e}}\,\Lambda_l^{j_1}+\frac{\partial^e(z^\alpha)}{\partial z_{j_1}\partial z_l\cdots\partial z_{j_e}}\,\Lambda_l^{j_2}+\cdots+\frac{\partial(z^\alpha)}{\partial z_{j_1}\cdots\partial z_{j_{e-1}}\partial z_l}\,\Lambda_l^{j_e}\Bigg),$$

where $j_1,\ldots,j_e=1,\ldots,n+1$ are arbitrary.

The equations for the unknows $\mathscr{L}_\alpha^\beta$ shall then be obtained by identifying the coefficients of the monomials z^ρ in the above equations $(\mathbf{0})$, $(\mathbf{1}_{j_1})$, $\ldots$, $(\mathbf{n}_{j_1\cdots j_n})$.

At first, we observe that since the degrees in z of the second terms of $(\mathbf{1}_{j_1})$, $(\mathbf{2}_{j_1j_2})$, *etc.* are at most $d-1$, $d-2$, etc., we can, without loss of generality, suppose that the $\mathscr{L}_\alpha^\beta$ are zero for $|\alpha|+|\beta|\geqslant d+1$, as it is written in the proposition. Next (cf. [26]), using the equation of $\mathscr{X}_0$, we may replace the occurence of z_1^d in the equation $(\mathbf{0})$ by $-\sum_{|\alpha|\leqslant d,\,\alpha_1<d}a_\alpha\,z^\alpha$, so that the degree in the z_1 variable is at most $d-1$ (as in [26], this will insure that the linear systems we have to solve are not overdetermined, and Cramer's basic rule will apply).

Now, the coefficient of each monomial z^ρ in the equation $(\mathbf{0})$ should vanish:

$$(\mathbf{0}_\rho)\qquad 0=\sum_{\alpha+\beta=\rho}\mathscr{L}_\alpha^\beta.$$

Next, if as usual $\delta_{j_2}^{j_1}$ denotes the Kronecker symbol, equal to 1 if $j_1=j_2$ and to 0 otherwise, we can shortly the various occuring partial derivatives of the monomial z^α as:

$$\frac{\partial(z^\alpha)}{\partial z_{j_1}}=\alpha_{j_1}\,z^{\alpha-\epsilon_{j_1}},\quad\frac{\partial^2(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}}=\alpha_{j_1}\big(\alpha_{j_2}-\delta_{j_2}^{j_1}\big)\,z^{\alpha-\epsilon_{j_1}-\epsilon_{j_2}},\ \cdots\cdots,$$
$$\frac{\partial^e(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}\cdots\partial z_{j_e}}=\alpha_{j_1}\big(\alpha_{j_2}-\delta_{j_2}^{j_1}\big)\cdots\big(\alpha_{j_e}-\delta_{j_e}^{j_1}-\cdots-\delta_{j_e}^{j_{e-1}}\big)\,z^{\alpha-\epsilon_{j_1}-\cdots-\epsilon_{j_e}}.$$

It follows that, for every $e\leqslant n$, the $(e+1)$-th family of equations, after equating to zero the coefficients of the monomial $z^{\rho-\epsilon_{j_1}-\cdots-\epsilon_{j_e}}$ and replacing α by $\rho-\beta$, identifies to the collection:

$(\mathbf{e}_{j_1j_2\cdots j_e\rho})$

$$0=\sum_{|\beta|\leqslant n}(\rho_{j_1}-\beta_{j_1})\big(\rho_{j_2}-\beta_{j_2}-\delta_{j_2}^{j_1}\big)\cdots\big(\rho_{j_e}-\beta_{j_e}-\delta_{j_e}^{j_1}-\cdots-\delta_{j_e}^{j_{e-1}}\big)\,\mathscr{L}_{\rho-\beta}^\beta+$$
$$+\,\mathsf{R}_{j_1j_2\cdots j_e\rho}(a,\Lambda),$$

where each second term $\mathsf{R}_{j_1 j_2 \cdots j_e \rho}(a, \Lambda)$, here considered as being just a remainder, is the coefficient of $z^{\rho - \epsilon_{j_1} - \cdots - \epsilon_{j_e}}$ in the second term of the equation $(\mathbf{e}_{j_1 j_2 \cdots j_e})$ and hence is clearly bilinear in $\big(a_\gamma, \Lambda_k^l\big)$.

Thus, we have written a constant coefficient system of linear equations having the $\mathscr{L}_{\rho-\beta}^{\beta}$ as unknowns, $|\beta| \leqslant n$. As in [26, 31], we now claim that the determinant of its matrix is nonzero.

Indeed, for each fixed multiindex ρ, the matrix whose column C_β consists of the partial derivatives of order at most n of the monomial $z^{\rho-\beta}$ has the same determinant, at the point $(1, 1, \dots, 1)$ as the linear subsystem $(\mathbf{0}_\rho)$, $(\mathbf{1}_{j_1 \rho})$, $\dots$, $(\mathbf{e}_{j_1 \cdots j_n \rho})$ we want to solve, where $j_1, \dots, j_n = 1, \dots, n+1$. Therefore, if the determinant would be zero, we would by linear combination, derive the existence of a *not* identically zero polynomial:

$$Q(z) := \sum_\beta c_\beta \, z^{\rho-\beta}$$

all of whose partial derivatives of order $\leqslant n$ vanish at $(1, \dots, 1)$. Hence the same would be true of:

$$P(z) := z^\rho \, Q\big(1/z_1, \dots, 1/z_{n+1}\big) = \sum_\beta c_\beta \, z^\beta,$$

and this would imply $P \equiv 0$, in contradiction to the assumption.

Thus for each fixed ρ, Cramer's rule solves the system for the $\mathscr{L}_\alpha^\beta$ with $\alpha + \beta = \rho$, and the solution is then obviously bilinear in (a, Λ). $\square$

Invariance under reparametrization and logarithmic versions. We would like to make two final remarks, useful in applications. At first, similarly as it was pointed out in [26, 31], we claim that all vector fields constructed above are invariant under the group G_n of n-jets at the origin of local reparametrizations $\phi(\zeta) = \zeta + \phi''(0) \, \frac{\zeta^2}{2!} + \cdots + \phi^{(n)}(0) \, \frac{\zeta^n}{n!} + \cdots$ of $(\mathbb{C}, 0) \ni \zeta$ that are tangent to the identity[32], which acts on the jets $(z'_{i_1}, z''_{i_2}, z'''_{i_3}, \dots)$ by transforming them to:

$$w'_{i_1} := z'_{i_1}, \qquad w''_{i_2} := z''_{i_2} + \phi'' \, z'_{i_2}, \qquad w'''_{i_3} := z'''_{i_3} + 3\phi'' z''_{i_3} + \phi''' z'_{i_3}, \ \dots.$$

Such a transformation makes a diffeomorphism of $J^n_{\mathsf{vert}}(\mathscr{X})$, its inverse being associated to $\phi^{-1}(\zeta)$, and we must verify that our 4 families of tangent vector fields T_α, $\mathsf{T}_\alpha^{\ell_1, \dots, \ell_{n+1}}$, T_i and T_Λ are left unchanged under this diffeomorphism. Indeed, the claim trivially holds true for both the $\mathsf{T}_\alpha^{\ell_1, \dots, \ell_{n+1}}$ and the T_i, because they incorporate absolutely no $z'_{i_1}, z''_{i_2}, \dots, z^{(n)}_{i_n}$. Next, the $\frac{\partial}{\partial a_\beta}$ in the T_α are clearly left unchanged, while their coefficients are all, say

[32] As a result, our two theorems can be applied in the framework of Demailly-Semple jets ([11]).

in the case $n = 3$ to fix ideas, of the Wronskian-like form:

$$\begin{vmatrix} f' & g' & h' \\ f'' & g'' & h'' \\ f''' & g''' & h''' \end{vmatrix} \equiv \begin{vmatrix} f' & g' & h' \\ f'' + \phi'' f' & g'' + \phi'' g' & h'' + \phi'' h' \\ f''' + 3\phi'' f'' + \phi''' f' & g''' + 3\phi'' g'' + \phi''' g' & h''' + 3\phi'' h'' + \phi''' h' \end{vmatrix},$$

where $f, g, h \in \mathbb{C}[z_1, z_2, z_3]$ are some polynomials, but then such a determinant remains unchanged, thanks to obvious line manipulations. The general case $n \geqslant 3$ is similar. Finally, erasing indices and again for $n = 3$, we give the formal reason why the T_Λ are also invariant. The transformation is $w' = z'$, $w'' = z'' + \phi'' z'$, $w''' = z''' + 3\phi'' z'' + \phi''' z'$ and it replaces the basic vector fields by:

$$\tfrac{\partial}{\partial z'} = \tfrac{\partial}{\partial w'} + \phi'' \tfrac{\partial}{\partial w''} + \phi''' \tfrac{\partial}{\partial w'''}, \qquad \tfrac{\partial}{\partial z''} = \tfrac{\partial}{\partial w''} + 3\phi'' \tfrac{\partial}{\partial w'''}, \qquad \tfrac{\partial}{\partial z'''} = \tfrac{\partial}{\partial w'''},$$

so that $z' \frac{\partial}{\partial z'} + z'' \frac{\partial}{\partial z''} + z''' \frac{\partial}{\partial z'''} = w' \frac{\partial}{\partial w'} + w'' \frac{\partial}{\partial w''} + w''' \frac{\partial}{\partial w'''}$ is invariant.

Another argument (transmitted to us by Erwan Rousseau) for invariance under reparametrization would be to say that the system of linear equations that the coefficients Z_i, A_α, Z'_k, Z''_k, $\ldots$, $\mathsf{Z}^{(n)}_k$ of a general tangent vector field T have to satisfy:

$$0 = \mathsf{T}\big[\textstyle\sum_\alpha a_\alpha z^\alpha\big] = \mathsf{T}\big[\textstyle\sum_\alpha a_\alpha (z^\alpha)'\big] = \mathsf{T}\big[\textstyle\sum_\alpha a_\alpha (z^\alpha)''\big] = \mathsf{T}\big[\textstyle\sum_\alpha a_\alpha (z^\alpha)'''\big] = \cdots,$$

is transformed, after reparametrization, into a system:

$$\begin{aligned} 0 &= \mathsf{T}\big[\textstyle\sum_\alpha a_\alpha z^\alpha\big] = \mathsf{T}\big[\textstyle\sum_\alpha a_\alpha (z^\alpha)'\big] = \mathsf{T}\big[\textstyle\sum_\alpha a_\alpha (z^\alpha)'' + \phi'' \textstyle\sum_\alpha a_\alpha (z^\alpha)'\big] \\ 0 &= \mathsf{T}\big[\textstyle\sum_\alpha a_\alpha (z^\alpha)''' + 3\phi'' \textstyle\sum_\alpha a_\alpha (z^\alpha)'' + \phi''' \textstyle\sum_\alpha a_\alpha (z^\alpha)'\big] = \cdots \end{aligned}$$

which is completely equivalent to the first one, thanks to obvious linear combinations, so that any solution to this linear system is *a priori* forced to be invariant.

The second remark is that one may adapt the formalism provided here to show that the global generation property holds in a logarithmic setting with the *same* specific pole orders $c_n = \frac{n^2+5n}{2}$ (*cf.* [32] for $n = 3$) or $c_n = n^2+2n$. Application to effective algebraic degeneracy of entire holomorphic maps in the complement of a generic hypersurface $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ are therefore also possible.

§5. Appendix: a determinantal identity

Proof of the combinatorial lemma. At the beginning of Section 3, a determinant left aside had to be computed. We drop the index i and we denote it shortly by:

$$\Delta := \begin{vmatrix} z' & (z^2)' & \cdots & (z^n)' \\ z'' & (z^2)'' & \cdots & (z^n)'' \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ z^{(n)} & (z^2)^{(n)} & \cdots & (z^n)^{(n)} \end{vmatrix}.$$

On the first line, the entry of the k-th column is $(z^k)' = kz^{k-1}z'$. The trick is then to write the entry of the second line inside the same column as $k(z^{k-1}z')'$, *etc.*, and generally the entry of the κ-th line as $k(z^{k-1}z')^{(\kappa-1)}$, so that:

$$\Delta = \begin{vmatrix} z' & 2zz' & 3z^2z' & \cdots & nz^{n-1}z' \\ z'' & 2(zz')' & 3(z^2z')' & \cdots & n(z^{n-1}z')' \\ z''' & 2(zz')'' & 3(z^2z')'' & \cdots & n(z^{n-1}z')'' \\ .. & .. & .. & \cdots & .. \\ z^{(n)} & 2(zz')^{(n-1)} & 3(z^2z')^{(n-1)} & \cdots & n(z^{n-1}z')^{(n-1)} \end{vmatrix}.$$

We see that $2\cdot 3\cdots n$ from the columns comes into factor. Next, using Leibniz's formula for the derivative of a product, we may expand $(z^{k-1}z')^{(\kappa-1)}$ just as $\sum_{0\leqslant\lambda_1\leqslant\kappa-1}\binom{\kappa-1}{\lambda_1}(z^{k-1})^{(\lambda_1)}z^{(1+\kappa-1-\lambda_1)}$. By subtracting to the k-th column the first one multiplied by z^{k-1}, the (κ,k)-entry then becomes $\sum_{1\leqslant\lambda_1\leqslant\kappa-1}\binom{\kappa-1}{\lambda_1}(z^{k-1})^{(\lambda_1)}z^{(\kappa-\lambda_1)}$, where now the sum starts from $\lambda_1 = 1$. In particular, the $(1,2)$-, $(1,3)$-, $\ldots$, $(0,n)$-entries all become null. By expanding the determinant along its first line, we are therefore left with an $(n-1)\times(n-1)$ determinant (notice the necessary shift of indices):

$$\frac{\Delta}{n!z'} = \Big|\sum_{1\leqslant\lambda_1\leqslant\kappa}\tbinom{\kappa}{\lambda_1}(z^k)^{(\lambda_1)}z^{(1+\kappa-\lambda_1)}\Big|_{1\leqslant\kappa\leqslant n-1}^{1\leqslant k\leqslant n-1}.$$

Iterating the trick, we again write:

$$(z^k)^{(\lambda_1)} = k(z^{k-1}z')^{(\lambda_1-1)} = k\sum_{0\leqslant\lambda_2\leqslant\lambda_1-1}\tbinom{\lambda_1-1}{\lambda_2}\,(z^{k-1})^{(\lambda_2)}\,z^{(1+\lambda_1-1-\lambda_2)}.$$

We again see that $2\cdot 3\cdots(n-1)$ from the columns comes into factor, and then substituting the computed value of $(z^k)^{(\lambda_1)}$, we get:

$$\frac{\Delta}{n!z'} = (n-1)!\cdot\Big|\sum_{1\leqslant\lambda_1\leqslant\kappa}\sum_{0\leqslant\lambda_2\leqslant\lambda_1-1}\tbinom{\kappa}{\lambda_1}\tbinom{\lambda_1-1}{\lambda_2}\cdot \\ \cdot(z^{k-1})^{(\lambda_2)}\,z^{(1+\kappa-\lambda_1)}\,z^{(\lambda_1-\lambda_2)}\Big|_{1\leqslant\kappa\leqslant n-1}^{1\leqslant k\leqslant n-1}.$$

The $(\kappa,1)$-entry inside the first colum is equal to $\sum_{1\leqslant\lambda_1\leqslant\kappa}\binom{\kappa}{\lambda_1}z^{(1+\kappa-\lambda_1)}\,z^{(\lambda_1)}$, because the terms $(z^0)^{(\lambda_2)}$ with $\lambda_2\geqslant 1$ are null. By subtracting to the k-th column the first one multiplied by z^{k-1}, the (κ,k)-th entry written above is slightly modified: the sum involving λ_2 is then just replaced by $\sum_{1\leqslant\lambda_2\leqslant\lambda_1-1}$. Moreover, the $(1,2)$-, $(1,3)$-, $\ldots$, $(1,n-1)$- entries all become null, while the $(1,1)$ entry is $\binom{1}{1}z'z'$. By expanding the determinant along its first line, we are therefore left with an $(n-2)\times(n-2)$ determinant:

$$\frac{\Delta}{n!(n-1)!\,z'(z')^2} = \Big|\sum_{1\leqslant\lambda_1\leqslant\kappa}\sum_{1\leqslant\lambda_2\leqslant\lambda_1-1}\tbinom{\kappa}{\lambda_1}\tbinom{\lambda_1-1}{\lambda_2}\cdot \\ \cdot(z^{k-1})^{(\lambda_2)}\,z^{(1+\kappa-\lambda_1)}\,z^{(\lambda_1-\lambda_2)}\Big|_{2\leqslant\kappa\leqslant n-1}^{2\leqslant k\leqslant n-1}.$$

We now have to change the indices. We at first set $k' := k-1$ and $\kappa' := \kappa-1$ and the determinant just obtained becomes:

$$\Big| \sum_{1\leqslant\lambda_1\leqslant\kappa'+1} \sum_{1\leqslant\lambda_2\leqslant\lambda_1-1} \tbinom{\kappa'+1}{\lambda_1} \tbinom{\lambda_1-1}{\lambda_2} \cdot \\ \cdot \, (z^{k'})^{(\lambda_2)} \, z^{(2+\kappa'-\lambda_1)} \, z^{(\lambda_1-\lambda_2)} \Big|_{1\leqslant\kappa'\leqslant n-2}^{1\leqslant k'\leqslant n-2}.$$

Next, if we set $\lambda_1' := \lambda_1 - 1$ and if we observe the identification of sums:

$$\sum_{1\leqslant\lambda_1\leqslant\kappa'+1} \sum_{1\leqslant\lambda_2\leqslant\lambda_1-1} (\bullet) = \sum_{0\leqslant\lambda_1'\leqslant\kappa'} \sum_{1\leqslant\lambda_2\leqslant\lambda_1'} (\bullet) = \sum_{1\leqslant\lambda_2\leqslant\lambda_1'\leqslant\kappa'} (\bullet),$$

then our determinant simply becomes, after erasing the primes:

$$\Big| \sum_{1\leqslant\lambda_2\leqslant\lambda_1\leqslant\kappa} \tbinom{\kappa+1}{\lambda_1+1} \tbinom{\lambda_1}{\lambda_2} \cdot z^{(\kappa-\lambda_1+1)} \, z^{(\lambda_1-\lambda_2+1)} \cdot (z^k)^{(\lambda_2)} \Big|_{1\leqslant\kappa\leqslant n-2}^{1\leqslant k\leqslant n-2}.$$

Performing the same computational and transformational processes, the result of the next step will be:

$$\frac{\Delta}{n!(n-1)!(n-2)!\, z'(z')^2(z')^3} = \Big| \sum_{1\leqslant\lambda_3\leqslant\lambda_2\leqslant\lambda_1\leqslant\kappa} \tbinom{\kappa+2}{\lambda_1+2} \tbinom{\lambda_1+1}{\lambda_2+1} \tbinom{\lambda_2}{\lambda_3} \cdot \\ \cdot \, z^{(\kappa-\lambda_1+1)} \, z^{(\lambda_1-\lambda_2+1)} \, z^{(\lambda_2-\lambda_3+1)} \cdot (z^k)^{(\lambda_3)} \Big|_{1\leqslant\kappa\leqslant n-3}^{1\leqslant k\leqslant n-3}.$$

The induction is now clear, and at the end one obtains a 1×1 determinant $\big|(\bullet)\big|_{1\leqslant\kappa\leqslant 1}^{1\leqslant k\leqslant 1}$ with a sum $\sum_{1\leqslant\lambda_{n-1}\leqslant\cdots\leqslant\lambda_1\leqslant\kappa}(\bullet)$ inside which necessarily $k = \kappa = \lambda_1 = \cdots = \lambda_{n-1}$, so that this last 1×1 determinant equals:

$$\tbinom{n-1}{n-1} \cdots \tbinom{1}{1} \, z' \cdots z' \cdot z' = 1! \, (z')^n,$$

and this final observation completes the proof of the combinatorial lemma. □

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Repères méromorphes sur l'espace des jets verticaux de l'hypersurface universelle

[Repères méromorphes sur les jets de l'hypersurface universelle]

RÉSUMÉ. Pour des ordres de jets petits, on sait construire des repères méromorphes sur l'espace des des *jets verticaux* $J^k_{\mathsf{vert}}(\mathscr{X})$ de l'hypersurface universelle $\mathscr{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}}$ qui paramétrise toutes les hypersurfaces projectives $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ de degré d. En 2004, pour $k = n$, Siu a annoncé qu'il existe deux constantes $c_n \geqslant 1$ et $c'_n \geqslant 1$ telles que le fibré tangent tensorisé:

$$T_{J^n_{\mathsf{vert}}(\mathscr{X})} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}\big(c_n\big) \otimes \mathscr{O}_{\mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}}}\big(c'_n\big)$$

est engendré par ses sections globales. Dans cet article, nous établissons cette propriété hors d'un certain ensemble algébrique exceptionnel $\Sigma \subset J^n_{\mathsf{vert}}(\mathscr{X})$ défini par l'annulation de certains wronskiens, avec l'ordre de pôles *effectif* $c_n = \frac{n^2+5n}{2}$, retrouvant ainsi $c_2 = 7$ (Paŭn), $c_3 = 12$ (Rousseau), et avec $c'_n = 1$.

De plus, quitte à augmenter c_n jusqu'à $c_n = n^2 + 2n$, la même propriété d'engendrement est satisfaite hors du plus petit sous-ensemble $\widetilde{\Sigma} \subset \Sigma \subset J^n_{\mathsf{vert}}(\mathscr{X})$ qui est défini par l'annulation de tous les jets d'ordre 1. Des applications à la dégénérescence algébrique *faible* (avec Σ) et *forte* (avec $\widetilde{\Sigma}$) des courbes holomorphes entières $\mathbb{C} \to X$ en découleront prochainement.

Mots et phrases-clés. Formule de Faà di Bruno à plusieurs variables, Hypersurfaces projectives algébriques, Jets de courbes holomorphes, Dégénérescence algébrique faible et forte au sens de Green-Griffiths.

An algorithm to generate all polynomials in the κ-jet of a holomorphic disc $\mathbb{D} \to \mathbb{C}^n$ that are invariant under source reparametrization

Joël Merker

Abstract. A major unsolved problem (according to Demailly 1997) towards the Kobayashi hyperbolicity conjecture in optimal degree is to understand jet differentials of germs of holomorphic discs that are invariant under any reparametrization of the source. The underlying group action is not reductive, but we provide a complete algorithm to generate all invariants, in arbitrary dimension n and for jets of arbitrary order k.

Two main new situations are studied in great details. For jets of order 4 in dimension 4, we establish that the algebra of Demailly-Semple invariants is generated by 2835 polynomials, while the algebra of bi-invariants is generated by 16 mutually independent polynomials sharing 41 gröbnerized syzygies. Nonconstant entire holomorphic curves valued in an algebraic 3-fold (resp. 4-fold) $X^3 \subset \mathbb{P}^4(\mathbb{C})$ (resp. $X^4 \subset \mathbb{P}^5(\mathbb{C})$) of degree d satisfy global differential equations as soon as $d \geqslant 72$ (resp. $d \geqslant 259$). A useful asymptotic formula for the Euler-Poincaré characteristic of Schur bundles in terms of Giambelli's determinants is derived.

For jets of order 5 in dimension 2, we establish that the algebra of Demailly-Semple invariants is generated by 56 polynomials, while the algebra of bi-invariants is generated by 17 mutually independent polynomials sharing 105 gröbnerized syzygies.

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Table of contents

§1. Introduction

The Kobayashi hyperbolicity conjecture (1970), in optimal degree and taking account of Brody's theorem (1978), expects that all entire holomorphic curves $f : \mathbb{C} \to X$ into a complex projective (algebraic, smooth) hypersurface $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ must be constant if $\deg X \geqslant 2n+1$, provided X is generic. In 1980, Green and Griffiths conjectured that if $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ is of general type, which holds in degree $d \geqslant n+3$, then there is a proper algebraic subvariety $Y \subsetneq X$ which absorbs the image of all nonconstant entire holomorphic maps $f : \mathbb{C} \to X$, namely $f(\mathbb{C}) \subset Y$ necessarily. Correspondingly, an entire holomorphic $f : \mathbb{C} \to X$ will be called *algebraically degenerate* if its image is contained in some proper algebraic subvariety (which might depend on f).

Publications up to 2008 are still quite far from approaching the two optimal degrees $2n+1$ and $n+3$. For $X^2 \subset \mathbb{P}^3(\mathbb{C})$ very generic, such entire f's are known to be algebraically degenerate and even constant, in degree $d \geqslant 21$ (resp. $d \geqslant 18$) according to [6] (resp. [26]). For $X^3 \subset \mathbb{P}^4(\mathbb{C})$ very generic, algebraic degeneracy of such f's holds true in degree $d \geqslant 593$ according to [31]. For $X^4 \subset \mathbb{P}^5(\mathbb{C})$, a forthcoming work [11] applying the results of the present paper will obtain an effective degree lower bound for algebraic degeneracy; other applications to the logarithmic case also are imminent.

Quite unexpectedly, the two conjectures above and other similar problems as well in complex algebraic geometry happened in the last few years to pertain to purely algebraic problems, and not only to rely upon the scope of some soft techniques (pluripotential theory, currents, plurisubharmonic functions, *etc.*). Computational invariant theory should be expressly invoked here, as the present paper will show that what is at stake really is to find a complete description of the algebra of polynomials that are invariant under a certain Lie group action, which is *not* reductive.

Green-Griffiths Jet differentials. How can one figure out that a given nonconstant entire holomorphic map $f = \mathbb{C} \to X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ is constrained to be somehow degenerate by just being valued in X? Looking at its derivatives $f', f'', \ldots, f^{(k)}$ (in some jet-chart), one may expect at first to derive, by means of some suitable elimination process, sufficiently many *differential equations* which might presumably be due to the virtual guidance by some hidden $Y \subsetneq X$ absorbing $f(\mathbb{C})$.

For instance, for $X^2 \subset \mathbb{P}^3(\mathbb{C})$, the entire f's do satisfy (invariant) algebraic differential equations of order $k = 2$, resp. $k = 3$, resp. $k = 4$ when X has degree $d \geqslant 15$, resp. $d \geqslant 11$, resp. $d \geqslant 9$ according to [4, 6], resp. [29], resp. [21]. For $X^3 \subset \mathbb{P}^4(\mathbb{C})$, differential equations of order $k = 3$ enjoyed by any entire f exist when X has degree $d \geqslant 97$ ([30]).

Intrinsically speaking, consider the bundle J_k of k-jets of holomorphic curves $f : (\mathbb{D}, 0) \longrightarrow (X, x)$ centered at various points $x = f(0) \in X$. In the seminal article [17] (1980), Green and Griffiths introduced the fiber bundle $\mathsf{E}_{k,m}^{GG} T_X^* \to X$ of jet polynomials of order k and of weighted degree m whose fibers in some jet-chart are complex-valued polynomials $Q\big(f', f'', \dots, f^{(k)}\big)$ satisfying the weighted homogeneity:

$$Q\big(\lambda f',\, \lambda^2 f'',\, \dots,\, \lambda^k f^{(k)}\big) = \lambda^m\, Q\big(f',\, f'',\, \dots, f^{(k)}\big),$$

for every $\lambda \in \mathbb{C}^*$. Global sections of $\mathsf{E}_{k,m}^{GG} T_X^*$ over X are differential operators of order k. Elementary reasonings show ([17, 4, 32, 9]) that $\mathsf{E}_{k,m}^{GG} T_X^*$ is in fact a graded *vector* bundle isomorphic to the direct sum:

$$\bigoplus_{\ell_1 + 2\ell_2 + \cdots + k\ell_k = m} \mathrm{Sym}^{\ell_1} T_X^* \otimes \mathrm{Sym}^{\ell_2} T_X^* \otimes \cdots \otimes \mathrm{Sym}^{\ell_k} T_X^*.$$

Such a grading of $\mathsf{E}_{k,m}^{GG} T_X^*$ enables one ([17]) to derive from Hirzebruch's Riemann-Roch formula ([18]) a sharp asymptotic estimate for its Euler-Poincaré characteristic, namely:

$$\chi\left(X,\, \mathsf{E}_{k,m}^{GG} T_X^*\right) = \frac{m^{(k+1)n-1}}{(k!)^n\, \big((k+1)n-1\big)!} \left(\frac{(-1)^n}{n!}\, \mathsf{c}_1(X)^n\, (\log k)^n + \mathrm{O}\big((\log k)^{n-1}\big) \right) + \\ + \mathrm{O}\big(m^{(k+1)n-2}\big).$$

This formula and the knowledge of the expression of the n-th power of the first Chern class (implicitly integrated over X) in terms of the degree:

$$(-1)^n\, \mathsf{c}_1(X)^n = (d - n - 2)^n\, d$$

entails that, as the jet order k tends to ∞, the characteristic $\chi\big(X,\, \mathsf{E}_{k,m}^{GG} T_X^*\big)$ becomes eventually positive for m large enough, as soon as $\deg X \geqslant n + 3$. Thus, up to a constant factor, c_1^n *becomes the dominant term of* $\chi\left(X,\, \mathsf{E}_{k,m}^{GG} T_X^*\right)$ *when* $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ *is of general type*.

Demailly-Semple invariant jet differentials. In 1997, inspired also by an older paper of Semple, Demailly introduced a subbundle of $\mathsf{E}_{k,m}^{GG} T_X^*$ having better positivity properties and exhibiting a nice, stepwise compactification process.

With $\mathbb{D} \subset \mathbb{C}$ denoting any nonempty open disc centered at 0 (possibly $\mathbb{D} = \mathbb{C}$), consider a nonconstant holomorphic curve $f : \mathbb{D} \to X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$. Of course, $f'(\zeta)$ then belongs to the tangent space $T_{X,f(\zeta)}$ for every $\zeta \in \mathbb{D}$. The projectivization $[f'(\zeta)] \in PT_{X,f(\zeta)}$ therefore belongs to the projectivized bundle of tangent lines to X, so that one gratuitously obtains a lifting $f_{[1]} := \big(f,\, [f']\big) : \mathbb{D} \longrightarrow P(T_X)$, at least for all ζ with $f'(\zeta) \neq 0$. Here, $P(T_X)$ is $(2n-1)$-dimensional, but the so lifted holomorphic curve $f_{[1]}$ happens to be guided by a certain n-dimensional subbundle of $P(T_X)$, better seen as follows.

Abstractly and generally speaking, let Y be a complex manifold, let $V \subset T_Y$ be any vector subbundle and call (Y, V) a *directed manifold*. Define $Y' := P(V)$ the projectivized bundle of lines contained in the vector subbundle $V \subset TY$ with of course $\dim Y' = \dim Y + \operatorname{rk} V - 1$. It is equipped with a natural projection $\pi : Y' \to Y$ which enables one to introduce the *lifted subbundle* $V' \subset T_{Y'}$, the fiber of which, at an arbitrary point $(x, [v]) \in Y'$, is precisely defined by:

$$V'_{(x,[v])} := \big\{ v' \in T_{X'} : d\pi(v') \in \mathbb{C}v \big\},$$

and the rank of which is clearly untouched: $\operatorname{rk} V' = \operatorname{rk} V$. Most importantly, any nonconstant holomorphic $f : \mathbb{D} \to Y$ constrained to be V-tangent, namely to satisfy $f'(\zeta) \in V_{f(\zeta)}$ for all $\zeta \in \mathbb{D}$, may be shown ([4]) to lift automatically, even at points ζ where $f'(\zeta)$ vanishes, as a map $f_{[1]} : \mathbb{D} \to Y'$ which is also constrained to be V'-tangent, namely which necessarily satisfies $f'_{[1]}(\zeta) \in V'_{f_{[1]}(\zeta)}$ for all $\zeta \in \mathbb{D}$. So lifting to a higher dimensional manifold keeps memory of the original guidance on the base.

Starting therefore with $Y = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ and with $V = T_X$, setting $X_0 := X$, $V_0 := T_X$, one defines first ([4]) $X_1 := P(T_X)$, $V_1 = V'$ and then inductively $(X_l, V_l) := (X'_{l-1}, V'_{l-1})$ with natural projections $\pi_{l,l-1} : X_l \to X_{l-1}$. One then assembles everything for $l = 0$ to $l = k$ as a tower of projectivized bundles with total projection $\pi_{0,k} : X_k \to X$ and with intermediate projections $\pi_{j,l} : X_l \to X_j$, for any $0 \leqslant j \leqslant l \leqslant k$. By applying inductively the above lifting operator $f_{[l]} := \big(f_{[l-1]}\big)_{[1]}$, every nonconstant holomorphic curve $f : \mathbb{D} \to X$ gives rise to lifts $f_{[l]} : \mathbb{D} \to X_l$ for all $l = 0, 1, \ldots, k$. Each of these lifts is guided by V_l, namely $f'_{[l]}(\zeta) \in V_{l, f'_{[l]}(\zeta)}$ for all $\zeta \in \mathbb{D}$.

At each level $l \geqslant 1$, we have a tautological line bundle $\mathscr{O}_{X_l}(-1)$ over $X_l = P(V_{l-1})$ whose fiber at a point $\big(x_{l-1}, [v_{l-1}]\big) \in P(V_{l-1})$ just consists of the line $\mathbb{C} \cdot v_{l-1}$ directed by (a representative of) $[v_{l-1}]$, and similarly as in the projective spaces, one may build the basic bundles $\mathscr{O}_{X_l}(q)$ for every $q \in \mathbb{Z}$.

Now, the bundle of invariant jet differentials of order k and of weighted degree m is the subbundle[33] $\mathsf{E}^n_{k,m} T^*_X$ of $\mathsf{E}^{GG}_{k,m} T^*_X$ whose fibers at a point $x \in X$ consist of polynomial differential operators $Q\big(f', f'', \ldots, f^{(k)}\big)$ which, under arbitrary local reparametrization $\phi : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ of the source with $\phi(0) = 0$, satisfy the general invariancy condition:

$$Q\big((f \circ \phi)', (f \circ \phi)'', \ldots, (f \circ \phi)^{(k)}\big) = \phi'(0)^m \, Q\big(f', f'', \ldots, f^{(k)}\big),$$

not only under rescaling-like changes of coordinates $\zeta \mapsto \lambda \zeta$ with $\lambda \in \mathbb{C}^*$. This apparently neat definition hides several algebraic objects which will be

[33] Because the dimension n will vary often in our study, it must be indicated as an exponent in the notation of the Demailly-Semple bundle.

inspected and explored in length throughout the present article. Comparing the two bundles:

$$\begin{array}{ccc} X_k & \longrightarrow & \mathscr{O}_{X_k}(m) \\ {\scriptstyle \pi_{0,k}}\downarrow & & \downarrow \\ X & \longrightarrow & \mathsf{E}^n_{k,m}T^*_X\,, \end{array}$$

over X and over X_k, one establishes ([4]) the direct image formula

$$(\pi_{0,k})_*\mathscr{O}_{X_k}(m) = \mathscr{O}\big(\mathsf{E}^n_{k,m}T^*_X\big).$$

Existence of global algebraic differential equations. What then are the global algebraic differential equations that nonconstant entire maps $f : \mathbb{C} \to X$ could satisfy? As the hypersurface X lives in $\mathbb{P}^{n+1}(\mathbb{C})$, it carries many ample line bundles, *e.g.* any $\mathscr{O}_{X_k}(q)$ with $q \geqslant 1$.

([17, 4]) *Fix an ample line bundle $A \to X$ and assume that $\mathsf{E}^n_{k,m}T^*_X \otimes A^{-1}$ has nonzero sections, namely:*

$$h^0\big(X,\, \mathsf{E}^n_{k,m}T^*_X \otimes A^{-1}\big) = \dim H^0(\mathsf{same}) \geqslant 1.$$

Then for every global invariant operator $P \in \Gamma\big(X,\, \mathsf{E}^n_{k,m}T^*_X \otimes A^{-1}\big)$ valued in A^{-1}, any entire holomorphic curve f must satisfy the algebraic differential equation $P\big(f_{[k]}\big) \equiv 0$. A similar result also holds true for the larger bundle $\mathsf{E}^{GG}_{k,m}T^*_X$.

How then one can guarantee the existence of such sections P? Because X is elementarily seen to be of general type when $d \geqslant n+3$, it is expected ([4, 29]) in a first moment that the Euler-Poincaré characteristic of the Demailly-Semple subbundle $\mathsf{E}^n_{k,m}T^*_X$ should behave in a way quite similar to the Green-Griffiths bundle, with c_1^n becoming the dominant term (up to a constant factor) as k and m both tend to ∞, so that $\chi\big(X,\, \mathsf{E}^n_{k,m}T^*_X\big)$ should be eventually large, and furthermore in a second moment, it is also expected that the dimension of the space of global sections $H^0\big(X,\, \mathsf{E}^n_{k,m}T^*_X\big)$ should be eventually large, due to some vanishing or to some control of the higher order cohomology groups. The truth of such conjectural expectations would presumably open new routes towards a solution in optimal degree to the two above-mentioned conjectures.

Seeking Schur bundle decomposition of $\mathsf{E}^n_{k,m}T^*_X$. However, as is written in [4], it is a major unsolved problem to find the decomposition of $\mathsf{E}^n_{k,m}T^*_X$ into direct sums of the irreducible Schur bundles $\Gamma^{(\ell_1,\ell_2,\ldots,\ell_n)}T^*_X$ with $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n$ that are the basic bricks and whose cohomology is somehow currently available. According to a possible strategy developed for $k = n = 3$ mainly by Rousseau in [29, 30], such a decomposition would yield access to the Euler characteristic $\chi\big(X,\, \mathsf{E}^n_{k,m}T^*_X\big)$, and then afterwards, one would

attain an effective estimate of $h^0\big(X,\, \mathsf{E}_{k,m}^n T_X^*\big)$, provided one controls the other cohomology groups. In fact, the only decompositions known up to now are the following; the second one ([29]) already required a nontrivial argument based on a theorem of Popov about polarization of multilinear invariants.

- For $n = k = 2$ ([4]):

$$\mathsf{E}_{2,m}^2 T_X^* = \bigoplus_{a+3b=m} \Gamma^{(a+b,\, b)}\, T_X^*.$$

- For $n = k = 3$ and also for $n = 2$, $k = 3$ ([29]):

$$\mathsf{E}_{3,m}^3 T_X^* = \bigoplus_{a+3b+5c+6d=m} \Gamma^{(a+b+2c+d,\, b+c+d,\, d)}\, T_X^*,$$
$$\text{and} \quad \mathsf{E}_{3,m}^2 T_X^* = \bigoplus_{a+3b+5c=m} \Gamma^{(a+b+2c,\, b+c)}\, T_X^*.$$

- For $n = 2$, $k = 4$ ([21]):

$$\mathsf{E}_{4,m}^2 T_X^* = \bigoplus_{a+3b+5c+8e=m} \Gamma^{(a+b+2c+2e,\, b+c+2e)}\, T_X^* \bigoplus_{7+a+5c+7d+8e=m} \Gamma^{(3+a+2c+3d+2e,\, 1+c+d+2e)}\, T_X^*.$$

In this paper, we mainly attack the case $n = k = 4$. The complexity increases suddenly and we seem to be still quite far from being able to push the jet order k to ∞.

Theorem. *On a smooth complex algebraic hypersurface $X^4 \subset \mathbb{P}^5(\mathbb{C})$, the graduate m-th part $\mathsf{E}_{4,m}^4 T_X^*$ of the Demailly-Semple bundle $\mathsf{E}_4^4 T_X^* = \oplus_m \mathsf{E}_{4,m}^4 T_X^*$ has the following decomposition in direct sums of Schur bundles:*

$$\mathsf{E}_{4,m}^4 T_X^* = \bigoplus_{\substack{(a,b,\ldots,n)\in\mathbb{N}^{14}\setminus(\square_1\cup\cdots\cup\square_{41}) \\ o+3a+\cdots+21n+10p=m}} \Gamma\begin{pmatrix} o+a+2b+3c+d+2e+3f+2g+2h+3i+4j+3k+3l+4m'+5n+p \\ a+b+c+d+e+f+2g+2h+2i+2j+2k+3l+3m'+3n+p \\ d+e+f+h+i+j+2k+2l+2m'+2n+p \\ p \end{pmatrix} T_X^*,$$

where the 41 subsets $\square_i$, $i = 1, 2, \ldots, 41$, of $\mathbb{N}^{14} \ni (a, b, \ldots, l, m', n)$ are explicitly defined in the complete statement on p. 269.

It is known ([30]) that $\mathsf{E}_{k,m}^3 T_X^*$ has no nonzero sections for jet order $k = 1$ or $k = 2$. More generally ([9]), for jet order $k \leqslant n-1$ strictly smaller than the dimension, sections are never available: $H^0\big(X,\, \mathsf{E}_{k,m}^n T_X^*\big) = 0$. Consequently, even if one may easily deduce from the above theorem a Schur decomposition of $\mathsf{E}_{4,m}^3 T_X^*$, for applications to hyperbolicity in dimension

higher than 3, one should always start with jet order k at least equal to the dimension[34]. The case $n = k = 4$ was the first unknown one before.

Asymptotic expansion of Euler-Poincaré characteristic. Because the characteristic is just additive on direct sums of vector bundles, knowing a representation of $\mathsf{E}^n_{k,m}T^*_X$ (for certain values of n, k, *e.g.* for $n = k = 4$) as a direct sum of certain Schur bundle is very convenient, provided of course that one already possesses an asymptotic for the Euler-Poincaré characteristic of the $\Gamma^{(\ell_1,\dots,\ell_n)}T^*_X$ as $\ell_1+\cdots+\ell_n \to \infty$. Section 13 will derive an explicit asymptotic for which there seems to be no reference with a precise enunciation (compare [1, 28]). Because of the relations $\mathsf{c}_k\big(T^*_X\big) = (-1)^k\mathsf{c}_k\big(T_X\big)$, there is no loss of generality to express everything in terms of the Chern classes of the tangent bundle T_X.

Theorem. *The terms of highest order with respect to* $|\ell| = \max_{1\leqslant i\leqslant n}\ell_i$ *in the Euler-Poincaré characteristic of the Schur bundle* $\Gamma^{(\ell_1,\ell_2,\dots,\ell_n)}\,T_X$ *are homogeneous of order* $\mathrm{O}\big(|\ell|^{\frac{n(n+1)}{2}}\big)$ *and they are given by a sum of* ℓ'_i*-determinants indexed by all the partitions* $(\lambda_1,\dots,\lambda_n)$ *of* n*:*

$$\chi\Big(X,\ \Gamma^{(\ell_1,\ell_2,\dots,\ell_n)}\,T_X\Big) =$$

$$= \sum_{\lambda \text{ partition of } n} \frac{\mathsf{C}_{\lambda^c}}{(\lambda_1+n-1)!\cdots\lambda_n!}\begin{vmatrix} {\ell'_1}^{\lambda_1+n-1} & {\ell'_2}^{\lambda_1+n-1} & \cdots & {\ell'_n}^{\lambda_1+n-1} \\ {\ell'_1}^{\lambda_2+n-2} & {\ell'_2}^{\lambda_2+n-2} & \cdots & {\ell'_n}^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ {\ell'_1}^{\lambda_n} & {\ell'_2}^{\lambda_n} & \cdots & {\ell'_n}^{\lambda_n} \end{vmatrix} +$$

$$+\,\mathrm{O}\Big(|\ell|^{\frac{n(n+1)}{2}-1}\Big),$$

where $\ell'_i := \ell_i + n - i$ *for notational brevity, with coefficients* C_{λ^c} *being expressed in terms of the Chern classes* $\mathsf{c}_k\big(T_X\big) = \mathsf{c}_k$ *of* T_X *by means of* Giambelli's determinantal expression *depending upon the* conjugate *partition* λ^c*:*

$$\mathsf{C}_{\lambda^c} = \mathsf{C}_{(\lambda^c_1,\dots,\lambda^c_n)} = \begin{vmatrix} \mathsf{c}_{\lambda^c_1} & \mathsf{c}_{\lambda^c_1+1} & \mathsf{c}_{\lambda^c_1+2} & \cdots & \mathsf{c}_{\lambda^c_1+n-1} \\ \mathsf{c}_{\lambda^c_2-1} & \mathsf{c}_{\lambda^c_2} & \mathsf{c}_{\lambda^c_2+1} & \cdots & \mathsf{c}_{\lambda^c_2+n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathsf{c}_{\lambda^c_n-n+1} & \mathsf{c}_{\lambda^c_n-n+2} & \mathsf{c}_{\lambda^c_n-n+3} & \cdots & \mathsf{c}_{\lambda^c_n} \end{vmatrix},$$

on understanding by convention that $\mathsf{c}_k := 0$ *for* $k < 0$ *or* $k > n$*, and that* $\mathsf{c}_0 := 1$.

Effective calculations of characteristics in dimensions 3 and 4. We then perform electronically assisted computations to obtain the desired, quite complicated value of the characteristic of $\mathsf{E}^4_{4,m}T^*_X$.

[34] Nonetheless, we ignore whether the case $n = k = 5$ is accessible to us.

Theorem. *If $X^4 \subset \mathbb{P}^5(\mathbb{C})$ is a degree d smooth algebraic 4-fold, then as $m \to \infty$, one has the asymptotic:*

$$\begin{aligned}\chi\big(X,\, \mathsf{E}^4_{4,m}T^*_X\big) = {}& \frac{m^{16}}{1313317832303894333210335641600000000000000} \cdot d \cdot \\ & \cdot \big(5004851113579703425623 5\, d^4 - \\ & - 617060662250595525598878 6\, d^3 - \\ & - 928886901354141153880624704\, d + \\ & + 1411704752502476621473639 41\, d^2 + \\ & + 1624908955061039283976041114\big) \\ & + \mathrm{O}\big(m^{15}\big).\end{aligned}$$

Furthermore, the coefficient of m^{16} here, a factorized polynomial of degree 5 with respect to d, is positive in all degrees $d \geqslant 96$.

For $n = k = 3$, based on his above-mentioned Schur decomposition of $\mathsf{E}^3_{3,m}T^*_X$, Rousseau ([29]) showed that:

$$\chi\big(X,\, \mathsf{E}^3_{3,m}T^*_X\big) = \frac{m^9}{81648000000} \cdot d \cdot \big(389d^3 - 20739d^2 + 185559d - 358873\big) + \mathrm{O}\big(m^8\big),$$

and that the coefficient of m^9 is positive for all degrees $d \geqslant 43$. Furthermore, in [28], Rousseau showed that $h^2\big(X,\, \mathrm{Sym}^m\, T^*_X\big) = \big(-\frac{7}{24}d + \frac{1}{8}d^2\big)\, m^5 + \mathrm{O}\big(m^4\big)$ in any degree $d \geqslant 6$, so that one cannot expect second cohomology groups to vanish. Afterwards, as the main objective of the paper [30], he first established the general majoration:

$$h^2\big(X,\, \Gamma^{(\ell_1,\ell_2,\ell_3)}T^*_X\big) \leqslant d(d+13)\, \frac{3(\ell_1+\ell_2+\ell_3)^3}{2}\, (\ell_1-\ell_2)(\ell_1-\ell_3)(\ell_2-\ell_3) + \mathrm{O}\big(|\ell|^5\big).$$

he then deduced by summation from the cited decomposition $\mathsf{E}^3_{3,m}T^*_X = \bigoplus_{a+3b+5c+6d=m} \Gamma^{(a+b+2c+d,\, b+c+d,\, d)}T^*_X$ that:

$$h^2\big(X,\, \mathsf{E}^3_{3,m}T^*_X\big) \leqslant \frac{49403}{252 \cdot 10^7}\, d(d+13)\, m^9 + \mathrm{O}\big(m^8\big),$$

and finally, by applying the trivial minoration:

$$h^0\big(X,\, \mathsf{E}^3_{4,m}T^*_X\big) \geqslant \chi\big(X,\, \mathsf{E}^3_{4,m}T^*_X\big) - h^2\big(X,\, \mathsf{E}^3_{4,m}T^*_X\big),$$

stemming from the definition $\chi = h^0 - h^1 + h^2 - h^3$, he immediately deduced the minoration:

$$h^0\big(X,\, \mathsf{E}^3_{3,m}T^*_X\big) \geqslant \frac{m^9}{408240000000} \cdot d \cdot \big(1945\, d^3 - 103695\, d^2 - 7075491\, d - 105837083\big) + \mathrm{O}\big(m^8\big),$$

in which the coefficient of m^9 is checked (again electronically) to be positive in all degrees $d \geqslant 97$. As a result, nontrivial sections of $\mathsf{E}^3_{3,m}T^*_X$ exist when $\deg X \geqslant 97$.

For jets of order 4 in dimension 3, when applying in dimension 3 our decomposition of $\mathsf{E}^3_{4,m}T^*_X$ into Schur bundles which appears in the theorem on p. 269, a Maple computation using the cited majoration formula for

$h^2\big(X,\, \Gamma^{(\ell_1,\ell_2,\ell_3)}T^*_X\big)$ then provides:

$$h^2\big(X,\, \mathsf{E}^3_{4,m}T^*_X\big) \leqslant d(d+13)\,\frac{342988705758851}{29822568148961280000000}\,m^{11} + \mathrm{O}\big(m^{10}\big).$$

Theorem. *The asymptotic, as $m \to \infty$, of the Euler-Poincaré characteristic of the Demailly-Semple bundle $\mathsf{E}^4_{4,m}T^*_X$ on a degree d smooth projective algebraic 3-fold $X^3 \subset \mathbb{P}^4(\mathbb{C})$ is given by:*

$$\begin{aligned}\chi\big(X,\, \mathsf{E}^3_{4,m}T^*_X\big) = {}& \frac{m^{11}}{20613359104562036736000000000} \cdot d \cdot \big(1029286103034112\, d^3 - \\ & - 38980726828290305\, d^2 + 299551055917162501\, d - 561169562618151944\big) + \\ & + \mathrm{O}\big(m^{10}\big),\end{aligned}$$

and the coefficient of m^{11} here is positive in all degrees $d \geqslant 29$. Furthermore, subtracting to this asymptotic the above majorant of $h^2\big(X,\, \mathsf{E}^3_{4,m}T^*_X\big)$:

$$\begin{aligned}h^0\big(X,\, \mathsf{E}^3_{4,m}T^*_X\big) \geqslant {}& \frac{m^{11}}{20613359104562036736000000000} \cdot d \cdot \big(1029286103034112\, d^3 - \\ & - 38980726828290305\, d^2 + 2071186878288015611\, d - 31380762707285467400\big) + \\ & + \mathrm{O}\big(m^{10}\big),\end{aligned}$$

and the modified coefficient here of m^{11} is now positive in all degrees $d \geqslant 72$.

This last condition $d \geqslant 72$ on the degree insuring the existence of global invariant jet differentials of order $\kappa = 4$ on $X^3 \subset \mathbb{P}^4(\mathbb{C})$ improves the condition $d \geqslant 97$ obtained in [30] and appears to be slightly better than the condition $d \geqslant 74$ obtained more recently in [9] with another approach. A number of further numerical applications, especially to the logarithmic case, shall appear soon ([11]); as will be seen in a near future, in dimension 4, the lower bound on the degree $d \geqslant 259$ for the existence of sections which will based on the present approach will also improve the bound $d \geqslant 298$ obtained in [9]. Nonetheless, we must stop at this point in order to describe the main contribution of the present article. Last but not least, we cannot go beyond without mentioning that Siu's strategy for establishing algebraic degeneracy ([35, 26, 31]) will also bring further fruits thanks to the recent construction of a global meromorphic framing on the space of vertical n-jets tangent to the universal hypersurface in arbitrary dimension n ([22]).

A problem in invariant theory. Now, how does one obtain Schur decompositions of Demailly-Semple bundles? To begin with, we show how one can understand the condition of being invariant under reparametrization in terms of classical invariant theory.

Let us from now on denote by κ (instead of k) the jet order and let us abbreviate $j^\kappa f = \big(f', f'', \ldots, f^{(\kappa)}\big)$.

The group G_κ of κ-jets at the origin of local reparametrizations $\phi(\zeta) = \zeta + \phi''(0)\,\frac{\zeta^2}{2!} + \cdots + \phi^{(\kappa)}(0)\,\frac{\zeta^\kappa}{\kappa!} + \cdots$ that are tangent to the identity,

namely which satisfy $\phi'(0) = 1$, may be seen to act linearly on the $n\kappa$-tuples $\big(f'_{j_1}, f''_{j_2}, \dots, f^{(\kappa)}_{j_\kappa}\big)$ by plain matrix multiplication, *i.e.* when we set $g_i^{(\lambda)} := \big(f_i \circ \phi\big)^{(\lambda)}$, a computation applying the chain rule gives for each index i:

$$\begin{pmatrix} g_i' \\ g_i'' \\ g_i''' \\ g_i'''' \\ \vdots \\ g_i^{(\kappa)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \phi'' & 1 & 0 & 0 & \cdots & 0 \\ \phi''' & 3\phi'' & 1 & 0 & \cdots & 0 \\ \phi'''' & 4\phi''' + 3\phi''^2 & 6\phi'' & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi^{(\kappa)} & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} f_i' \circ \phi \\ f_i'' \circ \phi \\ f_i''' \circ \phi \\ f_i'''' \circ \phi \\ \vdots \\ f_i^{(\kappa)} \circ \phi \end{pmatrix} \qquad {\scriptstyle (i=1\cdots n).}$$

Polynomials $\mathsf{P}\big(j^\kappa f\big)$ invariant by reparametrization satisfy by definition for some integer m:

$$\mathsf{P}\big(j^\kappa g\big) = \mathsf{P}\big(j^\kappa (f \circ \phi)\big) = \phi'(0)^m \cdot \mathsf{P}\big((j^\kappa f) \circ \phi\big) = \mathsf{P}\big((j^\kappa f) \circ \phi\big),$$

for any ϕ. If we denote by $\mathsf{E}^n_{\kappa,m}$ the vector space consisting of such polynomials, the direct sum $\mathsf{E}^n_\kappa = \oplus_{m \geqslant 1} \mathsf{E}^n_{\kappa,m}$ forms an algebra graded by constancy of weights: $\mathsf{E}^n_{\kappa,m_1} \cdot \mathsf{E}^n_{\kappa,m_2} \subset \mathsf{E}^n_{\kappa,m_1+m_2}$.

Then obviously when $\phi'(0) = 1$, the algebra E^n_κ just coincides with the algebra of invariants for the linear group action represented by the group of matrices just written:

$$\mathsf{P}\big(j^\kappa g\big) = \mathsf{P}\big(\mathsf{M}_{\phi'',\phi''',\dots,\phi^{(\kappa)}} \cdot j^\kappa f\big) = \mathsf{P}\big(j^\kappa f\big),$$

with $\phi'', \phi''', \dots, \phi^{(\kappa)}$ interpreted as arbitrary complex constants. Such a group clearly has dimension $\kappa - 1$.

But unfortunately, this group of matrices is a subgroup of the full unipotent group, hence it is *non-reductive*, and for this reason, it is impossible to apply almost anything from the so well developed invariant theory of reductive actions ([7]). Moreover, though the invariants of the full unipotent group are well understood, as soon as one looks at a *proper* subgroup of it, formal harmonies happen to be rapidly destroyed.

We ignore whether the algebra of invariants is finitely generated, in general. But in all previously known cases (carefully reminded below) and in all further new cases studied in this paper, E^n_κ is finitely generated. We will establish that the (graded) algebra $\mathsf{E}^4_4 = \oplus_m \mathsf{E}^4_{4,m}$ is generated by 2835 invariant polynomials and that $\mathsf{E}^2_5 = \oplus_m \mathsf{E}^2_{5,m}$ is generated by 56 invariant polynomials. We will also provide, in the theorem stated in length on p. 239, a *general algorithm* which, *in arbitrary dimension n and for arbitrary jet order κ, generates all invariants by adding a new invariant only when it cannot be expressed as a polynomial with respect to the already known invariants, and which stops after a finite number of loops if and only if $\mathsf{E}^n_\kappa = \oplus_m \mathsf{E}^n_{\kappa,m}$ is finitely generated as an algebra*.

Insufficiency of bracketing. By definition, a polynomial $\mathsf{P}\big(j^\kappa f\big)$ in the κ-th order jet space which is invariant by reparametrization must satisfy $\mathsf{P}\big(j^\kappa((f\circ\phi)\big) = \phi'^m\,\mathsf{P}\big(j^\kappa f\big)\circ\phi\big)$ for every biholomorphism $\phi:(\mathbb{D},0)\longrightarrow(\mathbb{D},0)$, where the integer m is called the *weight* of P, and where it is implicitly understood that the base point is the origin. Also, suppose next that Q is another invariant of weight n in the τ-th order jet space, *i.e.* satisfying $\mathsf{Q}\big(j^\tau(f\circ\phi)\big) = \phi'^n\,\mathsf{Q}\big((j^\tau f)\circ\phi\big)$. If $\mathsf{D} := \sum_{k=1}^n \sum_{\lambda\in\mathbb{N}} \frac{\partial(\bullet)}{\partial f_k^{(\lambda)}}\cdot f_k^{(\lambda+1)}$ denotes the *total differentiation operator*, which acts on any polynomial in $f', f'', \dots, f^{(\kappa)}$ as if it differentiated it with respect to the (virtual) source variable $\zeta\in\mathbb{D}$, then the *bracket expression*:

$$\big[\mathsf{P},\,\mathsf{Q}\big] := n\,\mathsf{DP}\cdot\mathsf{Q} - m\,\mathsf{P}\cdot\mathsf{DQ}$$

will easily be checked (in §3) to provide gratuitously another invariant of weight $m+n+1$ in the jet space of order $1+\max(\kappa,\tau)$.

For jet order $\kappa=1$, the algebra of invariants is just $\mathbb{C}\big[f_1', f_2', \dots, f_n'\big]$. For $\kappa=2$, the algebra E_2^n is generated by the f_i' together with the two-dimensional Wronskians $f_i'f_j''-f_i''f_j'$ which identify to the brackets $\big[f_j',\,f_i'\big]$, where $1\leqslant i,j\leqslant n$.

For $\kappa=3$ in dimension $n=2$, the Demailly-Semple algebra E_3^2 is generated by 5 mutually independent invariants:

$$f_1', \quad f_2', \quad \Lambda^3 := \big[f_2',\,f_1'\big], \quad \Lambda_1^5 := \big[\Lambda^3,\,f_1'\big], \quad \Lambda_2^5 := \big[\Lambda^3,\,f_2'\big],$$

which all are furnished by just bracketing, according to [29]; (but bracketing did not enter the scene there).

In the next dimension $n=3$ for jets of the same order $\kappa=3$, the Demailly-Semple algebra E_3^3 is generated by 16 mutually independent invariants ([29]), namely the $3+3+9=15$ following ones:

$$f_i', \quad \Lambda_{i,j}^3 := \big[f_j',\,f_i'\big], \quad \Lambda_{i,j;\,k}^5 := \big[\Lambda_{i,j}^3,\,f_k'\big],$$

where $1\leqslant i<j\leqslant 3$ and where $1\leqslant k\leqslant 3$, which are clearly all obtained by bracketing some invariants from the preceding jet level, together with the three-dimensional Wronskian:

$$D_{1,2,3}^6 := \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix},$$

which also appears, though after some division by f_1', to come from the brackets, for one checks by direct calculation the three relations:

$$\big[\Lambda_{1,2}^3,\,\Lambda_{1,3}^3\big] = -3\,f_1'\,D_{1,2,3}^6, \qquad \big[\Lambda_{1,2}^3,\,\Lambda_{2,3}^3\big] = -3\,f_2'\,D_{1,2,3}^6,$$
$$\big[\Lambda_{1,3}^3,\,\Lambda_{2,3}^3\big] = -3\,f_3'\,D_{1,2,3}^6.$$

Here, as the reader may have observed already, we always put the weight of every invariant at the upper index place.

Lastly, coming back to the dimension $n = 2$, for jet order $\kappa = 4$, the algebra E_4^2 is generated by the 9 mutually independent invariants ([5, 21]):

$$\begin{aligned}
&f_1', \qquad f_2', \qquad \Lambda^3_{1,2}, \qquad \Lambda^5_{1,2;\,1}, \qquad \Lambda^5_{1,2;\,2},\\
&\Lambda^7_{1,1} := \big[\Lambda^5_{1,2;\,1},\, f_1'\big], \qquad \Lambda^7_{1,2} := \big[\Lambda^5_{1,2;\,1},\, f_2'\big] = \big[\Lambda^5_{1,2;\,2},\, f_1'\big] = \Lambda^7_{2,1},\\
&\Lambda^7_{2,2} := \big[\Lambda^5_{1,2;\,2},\, f_2'\big], \qquad M^8 := \frac{1}{f_1'}\,\big[\Lambda^5_{1,2;\,1},\, \Lambda^3_{1,2}\big],
\end{aligned}$$

coming again all from bracketing, possibly allowing a division by f_1'.

In view of all these positive results, one could believe that bracketing (with possible division) always generate all invariants when passing from one jet level to the subsequent one. In fact, the two so-called *sigma-* and *Omega-processes* are known to generate all the invariants of binary forms in any degree ([4, 7, 27]).

Unfortunately, in [21], we discovered that in dimension $n = 2$ for jet order $\kappa = 5$, many invariants exist which are totally independent from the ones obtained by bracketing the invariants existing at the inferior jet levels $\kappa \leqslant 4$. Section 8 will provide more explanations, emphasizing that *it is by no means possible to derive these further invariants by dividing any incoming bracket-invariant by any other already known (bracket) invariant*.

Nonetheless, there could exist a second (and even a third) algebraically uniform process which would generate gratuitously many other invariants, and which, in cooperation with the bracketing process, would be complete, but regarding such an idea, we must confess our ignorance.

Initial rational expression for invariants. Hopefully, the algorithm we already devised (and hid slightly?) in [21] provides another route. How does it work?

To begin with, we define $\Lambda^3_{1,i} := \big[f_i', f_1'\big]$ and then by induction for any λ with $3 \leqslant \lambda \leqslant \kappa$:

$$\Lambda^{2\lambda-1}_{1,i;\,1^{\lambda-2}} := \big[\Lambda^{2\lambda-3}_{1,i;\,1^{\lambda-3}},\, f_1'\big].$$

Being built by bracketing, these are invariants of weight $2\lambda - 1$ for any $i = 1, \ldots, n$. The power $\lambda - 2$ of 1 counts the number of brackets with f_1', starting from the Wronskian $\Lambda^3_{1,i}$.

The preliminary step is to establish a *rational* representation of any invariant polynomial as a sum of polynomials in terms of f_1' and of the $\Lambda^{2\lambda-1}_{1,i;\,1^{\lambda-2}}$, $2 \leqslant i \leqslant n$, $1 \leqslant \lambda \leqslant \kappa$, a representation in which f_1' is allowed to have possibly negative powers $(f_1')^a$ with $-\frac{\kappa-1}{\kappa} m \leqslant a \leqslant m$. The following basic statement will appear in §5.

Lemma. *In dimension $n \geqslant 1$ and for jets of order $\kappa \geqslant 1$, every polynomial $\mathsf{P} = \mathsf{P}\big(j^\kappa f\big)$ invariant by reparametrization writes under the form:*

$$\boxed{\mathsf{P}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa} m \leqslant a \leqslant m} (f_1')^a \, \mathsf{P}_a \begin{pmatrix} f_2', & f_3', & f_4', & \dots, & f_n', \\ \Lambda^3_{1,2}, & \Lambda^3_{1,3}, & \Lambda^3_{1,4}, & \dots, & \Lambda^3_{1,n}, \\ \dots & \dots & \dots & \dots & \dots \\ \Lambda^{2\kappa-1}_{1,2;\,1^{\kappa-2}}, & \Lambda^{2\kappa-1}_{1,3;\,1^{\kappa-2}}, & \Lambda^{2\kappa-1}_{1,4;\,1^{\kappa-2}}, & \dots, & \Lambda^{2\kappa-1}_{1,n;\,1^{\kappa-2}} \end{pmatrix},}$$

where the integer a takes all possibly negative values in the interval $\big[-\frac{\kappa-1}{\kappa}\, m, m\big]$, for certain weighted homogeneous polynomials:

$$\mathsf{P}_a = \sum_{\substack{b_2+\cdots+b_n+3c_2+\cdots+3c_n+ \\ +\cdots+(2\kappa-1)q_2+\cdots+(2\kappa-1)q_n=m-a}} \mathsf{coeff} \cdot \prod_{i=2}^n \big(F_i\big)^{b_i} \prod_{i=2}^n \big(A_i^3\big)^{c_i} \cdots \prod_{i=2}^n \big(A_i^{2\kappa-1}\big)^{q_i}$$

of weighted degree $m-a$, namely satisfying:

$$\mathsf{P}_a\Big(\delta\, F_i,\ \delta^3\, A_i^3,\ \dots,\ \delta^{2\kappa-1}\, A_i^{2\kappa-1}\Big) = \delta^{m-a} \cdot \mathsf{P}_a\Big(F_i,\ A_i^3,\ \dots,\ A_i^{2\kappa-1}\Big).$$

Conversely, for every collection of such weighted homogeneous polynomials P_a in $\mathbb{C}\big[F_i, A_i^3, \dots, A_i^{2\kappa-1}\big]$ of weighted degree $m-a$ indexed by an integer a running in $\big[-\frac{\kappa-1}{\kappa}\, m,\ m\big]$ such that the reduction to the same denominator and the simplification of the finite sum:

$$\mathsf{R}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa} m \leqslant a \leqslant m} (f_1')^a \, \mathsf{P}_a \begin{pmatrix} f_2', & f_3', & f_4', & \dots, & f_n', \\ \Lambda^3_{1,2}, & \Lambda^3_{1,3}, & \Lambda^3_{1,4}, & \dots, & \Lambda^3_{1,n}, \\ \dots & \dots & \dots & \dots & \dots \\ \Lambda^{2\kappa-1}_{1,2;\,1^{\kappa-2}}, & \Lambda^{2\kappa-1}_{1,3;\,1^{\kappa-2}}, & \Lambda^{2\kappa-1}_{1,4;\,1^{\kappa-2}}, & \dots, & \Lambda^{2\kappa-1}_{1,n;\,1^{\kappa-2}} \end{pmatrix}$$

yields a true *jet* polynomial *in $\mathbb{C}\big[j^\kappa f\big]$, then $\mathsf{R}\big(j^\kappa f\big)$ is a polynomial invariant by reparametrization belonging to $\mathsf{E}^n_{\kappa,m}$.*

Next, we summarize briefly the way how our algorithm works; mathematical causalities, motivations and "reasons-why" shall be transparent to any reader who will study the example E^2_4 detailed in Section 6.

Suppose that, setting aside the special invariant f_1', we already know a certain number of invariants $L^{l_1}, \dots, L^{l_{k_1}}$, for instance the very initial ones above $f_2', \dots, f_n'$ together with all the $\Lambda^{2\lambda-1}_{1,i;\,1^{\lambda-2}}$. The recipe is to compute the ideal of relations between these invariants after setting $f_1' = 0$ in them:

$$\mathsf{Ideal\text{-}Rel}\Big(L^{l_1}(j^\kappa f)\big|_{f_1'=0},\ \dots,\ L^{l_{k_1}}(j^\kappa f)\big|_{f_1'=0}\Big).$$

Using any symbolic package for computing Gröbner bases, suppose that, for some monomial ordering, we may dispose of a Gröbner basis for the ideal of relations between these restricted invariants which we shall represent shortly by the following collection of algebraic equations:

$$0 \equiv \mathsf{S}_i\Big(L^{l_1}\big|_0,\ \dots,\ L^{l_{k_1}}\big|_0\Big) \qquad {\scriptstyle (i=1\cdots N_1)}.$$

One checks that each S_i may be supposed to be of constant homogeneous weight μ_i, namely:

$$\mathsf{S}_i\big(\delta^{l_1}A_1,\dots,\delta^{l_{k_1}}A_{k_1}\big)=\delta^{\mu_i}\mathsf{S}_i\big(A_1,\dots,A_{k_1}\big)\qquad\qquad {\scriptstyle (i=1\cdots N_1)}.$$

Since $\mathsf{S}_i\big(j^\kappa f\big)$ vanishes identically after setting $f_1'=0$, when we do not set $f_1'=0$, there must exist certain (possibly zero) polynomial remainders $\mathsf{R}_i\big(j^\kappa f\big)$ such that we may write in $\mathbb{C}\big[j^\kappa f\big]$:

$$\mathsf{S}_i\big(L^{l_1},\dots,L^{l_{k_1}}\big)=(f_1')^{\nu_i}\,\mathsf{R}_i\big(j^\kappa f\big)\qquad\qquad {\scriptstyle (i=1\cdots N_1)},$$

with $\mathsf{R}_i\not\equiv 0$ when $1\leqslant\nu_i<\infty$ and with $\mathsf{R}_i=0$ by convention when $\nu_i=\infty$.

Then one easily convinces oneself that every remainder $\mathsf{R}_i\big(j^\kappa f\big)$ also is a polynomial invariant by reparametrization.

Afterwards, one then tests whether the first remainder R_1 belongs to the algebra generated by $L^{l_1},\dots,L^{l_{k_1}}$. If not, R_1 must be added to the list of invariants. Next, one tests whether R_2 belongs to the algebra generated by $\mathsf{L}^{l_1},\dots,L^{l_{k_1}},\mathsf{R}_1$. If not, one adds R_2 to the list, and so on.

At the end, one gets a new list of invariants $L^{l_1},\dots,L^{k_1},M^{m_1},\dots,M^{m_{k_2}}$ and then one restarts a second loop by computing a Gröbner basis for the ideal of relations:

$$\mathsf{Ideal\text{-}Rel}\Big(L^{l_1}\big|_0,\ \dots,\ L^{l_{k_1}}\big|_0,\ M^{m_1}\big|_0,\ \dots,\ M^{m_{k_2}}\big|_0\Big).$$

Theorem. *For a certain dimension n and for a certain jet order κ, suppose that, after performing a finite number of loops of the algorithm, one possesses a finite number $1+M$ of mutually independent invariants f_1', $\Lambda^{l_1},\dots,\Lambda^{l_M}\in\mathbb{C}\big[j^\kappa f_1,\dots,j^\kappa f_n\big]$ of weights $1,l_1,\dots,l_M$ belonging to E^n_κ, whose restrictions to $\{f_1'=0\}$ share an ideal of relations:*

$$\mathsf{Ideal\text{-}Rel}\Big(\ \Lambda^{l_1}\big|_0,\ \ \dots\dots,\ \Lambda^{l_M}\big|_0\ \Big)$$

generated by a finite number N (often large) of homogeneous syzygies:

$$0\equiv\mathsf{S}_i\big(\Lambda^{l_1}\big|_0,\ \dots,\ \Lambda^{l_M}\big|_0\big),\qquad\qquad {\scriptstyle (i=1\cdots N)}$$

of weight μ_i assumed to be represented by a certain reduced Gröbner basis $\big\langle\mathsf{S}_i\big\rangle_{1\leqslant i\leqslant N}$ for a certain monomial order, with the crucial property that no *new invariant appears behind f_1', namely with the property that, without setting $f_1'=0$, one has N identically satisfied relations:*

$$0\equiv\mathsf{S}_i\big(\Lambda^{l_1},\ \dots,\ \Lambda^{l_M}\big)-f_1'\,\mathsf{R}_i\big(f_1',\ \Lambda^{l_1},\ \dots,\ \Lambda^{l_M}\big)\qquad\qquad {\scriptstyle (i=1\cdots N)},$$

for some remainders R_i which all depend polynomially upon the same collection of invariants $f_1',\Lambda^{l_1},\dots,\Lambda^{l_M}$, *so that no new invariant appears at this stage.*

Then the algorithm terminates and the algebra of invariants coincides with:

$$\boxed{\mathsf{E}^n_\kappa = \mathbb{C}\big[f_1',\, \Lambda^{l_1},\, \ldots\ldots,\, \Lambda^{l_M}\big] \quad \text{modulo syzygies}}\,.$$

As a standard byproduct of basic Gröbner bases theory, one deduces a unique representation of any polynomial invariant under reparametrization modulo the syzygies.

Indeed, for these values of n and of κ, if one denotes the leading terms (with respect to the monomial order in question) of the above N syzygies by:

$$\mathsf{LT}\big(\mathsf{S}_i(\Lambda)\big) = \big(\Lambda^{l_1}\big)^{\alpha_1^i} \cdots \big(\Lambda^{l_M}\big)^{\alpha_M^i} \qquad\qquad (i = 1 \cdots N),$$

for certain specific multiindices $\big(\alpha_1^i, \ldots, \alpha_M^i\big) \in \mathbb{N}^M$, and if for $i = 1, \ldots, N$ one denotes by:

$$\square_i := \alpha^i + \mathbb{N}^M = \big\{\big(\alpha_1^i + b_1,\, \ldots,\, \alpha_M^i + b_M\big) : b_1, \ldots, b_M \in \mathbb{N}^M\big\}$$

the positive quadrant of $\mathbb{N}^M$ having vertex at α^i, then a general, arbitrary invariant in $\mathsf{E}^n_{\kappa,m}$ of weight m writes *uniquely* under the *normal form*:

$$\sum_{0 \leqslant a \leqslant m} (f_1')^a\, \widetilde{\mathsf{P}}_a\big(\Lambda^{l_1},\, \ldots,\, \Lambda^{l_M}\big),$$

with summation containing *only positive powers* of f_1', where each $\widetilde{P}_a$ is of weight $m - a$ and is put under *Gröbner-normalized form*:

$$\boxed{\widetilde{\mathsf{P}}_a = \sum_{\substack{(b_1,\ldots,b_M) \in \mathbb{N}^M \setminus (\square_1 \cup \cdots \cup \square_N) \\ l_1 b_1 + \cdots + l_M b_M = m - a}} \mathsf{coeff}_{a;\, b_1,\ldots,b_M} \cdot \big(\Lambda^{l_1}\big)^{b_1} \cdots \big(\Lambda^{l_M}\big)^{b_M}}\,,$$

with complex coefficients $\mathsf{coeff}_{a;\, b_1,\ldots,b_M}$ subjected to no restriction at all.

The kernel algorithm. We would like to mention that, after the paper [21] was completed and submitted, on the occasion of a Workshop about holomorphic extension of CR functions and their removable singularities organized by Berit Stensønes and John-Erik Fornæss at the university of Michigan (Ann Arbor, December 2007), Harm Derksen indicated to us the so-called *Van den Essen's kernel algorithm* for locally nilpotent derivations, the goal of which is to generate all invariants for certain one-dimensional non-reductive actions ([14, 7, 15]). Although applied here to actions of any dimension, our algorithm here is in substance the same, though some features will be dealt with here more explicitly in the quite nontrivial explorations to which the paper is devoted: homogeneity of syzygies; stepwise generation of relations; skirting of Gröbner bases when they fail (due to oversizeness) to compute of the remainders R_i; systematic restriction to $\{f_1' = 0\}$ to shorten time computation.

In a near future, we hope to set up a refined algorithm which would almost completely tame the disturbing expression swelling.

Action of $\mathsf{GL}_n(\mathbb{C})$ and unipotent reduction. Lastly, we come back to explaining how one obtains Schur decompositions of Demailly-Semple bundles.

On an arbitrary fiber $\mathsf{E}^n_{\kappa,m}$ of $\mathsf{E}^n_{\kappa,m}T^*_X$ consisting of polynomials $\mathsf{P}\big(j^\kappa f\big) = \mathsf{P}\big(f', f'', \dots, f^{(\kappa)}\big)$ invariant by reparametrization, one looks at the action of matrices $\mathsf{w} = (w_{ij}) \in \mathsf{GL}_n(\mathbb{C})$ which, for each jet level λ with $1 \leqslant \lambda \leqslant \kappa$, multiplies by w the n jet-components $f^\lambda := \big(f_1^{(\lambda)}, \dots, f_n^{(\lambda)}\big)$, namely which transforms them into $\mathsf{w} \cdot f^\lambda := \big(\sum_{j=1}^n w_{1j} f_j^{(\lambda)}, \dots, \sum_{j=1}^n w_{nj} f_j^{(\lambda)}\big)$ with the *same* matrix for each jet level $\lambda = 1, 2, \dots, \kappa$.

According to elementary representation theory, $\mathsf{E}^n_{k,m}$ then decomposes into a certain direct sum of irreducible GL_n-representations, which are nothing but the Schur representations $\Gamma^{(\ell_1,\ell_2,\dots,\ell_n)}$ indexed by integers $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n$. General reasons ([4]) insure that such a decomposition on fibers globalizes coherently as a decomposition between bundles over $X \subset \mathbb{P}^{n+1}(\mathbb{C})$. How then does one determine the appearing Schur components? It suffices to look at the so-called *vectors of highest weight*, which in our situation are just the polynomials invariant by reparametrization $\mathsf{P} \in \mathsf{E}^n_{\kappa,m}$ which are *unipotent-invariant*, namely which are left untouched after multiplication by any unipotent matrix:

$$\mathsf{u} \cdot \mathsf{P}\big(j^\kappa f\big) = \mathsf{P}\big(j^\kappa f\big) \qquad \text{for every } \mathsf{u} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ u_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & 1 \end{pmatrix}.$$

Then the full space $\mathsf{E}^n_{\kappa,m}$ is obtained as just the $\mathsf{GL}_n(\mathbb{C})$-orbit of $\mathsf{UE}^n_{\kappa,m}$, and this will correspond to somehow *polarizing* the lower indices of bi-invariants, *see* below. We then call *bi-invariants* the polynomials which are both invariant under reparametrization and under the unipotent action:

$$\boxed{\mathsf{P}\big(j^\kappa(f \circ \phi)\big) = (\phi')^m \cdot \mathsf{P}\big((j^\kappa f) \circ \phi\big) \qquad \text{and} \qquad \mathsf{P}\big(\mathsf{u} \cdot j^\kappa f\big) = \mathsf{P}\big(j^\kappa f\big)}.$$

Thus, the bi-invariants are nothing but vectors of highest weight for this representation of $\mathsf{GL}_n(\mathbb{C})$. According to the general theory, to each vector of highest weight corresponds one and only one irreducible Schur representation $\Gamma^{(\ell_1,\ell_2,\dots,\ell_n)}$. How does one finds the integers ℓ_i?

Suppose that, after executing the algorithm, one already knows that UE^n_κ is generated by a finite number $f'_1, \Lambda^{l_1}, \dots, \Lambda^{l_M}$ of bi-invariants of weights

$1, l_1, \dots, l_M$, and suppose that we have a *unique* writing:

$$\sum_{(a,b_1,\dots,b_M)\in\mathscr{N}} \mathsf{coeff}_{a,b_1,\dots,b_M}\,(f_1')^a\,\big(\Lambda^{l_1}\big)^{b_1}\cdots\big(\Lambda^{l_M}\big)^{b_M}$$

of an arbitrary, general bi-invariant modulo the syzygies, for a certain monomial order, where $\mathscr{N}\subset\mathbb{N}^{1+M}$ denotes the complement of the union of quadrants having vertex at leading exponents. Then for every $(a, b_1, \dots, b_M)$, the single monomial $(f_1')^a\,\big(\Lambda^{l_1}\big)^{b_1}\cdots\big(\Lambda^{l_M}\big)^{b_M}$ is a vector of highest weight, and if one lets a general diagonal matrix:

$$\mathsf{x} := \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n \end{pmatrix}$$

act on it, the theory says it necessarily is an eigenvector, and the eigenvalue:

$$\mathsf{x}\cdot(f_1')^a\,\big(\Lambda^{l_1}\big)^{b_1}\cdots\big(\Lambda^{l_M}\big)^{b_M} = x_1^{\ell_1}\cdots x_n^{\ell_n}\,(f_1')^a\,\big(\Lambda^{l_1}\big)^{b_1}\cdots\big(\Lambda^{l_M}\big)^{b_M},$$

exhibits the wanted ℓ_i's which necessarily satisfy $\ell_1\geqslant\cdots\geqslant\ell_n$.

In conclusion, *both in order to understand invariants and in order to make Euler-characteristic computations, the very main goal is to explore algebras of bi-invariants*.

By requiring unipotent-invariance, the initial rational expression for bi-invariants will depend upon certain determinants defined as follows in terms of the initial invariants $\Lambda^{2\lambda-1}_{1,i:\,1^{\lambda-2}}$.

Theorem. *In dimension $n\geqslant 1$ and for jets of arbitrary order $\kappa\geqslant 1$, every bi-invariant polynomial $\mathsf{BP}=\mathsf{BP}\big(j^\kappa f\big)$ invariant by reparametrization and invariant under the unipotent action writes under the form:*

$$\mathsf{BP}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa}m\leqslant a\leqslant m} (f_1')^a\,\mathsf{BP}_a\Bigg(\left|\begin{matrix} \Lambda^{2\lambda_2-1}_{1,2} & \Lambda^{2\lambda_2-1}_{1,3} & \cdots & \Lambda^{2\lambda_2-1}_{1,n_1} \\ \Lambda^{2\lambda_3-1}_{1,2} & \Lambda^{2\lambda_3-1}_{1,3} & \cdots & \Lambda^{2\lambda_3-1}_{1,n_1} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda^{2\lambda_3-1}_{1,2} & \Lambda^{2\lambda_3-1}_{1,3} & \cdots & \Lambda^{2\lambda_3-1}_{1,n_1} \end{matrix}\right|^{2\leqslant\lambda_2,\dots,\lambda_{n_1}\leqslant\kappa}_{n_1=1,2\dots,n}\Bigg),$$

for certain specific polynomials BP_a which depend upon $\mathsf{BP}(j^\kappa f)$.

Algebras of bi-invariants. As announced in the abstract, we finalized two main applications of our algorithm. Only one bi-invariant, namely Y^{27}, was missed in [21], an article which pointed out that bracketing was insufficient.

Theorem. *In dimension $n = 2$ for jet order $\kappa = 5$, the algebra UE_5^2 of jet polynomials $\mathsf{P}\big(j^5 f_1, j^5 f_2\big)$ invariant by reparametrization and invariant under the unipotent action is generated by 17 mutually independent bi-invariants explicitly defined in Section 10:*

$$\boxed{\begin{array}{ccccccccc} f_1', & \Lambda^3, & \Lambda^5, & \Lambda^7, & \Lambda^9, & M^8, & M^{10}, & K^{12}, & \\ N^{12}, & H^{14}, & F^{16}, & X^{18}, & X^{19}, & X^{21}, & X^{23}, & X^{25}, & Y^{27} \end{array}}.$$

As a consequence, the full algebra E_5^2 of jet polynomials $\mathsf{P}\big(j^5 f\big)$ invariant by reparametrization is generated by the polarizations:

$$\boxed{\begin{array}{ccccccccc} f_i', & \Lambda^3, & \Lambda_i^5, & \Lambda_{i,j}^7, & \Lambda_{i,j,k}^9, & M^8, & M_i^{10}, & K_{i,j}^{12}, & \\ N^{12}, & H_i^{14}, & F_{i,j}^{16}, & X_{i,j,k}^{18}, & X_i^{19}, & X^{21}, & X_i^{23}, & X_{i,j}^{25}, & Y_{i,j,k}^{27} \end{array}}$$

of these 17 bi-invariants, where the indices i, j, k vary in $\{1, 2\}$, whence the total number of these invariants equals:

$$2+1+2+4+8+1+2+4+1+2+4+8+2+1+2+4+8 = \boxed{\mathbf{56}}.$$

Secondly, we obtain the following new result in dimension 4. We must confess that we were unable to discover some harmonious algebraic structures which could probably (in)exist?

Theorem. *In dimension $n = 4$ for jets of order $\kappa = 4$, the algebra UE_4^4 of jet polynomials $\mathsf{P}\big(j^4 f_1, j^4 f_2, j^4 f_3, j^4 f_4\big)$ invariant by reparametrization and invariant under the unipotent action is generated by 16 mutually independent bi-invariants explicitly defined*[35] *in Section 11:*

$$\boxed{\begin{array}{cccccccc} W^{10}, & f_1', & \Lambda^3, & \Lambda^5, & \Lambda^7, & D^6, & D^8, & N^{10}, \\ M^8, & E^{10}, & L^{12}, & Q^{14}, & R^{15}, & U^{17}, & V^{19}, & X^{21}, \end{array}}$$

whose restriction to $\{f_1' = 0\}$ has a reduced gröbnerized ideal of relations, for the Lexicographic ordering, which consists of the 41 syzygies written on p. 256.

Furthermore, any bi-invariant of weight m writes uniquely in the finite polynomial form:

$$\begin{aligned} \mathsf{P}\big(j^\kappa f\big) = \sum_{o,p} (f_1')^o \left(W^{10}\right)^p \sum_{\substack{(a,\dots,n)\in\mathbb{N}^{14}\setminus(\square_1\cup\cdots\cup\square_{41}) \\ 3a+\cdots+21n=m-o-10p}} \mathsf{coeff}_{a,\dots,n,o,p} \cdot \\ \cdot \left(\Lambda^3\right)^a \left(\Lambda^5\right)^b \left(\Lambda^7\right)^c \left(D^6\right)^d \left(D^8\right)^e \left(N^{10}\right)^f \left(M^8\right)^g \left(E^{10}\right)^h \\ \left(L^{12}\right)^i \left(Q^{14}\right)^j \left(R^{15}\right)^k \left(U^{17}\right)^l \left(V^{19}\right)^m \left(X^{21}\right)^n, \end{aligned}$$

[35] The bi-invariant X^{21} here is different from the X^{21} of the preceding theorem.

with coefficients $\mathsf{coeff}_{a,\ldots,n,o,p}$ *subjected to no restriction, where* $\square_1, \ldots, \square_{41}$ *denote the quadrants in* $\mathbb{N}^{14}$ *having vertex at the leading terms of the 41 syzygies in question.*

As a consequence, the full algebra E_4^4 *of jet polynomials* $\mathsf{P}\big(j^4 f\big)$ *invariant by reparametrization is generated by the polarizations of the 16 bi-invariants:*

$$\boxed{\begin{array}{c} W^{10}, \quad f_i', \quad \Lambda^3_{[i,j]}, \quad \Lambda^5_{[i,j];\,\alpha}, \quad \Lambda^7_{[i,j];\,\alpha,\beta}, \quad D^6_{[i,j,k]}, \\ D^8_{[i,j,k];\,\alpha}, \quad N^{10}_{[i,j,k];\,\alpha,\beta}, \quad M^8_{[i,j],[k,l]}, \quad E^{10}_{[i,j,k],[p,q]}, \quad L^{12}_{[i,j,k],[p,q];\,\alpha}, \\ Q^{14}_{[i,j,k],[p,q];\,\alpha,\beta}, \quad R^{15}_{[i,j,k],[p,q,r];\,\alpha}, \quad U^{17}_{[i,j,k],[p,q,r],[s,t]}, \\ V^{19}_{[i,j,k],[p,q,r],[s,t];\,\alpha}, \quad X^{21}_{[i,j,k],[p,q,r],[s,t];\,\alpha,\beta}. \end{array}}$$

These polarized invariants are skew-symmetric with respect to each collection of bracketed indices $[i,j,k]$, $[p,q,r]$, $[s,t]$, *where the roman indices satisfy* $1 \leqslant i < j < k \leqslant 4$, *where* $1 \leqslant p < q < r \leqslant 4$, *where* $1 \leqslant s < r \leqslant 4$ *and where the two greek indices* α, β *satisfy* $1 \leqslant \alpha, \beta \leqslant 4$ *without restriction and finally the total number of these invariants generating the Demailly-Semple algebra* E_4^4 *equals:*

$$\begin{aligned} &1 + 4 + 6 + 24 + 96 + 4 + 16 + 64 + \\ &+ 36 + 24 + 96 + 384 + 64 + 96 + 384 + 1536 = \boxed{\mathbf{2835}}. \end{aligned}$$

Acknowledgments. The author heartfully thanks Jacques Beigbeder (SPI, École Normale Supérieure) for having installed the package FGb (Spiral Team, LIP6) containing efficient Gröbner basis algorithms that were used to capture the quite huge ideals of relations exhibited in the present arxiv.org electronic (pre)publication. He is also grateful to Jean-Pierre Demailly, to Jahwer El Goul, to Erwan Rousseau, and to Simone Diverio for friend(ful)ly sharing thoughts about the puzzling complexity of invariant jet differentials. In March 2007, Tien-Cuong Dinh and Nessim Sibony suggested the question to our expertise.

Finally, the theorem on p. 258 which describes the structure of the algebra of bi-invariants for $n = k = 4$ was firmly gained during the author's stay at the Mittag-Leffler Institute in April 2008.

§2. Invariant polynomials and composite differentation

Fixing basic notations. Let X be a smooth n-dimensional complex algebraic hypersurface of $\mathbb{P}^{n+1}(\mathbb{C})$, let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and consider an arbitrary holomorphic disc $f : \mathbb{D} \to X$ valued in X, for instance the restriction to $\mathbb{D}$ of some entire holomorphic curve $\mathbb{C} \to X$. In some local chart on $X \simeq \mathbb{D}^n$ centered at $f(0)$, the κ-jet $j_0^\kappa f$ of f at $0 \in \mathbb{D}$ is represented by the

collection of all the derivatives, with respect to the variable $\zeta \in \mathbb{D}$, of the n components $f_1, \dots, f_n$ of f, up to order κ, that is to say:

$$j^\kappa f = \big(f_1', \dots, f_n',\, f_1'', \dots, f_n'',\, \dots\dots,\, f_1^{(\kappa)}, \dots, f_n^{(\kappa)}\big);$$

from the beginning and throughout this study, we shall in fact constantly omit to denote the base point $0 \in \mathbb{D}$.

Polynomials invariant by reparametrization. For $\kappa \geqslant 1$, we consider polynomials in all the jet variables:

$$\mathsf{P} = \mathsf{P}\big(j^\kappa f\big) = \mathsf{P}\big(f_{j_1}',\, f_{j_2}'', \dots,\, f_{j_\kappa}^{(\kappa)}\big),$$

where the indices $j_1, j_2, \dots, j_\kappa$ run in $\{1, \dots, n\}$. An open problem in Demailly's strategy towards the Kobayashi hyperbolicity conjecture ([4, 6]) was to describe those polynomials $\mathsf{P}\big(j^\kappa f\big)$ enjoying the property that a change of variable $\mathbb{D} \ni \zeta \longmapsto \phi(\zeta) \in \mathbb{C}$ in the source affects the polynomial only through multiplication by some power of the first derivative of ϕ:

$$\mathsf{P}\big(j^\kappa(f \circ \phi)\big) = (\phi')^m \cdot P\big((j^\kappa f) \circ \phi\big),$$

where $m \geqslant 1$ is an integer which shall be called here the *weight* of P.

Choosing in particular ϕ to be simply a dilation $\zeta \mapsto \delta \cdot z$ by a constant nonzero complex factor δ, one sees that such polynomials must at least (*cf.* [17]) be *weighted homogeneous of order* m with respect to the weighted anisotropic dilations:

$$\mathsf{P}\big(\delta \cdot f_{j_1'},\, \delta^2 \cdot f_{j_2}'',\, \dots,\, \delta^\kappa \cdot f_{j_\kappa}^{(\kappa)}\big) \equiv \delta^m \cdot \mathsf{P}\big(f_{j_1}',\, f_{j_2}'',\, \dots,\, f_{j_\kappa}^{(\kappa)}\big).$$

As a useful mnemonic, weight therefore always counts the total number of primes.

By $\mathsf{E}^n_{\kappa,m}$, we will thus denote the vector space consisting of all such polynomials. The direct sum $\mathsf{E}^n_\kappa := \bigoplus_{m \geqslant 1} \mathsf{E}^n_{\kappa,m}$ forms an algebra which is graded by constancy of weights, for the definition yields:

$$\mathsf{E}^n_{\kappa,m_1} \cdot \mathsf{E}^n_{\kappa,m_2} \subset \mathsf{E}^n_{\kappa,m_1+m_2}.$$

Following a nowadays established terminology, a polynomial $\mathsf{P}\big(j^\kappa f\big)$ in this algebra will be said to be *invariant by reparametrization*. The present article aims to describe a complete algorithm generating all such polynomials, sometimes briefly called *invariants*.

Example. For $\kappa = 1$, the components f_i' for $i = 1, \dots, n$ of the jet satisfy:

$$\big(f_i \circ \phi\big)' = \phi' \cdot f_i',$$

hence every polynomial $\mathsf{P} = \mathsf{P}\big(f_1', \dots, f_n'\big)$ which depends only upon the first order jet $j^1 f$ is invariant by reparametrization. So E^n_1 coincides with the plain polynomial algebra $\mathbb{C}\big[f_1', \dots, f_n'\big]$.

Example. For $\kappa = 2$, aside from the monomials $f_1', \dots, f_n'$ coming from the preceding jet level $\kappa = 1$, there are yet the 2×2 determinants (clearly of weight 3):

$$\Delta_{i,j}^{',''} := \begin{vmatrix} f_i' & f_j' \\ f_i'' & f_j'' \end{vmatrix},$$

for one easily checks, thanks to row linear dependence, that:

$$\begin{vmatrix} (f_i \circ \phi)' & (f_j \circ \phi)' \\ (f_i \circ \phi)'' & (f_j \circ \phi)'' \end{vmatrix} = \begin{vmatrix} \phi' f_i' & \phi' f_j' \\ \phi'' f_i' + \phi'^2 f_i'' & \phi'' f_j' + \phi'^2 f_j'' \end{vmatrix} = \phi'^3 \cdot \begin{vmatrix} f_i' & f_j' \\ f_i'' & f_j'' \end{vmatrix}.$$

It is a theorem, to be stated below, that the f_i' and the $\Delta_{j,k}^{',''}$ generate the algebra E_n^2.

Composite differentiation up to order $\kappa = 5$. Setting $g_i := f_i \circ \phi$ for $i = 1, \dots, n$, the elementary chain rule provides derivatives of g_i with respect to the source variable $\zeta \in \mathbb{D}$:

$$\begin{aligned}
g_i' &= \phi' f_i', \\
g_i'' &= \phi'' f_i' + \phi'^2 f_i'', \\
g_i''' &= \phi''' f_i' + 3\,\phi'' \phi' f_i'' + \phi'^3 f_i''', \\
g_i'''' &= \phi'''' f_i' + 4\,\phi''' \phi' f_i'' + 3\,\phi''^2 f_i'' + 6\,\phi'' \phi'^2 f_i''' + \phi'^4 f_i'''', \\
g_i''''' &:= \phi''''' f_i' + 5\,\phi'''' \phi' f_i'' + 10\,\phi''' \phi'' f_i'' + 15\,\phi''^2 \phi' f_i''' + \\
&\qquad + 10\,\phi''' \phi'^2 f_i''' + 10\,\phi'' \phi'^3 f_i'''' + \phi'^5 f_i'''''.
\end{aligned}$$

Thus with $\kappa = 5$ for instance, the goal is to find all polynomials $\mathsf{P} = \mathsf{P}\big(j^5 g\big)$ which, after replacing g_i', g_i'', g_i''', g_i'''' and g_i''''' by these expressions, have the property of *cancelling* the derivatives ϕ'', ϕ''', ϕ'''' and ϕ''''' of ϕ whose order is $\geqslant 2$, so that $\mathsf{P}\big(j^5 g\big) = \phi'^m \mathsf{P}\big(j^5 f\big)$ for a certain $m \in \mathbb{N}$.

For the sake of completeness, let us present the classical *Faà di Bruno*, well known in the case of one variable $\zeta \in \mathbb{C}$.

Theorem. *For every integer $\kappa \geqslant 1$, the derivative of order κ of each composite function $g_i(z) := f_i \circ \phi(z)$ $(1 \leqslant i \leqslant n)$ with respect to the variable $\zeta \in \mathbb{C}$ is a polynomial with integer coefficients in the derivatives of f_i (same index i) and in the derivatives of ϕ*:

$$\boxed{\begin{aligned} g_i^{(\kappa)} = \sum_{e=1}^{\kappa} \sum_{1 \leqslant \lambda_1 < \cdots < \lambda_e \leqslant \kappa} \sum_{\mu_1 \geqslant 1, \dots, \mu_e \geqslant 1} \sum_{\mu_1 \lambda_1 + \cdots + \mu_e \lambda_e = \kappa} \\ \frac{\kappa!}{(\lambda_1!)^{\mu_1}\, \mu_1! \cdots (\lambda_e!)^{\mu_e}\, \mu_e!} \big(\phi^{(\lambda_1)}\big)^{\mu_1} \cdots\cdots \big(\phi^{(\lambda_e)}\big)^{\mu_e}\, f_i^{(\mu_1 + \cdots + \mu_e)} \end{aligned}}.$$

To read this general formula with the help of the formulas specialized above, let us observe that the general monomial $\big(\phi^{(\lambda_1)}\big)^{\mu_1} \cdots\cdots \big(\phi^{(\lambda_e)}\big)^{\mu_e}$ in

the reparametrization jet gathers derivatives of increasing orders $\lambda_1 < \lambda_2 < \cdots < \lambda_e$, with $\mu_1, \mu_2, \ldots, \mu_e$ counting their respective numbers. Then the function f_i is subjected to a partial differentiation of order $\mu_1 + \mu_2 + \cdots + \mu_e$, the total number of derivatives $\phi^{(\lambda_k)}$ in the monomial in question. Finally, in the permutation group $\mathfrak{S}_\kappa$ of $\{1, 2, \ldots, \kappa\}$ whose cardinality clearly equals $\kappa!$, the quantity $(\lambda_1!)^{\mu_1}\mu_1! \cdots (\lambda_e!)^{\mu_e}\mu_e!$ counts the number of permutations which possess μ_1 cycles of length λ_1, μ_2 cycles of length λ_2, *etc.*, μ_e cycles of length λ_e, so that the fractional coefficient $\frac{\kappa!}{(\lambda_1!)^{\mu_1}\,\mu_1! \cdots (\lambda_e!)^{\mu_e}\,\mu_e!}$ with $\kappa = \mu_1\lambda_1 + \mu_2\lambda_2 + \cdots + \mu_e\lambda_e$ is an integer which provides the cardinality of the (left or right) coset of $\mathfrak{S}_\kappa$ modulo such a subgroup permutations. Notice that all these observations are confirmed by the formulas developed above up to $\kappa = 5$.

With such a formula, the problem of finding all polynomials invariant by reparametrization can be interpreted in terms of invariant theory ([4, 29]).

Indeed, the group G_κ of κ-jets at the origin of local reparametrizations:

$$\phi(\zeta) = \zeta + \phi''(0)\,\frac{\zeta^2}{2!} + \cdots + \phi^{(\kappa)}(0)\,\frac{\zeta^\kappa}{\kappa!} + \cdots$$

that are tangent to the identity, namely $\phi'(0) = 1$, may be seen, thanks to the above formulas, to act linearly on the $n\kappa$-tuples $\big(f'_{j_1}, f''_{j_2}, \ldots, f^{(\kappa)}_{j_\kappa}\big)$ just by matrix multiplication:

$$\begin{pmatrix} g'_i \\ g''_i \\ g'''_i \\ g''''_i \\ \vdots \\ g^{(\kappa)}_i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \phi'' & 1 & 0 & 0 & \cdots & 0 \\ \phi''' & 3\phi'' & 1 & 0 & \cdots & 0 \\ \phi'''' & 4\phi''' + 3\phi''^2 & 6\phi'' & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi^{(\kappa)} & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} f'_i \\ f''_i \\ f'''_i \\ f''''_i \\ \vdots \\ f^{(\kappa)}_i \end{pmatrix} \qquad (i = 1 \cdots n).$$

Polynomials $\mathsf{P}\big(j^\kappa f\big)$ invariant by reparametrization coincide with the invariants for this linear group action, an action which is clearly unipotent, hence non-reductive. In such a context, no general theory or algorithm exists to decide whether the algebra of invariants is finitely generated (*cf.* Problem 2 p. 2 in [8]). In fact, we will attack the problem from another point of view.

§3. Bracketing process and syzygies: Jacobi, Plücker 1 and Plücker 2

Cross product between two invariants polynomials. A natural process known to Demailly and to El Goul (*cf.* [5] and [21]) is as follows. Suppose that we know two reparametrization-invariant polynomials $\mathsf{P} = \mathsf{P}\big(j^\kappa g\big)$ of

weight m and $\mathsf{Q} = \mathsf{Q}\big(j^\tau f\big)$ of weight n, namely we have:

$$\begin{aligned}\mathsf{P}\big(j^\kappa g\big) &= \phi'^m\, \mathsf{P}\big((j^\kappa f)\circ\phi\big),\\ \mathsf{Q}\big(j^\tau g\big) &= \phi'^n\, \mathsf{Q}\big((j^\tau f)\circ\phi\big),\end{aligned}$$

where we have again set $g := f\circ\phi$. To differentiate a polynomial with respect to the source variable $\zeta\in\mathbb{C}$ amounts to apply to it the *total differentiation operator*:

$$\mathsf{D} := \sum_{k=1}^n \sum_{\lambda\in\mathbb{N}} \frac{\partial(\bullet)}{\partial f_k^{(\lambda)}}\cdot f_k^{(\lambda+1)},$$

which gives here:

$$\begin{aligned}\big[\mathsf{DP}\big]\big(j^{\kappa+1}g\big) &= m\,\phi''\,\phi'^{m-1}\,\mathsf{P}\big((j^\kappa f)\circ\phi\big) + \phi'^m\,\phi'\,\big[\mathsf{DP}\big]\big((j^{\kappa+1}f)\circ\phi\big),\\ \big[\mathsf{DQ}\big]\big(j^{\tau+1}g\big) &= n\,\phi''\,\phi'^{n-1}\,\mathsf{Q}\big((j^\kappa f)\circ\phi\big) + \phi'^n\,\phi'\,\big[\mathsf{DQ}\big]\big((j^{\tau+1}f)\circ\phi\big),\end{aligned}$$

and in order to remove the second order derivative ϕ'', it suffices to perform a *cross-product*, namely to form the 2×2 determinant:

$$\begin{aligned}&\begin{vmatrix}\big[\mathsf{DP}\big]\big(j^{\kappa+1}g\big) & m\,\mathsf{P}\big(j^\kappa g\big)\\ \big[\mathsf{DQ}\big]\big(j^{\tau+1}g\big) & n\,\mathsf{Q}\big(j^\tau g\big)\end{vmatrix} =\\ &= \begin{vmatrix} m\,\phi''\,\phi'^{m-1}\,\mathsf{P}\big((j^\kappa f)\circ\phi\big) + \phi'^{m+1}\,\big[\mathsf{DP}\big]\big((j^{\kappa+1}f)\circ\phi\big) & m\,\phi'^m\,\mathsf{P}\big((j^\kappa f)\circ\phi\big)\\ n\,\phi''\,\phi'^{n-1}\,\mathsf{Q}\big((j^\kappa f)\circ\phi\big) + \phi'^{n+1}\,\big[\mathsf{DQ}\big]\big((j^{\tau+1}f)\circ\phi\big) & n\,\phi'^n\,\mathsf{Q}\big((j^\kappa f)\circ\phi\big)\end{vmatrix}\\ &= \begin{vmatrix}\phi'^{m+1}\,\big[\mathsf{DP}\big]\big((j^{\kappa+1}f)\circ\phi\big) & m\,\phi'^m\,\mathsf{P}\big((j^\kappa f)\circ\phi\big)\\ \phi'^{n+1}\,\big[\mathsf{DQ}\big]\big((j^{\tau+1}f)\circ\phi\big) & n\,\phi'^n\,\mathsf{Q}\big((j^\kappa f)\circ\phi\big)\end{vmatrix}\\ &= \phi'^{m+n+1}\begin{vmatrix}\big[\mathsf{DP}\big]\big(j^{\kappa+1}f\big) & m\,\mathsf{P}\big(j^\kappa f\big)\\ \big[\mathsf{DQ}\big]\big(j^{\tau+1}f\big) & n\,\mathsf{Q}\big(j^\tau f\big)\end{vmatrix},\end{aligned}$$

which therefore happens to constitute a new invariant of weight $m+n+1$ in the jet space of order $1+\max(\kappa,\tau)$ *increased by one unit*.

Bracket operator $\big[\cdot,\,\cdot\big]$ and its accompanying syzygies. Thus, *every pair of invariants automatically produces a new invariant*:

$$\boxed{\big[\mathsf{P},\,\mathsf{Q}\big] := n\,\mathsf{DP}\cdot\mathsf{Q} - m\,\mathsf{P}\cdot\mathsf{DQ}},$$

which is obviously skew-symmetric with respect to the pair $\big(\mathsf{P},\mathsf{Q}\big)$. For instance, we recover in this way *all* the invariants of (jet) order 2 mentioned above:

$$\big[f_i',\,f_j'\big] = \mathsf{D}f_i'\cdot f_j' - f_i'\mathsf{D}f_j' = f_i''f_j' - f_i'f_j'' = -\Delta_{i,j}^{',''},$$

and again, we notice that bracketing increases jet order by one unit.

Certainly, as soon as at least 3 pairwise distinct invariants P, Q and R are known, a *Jacobi-type* identity (checked on pp. 867–868 of [21]) must hold:

$$\boxed{(\mathscr{J}ac): \qquad 0 \equiv \big[\big[\mathsf{P},\,\mathsf{Q}\big],\,\mathsf{R}\big] + \big[\big[\mathsf{R},\,\mathsf{P}\big],\,\mathsf{Q}\big] + \big[\big[\mathsf{Q},\,\mathsf{R}\big],\,\mathsf{P}\big]}.$$

Although such relations give nothing for jet order $\kappa = 2$, because the jet order of an iterated bracket $\big[[\cdot,\,\cdot],\,\cdot\big]$ is in any case $\geqslant 3$, if we introduce the following new bracket-type invariants:

$$\begin{aligned}\big[\Delta_{i,j}^{',''},\,f'_k\big] &= \mathsf{D}\big(f'_i f''_j - f''_i f'_j\big)\cdot f'_k - 3\,\big(f'_i f''_j - f''_i f'_j\big)\cdot f''_k \\ &= \big(f'_i f'''_j - f'''_i f'_j\big)\cdot f'_k - 3\,\big(f'_i f''_j - f''_i f'_j\big)\cdot f''_k,\end{aligned}$$

then we gratuitously have Jacobi-type relations which will hold true at the next jet level $\kappa = 3$:

$$0 \equiv \big[\Delta_{i,j}^{',''},\,f'_k\big] + \big[\Delta_{k,i}^{',''},\,f'_j\big] + \big[\Delta_{j,k}^{',''},\,f'_i\big].$$

On the other hand, we remind that the 2×2 minors $\mathsf{a}_{1,2}^{j_1,j_2} := \det\big(\mathsf{a}_i^j\big)_{i=1,2}^{j=j_1,j_2}$ of an arbitrary $2\times N$ complex-valued matrix $\big(\mathsf{a}_i^j\big)_{i=1,2}^{1\leqslant j\leqslant N}$ are known to enjoy ([21], p. 883) the so-called *quadratic Plücker relations* which are usually organized in two families[36]:

$$\begin{aligned}&(\mathscr{P}lck_1): \qquad 0 \equiv \mathsf{a}_1^{j_1}\cdot\mathsf{a}_{1,2}^{j_2,j_3} + \mathsf{a}_1^{j_3}\cdot\mathsf{a}_{1,2}^{j_1,j_2} + \mathsf{a}_1^{j_2}\cdot\mathsf{a}_{1,2}^{j_3,j_1},\\ &(\mathscr{P}lck_2): \qquad 0 \equiv \mathsf{a}_{1,2}^{j_1,j_2}\cdot\mathsf{a}_{1,2}^{j_3,j_4} + \mathsf{a}_{1,2}^{j_1,j_2}\cdot\mathsf{a}_{1,2}^{j_3,j_4} + \mathsf{a}_{1,2}^{j_1,j_2}\cdot\mathsf{a}_{1,2}^{j_3,j_4},\end{aligned}$$

and which may be checked by expanding the minors, just observing cancellations[37]. We then deduce that our bracketing process, when interpreted as computing the minors of an auxiliary matrix:

$$\begin{pmatrix} m\,\mathsf{P} & n\,\mathsf{Q} & o\,\mathsf{R} & p\,\mathsf{S} & \cdots \\ \mathsf{DP} & \mathsf{DQ} & \mathsf{DR} & \mathsf{DS} & \cdots \end{pmatrix},$$

whose first line lists known invariants multiplied by their own weight, and whose second line lists their total derivatives, we immediately deduce that our bracketing process introduces the following two supplementary families of identically satisfied *Plückerian-like* relations:

$$\boxed{\begin{aligned}&(\mathscr{P}lck_1): \qquad 0 \equiv m\,\mathsf{P}\,\big[\mathsf{Q},\,\mathsf{R}\big] + o\,\mathsf{R}\,\big[\mathsf{P},\,\mathsf{Q}\big] + n\,\mathsf{Q}\,\big[\mathsf{R},\,\mathsf{P}\big],\\ &(\mathscr{P}lck_2): \qquad 0 \equiv \big[\mathsf{P},\,\mathsf{Q}\big]\cdot\big[\mathsf{R},\,\mathsf{S}\big] + \big[\mathsf{S},\,\mathsf{P}\big]\cdot\big[\mathsf{R},\,\mathsf{Q}\big] + \big[\mathsf{Q},\,\mathsf{S}\big]\cdot\big[\mathsf{R},\,\mathsf{P}\big]\end{aligned}}.$$

[36] In the first line, the sum bears upon circular permutations of (j_1, j_2, j_3); in the second line, j_3 is fixed and the sum bears upon circular permutations of (j_1, j_2, j_4). Equivalently, one could have fixed j_4 and considered circular permutations of (j_1, j_2, j_3).

[37] In fact, only these relations appear in the ideal of syzygies between the a_1^j and the $\mathsf{a}_{1,2}^{j_1,j_2}$, for an appropriate monomial order ([24], p. 277).

Throughout the text, identically satisfied relations between polynomials will often be called *syzygies*, following the terminology of classical invariant theory ([4]).

For instance, at the jet level $\kappa = 2$, we plainly have:

$$0 \equiv \Delta_{i,j}^{',''} \cdot f'_k + \Delta_{k,i}^{',''} \cdot f'_j + \Delta_{j,k}^{',''} \cdot f'_i,$$
$$0 \equiv \Delta_{i,j}^{',''} \cdot \Delta_{k,l}^{',''} + \Delta_{l,i}^{',''} \cdot \Delta_{k,j}^{',''} + \Delta_{j,l}^{',''} \cdot \Delta_{k,i}^{',''},$$

for all indices $i, j, k, l = 1, \ldots, n$.

A general notation for Wronskian-like determinants. It will be quite useful to abbreviate the explicit denotation of the further, rather complicated invariants that we shall have to deal with in the sequel by introducing the minors:

$$\Delta_{i,\,j}^{(\alpha),(\beta)} := \begin{vmatrix} f_i^{(\alpha)} & f_j^{(\alpha)} \\ f_i^{(\beta)} & f_j^{(\beta)} \end{vmatrix}, \qquad \Delta_{i,\,j,\,k}^{(\alpha),(\beta),(\gamma)} := \begin{vmatrix} f_i^{(\alpha)} & f_j^{(\alpha)} & f_k^{(\alpha)} \\ f_i^{(\beta)} & f_j^{(\beta)} & f_k^{(\beta)} \\ f_i^{(\gamma)} & f_j^{(\gamma)} & f_k^{(\gamma)} \end{vmatrix}, \qquad \textit{etc.}$$

extracted from the jet matrix $\big(f_i^{(\lambda)}\big)$. Top indices list derivative orders, appearing in rows.

Thanks to skew-symmetry, after some row or column permutations, one can always write these determinants in such a way that the lower, dimensional indices satisfy $1 \leqslant i < j < k \leqslant n$ and similarly, the upper, derivative indices also satisfy $1 \leqslant \alpha < \beta < \gamma \leqslant \kappa$ at the same time.

In fact, the already mentioned observation that $\Delta_{i,j}^{',''}$ always provides an invariant easily generalizes, for if we set:

$$g_i^{(\lambda)} := \big(f_i \circ \phi\big)^{(\lambda)},$$

then by either manipulating the Faà di Bruno formula written above, or by using a less explicit intermediate inductive assertion in order to pass from one jet level to the next jet level, one may subject the determinants to row linear combinations in order to establish the following:

Lemma. *For every λ with $1 \leqslant \lambda \leqslant \kappa$ and for all indices $i_1, i_2, \ldots, i_\lambda = 1, \ldots, n$, one has:*

$$\begin{vmatrix} g'_{i_1} & g'_{i_2} & \cdots & g'_{i_\lambda} \\ g''_{i_1} & g''_{i_2} & \cdots & g''_{i_\lambda} \\ \vdots & \vdots & \ddots & \vdots \\ g_{i_1}^{(\lambda)} & g_{i_2}^{(\lambda)} & \cdots & g_{i_\lambda}^{(\lambda)} \end{vmatrix} = (\phi')^{2\lambda-1} \cdot \begin{vmatrix} f'_{i_1} & f'_{i_2} & \cdots & f'_{i_\lambda} \\ f''_{i_1} & f''_{i_2} & \cdots & f''_{i_\lambda} \\ \vdots & \vdots & \ddots & \vdots \\ f_{i_1}^{(\lambda)} & f_{i_2}^{(\lambda)} & \cdots & f_{i_\lambda}^{(\lambda)} \end{vmatrix} = (\phi')^{2\lambda-1} \cdot \Delta_{i_1,i_2,\ldots,i_\lambda}^{',\,'',\ldots,(\lambda)},$$

hence all the Wronskian-like determinants $\Delta_{i_1,i_2,\ldots,i_\lambda}^{',\,'',\ldots,(\lambda)}$ always are invariant by reparametrization.

Here, it is crucial that the derivative order starts from 1 at the first row and increases by one unit exactly while descending stepwise along the rows;

otherwise, we would *not in any case* get a true invariant; for instance in the expression:

$$\begin{vmatrix} g_i'' & g_j'' \\ g_i''' & g_j''' \end{vmatrix} = \begin{vmatrix} \phi'' f_i' + \phi'^2 f_i'' & \phi'' f_j' + \phi'^2 f_j'' \\ \phi''' f_i' + 3\,\phi''\phi' f_i'' + \phi'^3 f_i''' & \phi''' f_j' + 3\,\phi''\phi' f_j'' + \phi'^3 f_j''' \end{vmatrix}$$
$$= \begin{vmatrix} \phi'' f_i' + \phi'^2 f_i'' & \phi'' f_j' + \phi'^2 f_j'' \\ \phi''' f_i' - 2\phi'^3 f_i''' & \phi''' f_j' - 2\phi'^3 f_j''' \end{vmatrix},$$

no further simplification enables to get rid of ϕ'', ϕ''' and such an obstruction happens to hold generally.

Combinatorics of the subalgebra generated by the Wronskians. Thus at least, we know a large family of invariants. The following statement goes back to the nineteenth century.

Proposition. ([19, 4, 24]) *For jets of order $\kappa = 2$ in arbitrary dimension $n \geqslant 2$, the algebra E_2^n consists of the algebra generated by the $n + \frac{n(n-1)}{2}$ fundamental invariants:*

$$f_k' \qquad \text{and} \qquad \Delta_{i,j}^{',''} = \begin{vmatrix} f_i' & f_j' \\ f_i'' & f_j'' \end{vmatrix}$$

and their syzygy ideal is generated by the two families of Plückerian relations written above:

$$\begin{cases} 0 \equiv \Delta_{i,j}^{',''} \cdot f_k' + \Delta_{k,i}^{',''} \cdot f_j' + \Delta_{j,k}^{',''} \cdot f_i', \\ 0 \equiv \Delta_{i,j}^{',''} \cdot \Delta_{k,l}^{',''} + \Delta_{l,i}^{',''} \cdot \Delta_{k,j}^{',''} + \Delta_{j,l}^{',''} \cdot \Delta_{k,i}^{',''}. \end{cases}$$

§4. Survey of known descriptions of E_κ^n in low dimensions for small jet levels

The above-defined algebra E_κ^n of jet polynomials $\mathsf{P}\big(j^\kappa f\big)$ invariant by reparametrization is understood only in certain specific situations.

Demailly 1997. At first, in dimension $n \geqslant 2$ for jet level $\kappa = 2$, the $n + \frac{n(n-1)}{2}$ generators of the proposition just above appear on p. 341 of [4], namely every polynomial P in E_2^n writes:

$$\mathsf{P}\big(j^2 f\big) \equiv \mathscr{P}_{\mathsf{P}}\big(f_1', \ldots, f_n', \Delta_{1,2}^{',''}, \ldots, \Delta_{n-1,n}^{',''}\big)$$

having as arguments the basic invariants in question.

In the particular case of surfaces, namely for $n = 2$, no syzygy exists between f_1', f_2' and $\Delta_{1,2}^{',''}$, hence E_2^2 coincides with a plain polynomial algebra:

$$\mathsf{E}_2^2 = \mathbb{C}\big[f_1', f_2', \Delta_{1,2}^{',''}\big].$$

Basic notions of invariant theory. For higher n's and κ's, unpredictable syzygies will obscure the picture, but before pursuing, we must fix a suitable terminology. We formulate these concepts for E^n_κ, but they hold quite more generally.

Definition. If, for certain values of n and κ, there are finitely many invariants $\Lambda_1, \dots, \Lambda_{\mathsf{last}}$ in E^n_κ with the property that every polynomial $\mathsf{P}\big(j^\kappa f\big) \in \mathsf{E}^n_\kappa$ invariant by reparametrization can be written as a polynomial:

$$\mathsf{P}\big(j^\kappa f\big) \equiv \mathscr{P}_{\mathsf{P}}\big(\Lambda_1, \dots, \Lambda_{\mathsf{last}}\big)$$

having $\Lambda_1, \dots, \Lambda_{\mathsf{last}}$ as arguments, we shall say that E^n_κ is *generated* (as an algebra) *by* $\Lambda_1, \dots, \Lambda_{\mathsf{last}}$.

Definition. Further, we shall say that $\Lambda_1, \dots, \Lambda_{\mathsf{last}}$ are *mutually independent* if, for every middle index with $1 \leqslant \mathsf{middle} \leqslant \mathsf{last}$, there does not exist any polynomial $\mathscr{P}$ such that $\Lambda_{\mathsf{middle}}$ identifies to a polynomial:

$$\Lambda_{\mathsf{middle}} = \mathscr{P}\big(\Lambda_1, \dots, \widehat{\Lambda_{\mathsf{middle}}}, \dots, \Lambda_{\mathsf{last}}\big)$$

in the other remaining invariants. Then $\Lambda_1, \dots, \Lambda_{\mathsf{last}}$ will be called *fundamental invariants generating* E^n_κ (for such values of n, κ) and an indivivual $\Lambda_{\mathsf{middle}}$ will be called a *basic invariant*.

For a fixed E^n_κ, all sets of fundamental invariants, either finite or infinite, have the same cardinality.

Weights always appear as upper indices. Also, we want for later use to introduce the new notation:

$$\Lambda^3_{1,2} := \Delta^{',''}_{1,2},$$

where we specify the row indices $1, 2$ and where we specially emphasize the weight 3, counting the total number of primes. In fact, throughout the whole paper, *we shall systematically write the weight of every basic invariant as its upper index*. We thus can continue the survey.

Demailly 2004; Rousseau 2006. Next, in dimension $n = 2$ for jet level $\kappa = 3$, it is shown[38] in [29] that the algebra E^2_3 is generated by the three invariants f'_1, f'_2 and $\Delta^{',''}_{1,2}$ (already known from the preceding jet level) to which one adds the two further invariants of weight 5:

$$\begin{aligned} \Lambda^5_{1,2;\,1} &:= \big[\Lambda^3_{1,2},\, f'_1\big] \qquad\qquad \text{and} \qquad\qquad \Lambda^5_{1,2;\,2} := \big[\Lambda^3_{1,2},\, f'_2\big] \\ &= \Delta^{',''}_{1,3}\, f'_1 - 3\,\Delta^{',''}_{1,2}\, f''_1 \qquad\qquad\qquad\qquad = \Delta^{',''}_{1,3}\, f'_2 - 3\,\Delta^{',''}_{1,2}\, f''_2, \end{aligned}$$

the only possible brackets, as one checks. Moreover, these five invariants f'_1, f'_2, $\Lambda^3_{1,2}$, $\Lambda^5_{1,2;\,1}$ and $\Lambda^5_{1,2;\,2}$ are mutually independent and their syzygy ideal is

[38] The result was known to Demailly (unpublished).

principal, generated by the single quadratic relation:

$$0 \equiv 3\,\Lambda^3_{1,2}\Lambda^3_{1,2} - f'_2\Lambda^5_{1,2;\,1} + f'_1\Lambda^5_{1,2;\,2}.$$

One sees that this syzygy just comes $(\mathscr{P}lck_1)$. In fact, $(\mathscr{J}ac)$ and $(\mathscr{P}lck_1)$ give nothing.

Rousseau 2006. Now, in dimension $n = 3$ and for jet level $\kappa = 3$, applying a theorem of Popov, Rousseau ([29], p. 403) deduced that the algebra E^3_3 is generated by all the invariants known in dimension $n - 1 = 2$ whose lower indices are *polarized* in all possible ways, namely the 15 invariants:

$$\begin{array}{ccccccccc} & & f'_1, & f'_2, & f'_3, & \Lambda^3_{1,2}, & \Lambda^3_{1,3}, & \Lambda^3_{2,3}, & \\ \Lambda^5_{1,2;\,1}, & \Lambda^5_{1,3;\,1}, & \Lambda^5_{2,3;\,1}, & \Lambda^5_{1,2;\,2}, & \Lambda^5_{1,3;\,2}, & \Lambda^5_{2,3;\,2}, & \Lambda^5_{1,2;\,3}, & \Lambda^5_{1,3;\,3}, & \Lambda^5_{2,3;\,3}, \end{array}$$

together with a single further invariant, the Wronskian:

$$D^6_{1,2,3} := \begin{vmatrix} f'_1 & f'_2 & f'_3 \\ f''_1 & f''_2 & f''_3 \\ f'''_1 & f'''_2 & f'''_3 \end{vmatrix}.$$

This makes 16 invariants in sum. An alternative, direct proof of this result may be found in [21], and will also pop up again in the present paper.

We must mention that the Wronskian $D^6_{1,2,3}$ also appears in fact in terms of brackets, for one checks the following three relations:

$$\big[\Lambda^3_{1,2},\,\Lambda^3_{1,3}\big] = -3\,f'_1\,D^6_{1,2,3}, \qquad \big[\Lambda^3_{1,2},\,\Lambda^3_{2,3}\big] = -3\,f'_2\,D^6_{1,2,3},$$
$$\big[\Lambda^3_{1,3},\,\Lambda^3_{2,3}\big] = -3\,f'_3\,D^6_{1,2,3}.$$

A Maple computation ([28]) also provided the ideal of relations between these 16 invariants. Among the 62 generators of the reduced Gröbner basis supplied by Maple after ~ 15 hours of symbolic computations, 30 appear to be minimal generators of the ideal of relations, the 32 remaining ones being further automatically generated S-polynomials which are required to complete the basis. Remarkably, it may be checked ([21]) that *each one of the 30 minimal syzygies* in question is included among the collection of syzygies deduced from our three fundamental families by inserting f'_i and $\Lambda^3_{j,k}$ in all possible ways in place of P, Q, R and T:

$$(\mathscr{J}ac): \quad \Big\{ 0 \equiv \big[[f'_i,\,f'_j],\,f'_k\big] + \big[[f'_k,\,f'_i],\,f'_j\big] + \big[[f'_j,\,f'_k],\,f'_i\big],$$

$$(\mathscr{P}lck_1): \quad \begin{cases} 0 \equiv f'_i\big[f'_j,\,f'_k\big] + f'_k\big[f'_i,\,f'_j\big] + f'_j\big[f'_k,\,f'_i\big], \\ 0 \equiv f'_i\big[f'_j,\,\Lambda^3_{k,l}\big] + 3\,\Lambda^3_{k,l}\big[f'_i,\,f'_j\big] + f'_j\big[\Lambda^3_{k,l},\,f'_i\big], \\ 0 \equiv f'_i\big[\Lambda^3_{j,k},\,\Lambda^3_{l,m}\big] + \Lambda^3_{l,m}\big[f'_i,\,\Lambda^3_{j,k}\big] + \Lambda^3_{j,k}\big[\Lambda^3_{l,m},\,f'_i\big], \\ 0 \equiv \Lambda^3_{i,j}\big[\Lambda^3_{k,l},\,\Lambda^3_{m,n}\big] + \Lambda^3_{m,n}\big[\Lambda^3_{i,j},\,\Lambda^3_{k,l}\big] + \Lambda^3_{k,l}\big[\Lambda^3_{m,n},\,\Lambda^3_{i,j}\big], \end{cases}$$

$$(\mathscr{P}lck_2):\quad \begin{cases} 0\equiv [f'_i,\,f'_j]\cdot[f'_k,\Lambda^3_{l,m}]+[\Lambda^3_{l,m},\,f'_i]\cdot[f'_k,f'_j]+[f'_j,\,\Lambda^3_{l,m}]\cdot[f'_k,f'_i],\\ 0\equiv [f'_i,\,f'_j]\cdot[\Lambda^3_{k,l},\,\Lambda^3_{m,n}]+[\Lambda^3_{m,n},\,f'_i]\cdot[\Lambda^3_{k,l},\,f'_j]+[f'_j,\,\Lambda^3_{m,n}]\cdot[\Lambda^3_{k,l},\,f'_i],\\ 0\equiv [f'_i,\,\Lambda^3_{j,k}]\cdot[\Lambda^3_{l,m},\,\Lambda^3_{n,p}]+[\Lambda^3_{n,p},\,f'_i]\cdot[\Lambda^3_{l,m},\,\Lambda^3_{j,k}]+[\Lambda^3_{j,k},\,\Lambda^3_{n,p}]\cdot[\Lambda^3_{l,m},\,f'_i], \end{cases}$$

where the indices i, j, k, l, m, n, and p take all the values $1,2,3$.

Demailly-El Goul 2004; Rousseau 2007; M. 2007. Finally[39], for jets of order $\kappa=4$ in dimension $n=2$, the algebra E^2_4 is generated by the five invariants:

$$f'_1,\qquad f'_2,\qquad \Lambda^3_{1,2},\qquad \Lambda^5_{1,2;\,1},\qquad \Lambda^5_{1,2;\,2}$$

already known from the preceding jet level, to which one adds the four further invariants gently provided by bracketing:

$$\Lambda^7_{1,1}:=[\Lambda^5_{1,2;\,1},\,f'_1],\qquad \Lambda^7_{1,2}:=[\Lambda^5_{1,2;\,1},\,f'_2]=[\Lambda^5_{1,2;\,2},\,f'_1]=\Lambda^7_{2,1},\qquad \Lambda^7_{2,2}:=[\Lambda^5_{1,2;\,2},\,f'_2],$$
$$M^8:=\frac{1}{f'_1}\,[\Lambda^5_{1,2;\,1},\,\Lambda^3_{1,2}].$$

This in sum makes 9 fundamental invariants. Notice the (necessary) division by f'_1 to get M^8. The two missing brackets[40]:

$$\big[\Lambda^5_{1,2;\,2},\,\Lambda^3_{1,2}\big]=f'_2\,M^8\qquad\text{and}\qquad\big[\Lambda^5_{1,2;\,1},\,\Lambda^5_{1,2;\,2}\big]=\Lambda^3_{1,2}\,M^8$$

appear to in fact belong already to the algebra generated by these 9 invariants.

Now, we lighten a little the notation by dropping some of the lower indices, especially in the $\Delta^{(\alpha),(\beta)}_{1,2}\equiv:\Delta^{(\alpha),(\beta)}$, because in dimension $n=2$, by skew-symmetry of determinants, only $(1,2)$ can appear at the bottom.

Theorem. ([21]) *For jets of order $\kappa=4$ in dimension $n=2$, the algebra E^2_4 is generated by 9 mutually independent fundamental invariants explicitly defined by:*

$$f'_1,\qquad f'_2,\qquad \Lambda^3:=\Delta^{',''},$$
$$\Lambda^5_1:=\Delta^{','''}\,f'_1-3\,\Delta^{',''}\,f''_1,$$
$$\Lambda^5_2:=\Delta^{','''}\,f'_2-3\,\Delta^{',''}\,f''_2,$$
$$\Lambda^7_{1,1}:=\big(\Delta^{',''''}+4\,\Delta^{'','''}\big)\,f'_1f'_1-10\,\Delta^{','''}\,f'_1f''_1+15\,\Delta^{',''}\,f''_1f''_1,$$
$$\Lambda^7_{1,2}:=\big(\Delta^{',''''}+4\,\Delta^{'','''}\big)\,f'_1f'_2-5\,\Delta^{','''}\big(f''_1f'_2+f''_2f'_1\big)+15\,\Delta^{',''}\,f''_1f''_2,$$
$$\Lambda^7_{2,2}:=\big(\Delta^{',''''}+4\,\Delta^{'','''}\big)\,f'_2f'_2-10\,\Delta^{','''}\,f'_2f''_2+15\,\Delta^{',''}\,f''_2f''_2,$$
$$M^8:=3\,\Delta^{',''''}\,\Delta^{',''}+12\,\Delta^{'','''}\,\Delta^{',''}-5\,\Delta^{','''}\,\Delta^{','''}$$

whose ideal of relations is generated by 9 fundamental syzygies:

$$\Big[0\overset{1}{\equiv}f'_2\,\Lambda^5_1-f'_1\,\Lambda^5_2-3\,\Lambda^3\,\Lambda^3,$$

[39] The result was known (unpublished) to experts; a proof appears in [21].

[40] Details of computations may be found in [21], pp. 870–871 and also pp. 882–886.

$$\left[\begin{array}{l} 0 \overset{2}{\equiv} f'_2\,\Lambda^7_{1,1} - f'_1\,\Lambda^7_{1,2} - 5\,\Lambda^3\,\Lambda^5_1, \\ 0 \overset{3}{\equiv} f'_2\,\Lambda^7_{1,2} - f'_1\,\Lambda^7_{2,2} - 5\,\Lambda^3\,\Lambda^5_2, \end{array}\right.$$

$$\left[\begin{array}{l} 0 \overset{4}{\equiv} f'_1\,f'_1\,M^8 - 3\,\Lambda^3\,\Lambda^7_{1,1} + 5\,\Lambda^5_1\,\Lambda^5_1, \\ 0 \overset{5}{\equiv} f'_1\,f'_2\,M^8 - 3\,\Lambda^3\,\Lambda^7_{1,2} + 5\,\Lambda^5_1\,\Lambda^5_2, \\ 0 \overset{6}{\equiv} f'_2\,f'_2\,M^8 - 3\,\Lambda^3\,\Lambda^7_{2,2} + 5\,\Lambda^5_2\,\Lambda^5_2, \end{array}\right.$$

$$\left[\begin{array}{l} 0 \overset{7}{\equiv} f'_1\,\Lambda^3\,M^8 - \Lambda^5_1\,\Lambda^7_{1,2} + \Lambda^5_2\,\Lambda^7_{1,1}, \\ 0 \overset{8}{\equiv} f'_2\,\Lambda^3\,M^8 - \Lambda^5_1\,\Lambda^7_{2,2} + \Lambda^5_2\,\Lambda^7_{1,2}, \end{array}\right.$$

$$\left[\, 0 \overset{9}{\equiv} 5\,\Lambda^3\,\Lambda^3\,M^8 - \Lambda^7_{2,2}\,\Lambda^7_{1,1} + \Lambda^7_{1,2}\,\Lambda^7_{1,2},\right.$$

which are all obtained by means of the three families of automatic relations $(\mathscr{J}ac)$, $(\mathscr{P}lck_1)$ *and* $(\mathscr{P}lck_2)$.

Summary and induction. Thus, all known descriptions of algebras of jet polynomials invariant by reparametrization were obtained by starting with the trivial list:

$$f'_1,\ f'_2,\ \ldots,\ f'_n$$

of invariants of order 1, and bracketing them again and again in order to lift oneself to higher jet levels. The principle of induction then dictates to continue such a process.

Jets of order $\kappa = 5$ in dimension $n = 2$. Bracketing all invariants from the preceding jet level $\kappa = 4$ amounts to compute all the 2×2 minors of the following 2×9 matrix:

$$\left\| \begin{array}{ccccccccc} f'_1 & f'_2 & 3\,\Lambda^3 & 5\,\Lambda^5_1 & 5\,\Lambda^5_2 & 7\,\Lambda^7_{1,1} & 7\,\Lambda^7_{1,2} & 7\,\Lambda^7_{2,2} & 8\,M^8 \\ \mathsf{D}f'_1 & \mathsf{D}f'_2 & \mathsf{D}\Lambda^3 & \mathsf{D}\Lambda^5_1 & \mathsf{D}\Lambda^5_2 & \mathsf{D}\Lambda^7_{1,1} & \mathsf{D}\Lambda^7_{1,2} & \mathsf{D}\Lambda^7_{2,2} & \mathsf{D}M^8 \end{array} \right\|,$$

which in sum makes a total of $\frac{9!}{2!\,7!} = 36$ brackets. But taking account of the fact that the $\frac{5!}{2!\,3!} = 10$ minors of the first 5 columns correspond to the already known passage from $\kappa = 3$ to $\kappa = 4$, just a few less brackets, namely $36 - 10 = 26$ have to be computed, namely the eight families:

$$\begin{array}{cc} \big[\Lambda^7_{i,j},\ f'_k\big], & \big[M^8,\ f'_i\big], \\ \big[\Lambda^7_{i,j},\ \Lambda^3\big], & \big[M^8,\ \Lambda^3\big], \\ \big[\Lambda^7_{i,j},\ \Lambda^5_k\big], & \big[M^8,\ \Lambda^5_i\big], \\ \big[\Lambda^7_{i,j},\ \Lambda^7_{k,l}\big], & \big[M^8,\ \Lambda^7_{i,j}\big]. \end{array}$$

In [21], this task was achieved, thoroughly and in great details, the obtained brackets being all written in terms of the $\Delta^{(\alpha),(\beta)}$. Furthermore, by

inspecting systematically the first fundamental family[41] of syzygies $(\mathscr{J}ac)$, some superfluous brackets that are certain polynomials in terms of previously known invariants were left out.

Theorem. ([21]) *For jets of order $\kappa = 5$ in dimension $n = 2$, the algebra of bracket invariants in E_5^2 is generated by exactly* **24** *mutually independent fundamental invariants:*

$$\boxed{\begin{array}{r} f_1',\ f_2',\ \Lambda^3,\ \Lambda_1^5,\ \Lambda_2^5,\ \Lambda_{1,1}^7,\ \Lambda_{1,2}^7,\ \Lambda_{2,2}^7,\ M^8, \\ \Lambda_{1,1,1}^9,\ \Lambda_{1,2,1}^9,\ \Lambda_{2,1,2}^9,\ \Lambda_{2,2,2}^9,\ M_1^{10},\ M_2^{10}, \\ N^{12},\ K_{1,1}^{12},\ K_{1,2}^{12} = K_{2,1}^{12},\ K_{2,2}^{12}, \\ H_1^{14},\ H_2^{14},\ F_{1,1}^{16},\ F_{1,2}^{16},\ F_{2,2}^{16}, \end{array}}$$

among which the pure order 5 brackets are defined by:

$$\begin{aligned} \Lambda_{i,j,k}^9 &:= \big[\Lambda_{i,j}^7,\ f_k'\big] \\ M_i^{10} &:= \big[M^8,\ f_i'\big] \\ N^{12} &:= \big[M^8,\ \Lambda^3\big] \\ K_{i,j}^{12} &:= \big[\Lambda_{i,j}^7,\ \Lambda_1^5\big] \big/ f_i' \\ H_i^{14} &:= \big[M^8,\ \Lambda_i^5\big] \\ F_{i,j}^{16} &:= \big[M^8,\ \Lambda_{i,j}^7\big] \end{aligned}$$

and are explicitly given by the following normalized formulas:

$$\begin{aligned} \Lambda_{i,j,k}^9 := {} & \Delta^{',''''''}\, f_i' f_j' f_k' + 5\,\Delta^{'',''''}\, f_i' f_j' f_k' - \\ & - 4\,\Delta^{',''''}\big(f_i'' f_j' + f_i' f_j''\big)\, f_k' - 7\,\Delta^{',''''}\, f_i' f_j' f_k'' - \\ & - 16\,\Delta^{'','''}\big(f_i'' f_j' + f_i' f_j''\big) f_k' - 28\,\Delta^{'','''}\, f_i' f_j' f_k'' - \\ & - 5\,\Delta^{','''}\big(f_i''' f_j' + f_i' f_j'''\big)\, f_k' + 35\,\Delta^{','''}\big(f_i'' f_j'' f_k' + f_i'' f_j' f_k'' + f_i' f_j'' f_k''\big) - \\ & - 105\,\Delta^{',''}\, f_i'' f_j'' f_k'', \end{aligned}$$

$$\begin{aligned} M_i^{10} := {} & \big[3\,\Delta^{',''''''}\,\Delta^{',''} + 15\,\Delta^{'',''''}\,\Delta^{',''} - 7\,\Delta^{',''''}\,\Delta^{','''} + 2\,\Delta^{'','''}\,\Delta^{','''}\big]\, f_i' - \\ & - \big[24\,\Delta^{',''''}\,\Delta^{',''} + 96\,\Delta^{'','''}\,\Delta^{',''} - 40\,\Delta^{','''}\,\Delta^{','''}\big] f_i'', \end{aligned}$$

$$\begin{aligned} N^{12} := {} & 9\,\Delta^{',''''''}\,\Delta^{',''}\,\Delta^{',''} + 45\,\Delta^{'',''''}\,\Delta^{',''}\,\Delta^{',''} - 45\,\Delta^{',''''}\,\Delta^{','''}\,\Delta^{',''} - \\ & - 90\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} + 40\,\Delta^{','''}\,\Delta^{','''}\,\Delta^{','''}, \end{aligned}$$

[41] The other two families of syzygies $(\mathscr{P}lck_1)$ and $(\mathscr{P}lck_2)$ having all their terms quadratic, no resolved relation for any bracket invariant Π of the form $\Pi = \mathsf{polynomial}\,(\Lambda^1, \ldots, \Lambda_{\mathsf{last}})$ can arise from them.

$$
\begin{aligned}
K^{12}_{i,j} := {} & f'_i f'_j \Big(5\,\Delta^{',''''}\,\Delta^{','''} + 25\,\Delta^{'',''''}\,\Delta^{','''} - 7\,\Delta^{',''''}\,\Delta^{',''''} - 56\,\Delta^{'','''}\,\Delta^{',''''} - 112\,\Delta^{'','''}\,\Delta^{'','''} \Big) + \\
& + \frac{(f'_i f''_j + f''_i f'_j)}{2} \Big(-15\,\Delta^{','''''}\,\Delta^{',''} - 75\,\Delta^{'',''''}\,\Delta^{',''} + 65\,\Delta^{',''''}\,\Delta^{','''} + 110\,\Delta^{'','''}\,\Delta^{','''} \Big) + \\
& + \frac{(f'_i f'''_j + f'''_i f'_j)}{2} \Big(-50\,\Delta^{','''}\,\Delta^{','''} \Big) + \\
& + f''_i f''_j \Big(-25\,\Delta^{','''}\,\Delta^{','''} + 15\,\Delta^{',''''}\,\Delta^{',''} + 60\,\Delta^{'','''}\,\Delta^{',''} \Big),
\end{aligned}
$$

$$
\begin{aligned}
H^{14}_{i} := {} & \Big(15\,\Delta^{','''''}\,\Delta^{','''}\,\Delta^{',''} + 75\,\Delta^{'',''''}\,\Delta^{','''}\,\Delta^{',''} + 5\,\Delta^{',''''}\,\Delta^{','''}\,\Delta^{','''} + \\
& + 170\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{','''} - 24\,\Delta^{',''''}\,\Delta^{',''''}\,\Delta^{',''} - 192\,\Delta^{',''''}\,\Delta^{'','''}\,\Delta^{',''} - \\
& - 384\,\Delta^{'','''}\,\Delta^{'','''}\,\Delta^{',''} \Big)\, f'_i + \Big(-45\,\Delta^{','''''}\,\Delta^{',''}\,\Delta^{',''} - 225\,\Delta^{'',''''}\,\Delta^{',''}\,\Delta^{',''} + \\
& + 225\,\Delta^{',''''}\,\Delta^{','''}\,\Delta^{',''} + 450\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} - 200\,\Delta^{','''}\,\Delta^{','''}\,\Delta^{','''} \Big)\, f''_i,
\end{aligned}
$$

$$
\begin{aligned}
F^{16}_{i,j} := {} & \Big(-3\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{',''} - 15\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{',''} - 12\,\Delta^{','''''}\,\Delta^{'','''}\,\Delta^{',''} + \\
& + 40\,\Delta^{','''''}\,\Delta^{','''}\,\Delta^{','''} - 60\,\Delta^{'',''''}\,\Delta^{'','''}\,\Delta^{',''} + 200\,\Delta^{'',''''}\,\Delta^{','''}\,\Delta^{','''} - \\
& - 49\,\Delta^{',''''}\,\Delta^{',''''}\,\Delta^{','''} - 422\,\Delta^{',''''}\,\Delta^{'','''}\,\Delta^{','''} - 904\,\Delta^{'','''}\,\Delta^{'','''}\,\Delta^{','''} \Big) f'_i f'_j + \\
& + \Big(-105\,\Delta^{','''''}\,\Delta^{','''}\,\Delta^{',''} - 525\,\Delta^{'',''''}\,\Delta^{','''}\,\Delta^{',''} + 205\,\Delta^{',''''}\,\Delta^{','''}\,\Delta^{','''} - \\
& - 230\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{','''} + 96\,\Delta^{',''''}\,\Delta^{',''''}\,\Delta^{',''} + 768\,\Delta^{',''''}\,\Delta^{'','''}\,\Delta^{',''} + \\
& + 1536\,\Delta^{'','''}\,\Delta^{'','''}\,\Delta^{',''} \Big) (f''_i f'_j + f'_i f''_j) + \\
& + \Big(-200\,\Delta^{','''}\,\Delta^{','''}\,\Delta^{','''} \Big) (f'''_i f'_j + f'_i f'''_j) + \\
& + \Big(315\,\Delta^{','''''}\,\Delta^{',''}\,\Delta^{',''} + 1575\,\Delta^{'',''''}\,\Delta^{',''}\,\Delta^{',''} - 1575\,\Delta^{',''''}\,\Delta^{','''}\,\Delta^{',''} - \\
& - 3150\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} + 1400\,\Delta^{','''}\,\Delta^{','''}\,\Delta^{','''} \Big) f''_i f''_j,
\end{aligned}
$$

where the indices i, j and k run in $\{1, 2\}$. Furthermore, the ideal of relations between these 24 fundamental bracket invariants consists of all the syzygies that one obtains[42] *by substituting in $(\mathscr{P}lck_1)$ or in $(\mathscr{P}lck_2)$ for* P, Q, R, T *three or four among the nine invariants f'_1, f'_2, Λ^3, Λ^5_1, Λ^5_2, $\Lambda^7_{1,1}$, $\Lambda^7_{1,2}$, $\Lambda^7_{2,2}$, M^8, in all possible ways, which makes in sum:*

$$\tfrac{9!}{3!\,6!} + \tfrac{9!}{4!\,5!} = 84 + 126 = 210$$

generating syzygies.

It is now great time to offer ideas, arguments, principles of computations, and also proofs.

[42] The data of our manuscript are not reproduced here.

§5. Initial invariants in dimension n for arbitrary jet level $\kappa \geqslant 1$

Reparametrizing by f_1^{-1}. To fix ideas and to better offer the intuition of our computations, we shall firstly work in dimension $n = 2$ until everything about the first basic step becomes clear, so that afterwards, the description of the birth of the initial invariants in the higher dimensions $n = 3$ and $n = 4$ shall present no real difficulty.

Thus, let $\mathsf{P}\big(j^\kappa f_1,\, j^\kappa f_2\big)$ be a polynomial of weight m that is invariant by reparametrization. By definition,

$$(*) \qquad \mathsf{P}\big(j^\kappa(f \circ \phi)\big) = \phi'^m\, \mathsf{P}\big((j^\kappa f) \circ \phi\big),$$

for every local biholomorphism of $\mathbb{C}$ fixing 0. Following a trick of Rousseau ([29]), we will apply this formula to the inverse mapping $\phi := f_1^{-1}$ of the first coordinate map $f_1 : \mathbb{C} \to \mathbb{C}$, assuming that $f_1'(0) \neq 0$, whence $\phi' = \frac{1}{f_1'} \circ f_1^{-1}$. We will explain in a moment that the assumption $f_1'(0) \neq 0$ is harmless for the result.

At first, we trivially have $f_1 \circ f_1^{-1} = \mathrm{Id}$, whence $\big(f_1 \circ f_1^{-1}\big)' = 1$ and $\big(f_1 \circ f_1^{-1}\big)^{(\lambda)} = 0$ for all $\lambda \geqslant 2$. Next, by some direct computations, the derivatives of the reparametrization of f_2 happen to be:

$$\begin{aligned}
(f_2 \circ f_1^{-1})' &= \left(\frac{f_2'}{f_1'}\right) \circ f_1^{-1}, \\
(f_2 \circ f_1^{-1})'' &= \left[\frac{f_2''}{(f_1')^2} - \frac{f_1'' f_2'}{(f_1')^3}\right] \circ f_1^{-1} = \left[\frac{f_1' f_2'' - f_1'' f_2'}{(f_1')^3}\right] \circ f_1^{-1} \\
&= \frac{\Lambda^3}{(f_1')^3} \circ f_1^{-1},
\end{aligned}$$

where we recognize here our favorite Wronskian $\Lambda^3 = \Delta_{1,2}^{',''}$. Furthermore, by pursuing as we should the computations with the help of our beloved total differentiation operator, we next get:

$$\begin{aligned}
(f_2 \circ f_1^{-1})''' &= \left(\frac{\mathsf{D}\Lambda^3}{(f_1')^4} - 3\,\frac{\Lambda^3 f_1''}{(f_1')^5}\right) \circ f_1^{-1} = \frac{[\Lambda^3,\, f_1']}{(f_1')^5} \circ f_1^{-1} \\
&= \frac{\Lambda_1^5}{(f_1')^5} \circ f_1^{-1}, \\
(f_2 \circ f_1^{-1})'''' \circ f_1^{-1} &= \frac{[\Lambda_1^5,\, f_1']}{(f_1')^7} \circ f_1^{-1} \\
&= \frac{\Lambda_{1,1}^7}{(f_1')^7} \circ f_1^{-1},
\end{aligned}$$

and so on, with the now clear formal facts that numerators should be constructed by successively bracketing with f_1', their weight being visible as just the power of f_1' in the denominator.

With indices, we may therefore define inductively the collection of *initial invariants* (including f_1' and Λ^3):

$$\begin{aligned}\Lambda^{2\lambda-1}_{1^{\lambda-2}} &:= \big[\Lambda^{2\lambda-3}_{1^{\lambda-3}},\, f_1'\big] \\ &= \mathsf{D}\Lambda^{2\lambda-3}_{1^{\lambda-3}} \cdot f_1' - (2\lambda-3)\Lambda^{2\lambda-3}_{1^{1-\lambda_3}} \cdot f_1'',\end{aligned}$$

for all λ with $3 \leqslant \lambda \leqslant \kappa$, where at the bottom of $\Lambda^{\bullet}_{1^\ell}$, the notation 1^ℓ stands for ℓ copies of 1. We then get by induction:

$$\boxed{\big(f_2 \circ f_1^{-1}\big)^{(\lambda)} = \frac{\Lambda^{2\lambda-1}_{1^{\lambda-2}}}{(f_1')^{2\lambda-1}} \circ f_1^{-1}}.$$

It would not be a so straightforward task to find a general explicit expression of these invariants $\Lambda^{2\kappa-1}_{1^{\kappa-2}}$ in terms of $j^\kappa f$ for arbitrary jet order. For instance, the invariant $\Lambda^9_{1,1,1}$, obtained by specializing $i = j = k = 1$ in the expression given in the theorem stated above (and by simplifying) reads:

$$\begin{aligned}\Lambda^9_{1,1,1} = \big(\Delta^{',''''} + 5\,\Delta^{',''''}\big)\, f_1' f_1' f_1' - \big(15\,\Delta^{',''''} + 60\,\Delta^{'','''}\big)\, f_1' f_1' f_1'' - \\ - 10\,\Delta^{','''}\, f_1' f_1' f_1''' + 105\,\Delta^{','''}\, f_1' f_1'' f_1'' - 105\,\Delta^{',''}\, f_1'' f_1'' f_1''.\end{aligned}$$

Nonetheless, we will in fact not really need to expand the expressions of these initial invariants.

Fact. *The invariants f_1', Λ^3, Λ^5_1, $\Lambda^7_{1,1}$, $\ldots$, $\Lambda^{2\kappa-1}_{1^{\kappa-2}}$ are mutually algebraically independent.*

This is just because $\Lambda^{2\lambda-1}_{1^{\lambda-2}}$ is a polynomial in $\big(j^\lambda f_1, j^\lambda f_2\big)$ while $\Lambda^{2\lambda+1}_{1^{\lambda-1}}$ contains the higher jet monomial $f_2^{(\lambda+1)}[f_1']^\lambda$.

Initial rational expression for invariant polynomials. A general polynomial $\mathsf{P}\big(j^\kappa f\big)$ of weight m in $\mathsf{E}^2_{\kappa,m}$ writes in expanded form:

$$\begin{aligned}\mathsf{P}\big(j^\kappa f_1,\, j^\kappa f_2\big) = \sum_{a_1^1+a_2^1+2a_1^2+2a_2^2+\cdots+\kappa a_1^\kappa+\kappa a_2^\kappa=m} \mathsf{coeff}\cdot \big(f_1'\big)^{a_1^1}\, \big(f_2'\big)^{a_2^1}\, \big(f_1''\big)^{a_1^2}\, \big(f_2''\big)^{a_2^2} \\ \cdots\, \big(f_1^{(\kappa)}\big)^{a_1^\kappa}\, \big(f_2^{(\kappa)}\big)^{a_2^\kappa},\end{aligned}$$

where by "coeff" we mean varying, but notationally unspecified complex numbers. Reparametrizing by $\phi := f_1^{-1}$ by an application of the definition $(*)$, we should have the relation:

$$\frac{1}{(f_1' \circ f_1^{-1})^m} \cdot \mathsf{P}\big(j^\kappa f_1,\, j^\kappa f_2\big) \circ f_1^{-1} = \mathsf{P}\big(j^\kappa(f_1 \circ f_1^{-1}),\, j^\kappa(f_2 \circ f_1^{-1})\big),$$

in the open subset $\{f_1' \neq 0\}$ of the jet space $J^\kappa(\mathbb{C}, \mathbb{C}^n)$. Thanks to the preparatory computations above, we may replace each monomial in the right

hand side, and this gives us a quite interesting representation:

$$\frac{1}{(f_1'\circ f_1^{-1})^m}\cdot \mathsf{P}\big(j^\kappa f_1,\, j^\kappa f_2\big)\circ f_1^{-1} =$$
$$= \Bigg[\sum_{a_1^1+a_2^1+2a_1^2+2a_2^2+\cdots+\kappa a_1^\kappa+\kappa a_2^\kappa=m} \mathsf{coeff}\cdot (1)^{a_1^1}\left(\frac{f_2'}{f_1'}\right)^{a_2^1}(0)^{a_1^2}\left(\frac{\Lambda^3}{(f_1')^3}\right)^{a_2^2}$$
$$\cdots\ (0)^{a_1^\kappa}\left(\frac{\Lambda_{1^{\kappa-2}}^{2\kappa-1}}{(f_1')^{2\kappa-1}}\right)^{a_2^\kappa}\Bigg]\circ f_1^{-1}.$$

Immediately, we reparametrize this identity by f_1, which then simply erases all the appearing f^{-1}, we see that monomials with positive exponent $a_1^\lambda \geqslant 1$ for some λ with $2\leqslant\lambda\leqslant\kappa$ automatically vanish, and we reduce monomials to the same denominator:

$$\mathsf{P}\big(j^\kappa f_1,\, j^\kappa f_2\big) = \sum_{a_1^1+a_2^1+2a_2^2+\cdots+\kappa a_2^\kappa=m} \mathsf{coeff}\cdot\frac{\big(f_2'\big)^{a_2^1}\big(\Lambda^3\big)^{a_2^2}\cdots\big(\Lambda_{1^{\kappa-2}}^{2\kappa-1}\big)^{a_2^\kappa}}{(f_1')^{-m+a_2^1+3a_2^2+\cdots+(2\kappa-1)a_2^\kappa}}.$$

What is the largest power of f_1' as a denominator in the monomials of the right hand side? Supposing for a while that the quantities a_i^j are nonnegative real numbers, instead of integers, we may simplify step by step the definition of this maximum:

$$\max_{a_1^1+a_2^1+2a_2^2+\cdots+\kappa a_2^\kappa=m}\Big(-m+a_2^1+3a_2^2+\cdots+(2\kappa-1)a_2^\kappa\Big) =$$
$$= \max\Big|_{a_1^1=0}\Big(\text{substitute } a_2^1=m-2a_2^2-\cdots-\kappa a_2^\kappa \text{ in the same quantity}\Big)$$
$$= \max_{a_2^1+2a_2^2+3a_2^3+\cdots+\kappa a_2^\kappa=m}\Big(a_2^2+2a_2^3+\cdots+(\kappa-1)a_2^\kappa\Big) \qquad \text{[divide and multiply by 2]}$$
$$= \frac{1}{2}\cdot\max_{2a_2^2+3a_2^3+\cdots+\kappa a_2^\kappa=m}\Big(2a_2^2+4a_2^3+\cdots+(2\kappa-2)a_2^\kappa\Big) \qquad \text{[substitute } 2a_2^2\text{]}$$
$$= \frac{m}{2}+\frac{1}{2}\cdot\max_{3a_2^3+4a_2^4+\cdots+\kappa a_2^\kappa=m}\Big(a_2^3+2a_2^4+\cdots+(\kappa-2)a_2^\kappa\Big) \qquad \text{[divide and multiply by 3]}$$
$$= \frac{m}{2}+\frac{1}{2\cdot 3}\cdot\max_{3a_2^3+4a_2^4+\cdots+\kappa a_2^\kappa=m}\Big(3a_2^3+6a_2^4+\cdots+3(\kappa-2)a_2^\kappa\Big) \qquad \text{[substitute } 3a_2^3\text{]}$$
$$= \frac{m}{2}+\frac{m}{6}+\frac{1}{3\cdot 4}\cdot\max_{4a_2^4+5a_2^5+\cdots+\kappa a_2^\kappa=m}\Big(4a_2^4+8a_2^5+\cdots+4(\kappa-3)a_2^\kappa\Big)$$
$$= \frac{2}{3}\,m+\frac{1}{12}\,m+\frac{1}{4\cdot 5}\cdot\max_{5a_2^5+6a_2^6+\cdots+\kappa a_2^\kappa=m}\Big(5a_2^6+10a_2^6+\cdots+5(\kappa-4)a_2^\kappa\Big)$$
$$= \frac{3}{4}\,m+\frac{1}{20}\,m+\frac{1}{5\cdot 6}\cdot\max_{6a_2^6+7a_2^7+\cdots+\kappa a_2^\kappa=m}\Big(6a_2^6+12a_2^7+\cdots+6(\kappa-5)a_2^\kappa\Big)$$
$$= \frac{4}{5}\,m+\cdots \qquad \text{[observe the induction]}$$
$$= \frac{\kappa-1}{\kappa}\,m.$$

Thus, when the a_2^i are restricted to be integers, we in any case deduce that the maximally negative power of f_1' is $\geqslant -\frac{(\kappa-1)}{\kappa}\, m$. Reorganizing the result, we then obtain a representation of $\mathsf{P}\big(j^\kappa f\big)$, valid by construction in the subset $\{f_1' \neq 0\}$ of the jet space $J^\kappa\big(C,\, \mathbb{C}^n\big)$, as a sum of powers of f_1' :

$$\mathsf{P}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa}\, m \leqslant a \leqslant m} (f_1')^a \cdot \mathsf{P}_a\Big(f_2',\, \Lambda^3,\, \Lambda_1^5,\, \Lambda_{1,1}^7,\, \ldots,\, \Lambda_{1^{\kappa-2}}^{2\kappa-1}\Big),$$

multiplied by certain polynomials P_a which depend upon P and are *not* arbitrary. In fact, by reduction to the same denominator, we may write:

$$\mathsf{P}\big(j^\kappa f\big) = \frac{\mathsf{Q}\big(f_1',\, f_2',\, \Lambda^3,\, \ldots,\, \Lambda_{1^{\kappa-2}}^{2\kappa-1}\big)}{(f_1')^{-a_0}},$$

where a_0 is the smallest exponent a of f_1' above. Chasing the denominator in the case where a_0 is negative (this would be unnecessary if $a_0 \geqslant 0$), we get an identity $(f_1')^{a_0} \cdot \mathsf{P} \equiv \mathsf{Q}$ between two polynomials valid in $\{f_1' \neq 0\}$, hence everywhere by the principle of analytic continuation. Thus, the restriction $f_1' \neq 0$ is removed.

Weighted homogeneities. Let $\mu \in \mathbb{Z}$ be an integer, possibly negative. A rational expression $\mathsf{R}\big(j^\kappa f\big) \in \mathrm{Frac}\big(\mathbb{C}[j^\kappa f]\big)$ will be said to be of *weighted homogeneous degree* μ when for every complex weighted δ-dilation which acts in accordance with the number of primes:

$$\delta \cdot j^\kappa f := \big(\delta\, f_1',\, \delta f_2',\, \delta^2\, f_1'',\, \delta^2\, f_2'', \cdots,\, \delta^\kappa\, f_1^{(\kappa)},\, \delta^\kappa\, f_2^{(\kappa)}\big),$$

the dilation factor escapes the parentheses to exactly the μ-th power:

$$\mathsf{R}\big(\delta \cdot j^\kappa f\big) = \delta^\mu \cdot \mathsf{R}\big(j^\kappa f\big).$$

When R is a polynomial, μ is then the total, constant number of primes of each monomial.

By choosing the reparametrization ϕ to just be a δ-dilation in the source, with nonzero $\delta \in \mathbb{C}$, we immediataly see that our original jet polynomial $\mathsf{P} \in \mathsf{E}_{\kappa,m}^2$ — hence also its rational expression obtained above — must in particular be weighted homogeneous of degree m:

$$\mathsf{P}\big(\delta \cdot j^\kappa f\big) = \delta^m \cdot \mathsf{P}\big(j^\kappa f\big).$$

In addition and in particular, using the definition $\Lambda_{1^{\lambda-2}}^{2\lambda-1} = \big[\Lambda_{1^{\lambda-3}}^{2\lambda-3},\, f_1'\big]$, one easily verifies by induction that the invariant $\Lambda_{1^{\lambda-2}}^{2\lambda-1}$ is homogeneous of degree equal to its weight $2\lambda - 1$, an integer which we had already specified as the upper index:

$$\Lambda_{1^{\lambda-2}}^{2\lambda-1}\big(\delta \cdot j^\kappa f\big) = \delta^{2\lambda-1} \cdot \Lambda_{1^{\lambda-2}}^{2\lambda-1}\big(j^\lambda f\big).$$

In an analogous fashion, introducing some new extra independent variables $F_1,\, F_2,\, A^3,\, \ldots,\, A^{2\kappa-1}$ corresponding to $f_1', f_2', \Lambda^3,\, \ldots,\, \Lambda^{2\kappa-1}_{1^{\kappa-2}}$, a rational expression $\mathsf{T} \in \mathrm{Frac}\big(\mathbb{C}[F_1, F_2, A^3, \ldots, A^{2\kappa-1}]\big)$ will be said to be of *weighted homogeneous degree* μ when it enjoys:

$$\mathsf{T}\big(\delta F_1,\, \delta F_2,\, \delta^3 A^3,\, \delta^5 A^5,\, \ldots,\, \delta^{2\kappa-1} A^{2\kappa-1}\big) = \delta^\mu \cdot \mathsf{T}\big(F_1,\, F_2,\, A^3,\, A^5,\, \ldots,\, A^{2\kappa-1}\big),$$

for every $\delta \in \mathbb{C}$.

Lemma. *In dimension $n = 2$ for jets of order $\kappa \geqslant 2$, every jet polynomial $\mathsf{P} = \mathsf{P}\big(j^\kappa f\big)$ invariant by reparametrization writes under the form:*

$$\boxed{\mathsf{P}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa} m \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a\Big(f_2',\, \Lambda^3,\, \Lambda^5_1,\, \Lambda^7_{1,1},\, \cdots,\, \Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big)},$$

where the integer a takes possibly negative values in the interval $\big[-\frac{\kappa-1}{\kappa}\, m,\, m\big]$, for certain weighted homogeneous polynomials:

$$\mathsf{P}_a = \sum_{b_2+3c_3+\cdots+(2\kappa-1)c_{2\kappa-1}=m-a} \mathsf{coeff} \cdot \big(F_2\big)^{b_2} \big(A^3\big)^{c_3} \cdots \big(A^{2\kappa-1}\big)^{c_{2\kappa-1}}$$

of weighted degree $m - a$.

Conversely, for every collection of such weighted homogeneous polynomials P_a in $\mathbb{C}\big[F_2,\, A^3,\, \ldots,\, A^{2\kappa-1}\big]$ of weighted degree $m - a$ indexed by an integer a running in $\big[-\frac{\kappa-1}{\kappa}\, m,\, m\big]$ such that the reduction to the same denominator and the simplification of the finite sum:

$$\mathsf{R}\big(j^\kappa f\big) := \sum_{-\frac{\kappa-1}{\kappa} m \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a\Big(f_2',\, \Lambda^3,\, \Lambda^5_1,\, \Lambda^7_{1,1},\, \cdots,\, \Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big)$$

yields a true *jet* polynomial *in $\mathbb{C}\big[j^\kappa f\big]$, then $\mathsf{R}(j^\kappa f)$ is a polynomial invariant by reparametrization belonging to $\mathsf{E}^2_{\kappa,m}$.*

Proof. We saw that P is homogeneous of degree m, namely:

$$\sum_{-\frac{\kappa-1}{\kappa} m \leqslant a \leqslant m} \delta^a \big(f_1'\big)^a\, \mathsf{P}_a\Big(\delta f_2',\, \delta^3 \Lambda^3,\, \ldots,\, \delta^{2\kappa-1} \Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big) =$$

$$= \delta^m \cdot \sum_{-\frac{\kappa-1}{\kappa} m \leqslant a \leqslant m} \big(f_1'\big)^a\, \mathsf{P}_a\Big(f_2',\, \Lambda^3,\, \ldots,\, \Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big).$$

By algebraic independency of f_1' with respect to $\mathbb{C}\big[f_2', \Lambda^3, \ldots, \Lambda^{2\kappa-1}_{1^{\kappa-2}}\big]$, we then may identify powers of f_1', getting for each a:

$$\mathsf{P}_a\Big(\delta f_2',\, \delta^3 \Lambda^3, \ldots, \delta^{2\kappa-1} \Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big) = \delta^{m-a}\, \mathsf{P}_a\Big(f_2', \Lambda^3, \ldots, \Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big).$$

Further, the algebraic independency of $f_2', \Lambda^3, \dots, \Lambda^{2\kappa-1}_{1^{\kappa-2}}$ then entails that the homogeneities:

$$\mathsf{P}_a\Big(\delta F_2, \delta^3 A^3, \dots, \delta^{2\kappa-1} A^{2\kappa-1}\Big) = \delta^{m-a} \cdot \mathsf{P}_a\Big(F_2, A^3, \dots, A^{2\kappa-1}\Big)$$

hold in the polynomial algebra $\mathbb{C}\big[F_2, A^3, \dots, A^{2\kappa-1}\big]$. This gives the claimed representation of any $\mathsf{P} \in \mathsf{E}^2_{\kappa,m}$.

Conversely, assuming that the P_a are homogeneous in this way, then for any reparametrization ϕ, setting $g_i = f_i \circ \phi$ for $i = 1, 2$ and recalling:

$$\Lambda^{2\lambda-1}_{1^{\lambda-2}}\big(j^\lambda g\big) = (\phi')^{2\lambda-1}\, \Lambda^{2\lambda-1}_{1^{\lambda-2}}\big(j^\lambda f\big),$$

we immediately deduce that:

$$\begin{aligned}
&\mathsf{P}_a\Big(g_2', \Lambda^3\big(j^3 g\big), \dots, \Lambda^{2\kappa-1}_{1^{\kappa-2}}\big(j^\kappa g\big)\Big) = \\
&\qquad = \mathsf{P}_a\Big(\phi' \cdot f_2' \circ \phi,\ (\phi')^3 \cdot \Lambda^3 \circ \phi,\ \dots,\ (\phi')^{2\kappa-1} \cdot \Lambda^{2\kappa-1}_{1^{\kappa-2}} \circ \phi\Big) \\
&\qquad = (\phi')^{m-a} \cdot \mathsf{P}_a\Big(f_2',\ \Lambda^3,\ \dots,\ \Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big),
\end{aligned}$$

whence multiplication by $(g_1')^a = (\phi')^a\, (f_1')^a$ and summation gives:

$$\begin{aligned}
&\sum_{-\frac{\kappa-1}{\kappa}m \leqslant a \leqslant m} {g_1'}^a\, \mathsf{P}_a\Big(g_2',\ \Lambda^3\big(j^3 g\big),\ \dots,\ \Lambda^{2\kappa-1}_{1^{\kappa-2}}\big(j^\kappa g\big)\Big) = \\
&\qquad\qquad (\phi')^m \cdot \sum_{-\frac{\kappa-1}{\kappa}m \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a\Big(f_2',\ \Lambda^3,\ \dots,\ \Lambda^{2\kappa-1}_{1^{\kappa-2}}\Big) \circ \phi,
\end{aligned}$$

which exactly means, as soon as such a rational sum represents a true polynomial, that it belongs to $\mathsf{E}^2_{\kappa,m}$, *quod erat demonstrandum*. $\square$

Arbitrary dimension. To generalize the preceding proposition, suppose now that $n \geqslant 2$ is arbitrary. The same trick of reparametrizing each f_i by $\phi = f_1^{-1}$ gives birth to a collection of *initial invariants* appearing as numerators of:

$$\big(f_i \circ f_1^{-1}\big)^{(\lambda)} = \frac{\Lambda^{2\lambda-1}_{1,i;\,1^{\lambda-2}}\big(j^\lambda f\big)}{(f_1')^{2\lambda-1}},$$

for $i = 2, 3, \dots, n$, where the Λ-invariants depending on i and on λ are defined inductively by successively bracketing with f_1':

$$\Lambda^3_{1,i} := \big[f_i',\ f_1'\big], \qquad\qquad \Lambda^5_{1,i;\,1} := \big[\Lambda^3_{1,i},\ f_1'\big],$$

$$\text{and generally:} \qquad \Lambda^{2\lambda-1}_{1,i;\,1^{\lambda-2}} := \big[\Lambda^{2\lambda-3}_{1,i;\,1^{\lambda-3}},\ f_1'\big], \qquad \text{for } 3 \leqslant \lambda \leqslant \kappa.$$

Our considerations about brackets show that these polynomials are effectively invariant by reparametrization. Furthermore:

Fact. *The $n+(n-1)(\kappa-1)$ invariants:*

$$\begin{array}{cccccc} f_1', & f_2', & f_3', & f_4', & \dots, & f_n', \\ & \Lambda^3_{1,2}, & \Lambda^3_{1,3}, & \Lambda^3_{1,4}, & \dots, & \Lambda^3_{1,n}, \\ & \Lambda^5_{1,2;\,1}, & \Lambda^5_{1,3;\,1}, & \Lambda^5_{1,4;\,1}, & \dots, & \Lambda^5_{1,n;\,1}, \\ & \dots & \dots & \dots & \dots & \dots \\ & \Lambda^{2\kappa-1}_{1,2;\,1^{\kappa-2}}, & \Lambda^{2\kappa-1}_{1,3;\,1^{\kappa-2}}, & \Lambda^{2\kappa-1}_{1,4;\,1^{\kappa-2}}, & \dots, & \Lambda^{2\kappa-1}_{1,n;\,1^{\kappa-2}} \end{array}$$

are mutually algebraically independent.

Indeed, $\Lambda^{2\lambda-1}_{1,i;\,1^{\lambda-2}}$ contains the monomial $f_i^{(\lambda)} f_1'^{\,\lambda-1}$, while the invariants $\Lambda^{2\lambda-3}_{1,j;\,1^{\lambda-3}}$ only depend upon $j^{\lambda-1}f$.

Reasonings similar to the ones developed above yield the following lemma, valuable for any $n \geqslant 1$ and any $\kappa \geqslant 1$.

Lemma. *In dimension $n \geqslant 1$ and for jets of order $\kappa \geqslant 1$, every polynomial $\mathsf{P} = \mathsf{P}\big(j^\kappa f\big)$ invariant by reparametrization writes under the form:*

$$\boxed{\mathsf{P}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa}m \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a \begin{pmatrix} f_2', & f_3', & f_4', & \dots, & f_n', \\ \Lambda^3_{1,2}, & \Lambda^3_{1,3}, & \Lambda^3_{1,4}, & \dots, & \Lambda^3_{1,n}, \\ \dots & \dots & \dots & \dots & \dots \\ \Lambda^{2\kappa-1}_{1,2;\,1^{\kappa-2}}, & \Lambda^{2\kappa-1}_{1,3;\,1^{\kappa-2}}, & \Lambda^{2\kappa-1}_{1,4;\,1^{\kappa-2}}, & \dots, & \Lambda^{2\kappa-1}_{1,n;\,1^{\kappa-2}} \end{pmatrix},}$$

where the integer a takes all possibly negative values in the interval $\big[-\frac{\kappa-1}{\kappa}m, m\big]$, for certain weighted homogeneous polynomials:

$$\mathsf{P}_a = \sum_{\substack{b_2+\dots+b_n+3c_2+\dots+3c_n+ \\ +\dots+(2\kappa-1)q_2+\dots+(2\kappa-1)q_n=m-a}} \mathsf{coeff} \cdot \prod_{i=2}^n \big(F_i\big)^{b_i} \prod_{i=2}^n \big(A_i^3\big)^{c_i} \cdots \prod_{i=2}^n \big(A_i^{2\kappa-1}\big)^{q_i}$$

of weighted degree $m-a$, namely satisfying:

$$\mathsf{P}_a\Big(\delta\, F_i,\ \delta^3\, A_i^3,\ \dots,\ \delta^{2\kappa-1}\, A_i^{2\kappa-1}\Big) = \delta^{m-a} \cdot \mathsf{P}_a\Big(F_i,\ A_i^3,\ \dots,\ A_i^{2\kappa-1}\Big).$$

Conversely, for every collection of such weighted homogeneous polynomials P_a in $\mathbb{C}\big[F_i, A_i^3, \dots, A_i^{2\kappa-1}\big]$ of weighted degree $m-a$ indexed by an integer a running in $\big[-\frac{\kappa-1}{\kappa}m,\ m\big]$ such that the reduction to the same denominator and the simplification of the finite sum:

$$\mathsf{R}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa}m \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a \begin{pmatrix} f_2', & f_3', & f_4', & \dots, & f_n', \\ \Lambda^3_{1,2}, & \Lambda^3_{1,3}, & \Lambda^3_{1,4}, & \dots, & \Lambda^3_{1,n}, \\ \dots & \dots & \dots & \dots & \dots \\ \Lambda^{2\kappa-1}_{1,2;\,1^{\kappa-2}}, & \Lambda^{2\kappa-1}_{1,3;\,1^{\kappa-2}}, & \Lambda^{2\kappa-1}_{1,4;\,1^{\kappa-2}}, & \dots, & \Lambda^{2\kappa-1}_{1,n;\,1^{\kappa-2}} \end{pmatrix}$$

yields a true *jet* polynomial *in $\mathbb{C}\big[j^\kappa f\big]$, then $\mathsf{R}\big(j^\kappa f\big)$ is a polynomial invariant by reparametrization belonging to $\mathsf{E}^n_{\kappa,m}$.*

§6. Description of the algorithm in dimension $n = 2$ for jet level $\kappa = 4$

Necessity of negative powers of f_1'. Our aim now is to prove[43] the theorem which describes the algebraic structure of E_4^2, *see* p. 207. We will thus illustrate in a concrete case the general algorithm which will be presented in Section 9 below. We hope this will make the general considerations intuitively clearer.

Proof. Compared to the initial rational representation:

$$\mathsf{P}\big(j^4 f\big) = \sum_{-\frac{3}{4}m \leqslant a \leqslant m} (f_1')^a \, \mathsf{P}_a\big(f_2',\, \Lambda^3,\, \Lambda_1^5,\, \Lambda_{1,1}^7\big)$$

of an arbitrary polynomial $\mathsf{P} \in \mathsf{E}_4^2$ that was furnished by the lemma on p. 215, the theorem on p. 207 states that 4 further invariants, namely Λ_2^5, $\Lambda_{1,2}^7$, $\Lambda_{2,2}^7$ and M^8, are necessary to generate the full algebra E_4^2. In fact, by looking at the 9 syzygies listed in the theorem in question, one may easily obtain the expression of these 4 further invariants in $\mathbb{C}\big[f_2',\, \Lambda^3,\, \Lambda_1^5,\, \Lambda_{1,1}^7\big]\big[\frac{1}{f_1'}\big]$, namely:

$$\begin{aligned}
\Lambda_2^5 &= \frac{f_2'\Lambda_1^5 - 3\,\Lambda^3\Lambda^3}{f_1'},\\
\Lambda_{1,2}^7 &= \frac{f_2'\Lambda_{1,1}^7 - 5\,\Lambda^3\Lambda_1^5}{f_1'},\\
\Lambda_{2,2}^7 &= \frac{f_2'f_2'\Lambda_{1,1}^7 - 10\,f_2'\Lambda^3\Lambda_1^5 + 15\,\Lambda^3\Lambda^3\Lambda^3}{f_1'f_1'f_1'},\\
M^8 &= \frac{3\,\Lambda^3\Lambda_{1,1}^7 - 5\,\Lambda_1^5\Lambda_1^5}{f_1'f_1'}.
\end{aligned}$$

Crucial observation. *So when one substitutes, in an arbitrary polynomial:*

$$\mathsf{P}\big(f_1',\, f_2',\, \Lambda^3,\, \Lambda_1^5,\, \Lambda_{1,1}^7,\, \Lambda_{1,2}^7,\, \Lambda_{2,2}^7,\, M^8\big)$$

these rational representations of Λ_2^5, $\Lambda_{1,2}^7$, $\Lambda_{2,2}^7$, M^8, one indeed obtains a rational expression as the one above which necessarily and unavoidably *incorporates negative powers of f_1'.*

Well, how then should we interpret our initial rational expression? Why are the 4 further invariants Λ_2^5, $\Lambda_{1,2}^7$, $\Lambda_{2,2}^7$ and M^8 invisible in it?

First of all, as a preliminary, we must at least show that the 9 fundamental invariants f_1', f_2', Λ^3, Λ_1^5, Λ_2^5, $\Lambda_{1,1}^7$, $\Lambda_{1,2}^7$, $\Lambda_{2,2}^7$ and M^8 are mutually independent.

[43] An alternative proof was provided in [21].

On this purpose, we set $f_1' = 0$ in these 9 fundamental invariants, and this then leaves us with the 8 invariants:

$$f_2', \quad \Lambda^3\big|_0, \quad \Lambda_1^5\big|_0, \quad \Lambda_2^5\big|_0, \quad \Lambda_{1,1}^7\big|_0, \quad \Lambda_{1,2}^7\big|_0, \quad \Lambda_{2,2}^7\big|_0, \quad M^8\big|_0,$$

which we shall shortly call *restricted invariants*, our notation being self-evident. The following assertion is simply checked by inspecting the explicit expressions.

Fact. *The four restricted invariants:*

$$f_2', \qquad \Lambda^3\big|_0 = -f_1''f_2', \qquad \Lambda_2^5\big|_0 = 3\, f_1''f_2'f_1'' \qquad \text{and}$$
$$\Lambda_{2,2}^7\big|_0 = \big(-f_1''''f_2' + \Delta^{'','''}\big)\, f_2'f_2' + 10\, f_1'''f_2'f_2'f_2'' - 15\, f_1''f_2'f_2''f_2''$$

are mutually algebraically independent.

It follows that f_1', f_2', Λ^3, Λ_2^5 and $\Lambda_{2,2}^7$ are algebraically independent. Switching lower indices, f_1', f_2', Λ^3, Λ_1^5 and $\Lambda_{1,1}^7$ are also algebraically independent.

Next, by looking at the 9 syzygies listed in the theorem, we may express each one of the 8 restricted invariants by means of the above four algebraically independent restricted invariants, provided that one allows a division by f_2':

$$\begin{aligned}
&f_2',\\
&\Lambda^3\big|_0,\\
&\Lambda_1^5\big|_0 = 3\,\frac{\Lambda^3|_0\,\Lambda^3|_0}{f_2'},\\
&\Lambda_2^5\big|_0,\\
&\Lambda_{1,1}^7\big|_0 = 15\,\frac{\Lambda^3|_0\,\Lambda^3|_0\,\Lambda^3\,|_0}{f_2'f_2'},\\
&\Lambda_{1,2}^7\big|_0 = 5\,\frac{\Lambda^3|_0\,\Lambda_2^5|_0}{f_2'},\\
&\Lambda_{2,2}^7\big|_0,\\
&M^8\big|_0 = \frac{3\,\Lambda^3|_0\,\Lambda_{2,2}^7 - 5\,\Lambda_2^5|_0\,\Lambda_2^5|_0}{f_2'f_2'}.
\end{aligned}$$

In fact, all divisions by f_2' or by $f_2'f_2'$ do cancel out after simplification.

Lemma. *The 9 fundamental invariants f_1', f_2', Λ^3, Λ_1^5, Λ_2^5, $\Lambda_{1,1}^7$, $\Lambda_{1,2}^7$, $\Lambda_{2,2}^7$ and M^8 are mutually independent. More precisely, f_1', f_2', Λ^3, Λ_1^5 and $\Lambda_{1,1}^7$ are algebraically independent and there exist no polynomial representation*

of either one of the following four forms:

$$\begin{aligned}
\Lambda^5_2 &= \mathsf{polynomial}\big(f'_1,\, f'_2,\, \Lambda^3,\, \Lambda^5_1,\, \Lambda^7_{1,1}\big),\\
\Lambda^7_{1,2} &= \mathsf{polynomial}\big(f'_1,\, f'_2,\, \Lambda^3,\, \Lambda^5_1,\, \Lambda^5_2,\, \Lambda^7_{1,1}\big),\\
\Lambda^7_{2,2} &= \mathsf{polynomial}\big(f'_1,\, f'_2,\, \Lambda^3,\, \Lambda^5_1,\, \Lambda^5_2,\, \Lambda^7_{1,1},\, \Lambda^7_{1,2}\big),\\
M^8 &= \mathsf{polynomial}\big(f'_1,\, f'_2,\, \Lambda^3,\, \Lambda^5_1,\, \Lambda^5_2,\, \Lambda^7_{1,1},\, \Lambda^7_{1,2},\, \Lambda^7_{2,2}\big).
\end{aligned}$$

Proof. By setting $f'_1 = 0$ in a polynomial representation such as the first one and by replacing the values of some of the restricted invariants, we get:

$$\begin{aligned}
\Lambda^5_2\big|_0 &= \sum \mathsf{coeff} \cdot \big(f'_2\big)^b \big(\Lambda^3\big)^c \big(\Lambda^5_1\big)^d \big(\Lambda^7_{1,1}\big)^e\Big|_0\\
&= \sum \mathsf{coeff} \cdot \big(f'_2\big)^b \big(\Lambda^3\big)^c \left(3\,\frac{\Lambda^3\,\Lambda^3}{f'_2}\right)^d \left(15\,\frac{\Lambda^3\,\Lambda^3\,\Lambda^3}{f'_2 f'_2}\right)^e\Bigg|_0\\
&= \sum \mathsf{coeff} \cdot \big(f'_2\big)^{b-d-2e} \big(\Lambda^3\big|_0\big)^{c+2d+3e},
\end{aligned}$$

where the exponents $b, c, d, e \geqslant 0$ are nonnegative integers. But this is impossible, because $\Lambda^5_2\big|_0$ is transcendental over $\mathbb{C}\big[f'_2,\, \Lambda^3\big|_0\big]$.

Next, for the second hypothetical representation, the same kind of replacement yields:

$$5\,\frac{\Lambda^3\,\Lambda^5_2}{f'_2}\Big|_0 = \sum \mathsf{coeff} \cdot \big(f'_2\big)^{b-d-2f} \big(\Lambda^3\big)^{c+2d+3f} \big(\Lambda^5_2\big)^e\Big|_0.$$

So, by identifying the powers of the restricted algebraically independent invariants f'_2, $\Lambda^3\big|_0$, $\Lambda^5_2\big|_0$, we get three equations between integers:

$$-1 = b - d - 2f, \qquad 1 = c + 2d + 3f, \qquad 1 = e,$$

which are seen to be impossible, since $b, c, d, e, f \geqslant 0$, the second one yielding $d = f = 0$, while the first one then reads $-1 = b$.

Similarly as for the first one, the third hypothetical representation is *a priori* excluded, because the right hand side does not depend upon $\Lambda^7_{2,2}\big|_0$ at all.

Finally, the fourth hypothetical representation amounts to:

$$\frac{3\,\Lambda^3\,\Lambda^7_{2,2} - 5\,\Lambda^5_2\,\Lambda^5_2}{f'_2 f'_2}\Big|_0 = \sum \mathsf{coeff}\cdot\big(f'_2\big)^{b-d-2f-g} \big(\Lambda^3\big)^{c+2d+3f+g} \big(\Lambda^5_2\big)^{e+g} \big(\Lambda^7_{2,2}\big)^h\Big|_0,$$

hence looking at the representation of the first term $\frac{3\,\Lambda^3\,\Lambda^7_{2,2}}{f'_2 f'_2}$ of the left hand side, and identifying powers, we get three equations:

$$-2 = b - d - 2f - g, \qquad 1 = c + 2d + 3f + g, \qquad 1 = h.$$

The second one implies $d = f = 0$ and $g = 0$ or 1, whence the first one then becomes impossible. $\square$

First loop of the algorithm. The initial expression:

$$\mathsf{P}\big(j^4 f\big) = \sum_{-\frac{3}{4}m \leqslant a \leqslant m} (f_1')^a \, \mathsf{P}_a\big(f_2',\, \Lambda^3,\, \Lambda_1^5,\, \Lambda_{1,1}^7\big)$$

shows five invariants f_1', f_2', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$, and four restricted invariants f_2', $\Lambda^3\big|_0$, $\Lambda_1^5\big|_0$, $\Lambda_{1,1}^7\big|_0$. To determine the structure of E_4^2, here is the first loop of our algorithm.

• Compute the ideal of relations[44] between the 4 known restricted invariants:

$$\mathsf{Ideal} - \mathsf{Rel}\Big(f_2'\big|_0,\, \Lambda^3\big|_0,\, \Lambda_1^5\big|_0,\, \Lambda_{1,1}^7\big|_0\Big),$$

namely a generating set of the ideal of all polynomials $\mathscr{Q}\big(F_2,\, A^3,\, A^5,\, A^7\big)$ in four variables that give zero, identically, after substituting these four restricted invariants.

• Get as generators of this ideal of relations the three relations, valuable for $f_1' = 0$:

$$\begin{aligned}
0 &\equiv 3\,\Lambda^3\Lambda^3 - f_2'\Lambda_1^5\Big|_0,\\
0 &\equiv 5\,\Lambda^3\Lambda_1^5 - f_2'\Lambda_{1,1}^7\Big|_0,\\
0 &\equiv 5\,\Lambda_1^5\Lambda_1^5 - 3\,\Lambda^3\Lambda_{1,1}^7\Big|_0.
\end{aligned}$$

• Consequently, without setting $f_1' = 0$, there should exist remainders that are a multiple of f_1':

$$\begin{aligned}
0 &\equiv 3\,\Lambda^3\Lambda^3 - f_2'\Lambda_1^5 + f_1' \times \mathsf{something},\\
0 &\equiv 5\,\Lambda^3\Lambda_1^5 - f_2'\Lambda_{1,1}^7 + f_1' \times \mathsf{something},\\
0 &\equiv 5\,\Lambda_1^5\Lambda_1^5 - 3\,\Lambda^3\Lambda_{1,1}^7 + f_1' \times \mathsf{something}.
\end{aligned}$$

• Each "something" necessarily also is an invariant belonging to E_4^2, because it is a polynomial and we can write it as $\frac{1}{f_1'}$ times a corresponding quadratic expression in the already known invariants f_2', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$.

• Find the maximal power by which f_1' factors each remaining "something".

[44] We will discuss in a while two ways of computing ideal of relations. The data reproduced here are obtained by means of Gröbner bases computations.

• Get the three identically satisfied relations:

$$\begin{aligned}
0 &\equiv 3\,\Lambda^3\Lambda^3 - f_2'\Lambda_1^5 + f_1'\Lambda_2^5,\\
0 &\equiv 5\,\Lambda^3\Lambda_1^5 - f_2'\Lambda_{1,1}^7 + f_1'\Lambda_{1,2}^7,\\
0 &\equiv 5\,\Lambda_1^5\Lambda_1^5 - 3\,\Lambda^3\Lambda_{1,1}^7 + f_1'f_1'M^8,
\end{aligned}$$

where the appearing new invariants are already known from the statement of the theorem.

• Test whether or not the so obtained three invariants:

$$\Lambda_2^5, \qquad \Lambda_{1,2}^7, \qquad M^8,$$

belong or do not belong to the algebra generated by the previously known invariants. Here in fact, neither Λ_2^5 nor $\Lambda_{1,2}^7$, nor M^8 belongs to $\mathbb{C}\big[f_1', f_2', \Lambda^3, \Lambda_1^5, \Lambda_{1,1}^7\big]$, as we have already verified.

Second loop of the algorithm. We now restart the process with our new, extended list of 7 invariants f_1', f_2', Λ^3, Λ_1^5, Λ_2^5, $\Lambda_{1,1}^7$, $\Lambda_{1,2}^7$ and M^8.

• Compute the ideal of relations between the 6 restricted invariants known at this stage:

$$\mathsf{Ideal-Rel}\Big(f_2'\big|_0,\ \Lambda^3\big|_0,\ \Lambda_1^5\big|_0,\ \Lambda_2^5\big|_0,\ \Lambda_{1,1}^7\big|_0,\ \Lambda_{1,2}^7\big|_0,\ M^8\big|_0\Big).$$

• Get the 6 equations:

$$\begin{aligned}
0 &\equiv 3\,\Lambda^3\Lambda^3 - f_2'\Lambda_1^5\Big|_0,\\
0 &\equiv 5\,\Lambda^3\Lambda_1^5 - f_2'\Lambda_{1,1}^7\Big|_0,\\
0 &\equiv 5\,\Lambda_1^5\Lambda_1^5 - 3\,\Lambda^3\Lambda_{1,1}^7\Big|_0,\\
0 &\equiv 5\,\Lambda^3\Lambda_2^5 - f_2'\Lambda_{1,2}^7\Big|_0,\\
0 &\equiv 5\,\Lambda_1^5\Lambda_2^5 - 3\,\Lambda^3\Lambda_{1,2}^7\Big|_0,\\
0 &\equiv \Lambda_{1,1}^7\Lambda_2^5 - \Lambda_1^5\Lambda_{1,2}^7\Big|_0.
\end{aligned}$$

• Compute the remainders behind a power of f_1':

$$\begin{aligned}
0 &\equiv 3\,\Lambda^3\Lambda^3 - f_2'\Lambda_1^5 + f_1'\Lambda_2^5,\\
0 &\equiv 5\,\Lambda^3\Lambda_1^5 - f_2'\Lambda_{1,1}^7 + f_1'\Lambda_{1,2}^7,\\
0 &\equiv 5\,\Lambda_1^5\Lambda_1^5 - 3\,\Lambda^3\Lambda_{1,1}^7 + f_1'f_1'M^8,
\end{aligned}$$

$$\begin{aligned}
0 &\equiv 5\,\Lambda^3\Lambda_2^5 - f_2'\Lambda_{1,2}^7 + f_1'\Lambda_{2,2}^7,\\
0 &\equiv 5\,\Lambda_1^5\Lambda_2^5 - 3\,\Lambda^3\Lambda_{1,2}^7 + f_1'f_2'M^8,\\
0 &\equiv \Lambda_{1,1}^7\Lambda_2^5 - \Lambda_1^5\Lambda_{1,2}^7 + f_1'\Lambda^3M^8.
\end{aligned}$$

• Get only one new invariant $\Lambda^7_{2,2}$ not belonging to the algebra generated by already known invariants $\mathbb{C}\big[f_1', f_2', \Lambda^3, \Lambda^5_1, \Lambda^5_2, \Lambda^7_{1,1}, \Lambda^7_{1,2}, M^8\big]$.

Third loop of the algorithm. The final list of syzygies, after filling in the remainders and testing whether new invariants appear, reads:

$$\begin{aligned}
0 &\equiv 3\,\Lambda^3\Lambda^3 - f_2'\Lambda^5_1 + f_1'\Lambda^5_2,\\
0 &\equiv 5\,\Lambda^3\Lambda^5_1 - f_2'\Lambda^7_{1,1} + f_1'\Lambda^7_{1,2},\\
0 &\equiv 5\,\Lambda^5_1\Lambda^5_1 - 3\,\Lambda^3\Lambda^7_{1,1} + f_1'f_1'M^8,\\
0 &\equiv 5\,\Lambda^3\Lambda^5_2 - f_2'\Lambda^7_{1,2} + f_1'\Lambda^7_{2,2},\\
0 &\equiv 5\,\Lambda^5_1\Lambda^5_2 - 3\,\Lambda^3\Lambda^7_{1,2} + f_1'f_2'M^8,\\
0 &\equiv \Lambda^7_{1,1}\Lambda^5_2 - \Lambda^5_1\Lambda^7_{1,2} + f_1'\Lambda^3M^8,\\
0 &\equiv 5\,f_2'\Lambda^5_1M^8 + 3\,\Lambda^7_{1,2}\Lambda^7_{1,2} - 3\,\Lambda^7_{1,1}\Lambda^7_{2,2} + 0,\\
0 &\equiv f_2'\Lambda^3M^8 + \Lambda^5_2\Lambda^7_{1,2} - \Lambda^5_1\Lambda^7_{2,2} + 0,\\
0 &\equiv f_2'f_2'M^8 + 5\,\Lambda^5_2\Lambda^5_2 - 3\,\Lambda^3\Lambda^7_{2,2} + 0.
\end{aligned}$$

Three new syzygies only appear, namely the last three ones above, and for each of them, the remainders that are a multiple of f_1' are identically zero, which we specify explicitly by writing "$+0$". Importantly, *no new invariant appears at this stage*.

We then claim that the algorithm stops (*cf.* also Section 9), and that the following proposition holds true. In fact, the arguments of proof will follow from the general theorem of §9.

Proposition. *An arbitrary polynomial* $\mathsf{P} = \mathsf{P}\big(j^4j\big)$ *in* E^2_4 *invariant by reparametrization writes uniquely under the form:*

$$\begin{aligned}
\mathsf{P}\big(j^4j\big) = {}& \mathscr{Q}\big(f_1',\, f_2',\, \Lambda^7_{1,1},\, \Lambda^7_{2,2},\, M^8\big) + \Lambda^3\,\mathscr{R}\big(f_1',\, f_2',\, \Lambda^7_{1,1},\, \Lambda^7_{2,2},\, M^8\big) +\\
&+ \Lambda^5_1\,\mathscr{S}\big(f_1',\, f_2',\, \Lambda^7_{1,1},\, \Lambda^7_{2,2},\, M^8\big) + \Lambda^5_2\,\mathscr{T}\big(f_1',\, f_2',\, \Lambda^7_{1,1},\, \Lambda^7_{2,2},\, M^8\big) +\\
&+ \Lambda^7_{1,2}\,\mathscr{U}\big(f_1',\, f_2',\, \Lambda^7_{1,1},\, \Lambda^7_{2,2},\, M^8\big) + \Lambda^3\Lambda^7_{1,2}\,\mathscr{V}\big(f_1',\, f_2',\, \Lambda^7_{1,1},\, \Lambda^7_{2,2},\, M^8\big),
\end{aligned}$$

where $\mathscr{Q}$, $\mathscr{R}$, $\mathscr{S}$, $\mathscr{T}$, $\mathscr{U}$ *and* $\mathscr{V}$ *are complex polynomials in five variables subjected to no restriction.*

§7. Action of $\mathsf{GL}_n(\mathbb{C})$ and unipotent reduction

Sums of irreducible Schur representations. The cohomology of Schur bundles $\Gamma^{(\ell_1,\ell_2,\dots,\ell_n)}\,T^*_X$ on a complex algebraic projective hypersurface $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ being available through Hirzebruch's Riemann-Roch formula (§13 below), we should look for a decomposition of the Demailly-Semple bundle $\mathsf{E}^n_{\kappa,m}T^*_X$ as a direct sum of Schur bundles, at least in the cases where we understand the algebraic structure of the fiber algebras $\mathsf{E}^n_{\kappa,m}$. We recall that according to a fundamental theorem of representation theory ([16]), any group action of $\mathsf{GL}_n(\mathbb{C})$ on a space of polynomials is isomorphic to a certain direct sum of irreducible Schur representations.

Action of $\mathsf{GL}_n(\mathbb{C})$ on the jet space. On this purpose, similarly as in [29], we therefore define an appropriate linear action of $\mathsf{GL}_n(\mathbb{C})$ on the κ-th jet space $J^\kappa(\mathbb{C},\, \mathbb{C}^n)$. By definition, an arbitrary element w of $\mathsf{GL}_n(\mathbb{C})$ written in matrix form:

$$\mathsf{w} = \begin{pmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{pmatrix}$$

shall transform the collection $\big(f_1^{(\lambda)}, \dots, f_n^{(\lambda)}\big)$ of the n components of a κ-jet $j^\kappa f$ at each λ-th jet level, just by matrix multiplication:

$$\begin{cases} \mathsf{w} \cdot f_1^{(\lambda)} = w_{11}\, f_1^{(\lambda)} + \cdots + w_{1n}\, f_n^{(\lambda)} \\ \quad \cdots \qquad\qquad \cdots \qquad\qquad\qquad \cdots \\ \mathsf{w} \cdot f_n^{(\lambda)} = w_{n1}\, f_1^{(\lambda)} + \cdots + w_{nn}\, f_n^{(\lambda)}, \end{cases}$$

with the same matrix w at each jet level λ with $1 \leqslant \lambda \leqslant \kappa$.

Definition. A polynomial $\mathsf{P}\big(j^\kappa f\big)$ invariant by reparametrization will be called a <u>*bi-invariant*</u> if it is a *vector of highest weight* for this representation of $\mathsf{GL}_n(\mathbb{C})$, namely if it is invariant by the *unipotent subgroup* $\mathsf{U}_n(\mathbb{C}) \subset \mathsf{GL}_n(\mathbb{C})$ constituted by (unipotent) matrices of the form:

$$\mathsf{u} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ u_{21} & 1 & 0 & \cdots & 0 \\ u_{31} & u_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & u_{n3} & \cdots & 1 \end{pmatrix}.$$

The vector space of bi-invariant polynomials P thus satisfies:

$$\boxed{\mathsf{P}\big(j^\kappa(f \circ \phi)\big) = (\phi')^m \cdot \mathsf{P}\big((j^\kappa f) \circ \phi\big) \qquad \text{and} \qquad \mathsf{P}\big(\mathsf{u} \cdot j^\kappa f\big) = \mathsf{P}\big(j^\kappa f\big)}.$$

In the sequel, the vector space of bi-invariants of weight m will be denoted by $\mathsf{UE}^n_{\kappa,m}$. Also, one defines the graded algebra of bi-invariants $\mathsf{UE}^n_\kappa := \bigoplus_{m \geqslant 1} \mathsf{UE}^n_{\kappa,m}$ with of course $\mathsf{UE}^n_{\kappa,m_1} \cdot \mathsf{UE}^n_{\kappa,m_2} \subset \mathsf{UE}^n_{\kappa,m_1+m_2}$.

Without delay, we emphasize four fundamental observations.

• The full space $\mathsf{E}^n_{\kappa,m}$ is obtained as just the $\mathsf{GL}_n(\mathbb{C})$-orbit of $\mathsf{UE}^n_{\kappa,m}$.

• The algebraic structure of UE^n_κ is *always* much simpler than that of E^n_κ. For instance:

— UE^3_3 is generated by only 4 bi-invariant polynomials[45] f_1', $\Lambda^3_{1,2}$, $\Lambda^5_{1,2;\,1}$ and $D^6_{1,2,3}$ which are *algebraically independent* (no syzygy!), whereas, according to [28, 29] or to the description given on p. 206 here, the full algebra E^3_3 is generated by 16 invariants, submitted to the three complicated families of syzygies developed on p. 206.

[45] *See* the proposition on p. 229 below, or the considerations on pp. 931–932 in [21].

— UE_4^2 is generated by the 5 bi-invariant polynomials f_1', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$ and M^8, whose ideal of relations is principal, generated by the single syzygy:

$$0 \overset{4}{\equiv} f_1' f_1' M^8 - 3\,\Lambda^3 \Lambda_{1,1}^7 + 5\,\Lambda_1^5 \Lambda_1^5,$$

while, according to the theorem on p. 207, the full algebra E_4^2 is generated by 9 invariants submitted to 9 fundamental syzygies.

— We will establish that UE_4^4 is generated by 16 mutually independent bi-invariant polynomials, while E_4^4 is generated by 2835 polynomials invariant by reparametrization. Also, we will show 41 syzygies generate the ideal of relations between (the restriction to $\{f_1' = 0\}$ of) these 16 generators of UE_4^4, while we ignore the structure of the (presumably out of human scale) ideal of relations between the 2835 generators of E_4^4.

— We will establish that UE_5^2 is generated by 17 mutually independent bi-invariant polynomials, while E_5^2 is generated by 56 polynomials invariant by reparametrization. We will show 66 syzygies generating the ideal of relations between (the restriction to $\{f_1' = 0\}$ of) these 17 generators of UE_5^2.

• In any case, if we can show that UE_κ^n is, for a certain n and for a certain κ, generated as an algebra by a finite number of bi-invariants, we may easily deduce as a corollary finite generation of the full algebra E_κ^n. For instance:

— For $n = \kappa = 3$, computing the $\mathsf{GL}_3(\mathbb{C})$-orbit of the 4 bi-invariants f_1', $\Lambda_{1,2}^3$, $\Lambda_{1,2;\,1}^5$ and $D_{1,2,3}^6$ amounts to polarize their lower indices, which yields the invariants f_i', $\Lambda_{i,j}^3$, $\Lambda_{i,j;\,k}^5$ and $D_{i,j,k}^6$ generating E_3^3.

— For $n = 2$ and $\kappa = 4$, computing the $\mathsf{GL}_2(\mathbb{C})$-orbit of the 5 bi-invariants f_1', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$ and M^8 again amounts to polarize their lower indices, which yields the invariants f_i', Λ^3, Λ_i^5, $\Lambda_{i,j}^7$ and M^8 generating E_4^2.

• Finally, for applications to Kobayashi hyperbolicity (which involves estimating the Euler-Poincaré characteristic of $\mathsf{E}_{\kappa,m}^n T_X^*$), it is useless to look for a complete understanding of the algebraic structure of E_κ^n, and it only suffices to possess a complete description of the algebra of bi-invariants UE_κ^n. In fact, as will be (re)explained in §12, each bi-invariant will correspond to one and to only one Schur bundle.

$\boxed{\text{So from now on, we focus our attention on bi-invariants}}$

Initial representation of bi-invariants. We now restart with the initial, rational expression of any polynomial invariant by reparametrization provided by the lemma on p. 217 and we want to determine when such a polynomial is, in addition, invariant by the unipotent action.

To begin with, we consider the subgroup $\mathsf{U}_n^*(\mathbb{C})$ of $\mathsf{U}_n(\mathbb{C})$ generated by matrices of the form:

$$\mathsf{u}^* = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ u_{21} & 1 & 0 & \cdots & 0 \\ u_{31} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{n1} & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Clearly, the components of the first order jet $j^1 f$ are modified by the action of u^*:

$$\begin{cases} \mathsf{u}^* \cdot f_1' = f_1', \\ \mathsf{u}^* \cdot f_2' = f_2' + u_{21} f_1', \\ \mathsf{u}^* \cdot f_3' = f_3' + u_{31} f_1', \\ \quad \cdots \qquad\qquad \cdots \\ \mathsf{u}^* \cdot f_n' = f_n' + u_{n1} f_1'. \end{cases}$$

On the other hand, all the $\Lambda_{1,i}^3$ are left invariant:

$$\mathsf{u}^* \cdot \Lambda_{1,i}^3 = \mathsf{u}^* \cdot \big[f_i',\, f_1'\big] = \big[f_i' + u_{i1} f_1',\, f_1'\big] = \big[f_i',\, f_1'\big] = \Lambda_{1,i}^3,$$

and in fact, more generally, one may verify that the same is true of higher Λ's:

$$\mathsf{u}^* \cdot \Lambda_{1,i;\,1}^5 = \Lambda_{1,i;\,1}^5, \qquad \mathsf{u}^* \cdot \Lambda_{1,i;\,1,1}^7 = \Lambda_{1,i;\,1,1}^7, \ \ldots\ldots, \ \mathsf{u}^* \cdot \Lambda_{1,i;\,1^{\kappa-2}}^{2\kappa-1} = \Lambda_{1,i;\,1^{\kappa-2}}^{2\kappa-1},$$

for any $i = 2, 3, \ldots, n$. Consequently, the requirement that a polynomial invariant by reparametrization $\mathsf{P}\big(j^\kappa f\big) \in \mathsf{E}_{\kappa,m}^n$ be in addition also invariant by the unipotent subgroup $\mathsf{U}_n^*(\mathbb{C}) \subset \mathsf{U}_n(\mathbb{C})$, namely $\mathsf{u}^* \cdot \mathsf{P}\big(j^\kappa f\big) = \mathsf{P}\big(j^\kappa f\big)$, shall be written in length as follows, when employing the mentioned representation given on p. 217:

$$\begin{aligned} &\sum_a (f_1')^a\, \mathsf{P}_a\Big(f_2' + u_{21} f_1',\, f_3' + u_{31} f_1',\, \ldots,\, f_n' + u_{n1} f_1', \\ &\qquad\qquad \Lambda_{1,2}^3,\, \ldots,\, \Lambda_{1,n}^3,\, \ldots\ldots,\, \Lambda_{1,2;\,1^{\kappa-2}}^{2\kappa-1},\, \ldots,\, \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1}\Big) = \\ &= \sum_a (f_1')^a\, \mathsf{P}_a\Big(f_2',\, f_3',\, \ldots,\, f_n', \\ &\qquad\qquad \Lambda_{1,2}^3,\, \ldots,\, \Lambda_{1,n}^3,\, \ldots\ldots,\, \Lambda_{1,2;\,1^{\kappa-2}}^{2\kappa-1},\, \ldots,\, \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1}\Big). \end{aligned}$$

Because the $n + (n-1)(\kappa-1)$ invariants $f_1', \ldots, f_n'$, $\Lambda_{1,i;\,1^{\lambda-2}}^{2\lambda-1}$, $2 \leqslant i \leqslant n$, $2 \leqslant \lambda \leqslant \kappa$, are algebraically independent, we deduce that each P_a must be independent of $f_2', f_3', \ldots, f_n'$, so that we come to the simpler rational expression:

$$\mathsf{R} = \sum_a (f_1')^a\, \mathsf{P}_a\Big(\Lambda_{1,2}^3,\, \ldots,\, \Lambda_{1,n}^3,\, \ldots\ldots,\, \Lambda_{1,2;\,1^{\kappa-2}}^{2\kappa-1},\, \ldots,\, \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1}\Big),$$

which is however not yet invariant under the full unipotent action.

Second unipotent subgroup. Next, we consider the subgroup $\mathsf{U}_n^\sharp(\mathbb{C}) \subset \mathsf{U}_n(\mathbb{C})$ constituted by matrices of the form:

$$\mathsf{u}^\sharp = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & u_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & u_{n2} & u_{n3} & \cdots & 1 \end{pmatrix}.$$

Since $\mathsf{U}_n^*(\mathbb{C})$ and $\mathsf{U}_n^\sharp(\mathbb{C})$ clearly generate the full unipotent group $\mathsf{U}_n(\mathbb{C})$, it now only remains to require the $\mathsf{U}_n^\sharp(\mathbb{C})$-invariance for the rational expression R obtained just above.

The requirement $\mathsf{u}^\sharp \cdot \mathsf{R} = \mathsf{R}$ can in turn be written in length as follows: :

$$\begin{aligned} \mathsf{u}^\sharp\Big(\sum_a (f_1')^a\, \mathsf{P}_a\Big(\Lambda_{1,2}^3,\, \ldots,\, \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1}\Big)\Big) &= \sum_a (f_1')^a\, \mathsf{P}_a\Big(\mathsf{u}^\sharp \cdot \Lambda_{1,2}^3,\, \ldots,\, \mathsf{u}^\sharp \cdot \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1}\Big) \\ &= \sum_a (f_1')^a\, \mathsf{P}_a\Big(\Lambda_{1,2}^3,\, \ldots,\, \Lambda_{1,n;\,1^{\kappa-2}}^{2\kappa-1}\Big). \end{aligned}$$

But on the other hand, for any λ with $2 \leqslant \lambda \leqslant \kappa$, one may verify that the action of $\mathsf{u}^\sharp$ on the initial Λ-invariants appearing as arguments of R is given by the triangular formulas:

$$\begin{aligned} \mathsf{u}^\sharp \cdot \Lambda_{1,2;\,1^{\lambda-2}}^{2\lambda-1} &= \Lambda_{1,2;\,1^{\lambda-2}}^{2\lambda-1}, \\ \mathsf{u}^\sharp \cdot \Lambda_{1,3;\,1^{\lambda-2}}^{2\lambda-1} &= \Lambda_{1,3;\,1^{\lambda-2}}^{2\lambda-1} + u_{32}\, \Lambda_{1,2;\,1^{\lambda-2}}^{2\lambda-1}, \\ \mathsf{u}^\sharp \cdot \Lambda_{1,4;\,1^{\lambda-2}}^{2\lambda-1} &= \Lambda_{1,4;\,1^{\lambda-2}}^{2\lambda-1} + u_{43}\, \Lambda_{1,3;\,1^{\lambda-2}}^{2\lambda-1} + u_{42}\, \Lambda_{1,2;\,1^{\lambda-2}}^{2\lambda-1}, \\ &\cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ \mathsf{u}^\sharp \cdot \Lambda_{1,n;\,1^{\lambda-2}}^{2\lambda-1} &= \Lambda_{1,n;\,1^{\lambda-2}}^{2\lambda-1} + u_{n,n-1}\, \Lambda_{1,n-1;\,1^{\lambda-2}}^{2\lambda-1} + \cdots + u_{n2}\, \Lambda_{1,2;\,1^{\lambda-2}}^{2\lambda-1}. \end{aligned}$$

The algebraic independency of f_1', $\Lambda_{1,i;\,1^{\lambda-2}}^{2\lambda-1}$ then implies that such an R is $\mathsf{U}_n^\sharp(\mathbb{C})$-invariant if and only if every P_a is so, namely if and only if the following identity holds:

$$\begin{aligned} &\mathsf{P}_a\Big(A_{1,2}^3,\, A_{1,3}^3 + u_{32}\, A_{1,2}^3,\, \ldots,\, A_{1,n}^3 + u_{n,n-1}\, A_{1,n-1}^3 + \cdots + u_{n2}\, A_{1,2}^3, \\ &\quad A_{1,2}^5,\, A_{1,3}^5 + u_{32}\, A_{1,2}^5,\, \ldots,\, A_{1,n}^5 + u_{n,n-1}\, A_{1,n-1}^5 + \cdots + u_{n2}\, A_{1,2}^5, \\ &\quad \cdots \qquad \cdots \qquad \cdots \\ &\quad A_{1,2}^{2\kappa-1},\, A_{1,3}^{2\kappa-1} + u_{32}\, A_{1,2}^{2\kappa-1},\, \ldots,\, A_{1,n}^{2\kappa-1} + u_{n,n-1}\, A_{1,n-1}^{2\kappa-1} + \cdots + u_{n2}\, A_{1,2}^{2\kappa-1}\Big) = \end{aligned}$$

$$
\begin{aligned}
= \mathsf{P}_a\Big(&A^3_{1,2},\, A^3_{1,3},\, \dots,\, A^3_{1,n},\\
&A^5_{1,2},\, A^5_{1,3},\, \dots,\, A^5_{1,n},\\
&\dots \quad \dots \qquad \dots\\
&A^{2\kappa-1}_{1,2},\, A^{2\kappa-1}_{1,3},\, \dots,\, A^{2\kappa-1}_{1,n}\Big),
\end{aligned}
$$

as polynomials in $\mathbb{C}\big[A^3_{1,2}, \dots, A^3_{1,n}, \dots\dots, A^{2\kappa-1}_{1,2}, \dots, A^{2\kappa-1}_{1,n}\big]$, for every $\mathsf{u}^\sharp$, and for every a with $-\frac{m-1}{m}\kappa \leqslant a \leqslant m$.

Here, we recognize a full unipotent action, acted by means of a general $(n-1)\times(n-1)$ unipotent matrice of the form:

$$
\begin{pmatrix}
1 & 0 & \cdots & 0\\
u_{32} & 1 & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots\\
u_{n2} & u_{n3} & \cdots & 1
\end{pmatrix} \in \mathsf{U}_{n-1}(\mathbb{C}),
$$

on the set of the $\kappa-1$ vectors of $\mathbb{C}^{n-1}$ defined by:

$$
\big(A^{2\lambda-1}_{1,2},\, A^{2\lambda-1}_{1,3},\, \dots,\, A^{2\lambda-1}_{1,n}\big) \qquad\qquad (2\leqslant\lambda\leqslant\kappa).
$$

It is known ([19, 27]) that the invariants for such an action are constituted by all the minors:

$$
\Pi^{\lambda_2}_2 := A^{2\lambda_2-1}_{1,2}, \qquad \Pi^{\lambda_2,\lambda_3}_{2,3} := \begin{vmatrix} A^{2\lambda_2-1}_{1,2} & A^{2\lambda_2-1}_{1,3}\\ A^{2\lambda_3-1}_{1,2} & A^{2\lambda_3-1}_{1,3}\end{vmatrix},
$$

$$
\Pi^{\lambda_2,\lambda_3,\lambda_4}_{2,3,4} := \begin{vmatrix} A^{2\lambda_2-1}_{1,2} & A^{2\lambda_2-1}_{1,3} & A^{2\lambda_2-1}_{1,4}\\ A^{2\lambda_3-1}_{1,2} & A^{2\lambda_3-1}_{1,3} & A^{2\lambda_3-1}_{1,4}\\ A^{2\lambda_4-1}_{1,2} & A^{2\lambda_4-1}_{1,3} & A^{2\lambda_4-1}_{1,4}\end{vmatrix},
$$

and generally:

$$
\Pi^{\lambda_2,\lambda_3,\lambda_4,\dots,\lambda_{n_1}}_{2,3,4,\dots,n_1} := \begin{vmatrix}
A^{2\lambda_2-1}_{1,2} & A^{2\lambda_2-1}_{1,3} & A^{2\lambda_2-1}_{1,4} & \cdots & A^{2\lambda_2-1}_{1,n_1}\\
A^{2\lambda_3-1}_{1,2} & A^{2\lambda_3-1}_{1,3} & A^{2\lambda_3-1}_{1,4} & \cdots & A^{2\lambda_3-1}_{1,n_1}\\
A^{2\lambda_4-1}_{1,2} & A^{2\lambda_4-1}_{1,3} & A^{2\lambda_4-1}_{1,4} & \cdots & A^{2\lambda_4-1}_{1,n_1}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
A^{2\lambda_{n_1}-1}_{1,2} & A^{2\lambda_{n_1}-1}_{1,3} & A^{2\lambda_{n_1}-1}_{1,4} & \cdots & A^{2\lambda_{n_1}-1}_{1,n_1}
\end{vmatrix},
$$

for all n_1 from $n_1 = 1$ up to $n_1 = n$, and for arbitrary λ_j with $2 \leqslant \lambda_j \leqslant \kappa$. In fact, one immediately sees that these minors are obviously invariant by the unipotent action of $\mathsf{U}_{n-1}(\mathbb{C})$, thanks to the fact that column linear dependence leaves untouched any determinant.

THEOREM *In dimension $n \geqslant 1$ and for jets of arbitrary order $\kappa \geqslant 1$, every bi-invariant polynomial $\mathsf{BP} = \mathsf{BP}\big(j^\kappa f\big)$ invariant by*

reparametrization and invariant under the unipotent action writes under the form:

$$\mathsf{BP}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa}m\leqslant a\leqslant m} (f_1')^a\,\mathsf{BP}_a\bigg(\left|\begin{matrix} \Lambda_{1,2}^{2\lambda_2-1} & \Lambda_{1,3}^{2\lambda_2-1} & \cdots & \Lambda_{1,n_1}^{2\lambda_2-1} \\ \Lambda_{1,2}^{2\lambda_3-1} & \Lambda_{1,3}^{2\lambda_3-1} & \cdots & \Lambda_{1,n_1}^{2\lambda_3-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{1,2}^{2\lambda_3-1} & \Lambda_{1,3}^{2\lambda_3-1} & \cdots & \Lambda_{1,n_1}^{2\lambda_3-1} \end{matrix}\right|_{n_1=1,2\ldots,n}^{2\leqslant\lambda_2,\ldots,\lambda_{n_1}\leqslant\kappa}\bigg),$$

for certain specific polynomials BP_a *which depend upon* $\mathsf{BP}(j^\kappa f)$.

The case $n=\kappa=3$**.** After $\mathsf{U}_3^*(\mathbb{C})$-reduction, an arbitrary element of $\mathsf{UE}_{3,m}^3$ writes:

$$\mathsf{R} = \sum_{-\frac{2}{3}m\leqslant a\leqslant m} (f_1')^a\,\mathsf{P}_a\Big(\Lambda_{1,2}^3,\,\Lambda_{1,3}^3,\,\Lambda_{1,2;\,1}^5,\,\Lambda_{1,3;\,1}^5\Big).$$

Then the $\mathsf{U}_3^\sharp(\mathbb{C})$-reduction presented above shows that there are four initial bi-invariants, namely the three obvious ones f_1', $\Lambda_{1,2}^3$, $\Lambda_{1,2;\,1}^5$ together with:

$$\left|\begin{matrix} \Lambda_{1,2}^3 & \Lambda_{1,3}^3 \\ \Lambda_{1,2;\,1}^5 & \Lambda_{1,3;\,1}^5 \end{matrix}\right| = f_1'f_1'\cdot\left|\begin{matrix} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{matrix}\right| =: f_1'f_1'\cdot D_{1,2,3}^6,$$

where the first equality, which follows from a direct calculation, gives birth to the three-dimensional Wronskian. By pluging this minor in the above rational expression of R, we obtain that any bi-invariant polynomial in $\mathsf{UE}_{3,m}^3$ writes under the form:

$$\mathsf{BP}\big(j^3 f\big) = \sum_{-\frac{2}{3}m\leqslant a\leqslant m} (f_1')^a\,\widetilde{\mathsf{P}}_a\Big(\Lambda_{1,2}^3,\,\Lambda_{1,2;\,1}^5,\,D_{1,2,3}^6\Big),$$

for certain (new) polynomials $\widetilde{\mathsf{P}}_a$. More is true, for we claim that there are no negative powers of f_1' anymore in such a rational representation.

Proposition. *Any bi-invariant polynomial* $\mathsf{BP}\in\mathsf{UE}_{3,m}^3$ *writes uniquely under the form:*

$$\mathsf{BP}\big(j^3 f\big) = \sum_{0\leqslant a\leqslant m} (f_1')^a\,\mathsf{BP}_a\Big(\Lambda_{1,2}^3,\,\Lambda_{1,2;\,1}^5,\,D_{1,2,3}^6\Big),$$

where the BP_a *are arbitrary polynomials. In fact:*

$$\boxed{\mathsf{UE}\big(j^3 f\big) = \mathbb{C}\big[f_1',\,\Lambda_{1,2}^3,\,\Lambda_{1,2;\,1}^5,\,D_{1,2,3}^6\big]}.$$

Proof. One verifies at first sight that, after setting $f_1' = 0$, the 3 restricted invariants:

$$\Lambda^3_{1,2}\big|_0 = -f_1''f_2', \qquad \Lambda^5_{1,2;\,1}\big|_0 = 3\,f_1''f_2'f_1'' \quad \text{and} \quad D^6_{1,2,3}\big|_0 = \begin{vmatrix} 0 & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix}$$

are mutually algebraically independent. Suppose then by contradiction that the expression:

$$\mathsf{BP}\big(j^3 f\big) = \sum_{-a_0 \leqslant a \leqslant m} (f_1')^a\, \widetilde{\mathsf{P}}_a\Big(\Lambda^3_{1,2},\, \Lambda^5_{1,2;\,1},\, D^6_{1,2,3}\Big),$$

starts with a not identically zero $\widetilde{\mathsf{P}}_{-a_0}\big(A^3, A^5, \Delta^6\big) \not\equiv 0$ for some smallest negative power $-a_0 < 0$ of f_1'. Multiplying both sides by $(f_1')^{a_0}$ and setting $f_1' = 0$ afterwards, the left term $(f_1')^{a_0}\,\mathsf{BP}\big(j^3 f\big)$ then vanishes, hence one would derive an identity:

$$0 \equiv \widetilde{\mathsf{P}}_{-a_0}\Big(\Lambda^3_{1,2}\big|_0,\, \Lambda^5_{1,2;\,1}\big|_0,\, D^6_{1,2,3}\big|_0\Big)$$

between restricted bi-invariants which would then entail $\widetilde{\mathsf{P}}_{-a_0} \equiv 0$ because the arguments are algebraically independent, a contradiction.

Consequently, the rational expression for $\mathsf{BP}\big(j^3 f\big)$ was already polynomial and inversely, every arbitrary polynomial in $\mathbb{C}\big[f_1',\, \Lambda^3_{1,2},\, \Lambda^5_{1,2;\,1},\, D^6_{1,2,3}\big]$ obviously is a bi-invariant. $\square$

The case $n = \kappa = 4$. After $\mathsf{U}^*_4(\mathbb{C})$-reduction, an arbitrary element of $\mathsf{UE}^4_{4,m}$ writes under the form:

$$\mathsf{R} = \sum_{-\frac{3}{4}m \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a\Big(\Lambda^3_{1,2},\, \Lambda^3_{1,3},\, \Lambda^3_{1,4},\, \Lambda^5_{1,2;\,1},\, \Lambda^5_{1,3;\,1},\, \Lambda^5_{1,4;\,1},\, \Lambda^7_{1,2;\,1,1},\, \Lambda^7_{1,3;\,1,1},\, \Lambda^7_{1,4;\,1,1}\Big).$$

Then the $\mathsf{U}^\sharp_4(\mathbb{C})$-reduction presented above shows that there are the 4 obvious initial bi-invariants:

$$f_1', \qquad \Lambda^3_{1,2}, \qquad \Lambda^5_{1,2;\,1} \qquad \text{and} \qquad \Lambda^7_{1,2;\,1,1},$$

together with the 4 further ones:

$$D^6, \qquad D^8 = \big[D^6,\, f_1'\big], \qquad N^{10} \qquad \text{and} \qquad W^{10},$$

that are obtained by dividing the 4 minors involving the Λ's by the maximal power of f_1' which appears in factor, namely:

$$\begin{vmatrix} \Lambda^3_{1,2} & \Lambda^3_{1,3} \\ \Lambda^5_{1,2;1} & \Lambda^5_{1,3;1} \end{vmatrix} \equiv f_1'f_1'D^6,$$

$$\begin{vmatrix} \Lambda^3_{1,2} & \Lambda^3_{1,3} \\ \Lambda^7_{1,2;1,1} & \Lambda^7_{1,3;1,1} \end{vmatrix} \equiv f_1'f_1'D^8,$$

$$\begin{vmatrix} \Lambda^5_{1,2;1} & \Lambda^5_{1,3;1} \\ \Lambda^7_{1,2;1,1} & \Lambda^7_{1,3;1,1} \end{vmatrix} \equiv f_1' f_1' N^{10},$$

$$\begin{vmatrix} \Lambda^3_{1,2} & \Lambda^3_{1,3} & \Lambda^3_{1,4} \\ \Lambda^5_{1,2;1} & \Lambda^5_{1,3;1} & \Lambda^5_{1,4;1} \\ \Lambda^7_{1,2;1,1} & \Lambda^7_{1,3;1,1} & \Lambda^7_{1,4;1,1} \end{vmatrix} \equiv f_1' f_1' f_1' f_1' f_1' \, W^{10},$$

where the last one behind $(f_1')^5$ appears to be equal to the four-dimensional *Wronskian*:

$$W^{10} := \begin{vmatrix} f_1' & f_2' & f_3' & f_4' \\ f_1'' & f_2'' & f_3'' & f_4'' \\ f_1''' & f_2''' & f_3''' & f_4''' \\ f_1'''' & f_2'''' & f_3'''' & f_4'''' \end{vmatrix},$$

and where the first three ones are explicitly defined by:

$$D^6 := \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix},$$

$$D^8 := f_1' \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1'''' & f_2'''' & f_3'''' \end{vmatrix} - 6\, f_1'' \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix},$$

$$N^{10} := f_1' f_1' \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1''' & f_2''' & f_3''' \\ f_1'''' & f_2'''' & f_3'''' \end{vmatrix} - 3\, f_1' f_1'' \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1'''' & f_2'''' & f_3'''' \end{vmatrix} +$$

$$+ 4\, f_1' f_1''' \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix} + 3\, f_1'' f_1'' \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix}.$$

By pluging these 8 bi-invariants in the rational expression written on p. 228, we obtain that any bi-invariant polynomial in $\mathsf{UE}^4_{4,m}$ writes under the form:

$$\boxed{\mathsf{BP}\big(j^4 j\big) = \sum_{-\frac{3}{4}m \leqslant a \leqslant m} (f_1')^a \, \widetilde{\mathsf{P}}_a\Big(\Lambda^3,\, \Lambda^5,\, \Lambda^7,\, D^6,\, D^8,\, N^{10},\, W^{10}\Big)}.$$

This expression will be the very starting point for the application of our general algorithm, to be presented in Section 9 below. In fact, as in the case $n = 2$, $\kappa = 4$ of Section 6, there will appear further independent *ghost bi-invariants hidden behind powers of* f_1'.

§8. Counterexpectation: insufficiency of bracket invariants

According to the unexpected, main outcome of [21], the theorem for $n = 2$ and $\kappa = 5$ on p. 209 about bracket invariants does *not* capture all Demailly-Semple (bi-)invariants. This was striking, because brackets were sufficient

to capture all invariants in all previously known studies[46], namely for E_2^n, for E_3^2, for E_3^3 and for E_4^2.

Aside from the 11 bi-invariants f_1', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$, M^8, $\Lambda_{1,1,1}^9$, M_1^{10}, N^{12}, $K_{1,1}^{12}$, H_1^{14} and $F_{1,1}^{16}$, there are yet the following 6 bi-invariants X^{18}, X^{19}, X^{21}, X^{23}, X^{25} and Y^{27} that are defined by dividing by f_1' some appropriate quadratic combinations between already known bi-invariants. We provide here the complete explicit expressions. It is shown in [21] that the 16 first bi-invariants are mutually independent and it would be easy, by using the same method, to verify that when one adds the last, 17-th bi-invariant Y^{27}, one still gets a list of 17 mutually independent bi-invariants.

Importantly, we emphasize that by no means any of these 6 further bi-invariants can come from inspecting the bracket invariants, by dividing them either by f_1', or by Λ^3 or by anything based in brackets, because in [21], *all the possible bracket invariants* were computed thoroughly, were simplified and were analyzed at a piece. *The existence of X^{18}, X^{19}, X^{21}, X^{23}, X^{25}, Y^{27} really shows that bracketing does not generate the algebra of bi-invariants* UE_5^2. A similar phenomenon will appear to take place in dimension $n = 3$ for jet order $\kappa = 4$.

Before reading the formulas, we would like to mention that the invariant X^{21} of UE_5^2 below is not the same as the invariant X^{21} of UE_3^4 appearing in §11. Our manuscript sheets used the same notation, and we hope this should not cause any confusion.

$$
\begin{aligned}
X^{18} &:= \frac{-5\,\Lambda_{1,1,1}^9\,M_1^{10} + 56\,\Lambda_{1,1}^7\,K_{1,1}^{12}}{f_1'} \\
&= f_1' f_1' f_1' \Big(-18816\,\Delta^{',''''}\,[\Delta^{'','''}]^2 - 25088\,[\Delta^{'','''}]^3 - 15\,[\Delta^{','''''}]^2\,\Delta^{',''} - 150\,\Delta^{','''''}\,\Delta^{'',''''}\,\Delta^{',''} \\
&\quad + 315\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{','''} + 960\,\Delta^{','''''}\,\Delta^{'','''}\,\Delta^{','''} - 375\,[\Delta^{'',''''}]^2\,\Delta^{',''} + 1575\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{','''} \\
&\quad + 4800\,\Delta^{'',''''}\,\Delta^{'','''}\,\Delta^{','''} - 392\,[\Delta^{',''''}]^3 - 4704\,[\Delta^{',''''}]^2\,\Delta^{'','''} \Big) - f_1' f_1' f_1'' \Big(-2475\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{',''} \\
&\quad - 9900\,\Delta^{'',''''}\,\Delta^{'','''}\,\Delta^{',''} - 2850\,\Delta^{','''''}\,[\Delta^{','''}]^2 + 51330\,\Delta^{',''''}\,\Delta^{'','''}\,\Delta^{','''} \\
&\quad + 92760\,[\Delta^{'','''}]^2\,\Delta^{','''} - 14250\,\Delta^{'',''''}\,[\Delta^{','''}]^2 + 7035\,[\Delta^{',''''}]^2\,\Delta^{','''} - 495\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{',''} \\
&\quad - 1980\,\Delta^{','''''}\,\Delta^{'','''}\,\Delta^{',''} \Big) - f_1' f_1' f_1''' \Big(-11100\,\Delta^{'','''}\,[\Delta^{','''}]^2 - 3150\,\Delta^{',''''}\,[\Delta^{','''}]^2 \Big) \\
&\quad + f_1' f_1'' f_1'' \Big(-109440\,[\Delta^{'','''}]^2\,\Delta^{',''} - 19050\,\Delta^{'','''}\,[\Delta^{','''}]^2 - 32325\,\Delta^{',''''}\,[\Delta^{','''}]^2 \\
&\quad + 11025\,\Delta^{','''''}\,\Delta^{','''}\,\Delta^{',''} + 55125\,\Delta^{'',''''}\,\Delta^{','''}\,\Delta^{',''} - 6840\,[\Delta^{',''''}]^2\,\Delta^{',''} \\
&\quad - 54720\,\Delta^{',''''}\,\Delta^{'','''}\,\Delta^{',''} \Big) - f_1' f_1'' f_1''' \Big(+30000\,[\Delta^{','''}]^3 \Big) - f_1'' f_1'' f_1'' \Big(11025\,\Delta^{','''''}\,[\Delta^{',''}]^2 \\
&\quad - 55125\,\Delta^{'',''''}\,[\Delta^{',''}]^2 + 55125\,\Delta^{',''''}\,\Delta^{','''}\,\Delta^{',''} + 110250\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} \\
&\quad - 49000\,[\Delta^{','''}]^3 \Big).
\end{aligned}
$$

[46] On observes that UE_3^3 is *not* obtained by bracketing bi-invariants in UE_2^3 (think of D^6), but nevertheless UE_3^3 is the unipotent-invariant subalgebra of E_3^3, and E_3^3 itself is obtained by bracketing invariants from the preceding jet level.

$$
\begin{aligned}
X^{19} &:= \frac{-5\,M_1^{10}\,M_1^{10} + 64\,M^8\,K_{1,1}^{12}}{f_1'} \\
&= f_1'\,\Big(1170\,\Delta^{',''''''}\,\Delta^{',''''}\,\Delta^{','''}\,\Delta^{',''} - 45\,[\Delta^{',''''''}]^2\,[\Delta^{',''}]^2 - 450\,\Delta^{',''''''}\,\Delta^{'',''''}\,[\Delta^{',''}]^2 \\
&\quad + 74220\,[\Delta^{'','''}]^2\,[\Delta^{','''}]^2 + 3780\,\Delta^{',''''''}\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} - 1600\,\Delta^{',''''''}\,[\Delta^{','''}]^3 \\
&\quad - 1125\,[\Delta^{'',''''}]^2\,[\Delta^{',''}]^2 + 5850\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{','''}\,\Delta^{',''} + 18900\,\Delta^{'',''''}\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} \\
&\quad - 8000\,\Delta^{'',''''}\,[\Delta^{','''}]^3 - 1344\,[\Delta^{',''''}]^3\,\Delta^{',''} - 16128\,[\Delta^{',''''}]^2\,\Delta^{'','''}\,\Delta^{',''} + 1995\,[\Delta^{',''''}]^2\,[\Delta^{','''}]^2 \\
&\quad - 64512\,\Delta^{',''''}\,[\Delta^{'','''}]^2\,\Delta^{',''} + 27660\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^2 - 86016\,[\Delta^{'','''}]^3\,\Delta^{',''}\Big) \\
&\quad + f_1''\,\Big(-74400\,\Delta^{'','''}\,[\Delta^{','''}]^3 - 10800\,\Delta^{'',''''}\,\Delta^{',''''}\,[\Delta^{',''}]^2 - 2160\,\Delta^{',''''''}\,\Delta^{',''''}\,[\Delta^{',''}]^2 \\
&\quad - 8640\,\Delta^{',''''''}\,\Delta^{'','''}\,[\Delta^{',''}]^2 + 3600\,\Delta^{',''''''}\,[\Delta^{','''}]^2\,\Delta^{',''} + 64800\,\Delta^{',''''}\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} \\
&\quad - 43200\,\Delta^{'',''''}\,\Delta^{'','''}\,[\Delta^{',''}]^2 + 18000\,\Delta^{'',''''}\,[\Delta^{','''}]^2\,\Delta^{',''} + 10800\,[\Delta^{',''''}]^2\,\Delta^{','''}\,\Delta^{',''} \\
&\quad - 27600\,\Delta^{',''''}\,[\Delta^{','''}]^3 + 86400\,[\Delta^{'','''}]^2\,\Delta^{','''}\,\Delta^{',''}\Big) + f_1'''\,\Big(16000\,[\Delta^{','''}]^4\Big).
\end{aligned}
$$

$$
\begin{aligned}
X^{21} &:= \frac{-5\,M_1^{10}\,N^{12} + 8\,M^8\,H_1^{14}}{f_1'} \\
&= -135\,[\Delta^{',''''''}]^2\,[\Delta^{',''}]^3 - 1350\,\Delta^{',''''''}\,\Delta^{'',''''}\,[\Delta^{',''}]^3 + 1350\,\Delta^{',''''''}\,\Delta^{',''''}\,\Delta^{','''}\,[\Delta^{',''}]^2 \\
&\quad + 2700\,\Delta^{',''''''}\,\Delta^{'','''}\,\Delta^{','''}\,[\Delta^{',''}]^2 - 1200\,\Delta^{',''''''}\,[\Delta^{','''}]^3\,\Delta^{',''} - 3375\,[\Delta^{'',''''}]^2\,[\Delta^{',''}]^3 \\
&\quad + 6750\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{','''}\,[\Delta^{',''}]^2 + 13500\,\Delta^{'',''''}\,\Delta^{'','''}\,\Delta^{','''}\,[\Delta^{',''}]^2 - 6000\,\Delta^{'',''''}\,[\Delta^{','''}]^3\,\Delta^{',''} \\
&\quad - 576\,[\Delta^{',''''}]^3\,[\Delta^{',''}]^2 - 6912\,[\Delta^{',''''}]^2\,\Delta^{'','''}\,[\Delta^{',''}]^2 - 495\,[\Delta^{',''''}]^2\,[\Delta^{','''}]^2\,\Delta^{',''} \\
&\quad - 27648\,\Delta^{',''''}\,[\Delta^{'','''}]^2\,[\Delta^{',''}]^2 + 9540\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^2\,\Delta^{',''} + 1200\,\Delta^{',''''}\,[\Delta^{','''}]^4 \\
&\quad - 36864\,[\Delta^{'','''}]^3\,[\Delta^{',''}]^2 + 32580\,[\Delta^{'','''}]^2\,[\Delta^{','''}]^2\,\Delta^{',''} - 7200\,\Delta^{'','''}\,[\Delta^{','''}]^4.
\end{aligned}
$$

$$
\begin{aligned}
X^{23} &:= \frac{-7\,N^{12}\,K_{1,1}^{12} + M^8\,F_{1,1}^{16}}{f_1'} \\
&= f_1'\,\Big(432\,\Delta^{',''''''}\,[\Delta^{',''''}]^2\,[\Delta^{',''}]^2 + 3456\,\Delta^{',''''''}\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{',''}]^2 + 1710\,\Delta^{',''''''}\,\Delta^{',''''}\,[\Delta^{','''}]^2\,\Delta^{',''} \\
&\quad - 3150\,\Delta^{',''''''}\,\Delta^{'',''''}\,\Delta^{','''}\,[\Delta^{',''}]^2 + 540\,\Delta^{',''''''}\,\Delta^{'','''}\,[\Delta^{','''}]^2\,\Delta^{',''} - 1600\,\Delta^{',''''''}\,[\Delta^{','''}]^4 \\
&\quad - 7875\,[\Delta^{'',''''}]^2\,\Delta^{','''}\,[\Delta^{',''}]^2 + 6912\,\Delta^{',''''''}\,[\Delta^{'','''}]^2\,[\Delta^{',''}]^2 - 8000\,\Delta^{'',''''}\,[\Delta^{','''}]^4 \\
&\quad - 2352\,[\Delta^{',''''}]^3\,\Delta^{','''}\,\Delta^{',''} - 23904\,[\Delta^{',''''}]^2\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} + 2205\,[\Delta^{',''''}]^2\,[\Delta^{','''}]^3 \\
&\quad - 78336\,\Delta^{',''''}\,[\Delta^{'','''}]^2\,\Delta^{','''}\,\Delta^{',''} + 34740\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^3 - 81408\,[\Delta^{'','''}]^3\,\Delta^{','''}\,\Delta^{',''} \\
&\quad + 72180\,[\Delta^{'','''}]^2\,[\Delta^{','''}]^3 + 2160\,\Delta^{'',''''}\,[\Delta^{',''''}]^2\,[\Delta^{',''}]^2 + 17280\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{',''}]^2 \\
&\quad + 8550\,\Delta^{'',''''}\,\Delta^{',''''}\,[\Delta^{','''}]^2\,\Delta^{',''} + 34560\,\Delta^{'',''''}\,[\Delta^{'','''}]^2\,[\Delta^{',''}]^2 + 2700\,\Delta^{'',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^2\,\Delta^{',''} \\
&\quad - 315\,[\Delta^{',''''''}]^2\,\Delta^{','''}\,[\Delta^{',''}]^2\Big) + f_1''\,\Big(23625\,[\Delta^{'',''''}]^2\,[\Delta^{',''}]^3 - 47250\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{','''}\,[\Delta^{',''}]^2 \\
&\quad - 94500\,\Delta^{'',''''}\,\Delta^{'','''}\,\Delta^{','''}\,[\Delta^{',''}]^2 + 42000\,\Delta^{'',''''}\,[\Delta^{','''}]^3\,\Delta^{',''} + 576\,[\Delta^{',''''}]^3\,[\Delta^{',''}]^2 \\
&\quad + 6912\,[\Delta^{',''''}]^2\,\Delta^{'','''}\,[\Delta^{',''}]^2 + 20745\,[\Delta^{',''''}]^2\,[\Delta^{','''}]^2\,\Delta^{',''} + 27648\,\Delta^{',''''}\,[\Delta^{'','''}]^2\,[\Delta^{',''}]^2 \\
&\quad + 945\,[\Delta^{',''''''}]^2\,[\Delta^{',''}]^3 + 9450\,\Delta^{',''''''}\,\Delta^{'',''''}\,[\Delta^{',''}]^3 - 9450\,\Delta^{',''''''}\,\Delta^{',''''}\,\Delta^{','''}\,[\Delta^{',''}]^2 \\
&\quad - 18900\,\Delta^{',''''''}\,\Delta^{'','''}\,\Delta^{','''}\,[\Delta^{',''}]^2 + 8400\,\Delta^{',''''''}\,[\Delta^{','''}]^3\,\Delta^{',''} + 71460\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^2\,\Delta^{',''} \\
&\quad - 37200\,\Delta^{',''''}\,[\Delta^{','''}]^4 + 36864\,[\Delta^{'','''}]^3\,[\Delta^{',''}]^2 + 48420\,[\Delta^{'','''}]^2\,[\Delta^{','''}]^2\,\Delta^{',''} \\
&\quad - 64800\,\Delta^{'','''}\,[\Delta^{','''}]^4\Big) + f_1'''\,\Big(16000\,[\Delta^{','''}]^5\Big).
\end{aligned}
$$

$$
\begin{aligned}
X^{25} := {} & \frac{-56\,K_{1,1}^{12}\,H_1^{14} + 5\,M_1^{10}\,F_{1,1}^{16}}{f_1'} \\
= {} & f_1'\,f_1'\,\Big(-45\,[\Delta^{',''''}]^2\,\Delta^{',''''}\,[\Delta^{',''}]^2 - 180\,[\Delta^{','''''}]^2\,\Delta^{'','''}\,[\Delta^{',''}]^2 - 3600\,[\Delta^{','''''}]^2\,[\Delta^{','''}]^2\,\Delta^{',''} \\
& - 2800\,\Delta^{','''''}\,\Delta^{',''''}\,[\Delta^{','''}]^3 - 83200\,\Delta^{','''''}\,\Delta^{'','''}\,[\Delta^{','''}]^3 - 1125\,[\Delta^{'',''''}]^2\,\Delta^{',''''}\,[\Delta^{',''}]^2 \\
& - 4500\,[\Delta^{'',''''}]^2\,\Delta^{'','''}\,[\Delta^{',''}]^2 - 90000\,[\Delta^{'',''''}]^2\,[\Delta^{','''}]^2\,\Delta^{',''} - 14000\,\Delta^{'',''''}\,\Delta^{',''''}\,[\Delta^{','''}]^3 \\
& - 416000\,\Delta^{'',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^3 - 150528\,[\Delta^{',''''}]^3\,\Delta^{'','''}\,\Delta^{',''} - 903168\,[\Delta^{',''''}]^2\,[\Delta^{'','''}]^2\,\Delta^{',''} \\
& + 163800\,[\Delta^{',''''}]^2\,\Delta^{'','''}\,[\Delta^{','''}]^2 - 2408448\,\Delta^{',''''}\,[\Delta^{'','''}]^3\,\Delta^{',''} + 1129500\,\Delta^{',''''}\,[\Delta^{'','''}]^2\,[\Delta^{','''}]^2 \\
& - 9408\,[\Delta^{',''''}]^4\,\Delta^{',''} + 3675\,[\Delta^{',''''}]^3\,[\Delta^{','''}]^2 - 2408448\,[\Delta^{'','''}]^4\,\Delta^{',''} + 2132400\,[\Delta^{'','''}]^3\,[\Delta^{','''}]^2 \\
& - 450\,\Delta^{','''''}\,\Delta^{'',''''}\,\Delta^{',''''}\,[\Delta^{',''}]^2 - 1800\,\Delta^{','''''}\,\Delta^{'',''''}\,\Delta^{'','''}\,[\Delta^{',''}]^2 \\
& - 36000\,\Delta^{','''''}\,\Delta^{'',''''}\,[\Delta^{','''}]^2\,\Delta^{',''} + 11970\,\Delta^{','''''}\,[\Delta^{',''''}]^2\,\Delta^{','''}\,\Delta^{',''} \\
& + 187920\,\Delta^{','''''}\,[\Delta^{'','''}]^2\,\Delta^{','''}\,\Delta^{',''} + 59850\,\Delta^{'',''''}\,[\Delta^{',''''}]^2\,\Delta^{','''}\,\Delta^{',''} \\
& + 939600\,\Delta^{'',''''}\,[\Delta^{'','''}]^2\,\Delta^{','''}\,\Delta^{',''} + 474300\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} \\
& + 94860\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''}\Big) + f_1'\,f_1''\,\Big(-2556600\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^3 \\
& - 5014200\,[\Delta^{'','''}]^2\,[\Delta^{','''}]^3 - 187950\,[\Delta^{',''''}]^2\,[\Delta^{','''}]^3 + 5621760\,\Delta^{',''''}\,[\Delta^{'','''}]^2\,\Delta^{','''}\,\Delta^{',''} \\
& + 5652480\,[\Delta^{'','''}]^3\,\Delta^{','''}\,\Delta^{',''} - 2764800\,\Delta^{'',''''}\,[\Delta^{'','''}]^2\,[\Delta^{',''}]^2 \\
& + 99000\,\Delta^{'',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^2\,\Delta^{',''} + 500000\,\Delta^{'',''''}\,[\Delta^{','''}]^4 + 174720\,[\Delta^{',''''}]^3\,\Delta^{','''}\,\Delta^{',''} \\
& + 1751040\,[\Delta^{',''''}]^2\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} - 276480\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{',''}]^2 \\
& - 105300\,\Delta^{','''''}\,\Delta^{',''''}\,[\Delta^{','''}]^2\,\Delta^{',''} - 552960\,\Delta^{','''''}\,[\Delta^{'','''}]^2\,[\Delta^{',''}]^2 \\
& + 19800\,\Delta^{','''''}\,\Delta^{'','''}\,[\Delta^{','''}]^2\,\Delta^{',''} + 100000\,\Delta^{','''''}\,[\Delta^{','''}]^4 + 551250\,[\Delta^{'',''''}]^2\,\Delta^{','''}\,[\Delta^{',''}]^2 \\
& - 172800\,\Delta^{'',''''}\,[\Delta^{',''''}]^2\,[\Delta^{',''}]^2 - 1382400\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{',''}]^2 \\
& - 526500\,\Delta^{'',''''}\,\Delta^{',''''}\,[\Delta^{','''}]^2\,\Delta^{',''} - 34560\,\Delta^{','''''}\,[\Delta^{',''''}]^2\,[\Delta^{',''}]^2 + 22050\,[\Delta^{','''''}]^2\,\Delta^{','''}\,[\Delta^{',''}]^2 \\
& + 220500\,\Delta^{','''''}\,\Delta^{'',''''}\,\Delta^{','''}\,[\Delta^{',''}]^2\Big) \\
& + f_1'\,f_1'''\,\Big(28000\,\Delta^{',''''}\,[\Delta^{','''}]^4 + 472000\,\Delta^{'','''}\,[\Delta^{','''}]^4\Big) \\
& + f_1''\,f_1''\,\Big(330750\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{','''}\,[\Delta^{',''}]^2 + 661500\,\Delta^{','''''}\,\Delta^{'','''}\,\Delta^{','''}\,[\Delta^{',''}]^2 \\
& - 294000\,\Delta^{','''''}\,[\Delta^{','''}]^3\,\Delta^{',''} - 330750\,\Delta^{','''''}\,\Delta^{'',''''}\,[\Delta^{',''}]^3 \\
& + 1653750\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{','''}\,[\Delta^{',''}]^2 + 3307500\,\Delta^{'',''''}\,\Delta^{'','''}\,\Delta^{','''}\,[\Delta^{',''}]^2 \\
& - 1470000\,\Delta^{'',''''}\,[\Delta^{','''}]^3\,\Delta^{',''} - 2880\,[\Delta^{',''''}]^3\,[\Delta^{',''}]^2 - 34560\,[\Delta^{',''''}]^2\,\Delta^{'','''}\,[\Delta^{',''}]^2 \\
& - 812475\,[\Delta^{',''''}]^2\,[\Delta^{','''}]^2\,\Delta^{',''} - 138240\,\Delta^{',''''}\,[\Delta^{'','''}]^2\,[\Delta^{',''}]^2 - 33075\,[\Delta^{','''''}]^2\,[\Delta^{',''}]^3 \\
& + 1446000\,\Delta^{',''''}\,[\Delta^{','''}]^4 - 184320\,[\Delta^{'','''}]^3\,\Delta^{',''}]^2 - 3077100\,[\Delta^{'','''}]^2\,[\Delta^{','''}]^2\,\Delta^{',''} \\
& + 2844000\,\Delta^{'','''}\,[\Delta^{','''}]^4 - 826875\,[\Delta^{'',''''}]^2\,[\Delta^{',''}]^3 - 3192300\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^2\,\Delta^{',''}\Big) \\
& + f_1''\,f_1'''\,\Big(-640000\,[\Delta^{','''}]^5\Big).
\end{aligned}
$$

$$
\begin{aligned}
Y^{27} &:= \frac{-56\,K_{1,1}^{12}F_{1,1}^{16} + M_1^{10}X^{18}}{f_1'} \\
&= f_1'f_1'f_1'\Big(572820\,\Delta^{',''''}\,[\Delta^{'','''}]^2\,[\Delta^{','''}]^2 - 5343744\,\Delta^{',''''}\,[\Delta^{'','''}]^3\,\Delta^{','''} - 752640\,\Delta^{'',''''}\,[\Delta^{'','''}]^3\,\Delta^{',''} - \\
&\quad - 286944\,[\Delta^{',''''}]^3\,\Delta^{'','''}\,\Delta^{','''} + 2864100\,\Delta^{'',''''}\,[\Delta^{'','''}]^2\,[\Delta^{','''}]^2 - 1862784\,[\Delta^{',''''}]^2\,[\Delta^{'','''}]^2\,\Delta^{','''} + \\
&\quad + 27195\,\Delta^{','''''}\,[\Delta^{',''''}]^2\,[\Delta^{','''}]^2 - 112000\,\Delta^{','''''}\,\Delta^{'',''''}\,[\Delta^{','''}]^3 - 150528\,\Delta^{','''''}\,[\Delta^{'','''}]^3\,\Delta^{',''} - \\
&\quad - 2352\,\Delta^{','''''}\,[\Delta^{',''''}]^3\,\Delta^{',''} + 135975\,\Delta^{'',''''}\,[\Delta^{',''''}]^2\,[\Delta^{','''}]^2 - 11760\,\Delta^{'',''''}\,[\Delta^{',''''}]^3\,\Delta^{',''} - \\
&\quad - 3375\,\Delta^{','''''}\,[\Delta^{'',''''}]^2\,[\Delta^{',''}]^2 - 675\,[\Delta^{','''''}]^2\,\Delta^{'',''''}\,[\Delta^{',''}]^2 - 45\,[\Delta^{','''''}]^3\,[\Delta^{',''}]^2 - \\
&- 11200\,[\Delta^{','''''}]^2\,[\Delta^{','''}]^3 - 5625\,[\Delta^{'',''''}]^3\,[\Delta^{',''}]^2 - 280000\,[\Delta^{'',''''}]^2\,[\Delta^{','''}]^3 - \\
&- 16464\,[\Delta^{',''''}]^4\,\Delta^{','''} - 5720064\,[\Delta^{'','''}]^4\,\Delta^{','''} + 1890\,[\Delta^{','''''}]^2\,\Delta^{',''''}\,\Delta^{','''}\,\Delta^{',''} + \\
&+ 6210\,[\Delta^{','''''}]^2\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} - 28224\,\Delta^{','''''}\,[\Delta^{',''''}]^2\,\Delta^{'','''}\,\Delta^{',''} - 112896\,\Delta^{','''''}\,\Delta^{',''''}\,[\Delta^{'','''}]^2\,\Delta^{',''} + \\
&+ 255360\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^2 + 47250\,[\Delta^{'',''''}]^2\,\Delta^{',''''}\,\Delta^{','''}\,\Delta^{',''} + 155250\,[\Delta^{'',''''}]^2\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} - \\
&- 141120\,\Delta^{'',''''}\,[\Delta^{',''''}]^2\,\Delta^{'','''}\,\Delta^{',''} - 564480\,\Delta^{'',''''}\,\Delta^{',''''}\,[\Delta^{'','''}]^2\,\Delta^{',''} + 1276800\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^2 + \\
&+ 18900\,\Delta^{','''''}\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{','''}\,\Delta^{',''} + 62100\,\Delta^{','''''}\Big) + \\
&+ \Big(\Big(-36450\,\Delta^{','''''}\,\Delta^{'',''''}\,\Delta^{',''''}\,[\Delta^{',''}]^2 - 145800\,\Delta^{','''''}\,\Delta^{'',''''}\,\Delta^{'','''}\,[\Delta^{',''}]^2 + 832500\,\Delta^{','''''}\,[\Delta^{','''}]^2\,\Delta^{',''} - \\
&\quad - 149310\,\Delta^{','''''}\,[\Delta^{',''''}]^2 - 2680560\,\Delta^{','''''}\,[\Delta^{'','''}]^2\,\Delta^{','''}\,\Delta^{',''} - 746550\,\Delta^{'',''''}\,[\Delta^{',''''}]^2\,\Delta^{','''}\,\Delta^{',''} - \\
&\quad - 13402800\,\Delta^{'',''''}\,[\Delta^{'','''}]^2\,\Delta^{','''}\,\Delta^{',''} - 3645\,[\Delta^{','''''}]^2\,\Delta^{',''''}\,[\Delta^{',''}]^2 - 14580\,[\Delta^{','''''}]^2\,\Delta^{'','''}\,[\Delta^{',''}]^2 + \\
&\quad + 83250\,[\Delta^{','''''}]^2\,[\Delta^{','''}]^2\,\Delta^{',''} - 245700\,\Delta^{','''''}\,\Delta^{',''''}\,[\Delta^{','''}]^3 + 682200\,\Delta^{','''''}\,\Delta^{'','''}\,[\Delta^{','''}]^3 - \\
&- 91125\,[\Delta^{'',''''}]^2\,\Delta^{',''''}\,[\Delta^{',''}]^2 - 364500\,[\Delta^{'',''''}]^2\,\Delta^{'','''}\,[\Delta^{',''}]^2 + 2081250\,[\Delta^{'',''''}]^2\,[\Delta^{','''}]^2\,\Delta^{',''} - \\
&- 1228500\,\Delta^{'',''''}\,\Delta^{',''''}\,[\Delta^{','''}]^3 + 3411000\,\Delta^{'',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^3 + 1354752\,[\Delta^{',''''}]^3\,\Delta^{'','''}\,\Delta^{',''} + \\
&+ 8128512\,[\Delta^{',''''}]^2\,[\Delta^{'','''}]^2\,\Delta^{',''} + 1796760\,[\Delta^{',''''}]^2\,\Delta^{'','''}\,[\Delta^{','''}]^2 + 21676032\,\Delta^{',''''}\,[\Delta^{'','''}]^3\,\Delta^{',''} + \\
&+ 850140\,\Delta^{',''''}\,[\Delta^{'','''}]^2\,[\Delta^{','''}]^2 + 84672\,[\Delta^{',''''}]^4\,\Delta^{',''} + 274155\,[\Delta^{',''''}]^3\,[\Delta^{','''}]^2 + \\
&+ 21676032\,[\Delta^{'','''}]^4\,\Delta^{',''} - 7801680\,[\Delta^{'','''}]^3\,[\Delta^{','''}]^2 - 6336900\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} - \\
&- 1267380\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''}\Big)\,f_1'' + \\
&+ \Big(1120000\,\Delta^{'',''''}\,[\Delta^{','''}]^4 - 5062200\,[\Delta^{'','''}]^2\,[\Delta^{','''}]^3 + 224000\,\Delta^{','''''}\,[\Delta^{','''}]^4 - \\
&- 271950\,[\Delta^{',''''}]^2\,[\Delta^{','''}]^3 - 2364600\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^3\Big)\,f_1'''\Big)\,f_1'f_1' + \\
&+ \Big(\Big(34044300\,[\Delta^{'','''}]^2\,[\Delta^{','''}]^3 + 108675\,[\Delta^{',''''}]^2\,[\Delta^{','''}]^3 - 231525\,[\Delta^{','''''}]^2\,\Delta^{','''}\,[\Delta^{',''}]^2 + \\
&+ 278640\,\Delta^{','''''}\,[\Delta^{',''''}]^2\,[\Delta^{',''}]^2 + 4458240\,\Delta^{','''''}\,[\Delta^{'','''}]^2\,[\Delta^{',''}]^2 - 5788125\,[\Delta^{'',''''}]^2\,\Delta^{','''}\,[\Delta^{',''}]^2 + \\
&+ 1393200\,\Delta^{'',''''}\,[\Delta^{',''''}]^2\,[\Delta^{',''}]^2 + 22291200\,\Delta^{'',''''}\,[\Delta^{'','''}]^2\,[\Delta^{',''}]^2 - 1396080\,[\Delta^{',''''}]^3\,\Delta^{','''}\,\Delta^{',''} + \\
&+ 14733900\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^3 - 44766720\,[\Delta^{'','''}]^3\,\Delta^{','''}\,\Delta^{',''} - 1284000\,\Delta^{','''''}\,[\Delta^{','''}]^4 - \\
&- 6420000\,\Delta^{'',''''}\,[\Delta^{','''}]^4 - 2315250\,\Delta^{','''''}\,\Delta^{'',''''}\,\Delta^{','''}\,[\Delta^{',''}]^2 + 2229120\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{',''}]^2 + \\
&+ 1386450\,\Delta^{','''''}\,\Delta^{',''''}\,[\Delta^{','''}]^2\,\Delta^{',''} + 915300\,\Delta^{','''''}\,\Delta^{'','''}\,[\Delta^{','''}]^2\,\Delta^{',''} + 11145600\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{',''}]^2 + \\
&+ 6932250\,\Delta^{'',''''}\,\Delta^{',''''}\,[\Delta^{','''}]^2\,\Delta^{',''} + 4576500\,\Delta^{'',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^2\,\Delta^{',''} - 13966560\,[\Delta^{',''''}]^2\,\Delta^{'','''}\,\Delta^{','''}\,\Delta^{',''} - \\
&- 44720640\,\Delta^{',''''}\,[\Delta^{'','''}]^2\,\Delta^{','''}\,\Delta^{',''}\Big)\,f_1''f_1'' + \Big(2268000\,\Delta^{',''''}\,[\Delta^{','''}]^4 + 792000\,\Delta^{'','''}\,[\Delta^{','''}]^4\Big)\,f_1''f_1''' - \\
&- 1120000\,[\Delta^{','''}]^5\,f_1'''f_1'''\Big)\,f_1' +
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(-4630500\,\Delta^{',''''}\,\Delta^{'','''}\,\Delta^{','''}\,[\Delta^{',''}]^2+2058000\,\Delta^{','''''}\,[\Delta^{','''}]^3\,\Delta^{',''}-11576250\,\Delta^{'',''''}\,\Delta^{',''''}\,\Delta^{','''}\,[\Delta^{',''}]^2-\\
&-23152500\,\Delta^{'',''''}\,\Delta^{'','''}\,\Delta^{','''}\,[\Delta^{',''}]^2+10290000\,\Delta^{'',''''}\,[\Delta^{','''}]^3\,\Delta^{',''}+2880\,[\Delta^{',''''}]^3\,[\Delta^{',''}]^2+\\
&+34560\,[\Delta^{',''''}]^2\,\Delta^{'','''}\,[\Delta^{',''}]^2+5773725\,[\Delta^{',''''}]^2\,[\Delta^{','''}]^2\,\Delta^{',''}+138240\,\Delta^{',''''}\,[\Delta^{'','''}]^2\,[\Delta^{',''}]^2+\\
&+231525\,[\Delta^{','''''}]^2\,[\Delta^{',''}]^3+2315250\,\Delta^{','''''}\,\Delta^{'',''''}\,[\Delta^{',''}]^3-2315250\,\Delta^{','''''}\,\Delta^{',''''}\,\Delta^{','''}\,[\Delta^{',''}]^2+\\
&+22922100\,[\Delta^{'','''}]^2\,[\Delta^{','''}]^2\,\Delta^{',''}-20484000\,\Delta^{'','''}\,[\Delta^{','''}]^4+5788125\,[\Delta^{'',''''}]^2\,[\Delta^{',''}]^3-\\
&-10266000\,\Delta^{',''''}\,[\Delta^{','''}]^4+184320\,[\Delta^{'','''}]^3\,[\Delta^{',''}]^2+23037300\,\Delta^{',''''}\,\Delta^{'','''}\,[\Delta^{','''}]^2\,\Delta^{',''}\Big)\,f_1''f_1''f_1''+\\
&+4560000\,[\Delta^{','''}]^5\,f_1''f_1''f_1'''.
\end{aligned}
$$

It will be a theorem, to be established in §10 below, that the 17 mutually independent bi-invariants f_1', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$, M^8, $\Lambda_{1,1,1}^9$, M_1^{10}, N^{12}, $K_{1,1}^{12}$, H_1^{14}, $F_{1,1}^{16}$, X^{18}, X^{19}, X^{21}, X^{23}, X^{25} and Y^{27} generate the algebra UE_5^2.

§9. Principle of the general algorithm

Initializing the algorithm. We now explain a general algorithm which generates all bi-invariants, which stops after a finite number of steps if and only if the algebra of bi-invariants is finitely generated and which, in such a circumstance, yields a complete generating family of mutually independent bi-invariants together with a complete generating family of syzygies between these bi-invariants. The same algorithm would work equally well for Demailly-Semple invariants, but as we already observed, in the desired applications, the complexity and the cardinality of generators and of syzygies being much higher, only the exploration of bi-invariants seems accessible.

Fix the dimension n and the jet order κ, both arbitrary. Start from the representation of an arbitrary bi-invariant of weight m gained previously thanks to the proposition on p. 228:

$$
\mathsf{P}=\mathsf{P}\big(j^\kappa f\big)=\sum_{-\frac{\kappa-1}{\kappa}m\leqslant a\leqslant m}(f_1')^a\,\mathsf{P}\big(L^{l_1},\dots,L^{l_{k_1}}\big),
$$

where the L^{l_i}, $i=1,\dots,k_1$, have weight l_i and come from the Λ-minors written there, after a division by an appropriate maximal factoring power of f_1', *cf.* the two special cases analyzed after the general proposition. Call $f_1', L^{l_1},\dots,L^{l_{k_1}}$ the *initial bi-invariants.*

First loop of the algorithm. The first step of the algorithm consists in computing a reduced Gröbner basis (for a certain monomial order) of the ideal of relations of the restrictions to $\{f_1'=0\}$ of these initial bi-invariants:

$$
\mathsf{Ideal\text{-}Rel}\Big(L^{l_1}\big|_0,\ \dots,\ L^{l_{k_1}}\big|_0\Big).
$$

In some favorables circumstances, this task may be done by symbolic Gröbner bases packages, although it is well known that due to exponentiality of time computation and to expression swelling, Gröbner bases often appear

to be frustratingly unusable. Write as follows the so obtained gröbnerized syzygies:

$$0 \equiv \mathsf{S}_i\Big(L^{l_1}(j^\kappa f)\big|_0,\ \dots,\ L^{l_{k_1}}(j^\kappa f)\big|_0\Big) \qquad {\scriptstyle (i=1\cdots N_1)}.$$

At first, we claim that, without loss of generality, one may assume that each syzygy polynomial S_i is *weighted homogeneous*, say of weight μ_i, namely satisfies:

$$\mathsf{S}_i\big(\delta^{l_1}A_1,\ \dots,\ \delta^{l_{k_1}}A_{k_1}\big) = \delta^{\mu_i}\,\mathsf{S}_i\big(A_1,\ \dots,\ A_{k_1}\big),$$

in $\mathbb{C}\big[A_1,\dots,A_{k_1}\big]$ for every weighted dilation factor $\delta\in\mathbb{C}$. Indeed, dilating $j^\kappa f$ as usual:

$$\delta\cdot j^\kappa f := \big(\delta^\lambda f_i^{(\lambda)}\big)_{1\leqslant i\leqslant n}^{1\leqslant\lambda\leqslant\kappa},$$

since the syzygies hold for any collection of $n\kappa$ components $\big(f_i^{(\lambda)}\big)_{1\leqslant i\leqslant n}^{1\leqslant\lambda\leqslant\kappa}$ in the jet space, they must hold too with $\big(\delta^\lambda f_i^{(\lambda)}\big)_{1\leqslant i\leqslant n}^{1\leqslant\lambda\leqslant\kappa}$, namely:

$$\begin{aligned} 0 &\equiv \mathsf{S}_i\Big(L^{l_1}(\delta\cdot j^\kappa f)\big|_0,\ \dots,\ L^{l_{k_1}}(\delta\cdot j^\kappa f)\big|_0\Big) \\ &= \mathsf{S}_i\Big(\delta^{l_1}\,L^{l_1}(j^\kappa f)\big|_0,\ \dots,\ \delta^{l_{k_1}}\,L^{l_{k_1}}(j^\kappa f)\big|_0\Big) \qquad {\scriptstyle (i=1\cdots N_1)}, \end{aligned}$$

and we may use the fact that the L^{l_i} are invariant under reparametrization. Therefore, if we gather together, in each syzygy polynomial S_i, all terms which have equal, constant weight μ:

$$\mathsf{S}_i = \sum_\mu \mathsf{S}_i^\mu, \qquad \text{with} \qquad \mathsf{S}_i^\mu\big(\delta^{l_1}A_1,\ \dots,\ \delta^{l_{k_1}}A_{k_1}\big) = \delta^\mu\,\mathsf{S}_i^\mu\big(A_1,\ \dots,\ A_{k_1}\big),$$

we may expand according to weight the obtained relations under the specific form:

$$0 \equiv \sum_\mu \delta^\mu\,\mathsf{S}_i^\mu\Big(L^{l_1}(j^\kappa f)\big|_0,\ \dots,\ L^{l_{k_1}}(j^\kappa f)\big|_0\Big) \qquad {\scriptstyle (i=1\cdots N_1)}.$$

Because these identities then hold in $\mathbb{C}\big[\delta,\ j^\kappa f\big]$, they are equivalent to the (possibly larger) collection of *constantly weighted* syzygies:

$$0 \equiv \mathsf{S}_i^\mu\Big(L^{l_1}(j^\kappa f)\big|_0,\ \dots,\ L^{l_{k_1}}(j^\kappa f)\Big) \qquad {\scriptstyle (i=1\cdots N_1\,;\forall\,\mu)},$$

and this justifies the claim.

So let μ_i be the weight of the (homogeneous) syzygy S_i, for $i=1,\dots,N_1$. Because by assumption each polynomial $\mathsf{S}_i\big(L^{l_1}(j^\kappa f),\dots,L^{l_{k_1}}(j^\kappa f)\big)$ vanishes identically in $\mathbb{C}\big[j^\kappa f\big]$ after setting $f_1'=0$, there are maximal factoring powers $(f_1')^{\nu_i}$ of f_1', with $1\leqslant\nu_i\leqslant\infty$, and there are certain (possibly zero) polynomial remainders $\mathsf{R}_i\big(j^\kappa f\big)$ such that we may write in $\mathbb{C}\big[j^\kappa f\big]$:

$$\mathsf{S}_i\big(L^{l_1},\ \dots,\ L^{l_{k_1}}\big) = (f_1')^{\nu_i}\,\mathsf{R}_i\big(j^\kappa f\big) \qquad {\scriptstyle (i=1\cdots N_1)},$$

with $\mathsf{R}_i \not\equiv 0$ when $1 \leqslant \nu_i < \infty$ and with $\mathsf{R}_i = 0$ by convention when $\nu_i = \infty$.

We claim that each such $\mathsf{R}_i\big(j^\kappa f\big)$ is then a bi-invariant. In fact, it is a polynomial by definition, and its representation as a quotient:

$$\mathsf{R}_i\big(j^\kappa f\big) = \frac{\mathsf{S}_i\big(L^{l_1}, \dots, L^{l_{k_1}}\big)}{(f_1')^{\nu_i}}$$

of two polynomials invariant by reparametrizations and invariant under the unipotent action shows at once that R_i too enjoys bi-invariancy.

The second step of the algorithm consists in testing, for each i, whether or not R_i belongs to the algebra $\mathbb{C}\big[f_1', L^{l_1}, \dots, L^{l_{k_1}}\big]$ generated by the initial bi-invariants. In the case where no new bi-invariant appears, the algorithm will be shown to terminate, so let us assume that at least one R_i provides a new bi-invariant, independent of $f_1', L^{l_1}, \dots, L^{l_{k_1}}$. It is then clear that after renumbering the R_i if necessary, one may assume that:

$$\begin{cases} \mathsf{R}_1 \quad \text{is independent of } \ f_1', L^{l_1}, \dots, L^{l_{k_1}}, \\ \mathsf{R}_2 \quad \text{is independent of } \ f_1', L^{l_1}, \dots, L^{l_{k_1}}, \mathsf{R}_1, \\ \dots\dots\dots \\ \mathsf{R}_{k_2} \quad \text{is independent of } \ f_1', L^{l_1}, \dots, L^{l_{k_1}}, \mathsf{R}_1, \dots, \mathsf{R}_{k_2-1}, \end{cases}$$

while for the next indices $i = k_2 + 1, \dots, N_1$:

$$\Big\{ \mathsf{R}_i \quad \text{belongs to the algebra } \ \mathbb{C}\big[f_1', L^{l_1}, \dots, L^{l_{k_1}}, \mathsf{R}_1, \dots, \mathsf{R}_{k_2}\big].$$

Denoting instead by $M^{m_1}, \dots, M^{m_{k_2}}$ these R_i for $i = 1, \dots, k_2$ which provide new mutually independent bi-invariants, where as usual the weights $m_i := \mu_i - \nu_i$, for $i = 1, \dots, k_2$ are put in exponent place, we can therefore write down in more explicit form the filled syzygy polynomials (without setting $f_1' = 0$):

$$\begin{cases} 0 \equiv \mathsf{S}_i\big(L^{l_1}, \dots, L^{l_{k_1}}\big) + (f_1')^{\nu_i}\, M^{m_i} & (i = 1 \cdots k_2), \\ 0 \equiv \mathsf{S}_i\big(L^{l_1}, \dots, L^{l_{k_1}}\big) + (f_1')^{\nu_i}\, \mathsf{R}_i\big(L^{l_1}, \dots, L^{l_{k_1}}, M^{m_1}, \dots, M^{m_{k_2}}\big) & (i = k_2+1 \cdots N_1), \end{cases}$$

from which we recover at once, by setting f_1', the original syzygies:

$$0 \equiv \mathsf{S}_i\big(L^{l_1}\big|_0, \ \dots, \ L^{l_{k_1}}\big|_0\big) \qquad (i = 1 \cdots N_1).$$

So the equations above, when written explicitly in specific applications below, shall show both the collection of new appearing bi-invariants $M^{m_1}, \dots, M^{m_{k_2}}$ (without setting $f_1' = 0$) and (after setting f_1') a reduced Gröbner basis for the ideal of relations between the initial bi-invariants $L^{l_1}\big|_0, \dots, L^{L_{k_1}}\big|_0$.

Second and further loops of the algorithm. Next, we restart the process with the new, larger collection of bi-invariants, namely we compute a reduced Gröbner basis (for a certain monomial order compatible with the preceding loop):

$$\mathsf{Ideal\text{-}Rel}\Big(L^{l_1}\big|_0,\ \dots,\ L^{l_{k_1}}\big|_0,\ M^{m_1}\big|_0,\ \dots,\ M^{m_{k_2}}\big|_0\Big).$$

Write as follows the so obtained gröbnerized syzygies, after filling the remainders behind a power of f_1' and after testing whether these remainders provide new bi-invariants:

$$\begin{cases} 0 \equiv \mathsf{S}_i\big(L^{l_1},\dots,L^{l_{k_1}}\big) + (f_1')^{\nu_i}\, M^{m_i} & (i=1\cdots k_2), \\ 0 \equiv \mathsf{S}_i\big(L^{l_1},\dots,L^{l_{k_1}}\big) + (f_1')^{\nu_i}\, \mathsf{R}_i\big(L^{l_1},\dots,L^{l_{k_1}},M^{m_1},\dots,M^{m_{k_2}}\big) & (i=k_2+1\cdots N_1), \\ 0 \equiv \mathsf{T}_j\big(L^{l_1},\dots,L^{l_{k_1}},M^{m_1},\dots,M^{m_{k_2}}\big) + (f_1')^{\nu_j}\, N^{n_j} & (j=1\cdots k_3), \\ 0 \equiv \mathsf{T}_j\big(L^{l_1},\dots,L^{l_{k_1}},M^{m_1},\dots,M^{m_{k_2}}\big) + & \\ \qquad + (f_1')^{\nu_j}\, R_j\big(L^{l_1},\dots,L^{l_{k_1}},M^{m_1},\dots,M^{m_{k_2}},N^{n_1},\dots,N^{n_{k_3}}\big) & (j=k_3+1\cdots N_2). \end{cases}$$

with $N^{n_1},\dots,N^{n_{k_3}}$ denoting the new appearing bi-invariants, of weight $n_1,\dots,n_{k_3}$.

Successively, continue to perform further loops as long as new bi-invariants appear which do not belong to the algebra generated by already known bi-invariants.

Termination of the algorithm. Either there always appear new bi-invariants or, after a finite number of loops, we come to a situation which falls under the scope of the following important statement.

THEOREM *For a certain dimension n and for a certain jet order κ, suppose that, after performing a finite number of loops of the algorithm, one possesses a finite number $1+M$ of mutually independent bi-invariants f_1', Λ^{ℓ_1}, ..., $\Lambda^{\ell_M} \in \mathbb{C}\big[j^\kappa f_1,\dots,j^\kappa f_n\big]$ of weights $1,\ell_1,\dots,\ell_M$ belonging to UE^n_κ, whose restrictions to $\{f_1'=0\}$ share an ideal of relations:*

$$\mathsf{Ideal\text{-}Rel}\Big(\ \Lambda^{\ell_1}\big|_0,\ \ \dots\dots,\ \Lambda^{\ell_M}\big|_0\ \Big)$$

generated by a finite number N (often large) of homogeneous syzygies:

$$0 \equiv \mathsf{S}_i\big(\Lambda^{\ell_1}\big|_0,\ \dots,\ \Lambda^{\ell_M}\big|_0\big), \qquad (i=1\cdots N)$$

of weight μ_i assumed to be represented by a certain reduced Gröbner basis $\big\langle \mathsf{S}_i\big\rangle_{1\leqslant i\leqslant N}$ for a certain monomial order, with the crucial property that no *new bi-invariant appears behind f_1', namely with the property that, without setting $f_1'=0$, one has N identically satisfied relations:*

$$0 \equiv \mathsf{S}_i\big(\Lambda^{\ell_1},\ \dots,\ \Lambda^{\ell_M}\big) - f_1'\,\mathsf{R}_i\big(f_1',\ \Lambda^{\ell_1},\ \dots,\ \Lambda^{\ell_M}\big) \qquad (i=1\cdots N),$$

for some remainders R_i which all depend polynomially upon the same collection of invariants $f_1', \Lambda^{\ell_1}, \ldots, \Lambda^{\ell_M}$*, so that no new bi-invariant appears at this stage.*

Then the algorithm terminates and the algebra of bi-invariants coincides with:

$$\boxed{\mathsf{UE}^n_\kappa = \mathbb{C}\big[f_1',\, \Lambda^{\ell_1},\, \ldots\ldots,\, \Lambda^{\ell_M}\big] \quad \text{modulo syzygies}}\,.$$

In addition, for these values of n *and of* κ*, if one denotes the leading terms of the syzygies by:*

$$\mathsf{LT}\big(\mathsf{S}_i(\Lambda)\big) = \big(\Lambda^{\ell_1}\big)^{\alpha_1^i} \cdots \big(\Lambda^{\ell_M}\big)^{\alpha_M^i} \qquad\qquad (i = 1 \cdots N),$$

for certain specific multiindices $\big(\alpha_1^i, \ldots, \alpha_M^i\big) \in \mathbb{N}^M$*, and if for* $i = 1, \ldots, N$ *one denotes by:*

$$\square_i := \alpha^i + \mathbb{N}^M = \big\{\big(\alpha_1^i + b_1,\, \ldots,\, \alpha_M^i + b_M\big) :\, b_1, \ldots, b_M \in \mathbb{N}^M\big\}$$

the positive quadrant of $\mathbb{N}^M$ *having vertex at* α^i*, then a general, arbitrary bi-invariant in* $\mathsf{UE}^n_{\kappa,m}$ *of weight* m *writes* uniquely *under the* normal form*:*

$$\sum_{0 \leqslant a \leqslant m} (f_1')^a\, \widetilde{\mathsf{P}}_a\big(\Lambda^{\ell_1},\, \ldots,\, \Lambda^{\ell_M}\big),$$

with summation containing only positive powers *of* f_1'*, where each* $\widetilde{P}_a$ *is of weight* $m - a$ *and is put under* Gröbner-normalized form*:*

$$\boxed{\widetilde{\mathsf{P}}_a = \sum_{\substack{(b_1,\ldots,b_M) \in \mathbb{N}^M \setminus (\square_1 \cup \cdots \cup \square_N) \\ \ell_1 b_1 + \cdots + \ell_M b_M = m - a}} \mathsf{coeff}_{a;\, b_1,\ldots,b_M} \cdot \big(\Lambda^{\ell_1}\big)^{b_1} \cdots \big(\Lambda^{\ell_M}\big)^{b_M}}\,,$$

with complex coefficients $\mathsf{coeff}_{a;\, b_1,\ldots,b_M}$ *subjected to no restriction at all.*

Proof. We start with the list of initial bi-invariants $f_1', L^{l_1}, \ldots, L^{l_{k_1}}$ and with the initial, rational representation of an arbitrary bi-invariant $\mathsf{P}\big(j^\kappa f\big) \in \mathsf{UE}^n_{\kappa,m}$ which was obtained previously:

$$\begin{aligned} \mathsf{P}\big(j^\kappa f\big) &= \sum_{-\frac{\kappa-1}{\kappa} m \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a\big(L^{l_1}, \ldots, L^{l_{k_1}}\big) \\ &= (f_1')^{a_0}\, \mathsf{P}_{a_0} + \sum_{a_0 + 1 \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a, \end{aligned}$$

and we denote by a_0 the smallest appearing exponent of f_1'. Clearly, the final list of bi-invariants $\Lambda^{\ell_1}, \ldots, \Lambda^{\ell_M}$ stabilized after a finite number of loops of the algorithm contains $L^{l_1}, \ldots, L^{l_{k_1}}$ as its first k_1 terms. Working in the

polynomial ring $\mathbb{C}\big[A^1,\dots,A^{k_1},\dots,A^M\big]$, we may then divide P_{a_0} by the ideal of relations $\big\langle \mathsf{S}_i(A)\big\rangle_{1\leqslant i\leqslant N}$:

$$\mathsf{P}_{a_0}\big(A^1,\dots,A^{k_1}\big)=\widetilde{\mathsf{P}}_{a_0}\big(A^1,\dots,A^{k_1},\dots,A^M\big)+\sum_{i=1}^{N}q_i(A)\cdot\mathsf{S}_i(A),$$

with multiplicands $q_i(A)$ of weight $m-a_0-\mu_i$, getting a remainder $\widetilde{\mathsf{P}}_{a_0}$ of weight $m-a_0$ which in general will depend upon all the variables $A^1,\dots,A^{k_1},\dots,A^M$ and which is *unique* (while the multiplicands q_i cannot be unique, as soon as $N\geqslant 2$), by virtue of a classical feature of Gröbner bases. Consequently, replacing the independent variables A^l by the bi-invariants in the arguments and then substituting each $\mathsf{S}_i(\Lambda)$ by $f_1'\,\mathsf{R}_i(f_1',\Lambda)$ — thanks to the main assumption that in filled syzygies, all the remainders behind f_1' depend polynomially upon the same bi-invariants $f_1',\Lambda^{\ell_1},\dots,\Lambda^{\ell_M}$ —, we then get a normalized representation of $\widetilde{\mathsf{P}}_{a_0}$:

$$\begin{aligned}\mathsf{P}_{a_0}\big(L^{l_1},\dots,L^{l_{k_1}}\big)&=\widetilde{\mathsf{P}}_{a_0}\big(\Lambda^{\ell_1},\dots,\Lambda^{\ell_M}\big)+\sum_{i=1}^{N}q_i(\Lambda)\cdot\mathsf{S}_i(\Lambda)\\&=\widetilde{\mathsf{P}}_{a_0}\big(\Lambda^{\ell_1},\dots,\Lambda^{\ell_M}\big)+\sum_{i=1}^{N}q_i(\Lambda)\cdot f_1'\,\mathsf{R}_i\big(f_1',\Lambda\big)\\&=\widetilde{\mathsf{P}}_{a_0}\big(\Lambda^{\ell_1},\dots,\Lambda^{\ell_M}\big)+f_1'\,\widetilde{\mathsf{R}}_{a_0}\big(f_1',\,\Lambda^{\ell_1},\dots,\Lambda^{\ell_M}\big),\end{aligned}$$

(modulo an uncontrolled remainder $\widetilde{\mathsf{R}}_{a_0}$ which hopefully, lies behind f_1') which we may therefore inject in our rational representation:

$$\mathsf{P}\big(j^\kappa f\big)=(f_1')^{a_0}\,\widetilde{\mathsf{P}}_{a_0}(\Lambda)+(f_1')^{a_0+1}\,\widetilde{\mathsf{R}}_{a_0}\big(f_1',\,\Lambda\big)+\sum_{a_0+1\leqslant a\leqslant m}(f_1')^a\,\mathsf{P}_a(L).$$

But both $\widetilde{\mathsf{P}}_{a_0}$ and $f_1'\,\widetilde{\mathsf{R}}_{a_0}$ being of weight $m-a_0$ as was P_{a_0}, it follows that, when developping the perturbing term $(f_1')^{a_0+1}\,\widetilde{\mathsf{R}}_{a_0}\big(f_1',\Lambda\big)$ in powers of f_1', the fact that this remainder is of weight m guarantees that the sum does not go beyond $(f_1')^m$, and thus, we come to an expression:

$$\mathsf{P}\big(j^\kappa f\big)=(f_1')^{a_0}\,\widetilde{\mathsf{P}}_{a_0}(\Lambda)+\sum_{a_0+1\leqslant a\leqslant m}(f_1')^a\,\mathsf{Q}_a(\Lambda)$$

entirely similar to the one we started with, whose first term:

$$\widetilde{\mathsf{P}}_{a_0}=\sum_{\substack{(b_1,\dots,b_M)\in\mathbb{N}^M\setminus(\square_1\cup\cdots\cup\square_N)\\ \ell_1b_1+\cdots+\ell_Mb_M=m-a_0}}\mathsf{coeff}_{a_0;\,b_1,\dots,b_M}\cdot\big(\Lambda^{\ell_1}\big)^{b_1}\cdots\big(\Lambda^{\ell_M}\big)^{b_M},$$

is normalized modulo the syzygies. But we can then subject the next term $\mathsf{Q}_{a_0+1}(\Lambda)$ to the same process, and consequently by induction, after a finite

number of steps, we come to an expression in which *all* multiplicands of a power of f_1' have been normalized:

$$\mathsf{P}\big(j^\kappa f\big) = \sum_{a_0' \leqslant a \leqslant m} (f_1')^a \sum_{\substack{(b_1,\dots,b_M)\in\mathbb{N}^M\setminus(\square_1\cup\cdots\cup\square_N)\\ \ell_1 b_1+\cdots+\ell_M b_M=m-a}} \mathsf{coeff}_{a;\,b_1,\dots,b_M}\cdot\big(\Lambda^{\ell_1}\big)^{b_1}\cdots\big(\Lambda^{\ell_M}\big)^{b_M},$$

with a possibly larger a_0' (in case $\widetilde{P}_{a_0}$ vanishes identically). However, the smallest a_0 in the initial expression for $\mathsf{P}\big(j^\kappa f\big)$ was possibly negative and hence our a_0' here can still be negative too, and our gained representation of $\mathsf{P}\big(j^\kappa f\big)$ can still be not polynomial.

Hopefully, we may now claim that there are no negative powers of f_1' anymore in such a normalized expression, so that the right hand side is a true polynomial.

Indeed, suppose that $a_0' < 0$ with $\widetilde{P}_{a_0'} \not\equiv 0$. Multiply both sides by $(f_1')^{-a_0'}$, set afterwards $f_1' = 0$ and then get in such a way a nontrivial identity:

$$0 \equiv \sum_{\substack{(b_1,\dots,b_M)\in\mathbb{N}^M\setminus(\square_1\cup\cdots\cup\square_N)\\ \ell_1 b_1+\cdots+\ell_M b_M=m-a_0'}} \mathsf{coeff}_{a_0';\,b_1,\dots,b_M}\cdot\big(\Lambda^{\ell_1}\big|_0\big)^{b_1}\cdots\big(\Lambda^{\ell_M}\big|_0\big)^{b_M}.$$

This equation would then represent a syzygy between bi-invariants restricted to $\{f_1' = 0\}$ whose leading term is strictly smaller than the leadings terms of the syzygies S_i. This would contradict the assumption that the collection $\big\langle \mathsf{S}_i\big\rangle_{1\leqslant i\leqslant N}$ is a Gröbner basis for the ideal of relations between $\Lambda^{\ell_1}\big|_0, \dots, \Lambda^{\ell_M}\big|_0$. So $a_0' \geqslant 0$, namely the normalized representation is polynomial.

The same argument shows that the normalized representation is unique.

Finally, it suffices to say, if not remarked stealthily before, that any polynomial in f_1', $\Lambda^{\ell_1}, \dots, \Lambda^{\ell_M}$ obviously is a bi-invariant. The proof is now complete. $\square$

§10. Seventeen bi-invariant generators in dimension $n = 2$ for jet level $\kappa = 5$

First loop of the algorithm. According to these general principles, in the case $n = 2$, $\kappa = 5$, we should therefore start with the initial rational representation:

$$\mathsf{P}\big(j^5 f\big) = \sum_{-\frac{4}{5}m \leqslant a \leqslant m} (f_1')^a\, \mathsf{P}_a\big(\Lambda^3,\, \Lambda^5,\, \Lambda^7,\, \Lambda^9\big)$$

of an arbitrary bi-invariant $\mathsf{P}\big(j^5 f\big) \in \mathsf{UE}_5^2$. Here for simplicity, we shall denote without any lower index each one of the appearing bi-invariants. In fact, among all the invariants explicitly defined in the theorem on p. 209,

bi-invariants correspond to lower indices being constantly equal to 1, and one has also to consider the *non-bracket* bi-invariants introduced in §8.

So according to the general algorithm, we have to start by computing the ideal of relations:

$$\mathsf{Ideal\text{-}Rel}\Big(\Lambda^3\big|_0,\ \Lambda^5\big|_0,\ \Lambda^7\big|_0,\ \Lambda^9\big|_0\Big).$$

For this easy first step, we may use any Gröbner bases package[47]. For the Reverse Degree Lexicographic Ordering, the result provided is:

$$\begin{aligned}
0 &\equiv -7\,\Lambda^7\big|_0\Lambda^7\big|_0 + 5\,\Lambda^5\big|_0\Lambda^9\big|_0,\\
0 &\equiv -7\,\Lambda^5\big|_0\Lambda^7\big|_0 + 3\,\Lambda^3\big|_0\Lambda^9\big|_0,\\
0 &\equiv -5\,\Lambda^5\big|_0\Lambda^5\big|_0 + 3\,\Lambda^3\big|_0\Lambda^7\big|_0.
\end{aligned}$$

Then we compute the remainder bi-invariants appearing behind a power of f_1'. Here, for the three syzygies, the maximal factoring power of f_1' is the same, equal to 2, and three new bi-invariants appear:

$$\begin{aligned}
0 &\equiv -7\,\Lambda^7\Lambda^7 + 5\,\Lambda^5\Lambda^9 - f_1'f_1'K^{12},\\
0 &\equiv -7\,\Lambda^5\Lambda^7 + 3\,\Lambda^3\Lambda^9 - f_1'f_1'M^{10},\\
0 &\equiv -5\,\Lambda^5\Lambda^5 + 3\,\Lambda^3\Lambda^7 - f_1'f_1'M^8,
\end{aligned}$$

namely: M^8, M^{10} and K^{12}. Either looking at the syzygies of the second loop below, or computing directly by hand, or playing a bit with Maple, we find the values of the restrictions to $\{f_1' = 0\}$ of all the bi-invariants obtained so far, expressed in (rational) terms of the three restricted bi-invariants, $\Lambda^3|_0$, $\Lambda^5|_0$ and $M^8|_0$ which are easily checked to be algebraically independent:

$$\begin{aligned}
&\underline{\Lambda^3}\big|_0\\
&\underline{\Lambda^5}\big|_0\\
&\Lambda^7\big|_0 = \tfrac{5}{3}\,\tfrac{\Lambda^5|_0\,\Lambda^5|_0}{\Lambda^3|_0},\\
&\Lambda^9\big|_0 = \tfrac{35}{9}\,\tfrac{\Lambda^5|_0\,\Lambda^5|_0\,\Lambda^5|_0}{\Lambda^3|_0\,\Lambda^3|_0},\\
&M^8\big|_0\\
&M^{10}\big|_0 = \tfrac{8}{3}\,\tfrac{\Lambda^5|_0\,M^8|_0}{\Lambda^3|_0},\\
&K^{12}\big|_0 = \tfrac{5}{9}\,\tfrac{\Lambda^5|_0\,\Lambda^5|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0}.
\end{aligned}$$

Proceeding then as in the lemma on p. 219 and using these rational expressions, one may establish that the 8 bi-invariants known so far, namely f_1', Λ^3, Λ^5, Λ^7, Λ^9, M^8, M^{10} and K^{12}, are mutually independent.

[47] *See* dim-2-order-5-step-1-with-FGb.mw at [23].

Second loop of the algorithm. Afterwards, we must compute the ideal of relations between the 7 restricted bi-invariants in question:

$$\mathsf{Ideal\text{-}Rel}\Big(\Lambda^3\big|_0,\ \Lambda^5\big|_0,\ \Lambda^7\big|_0,\ \Lambda^9\big|_0,\ M^8\big|_0,\ M^{10}\big|_0,\ K^{12}\big|_0\Big).$$

For the Degree Reverse Lexicographic Ordering, a Gröbner basis for this ideal of relations consists of the following 10 polynomials[48] (in which the remainders behind a power of f_1' have already been filled):

$$\begin{aligned}
0 &\equiv -5\,M^{10}M^{10} + 64\,M^8K^{12} - f_1'X^{19},\\
0 &\equiv -5\,\Lambda^9M^{10} + 56\,\Lambda^7K^{12} - f_1'X^{18},\\
0 &\equiv -8\,\Lambda^9M^8 + 7\,\Lambda^7M^{10} - f_1'F^{16},\\
0 &\equiv -\Lambda^9M^8 + 7\,\Lambda^5K^{12} - f_1'F^{16},\\
0 &\equiv -8\,\Lambda^7M^8 + 5\,\Lambda^5M^{10} - f_1'H^{14},\\
0 &\equiv -\Lambda^7M^8 + 3\,\Lambda^3K^{12} - f_1'H^{14},\\
0 &\equiv -8\,\Lambda^5M^8 + 3\,\Lambda^3M^{10} - f_1'N^{12},\\
0 &\equiv -7\,\Lambda^7\Lambda^7 + 5\,\Lambda^5\Lambda^9 - f_1'f_1'K^{12},\\
0 &\equiv -7\,\Lambda^5\Lambda^7 + 3\,\Lambda^3\Lambda^9 - f_1'f_1'M^{10},\\
0 &\equiv -5\,\Lambda^5\Lambda^5 + 3\,\Lambda^3\Lambda^7 - f_1'f_1'M^8.
\end{aligned}$$

How exactly do we manage to fill in what appears at the end of each syzygy behind any power of f_1'?

A standard obstacle: unavailability because of size computations. A natural idea would be to automatically apply the *Algebra Membership Algorithm* based on Gröbner bases ([14], p. 289), but this would be (at least for us) impossible, because this test would rely upon the (unavalaible to us) knowledge of a full Gröbner basis for the ideal generated by the 8 equations:

$$t_1 - f_1',\ l_3 - \Lambda^3,\ l_5 - \Lambda^5,\ l_7 - \Lambda^7,\ l_9 - \Lambda^9,\ m_8 - M^8,\ m_{10} - M^{10},\ k_{12} - K^{12},$$

in the ring of 18 variables:

$$\mathbb{C}\big[j^5f_1, j^5f_2,\ t_1, l_3, l_5, l_7, l_9, m_8, m_{10}, k_{12}\big]$$

with any monomial ordering having the only property that each jet variable $f_i^{(\lambda)}$ is bigger than any monomial written with only the 8 auxiliary variables $t_1, l_3, l_5, l_7, l_9, m_8, m_{10}, k_{12}$. Indeed, according to Proposition C.2.3 in the reference cited, any remainder behind a power of f_1', for instance the one appearing in the sixth syzygy above:

$$\mathsf{rem}_6 := \frac{1}{f_1'}\,\big(8\,\Lambda^5M^8 - 3\,\Lambda^3M^{10}\big),$$

[48] *See* dim-2-order-5-step-2-with-FGb.mw at [23].

would then belong to the algebra generated by the 8 already known bi-invariants: f_1', Λ^3, Λ^5, Λ^7, Λ^9, M^8, M^{10}, K^{12}, if and only if the *normal form* of rem_6 with respect to such a Gröbner basis would belong to $\mathbb{C}\big[t_1, \ldots, k_{12}\big]$, and in such a case, the (unique) normal form in question rem_6 would provide without any further effort the corresponding polynomial.

However, Gröbner bases here are blocked due to oversizeness

Hence to bypass such a (usual, forseeable) drawback of Gröbner bases, we have to proceed differently.

What we do using Maple is a little bit tricky, and it works well. After division by f_1' (most often, and sometimes also by $(f_1')^2$, but never by $(f_1')^3$), we start by computing each one of the 10 remainder; in fact, since 3 of them were already treated in the first loop, only 7 remainders have to be studied here. On the other hand and as an independent preparation, we may check by inspecting the explicit expressions given at the end of §4, that $\Lambda^3\big|_0$, $\Lambda^5\big|_0$, $M^8\big|_0$ and $N^{12}\big|_0$ (our rem_6 itself!) are mutually algebraically independent. Subsequently, we compute a Gröbner basis for the four polynomial:

$$l_{30} - \Lambda^3\big|_0,\ l_{50} - \Lambda^5\big|_0,\ m_{80} - M^8\big|_0,\ n_{120} - N^{12}\big|_0,$$

in the ring $\mathbb{C}\big[j^5 f_1|_0, j^5 f_2, l_{30}, l_{50}, m_{80}, n_{120}\big]$, where l_{30}, l_{50}, m_{80} and n_{120} denote auxiliary, supplementary variables, with any monomial order having the property that each jet variable $f_i^{(\lambda)}$ is bigger than any monomial written with only the 4 auxiliary variables l_{30}, l_{50}, m_{80} and n_{120}. This then is available to the computer: size is reasonable and it costs less than 5 minutes on any computer. Then we set $f_1' = 0$ in each remainder rem_k, getting $\mathsf{rem}_k\big|_0$. We then multiply each restricted remainder $\mathsf{rem}_k\big|_0$ for $k = 1, 2, \ldots, 10$ by a suitable power of $\Lambda^3\big|_0$ choosen by head, for instance if one looks at the third remainder:

$$\Lambda^3\big|_0 \Lambda^3\big|_0 \cdot \mathsf{rem}_3\big|_0 = \Lambda^3\big|_0 \Lambda^3\big|_0 \cdot \left[\frac{1}{f_1'}\big(8\,\Lambda^9 M^8 - 7\,\Lambda^7 M^{10}\big)\right]_{f_1'=0}.$$

Then we compute the normal form of this latter polynomial with respect to the mentioned auxiliary Gröbner basis. For instance, our computer yields for the third remainder the normal form:

$$\tfrac{35}{9}\, l_{50} l_{50} n_{120}.$$

This result therefore means that the third unknown remainder rem_3 (appearing in the third syzygy) which we denoted in advance by F^{16}, has the following value after setting $f_1' = 0$:

$$F^{16}\big|_0 = \tfrac{35}{9}\, \tfrac{\Lambda^5|_0\, \Lambda^5|_0\, N^{12}|_0}{\Lambda^3|_0\, \Lambda^3|_0}.$$

Then we test by hand and by head whether such a value for $f_1' = 0$ can be obtained as a polynomial in terms of the 7 previously known restricted

bi-invariants $\Lambda^3|_0, \ldots, K^{12}|_0$. Here, it is easy to convince oneself that this cannot be the case, so that F^{16} really is a new bi-invariant.

On the other hand, we should do the same work for the fourth remainder rem_4. It then happens that we find the *same* value at $f_1' = 0$ in terms of $\Lambda^3|_0, \Lambda^5|_0, M^8|_0, N^{12}|_0$. So we suspect that *without setting* $f_1' = 0$, the two remainders rem_3 and rem_4 could be identical and finally, a simple computation with Maple verifies that this is indeed the case. Other remainders are computed similary, and we thus have fully explained all our trick to bypass the unavailability of full Gröbner bases due to oversizeness in this problem.

However, we would like to mention that achieving such a kind of task took hours and days of patience. Hopefully, checking *a posteriori* with Maple that a syzygy effectively holds is much, much more rapid and the reader will find in the Maple worksheets referenced here the declaration of new bi-invariants at each step and the checking (at a piece) of all syzygies by means of the basic "simplify" command of Maple.

Finally, to finish with the second loop, we give the values, restricted to $\{f_1' = 0\}$, of the 5 appearing new bi-invariants at this stage:

$$\begin{aligned}
&N^{12}\big|_0 \\
&H^{14}\big|_0 = \tfrac{5}{3}\,\tfrac{\Lambda^5|_0\,N^{12}|_0}{\Lambda^3|_0}, \\
&F^{16}\big|_0 = \tfrac{35}{9}\,\tfrac{\Lambda^5|_0\,\Lambda^5|_0\,N^{12}|_0}{\Lambda^3|_0\,\Lambda^3|_0}, \\
&X^{18}\big|_0 = \tfrac{1225}{27}\,\tfrac{\Lambda^5|_0\,\Lambda^5|_0\,\Lambda^5|_0\,N^{12}|_0}{\Lambda^3|_0\,\Lambda^3|_0\,\Lambda^3|_0}, \\
&X^{19}\big|_0 = \tfrac{80}{3}\,\tfrac{\Lambda^5|_0\,M^8|_0\,N^{12}|_0}{\Lambda^3|_0\,\Lambda^3|_0}.
\end{aligned}$$

Third loop of the algorithm. Now that we have explained how we proceed, we can offer directly the 32 filled syzygies appearing at the next step[49], again for the Degree Reverse Lexicographic Ordering.

$$\begin{aligned}
0 &\equiv -5\,F^{16}F^{16} + H^{14}X^{18} - f_1'K^{12}X^{19}, \\
0 &\equiv -7\,H^{14}F^{16} + N^{12}X^{18} - f_1'M^{10}X^{19}, \\
0 &\equiv -7\,H^{14}H^{14} + 5\,N^{12}F^{16} - f_1'M^{8}X^{19}, \\
0 &\equiv -56\,K^{12}F^{16} + M^{10}X^{18} - f_1'Y^{27}, \\
0 &\equiv -56K^{12}H^{14} + 5\,M^{10}F^{16} - f_1'X^{25}, \\
0 &\equiv -8\,K^{12}N^{12} + M^{10}H^{14} - f_1'X^{23}, \\
0 &\equiv -49\,K^{12}H^{14} + M^{8}X^{18} - f_1'X^{25}, \\
0 &\equiv -7\,K^{12}N^{12} + M^{8}F^{16} - f_1'X^{23}, \\
0 &\equiv -5\,M^{10}N^{12} + 8\,M^{8}H^{14} - f_1'X^{21},
\end{aligned}$$

[49] *See* dim-2-order-5-step-3-with-FGb.mw at [23].

$$
\begin{aligned}
0 &\equiv -48\,K^{12}F^{16} + \Lambda^9 X^{19} - f_1' Y^{27},\\
0 &\equiv -48\,K^{12}H^{14} + \Lambda^7 X^{19} - f_1' X^{25}\\
0 &\equiv -5\,\Lambda^9 F^{16} + \Lambda^7 X^{18} + 8 f_1' K^{12} K^{12},\\
0 &\equiv -\Lambda^9 H^{14} + \Lambda^7 F^{16} + f_1' M^{10} K^{12},\\
0 &\equiv -5\,\Lambda^9 N^{12} + 7\,\Lambda^7 H^{14} + 56 f_1' M^8 K^{12} - f_1' f_1' X^{19},\\
0 &\equiv -48 K^{12} N^{12} + \Lambda^5 X^{19} - 7\, f_1' X^{23},
\end{aligned}
$$

$$
\begin{aligned}
0 &\equiv -7\,\Lambda^9 H^{14} + \Lambda^5 X^{18} + 8\, f_1' M^{10} K^{12},\\
0 &\equiv -\Lambda^9 N^{12} + \Lambda^5 F^{16} + f_1' M^{10} M^{10},\\
0 &\equiv -\Lambda^7 N^{12} + \Lambda^5 H^{14} + f_1' M^8 M^{10},\\
0 &\equiv -10\,M^{10} N^{12} + \Lambda^3 X^{19} - \tfrac{7}{3}\, f_1' X^{21},\\
0 &\equiv -35\,\Lambda^9 N^{12} + 3\,\Lambda^3 X^{18} + \tfrac{285}{8}\, f_1' M^{10} M^{10} - \tfrac{7}{8}\, f_1' f_1' X^{19},\\
0 &\equiv -7\,\Lambda^7 N^{12} + 3\,\Lambda^3 F^{16} + 8 f_1' M^8 M^{10},\\
0 &\equiv -5\,\Lambda^5 N^{12} + 3\,\Lambda^3 H^{14} + 8\, f_1' M^8 M^8,
\end{aligned}
$$

$$
\begin{aligned}
0 &\equiv -5\,M^{10} M^{10} + 64\,M^8 K^{12} - f_1' X^{19},\\
0 &\equiv -5\,\Lambda^9 M^{10} + 56\,\Lambda^7 K^{12} - f_1' X^{18},\\
0 &\equiv -8\,\Lambda^9 M^8 + 7\,\Lambda^7 M^{10} - f_1' F^{16},\\
0 &\equiv -\Lambda^9 M^8 + 7\,\Lambda^5 K^{12} - f_1' F^{16},\\
0 &\equiv -8\,\Lambda^7 M^8 + 5\,\Lambda^5 M^{10} - f_1' H^{14},
\end{aligned}
$$

$$
\begin{aligned}
0 &\equiv -\Lambda^7 M^8 + 3\,\Lambda^3 K^{12} - f_1' H^{14},\\
0 &\equiv -8\,\Lambda^5 M^8 + 3\,\Lambda^3 M^{10} - f_1' N^{12},\\
0 &\equiv -7\,\Lambda^7 \Lambda^7 + 5\,\Lambda^5 \Lambda^9 - f_1' f_1' K^{12},\\
0 &\equiv -7\,\Lambda^5 \Lambda^7 + 3\,\Lambda^3 \Lambda^9 - f_1' f_1' M^{10},\\
0 &\equiv -5\,\Lambda^5 \Lambda^5 + 3\,\Lambda^3 \Lambda^7 - f_1' f_1' M^8.
\end{aligned}
$$

Here, 4 new bi-invariants appear:

$$
X^{21}, \quad X^{23}, \quad X^{25}, \quad Y^{17}.
$$

Their values restricted to $\{f_1' = 0\}$ are:

$$X^{21}\big|_0 = -\frac{5}{3}\,\frac{N^{12}|_0\,N^{12}|_0}{\Lambda^3|_0} - \frac{64}{3}\,\frac{M^8|_0\,M^8|_0\,M^8|_0}{\Lambda^3|_0},$$
$$X^{23}\big|_0 = -\frac{35}{3}\,\frac{\Lambda^5|_0\,N^{12}|_0\,N^{12}|_0}{\Lambda^3|_0\,\Lambda^3|_0} - \frac{64}{9}\,\frac{\Lambda^5|_0\,M^8|_0\,M^8|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0},$$
$$X^{25}\big|_0 = -\frac{1225}{27}\,\frac{\Lambda^5|_0\,\Lambda^5|_0\,N^{12}|_0\,N^{12}|_0}{\Lambda^3|_0\,\Lambda^3|_0\,\Lambda^3|_0} - \frac{320}{27}\,\frac{\Lambda^5|_0\,\Lambda^5|_0\,M^8|_0\,M^8|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0\,\Lambda^3|_0},$$
$$Y^{27}\big|_0 = -\frac{8575}{81}\,\frac{\Lambda^5|_0\,\Lambda^5|_0\,\Lambda^5|_0\,N^{12}|_0\,N^{12}|_0}{\Lambda^3|_0\,\Lambda^3|_0\,\Lambda^3|_0\,\Lambda^3|_0} - \frac{320}{81}\,\frac{\Lambda^5|_0\,\Lambda^5|_0\,\Lambda^5|_0\,M^8|_0\,M^8|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0\,\Lambda^3|_0\,\Lambda^3|_0}.$$

Fourth loop of the algorithm. The Gröbner basis of syzygies between the restriction to $\{f_1' = 0\}$ of the 17 bi-invariants known so far consists here of 105 equations. By an independent calculation, we checked that 39 among these 105 generators belong to the ideal of the 66 remaining ones. We could fill in the remainders behind a power of f_1'. To test whether there appear new bi-invariants, it is in fact useless to fill in the 39 left out remainders. Here are the 66 syzygies[50] in question:

$$0 \equiv X^{18}X^{23} - 8\,F^{16}X^{25} + 7\,H^{14}Y^{27} + 0,$$
$$0 \equiv 5\,F^{16}X^{23} - 8\,H^{14}X^{25} + 5\,N^{12}Y^{27} + f_1'\,X^{19}X^{19},$$
$$0 \equiv 7\,K^{12}X^{23} - M^{10}X^{25} + M^8Y^{27} + 0,$$
$$0 \equiv 5\,\Lambda^9X^{23} - 8\,\Lambda^7X^{25} + 5\,\Lambda^5Y^{27} - 8\,f_1'\,K^{12}X^{19},$$
$$0 \equiv 7\,\Lambda^7X^{23} - 8\,\Lambda^5X^{25} + 3\,\Lambda^3Y^{27} - f_1'\,M^{10}X^{19},$$
$$0 \equiv X^{18}X^{21} - 57\,H^{14}X^{25} + 40\,N^{12}Y^{27} + 7\,f_1'\,X^{19}X^{19},$$
$$0 \equiv F^{16}X^{21} - 8\,H^{14}X^{23} + N^{12}X^{25} + 0,$$
$$0 \equiv 7\,K^{12}X^{21} - 5\,M^{10}X^{23} + M^8X^{25} + 0,$$
$$0 \equiv 7\,\Lambda^9X^{21} - 57\,\Lambda^5X^{25} + 24\,\Lambda^3Y^{27} - 15\,f_1'\,M^{10}X^{19},$$
$$0 \equiv 7\,\Lambda^7X^{21} - 40\,\Lambda^5X^{23} + 3\,\Lambda^3X^{25} - 8\,f_1'\,M^8X^{19},$$
$$0 \equiv X^{18}X^{19} - 8\,K^{12}X^{25} + 5\,M^{10}Y^{27} + 0,$$
$$0 \equiv 7\,F^{16}X^{19} - M^{10}X^{25} + 8\,M^8Y^{27} + 0,$$
$$0 \equiv 7\,H^{14}X^{19} - 5\,M^{10}X^{23} + 8\,M^8X^{25} + 0,$$
$$0 \equiv N^{12}X^{19} - M^{10}X^{21} + 8\,M^8X^{23} + 0,$$
$$0 \equiv 6\,F^{16}X^{18} - \Lambda^9X^{25} + 7\,\Lambda^7Y^{27} + 0,$$
$$0 \equiv 6\,H^{14}X^{18} - \Lambda^7X^{25} + 5\,\Lambda^5Y^{27} - 7\,f_1'\,K^{12}X^{19},$$
$$0 \equiv 6\,N^{12}X^{18} - \Lambda^5X^{25} + 3\,\Lambda^3Y^{27} - 7\,f_1'\,M^{10}X^{19},$$
$$0 \equiv 6\,M^{10}X^{18} - 7\,\Lambda^9X^{19} + f_1'Y^{27},$$

[50] *See* dim-2-order-5-step-4-with-FGb.mw at [23].

$$0 \equiv 48\, M^8 X^{18} - 49\, \Lambda^7 X^{19} + f_1' X^{25},$$
$$0 \equiv 30\, F^{16} F^{16} - \Lambda^7 X^{25} + 5\, \Lambda^5 Y^{27} - f_1'\, K^{12} X^{19},$$
$$0 \equiv 42\, H^{14} F^{16} - \Lambda^5 X^{25} + 3\, \Lambda^3 Y^{27} - f_1'\, M^{10} X^{19},$$
$$0 \equiv 30\, N^{12} F^{16} - 5\, \Lambda^5 X^{23} + 3\, \Lambda^3 X^{25} - f_1'\, M^8 X^{19},$$
$$0 \equiv 48\, K^{12} F^{16} - \Lambda^9 X^{19} + f_1' Y^{27},$$
$$0 \equiv 30\, M^{10} F^{16} - 7\, \Lambda^7 X^{19} + f_1' X^{25},$$

$$0 \equiv 48\, M^8 F^{16} - 7\, \Lambda^5 X^{19} + f_1' X^{23},$$
$$0 \equiv 5\, \Lambda^9 F^{16} - \Lambda^7 X^{18} - 8\, f_1'\, K^{12} K^{12},$$
$$0 \equiv 7\, \Lambda^7 F^{16} - \Lambda^5 X^{18} - f_1'\, M^{10} K^{12},$$
$$0 \equiv 35\, \Lambda^5 F^{16} - 3\, \Lambda^3 X^{18} - 8\, f_1'\, M^8 K^{12} + f_1' f_1'\, X^{19},$$
$$0 \equiv 42\, H^{14} H^{14} - 5\, \Lambda^5 X^{23} + 3\, \Lambda^3 X^{25} - f_1' M^8 X^{19},$$
$$0 \equiv 6\, N^{12} H^{14} - \Lambda^5 X^{21} + 3\, \Lambda^3 X^{23} + 0,$$

$$0 \equiv 48\, K^{12} H^{14} - \Lambda^7 X^{19} + f_1' X^{25},$$
$$0 \equiv 6\, M^{10} H^{14} - \Lambda^5 X^{19} + f_1' X^{23},$$
$$0 \equiv 16\, M^8 H^{14} - \Lambda^3 X^{19} + \tfrac{1}{3}\, f_1'\, X^{21},$$
$$0 \equiv 7\, \Lambda^9 H^{14} - \Lambda^5 X^{18} - 8\, f_1'\, M^{10} K^{12},$$
$$0 \equiv 49\, \Lambda^7 H^{14} - 3\, \Lambda^3 X^{18} - 5\, f_1'\, M^{10} M^{10},$$
$$0 \equiv 7\, \Lambda^5 H^{14} - 3\, \Lambda^3 F^{16} - f_1' M^8 M^{10},$$

$$0 \equiv 48\, K^{12} N^{12} - \Lambda^5 X^{19} + 7\, f_1' X^{23},$$
$$0 \equiv 10\, M^{10} N^{12} - \Lambda^3 X^{19} + \tfrac{7}{3}\, f_1'\, X^{21},$$
$$0 \equiv 35\, \Lambda^9 N^{12} - 3\, \Lambda^3 X^{18} - \tfrac{285}{8}\, f_1'\, M^{10} M^{10} + \tfrac{7}{8}\, f_1' f_1'\, X^{19},$$
$$0 \equiv 7\, \Lambda^7 N^{12} - 3\, \Lambda^3 F^{16} - 8\, f_1'\, M^8 M^{10},$$
$$0 \equiv 5\, \Lambda^5 N^{12} - 3\, \Lambda^3 H^{14} - 8\, f_1'\, M^8 M^8,$$
$$0 \equiv 5\, M^{10} M^{10} - 64\, M^8 K^{12} + f_1' X^{19},$$

$$0 \equiv 5\, \Lambda^9 M^{10} - 56\, \Lambda^7 K^{12} + f_1' X^{18},$$
$$0 \equiv \Lambda^7 M^{10} - 8\, \Lambda^5 K^{12} + f_1' F^{16},$$
$$0 \equiv 5\, \Lambda^5 M^{10} - 24\, \Lambda^3 K^{12} + f_1' H^{14},$$
$$0 \equiv \Lambda^9 M^8 - 7\, \Lambda^5 K^{12} + f_1' F^{16},$$
$$0 \equiv \Lambda^7 M^8 - 3\, \Lambda^3 K^{12} + f_1' H^{14},$$
$$0 \equiv 8\, \Lambda^5 M^8 - 3\, \Lambda^3 M^{10} + f_1' N^{12},$$

$$
\begin{aligned}
0 &\equiv 7\,\Lambda^7\Lambda^7 - 5\,\Lambda^5\Lambda^9 + f_1' f_1'\, K^{12},\\
0 &\equiv 7\,\Lambda^5\Lambda^7 - 3\,\Lambda^3\Lambda^9 + f_1' f_1' M^{10},\\
0 &\equiv 5\,\Lambda^5\Lambda^5 - 3\,\Lambda^3\Lambda^7 + f_1' f_1'\, M^8,\\
0 &\equiv 7\,K^{12}X^{19}X^{19} + X^{25}X^{25} - 5\,X^{23}Y^{27} + 0,\\
0 &\equiv M^{10}X^{19}X^{19} + X^{23}X^{25} - X^{21}Y^{27} + 0,\\
0 &\equiv M^8X^{19}X^{19} + 5\,X^{23}X^{23} - X^{21}X^{25} + 0,
\end{aligned}
$$

$$
\begin{aligned}
0 &\equiv 56\,K^{12}K^{12}X^{19} + X^{18}X^{25} - 5\,F^{16}Y^{27} + 0,\\
0 &\equiv M^{10}K^{12}X^{19} + F^{16}X^{25} - H^{14}Y^{27} + 0,\\
0 &\equiv 8\,M^8K^{12}X^{19} + 7\,H^{14}X^{25} - 5\,N^{12}Y^{27} - f_1' X^{19}X^{19},\\
0 &\equiv M^8M^{10}X^{19} + 7\,H^{14}X^{23} - N^{12}X^{25} + 0,\\
0 &\equiv 8\,M^8M^8X^{19} + 7\,H^{14}X^{21} - 5\,N^{12}X^{23} + 0,\\
0 &\equiv 448\,K^{12}K^{12}K^{12} + X^{18}X^{18} + 5\,\Lambda^9Y^{27} + 0,
\end{aligned}
$$

$$
\begin{aligned}
0 &\equiv 48\,M^{10}K^{12}K^{12} + \Lambda^9X^{25} - \Lambda^7Y^{27} + 0,\\
0 &\equiv 384\,M^8K^{12}K^{12} + 7\,\Lambda^7X^{25} - 5\,\Lambda^5Y^{27} + f_1' K^{12}X^{19},\\
0 &\equiv 48\,M^8M^{10}K^{12} + 7\,\Lambda^5X^{25} - 3\,\Lambda^3Y^{27} + f_1'\, M^{10}X^{19},\\
0 &\equiv 384\,M^8M^8K^{12} + 35\,\Lambda^5X^{23} - 3\,\Lambda^3X^{25} + f_1' M^8X^{19},\\
0 &\equiv 48\,M^8M^8M^{10} + 7\,\Lambda^5X^{21} - 3\,\Lambda^3X^{23} + 0,\\
0 &\equiv 64\,M^8M^8M^8 + 5\,N^{12}N^{12} + 3\,\Lambda^3X^{21} + 0.
\end{aligned}
$$

Remarkably, no new bi-invariant appears at this fourth stage. According to the general principle, we may therefore conclude that the algorithm stops.

THEOREM *In dimension $n = 2$ for jet order $\kappa = 5$, the algebra UE_5^2 of jet polynomials $\mathsf{P}\big(j^5 f_1, j^5 f_2\big)$ invariant by reparametrization and invariant under the unipotent action is generated by the 17 mutually independent bi-invariants explicitly defined above:*

$$
\begin{array}{|ccccccccc|}
\hline
 & f_1', & \Lambda^3, & \Lambda^5, & \Lambda^7, & \Lambda^9, & M^8, & M^{10}, & K^{12}, \\
N^{12}, & H^{14}, & F^{16}, & X^{18}, & X^{19}, & X^{21}, & X^{23}, & X^{25}, & Y^{27} \\
\hline
\end{array}
$$

whose restriction to $\{f_1' = 0\}$ has a reduced gröbnerized ideal of relations for the Degree Reverse Lexicographic ordering which consists of 105 equations, 66 of which generate the ideal in question and whose remainders behind a power of f_1' have been filled just above.

As a consequence, the full algebra E_5^2 *of jet polynomials* $\mathsf{P}\big(j^5 f\big)$ *invariant by reparametrization is generated by the polarizations:*

$$\boxed{\begin{array}{ccccccccc} f_i', & \Lambda^3, & \Lambda_i^5, & \Lambda_{i,j}^7, & \Lambda_{i,j,k}^9, & M^8, & M_i^{10}, & K_{i,j}^{12}, & \\ N^{12}, & H_i^{14}, & F_{i,j}^{16}, & X_{i,j,k}^{18}, & X_i^{19}, & X^{21}, & X_i^{23}, & X_{i,j}^{25}, & Y_{i,j,k}^{27} \end{array}}$$

of these 17 bi-invariants, where the indices i, j, k *vary in* $\{1, 2\}$*, whence the* **total number** *of these invariants equals:*

$$2+1+2+4+8+1+2+4+1+2+4+8+2+1+2+4+8 = \boxed{\mathbf{56}}.$$

§11. Sixteen (fifteen) bi-invariant in dimension $n = 4$ ($n = 3$) for jet level $\kappa = 4$

First loop of the algorithm. Coming back to the end of §7, we start with the seven initial bi-invariants:

$$\begin{aligned} \Lambda^3 &= \Delta_{1,2}^{',''}, \\ \Lambda^5 &= \Delta_{1,2}^{','''}\, f_1' - 3\,\Delta_{1,2}^{',''}\, f_1'', \\ \Lambda^7 &= \Delta_{1,2}^{',''''}\, f_1' f_1' + \Delta_{1,2}^{'','''}\, f_1' f_1' - 10\,\Delta_{1,2}^{','''}\, f_1' f_1'' + 15\,\Delta_{1,2}^{',''}\, f_1'' f_1'', \end{aligned}$$

$$\begin{aligned} D^6 &= \Delta_{1,2,3}^{','','''}, \\ D^8 &= \Delta_{1,2,3}^{',''',''''}\, f_1' - 3\,\Delta_{1,2,3}^{','',''''}\, f_1'', \\ N^{10} &= \Delta_{1,2,3}^{',''',''''}\, f_1' f_1' - 3\,\Delta_{1,2,3}^{','',''''}\, f_1' f_1'' + 4\,\Delta_{1,2,3}^{','','''}\, f_1' f_1''' + 3\,\Delta_{1,2,3}^{','','''}\, f_1'' f_1'', \\ W^{10} &= \Delta_{1,2,3,4}^{','',''',''''}. \end{aligned}$$

Then we compute the ideal of relations between these bi-invariants, after setting $f_1' = 0$ in them:

$$\mathsf{Ideal-Rel}\Big(\Lambda^3\big|_0,\ \Lambda^5\big|_0,\ \Lambda^7\big|_0,\ D^6\big|_0,\ D^8\big|_0,\ N^{10}\big|_0,\ W^{10}\big|_0\Big).$$

We should observe that the first six initial bi-invariants Λ^3, Λ^5, Λ^7, D^6, D^8 and N^{10} depend only upon the first three jet components $\big(j^4 f_1,\, j^4 f_2,\, j^4 f_3\big)$ of $j^4 f$, while W^{10} and $W^{10}\big|_0$ — which both contain the monomial $-f_4'''' f_3' f_2'' f_1'''$ — really depend upon the fourth jet component $j^4 f_4$. It follows that $W^{10}\big|_0$ is algebraically independent of $\Lambda^3\big|_0$, $\Lambda^5\big|_0$, $\Lambda^7\big|_0$, $D^6\big|_0$, $D^8\big|_0$, $N^{10}\big|_0$, so it cannot intervene in the ideal of relations. Without loss of generality, we therefore have to consider:

$$\mathsf{Ideal-Rel}\Big(\Lambda^3\big|_0,\ \Lambda^5\big|_0,\ \Lambda^7\big|_0,\ D^6\big|_0,\ D^8\big|_0,\ N^{10}\big|_0\Big).$$

A Maple computation with the Degree Reverse Lexicographic ordering yields a reduced Gröbner basis for this ideal consisting of the following 6 generators[51]:

$$
\begin{aligned}
0 &\overset{a}{\equiv} 5\,\Lambda^5\Lambda^5 - 3\,\Lambda^3\Lambda^7 + f_1' f_1' M^8,\\
0 &\overset{b}{\equiv} 2\,\Lambda^5 D^6 - \Lambda^3 D^8 + \tfrac{1}{3}\, f_1' E^{10},\\
0 &\overset{c}{\equiv} \Lambda^7 D^6 - 5\,\Lambda^3 N^{10} + f_1' L^{12},\\
0 &\overset{d}{\equiv} \Lambda^5 D^8 - 6\,\Lambda^3 N^{10} + f_1' L^{12},\\
0 &\overset{e}{\equiv} \Lambda^7 D^8 - 10\,\Lambda^5 N^{10} - f_1' Q^{14},\\
0 &\overset{f}{\equiv} D^8 D^8 - 12\,D^6 N^{10} - f_1' R^{15}.
\end{aligned}
$$

To read these equations (*cf.* §9), one should at first set $f_1' = 0$ virtually in one's head and then consider that further computations show what are the remainders behind a power of f_1'. Five new bi-invariants appear which are implicitly defined by five among these six sizygies and we provide their explicit expression in terms of Δ determinants, after mild simplifications:

$$
\begin{aligned}
M^8 &:= \frac{-5\,\Lambda^5\Lambda^5 + 3\,\Lambda^3\Lambda^7}{f_1' f_1'}\\
&= 3\,\Delta_{1,2}^{','''''}\,\Delta_{1,2}^{',''} + 12\,\Delta_{1,2}^{'','''}\,\Delta_{1,2}^{',''} - 5\,\Delta_{1,2}^{','''}\,\Delta_{1,2}^{','''},
\end{aligned}
$$

$$
\begin{aligned}
E^{10} &:= \frac{-6\,\Lambda^5\,D^6 + 3\,\Lambda^3\,D^8}{f_1'}\\
&= 3\,\Delta_{1,2,3}^{','',''''}\,\Delta_{1,2}^{',''} - 6\,\Delta_{1,2,3}^{','','''}\,\Delta_{1,2}^{','''},
\end{aligned}
$$

$$
\begin{aligned}
L^{12} &:= \frac{-\Lambda^7 D^6 + 5\,\Lambda^3 N^{10}}{f_1'}\\
&= -\Delta_{1,2,3}^{','','''}\,\Delta_{1,2}^{',''''}\,f_1' - 4\,\Delta_{1,2,3}^{','','''}\,\Delta_{1,2}^{'','''}\,f_1' + 5\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{',''}\,f_1' + 10\,\Delta_{1,2,3}^{','','''}\,\Delta_{1,2}^{','''}\,f_1'' -\\
&\qquad - 15\,\Delta_{1,2,3}^{','',''''}\,\Delta_{1,2}^{',''}\,f_1'' + 20\,\Delta_{1,2,3}^{','','''}\,\Delta_{1,2}^{',''}\,f_1''',
\end{aligned}
$$

$$
\begin{aligned}
Q^{14} &:= \frac{\Lambda^7 D^8 - 10\,\Lambda^5 N^{10}}{f_1'}\\
&= -10\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{','''}\,f_1' f_1' + \Delta_{1,2,3}^{','',''''}\,\Delta_{1,2}^{',''''}\,f_1' f_1' + 4\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{'','''}\,f_1' f_1' +\\
&\quad + 20\,\Delta_{1,2,3}^{','',''''}\,\Delta_{1,2}^{','''}\,f_1' f_1'' + 30\,\Delta_{1,2,3}^{',''',''''}\,\Delta_{1,2}^{',''}\,f_1' f_1'' - 6\,\Delta_{1,2,3}^{','','''}\,\Delta_{1,2}^{',''''}\,f_1' f_1'' -\\
&\quad - 24\,\Delta_{1,2,3}^{','','''}\,\Delta_{1,2}^{'','''}\,f_1' f_1'' - 40\,\Delta_{1,2,3}^{','','''}\,\Delta_{1,2}^{','''}\,f_1' f_1''' - 75\,\Delta_{1,2,3}^{','',''''}\,\Delta_{1,2}^{',''}\,f_1'' f_1'' +\\
&\quad + 30\,\Delta_{1,2,3}^{','','''}\,\Delta_{1,2}^{','''}\,f_1'' f_1'' + 120\,\Delta_{1,2,3}^{','','''}\,\Delta_{1,2}^{',''}\,f_1'' f_1''',
\end{aligned}
$$

[51] *See* dim-3-order-4-step-1-with-FGb.mw at [23].

$$
\begin{aligned}
R^{15} &:= \frac{D^8 D^8 - 12\, D^6 N^{10}}{f_1'} \\
&= \Delta_{1,2,3}^{','',''''}\, \Delta_{1,2,3}^{','',''''}\, f_1' - 12\, \Delta_{1,2,3}^{',''',''''}\, \Delta_{1,2,3}^{',''',''''}\, f_1' + 24\, \Delta_{1,2,3}^{',''',''''}\, \Delta_{1,2,3}^{',''',''''}\, f_1'' - \\
&\quad - 48\, \Delta_{1,2,3}^{',''',''''}\, \Delta_{1,2,3}^{',''',''''}\, f_1''',
\end{aligned}
$$

and as usual, the weights are denoted by an upper index. Setting W^{10} apart, in order to verify that these 11 bi-invariants are mutually independent, one computes at first which value they have after setting $f_1' = 0$:

$$
\begin{aligned}
&\underline{\Lambda^3}\big|_0 \\
&\underline{\Lambda^5}\big|_0 \\
&\Lambda^7\big|_0 = \tfrac{5}{3}\, \tfrac{\Lambda^5|_0\, \Lambda^5|_0}{\Lambda^3|_0}
\end{aligned}
$$

$$
\begin{aligned}
&\underline{D^6}\big|_0 \\
&D^8\big|_0 \;\; = 2\, \tfrac{\Lambda^5|_0\, D^6|_0}{\Lambda^3|_0} \\
&N^{10}\big|_0 = \tfrac{1}{3}\, \tfrac{\Lambda^5|_0\, \Lambda^5|_0\, D^6|_0}{\Lambda^3|_0\, \Lambda^3|_0}
\end{aligned}
$$

$$
\begin{aligned}
&\underline{M^8}\big|_0 \\
&\underline{E^{10}}\big|_0 \\
&L^{12}\big|_0 = \tfrac{5}{3}\, \tfrac{\Lambda^5|_0\, E^{10}|_0}{\Lambda^3|_0}
\end{aligned}
$$

$$
\begin{aligned}
Q^{14}\big|_0 &= -\tfrac{25}{9}\, \tfrac{\Lambda^5|_0\, \Lambda^5|_0\, E^{10}|_0}{\Lambda^3|_0 \Lambda^3|_0} \\
R^{15}\big|_0 &= -\tfrac{8}{3}\, \tfrac{\Lambda^5|_0\, D^6|_0\, E^{10}|_0}{\Lambda^3|_0\, \Lambda^3|_0},
\end{aligned}
$$

with the 5 underlined bi-invariants being algebraically independent and being considered as a transcendence basis, while the value of $\Lambda^7\big|_0$ comes from "$\overset{a}{=}$" above; the value of $D^8\big|_0$ comes from "$\overset{b}{=}$" above; the value of $N^{10}\big|_0$ comes from "$\overset{d}{=}$" above; the value of $L^{12}\big|_0$ comes from "$\overset{r}{=}$" below; the value of $Q^{14}\big|_0$ comes from "$\overset{q}{=}$" below; and the value of $R^{15}\big|_0$ comes from "$\overset{p}{=}$" below. Then one proceeds as in the proof of the lemma on p. 219 to show mutual independence (details will not be provided).

Importantly, the five new bi-invariants M^8, E^{10}, L^{12}, Q^{14} and R^{15} again depend only upon the first three jet components $\big(j^4 f_1, j^4 f_2, j^4 f_3\big)$, so that $W^{10}\big|_0$ again will not intervene in the next ideal of relations. In fact, all bi-invariants except W^{10} live in dimension $n = 3$, and hence it is enough to explore the structure of UE_4^3.

Second loop of the algorithm. Setting therefore W^{10} apart, a Maple computation with the Degree Reverse Lexicographic Ordering offers a reduced Gröbner basis for the ideal of relations:

$$\mathsf{Ideal-Rel}\left(\begin{array}{c}\Lambda^3\big|_0,\ \Lambda^5\big|_0,\ \Lambda^7\big|_0,\ D^6\big|_0,\ D^8\big|_0,\ N^{10}\big|_0,\\ M^8\big|_0,\ E^{10}\big|_0,\ L^{12}\big|_0,\ Q^{14}\big|_0,\ R^{15}\big|_0\end{array}\right)$$

between our 11 bi-invariants restricted to $\{f_1'=0\}$, and this basis consists of the 6 generators above together with the following 14 generators [52]:

$$\begin{aligned}
0 &\overset{g}{\equiv} 4\,D^8Q^{14}-5\,\Lambda^7R^{15}-f_1'X^{21},\\
0 &\overset{h}{\equiv} 24\,D^6Q^{14}-25\,\Lambda^5R^{15}+f_1'V^{19},\\
0 &\overset{i}{\equiv} L^{12}L^{12}+E^{10}Q^{14}-f_1'M^8R^{15},\\
0 &\overset{j}{\equiv} 8\,N^{10}L^{12}+\Lambda^7R^{15}+f_1'X^{21},\\
0 &\overset{k}{\equiv} 4\,D^8L^{12}+5\,\Lambda^5R^{15}-f_1'V^{19},\\
\\
0 &\overset{l}{\equiv} 8\,D^6L^{12}+5\,\Lambda^3R^{15}-\tfrac{1}{3}\,f_1'U^{17},\\
0 &\overset{m}{\equiv} \Lambda^7L^{12}+\Lambda^5Q^{14}-2\,f_1'M^8N^{10},\\
0 &\overset{n}{\equiv} 5\,\Lambda^5L^{12}+3\,\Lambda^3Q^{14}-f_1'D^8M^8,\\
0 &\overset{o}{\equiv} 8\,N^{10}E^{10}+\Lambda^5\,R^{15}-f_1'V^{19},\\
0 &\overset{p}{\equiv} 4\,D^8E^{10}+3\,\Lambda^3R^{15}-f_1'U^{17},
\end{aligned}$$

$$\begin{aligned}
0 &\overset{q}{\equiv} 5\,\Lambda^7E^{10}+3\,\Lambda^3Q^{14}-6f_1'D^8M^8,\\
0 &\overset{r}{\equiv} 5\,\Lambda^5E^{10}-3\,\Lambda^3L^{12}-6\,f_1'D^6M^8,\\
0 &\overset{s}{\equiv} 8\,\Lambda^5N^{10}Q^{14}-\Lambda^7\Lambda^7R^{15}+f_1'Q^{14}Q^{14}+4\,f_1'N^{10}N^{10}M^8,\\
0 &\overset{t}{\equiv} 24\,\Lambda^3N^{10}Q^{14}-5\,\Lambda^5\Lambda^7R^{15}-5\,f_1'L^{12}Q^{14}+2\,f_1'M^8D^8N^{10}.
\end{aligned}$$

Here, three new bi-invariants appear: U^{17}, V^{19} and X^{21}, which are implicitly defined by the syzygies "$\overset{p}{\equiv}$", "$\overset{o}{\equiv}$", and "$\overset{g}{\equiv}$", and we provide their explicit

[52] *See* dim-3-order-4-step-2-with-FGb.mw at [23]. Here again, the remainders behind a power of f_1' have all been computed and tested to know whether they belong to the algebra of the already known 11 bi-invariants.

expression in terms of Δ determinants[53]:

$$
\begin{aligned}
U^{17} &= \frac{4\,D^8E^{10}+3\,\Lambda^3R^{15}}{f_1'}\\
&= 15\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2}^{',\,''}-36\,\Delta_{1,2,3}^{',\,''',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,''}-\\
&\quad -24\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,'''}+144\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{'',\,'''},
\end{aligned}
$$

$$
\begin{aligned}
V^{19} &= \frac{8\,N^{10}E^{10}+\Lambda^5R^{15}}{f_1'}\\
&= 24\,\Delta_{1,2,3}^{',\,''',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2}^{',\,''}\,f_1'-60\,\Delta_{1,2,3}^{',\,''',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,'''}\,f_1'+\\
&\quad +\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2}^{',\,'''}\,f_1'-75\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2}^{',\,''}\,f_1''+\\
&\quad +36\,\Delta_{1,2,3}^{',\,''',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,''}\,f_1''+168\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,'''}\,f_1''-\\
&\quad -144\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{'',\,'''}\,f_1''+96\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,''}\,f_1'''-\\
&\quad -240\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,'''}\,f_1''',
\end{aligned}
$$

$$
\begin{aligned}
X^{21} &= \frac{4\,D^8Q^{14}-5\,\Lambda^7R^{15}}{f_1'}\\
&= -40\,\Delta_{1,2,3}^{',\,''',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2}^{',\,'''}\,f_1'f_1'-4\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2}^{',\,''''}\,f_1'f_1'-\\
&\quad -4\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2}^{'',\,'''}\,f_1'f_1'+60\,\Delta_{1,2,3}^{',\,''',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,''''}\,f_1'f_1'+\\
&\quad +240\,\Delta_{1,2,3}^{',\,''',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{'',\,'''}\,f_1'f_1'+130\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2}^{',\,'''}\,f_1'f_1''+\\
&\quad +120\,\Delta_{1,2,3}^{',\,''',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2}^{',\,''}\,f_1'f_1''-168\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,''''}\,f_1'f_1''-\\
&\quad -668\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{'',\,'''}\,f_1'f_1''-360\,\Delta_{1,2,3}^{',\,''',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,'''}\,f_1'f_1''-\\
&\quad -160\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,'''}\,f_1'f_1'''+240\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,''''}\,f_1'f_1'''+\\
&\quad +960\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{'',\,'''}\,f_1'f_1'''-375\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2}^{',\,''}\,f_1''f_1''+\\
&\quad +840\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,'''}\,f_1''f_1''+180\,\Delta_{1,2,3}^{',\,''',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,''}\,f_1''f_1''+\\
&\quad +144\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,''''}\,f_1''f_1''+144\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{'',\,'''}\,f_1''f_1''-\\
&\quad -1440\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,'''}\,f_1''f_1'''+480\,\Delta_{1,2,3}^{',\,'',\,'''',\,''''}\,\Delta_{1,2,3}^{',\,'',\,''',\,'''}\,\Delta_{1,2}^{',\,''}\,f_1''f_1'''.
\end{aligned}
$$

Either a Maple computation or a glance at the syzygies "$\overset{7}{\equiv}$", "$\overset{8}{\equiv}$", "$\overset{9}{\equiv}$" below arriving in the third loop provides the values of these two bi-invariants after

[53] To be able do divide by f_1', as in [21], we sometimes need to replace $\Delta_{1,2}^{',\,''}\,f_1'''$ by $-\Delta_{1,2}^{'',\,'''}\,f_1'+\Delta_{1,2}^{',\,'''}\,f_1''$, using the immediately checked syzygy: $0\equiv\Delta_{1,2}^{'',\,'''}\,f_1'-\Delta_{1,2}^{',\,'''}\,f_1''+\Delta_{1,2}^{',\,''}\,f_1'''$.

setting $f_1' = 0$:

$$
\begin{aligned}
U^{17}\big|_0 &= 12\,\frac{D^6|_0\,D^6|_0\,M^8|_0}{\Lambda^3|_0} + \frac{5}{3}\,\frac{E^{10}|_0\,E^{10}|_0}{\Lambda^3|_0},\\
V^{19}\big|_0 &= \frac{25}{9}\,\frac{\Lambda^5|_0\,E^{10}|_0\,E^{10}|_0}{\Lambda^3|_0\,\Lambda^3|_0} + 4\,\frac{\Lambda^5|_0\,D^6|_0\,D^6|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0},\\
X^{21}\big|_0 &= -\frac{4}{3}\,\frac{\Lambda^5|_0\,\Lambda^5|_0\,D^6|_0\,D^6|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0\,\Lambda^3|_0} - \frac{125}{27}\,\frac{\Lambda^5|_0\,\Lambda^5|_0\,E^{10}|_0\,E^{10}|_0}{\Lambda^3|_0\,\Lambda^3|_0\,\Lambda^3|_0}.
\end{aligned}
$$

Proceeding as in the lemma on p. 219, one checks patiently by hand that the 16 bi-invariants known so far:

W^{10},	f_1',	Λ^3,	Λ^5,	Λ^7,	D^6,	D^8,	N^{10},
M^8,	E^{10},	L^{12},	Q^{14},	R^{15},	U^{17},	V^{19},	X^{21}

are mutually independent.

Third loop of the algorithm. Again for the Degree Reverse Lexicographic ordering, setting W^{10} apart, a Maple computation offers a reduced Gröbner basis for the ideal of relations between the $14 = 15 - 1$ (f_1' goes to zero) restricted bi-invariants. The result consists of 50 generators [54]. Taking the Lexicographic ordering instead:

$$
\begin{aligned}
\Lambda^3 > \Lambda^5 > \Lambda^7 > D^6 > D^8 > N^{10} > M^8 > E^{10} > L^{12} >\\
> Q^{14} > R^{15} > U^{17} > V^{19} > X^{21},
\end{aligned}
$$

one shows that the ideal of relations, in Gröbnerized form, contains less equations — which is convenient —, namely the following 41 equations[55], where we underline the Leading Term of each syzygy with the acronym "LT" appended[56]:

$$
\begin{aligned}
0 &\overset{1}{\equiv} -5\,\Lambda^5\Lambda^5 + 3\,\underline{\Lambda^3\Lambda^7}_{\mathsf{LT}} - f_1' f_1' M^8,\\
0 &\overset{2}{\equiv} -2\,\Lambda^5 D^6 + \underline{\Lambda^3 D^8}_{\mathsf{LT}} - \tfrac{1}{3} f_1'\,E^{10},\\
0 &\overset{3}{\equiv} -\Lambda^7 D^6 + 5\,\underline{\Lambda^3 N^{10}}_{\mathsf{LT}} - f_1' L^{12},\\
0 &\overset{4}{\equiv} -5\,\Lambda^5 E^{10} + 3\,\underline{\Lambda^3 L^{12}}_{\mathsf{LT}} + 6\,f_1' D^6 M^8,\\
0 &\overset{5}{\equiv} 5\,\Lambda^7 E^{10} + 3\,\underline{\Lambda^3 Q^{14}}_{\mathsf{LT}} - 6\,f_1' D^8 M^8,\\
0 &\overset{6}{\equiv} 4\,D^8 E^{10} + 3\,\underline{\Lambda^3 R^{15}}_{\mathsf{LT}} - f_1' U^{17},
\end{aligned}
$$

[54] *See* dim-3-order-4-step-3-with-FGb.mws at [23].

[55] *See* 41-syzygies-dim-3-order-4.mw at [23].

[56] We recall that, in order to appropriately read the ideal of relations between restricted bi-invariants, one should set $f_1' = 0$, namely disregard the last term(s) of each equation. We specify "$+0$" when the remainder behing a power of f_1' vanishes identically.

$$0 \overset{7}{\equiv} -36\,D^6D^6M^8 - 5\,E^{10}E^{10} + 3\,\underline{\Lambda^3U^{17}}_{\mathsf{LT}} + 0,$$
$$0 \overset{8}{\equiv} -5\,E^{10}L^{12} - 6\,D^6D^8M^8 + 3\,\underline{\Lambda^3V^{19}}_{\mathsf{LT}} + 0,$$
$$0 \overset{9}{\equiv} 5\,L^{12}L^{12} + 3\,\underline{\Lambda^3X^{21}}_{\mathsf{LT}} + M^8D^8D^8 + 0,$$
$$0 \overset{10}{\equiv} -6\,\Lambda^7D^6 + 5\,\underline{\Lambda^5D^8}_{\mathsf{LT}} - f_1'L^{12},$$
$$0 \overset{11}{\equiv} -\Lambda^7D^8 + 10\,\underline{\Lambda^5N^{10}}_{\mathsf{LT}} + f_1'Q^{14},$$
$$0 \overset{12}{\equiv} \underline{\Lambda^5L^{12}}_{\mathsf{LT}} - \Lambda^7E^{10} + f_1'D^8M^8,$$

$$0 \overset{13}{\equiv} \Lambda^7L^{12} + \underline{\Lambda^5Q^{14}}_{\mathsf{LT}} - 2\,f_1'M^8N^{10},$$
$$0 \overset{14}{\equiv} 8\,N^{10}E^{10} + \underline{\Lambda^5R^{15}}_{\mathsf{LT}} - f_1'V^{19},$$
$$0 \overset{15}{\equiv} \underline{\Lambda^5U^{17}}_{\mathsf{LT}} - E^{10}L^{12} - 6\,D^6D^8M^8 + 0,$$
$$0 \overset{16}{\equiv} \underline{\Lambda^5V^{19}}_{\mathsf{LT}} - M^8D^8D^8 - L^{12}L^{12} + f_1'M^8R^{15},$$
$$0 \overset{17}{\equiv} \underline{\Lambda^5X^{21}}_{\mathsf{LT}} - L^{12}Q^{14} + 2\,D^8N^{10}M^8 + 0,$$
$$0 \overset{18}{\equiv} 8\,N^{10}L^{12} + \underline{\Lambda^7R^{15}}_{\mathsf{LT}} + f_1'X^{21},$$

$$0 \overset{19}{\equiv} -L^{12}L^{12} + \underline{\Lambda^7U^{17}}_{\mathsf{LT}} - 5\,M^8D^8D^8 + 0,$$
$$0 \overset{20}{\equiv} L^{12}Q^{14} + \underline{\Lambda^7V^{19}}_{\mathsf{LT}} - 10\,D^8M^8N^{10} + 0,$$

$$0 \overset{21}{\equiv} 20\,N^{10}N^{10}M^8 + Q^{14}Q^{14} + \underline{\Lambda^7X^{21}}_{\mathsf{LT}} + 0,$$
$$0 \overset{22}{\equiv} 6\,\underline{D^6M^8R^{15}}_{\mathsf{LT}} + L^{12}U^{17} - E^{10}V^{19} + 0,$$
$$0 \overset{23}{\equiv} 5\,\underline{D^8M^8R^{15}}_{\mathsf{LT}} - Q^{14}U^{17} - L^{12}V^{19} + 0,$$
$$0 \overset{24}{\equiv} 10\,\underline{N^{10}M^8R^{15}}_{\mathsf{LT}} - Q^{14}V^{19} + L^{12}X^{21} + 0,$$

$$0 \overset{25}{\equiv} 5\,\underline{M^8R^{15}R^{15}}_{\mathsf{LT}} + V^{19}V^{19} + U^{17}X^{21} + 0,$$
$$0 \overset{26}{\equiv} -D^8D^8 + 12\,\underline{D^6N^{10}}_{\mathsf{LT}} + f_1'R^{15},$$
$$0 \overset{27}{\equiv} -5\,D^8E^{10} + 6\,\underline{D^6L^{12}}_{\mathsf{LT}} + f_1'U^{17},$$
$$0 \overset{28}{\equiv} 3\,\underline{D^6Q^{14}}_{\mathsf{LT}} + 25\,N^{10}E^{10} - 3\,f_1'V^{19},$$
$$0 \overset{29}{\equiv} 5\,E^{10}R^{15} - D^8U^{17} + 6\,\underline{D^6V^{19}}_{\mathsf{LT}} + 0,$$
$$0 \overset{30}{\equiv} -3\,L^{12}R^{15} + N^{10}U^{17} + 3\,\underline{D^6X^{21}}_{\mathsf{LT}} + 0,$$

$$0 \overset{31}{\equiv} -10\,N^{10}E^{10} + \underline{D^8L^{12}}_{\mathsf{LT}} + f_1'V^{19},$$
$$0 \overset{32}{\equiv} \underline{D^8Q^{14}}_{\mathsf{LT}} + 10\,N^{10}L^{12} + f_1'X^{21},$$
$$0 \overset{33}{\equiv} -2\,N^{10}U^{17} + \underline{D^8V^{19}}_{\mathsf{LT}} + L^{12}R^{15} + 0,$$
$$0 \overset{34}{\equiv} Q^{14}R^{15} + 2\,N^{10}V^{19} + \underline{D^8X^{21}}_{\mathsf{LT}} + 0,$$
$$0 \overset{35}{\equiv} -2\,L^{12}N^{10}U^{17} + R^{15}L^{12}L^{12} + 10\,\underline{V^{19}N^{10}E^{10}}_{\mathsf{LT}} - f_1'V^{19}V^{19},$$
$$0 \overset{36}{\equiv} 2\,N^{10}U^{17}Q^{14} - R^{15}L^{12}Q^{14} + 10\,\underline{V^{19}N^{10}L^{12}}_{\mathsf{LT}} + f_1'V^{19}X^{21},$$

$$0 \overset{37}{\equiv} 10\,\underline{N^{10}L^{12}X^{21}}_{\mathsf{LT}} - R^{15}Q^{14}Q^{14} - 2\,Q^{14}N^{10}V^{19} + f_1'X^{21}X^{21},$$
$$0 \overset{38}{\equiv} 2\,\underline{N^{10}U^{17}X^{21}}_{\mathsf{LT}} - X^{21}L^{12}R^{15} + V^{19}Q^{14}R^{15} + 2\,N^{10}V^{19}V^{19} + 0,$$
$$0 \overset{39}{\equiv} \underline{E^{10}Q^{14}}_{\mathsf{LT}} + L^{12}L^{12} - f_1'M^8R^{15},$$
$$0 \overset{40}{\equiv} Q^{14}U^{17} + 6\,L^{12}V^{19} + 5\,\underline{E^{10}X^{21}}_{\mathsf{LT}} + 0,$$
$$0 \overset{41}{\equiv} -6\,Q^{14}L^{12}V^{19} - Q^{14}Q^{14}U^{17} + 5\,\underline{X^{21}L^{12}L^{12}}_{\mathsf{LT}} - 5\,f_1'M^8R^{15}X^{21}.$$

Remarkably, each one of the 41 remainders behind a power of f_1' belongs to the algebra of already known bi-invariants. No new bi-invariant appears at this stage. In such a circumstance, according to the general theorem on p. 239, we know that our algorithm stops, so that we have gained the following complete, quite nontrivial result.

THEOREM *In dimension $n = 4$ for jets of order $\kappa = 4$, the algebra UE_4^4 of jet polynomials $\mathsf{P}\big(j^4f_1, j^4f_2, j^4f_3, j^4f_4\big)$ invariant by reparametrization and invariant under the unipotent action is generated by the 16 mutually independent bi-invariants defined above:*

W^{10},	f_1',	Λ^3,	Λ^5,	Λ^7,	D^6,	D^8,	N^{10},
M^8,	E^{10},	L^{12},	Q^{14},	R^{15},	U^{17},	V^{19},	X^{21},

whose restriction to $\{f_1' = 0\}$ has a reduced gröbnerized ideal of relations, for the Lexicographic ordering, which consists of the 41 syzygies written above.

Furthermore, any bi-invariant of weight m writes uniquely in the finite polynomial form:

$$\mathsf{P}\big(j^\kappa f\big) = \sum_{o,\,p} (f_1')^o \left(W^{10}\right)^p \sum_{\substack{(a,\dots,n)\in\mathbb{N}^{14}\setminus(\square_1\cup\cdots\cup\square_{41})\\ 3a+\cdots+21n=m-o-10p}} \mathsf{coeff}_{a,\dots,n,o,p}\,\cdot$$
$$\cdot\left(\Lambda^3\right)^a\left(\Lambda^5\right)^b\left(\Lambda^7\right)^c\left(D^6\right)^d\left(D^8\right)^e\left(N^{10}\right)^f\left(M^8\right)^g\left(E^{10}\right)^h$$
$$\left(L^{12}\right)^i\left(Q^{14}\right)^j\left(R^{15}\right)^k\left(U^{17}\right)^l\left(V^{19}\right)^m\left(X^{21}\right)^n,$$

with coefficients $\mathsf{coeff}_{a,\dots,n,o,p}$ *subjected to no restriction, where* $\square_1$, *...,* $\square_{41}$ *denote the quadrants in* $\mathbb{N}^{14}$ *having vertex at the leading terms of the 41 syzygies in question.*

Finally, in the preceding dimension $n=3$ *for jets of the same order* $\kappa=4$*, the algebra* UE_4^3 *is generated by the same list from which one removes only the four-dimensional Wronskian* W^{10}*, the ideal of relations for the 15 restricted bi-invariants being exactly the same, with an entirely similar normal form for a general bi-invariant of weight* m*.*

As a consequence, by looking at the $\mathsf{GL}_4(\mathbb{C})$-orbit of each one of these 16 bi-invariants, we deduce a system of **2835** generators for the algebra E_4^4 of polynomials which are invariant (only) by reparametrization.

THEOREM *In dimension* $n=4$ *for jets of order* $\kappa=4$*, the algebra* E_4^4 *of jet polynomials* $\mathsf{P}\big(j^4 f\big)$ *invariant by reparametrization is generated by the polarizations:*

$$\boxed{\begin{array}{c}
W^{10},\quad f_i',\quad \Lambda^3_{[i,j]},\quad \Lambda^5_{[i,j];\,\alpha},\quad \Lambda^7_{[i,j];\,\alpha,\beta},\quad D^6_{[i,j,k]},\\
D^8_{[i,j,k];\,\alpha},\quad N^{10}_{[i,j,k];\,\alpha,\beta},\quad M^8_{[i,j],[k,l]},\quad E^{10}_{[i,j,k],[p,q]},\quad L^{12}_{[i,j,k],[p,q];\,\alpha},\\
Q^{14}_{[i,j,k],[p,q];\,\alpha,\beta},\quad R^{15}_{[i,j,k],[p,q,r];\,\alpha},\quad U^{17}_{[i,j,k],[p,q,r],[s,t]},\\
V^{19}_{[i,j,k],[p,q,r],[s,t];\,\alpha},\quad X^{21}_{[i,j,k],[p,q,r],[s,t];\,\alpha,\beta},
\end{array}}$$

of the 16 bi-invariants W^{10}*,* f_1'*,* Λ^3*,* Λ^5*,* Λ^7*,* D^6*,* D^8*,* N^{10}*,* M^8*,* E^{10}*,* L^{12}*,* Q^{14}*,* R^{15}*,* U^{17}*,* V^{19}*,* X^{21} *generating the algebra* UE_4^4 *of bi-invariants; these polarized invariants are skew-symmetric with respect to each collection of bracketed indices* $[i,j,k]$*,* $[p,q,r]$*,* $[s,t]$*, and they are* explicitly *represented in terms of* Δ*-determinants by the following complete formulas:*

$$W^{10}_{1,2,3,4},$$
$$f_i',$$
$$\Lambda^3_{[i,j]} := \Delta^{',''}_{i,j},$$

$$\Lambda^5_{[i,j];\,\alpha} := \Delta^{','''}_{i,j}\, f'_\alpha - 3\,\Delta^{',''}_{i,j}\, f''_\alpha,$$

$$\begin{aligned}\Lambda^7_{[i,j];\,\alpha,\beta} := {}& \Delta^{',''''}_{i,j}\, f'_\alpha f'_\beta + 4\,\Delta^{'','''}_{i,j}\, f'_\alpha f'_\beta - 5\Delta^{','''}_{i,j}\,\big(f'_\alpha f''_\beta + f''_\alpha f'_\beta\big) + \\ & + 15\,\Delta^{',''}_{i,j}\, f''_\alpha f''_\beta,\end{aligned}$$

$$D^6_{[i,j,k]} := \Delta^{','','''}_{i,j,k},$$

$$D^8_{[i,j,k];\,\alpha} := \Delta^{','',''''}_{i,j,k}\, f'_\alpha - 6\,\Delta^{','','''}_{i,j,k}\, f''_\alpha,$$

$$\begin{aligned}N^{10}_{[i,j,k];\,\alpha,\beta} := {}& \Delta^{',''',''''}_{i,j,k}\, f'_\alpha f'_\beta - \frac{3}{2}\,\Delta^{','',''''}_{i,j,k}\,\big(f'_\alpha f''_\beta + f''_\alpha f'_\beta\big) + \\ & + 2\,\Delta^{','','''}_{i,j,k}\,\big(f'_\alpha f'''_\beta + f'''_\alpha f'_\beta\big) + 3\,\Delta^{','','''}_{i,j,k}\, f''_\alpha f''_\beta,\end{aligned}$$

$$\begin{aligned}M^8_{[i,j],[k,l]} := {}& 3\,\Delta^{',''''}_{i,j}\,\Delta^{',''}_{k,l} + 12\,\Delta^{'','''}_{i,j}\,\Delta^{',''}_{k,l} - \\ & - 5\,\Delta^{','''}_{i,j}\,\Delta^{','''}_{k,l},\end{aligned}$$

$$E^{10}_{[i,j,k],[p,q]} := 3\,\Delta^{','',''''}_{i,j,k}\,\Delta^{',''}_{l,m} - 6\,\Delta^{','','''}_{i,j,k}\,\Delta^{','''}_{l,m},$$

$$\begin{aligned}L^{12}_{[i,j,k],[l,m];\,\alpha} := {}& 5\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','''}_{p,q}\, f'_\alpha - 15\,\Delta^{','',''''}_{i,j,k}\,\Delta^{',''}_{p,q}\, f''_\alpha - 6\,\Delta^{','','''}_{i,j,k}\,\Delta^{',''''}_{p,q}\, f'_\alpha - \\ & - 24\,\Delta^{','','''}_{i,j,k}\,\Delta^{'','''}_{p,q}\, f'_\alpha + 30\,\Delta^{','','''}_{i,j,k}\,\Delta^{','''}_{p,q}\, f'_\alpha,\end{aligned}$$

$$\begin{aligned}Q^{14}_{[i,j,k],[p,q];\,\alpha,\beta} := {}& -10\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','''}_{p,q}\, f'_\alpha f'_\beta + \Delta^{','',''''}_{i,j,k}\,\Delta^{',''''}_{p,q}\, f'_\alpha f'_\beta + \\ & + 4\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{'','''}_{p,q}\, f'_\alpha f'_\beta + +20\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','''}_{p,q}\, f'_\alpha f''_\beta + \\ & + 30\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''}_{p,q}\, f'_\alpha f''_\beta - 6\,\Delta^{','','''}_{i,j,k}\,\Delta^{',''''}_{p,q}\, f'_\alpha f''_\beta - \\ & - 24\,\Delta^{','','''}_{i,j,k}\,\Delta^{'','''}_{p,q}\, f'_\alpha f''_\beta - 40\,\Delta^{','','''}_{i,j,k}\,\Delta^{','''}_{p,q}\, f'_\alpha f'''_\beta - \\ & - 75\,\Delta^{','',''''}_{i,j,k}\,\Delta^{',''}_{p,q}\, f''_\alpha f''_\beta + 30\,\Delta^{','','''}_{i,j,k}\,\Delta^{','''}_{p,q}\, f''_\alpha f''_\beta + \\ & + 120\,\Delta^{','','''}_{i,j,k}\,\Delta^{',''}_{p,q}\, f''_\alpha f'''_\beta,\end{aligned}$$

$$\begin{aligned}R^{15}_{[i,j,k],[p,q,r];\,\alpha} := {}& \Delta^{','',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\, f'_\alpha - 12\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\, f'_\alpha + \\ & + 24\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\, f''_\alpha - 48\,\Delta^{','','''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\, f'''_\alpha,\end{aligned}$$

$$\begin{aligned}U^{17}_{[i,j,k],[p,q,r],[s,t]} := {}& 15\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\,\Delta^{',''}_{s,t} - 36\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{',''}_{s,t} - \\ & - 24\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{','''}_{s,t} + 144\,\Delta^{','','''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{'','''}_{s,t},\end{aligned}$$

$$\begin{aligned}V^{19}_{[i,j,k],[p,q,r],[s,t];\,\alpha} := {}& 24\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\,\Delta^{',''}_{s,t}\, f'_\alpha - 60\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{','''}_{s,t}\, f'_\alpha + \\ & + \Delta^{','',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\,\Delta^{','''}_{s,t}\, f'_\alpha - 75\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','',''''}_{p,q,r}\,\Delta^{',''}_{s,t}\, f''_\alpha + \\ & + 36\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{',''}_{s,t}\, f''_\alpha + 168\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{','''}_{s,t}\, f''_\alpha - \\ & - 144\,\Delta^{','','''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{'','''}_{s,t}\, f''_\alpha + 96\,\Delta^{','',''''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{',''}_{s,t}\, f'''_\alpha - \\ & - 240\,\Delta^{','','''}_{i,j,k}\,\Delta^{','','''}_{p,q,r}\,\Delta^{','''}_{s,t}\, f'''_\alpha,\end{aligned}$$

$$
\begin{aligned}
&X^{21}_{[i,j,k],[p,q,r],[s,t];\,\alpha,\beta} := \\
&:= -40\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''',''''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f'_\alpha f'_\beta - 4\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''',''''}_{p,q,r}\,\Delta^{',''''}_{s,t}\,f'_\alpha f'_\beta - \\
&- 4\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''',''''}_{p,q,r}\,\Delta^{'','''}_{s,t}\,f'_\alpha f'_\beta + 60\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''','''}_{p,q,r}\,\Delta^{',''''}_{s,t}\,f'_\alpha f'_\beta + \\
&+ 240\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''','''}_{p,q,r}\,\Delta^{'','''}_{s,t}\,f'_\alpha f'_\beta + 130\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''',''''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f'_\alpha f''_\beta + \\
&+ 120\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''',''''}_{p,q,r}\,\Delta^{',''}_{s,t}\,f'_\alpha f''_\beta - 168\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''','''}_{p,q,r}\,\Delta^{',''''}_{s,t}\,f'_\alpha f''_\beta - \\
&- 668\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''','''}_{p,q,r}\,\Delta^{'','''}_{s,t}\,f'_\alpha f''_\beta - 360\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''','''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f'_\alpha f''_\beta - \\
&- 160\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''','''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f'_\alpha f'''_\beta + 240\,\Delta^{',''','''}_{i,j,k}\,\Delta^{',''','''}_{p,q,r}\,\Delta^{',''''}_{s,t}\,f'_\alpha f'''_\beta + \\
&+ 960\,\Delta^{',''','''}_{i,j,k}\,\Delta^{',''','''}_{p,q,r}\,\Delta^{'','''}_{s,t}\,f'_\alpha f'''_\beta - 375\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''',''''}_{p,q,r}\,\Delta^{',''}_{s,t}\,f''_\alpha f''_\beta + \\
&+ 840\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''','''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f''_\alpha f''_\beta + 180\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''','''}_{p,q,r}\,\Delta^{',''}_{s,t}\,f''_\alpha f''_\beta + \\
&+ 144\,\Delta^{',''','''}_{i,j,k}\,\Delta^{',''','''}_{p,q,r}\,\Delta^{',''''}_{s,t}\,f''_\alpha f''_\beta + 144\,\Delta^{',''','''}_{i,j,k}\,\Delta^{',''','''}_{p,q,r}\,\Delta^{'','''}_{s,t}\,f''_\alpha f''_\beta - \\
&- 1440\,\Delta^{',''','''}_{i,j,k}\,\Delta^{',''','''}_{p,q,r}\,\Delta^{','''}_{s,t}\,f''_\alpha f'''_\beta + 480\,\Delta^{',''',''''}_{i,j,k}\,\Delta^{',''','''}_{p,q,r}\,\Delta^{',''}_{s,t}\,f''_\alpha f'''_\beta,
\end{aligned}
$$

where the roman indices satisfy $1 \leqslant i < j < k \leqslant 4$*, where* $1 \leqslant p < q < r \leqslant 4$*, where* $1 \leqslant s < r \leqslant 4$ *and where the two greek indices* α, β *satisfy* $1 \leqslant \alpha, \beta \leqslant 4$ *without restriction and finally the* **total number** *of these invariants generating the Demailly-Semple algebra* E^4_4 *equals:*

$$
\begin{aligned}
&1 + 4 + 6 + 24 + 96 + 4 + 16 + 64 + \\
&+ 36 + 24 + 96 + 384 + 64 + 96 + 384 + 1536 = \boxed{\mathbf{2835}}.
\end{aligned}
$$

Furthermore, in the preceding dimension $n = 3$ *for jets of the same order* $\kappa = 4$*, the Demailly-Semple algebra* E^3_4 *is generated by the analogous list from which one removes the four-dimensional Wronskian* $W^{10}_{1,2,3,4}$ *and in which the triples of skew-symmetric indices* $[i, j, k]$ *and* $[p, q, r]$ *are set to* $[1, 2, 3]$ *while* $[p, q]$ *satisfy* $1 \leqslant p < q \leqslant 3$ *and* α, β *satisfy* $1 \leqslant \alpha, \beta \leqslant 3$ *without restriction, whence the total number of generators of* E^3_4 *equals:*

$$
3 + 3 + 9 + 27 + 1 + 3 + 9 + 9 + 3 + 9 + 27 + 3 + 3 + 9 + 27 = \boxed{\mathbf{145}}.
$$

§12. Approximate Schur bundle decomposition of $\mathsf{E}^4_{4,m}T^*_X$

Finite generation. Thus, we know from the preceding section that UE^4_4 is generated by the sixteen bi-invariant polynomials:

$$
\Lambda^3,\ \Lambda^5,\ \Lambda^7,\ D^6,\ D^8,\ N^{10},\ M^8,\ E^{10},\ L^{12},\ Q^{14},\ R^{15},\ U^{17},\ V^{19},\ X^{21},\ f'_1,\ W^{10},
$$

whose weight appears as an exponent. A general polynomial in these 16 invariants writes:

$$
\begin{aligned}
\sum \mathsf{coeff}\cdot\, &(\Lambda^3)^a\,(\Lambda^5)^b\,(\Lambda^7)^c\,(D^6)^d\,(D^8)^e\,(N^{10})^f\,(M^8)^g\,(E^{10})^h\,(L^{12})^i \\
&(Q^{14})^j\,(R^{15})^k\,(U^{17})^l\,(V^{19})^m\,(X^{21})^n\,(f'_1)^o\,(W^{10})^p,
\end{aligned}
$$

where $a, b, c, d, e, f, g, h, i, j, k, l, m, n, o$ and p are nonnegative integer exponents. We temporarily use the letter m which should not make confusion with the weighting m appearing in $\mathsf{UE}^n_{\kappa,m}$. When one requires that such a polynomial has weight m, the sum should be restricted to exponents satisfying:

$$m = 3a+5b+7c+6d+8e+10f+8g+10h+12i+14j+15k+17l+19m+21n+o+10p.$$

When one furthermore restricts such a general polynomial to $\{f_1' = 0\}$, one gets:

$$\sum_{3a+5b+\cdots+21n+10p=m} \mathsf{coeff}\cdot \left(\Lambda^3|_0\right)^a \left(\Lambda^5|_0\right)^b \left(\Lambda^7|_0\right)^c \left(D^6|_0\right)^d \left(D^8|_0\right)^e \left(N^{10}|_0\right)^f \left(M^8|_0\right)^g \left(E^{10}|_0\right)^h$$
$$\left(L^{12}|_0\right)^i \left(Q^{14}|_0\right)^j \left(R^{15}|_0\right)^k \left(U^{17}|_0\right)^l \left(V^{19}|_0\right)^m \left(X^{21}|_0\right)^n \left(W^{10}|_0\right)^p.$$

Next, let Syz_{41} denote the ideal of $\mathbb{C}\big[\Lambda^3|_0, \ldots, X^{21}|_0\big]$ generated by the 41 lexicographic syzygies written on p. 256 (in which one sets $f_1' = 0$) holding between the ordered variables:

$$\Lambda^3|_0 > \Lambda^5|_0 > \Lambda^7|_0 > D^6|_0 > D^8|_0 > N^{10}|_0 > M^8|_0 > E^{10}|_0 >$$
$$L^{12}|_0 > Q^{14}|_0 > R^{15}|_0 > U^{17}|_0 > V^{19}|_0 > X^{21}|_0.$$

We list in columns the 41 Leading Terms of these 41 syzygies:

$\underline{\Lambda^3|_0\Lambda^7|_0}_{\mathsf{LT}}$: $a \geqslant 1, \quad c \geqslant 1$

$\underline{\Lambda^3|_0D^8|_0}_{\mathsf{LT}}$: $a \geqslant 1, \quad e \geqslant 1$

$\underline{\Lambda^3|_0N^{10}|_0}_{\mathsf{LT}}$: $a \geqslant 1, \quad f \geqslant 1$

$\underline{\Lambda^3|_0L^{12}|_0}_{\mathsf{LT}}$: $a \geqslant 1, \quad i \geqslant 1$

$\underline{\Lambda^3|_0Q^{14}|_0}_{\mathsf{LT}}$: $a \geqslant 1, \quad j \geqslant 1$

$\underline{\Lambda^3|_0R^{15}|_0}_{\mathsf{LT}}$: $a \geqslant 1, \quad k \geqslant 1$

$\underline{\Lambda^3|_0U^{17}|_0}_{\mathsf{LT}}$: $a \geqslant 1, \quad l \geqslant 1$

$\underline{\Lambda^3|_0V^{19}|_0}_{\mathsf{LT}}$: $a \geqslant 1, \quad m \geqslant 1$

$\underline{\Lambda^3|_0X^{21}|_0}_{\mathsf{LT}}$: $a \geqslant 1, \quad n \geqslant 1$

$\underline{\Lambda^5|_0D^8|_0}_{\mathsf{LT}}$: $b \geqslant 1, \quad e \geqslant 1$

$\underline{\Lambda^5|_0N^{10}|_0}_{\mathsf{LT}}$: $b \geqslant 1, \quad f \geqslant 1$

$\underline{\Lambda^5|_0L^{12}|_0}_{\mathsf{LT}}$: $b \geqslant 1, \quad i \geqslant 1$

$\underline{\Lambda^5|_0Q^{14}|_0}_{\mathsf{LT}}$: $b \geqslant 1, \quad j \geqslant 1$

$\underline{\Lambda^5|_0R^{15}|_0}_{\mathsf{LT}}$: $b \geqslant 1, \quad k \geqslant 1$

$\underline{\Lambda^5|_0U^{17}|_0}_{\mathsf{LT}}$: $b \geqslant 1, \quad l \geqslant 1$

$\underline{\Lambda^5|_0V^{19}|_0}_{\mathsf{LT}}$: $b \geqslant 1, \quad m \geqslant 1$

$\underline{\Lambda^5|_0X^{21}|_0}_{\mathsf{LT}}$: $b \geqslant 1, \quad n \geqslant 1$

$\underline{\Lambda^7|_0R^{15}|_0}_{\mathsf{LT}}$: $c \geqslant 1, \quad k \geqslant 1$

$\underline{\Lambda^7|_0U^{17}|_0}_{\mathsf{LT}}$: $c \geqslant 1, \quad l \geqslant 1$

$\underline{\Lambda^7|_0V^{19}|_0}_{\mathsf{LT}}$: $c \geqslant 1, \quad m \geqslant 1$

$\underline{\Lambda^7|_0X^{21}|_0}_{\mathsf{LT}}$: $c \geqslant 1, \quad n \geqslant 1$

$\underline{D^6|_0N^{10}|_0}_{\mathsf{LT}}$: $d \geqslant 1, \quad f \geqslant 1$

$\underline{D^6|_0L^{12}|_0}_{\mathsf{LT}}$: $d \geqslant 1, \quad i \geqslant 1$

$\underline{D^6|_0Q^{14}|_0}_{\mathsf{LT}}$: $d \geqslant 1, \quad j \geqslant 1$

$\underline{D^6|_0V^{19}|_0}_{\mathsf{LT}}$: $d \geqslant 1, \quad m \geqslant 1$

$\underline{D^6|_0X^{21}|_0}_{\mathsf{LT}}$: $d \geqslant 1, \quad n \geqslant 1$

$$\underline{D^8|_0L^{12}|_0}_{\mathsf{LT}}: \quad e \geqslant 1, \quad i \geqslant 1 \qquad \underline{D^6|_0M^8|_0R^{15}|_0}_{\mathsf{LT}}: \quad d \geqslant 1, \quad g \geqslant 1, \quad k \geqslant 1$$
$$\underline{D^8|_0Q^{14}|_0}_{\mathsf{LT}}: \quad e \geqslant 1, \quad j \geqslant 1 \qquad \underline{D^8|_0M^8|_0R^{15}|_0}_{\mathsf{LT}}: \quad e \geqslant 1, \quad g \geqslant 1, \quad k \geqslant 1$$
$$\underline{D^8|_0V^{19}|_0}_{\mathsf{LT}}: \quad e \geqslant 1, \quad m \geqslant 1 \qquad \underline{N^{10}|_0M^8|_0R^{15}|_0}_{\mathsf{LT}}: \quad f \geqslant 1, \quad g \geqslant 1, \quad k \geqslant 1$$
$$\underline{D^8|_0X^{21}|_0}_{\mathsf{LT}}: \quad e \geqslant 1, \quad n \geqslant 1 \qquad \underline{M^8|_0R^{15}|_0R^{15}|_0}_{\mathsf{LT}}: \quad g \geqslant 1, \quad k \geqslant 2$$

$$\underline{E^{10}|_0Q^{14}|_0}_{\mathsf{LT}}: \quad h \geqslant 1, \quad j \geqslant 1 \qquad \underline{N^{10}|_0E^{10}|_0V^{19}|_0}_{\mathsf{LT}}: \quad f \geqslant 1, \quad h \geqslant 1, \quad m \geqslant 1$$
$$\underline{E^{10}|_0X^{21}|_0}_{\mathsf{LT}}: \quad h \geqslant 1, \quad n \geqslant 1 \qquad \underline{N^{10}|_0L^{12}|_0V^{19}|_0}_{\mathsf{LT}}: \quad f \geqslant 1, \quad i \geqslant 1, \quad m \geqslant 1$$
$$\underline{N^{10}|_0L^{12}|_0X^{21}|_0}_{\mathsf{LT}}: \quad f \geqslant 1, \quad i \geqslant 1, \quad n \geqslant 1$$
$$\underline{L^{12}|_0L^{12}|_0X^{21}|_0}_{\mathsf{LT}}: \quad i \geqslant 2, \quad n \geqslant 1 \qquad \underline{N^{10}|_0U^{17}|_0X^{21}|_0}_{\mathsf{LT}}: \quad f \geqslant 1, \quad l \geqslant 1, \quad n \geqslant 1$$

If, by $\mathsf{LT}\big(\mathsf{Syz}_{41}\big)$, we denote the monomial ideal of $\mathbb{C}\big[\Lambda^3\big|_0, \dots, X^{21}\big|_0\big]$ generated by these 41 Leading Terms, a known elementary property of reduced Gröbner bases shows that:

$$\mathbb{C}\big[\Lambda^3\big|_0, \dots, X^{21}\big|_0\big]/\mathsf{Syz}_{41} \simeq \mathbb{C}\big[\Lambda^3\big|_0, \dots, X^{21}\big|_0\big]/\mathsf{LT}\big(\mathsf{Syz}_{41}\big).$$

More suitably for our purposes, the theorem on p. 258 states that any bi-invariant of weight m writes uniquely under the form:

$$\begin{aligned}\mathsf{P}\big(j^\kappa f\big) = \sum_{o,\,p} (f_1')^o \, \big(W^{10}\big)^p \sum_{\substack{(a,b,\dots,n)\in\mathbb{N}^{14}\setminus(\square_1\cup\dots\cup\square_{41})\\ 3a+\dots+21n=m-o-10p}} \mathsf{coeff}_{a,\dots,n,o,p} \cdot \\ \cdot \big(\Lambda^3\big)^a \big(\Lambda^5\big)^b \big(\Lambda^7\big)^c \big(D^6\big)^d \big(D^8\big)^e \big(N^{10}\big)^f \big(M^8\big)^g \big(E^{10}\big)^h \\ \big(L^{12}\big)^i \big(Q^{14}\big)^j \big(R^{15}\big)^k \big(U^{17}\big)^l \big(V^{19}\big)^m \big(X^{21}\big)^n,\end{aligned}$$

with coefficients $\mathsf{coeff}_{a,\dots,n,o,p}$ subjected to no restriction, where $\square_1, \dots, \square_{41}$ denote the quadrants in $\mathbb{N}^{14}$ having vertex at the leading terms of our 41 syzygies.

Our goal now is to compute an approximation of this general sum of monomials which will suffice for our Euler-Poincaré characteristic computations below.

A general monomial in $\mathbb{C}\big[\Lambda^3, \dots, X^{21}\big]$ writes:

$$\begin{aligned}\mathsf{Monomial} = \big(\Lambda^3\big)^a \big(\Lambda^5\big)^b \big(\Lambda^7\big)^c \big(D^6\big)^d \big(D^8\big)^e \big(N^{10}\big)^f \big(M^8\big)^g \big(E^{10}\big)^h \\ \big(L^{12}\big)^i \big(Q^{14}\big)^j \big(R^{15}\big)^k \big(U^{17}\big)^l \big(V^{19}\big)^m \big(X^{21}\big)^n.\end{aligned}$$

Such a monomial *belongs* to the monomial ideal $\mathsf{LT}\big(\mathsf{Syz}_{41}\big)$ *if and only if* it is a multiple of at least one of the 41 Leading Terms. Equivalently, the 14-tuple of integers $(a, \dots, n)$ belongs to at least one quadrant $\square_i$ with vertex the exponent of the leading term of the i-th syzygy. For instance, being a multiple of $\Lambda^3\Lambda^7$ occurs when and only when $a \geqslant 1$ and $c \geqslant 1$. In fact, in our complete list of the 41 leading terms above, just after each leading

Term, we have in advance written the condition that such a Monomial be a multiple of it.

On the contrary, for Monomial *not to be a multiple* of $\Lambda^3\Lambda^7$, it is necessary and sufficient that $a = 0$ or $c = 0$, and more generally, for it to belong to the relevant quotient ideal:

$$\mathbb{C}\big[\Lambda^3, \dots, X^{21}\big]/\mathsf{LT}\big(\mathsf{Syz}_{41}\big),$$

it is necessary and sufficient that its 14-tuple exponent $\big(a, b, c, d, e, f, g, h, i, j, k, l, m, n\big) \in \mathbb{N}^{14}$ belongs to the following *intersection* of 41 subsets of $\mathbb{N}^{14}$:

$$\big\{a = 0\big\}\cup\big\{c = 0\big\} \bigcap \big\{a = 0\big\}\cup\big\{e = 0\big\} \bigcap \cdots \bigcap \big\{f = 0\big\}\cup\big\{l = 0\big\}\cup\big\{n = 0\big\}.$$

To compute this intersection, we shall abbreviate for instance $\big\{a = 0\big\} \cup \big\{c = 0\big\}$ by $(a + c)$ with the symbol "+" denoting union, and with the intersection being denoted by an unwritten multiplication symbol, so that we may develope for instance the product of the first two terms as follows:

$$\begin{aligned}\big\{a = 0\big\} \cup \big\{c = 0\big\} \bigcap \big\{a = 0\big\} \cup \big\{e = 0\big\} &\equiv (a + c)(a + e)\\ &= aa + ae + ca + ce\\ &= a + ce,\end{aligned}$$

and simplify it immediately, on understanding that the symbol a represents $\big\{a = 0\big\}$, hence contains both $ae \equiv \big\{a = e = 0\big\}$ and $ca \equiv \big\{c = a = 0\big\}$.

With such a convention, grouping by packages, we may compute the intersections colum by column, starting with the first column containing $\Lambda^3|_0$:

$$(a{+}c)(a{+}e)(a{+}f)(a{+}i)(a{+}j)(a{+}k)(a{+}l)(a{+}m)(a{+}n) = a{+}cefijklmn,$$

and getting in sum nine "words" that we should further "intersect":

$$\begin{aligned}&a + cefijklmn,\\ &b + efijklmn,\\ &c + klmn,\\ &d + fijmn,\\ &e + ijmn,\\ &(d + g + k)(e + g + k)(f + g + k)(g + k + k^1),\\ &h + jn,\\ &i + i^1 + n,\\ &(f + h + m)(f + i + m)(f + i + n)(f + l + n).\end{aligned}$$

Here, the "letter" k^1 appearing at the end of the sixth line means the subset $\big\{k = 1\big\}$ of $\mathbb{N}^{14}$, not to be confused with $k \equiv \big\{k = 0\big\}$. Let us develope

step by step the sixth and the ninth lines:

$$\begin{aligned}
&(d+g+k)(e+g+k)(f+g+k)(g+k+k^1) = \\
&(d+g+k)(e+g+k)(g+k+fk^1) = \\
&(d+g+k)(g+k+efk^1) = \\
&g+k+defk^1
\end{aligned}$$

$$\begin{aligned}
&(f+h+m)(f+i+m)(f+i+n)(f+l+n) = \\
&(f+h+m)(f+i+m)(f+n+il) = \\
&(f+h+m)(f+il+in+mn) = \\
&f+hil+hin+mn+ilm.
\end{aligned}$$

Now we compute the product of the lines 3, 4, 5, 7:

$$\begin{aligned}
&(c+klmn)(d+fijmn)(e+ijmn)(h+jn) = \\
&(c+klmn)(d+fijmn)(eh+ejn+ijmn) = \\
&(c+klmn)(deh+dejn+dijmn+fijmn) = \\
&cdeh+cdejn+cdijmn+cfijmn+dehklmn+dejklmn+dijklmn+fijklmn
\end{aligned}$$

and the product of the lines 1 and 2:

$$ab+aefijklmn+cefijklmn,$$

whence the product of the lines 1, 2, 3, 4, 5, 7 is:

$$\begin{aligned}
&abcdeh+abcdejn+abcdijmn+abcfijmn+abdehklmn+abdejklmn+ \\
&+abdijklmn+abfijklmn+aefijklmn+cefijklmn.
\end{aligned}$$

On the other hand, the product of the lines 9, 6, 8 is:

$$\begin{aligned}
&(f+hil+hin+mn+ilm)(g+k+defk^1)(i+i^1+n) = \\
&(f+hil+hin+mn+ilm)(gi+gi^1+gn+ik+i^1k+kn+defik^1+defi^1k^1+defk^1n) =
\end{aligned}$$

When developing the latter product, sometimes words containing the product ii^1 (or kk^1) might appear. But they denote the empty set $\{i=0\}\cap\{i=1\}$, so they should be left out. The direct result of the product, before any simplification, is:

$$\begin{aligned}
&= fgi+fgi^1+fgn+fik+fi^1k+fkn+defik^1+defi^1k^1+defk^1n+ \\
&\quad +ghil+\emptyset+ghiln+hikl+\emptyset+hikln+defik^1l+\emptyset+defhik^1ln+ \\
&\quad +ghin+\emptyset+ghin+hikn+\emptyset+hikn+defhik^1n+\emptyset+defhik^1n+ \\
&\quad +gimn+gi^1mn+gmn+ikmn+i^1kmn+kmn+defik^1mn+defi^1k^1mn+defk^1mn+ \\
&\quad +gilm+\emptyset+gilmn+iklm+\emptyset+iklmn+defik^1lm+\emptyset+defik^1lmn,
\end{aligned}$$

and after simplification:

$$= fgi + fgi^1 + fgn + fik + fi^1k + fkn + defik^1 + defi^1k^1 + defk^1n + \\ + ghil + hikl + ghin + hikn + gmn + kmn + gilm + iklm.$$

The final multiplication shall be:

$$\Big(abcdeh + abcdejn + abcdijmn + abcfijmn + abdehklmn + abdejklmn + \\ + abdijklmn + abfijklmnaefijklmn + cefijklmn\Big) \cdot \\ \cdot \Big(fgi + fgi^1 + fgn + fik + fi^1k + fkn + defik^1 + defi^1k^1 + defk^1n + \\ + ghil + hikl + ghin + hikn + gmn + kmn + gilm + iklm\Big),$$

but we will not expand it completely.

Twenty-four families of monomials. Instead, we will compute the product modulo words which contain more than 9 letters. The reason why we do so will be appearent later. The result then consists of 30 words of 9 letters:

A :	$abcdefghi$	J :	$abcdegjmn$
A′ :	$abcdefghi^1$	K :	$abcdehikl$
B :	$abcdefghn$	L :	$abcdehikn$
C :	$abcdefgjn$	M :	$abcdehkmn$
D :	$abcdefhik$	N :	$abcdejkmn$
D′ :	$abcdefhi^1k$	O :	$abcdgijmn$
D″ :	$abcdefhik^1$	P :	$abcdijkmn$
D‴ :	$abcdefhi^1k^1$	Q :	$abcfgijmn$
E :	$abcdefhkn$	R :	$abcfijkmn$
E′ :	$abcdefhk^1n$	S :	$abdehklmn$
F :	$abcdefjkn$	T :	$abdejklmn$
F′ :	$abcdefjk^1n$	U :	$abdijklmn$
G :	$abcdeghil$	V :	$abfijklmn$
H :	$abcdeghin$	W :	$aefijklmn$
I :	$abcdeghimn$	X :	$cefijklmn$

Recalling that the first word $abcdefghi$ for instance means the condition $\{a = b = c = d = e = f = g = h = i = 0\}$ on the exponents of a general monomial, we may therefore list in an extensive array the 24 families A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X of

corresponding monomials, the subsidiary families A'; D', D'', D'''; E'; F' being considered as similar to A; D; E; F:

A :	•	•	•	•	•	•	•	•	•	$(Q^{14})^j$	$(R^{15})^k$	$(U^{17})^l$	$(V^{19})^m$	$(X^{21})^n$	$(f_1')^o$	$(W^{10})^p$
B :	•	•	•	•	•	•	•	•	$(L^{12})^i$	$(Q^{14})^j$	$(R^{15})^k$	$(U^{17})^l$	$(V^{19})^m$	•	$(f_1')^o$	$(W^{10})^p$
C :	•	•	•	•	•	•	•	$(E^{10})^h$	$(L^{12})^i$	•	$(R^{15})^k$	$(U^{17})^l$	$(V^{19})^m$	•	$(f_1')^o$	$(W^{10})^p$
D :	•	•	•	•	•	•	$(M^8)^g$	•	•	$(Q^{14})^j$	•	$(U^{17})^l$	$(V^{19})^m$	$(X^{21})^n$	$(f_1')^o$	$(W^{10})^p$
E :	•	•	•	•	•	•	$(M^8)^g$	•	$(L^{12})^i$	$(Q^{14})^j$	•	$(U^{17})^l$	$(V^{19})^m$	•	$(f_1')^o$	$(W^{10})^p$
F :	•	•	•	•	•	•	$(M^8)^g$	$(E^{10})^h$	$(L^{12})^i$	•	•	$(U^{17})^l$	$(V^{19})^m$	•	$(f_1')^o$	$(W^{10})^p$
G :	•	•	•	•	•	$(N^{10})^f$	•	•	•	$(Q^{14})^j$	$(R^{15})^k$	•	$(V^{19})^m$	$(X^{21})^n$	$(f_1')^o$	$(W^{10})^p$
H :	•	•	•	•	•	$(N^{10})^f$	•	•	•	$(Q^{14})^j$	$(R^{15})^k$	$(U^{17})^l$	$(V^{19})^m$	•	$(f_1')^o$	$(W^{10})^p$
I :	•	•	•	•	•	$(N^{10})^f$	•	•	$(L^{12})^i$	$(Q^{14})^j$	$(R^{15})^k$	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
J :	•	•	•	•	•	$(N^{10})^f$	•	$(E^{10})^h$	$(L^{12})^i$	•	$(R^{15})^k$	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
K :	•	•	•	•	•	$(N^{10})^f$	$(M^8)^g$	•	•	$(Q^{14})^j$	•	•	$(V^{19})^m$	$(X^{21})^n$	$(f_1')^o$	$(W^{10})^p$
L :	•	•	•	•	•	$(N^{10})^f$	$(M^8)^g$	•	•	$(Q^{14})^j$	•	$(U^{17})^l$	$(V^{19})^m$	•	$(f_1')^o$	$(W^{10})^p$
M :	•	•	•	•	•	$(N^{10})^f$	$(M^8)^g$	•	$(L^{12})^i$	$(Q^{14})^j$	•	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
N :	•	•	•	•	•	$(N^{10})^f$	$(M^8)^g$	$(E^{10})^h$	$(L^{12})^i$	•	•	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
O :	•	•	•	•	$(D^8)^e$	$(N^{10})^f$	•	$(E^{10})^h$	•	•	$(R^{15})^k$	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
P :	•	•	•	•	$(D^8)^e$	$(N^{10})^f$	$(M^8)^g$	$(E^{10})^h$	•	•	•	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
Q :	•	•	•	$(D^6)^d$	$(D^8)^e$	•	•	$(E^{10})^h$	•	•	$(R^{15})^k$	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
R :	•	•	•	$(D^6)^d$	$(D^8)^e$	•	$(M^8)^g$	$(E^{10})^h$	•	•	•	$(U^{17})^l$	•	•	$(f_1')^o$	$(W^{10})^p$
S :	•	•	$(\Lambda^7)^c$	•	•	$(N^{10})^f$	$(M^8)^g$	•	$(L^{12})^i$	$(Q^{14})^j$	•	•	•	•	$(f_1')^o$	$(W^{10})^p$
T :	•	•	$(\Lambda^7)^c$	•	•	$(N^{10})^f$	$(M^8)^g$	$(E^{10})^h$	$(L^{12})^i$	•	•	•	•	•	$(f_1')^o$	$(W^{10})^p$
U :	•	•	$(\Lambda^7)^c$	•	$(D^8)^e$	$(N^{10})^f$	$(M^8)^g$	$(E^{10})^h$	•	•	•	•	•	•	$(f_1')^o$	$(W^{10})^p$
V :	•	•	$(\Lambda^7)^c$	$(D^6)^d$	$(D^8)^e$	•	$(M^8)^g$	$(E^{10})^h$	•	•	•	•	•	•	$(f_1')^o$	$(W^{10})^p$
W :	•	$(\Lambda^5)^b$	$(\Lambda^7)^c$	$(D^6)^d$	•	•	$(M^8)^g$	$(E^{10})^h$	•	•	•	•	•	•	$(f_1')^o$	$(W^{10})^p$
X :	$(\Lambda^3)^a$	$(\Lambda^5)^b$	•	$(D^6)^d$	•	•	$(M^8)^g$	$(E^{10})^h$	•	•	•	•	•	•	$(f_1')^o$	$(W^{10})^p$

General Schur bundle decomposition of $\mathsf{E}^4_{4,m}T^*_X$. By general representation theory, the polynomial action of $\mathsf{GL}_4(\mathbb{C})$ decomposes in a certain direct sum of irreducible Schur representations. What we call bi-invariants correspond to vectors of highest weight for the $\mathsf{GL}_4(\mathbb{C})$-representation. To each vector of highest weight corresponds one and only one irreducible Schur representation. Such a vector of highest weight is nothing else but a monomial:

$$(\Lambda^3)^a\,(\Lambda^5)^b\,(\Lambda^7)^c\,(D^6)^d\,(D^8)^e\,(N^{10})^f\,(M^8)^g\,(E^{10})^h$$
$$(L^{12})^i\,(Q^{14})^j\,(R^{15})^k\,(U^{17})^l\,(V^{19})^{m'}\,(X^{21})^n\,(f_1')^o\,(W^{10}),$$

with the usual condition on exponents: $3a + \cdots + 21n + o + 10p = m$ and $(a, \ldots, n)$ belonging to the complement $\mathbb{N}^{14}\backslash\big(\square_1 \cup \cdots \cup \square_{41}\big)$ of the 41 quadrants. From now on, we denote by m' the exponent of V^{19} to distinguish it from the weight m of the bi-invariant.

To know what are the four integers $\ell_1, \ell_2, \ell_3, \ell_4$ of the corresponding Schur representations $\Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)}\mathbb{C}^4$, it suffices to consider the diagonal matrices of $\mathsf{GL}_4(\mathbb{C})$ of the form:

$$\mathsf{x} := \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix},$$

for which all vectors of highest weight are then just eigenvectors having eigenvalue of the form $x_1^{\ell_1} x_2^{\ell_2} x_3^{\ell_3} x_4^{\ell_4}$.

Here in our situation, coming back to the theorem which describes the 2835 generators of E_4^4, we should at first write down our 16 bi-invariants under a form in which we emphasize the lower indices as we did for the general invariants. This gives us the following more informative list:

$$\boxed{\begin{array}{l} \ell^3_{[1,2]}, \quad \ell^5_{[1,2];\,1}, \quad \ell^7_{[1,2];\,1,1}, \quad D^6_{[1,2,3]}, \quad D^8_{[1,2,3];\,1}, \quad N^{10}_{[1,2,3];\,1,1}, \quad M^8_{[1,2],[1,2]}, \\ \qquad E^{10}_{[1,2,3],[1,2]}, \quad L^{12}_{[1,2,3],[1,2];\,1}, \quad Q^{14}_{[1,2,3],[1,2];\,1,1}, \quad R^{15}_{[1,2,3],[1,2,3];\,1}, \\ U^{17}_{[1,2,3],[1,2,3],[1,2]}, \quad V^{19}_{[1,2,3],[1,2,3],[1,2];\,1}, \quad X^{21}_{[1,2,3],[1,2,3],[1,2];\,1,1}, \quad f'_1, \quad W^{10}_{[1,2,3,4]} \end{array}}.$$

Then it is easy to realize that ℓ_1, ℓ_2, ℓ_3, ℓ_4 just count the number of indices 1, 2, 3, 4 respectively at the bottom of each invariant. Consequently, we have the sixteen correspondences:

$$\begin{aligned}
\big(\ell^3\big)^a &: \qquad \Gamma^{(a,a,0,0)}\mathbb{C}^4 \\
\big(\ell^5\big)^b &: \qquad \Gamma^{(2b,b,0,0)}\mathbb{C}^4 \\
\big(\ell^7\big)^c &: \qquad \Gamma^{(3c,c,0,0)}\mathbb{C}^4 \\
\big(D^6\big)^d &: \qquad \Gamma^{(d,d,d,0)}\mathbb{C}^4 \\
\big(D^8\big)^e &: \qquad \Gamma^{(2e,e,e,0)}\mathbb{C}^4 \\
\big(N^{10}\big)^f &: \qquad \Gamma^{(3f,f,f,0)}\mathbb{C}^4 \\
\big(M^8\big)^g &: \qquad \Gamma^{(2g,2g,0,0)}\mathbb{C}^4 \\
\big(E^{10}\big)^h &: \qquad \Gamma^{(2h,2h,h,0)}\mathbb{C}^4 \\
\big(L^{12}\big)^i &: \qquad \Gamma^{(3i,2i,i,0)}\mathbb{C}^4 \\
\big(Q^{14}\big)^j &: \qquad \Gamma^{(4j,2j,j,0)}\mathbb{C}^4 \\
\big(R^{15}\big)^k &: \qquad \Gamma^{(3k,2k,2k,0)}\mathbb{C}^4 \\
\big(U^{17}\big)^l &: \qquad \Gamma^{(3l,3l,2l,0)}\mathbb{C}^4 \\
\big(V^{19}\big)^{m'} &: \qquad \Gamma^{(4m',3m',2m',0)}\mathbb{C}^4 \\
\big(X^{21}\big)^n &: \qquad \Gamma^{(5n,3n,2n,0)}\mathbb{C}^4 \\
\big(f'_1\big)^o &: \qquad \Gamma^{(o,0,0,0)}\mathbb{C}^4 \\
\big(W^{10}\big)^p &: \qquad \Gamma^{(p,p,p,p)}\mathbb{C}^4
\end{aligned}$$

and it immediately follows that the Schur representation $\Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)}\mathbb{C}^4$ which corresponds to the general monomial written above has integers ℓ_i

given by:

$$\begin{cases} \ell_1 = o + a + 2b + 3c + d + 2e + 3f + 2g + 2h + 3i + 4j + 3k + 3l + 4m' + 5n + p, \\ \ell_2 = a + b + c + d + e + f + 2g + 2h + 2i + 2j + 2k + 3l + 3m' + 3n + p, \\ \ell_3 = d + e + f + h + i + j + 2k + 2l + 2m' + 2n + p, \\ \ell_4 = p. \end{cases}$$

By a direct application of the theorem on p. 258 of §11, we obtain an exact Schur bundle decompositition of the graduate m-th part $\mathsf{E}^4_{4,m}T^*_X$ of the Demailly-Semple bundle $\mathsf{E}^4_4T^*_X$ on a complex algebraic hypersurface $X \subset \mathbb{P}^5(\mathbb{C})$.

THEOREM *In dimension $n = 4$ for jet order $\kappa = 4$, graduate m-th part $\mathsf{E}^4_{4,m}T^*_X$ of the Demailly-Semple bundle $\mathsf{E}^4_4T^*_X = \oplus_m \mathsf{E}^4_{4,m}T^*_X$ on a complex algebraic hypersurface $X \subset \mathbb{P}^5(\mathbb{C})$ has the following decomposition in direct sums of Schur bundles:*

$$\mathsf{E}^4_{4,m}T^*_X = \bigoplus_{\substack{(a,b,\dots,n)\in\mathbb{N}^{14}\setminus(\square_1\cup\dots\cup\square_{41}) \\ o+3a+\dots+21n+10p=m}} \Gamma\begin{pmatrix} o + a + 2b + 3c + d + 2e + 3f + 2g + 2h + 3i + 4j + 3k + 3l + 4m' + 5n + p \\ a + b + c + d + e + f + 2g + 2h + 2i + 2j + 2k + 3l + 3m' + 3n + p \\ d + e + f + h + i + j + 2k + 2l + 2m' + 2n + p \\ p \end{pmatrix} T^*_X,$$

where the 41 subsets $\square_i$ of $\mathbb{N}^{14}$ are precisely defined by:

$$\begin{aligned} &\{a \geqslant 1, c \geqslant 1\}, \quad \{a \geqslant 1, e \geqslant 1\}, \quad \{a \geqslant 1, f \geqslant 1\}, \quad \{a \geqslant 1, i \geqslant 1\}, \\ &\{a \geqslant 1, j \geqslant 1\}, \quad \{a \geqslant 1, k \geqslant 1\}, \quad \{a \geqslant 1, l \geqslant 1\}, \quad \{a \geqslant 1, m' \geqslant 1\}, \\ &\{a \geqslant 1, n \geqslant 1\}, \quad \{b \geqslant 1, e \geqslant 1\}, \quad \{b \geqslant 1, f \geqslant 1\}, \quad \{b \geqslant 1, i \geqslant 1\}, \\ &\{b \geqslant 1, j \geqslant 1\}, \quad \{b \geqslant 1, k \geqslant 1\}, \quad \{b \geqslant 1, l \geqslant 1\}, \quad \{b \geqslant 1, m' \geqslant 1\}, \\ &\{b \geqslant 1, n \geqslant 1\}, \quad \{c \geqslant 1, k \geqslant 1\}, \quad \{c \geqslant 1, l \geqslant 1\}, \quad \{c \geqslant 1, m' \geqslant 1\}, \\ &\{c \geqslant 1, n \geqslant 1\}, \quad \{d \geqslant 1, f \geqslant 1\}, \quad \{d \geqslant 1, i \geqslant 1\}, \quad \{d \geqslant 1, j \geqslant 1\}, \\ &\{d \geqslant 1, m \geqslant 1\}, \quad \{d \geqslant 1, n \geqslant 1\}, \quad \{e \geqslant 1, i \geqslant 1\}, \quad \{e \geqslant 1, j \geqslant 1\}, \\ &\{e \geqslant 1, m' \geqslant 1\}, \quad \{e \geqslant 1, n \geqslant 1\}, \quad \{d \geqslant 1, g \geqslant 1, k \geqslant 1\}, \\ &\{e \geqslant 1, g \geqslant 1, k \geqslant 1\}, \quad \{f \geqslant 1, g \geqslant 1, k \geqslant 1\}, \quad \{g \geqslant 1, k \geqslant 2\}, \\ &\{h \geqslant 1, j \geqslant 1\}, \quad \{h \geqslant 1, n \geqslant 1\}, \quad \{i \geqslant 2, n \geqslant 1\}, \\ &\{f \geqslant 1, h \geqslant 1, m' \geqslant 1\}, \quad \{f \geqslant 1, i \geqslant 1, m' \geqslant 1\}, \quad \{f \geqslant 1, i \geqslant 1, n \geqslant 1\}, \\ &\{f \geqslant 1, l \geqslant 1, n \geqslant 1\}. \end{aligned}$$

In addition, in the preceding dimension $n = 3$ for jets of the same order $\kappa = 4$, one has an entirely similar Schur bundle decomposition of $\mathsf{E}^3_{4,m}T^*_X$ for any m in which one removes W^{10}, one sets $p = 0$ and

one removes the fourth component ℓ_4 *of* $\Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)}$*:*

$$\mathsf{E}^3_{4,m}T^*_X = \bigoplus_{\substack{(a,b,\ldots,n)\in\mathbb{N}^{14}\setminus(\square_1\cup\cdots\cup\square_{41})\\ o+3a+\cdots+21n=m}}$$

$$\Gamma\begin{pmatrix} o+a+2b+3c+d+2e+3f+2g+2h+3i+4j+3k+3l+4m'+5n \\ a+b+c+d+e+f+2g+2h+2i+2j+2k+3l+3m'+3n \\ d+e+f+h+i+j+2k+2l+2m'+2n \end{pmatrix} T^*_X.$$

Approximate Schur bundle decomposition. We now come back to our 24 words of 9 letters and we make three remarks which will simplify a bit the further computations.

• The full complement $\mathbb{N}^{14}\big\backslash\big(\square_1\cup\cdots\cup\square_{41}\big)$ is slightly larger than the union of the 30 subsets of $\mathbb{N}^{14}$ defined by A, A', B, $\ldots$, W X, in the sense that it contains also a finite number of subsets defined by equating to 0 (or to 1) more than 9 exponents. These subsets will not contribute to the dominant term m^{16} when calculating the Euler-Poincaré characteristic of $\mathsf{E}^4_{4,m}T^*_X$ and hence, they will at once be left out.

• The first family A corresponds to a general polynomial of the form:

$$\sum_{o+14j+15k+17l+19m'+21n+10p=m} \mathsf{A}_{j,k,l,m',n,o,p}\cdot\big(Q^{14}\big)^j\big(R^{15}\big)^k\big(U^{17}\big)^l\big(V^{19}\big)^{m'} \big(X^{21}\big)^n\big(f_1'\big)^o\big(W^{10}\big)^p.$$

The second family A' corresponds to a general polynomial of the form:

$$L^{12}\sum_{o+14j+15k+17l+19m'+21n+10p=m-12} \mathsf{A}'_{j,k,l,m',n,o,p}\cdot\big(Q^{14}\big)^j\big(R^{15}\big)^k\big(U^{17}\big)^l\big(V^{19}\big)^{m'} \big(X^{21}\big)^n\big(f_1'\big)^o\big(W^{10}\big)^p.$$

It is entirely of the same type as A, except that the weight m is replaced by $m-12$. We will see that its contribution to the dominant m^{16}-term of the Euler-Poincaré characteristic is exactly the same[57], hence we will remove A' and provide the family A with the multiplicity 2. Similarly, D, E and F will have multiplicity 4, 2 and 2.

• The third (now second) family B corresponds to a general polynomial of the form:

$$\sum_{o+12i+14j+15k+17l+19m'+10p=m} \mathsf{B}_{i,j,k,l,m',o,p}\cdot\big(L^{12}\big)^i\big(Q^{14}\big)^j\big(R^{15}\big)^k \big(U^{17}\big)^l\big(V^{19}\big)^{m'}\big(f_1'\big)^o\big(W^{10}\big)^p,$$

[57] The argument will simply be that $(m-\text{cst.})^{16}=m^{16}+\mathrm{O}(m^{15})$ as $m\to\infty$.

hence its intersection with the family A is nontrivial, consisting of polynomials of the form:

$$\sum_{o+14j+15k+17l+19m'+10p=m} \widetilde{\mathsf{B}}_{j,k,l,m',o,p} \cdot \big(Q^{14}\big)^j \big(R^{15}\big)^k \big(U^{17}\big)^l \big(V^{19}\big)^{m'} \big(f_1'\big)^o \big(W^{10}\big)^p.$$

In principle, we should write the union of two overlapping families $\mathsf{A} \cup \mathsf{B}$ in the form of two non-intersecting families: $\mathsf{A} \cup \big(\mathsf{B} \backslash \mathsf{A}\big)$, but here again, because the intersection $\mathsf{A} \cap \mathsf{B}$ is represented by the word $abcdefghin$ which has $10 > 9$ letters, this intersection will only contribute the Euler-Poincaré characteristic as an $\mathrm{O}(m^{15})$, which will not perturb the dominant term m^{16}, as $m \to \infty$. So we can consider the 24 remaining families (a bit of which have multiplicities) without caring about overlappings.

In summary, up to certain negligible sums of Schur bundles which will not contribute to the dominant m^{16}-term while calculating the Euler-Poincaré characteristic of $\mathsf{E}^4_{4,m} T^*_X$, we have to consider **24** direct sums of Schur bundles with multiplicities, indexed from A up to X in the roman alphabet:

$$\underline{\mathsf{A}:} \quad \mathbf{2}\cdot \bigoplus_{m=o+14j+15k+17l+19m+21n+10p} \Gamma \begin{pmatrix} o+4j+3k+3l+4m+5n+p \\ 2j+2k+3l+3m+3n+p \\ j+2k+2l+2m+2n+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{B}:} \quad \bigoplus_{m=o+12i+14j+15k+17l+19m+10p} \Gamma \begin{pmatrix} o+3i+4j+3k+3l+4m+p \\ 2i+2j+2k+3l+3m+p \\ i+j+2k+2l+2m+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{C}:} \quad \bigoplus_{m=o+10h+12i+15k+17l+19m+10p} \Gamma \begin{pmatrix} o+2h+3i+3k+3l+4m+p \\ 2h+2i+2k+3l+3m+p \\ h+i+2k+2l+2m+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{D}:} \quad \mathbf{4}\cdot \bigoplus_{m=o+8g+14j+17l+19m+21n+10p} \Gamma \begin{pmatrix} o+2g+4j+3l+4m+5n+p \\ 2g+2j+3l+3m+3n+p \\ j+2l+2m+2n+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{E}:} \quad \mathbf{2}\cdot \bigoplus_{m=o+8g+12i+14j+17l+19m+10p} \Gamma \begin{pmatrix} o+2g+3i+4j+3l+4m+p \\ 2g+2i+2j+3l+3m+p \\ i+j+2l+2m+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{F}:} \quad \mathbf{2}\cdot \bigoplus_{m=o+8g+10h+12i+17l+19m+10p} \Gamma \begin{pmatrix} o+2g+2h+3i+3l+4m+p \\ 2g+2h+2i+3l+3m+p \\ h+i+2l+2m+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{G}:}\qquad \bigoplus_{m=o+10f+14j+15k+19m+21n+10p} \Gamma\begin{pmatrix} o+3f+4j+3k+4m+5n+p \\ f+2j+2k+3m+3n+p \\ f+j+2k+2m+2n+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{H}:}\qquad \bigoplus_{m=o+10f+14j+15k+17l+19m+10p} \Gamma\begin{pmatrix} o+3f+4j+3k+3l+4m+p \\ f+2j+2k+3l+3m+p \\ f+j+2k+2l+2m+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{I}:}\qquad \bigoplus_{m=o+10f+12i+14j+15k+17l+10p} \Gamma\begin{pmatrix} o+3f+3i+4j+3k+3l+p \\ f+2i+2j+2k+3l+p \\ f+i+j+2k+2l+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{J}:}\qquad \bigoplus_{m=o+10f+10h+12i+15k+17l+10p} \Gamma\begin{pmatrix} o+3f+2h+3i+3k+3l+p \\ f+2h+2i+2k+3l+p \\ f+h+i+2k+2l+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{K}:}\qquad \bigoplus_{m=o+10f+8g+14j+19m+21n+10p} \Gamma\begin{pmatrix} o+3f+2g+4j+4m+5n+p \\ f+2g+2j+3m+3n+p \\ f+j+2m+2n+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{L}:}\qquad \bigoplus_{m=o+10f+8g+14j+17l+19m+10p} \Gamma\begin{pmatrix} o+3f+2g+4j+3l+4m+p \\ f+2g+2j+3l+3m+p \\ f+j+2l+2m+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{M}:}\qquad \bigoplus_{m=o+10f+8g+12i+14j+17l+10p} \Gamma\begin{pmatrix} o+3f+2g+3i+4j+3l+p \\ f+2g+2i+2j+3l+p \\ f+i+j+2l+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{N}:}\qquad \bigoplus_{m=o+10f+8g+10h+12i+17l+10p} \Gamma\begin{pmatrix} o+3f+2g+2h+3i+3l+p \\ f+2g+2h+2i+3l+p \\ f+h+i+2l+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{O}:}\qquad \bigoplus_{m=o+8e+10f+10h+15k+17l+10p} \Gamma\begin{pmatrix} o+2e+3f+2h+3k+3l+p \\ e+f+2h+2k+3l+p \\ e+f+h+2k+2l+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{P}:}\qquad \bigoplus_{m=o+8e+10f+8g+10h+17l+10p} \Gamma\begin{pmatrix} o+2e+3f+2g+2h+3l+p \\ e+f+2g+2h+3l+p \\ e+f+h+2l+p \\ p \end{pmatrix} T^*_X,$$

$$\underline{\mathsf{Q}:} \quad \bigoplus_{m=o+6d+8e+10h+15k+17l+10p} \Gamma \begin{pmatrix} o+d+2e+2h+3k+3l+p \\ d+e+2h+2k+3l+p \\ d+e+h+2k+2l+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{R}:} \quad \bigoplus_{m=o+6d+8e+8g+10h+17l+10p} \Gamma \begin{pmatrix} o+d+2e+2g+2h+3l+p \\ d+e+2g+2h+3l+p \\ d+e+h+2l+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{S}:} \quad \bigoplus_{m=o+7c+10f+8g+12i+14j+10p} \Gamma \begin{pmatrix} o+3c+3f+2g+3i+4j+p \\ c+f+2g+2i+2j+p \\ f+i+j+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{T}:} \quad \bigoplus_{m=o+7c+10f+8g+10h+12i+10p} \Gamma \begin{pmatrix} o+3c+3f+2g+2h+3i+p \\ c+f+2g+2h+2i+p \\ f+h+i+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{U}:} \quad \bigoplus_{m=o+7c+8e+10f+8g+10h+10p} \Gamma \begin{pmatrix} o+3c+2e+3f+2g+2h+p \\ c+e+f+2g+2h+p \\ e+f+h+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{V}:} \quad \bigoplus_{m=o+7c+6d+8e+8g+10h+10p} \Gamma \begin{pmatrix} o+3c+d+2e+2g+2h+p \\ c+d+e+2g+2h+p \\ d+e+h+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{W}:} \quad \bigoplus_{m=o+5b+7c+6d+8g+10h+10p} \Gamma \begin{pmatrix} o+2b+3c+d+2g+2h+p \\ b+c+d+2g+2h+p \\ d+h+p \\ p \end{pmatrix} T_X^*,$$

$$\underline{\mathsf{X}:} \quad \bigoplus_{m=o+3a+5b+6d+8g+10h+10p} \Gamma \begin{pmatrix} o+a+2b+d+2g+2h+p \\ a+b+d+2g+2h+p \\ d+h+p \\ p \end{pmatrix} T_X^*.$$

It is now time to speak of the asymptotic of the Euler characteristic of a single Schur bundle.

§13. Asymptotic expansion of the Euler characteristic $\chi\big(\Gamma^{(\ell_1,\ell_2,\ldots,\ell_n)}T_X^*\big)$

Euler-Poincaré characteristic of Schur bundles. Let $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a complex algebraic hypersurface and denote by $\mathsf{c}_1, \mathsf{c}_2, \ldots, \mathsf{c}_n$ be the Chern classes $\mathsf{c}_k\big(T_X\big)$ of the tangent bundle T_X. Each c_k may be represented by a smooth differential form of bidegree (k,k) on X. One thus assigns the weight k to c_k. Because the total degrees of these forms are all even, the commutation relations $\mathsf{c}_{k_1}\mathsf{c}_{k_2} = \mathsf{c}_{k_2}\mathsf{c}_{k_1}$ hold for the cup product.

Every polynomial in the Chern classes:

$$\sum_{k_1+\cdots+k_n=n} \mathsf{coeff} \cdot \mathsf{c}_{k_1}\, \mathsf{c}_{k_2} \cdots \mathsf{c}_{k_n}$$

which is homogeneous of degree $n = \dim X$ is represented by an (n,n)-form on X, hence may be integrated. By a standard abuse of language, such a polynomial is usually considered both as an (n,n)-form and as the purely numerical quantity:

$$\int_X \sum_{k_1+\cdots+k_n=n} \mathsf{coeff}\, \mathsf{c}_{k_1} \cdot \mathsf{c}_{k_2} \cdots \mathsf{c}_{k_n}.$$

For instance, if d denotes the degree of X, one shows $\int_X \mathsf{c}_1^n = d^{n+1}$, a kind of relation often abbreviated $\mathsf{c}_1^n = d^{n+1}$.

To speak in full generality ([4, 29, 9]), the short exact sequence:

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^{n+1}}\big|_X \longrightarrow \mathscr{O}_X(d) \longrightarrow 0$$

gives the relation $c_\bullet\big(T_{\mathbb{P}^{n+1}}\big|_X\big) = c_\bullet\big(T_X\big) \cdot c_\bullet\big(\mathscr{O}_X(d)\big)$ between total Chern classes of the middle term and of the two extreme ones, or more explicitly:

$$(1+h)^{n+2} = \big[1 + \mathsf{c}_1 + \cdots + \mathsf{c}_n\big](1+dh),$$

where $(1+h)^{n+2}$ is the total Chern class of $\mathbb{P}^{n+1}$ with $h = \mathsf{c}_1\big(\mathscr{O}_{\mathbb{P}^{n+1}}(1)\big)$ being a $(1,1)$-form. Consequently, by expanding both the left-hand and the right-hand sides and by identifying terms of the same bidegree, we get closed expressions for all the Chern classes.

Lemma. *In terms of the hyperplane divisor $h = \mathsf{c}_1\big(\mathscr{O}_{\mathbb{P}^{n+1}}(1)\big)$ which satisfies $\int_X h^n = d = \deg X$, the Chern classes c_k of T_X are given by:*

$$\mathsf{c}_k = (-1)^k\, h^k \Big(d^k - \tfrac{(n+2)!}{1!\,(n+1)!}\, d^{k-1} + \cdots + (-1)^k \tfrac{(n+2)!}{k!\,(n+2-k)!}\Big).$$

Proof. We indeed expand the two sides of the above relation between total Chern classes:

$$1 + \tfrac{(n+2)!}{1!\,(n+1)!}\, h + \cdots + \tfrac{(n+2)!}{n!\,2!}\, h^n = 1 + (c_1 + dh) + (\mathsf{c}_2 + d\mathsf{c}_1 h) + \cdots + (\mathsf{c}_n + d\mathsf{c}_{n-1} h),$$

on understanding that the forms h^{n+1}, h^{n+2} and $\mathsf{c}_n h$ of degree $> 2n$ vanish identically. Identifying forms of the same bidegree yields the binomial-type recurrence relations: $\mathsf{c}_k = \frac{(n+2)!}{k!\,(n+2-k)!}\, h^k - d\mathsf{c}_{k-1} h$. $\square$

It follows for instance as we said that $\mathsf{c}_1^n = (-1)^n d^{n+1}$ and that $\mathsf{c}_1^{n-2}\mathsf{c}_2 = (-1)^{n-2} d\big(d - \frac{(n+2)!}{(n+1)!\,1!}\big)^{n-2}\big(d^2 - \frac{(n+2)!}{(n+1)!\,1!}\, d + \frac{(n+2)!}{n!\,2!}\big)$ are numerical quantities.

Following [18], one introduces the formal factorization:

$$1 + \mathsf{c}_1\, x + \mathsf{c}_2\, x^2 + \cdots + \mathsf{c}_n\, x^n = \prod_{0 \leqslant i \leqslant n} \big(1 + \mathsf{a}_i\, x\big),$$

using new formal symbols a_i whose elementary symmetric functions regive the Chern classes c_k:

$$\mathsf{c}_k = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} \mathsf{a}_{i_1}\, \mathsf{a}_{i_2} \cdots \mathsf{a}_{i_k},$$

so that any polynomial $\mathsf{P}\big(\mathsf{a}_1, \ldots, \mathsf{a}_n\big)$ in the a_i which is invariant under all permutations of its arguments may in fact be expressed in terms of the c_k. Every such a symmetric $\mathsf{P}\big(\mathsf{a}_1, \ldots, \mathsf{a}_n\big)$ which is homogeneous of degree n may thus be considered as a numerical quantity, after integration.

Proposition. ([18, 28]) *The Euler-Poincaré characteristic:*

$$\chi\Big(X,\, \Gamma^{(\ell_1, \ldots, \ell_n)}\, T_X\Big) = \sum_{i=0}^{n} (-1)^i\, \dim H^i\big(X,\, \Gamma^{(\ell_1, \ldots, \ell_n)}\, T_X\big)$$

of an arbitrary Schur bundle $\Gamma^{(\ell_1, \ell_2, \ldots, \ell_n)}\, T_X$ *with* $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_n$ *is given as* (*the integral over* X *of*) *the rewriting by means of the* c_k *of all the terms which are homogeneous of degree* n *with respect to* $\mathsf{a}_1, \ldots, \mathsf{a}_n$ *in the expansion of the* (*symmetric*) *quotient:*

$$\begin{vmatrix} e^{\mathsf{a}_1 \ell'_1} & \cdots & e^{\mathsf{a}_1 \ell'_n} \\ \vdots & \ddots & \vdots \\ e^{\mathsf{a}_n \ell'_1} & \cdots & e^{\mathsf{a}_n \ell'_n} \end{vmatrix} \Bigg/ \begin{vmatrix} e^{(n-1)\mathsf{a}_1} & \cdots & 1 \\ \vdots & \ddots & \vdots \\ e^{(n-1)\mathsf{a}_n} & \cdots & 1 \end{vmatrix},$$

in which one has abbreviated for notational condensation:

$$\ell'_1 := \ell_1 + n - 1, \qquad \ell'_2 := \ell_2 + n - 2, \ \ldots\ldots, \quad \ell'_n := \ell_n.$$

We shall admit this result. In fact, the well known Van der Monde determinant yields an approximate expression of the denominator:

$$\begin{aligned} \begin{vmatrix} e^{(n-1)\mathsf{a}_1} & \cdots & 1 \\ \vdots & \ddots & \vdots \\ e^{(n-1)\mathsf{a}_n} & \cdots & 1 \end{vmatrix} &= \prod_{1 \leqslant i < j \leqslant n} \big(e^{\mathsf{a}_i} - e^{\mathsf{a}_j}\big) \\ &= \prod_{1 \leqslant i < j \leqslant n} \big(\mathsf{a}_i - \mathsf{a}_j\big) \cdot \big[1 + R(\mathsf{a}_1, \ldots, \mathsf{a}_n)\big], \end{aligned}$$

where the remainder $R(\mathsf{a})$ denotes a local holomorphic function which vanishes at the origin. Because the determinant at the numerator also visibly vanishes whenever one a_{i_1} is equal to another a_{i_2}, for some two distinct indices i_1 and i_2, this numerator also is a multiple, as a holomorphic function, of the same product $\prod_{1 \leqslant i < j \leqslant n} \big(\mathsf{a}_i - \mathsf{a}_j\big)$. Consequently, when one expands

simultaneously the numerator and the denominator, the two products should cancel out:

$$\frac{\prod_{i<j}(\mathsf{a}_i-\mathsf{a}_j)\big[S(\mathsf{a},\,\ell')\big]}{\prod_{i<j}(\mathsf{a}_i-\mathsf{a}_j)\big[1+R(\mathsf{a})\big]}=S(\mathsf{a},\,\ell')\Big[1-R(\mathsf{a})+R(\mathsf{a})^2-R(\mathsf{a})^3+\cdots\Big]$$

and one should obtain a power series in which only the homogeneous terms of degree n in the a_i are relevant. Getting a partial *explicit* expression of the result is our next goal.

Asymptotic expansion of the Euler-Poincaré characteristic of $\Gamma^{(\ell_1,\ell_2,\ldots,\ell_n)}T_X$. A *partition* of n is any sequence:

$$\lambda=\big(\lambda_1,\,\lambda_2,\,\ldots,\,\lambda_n\big)$$

of non-negative integers listed in decreasing order:

$$\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n,$$

whose total sum equals n:

$$\lambda_1+\lambda_2+\cdots+\lambda_n=n.$$

The *diagram* of a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ in the real plane consists of λ_1 squares of length one placed above λ_2 squares of length one, *etc.*, placed above λ_n squares of length one, all horizontal series of squares being justified to the left along a fixed vertical line; some figures appear below. The *conjugate* of a partition λ is the partition $\lambda^c=(\lambda_1^c,\lambda_2^c,\ldots,\lambda_n^c)$ whose diagram is obtained from the diagram of λ by reflecting it across its main diagonal. Hence λ_i^c is the number of squares in the i-th column of λ, or equivalently $\lambda_i^c=\mathrm{Card}\big\{j:\,\lambda_j\geqslant i\big\}$.

THEOREM *The terms of highest order with respect to $|\ell|=\max_{1\leqslant i\leqslant n}\ell_i$ in the Euler-Poincaré characteristic of the Schur bundle $\Gamma^{(\ell_1,\ell_2,\ldots,\ell_n)}T_X$ are homogeneous of order $\mathrm{O}\big(|\ell|^{\frac{n(n+1)}{2}}\big)$ and they are given by a sum of ℓ_i'-determinants indexed by all the partitions $(\lambda_1,\ldots,\lambda_n)$ of n:*

$$\chi\Big(X,\,\Gamma^{(\ell_1,\ell_2,\ldots,\ell_n)}T_X\Big)=$$

$$=\sum_{\lambda\text{ partition of }n}\frac{\mathsf{C}_{\lambda^c}}{(\lambda_1+n-1)!\cdots\lambda_n!}\begin{vmatrix}{\ell_1'}^{\lambda_1+n-1} & {\ell_2'}^{\lambda_1+n-1} & \cdots & {\ell_n'}^{\lambda_1+n-1}\\ {\ell_1'}^{\lambda_2+n-2} & {\ell_2'}^{\lambda_2+n-2} & \cdots & {\ell_n'}^{\lambda_2+n-2}\\ \vdots & \vdots & \ddots & \vdots\\ {\ell_1'}^{\lambda_n} & {\ell_2'}^{\lambda_n} & \cdots & {\ell_n'}^{\lambda_n}\end{vmatrix}+$$

$$+\mathrm{O}\Big(|\ell|^{\frac{n(n+1)}{2}-1}\Big),$$

where $\ell_i':=\ell_i+n-i$ for notational brevity, with coefficients C_{λ^c} being expressed in terms of the Chern classes $\mathsf{c}_k\big(T_X\big)=\mathsf{c}_k$ of T_X by

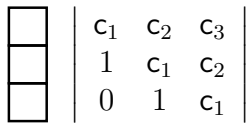
means of Giambelli's determinantal expression *depending upon the* conjugate *partition* λ^c*:*

$$\mathsf{C}_{\lambda^c} = \mathsf{C}_{(\lambda_1^c,\dots,\lambda_n^c)} = \begin{vmatrix} \mathsf{c}_{\lambda_1^c} & \mathsf{c}_{\lambda_1^c+1} & \mathsf{c}_{\lambda_1^c+2} & \cdots & \mathsf{c}_{\lambda_1^c+n-1} \\ \mathsf{c}_{\lambda_2^c-1} & \mathsf{c}_{\lambda_2^c} & \mathsf{c}_{\lambda_2^c+1} & \cdots & \mathsf{c}_{\lambda_2^c+n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathsf{c}_{\lambda_n^c-n+1} & \mathsf{c}_{\lambda_n^c-n+2} & \mathsf{c}_{\lambda_n^c-n+3} & \cdots & \mathsf{c}_{\lambda_n^c} \end{vmatrix},$$

on understanding by convention that $\mathsf{c}_k := 0$ *for* $k < 0$ *or* $k > n$*, and that* $\mathsf{c}_0 := 1$*.*

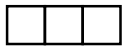
In fact, replacing the ℓ_i' by the ℓ_i everywhere in the framed formula would be harmless, because the difference between any two corresponding determinants is easily seen to be an $\mathrm{O}\big(|\ell|^{\frac{n(n+1)}{2}-1}\big)$, neglected in the remainder.

We give two expanded instances of this general formula. Firstly, in dimension $n = 3$, there are only three partitions of 3, namely $3 + 0 + 0$, $2 + 1 + 0$ and $1 + 1 + 1$, along which we draw the diagram of the conjugate partitions $1+1+1$, $2+1+1$ and $3+0+0$ together with the corresponding Giambelli determinants:



$$\begin{vmatrix} \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 \\ 1 & \mathsf{c}_1 & \mathsf{c}_2 \\ 0 & 1 & \mathsf{c}_1 \end{vmatrix}$$

$$\begin{vmatrix} \mathsf{c}_2 & \mathsf{c}_3 & 0 \\ 1 & \mathsf{c}_1 & \mathsf{c}_2 \\ 0 & 0 & 1 \end{vmatrix}$$



$$\begin{vmatrix} \mathsf{c}_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

so that we can write down in great details the leading terms, for $|\ell| \to \infty$, of the Euler-Poincaré characteristic:

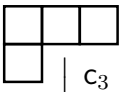
$$\begin{aligned} &\chi\big(X,\, \Gamma^{(\ell_1,\ell_2,\ell_3)}\, T_X\big) = \\ &= \frac{\mathsf{c}_1^3 - 2\,\mathsf{c}_1\mathsf{c}_2 + \mathsf{c}_3}{0!\ 1!\ 5!} \begin{vmatrix} 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 \\ \ell_1^5 & \ell_2^5 & \ell_3^5 \end{vmatrix} + \frac{\mathsf{c}_1\mathsf{c}_2 - \mathsf{c}_3}{0!\ 2!\ 4!} \begin{vmatrix} 1 & 1 & 1 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 \\ \ell_1^4 & \ell_2^4 & \ell_3^4 \end{vmatrix} + \\ &\quad + \frac{\mathsf{c}_3}{1!\ 2!\ 3!} \begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 \end{vmatrix} + \mathrm{O}\big(|\ell|^5\big). \end{aligned}$$

Secondly, in dimension $n = 4$, there are five partitions of 4, namely 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$ along which we again draw the diagram of the conjugate partition together with the corresponding Giambelli determinants:

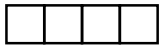
$$\begin{vmatrix} \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 & \mathsf{c}_4 \\ 1 & \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 \\ 0 & 1 & \mathsf{c}_1 & \mathsf{c}_2 \\ 0 & 0 & 1 & \mathsf{c}_1 \end{vmatrix}$$

$$\begin{vmatrix} \mathsf{c}_2 & \mathsf{c}_3 & \mathsf{c}_4 & 0 \\ 1 & \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 \\ 0 & 1 & \mathsf{c}_1 & \mathsf{c}_2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} \mathsf{c}_2 & \mathsf{c}_3 & \mathsf{c}_4 & 0 \\ \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 & \mathsf{c}_4 \\ 0 & 0 & 1 & \mathsf{c}_1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$



$$\begin{vmatrix} \mathsf{c}_3 & \mathsf{c}_4 & 0 & 0 \\ 1 & \mathsf{c}_1 & \mathsf{c}_2 & \mathsf{c}_3 \\ 0 & 0 & 1 & \mathsf{c}_1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$



$$\begin{vmatrix} \mathsf{c}_4 & 0 & 0 & 0 \\ 0 & 1 & \mathsf{c}_1 & \mathsf{c}_2 \\ 0 & 0 & 1 & \mathsf{c}_1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

so that we can write down in length the asymptotic of the Euler-Poincaré characteristic also in this case, of major interest to us:

$$\boxed{\begin{aligned} &\chi\big(X,\, \Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)}\, T_X\big) = \\ &= \frac{\mathsf{c}_1^4 - 3\,\mathsf{c}_1^2\mathsf{c}_2 + \mathsf{c}_2^2 + 2\,\mathsf{c}_1\mathsf{c}_3 - \mathsf{c}_4}{0!\ 1!\ 2!\ 7!} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^7 & \ell_2^7 & \ell_3^7 & \ell_4^7 \end{vmatrix} + \\ &+ \frac{\mathsf{c}_1^2\mathsf{c}_2 - \mathsf{c}_2^2 - \mathsf{c}_1\mathsf{c}_3 + \mathsf{c}_4}{0!\ 1!\ 3!\ 6!} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^6 & \ell_2^6 & \ell_3^6 & \ell_4^6 \end{vmatrix} + \frac{-\mathsf{c}_1\mathsf{c}_3 + \mathsf{c}_2^2}{0!\ 1!\ 4!\ 5!} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \\ \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \end{vmatrix} + \\ &+ \frac{\mathsf{c}_1\mathsf{c}_3 - \mathsf{c}_4}{0!\ 2!\ 3!\ 5!} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^5 & \ell_2^5 & \ell_3^5 & \ell_4^5 \end{vmatrix} + \frac{\mathsf{c}_4}{1!\ 2!\ 3!\ 4!} \begin{vmatrix} \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ \ell_1^2 & \ell_2^2 & \ell_3^2 & \ell_4^2 \\ \ell_1^3 & \ell_2^3 & \ell_3^3 & \ell_4^3 \\ \ell_1^4 & \ell_2^4 & \ell_3^4 & \ell_4^4 \end{vmatrix} + \mathrm{O}\big(|\ell|^9\big). \end{aligned}}$$

Proof of the general theorem. Taking the proposition for granted, we start by expanding plainly in Taylor series the exponentials of the numerator determinant:

$$\begin{aligned} \begin{vmatrix} e^{\mathsf{a}_1\ell'_1} & \cdots & e^{\mathsf{a}_1\ell'_n} \\ \vdots & \ddots & \vdots \\ e^{\mathsf{a}_n\ell'_1} & \cdots & e^{\mathsf{a}_n\ell'_n} \end{vmatrix} &= \begin{vmatrix} \sum_{\mu\geqslant 0} \frac{(\ell'_1)^\mu}{\mu!}\, \mathsf{a}_1^\mu & \cdots & \sum_{\mu\geqslant 0} \frac{(\ell'_n)^\mu}{\mu!}\, \mathsf{a}_1^\mu \\ \vdots & \ddots & \vdots \\ \sum_{\mu\geqslant 0} \frac{(\ell'_1)^\mu}{\mu!}\, \mathsf{a}_n^\mu & \cdots & \sum_{\mu\geqslant 0} \frac{(\ell'_n)^\mu}{\mu!}\, \mathsf{a}_n^\mu \end{vmatrix} \\ &= \sum_{\mu_1,\mu_2,\ldots,\mu_n\geqslant 0} \frac{(\ell'_1)^{\mu_1}}{\mu_1!} \frac{(\ell'_2)^{\mu_2}}{\mu_2!} \cdots \frac{(\ell'_n)^{\mu_n}}{\mu_n!} \begin{vmatrix} \mathsf{a}_1^{\mu_1} & \cdots & \mathsf{a}_1^{\mu_n} \\ \vdots & \ddots & \vdots \\ \mathsf{a}_n^{\mu_1} & \cdots & \mathsf{a}_n^{\mu_n} \end{vmatrix} \end{aligned}$$

and we then develope the result by multilinearity. According to what has already been noticed after the proposition, dividing this last sum by the determinant at the denominator amounts to multiplying it by $\big[1/\prod_{i<j}\big(\mathsf{a}_i-\mathsf{a}_j\big)\big]\cdot\big[1+\sum_{k\geqslant 1}(-1)^k\,R(\mathsf{a})^k\big]$, so we obtain:

$$\chi\Big(X,\,\Gamma^{(\ell_1,\ldots,\ell_n)}\,T_X\Big)=\sum_{\mu_1,\mu_2,\ldots,\mu_n\geqslant 0}\frac{(\ell_1')^{\mu_1}}{\mu_1!}\,\frac{(\ell_2')^{\mu_2}}{\mu_2!}\,\cdots\,\frac{(\ell_n')^{\mu_n}}{\mu_n!}\cdot$$

$$\cdot\,\text{homogeneous n-th part of}\bigg(\frac{1}{\prod_{i<j}\big(\mathsf{a}_i-\mathsf{a}_j\big)}\begin{vmatrix}\mathsf{a}_1^{\mu_1}&\cdots&\mathsf{a}_1^{\mu_n}\\ \vdots&\ddots&\vdots\\ \mathsf{a}_n^{\mu_1}&\cdots&\mathsf{a}_n^{\mu_n}\end{vmatrix}\Big[1+\mathrm{O}_1(\mathsf{a})\Big]\bigg),$$

where we have gathered all terms $-R(\mathsf{a})+R(\mathsf{a})^2-\cdots$ simply as a remainder $\mathrm{O}_1(\mathsf{a})$ vanishing at $\mathsf{a}=0$. The order at $\mathsf{a}=0$ of the Van der Monde denominator $\prod_{i<j}\big(\mathsf{a}_i-\mathsf{a}_j\big)$ is equal to $\frac{n(n-1)}{2}$, while the order of the determinant $\big|\mathsf{a}_i^{\mu_j}\big|$ equals $\mu_1+\cdots+\mu_n$. Consequently, when selecting in the sum $\sum_{\mu_1,\ldots,\mu_n\geqslant 0}$ only homogeneous terms of order n with respect to a, one must consider:

- all terms with $\mu_1+\cdots+\mu_n=n+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}$ if the determinant is multiplied by the term 1 inside the last brackets; with respect to the ℓ_i', this then gives terms which are homogeneous of degree $\frac{n(n+1)}{2}$;
- some appropriate terms with $\mu_1+\cdots+\mu_n<\frac{n(n+1)}{2}$ if the determinant is multiplied by some nonzero monomial belonging to the remainder $\mathrm{O}_1(\mathsf{a})$; with respect to the ℓ_i', this then gives terms in $\mathrm{O}\big(|\ell'|\big)^{\frac{n(n+1)}{2}-1}$, and we announced in the theorem that we should neglect them.

As a result, we may therefore represent as follows the principal terms of the Euler-Poincaré characteristic, considered asymptotically for $|\ell|\to\infty$:

$$\chi\Big(X,\,\Gamma^{(\ell_1,\ldots,\ell_n)}\,T_X\Big)=\sum_{\substack{\mu_1+\cdots+\mu_n=\frac{n(n+1)}{2}\\ \mu_1,\ldots,\mu_n\geqslant 0}}\frac{(\ell_1')^{\mu_1}}{\mu_1!}\,\frac{(\ell_2')^{\mu_2}}{\mu_2!}\,\cdots\,\frac{(\ell_n')^{\mu_n}}{\mu_n!}\cdot$$

$$\cdot\,\frac{1}{\prod_{i<j}\big(\mathsf{a}_i-\mathsf{a}_j\big)}\begin{vmatrix}\mathsf{a}_1^{\mu_1}&\cdots&\mathsf{a}_1^{\mu_n}\\ \vdots&\ddots&\vdots\\ \mathsf{a}_n^{\mu_1}&\cdots&\mathsf{a}_n^{\mu_n}\end{vmatrix}+\mathrm{O}\Big(|\ell'|^{\frac{n(n+1)}{2}-1}\Big).$$

Whenever there exist two equal exponents $\mu_{i_1}=\mu_{i_2}$ for two distinct indices $i_1\neq i_2$, the determinant obviously vanishes. So in the sum, one may assume the μ_i to be pairwise distinct. Furthermore, for any n-tuple $(\mu_1,\ldots,\mu_n)$ of pairwise distinct μ_i, there exists a unique permutation $\sigma\in\mathfrak{S}_n$ rearranging them in decreasing order: $\mu_{\sigma(1)}>\mu_{\sigma(2)}>\cdots>\mu_{\sigma(n)}$. Consequently, we

can split as follows the sum to be considered:

$$\chi\Big(X,\,\Gamma^{(\ell_1,\dots,\ell_n)}\,T_X\Big)=\sum_{\sigma\in\mathfrak{S}_n}\sum_{\substack{\mu_1+\cdots+\mu_n=\frac{n(n+1)}{2}\\ \mu_1>\cdots>\mu_n\geqslant 0}}\frac{(\ell'_1)^{\mu_{\sigma(1)}}}{\mu_{\sigma(1)}!}\cdots\frac{(\ell'_n)^{\mu_{\sigma(n)}}}{\mu_{\sigma(n)}!}\cdot$$
$$\cdot\,\frac{1}{\prod_{i<j}\big(\mathsf{a}_i-\mathsf{a}_j\big)}\begin{vmatrix}\mathsf{a}_1^{\mu_{\sigma(1)}} & \cdots & \mathsf{a}_1^{\mu_{\sigma(n)}}\\ \vdots & \ddots & \vdots\\ \mathsf{a}_n^{\mu_{\sigma(1)}} & \cdots & \mathsf{a}_n^{\mu_{\sigma(n)}}\end{vmatrix}+\mathrm{O}\Big(|\ell'|^{\frac{n(n+1)}{2}-1}\Big).$$

Finally, one easily convinces oneself that there is a one-to-one correspondence between the n-tuples $\mu=(\mu_1,\dots,\mu_n)$ as above with $\mu_1>\cdots>\mu_n\geqslant 0$ and $\mu_1+\cdots+\mu_n=\frac{n(n+1)}{2}$ on the one hand, and on the other hand, the partitions $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ of n, namely with $\lambda_1\geqslant\cdots\geqslant\lambda_n\geqslant 0$ and $\lambda_1+\cdots+\lambda_n=n$, a correspondence which is simply given by:

$$\mu_i\longmapsto\lambda_i:=\mu_i-n+i\quad\text{ and has obvious inverse }\quad\lambda_i\longmapsto\mu_i:=\lambda_i+n-i.$$

Taking account of the skew-symmetry $\big|\mathsf{a}_i^{\mu_{\sigma(j)}}\big|=\mathrm{sgn}(\sigma)\,\big|\mathsf{a}_i^{\mu_j}\big|$, we thus obtain an almost final asymptotic representation of the Euler-Poincaré characteristic:

$$\chi\Big(X,\,\Gamma^{(\ell_1,\dots,\ell_n)}\,T_X\Big)=\sum_{\sigma\in\mathfrak{S}_n}\sum_{\substack{\lambda_1+\cdots+\lambda_n=n\\ \lambda_1\geqslant\cdots\geqslant\lambda_n\geqslant 0}}\frac{(\ell'_{\sigma^{-1}(1)})^{\lambda_1+n-1}}{(\lambda_1+n-1)!}\cdots\frac{(\ell'_{\sigma^{-1}(n)})^{\lambda_n}}{\lambda_n!}\cdot\mathrm{sgn}(\sigma)\cdot$$
$$\cdot\,\frac{1}{\prod_{i<j}\big(\mathsf{a}_i-\mathsf{a}_j\big)}\begin{vmatrix}\mathsf{a}_1^{\lambda_1+n-1} & \cdots & \mathsf{a}_1^{\lambda_n}\\ \vdots & \ddots & \vdots\\ \mathsf{a}_n^{\lambda_1+n-1} & \cdots & \mathsf{a}_n^{\lambda_n}\end{vmatrix}+\mathrm{O}\Big(|\ell'|^{\frac{n(n+1)}{2}-1}\Big).$$

To conclude the proof of the theorem, using $\mathrm{sgn}(\sigma^{-1})=\mathrm{sgn}(\sigma)$, it now suffices only to observe the compulsory reconstitution of ℓ'-determinants:

$$\sum_{\sigma\in\mathfrak{S}_n}\mathrm{sgn}\,(\sigma^{-1})\cdot\frac{(\ell'_{\sigma^{-1}(1)})^{\lambda_1+n-1}}{(\lambda_1+n-1)!}\cdots\frac{(\ell'_{\sigma^{-1}(n)})^{\lambda_n}}{\lambda_n!}=$$
$$=\frac{1}{(\lambda_1+n-1)!\cdots\lambda_n!}\cdot\begin{vmatrix}{\ell'_1}^{\lambda_1+n-1} & \cdots & {\ell'_n}^{\lambda_1+n-1}\\ \vdots & \ddots & \vdots\\ {\ell'_1}^{\lambda_n} & \cdots & {\ell'_n}^{\lambda_n},\end{vmatrix},$$

and also to recognize the Schur polynomials:

$$S_\lambda(\mathsf{a})=S_{(\lambda_1,\dots,\lambda_n)}(\mathsf{a})=\frac{1}{\prod_{i<j}\big(\mathsf{a}_i-\mathsf{a}_j\big)}\begin{vmatrix}\mathsf{a}_1^{\lambda_1+n-1} & \mathsf{a}_1^{\lambda_2+n-2} & \cdots & \mathsf{a}_1^{\lambda_n}\\ \mathsf{a}_2^{\lambda_1+n-1} & \mathsf{a}_2^{\lambda_2+n-2} & \cdots & \mathsf{a}_2^{\lambda_n}\\ \vdots & \vdots & \ddots & \vdots\\ \mathsf{a}_n^{\lambda_1+n-1} & \mathsf{a}_n^{\lambda_2+n-2} & \cdots & \mathsf{a}_n^{\lambda_n}\end{vmatrix}$$

indexed by the partitions of n, which according to Giambelli's formulas (Appendix A of [16]), are expressed in terms of the elementary symmetric functions $\mathsf{c}_k = \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \mathsf{a}_{i_1}\cdots\mathsf{a}_{i_k}$ of the a_i by means of the specific determinants written and exemplified above. Thus, the proof is achieved. $\square$

Computation of the Euler-Poincaré characteristic of $\mathsf{E}^4_{4,m}T^*_X$. As is known, duality shows that the cotangent bundle T^*_X has Chern classes $\mathsf{c}_k\big(T^*_X\big)$ related to those of T_X by the relations:

$$\mathsf{c}^*_k := \mathsf{c}_k\big(T^*_X\big) = (-1)^k\,\mathsf{c}_k\big(T_X\big) = (-1)^k\,\mathsf{c}_k.$$

Consequently, the dual Giambelli determinants satisfy $\mathsf{C}^*_{\lambda^c} = (-1)^n\mathsf{C}_{\lambda^c}$, because all monomials $\mathsf{c}^*_{\mu_1}\cdots\mathsf{c}^*_{\mu_n}$ have total weight $\mu_1+\cdots+\mu_n = n$ and we therefore deduce:

$$\chi\big(X,\,\Gamma^{(\ell_1,\ldots,\ell_n)}T^*_X\big) = (-1)^n\,\chi\big(X,\,\Gamma^{(\ell_1,\ldots,\ell_n)}T_X\big).$$

When considering Demailly-Semple and Schur bundles, everything shall be expressed in terms of Chern classes of T_X (not of T^*_X).

§14. Euler characteristic calculations

Explaining the final calculations on an example. We may now come back to our 24 sums of Schur bundles (with multiplicities). Consider for instance the family A. In it, we have:

$$\begin{cases} \ell_1 = o + 4j + 3k + 3l + 4m' + 5n + p,\\ \ell_2 = 2j + 2k + 3l + 3m' + 3n + p,\\ \ell_3 = j + 2k + 2l + 2m' + 2n + p,\\ \ell_4 = p. \end{cases}$$

But since sums of weight should be equal to m:

$$o + 14j + 15k + 17l + 19m' + 21n + 10p = m,$$

we may eliminate o and this provides ℓ_1 with the value:

$$\ell_1 = m - 10j - 12k - 14l - 15m' - 16n - 9p,$$

while ℓ_2, ℓ_3 and ℓ_4 where at the beginning independent of o. The Euler-Poincaré characteristic being additive, we have:

$$\chi\Big(X,\,\oplus_{\mathsf{A}}\,\Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)}T^*_X\Big) = \sum_{o+14j+15k+17l+19m'+21n+10p=m} \chi\big(X,\,\Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)}T^*_X\big).$$

Furthermore, according to the formula written on p. 278, the dominant term of the Euler-Poincaré characteristic, as $|\ell|\to\infty$, of a single Schur bundle in

such a sum is given, in terms of the Chern classes c_k of T_X, by:

$$\begin{aligned}\chi\big(X,\,\Gamma^{(\ell_1,\ell_2,\ell_3,\ell_4)}T_X^*\big) &= \frac{\mathsf{c}_1^4-3\,\mathsf{c}_1^2\mathsf{c}_2+\mathsf{c}_2^2+2\,\mathsf{c}_1\mathsf{c}_3-\mathsf{c}_4}{0!\,1!\,2!\,7!}\,\Delta_{0127}+\\ &+\frac{\mathsf{c}_1^2\mathsf{c}_2-\mathsf{c}_2^2-\mathsf{c}_1\mathsf{c}_3+\mathsf{c}_4}{0!\,1!\,3!\,6!}\,\Delta_{0136}+\frac{-\mathsf{c}_1\mathsf{c}_3+\mathsf{c}_2^2}{0!\,1!\,4!\,5!}\,\Delta_{0145}+\\ &+\frac{\mathsf{c}_1\mathsf{c}_3-\mathsf{c}_4}{0!\,2!\,3!\,5!}\,\Delta_{0235}+\frac{\mathsf{c}_4}{1!\,2!\,3!\,4!}\,\Delta_{1234}\\ &+\mathrm{O}\big(|\ell|^9\big),\end{aligned}$$

on understanding that, in the five determinants:

$$\Delta_{0137}:=\begin{vmatrix}1&1&1&1\\ \ell_1&\ell_2&\ell_3&\ell_4\\ \ell_1^2&\ell_2^2&\ell_3^2&\ell_4^2\\ \ell_1^7&\ell_2^7&\ell_3^7&\ell_4^7\end{vmatrix},\qquad \Delta_{0136}:=\begin{vmatrix}1&1&1&1\\ \ell_1&\ell_2&\ell_3&\ell_4\\ \ell_1^3&\ell_2^3&\ell_3^3&\ell_4^3\\ \ell_1^6&\ell_2^6&\ell_3^6&\ell_4^6\end{vmatrix},$$

$$\Delta_{0145}:=\begin{vmatrix}1&1&1&1\\ \ell_1&\ell_2&\ell_3&\ell_4\\ \ell_1^4&\ell_2^4&\ell_3^4&\ell_4^4\\ \ell_1^5&\ell_2^5&\ell_3^5&\ell_4^5\end{vmatrix},\qquad \Delta_{0235}:=\begin{vmatrix}1&1&1&1\\ \ell_1^2&\ell_2^2&\ell_3^2&\ell_4^2\\ \ell_1^3&\ell_2^3&\ell_3^3&\ell_4^3\\ \ell_1^5&\ell_2^5&\ell_3^5&\ell_4^5\end{vmatrix},$$

$$\Delta_{1234}:=\begin{vmatrix}\ell_1&\ell_2&\ell_3&\ell_4\\ \ell_1^2&\ell_2^2&\ell_3^2&\ell_4^2\\ \ell_1^3&\ell_2^3&\ell_3^3&\ell_4^3\\ \ell_1^4&\ell_2^4&\ell_3^4&\ell_4^4\end{vmatrix},$$

one should substitute the above values for ℓ_1, ℓ_2, ℓ_3 and ℓ_4 in terms of j, k, l, m', n and p.

On the other hand, it is well known that the dominant term of a multiple sum is given by an integral, so that we have to compute[58]:

$$\int_0^{\frac{(m-15k-17l-19m'-21n-10p)}{14}} dj \int_0^{\frac{(m-17l-19m'-21n-10p)}{15}} dk \int_0^{\frac{(m-19m'-21n-10p)}{17}} dl$$

$$\int_0^{\frac{(m-21n-10p)}{19}} dm' \int_0^{\frac{(m-10p)}{21}} dn \int_0^{\frac{m}{10}} dp \begin{cases}\Delta_{0127}\\ \Delta_{0136}\\ \Delta_{0145}\\ \Delta_{0235}\\ \Delta_{1234}\end{cases}$$

[58] The Δ determinants being of degree 10 in the ℓ_i, the presence of six integrals entails that the result is m^{16} times a fractional constant plus an $\mathrm{O}\big(m^{15}\big)$. If there would be 5 or less integrals, this would leave us with an $\mathrm{O}\big(m^{15}\big)$, negligible in comparison with m^{16} as $m\to\infty$. By this remark we therefore justify why we considered only the approximate Schur bundle decomposition of $\mathsf{E}_{4,m}^4T_X^*$ in the §12.

It happens that all the five integrals are equal to m^{16} times a fractional number. A computation with the help of Maple yields the values of these five fractional numbers, which, we guess, would be quite uneasy to get by hand:

$$\mathsf{A}_{0127} = \frac{1574237547668636514821100639396316177136142 67}{747013044054984907099576266082278168554541241872000000000000},$$

$$\mathsf{A}_{0136} = \frac{28522461125390254458949101163845780853731504 7}{348606087225659623313135590838396478658785912873600000000000000}$$

$$\mathsf{A}_{0145} = \frac{10306128852122999807705628256770676631371801}{5229091308348943496970338625759471798817886931040000000000000}$$

$$\mathsf{A}_{0235} = \frac{20975222336265133050996115522925065371392 47}{2376859685629497431680469937534521445400813042320000000000000}$$

$$\mathsf{A}_{1234} = \frac{20051359515371820286197508247902844353}{2102485347748339169995992868230447983547822240000000000000}$$

End of the computation. Similarly, for the other 23 families, we compute these 5-tuples of rational numbers and at the end, we make the addition[59]:

$$\begin{aligned}\mathsf{Coeff}_{0127} &= 2\,\mathsf{A}_{0127} + \mathsf{B}_{0127} + \mathsf{C}_{0127} + 4\,\mathsf{D}_{0127} + 2\,\mathsf{E}_{0127} + 2\,\mathsf{F}_{0127} + \mathsf{G}_{0127} + \mathsf{H}_{0127} + \\ &\quad + \mathsf{I}_{0127} + \mathsf{J}_{0127} + \mathsf{K}_{0127} + \mathsf{L}_{0127} + \mathsf{M}_{0127} + \mathsf{N}_{0127} + \mathsf{O}_{0127} + \mathsf{P}_{0127} + \\ &\quad + \mathsf{Q}_{0127} + \mathsf{R}_{0127} + \mathsf{S}_{0127} + \mathsf{T}_{0127} + \mathsf{U}_{0127} + \mathsf{V}_{0127} + \mathsf{W}_{0127} + \mathsf{X}_{0127} \\ &= \frac{2127566277536547206644157}{6514473374523285382987776000000000000},\end{aligned}$$

$$\begin{aligned}\mathsf{Coeff}_{0136} &= 2\,\mathsf{A}_{0136} + \mathsf{B}_{0136} + \mathsf{C}_{0136} + 4\,\mathsf{D}_{0136} + 2\,\mathsf{E}_{0136} + 2\,\mathsf{F}_{0136} + \mathsf{G}_{0136} + \mathsf{H}_{0136} + \\ &\quad + \mathsf{I}_{0136} + \mathsf{J}_{0136} + \mathsf{K}_{0136} + \mathsf{L}_{0136} + \mathsf{M}_{0136} + \mathsf{N}_{0136} + \mathsf{O}_{0136} + \mathsf{P}_{0136} + \\ &\quad + \mathsf{Q}_{0136} + \mathsf{R}_{0136} + \mathsf{S}_{0136} + \mathsf{T}_{0136} + \mathsf{U}_{0136} + \mathsf{V}_{0136} + \mathsf{W}_{0136} + \mathsf{X}_{0136} \\ &= \frac{5267640708714311654799 7}{4053450099703377571636838400000000000},\end{aligned}$$

$$\begin{aligned}\mathsf{Coeff}_{0145} &= 2\,\mathsf{A}_{0145} + \mathsf{B}_{0145} + \mathsf{C}_{0145} + 4\,\mathsf{D}_{0145} + 2\,\mathsf{E}_{0145} + 2\,\mathsf{F}_{0145} + \mathsf{G}_{0145} + \mathsf{H}_{0145} + \\ &\quad + \mathsf{I}_{0145} + \mathsf{J}_{0145} + \mathsf{K}_{0145} + \mathsf{L}_{0145} + \mathsf{M}_{0145} + \mathsf{N}_{0145} + \mathsf{O}_{0145} + \mathsf{P}_{0145} + \\ &\quad + \mathsf{Q}_{0145} + \mathsf{R}_{0145} + \mathsf{S}_{0145} + \mathsf{T}_{0145} + \mathsf{U}_{0145} + \mathsf{V}_{0145} + \mathsf{W}_{0145} + \mathsf{X}_{0145} \\ &= \frac{1646852821245426649460 51}{5066812624629221964546048000000000000},\end{aligned}$$

$$\begin{aligned}\mathsf{Coeff}_{0235} &= 2\,\mathsf{A}_{0235} + \mathsf{B}_{0235} + \mathsf{C}_{0235} + 4\,\mathsf{D}_{0235} + 2\,\mathsf{E}_{0235} + 2\,\mathsf{F}_{0235} + \mathsf{G}_{0235} + \mathsf{H}_{0235} + \\ &\quad + \mathsf{I}_{0235} + \mathsf{J}_{0235} + \mathsf{K}_{0235} + \mathsf{L}_{0235} + \mathsf{M}_{0235} + \mathsf{N}_{0235} + \mathsf{O}_{0235} + \mathsf{P}_{0235} + \mathsf{Q}_{0235} + \\ &\quad + \mathsf{R}_{0235} + \mathsf{S}_{0235} + \mathsf{T}_{0235} + \mathsf{U}_{0235} + \mathsf{V}_{0235} + \mathsf{W}_{0235} + \mathsf{X}_{0235} \\ &= \frac{1222982407435661052177 37}{1140032840541574942022860800000000000},\end{aligned}$$

[59] *See* new-riemann-roch-4-4.mws at [23].

$$\begin{aligned}\mathsf{Coeff}_{1234} &= 2\,\mathsf{A}_{1234} + \mathsf{B}_{1234} + \mathsf{C}_{1234} + 4\,\mathsf{D}_{1234} + 2\,\mathsf{E}_{1234} + 2\,\mathsf{F}_{1234} + \mathsf{G}_{1234} + \mathsf{H}_{1234} + \\ &\quad + \mathsf{I}_{1234} + \mathsf{J}_{1234} + \mathsf{K}_{1234} + \mathsf{L}_{1234} + \mathsf{M}_{1234} + \mathsf{N}_{1234} + \mathsf{O}_{1234} + \mathsf{P}_{1234} + \mathsf{Q}_{1234} + \\ &\quad + \mathsf{R}_{1234} + \mathsf{S}_{1234} + \mathsf{T}_{1234} + \mathsf{U}_{1234} + \mathsf{V}_{1234} + \mathsf{W}_{1234} + \mathsf{X}_{1234} \\ &= \frac{1429957461022772407321}{1302894674904657076597555200000000000000}.\end{aligned}$$

Coming back to the Euler-Poincaré characteristic we therefore get:

$$\begin{aligned}\chi\big(X,\, \mathsf{E}^4_{4,m}T^*_X\big) &= \frac{\mathsf{c}_1^4 - 3\,\mathsf{c}_1^2\mathsf{c}_2 + \mathsf{c}_2^2 + 2\,\mathsf{c}_1\mathsf{c}_3 - \mathsf{c}_4}{0!\,1!\,2!\,7!}\,\mathsf{Coeff}_{0127} + \\ &\quad + \frac{\mathsf{c}_1^2\mathsf{c}_2 - \mathsf{c}_2^2 - \mathsf{c}_1\mathsf{c}_3 + \mathsf{c}_4}{0!\,1!\,3!\,6!}\,\mathsf{Coeff}_{0136} + \frac{-\mathsf{c}_1\mathsf{c}_3 + \mathsf{c}_2^2}{0!\,1!\,4!\,5!}\,\mathsf{Coeff}_{0145} + \\ &\quad + \frac{\mathsf{c}_1\mathsf{c}_3 - \mathsf{c}_4}{0!\,2!\,3!\,5!}\,\mathsf{Coeff}_{0235} + \frac{\mathsf{c}_4}{1!\,2!\,3!\,4!}\,\mathsf{Coeff}_{1234} + \mathrm{O}\big(m^{15}\big) \\ &= m^{16}\bigg(\frac{2127566277536547206644157}{6566589161519471666051678208000000000000000}\,\mathsf{c}_1^4 - \\ &\qquad - \frac{13991535132831030950420 9}{2084631479847451322556088320000000000000000}\,\mathsf{c}_1^2\mathsf{c}_2 + \\ &\qquad + \frac{18230301659778006701051}{1340120237044790135928913920000000000000000}\,\mathsf{c}_2^2 + \\ &\qquad + \frac{4055752965438099942704 29}{1313317832303894333210335641600000000000000}\,\mathsf{c}_1\mathsf{c}_3 - \\ &\qquad - \frac{6163697191750462398371}{6566589161519471666051678208000000000000000}\,\mathsf{c}_4\bigg) + \\ &\quad + \mathrm{O}\big(m^{15}\big).\end{aligned}$$

In terms of the degree:

$$\boxed{\begin{aligned}\chi\big(X,\, \mathsf{E}^4_{4,m}T^*_X\big) &= \frac{m^{16}}{1313317832303894333210335641600000000000000} \cdot d \cdot \\ &\quad \cdot \big(5004851113579703425623 5\, d^4 - \\ &\qquad - 617060662250595525598878 6\, d^3 - \\ &\qquad - 928886901354141153880624704\, d + \\ &\qquad + 1411704752502476621473639 41\, d^2 + \\ &\qquad + 1624908955061039283976041114\big) \\ &\quad + \mathrm{O}\big(m^{15}\big).\end{aligned}}$$

The four roots of the 4-th degree numerator in parentheses are:

$$2.794353346\cdots, \qquad 6.784939538\cdots, \qquad 17.86618823\cdots, \qquad 95.84703014\cdots,$$

hence in conclusion, the characteristic is positive for all degrees $d \geqslant 96$.

Jets of order $\kappa = 4$ in dimension $n = 3$. For a hypersurface $X^3 \subset \mathbb{P}_4(\mathbb{C})$ of degree d, thanks to a similar but quicker Maple computation[60], one obtains the asymptotic:

$$\begin{aligned}\chi\big(X,\, \mathsf{E}^3_{4,m}T^*_X\big) = m^{11}\bigg(&- \frac{78181453985171}{20130233500548864000000000}\,\mathsf{c}_1^3 + \\ &+ \frac{3780346214152789}{3435559850760339456000000000}\,\mathsf{c}_3 - \\ &- \frac{46223512567695359}{10306679552281018368000000000}\,\mathsf{c}_1\mathsf{c}_2\bigg) + \\ &+ \mathrm{O}\big(m^{10}\big),\end{aligned}$$

and then in terms of the degree d of the hypersurface X:

$$\boxed{\begin{aligned}\chi\big(X,\, \mathsf{E}^3_{4,m}T^*_X\big) = \frac{m^{11}}{20613359104562036736000000000}\cdot d\cdot \\ \cdot\big(1029286103034112\,d^3 - 38980726828290305\,d^2 + \\ + 299551055917162501\,d - 561169562618151944\big)\end{aligned}}\,.$$

The three roots of the third degree numerator in parentheses are:

$$2.852373090\cdots, \qquad 6.765004304\cdots, \qquad 28.25423742,$$

hence in conclusion, the Euler-Poincaré characteristic of $\mathsf{E}^3_{4,m}T^*_X$ is positive in all degrees $d \geqslant 29$ as $m \to \infty$. This condition improves the condition $d \geqslant 43$ obtained in [29] for the positivity of $\chi\big(X,\, \mathsf{E}^3_{3,m}T^*_X\big)$ as $m \to \infty$.

Existence of sections. Finally, in order to get positivity of the dimension h^0 of the vector space of sections of $\mathsf{E}^3_{4,m}T^*_X$, it would suffice, in the trivial minoration:

$$h^0\big(X,\, \mathsf{E}^3_{4,m}T^*_X\big) \geqslant \chi\big(X,\, \mathsf{E}^3_{4,m}T^*_X\big) - h^2\big(X,\, \mathsf{E}^3_{4,m}T^*_X\big),$$

stemming from the definition $\chi = h^0 - h^1 + h^2 - h^3$, to possess a good majoration of h^2. This main task is achieved in [30, 32]: for each Schur bundle, one has:

$$h^2\big(X,\, \Gamma^{(\ell_1,\ell_2,\ell_3)}T^*_X\big) \leqslant d(d{+}13)\,\frac{3(\ell_1+\ell_2+\ell_3)^3}{2}\,(\ell_1{-}\ell_2)(\ell_1{-}\ell_3)(\ell_2{-}\ell_3) + \mathrm{O}\big(|\ell|^5\big).$$

When summing up our 24 sums of Schur bundles (with multiplicities), a Maple computation provides:

$$h^2\big(X,\, \Gamma^{(\ell_1,\ell_2,\ell_3)}T^*_X\big) \leqslant d(d{+}13)\,\frac{342988705758851}{29822568148961280000000}\,m^{11} + \mathrm{O}\big(m^{10}\big).$$

[60] *See* new-riemann-roch-3-4.mws at [23].

Finally, one sees that χ minus this upper bound for h^2 is positive, for $m \to \infty$, in all degrees $d \geqslant 72$. This last condition on the degre insuring the existence of invariant jet differentials improves the condition $d \geqslant 97$ obtained in [30] and appears to be slightly better than the condition $d \geqslant 74$ obtained recently in [9].

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Jets de Demailly-Semple d'ordres 4 et 5 en dimension 2

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Table des matières

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> I do not mean to suggest that all mathematical relations can be perceived directly as obvious if they are visualised in the right way — or merely that they can always be perceived in some other way that is immediate to our intuitions. Far from it. Some mathematical relations require long chains of reasoning before they can be perceived with certainty. But the object of mathematical proof is, in effect, to provide such chains of reasoning where each step is indeed something that can be perceived as obvious. Consequently, the endpoint of the reasoning is something that must be accepted as true, even though it may not, in itself, be at all obvious. Sir Roger PENROSE, *Shadows of the Mind*, Oxford, 1994.

§1. INTRODUCTION

En dimension $\nu \leqslant 3$, la description de l'algèbre $\mathscr{DS}_\nu^\kappa$ des polynômes invariants de Demailly-Semple n'est explicitée dans la littérature que pour les jets d'ordre $\kappa \leqslant 3$ ([2, 5]) ; en dimension $\nu = 2$ et à l'ordre $\kappa = 3$, on sait que l'algèbre $\mathscr{DS}_2^3$ est engendrée par 5 polynômes fondamentaux liés entre eux par une unique syzygie ; en dimension 2 et à l'ordre 4, d'après un travail non publié de Demailly (*cf. e.g.* [3]), $\mathscr{DS}_2^4$ est engendrée par 9 polynômes invariants fondamentaux. Dans cet article, nous établissons ce résultat, nous explicitons les syzygies fondamentales entre ces 9 invariants — qui sont aussi au nombre de 9 —, nous effectuons un calcul de

Riemann-Roch pour estimer la caractéristique d'Euler du fibré correspondant, et nous en déduisons que toute courbe holomorphe entière à valeurs dans une surface projective algébrique complexe lisse $X^2 \subset P_3(\mathbb{C})$ (très) générique de degré $d \geqslant 9$ satisfait des équations algébriques globales non triviales d'ordre 4.

Pour $\kappa \geqslant 5$, la structure algébrique de $\mathscr{D}\mathscr{S}_2^\kappa$ explose en complexité. Nous exposons un procédé récursif : le *crochet entre deux invariants*, qui semble permettre (*cf.* [3]), en toute dimension et pour les jets d'ordre quelconque, d'engendrer un système fondamental de polynômes invariants, et aussi de trois autres procédés récursifs : *identités de Jacobi*, *identités plückeriennes d'ordre un*, et *identités plückeriennes d'ordre deux*, qui décrivent exhaustivement le gigantesque idéal des relations entre les invariants ainsi construits.

Ces quatre procédés (confirmés sur les cas connus) font apparaître de manière saillante une explosion symbolique incontrôlable. Par exemple, pour le cas $\nu = 2$ et $\kappa = 5$ étudié complètement ici, on reçoit 36 invariants bruts dont 12 exactement sont redondants, et on doit tenir compte de 210 syzygies non redondantes de degré $\leqslant 4$ entre ces invariants, lesquelles se déploient sur 13 pages manuscrites[61]. En ne considérant que les invariants stables par l'action d'un certain sous-groupe unipotent de $\mathsf{GL}_2(\mathbb{C})$, le nombre de syzygies fondamentales se réduit à 15, ce qui permet d'effectuer un calcul de Riemann-Roch au niveau $\kappa = 5$.

Si l'on s'en tenait seulement aux estimations ainsi obtenues pour la caractéristique d'Euler du fibré de Demailly-Semple $\mathscr{D}\mathscr{S}_{2,m}^\kappa T_X^*$, on pourrait penser qu'il faudrait connaître sa décomposition de Schur pour des niveaux κ au moins $\geqslant 20$, eu égard à la difficile conjecture d'hyperbolicité de Kobayashi concernant les surfaces complexes de degré $d \geqslant 5$ dans $P_3(\mathbb{C})$ qui sont (très) génériques ([2, 5, 3]). Mais l'avenir dira si cette approche ne devrait pas être amendée et réorientée dès le niveau $\kappa = 5$, en tenant compte de la structure spécifique de $\mathscr{D}\mathscr{S}_2^5$.

Nos résultats principaux apparaissent dans les Sections 4, 5, 6, 7 et 8.

§2. Polynômes invariants et différentiation composée

Notations initiales. En dimension $\nu \geqslant 2$, le jet strict d'ordre $\kappa \geqslant 1$ en un point fixé d'une application holomorphe locale $f = (f_1, f_2, \ldots, f_\nu)$ de $\mathbb{C}$ à valeurs dans $\mathbb{C}^\nu$ sera noté :

$$j^\kappa f := \big(f_1', \ldots, f_\nu', f_1'', \ldots, f_\nu'', \ldots\ldots, f_1^{(\kappa)}, \ldots, f_\nu^{(\kappa)}\big).$$

[61] Pour les jets d'ordre $\kappa = 6$, nous ne nous sommes pas risqué à calculer les 325 invariants attendus, ni à entreprendre d'écrire les 14 950 syzygies de degré $\leqslant 5$ qui nous sont données automatiquement.

Polynômes invariants par reparamétrisation. Pour $\kappa \geqslant 1$ entier, on recherche les polynômes $\mathsf{P} = \mathsf{P}(j^\kappa f)$ tels que :

$$\mathsf{P}\big(j^\kappa(f \circ \phi)\big) = (\phi')^m \, P\big((j^\kappa f) \circ \phi\big),$$

pour tout biholomorphisme local $\phi : U \to \phi(U)$, où $U \subset \mathbb{C}$ est ouvert, et où $m \geqslant 1$ est un entier que nous appellerons *poids* de P. On note $\mathscr{DS}^\kappa_{\nu,m}$ l'espace vectoriel constitué par ces *polynômes invariants par reparamétrisation* ([2]). La réunion $\mathscr{DS}^\kappa_\nu := \bigoplus_{m \geqslant 1} \mathscr{DS}^\kappa_{\nu,m}$ forme une algèbre graduée : $\mathscr{DS}^\kappa_{\nu,m_1} \cdot \mathscr{DS}^\kappa_{\nu,m_2} \subset \mathscr{DS}^\kappa_{\nu,m_1+m_2}$. On travaillera toujours dans une fibre en un point $z \in U$ qui n'apparaîtra pas dans les notations.

Lorsque $\kappa = 1$, les composantes f'_i $(i = 1, \ldots, \nu)$ du jet d'ordre un satisfont

$$\big(f_i \circ \phi\big)' = \phi' \, f'_i,$$

et par conséquent, tout polynôme $\mathsf{P} = \mathsf{P}\big(f'_1, \ldots, f'_\nu\big)$ qui ne dépend que du jet d'ordre 1 est invariant par reparamétrisation.

Différentiation composée jusqu'à l'ordre 5. En posant $g_i := f_i \circ \phi$ pour $i = 1, \ldots, \nu$, on calcule :

$$\begin{aligned}
g'_i &= \phi' f'_i, \\
g''_i &= \phi'' f'_i + \phi'^2 f''_i, \\
g'''_i &= \phi''' f'_i + 3\,\phi''\phi' f''_i + \phi'^3 f'''_i, \\
g''''_i &= \phi'''' f'_i + 4\,\phi'''\phi' f''_i + 3\,\phi''^2 f''_i + 6\,\phi''\phi'^2 f'''_i + \phi'^4 f''''_i, \\
g'''''_i &:= \phi''''' f'_i + 5\,\phi''''\phi' f''_i + 10\,\phi'''\phi'' f''_i + 15\,\phi''^2\phi' f'''_i + \\
&\qquad + 10\,\phi'''\phi'^2 f'''_i + 10\,\phi''\phi'^3 f''''_i + \phi'^5 f'''''_i.
\end{aligned}$$

Moralement, on recherche tous les polynômes possibles $\mathsf{P} = \mathsf{P}\big(j^5 g\big)$ qui, lorsqu'on remplace g'_i, g''_i, g'''_i, g''''_i et g'''''_i par ces valeurs, ont la vertu de faire disparaître toutes les dérivées intempestives ϕ'', ϕ''', ϕ'''' et ϕ''''' d'ordre $\geqslant 2$ de ϕ, de telle sorte que $\mathsf{P}\big(j^5 g\big) = \phi'^m \mathsf{P}\big(j^5 f\big)$ pour un $m \in \mathbb{N}$. Ce calcul d'élimination heuristique fonctionne pour $\kappa = 2$ et $\kappa = 3$ en dimensions $\nu = 2$ et $\nu = 3$, mais il se complexifie au-delà et nous ne poursuivrons pas la recherche en prenant cette optique.

Donnons toutefois la formule générale de dérivation composée, dite de *Faà di Bruno*, bien connue dans le cas classique d'une seule variable $z \in \mathbb{C}$.

Théorème. Pour tout entier $\kappa \geqslant 1$, la dérivée d'ordre κ de chaque fonction composée $g_i(z) := f_i \circ \phi(z)$ $(1 \leqslant i \leqslant \nu)$ par rapport à la variable $z \in \mathbb{C}$ s'exprime comme polynôme à coefficients entiers en les dérivées de f_i et de

ϕ :

$$\boxed{\begin{aligned} g_i^{(\kappa)} = \sum_{d=1}^{\kappa} \sum_{1\leqslant\lambda_1<\cdots<\lambda_d\leqslant\kappa} \sum_{\mu_1\geqslant 1,\ldots,\mu_d\geqslant 1} \sum_{\mu_1\lambda_1+\cdots+\mu_d\lambda_d=\kappa} \\ \frac{\kappa!}{(\lambda_1!)^{\mu_1}\,\mu_1!\cdots(\lambda_d!)^{\mu_d}\,\mu_d!}\,\big(\phi^{(\lambda_1)}\big)^{\mu_1}\cdots\cdots\big(\phi^{(\lambda_d)}\big)^{\mu_d}\,f_i^{(\mu_1+\cdots+\mu_d)}. \end{aligned}}$$

Dans tout ce qui va suivre, par souci de simplicité, nous nous restreindrons dorénavant à la dimension $\nu = 2$; des généralisations en dimension supérieure apparaîtront en temps voulu.

§3. Déterminants 2×2 et jets d'ordres 3 et 4

Wronskien et généralisations. En examinant g_1', g_2', g_1'' et g_2'', on constate l'invariance par reparamétrisation du *wronskien*, défini comme étant le déterminant 2×2 :

$$\Delta^{1,2} := \begin{vmatrix} f_1' & f_2' \\ f_1'' & f_2'' \end{vmatrix},$$

et ce, grâce au calcul élémentaire suivant :

$$\begin{aligned} \square^{1,2} := \begin{vmatrix} g_1' & g_2' \\ g_1'' & g_2'' \end{vmatrix} &= \begin{vmatrix} \phi' f_1' & \phi' f_2' \\ \phi'' f_1' + \phi'^2 f_1'' & \phi'' f_2' + \phi'^2 f_2'' \end{vmatrix} \\ &= \begin{vmatrix} \phi' f_1' & \phi' f_2' \\ \phi'^2 f_1'' & \phi'^2 f_2'' \end{vmatrix} \\ &= \big(\phi'\big)^3 \Delta^{1,2}. \end{aligned}$$

Son poids m est égal à 3. Ensuite, en éliminant de manière analogue ϕ''' et ϕ'' parmi les six équations donnant g_1', g_2', g_1'', g_2'', g_1''' et g_2''' — ou bien en procédant d'une manière alternative —, on trouve les deux invariants de poids $m = 5$:

$$\begin{aligned} [g_1' g_2''' - g_1''' g_2'] g_1' - 3\,[g_1' g_2'' - g_1'' g_2'] g_1'' &= \big(\phi'\big)^5 [f_1' f_2''' - f_1''' f_2'] f_1' - 3\,[f_1' f_2'' - f_1'' f_2'] f_1'', \\ [g_1' g_2''' - g_1''' g_2'] g_2' - 3\,[g_1' g_2'' - g_1'' g_2'] g_2'' &= \big(\phi'\big)^5 [f_1' f_2''' - f_1''' f_2'] f_2' - 3\,[f_1' f_2'' - f_1'' f_2'] f_2''. \end{aligned}$$

Déterminants 2×2 généralisant le wronskien. Il est commode de réécrire ces deux invariants de poids $m = 5$ sous une forme contractée en introduisant la notation :

$$\Delta^{\alpha,\beta} := \begin{vmatrix} f_1^{(\alpha)} & f_2^{(\alpha)} \\ f_1^{(\beta)} & f_2^{(\beta)} \end{vmatrix},$$

pour tous entiers $\alpha, \beta \geqslant 1$, ce qui donne, pour $k = 1, 2$:

$$\square^{1,3}\, g_k' - 3\,\square^{1,2}\, g_k'' = \big(\phi'\big)^5 \Big[\Delta^{1,3}\, f_k' - 3\,\Delta^{1,2}\, f_k''\Big].$$

Notons au passage la formule de dérivation bien connue qui sera utile par la suite :

$$\big[\Delta^{\alpha,\beta}\big]' = \begin{vmatrix} f_1^{(\alpha+1)} & f_2^{(\alpha)} \\ f_1^{(\beta+1)} & f_2^{(\beta)} \end{vmatrix} + \begin{vmatrix} f_1^{(\alpha)} & f_2^{(\alpha+1)} \\ f_1^{(\beta)} & f_2^{(\beta+1)} \end{vmatrix}.$$

Lemme . ([5]) Le degré de transcendance du corps engendré par les 5 polynômes invariants :

$$\boxed{\begin{array}{l} f_1', \quad f_2', \\ \Lambda^3 := \Delta^{1,2}, \\ \Lambda_1^5 := \Delta^{1,3}\, f_1' - 3\,\Delta^{1,2}\, f_1'' \qquad \textit{et} \qquad \Lambda_2^5 := \Delta^{1,3}\, f_2' - 3\,\Delta^{1,2}\, f_2'' \end{array}},$$

au-dessus de $\mathbb{C}\big[f_1', f_1'', f_1''', f_2', f_2'', f_2'''\big]$ est égal à 4, et pour préciser, les quatre polynômes f_1', f_2', Λ_1^5 et Λ_2^5 sont algébriquement indépendants, tandis que Λ^3 est quadratique sur $\mathbb{C}\big[f_1', f_2', \Lambda_1^5, \Lambda_2^5\big]$ via la relation algébrique immédiatement vérifiable :

$$0 \equiv f_2'\,\Lambda_1^5 - f_1'\,\Lambda_2^5 - 3\,\Lambda^3\,\Lambda^3,$$

et de plus, l'idéal des relations entre $f_1', f_2', \Lambda^3, \Lambda_1^5, \Lambda_2^5$ est principal et se réduit à cette unique relation.

Grâce à ladite syzygie, on peut éliminer toutes les puissances de Λ^3 supérieures ou égales à 2 qui apparaissent dans un polynôme général :

$$\mathscr{P}\big(f_1', f_2', \Lambda^3, \Lambda_1^5, \Lambda_2^5\big)$$

exprimé en fonction de ces cinq polynômes, et il ne reste alors que des puissances de Λ^3 égales à 0 ou à 1. C'est un fait remarquable que ces cinq polynômes forment un système générateurs, comme l'énonce précisément le résultat suivant.

Théorème. ([5, 3]) En dimension $\nu = 2$ et au niveau $\kappa = 3$, tout polynôme $\mathsf{P}\big(j^3 f\big)$ invariant par reparamétrisation s'écrit de manière unique :

$$\mathsf{P}\big(j^3 f\big) = \mathscr{P}\big(f_1', f_2', \Lambda_1^5, \Lambda_2^5\big) + \Lambda^3\,\mathscr{Q}\big(f_1', f_2', \Lambda_1^5, \Lambda_2^5\big),$$

avec des polynômes quelconques $\mathscr{P}$ et $\mathscr{Q}$.

Travaux de calcul pour passer aux jets d'ordre $\kappa = 4$ et $\kappa = 5$.

- Trouver un système de polynômes invariants fondamentaux.
- Connaître leur idéal des relations.
- Trouver une écriture unique de tout polynôme en les polynômes invariants fondamentaux.

Deux opérateurs de différentiation. Comment engendrer méthodiquement une liste appropriée de polynômes invariants fondamentaux pour les jets

d'ordre $\kappa = 4$ ou 5? Voici une première idée : si P est un polynôme invariant de *poids* m, définissons la différentiation "covariante" :

$$\mathsf{P}_{;\,k} := f'_k\,\mathsf{P}' - m\,f''_k\,\mathsf{P},$$

où $\mathsf{P}' = \mathsf{P}\big(j^{\kappa+1}f\big)$ s'obtient en différentiant $\mathsf{P} = \mathsf{P}\big(j^{\kappa}f\big)$ par rapport à la variable $z \in \mathbb{C}$.

Lemme. Ces deux opérateurs de différentiation $(\cdot)_{;\,1}$ et $(\cdot)_{;\,2}$ satisfont la règle de Leibniz :

$$\big(\mathsf{P}\cdot\mathsf{Q}\big)_{;\,k} = \mathsf{P}_{;\,k}\cdot\mathsf{Q} + \mathsf{P}\cdot\mathsf{Q}_{;\,k},$$

et ils produisent[62] des polynômes invariants par reparamétrisation $\mathsf{P}_{;1}$ et $\mathsf{P}_{;2}$ qui sont tous deux de poids $m+2$. $\square$

Exemple. On vérifie immédiatement :

$$\begin{aligned}\big(f'_2\big)_{;\,1} &= f'_1 f''_2 - f''_1 f'_2 \equiv \Lambda^3 \equiv -\big(f'_1\big)_{;\,2},\\ \Lambda^3{}_{;\,i} &= f'_i\,\Delta^{1,3} - 3\,f''_i\,\Delta^{1,2} \equiv \Lambda^5_i.\end{aligned}$$

Ensuite, à l'étage $\kappa = 4$, on est naturellement conduit à introduire les quatre nouveaux invariants :

$$\Lambda^7_{1,1} := \big(\Lambda^5_1\big)_{;\,1}, \qquad \Lambda^7_{1,2} := \big(\Lambda^5_1\big)_{;\,2}, \qquad \Lambda^7_{2,1} := \big(\Lambda^5_2\big)_{;\,1}, \qquad \Lambda^7_{2,2} := \big(\Lambda^5_2\big)_{;\,2},$$

dont l'expression explicite sera fournie dans un instant.

Produit croisé entre invariants. Comment donner corps à l'idée qu'il doit exister des différentiations covariantes, non seulement par rapport à f'_1 et f'_2, mais aussi par rapport à n'importe quel invariant?

Supposons donc connus deux polynômes homogènes invariants P de poids m et Q de poids n :

$$\begin{aligned}\mathsf{P}\big(j^{\kappa}g\big) &= \phi'^{m}\,\mathsf{P}\big((j^{\kappa}f)\circ\phi\big),\\ \mathsf{Q}\big(j^{\tau}g\big) &= \phi'^{n}\,\mathsf{Q}\big((j^{\tau}f)\circ\phi\big),\end{aligned}$$

où l'on a posé $g := f\circ\phi$. Différentier un polynôme par rapport à la variable $z \in \mathbb{C}$ revient à lui appliquer l'opérateur de *différentiation totale* :

$$\mathsf{D} := \sum_{\lambda\in\mathbb{N}} \frac{\partial(\bullet)}{\partial f^{(\lambda)}}\cdot f^{(\lambda+1)},$$

ce qui nous donne ici :

$$\begin{aligned}\big[\mathsf{DP}\big]\big(j^{\kappa+1}g\big) &= m\,\phi''\,\phi'^{m-1}\,\mathsf{P}\big((j^{\kappa}f)\circ\phi\big) + \phi'^{m}\,\phi'\,\big[\mathsf{DP}\big]\big((j^{\kappa+1}f)\circ\phi\big)\\ \big[\mathsf{DQ}\big]\big(j^{\tau+1}g\big) &= n\,\phi''\,\phi'^{n-1}\,\mathsf{Q}\big((j^{\kappa}f)\circ\phi\big) + \phi'^{n}\,\phi'\,\big[\mathsf{DQ}\big]\big((j^{\tau+1}f)\circ\phi\big)\end{aligned}$$

[62] La démonstration, laissée au lecteur qui désirerait anticiper, apparaîtra dans un instant comme absorbée par une observation plus générale.

et pour faire disparaître la dérivée seconde ϕ'', il suffit d'effectuer un produit croisé, autrement dit de former le déterminant 2×2 :

$$\begin{aligned}
&\begin{vmatrix} [\mathsf{DP}]\big(j^{\kappa+1}g\big) & m\,\mathsf{P}\big(j^{\kappa}g\big) \\ [\mathsf{DQ}]\big(j^{\tau+1}g\big) & n\,\mathsf{Q}\big(j^{\tau}g\big) \end{vmatrix} = \\
&= \begin{vmatrix} m\,\phi''\,\phi'^{m-1}\,\mathsf{P}\big((j^{\kappa}f)\circ\phi\big) + \phi'^{m+1}\,[\mathsf{DP}]\big((j^{\kappa+1}f)\circ\phi\big) & m\,\phi'^{m}\,\mathsf{P}\big((j^{\kappa}f)\circ\phi\big) \\ n\,\phi''\,\phi'^{n-1}\,\mathsf{Q}\big((j^{\kappa}f)\circ\phi\big) + \phi'^{n+1}\,[\mathsf{DQ}]\big((j^{\tau+1}f)\circ\phi\big) & n\,\phi'^{n}\,\mathsf{Q}\big((j^{\kappa}f)\circ\phi\big) \end{vmatrix} \\
&= \begin{vmatrix} \phi'^{m+1}\,[\mathsf{DP}]\big((j^{\kappa+1}f)\circ\phi\big) & m\,\phi'^{m}\,\mathsf{P}\big((j^{\kappa}f)\circ\phi\big) \\ \phi'^{n+1}\,[\mathsf{DQ}]\big((j^{\tau+1}f)\circ\phi\big) & n\,\phi'^{n}\,\mathsf{Q}\big((j^{\kappa}f)\circ\phi\big) \end{vmatrix} \\
&= \phi'^{m+n+1} \begin{vmatrix} [\mathsf{DP}]\big(j^{\kappa+1}f\big) & m\,\mathsf{P}\big(j^{\kappa}f\big) \\ [\mathsf{DQ}]\big(j^{\tau+1}f\big) & n\,\mathsf{Q}\big(j^{\tau}f\big) \end{vmatrix}
\end{aligned}$$

qui s'avère ainsi constituer un nouvel invariant de poids $m+n+1$.

Notation crochet $[\cdot,\,\cdot]$. Ainsi, toute paire d'invariants produit automatiquement un nouvel invariant :

$$\boxed{[\mathsf{P},\,\mathsf{Q}] := n\,\mathsf{DP}\cdot\mathsf{Q} - m\,\mathsf{P}\cdot\mathsf{DQ}},$$

qui est évidemment antisymétrique par rapport au couple (P,Q).

Observation. Ces crochets pondérés généralisent les deux opérateurs de différentiation précédents :

$$\mathsf{P}_{;k} \equiv [\mathsf{P},\,f'_k].$$

De plus, ils satisfont à la règle de Leibniz :

$$[\mathsf{P},\,\mathsf{QR}] = [\mathsf{P},\,\mathsf{Q}]\,\mathsf{R} + [\mathsf{P},\,\mathsf{R}]\,\mathsf{Q},$$

de telle sorte que l'opérateur $[\bullet,\mathsf{Q}]$, à savoir : $\mathsf{P} \longmapsto [\mathsf{P},\,\mathsf{Q}]$, peut être considéré comme un opérateur de dérivation.

Lemme. Pour tout triplet $(\mathsf{P},\,\mathsf{Q},\,\mathsf{R})$ d'invariants de poids m,n,o, l'identité suivante de type Jacobi est satisfaite :

$$(\mathscr{J}ac) \qquad \boxed{0 \equiv [[\mathsf{P},\,\mathsf{Q}],\,\mathsf{R}] + [[\mathsf{R},\,\mathsf{P}],\,\mathsf{Q}] + [[\mathsf{Q},\,\mathsf{R}],\,\mathsf{P}]}.$$

Preuve. Développons le premier double crochet :

$$\begin{aligned}
[[\mathsf{P},\,\mathsf{Q}],\,\mathsf{R}] &= [n\,\mathsf{DP}\cdot\mathsf{Q} - m\,\mathsf{P}\cdot\mathsf{DQ},\,\mathsf{R}] \\
&= o\big(n\,\mathsf{DDP}\cdot\mathsf{Q} + (n-m)\,\mathsf{DP}\cdot\mathsf{DQ} - m\,\mathsf{P}\cdot\mathsf{DDQ}\big)\mathsf{R} - \\
&\quad - (m+n+1)\big(n\,\mathsf{DP}\cdot\mathsf{Q} - m\,\mathsf{P}\cdot\mathsf{DQ}\big)\mathsf{DR} \\
&= no\,\mathsf{DDP}\cdot\mathsf{Q}\cdot\mathsf{R} + (n-m)o\,\mathsf{DP}\cdot\mathsf{DQ}\cdot\mathsf{R} - mo\,\mathsf{P}\cdot\mathsf{DDQ}\cdot\mathsf{R} - \\
&\quad - (m+n+1)n\,\mathsf{DP}\cdot\mathsf{Q}\cdot\mathsf{DR} + (m+n+1)m\,\mathsf{P}\cdot\mathsf{DQ}\cdot\mathsf{DR}.
\end{aligned}$$

Il suffit alors de constater que l'annulation identique de la somme suivante :

$$\begin{aligned}0 \equiv\ & no\,\mathsf{DDP}\cdot\mathsf{Q}\cdot\mathsf{R} + (n-m)o\,\mathsf{DP}\cdot\mathsf{DQ}\cdot\mathsf{R} - mo\,\mathsf{P}\cdot\mathsf{DDQ}\cdot\mathsf{R} - \\ & - (m+n+1)n\,\mathsf{DP}\cdot\mathsf{Q}\cdot\mathsf{DR} + (m+n+1)m\,\mathsf{P}\cdot\mathsf{DQ}\cdot\mathsf{DR} + \\ & + mn\,\mathsf{DDR}\cdot\mathsf{P}\cdot\mathsf{Q} + (m-o)n\,\mathsf{DR}\cdot\mathsf{DP}\cdot\mathsf{Q} - on\,\mathsf{R}\cdot\mathsf{DDP}\cdot\mathsf{Q} - \\ & - (o+m+1)m\,\mathsf{DR}\cdot\mathsf{P}\cdot\mathsf{DQ} + (o+m+1)o\,\mathsf{R}\cdot\mathsf{DP}\cdot\mathsf{DQ} + \\ & + om\,\mathsf{DDQ}\cdot\mathsf{R}\cdot\mathsf{P} + (o-n)m\,\mathsf{DQ}\cdot\mathsf{DR}\cdot\mathsf{P} - nm\,\mathsf{Q}\cdot\mathsf{DDR}\cdot\mathsf{P} - \\ & - (n+o+1)o\,\mathsf{DQ}\cdot\mathsf{R}\cdot\mathsf{DP} + (n+o+1)n\,\mathsf{Q}\cdot\mathsf{DR}\cdot\mathsf{DP},\end{aligned}$$

est effectivement satisfaite.

Exemple. Avec $\mathsf{P} := f_1'$, $\mathsf{Q} := f_2'$ et $\mathsf{R} := \Lambda^3$, nous obtenons :

$$\begin{aligned}0 &\equiv \big[\big[f_1',\, f_2'\big],\, \Lambda^3\big] + \big[\big[\Lambda^3,\, f_1'\big],\, f_2'\big] + \big[\big[f_2',\, \Lambda^3\big],\, f_1'\big] \\ &\equiv 0 + \big[\Lambda_1^5,\, f_2'\big] - \big[\Lambda_2^5,\, f_1'\big] \\ &\equiv \Lambda_{1,2}^7 - \Lambda_{2,1}^7.\end{aligned}$$

Cette relation sera confirmée par les expressions explicites de $\Lambda_{1,2}^7$ et $\Lambda_{2,1}^7$.

Genèse des invariants fondamentaux. Crucialement, il semblerait que l'on puisse engendrer tous les polynômes invariants en les jets d'un ordre $\kappa \geqslant 1$ quelconque, juste en calculant par récurrence tous les crochets possibles, d'un étage de jets λ à l'étage supérieur $\lambda + 1$. Cette idée conjecturale ([3]), sur laquelle nous donnerons plus de précision ultérieurement, est renforcée par le fait que dans la théorie classique des invariants pour une forme binaire $\sum_{i=0}^{\kappa} a_i\, x^i y^{\kappa-i}$ de degré κ par rapport à l'action linéaire standard de $\mathsf{SL}_2(\mathbb{C})$:

$$x \longmapsto \overline{x} = \alpha x + \beta y, \qquad y \longmapsto \overline{y} = \gamma x + \delta y, \qquad 1 = \alpha\delta - \beta\gamma,$$

$$\sum_{i=0}^{\kappa} a_i\, x^i y^{\kappa-i} \longmapsto \sum_{i=0}^{\kappa} \overline{a}_i\, \overline{x}^i \overline{y}^{\kappa-i},$$

$$a_i = \sum_{l=0}^{\kappa} \overline{a}_l \sum_{j=\max(0,i+l-\kappa)}^{\min(i,l)} C_i^j\, C_{\kappa-i}^{l-j}\, \alpha^j\, \beta^{l-j}\, \gamma^{i-j}\, \delta^{\kappa+j-i-l}, \qquad C_p^q = \tfrac{p!}{q!\,(p-q)!},$$

on sait établir que deux procédés algébriques élémentaires, à savoir le "processus Ω" et le "processus σ" (*cf.* [4]) permettent d'engendrer un système fondamental de polynômes $\mathsf{P} = \mathsf{P}\big(a_0, a_1, \ldots, a_\kappa\big)$ qui sont invariants :

$$\mathsf{P}\big(a_0, a_1, \ldots, a_\kappa\big) = \mathsf{P}\big(\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_\kappa\big).$$

Reconstitution par crochets des invariants connus. Pour passer des jets d'ordre 1 aux jets d'ordre 2, seul un crochet (au signe près) peut être formé :

$$\big[f_1',\, f_2'\big] = -\big[f_2',\, f_1'\big] = -\Lambda^3.$$

Pour passer des jets d'ordre 2 aux jets d'ordre 3, on peut former trois crochets :

$$\big[\Lambda^3,\, f_1'\big] \qquad \big[\Lambda^3,\, f_2'\big] \qquad \big[\Lambda^3,\, \Lambda^3\big],$$

le dernier étant trivialement nul, et l'on vérifie immédiatement que les deux premiers fournissent les deux invariants propres à l'étage $\kappa = 3$:

$$\big[\Lambda^3,\, f_i'\big] = \Delta^{1,3}\, f_i' - 3\,\Delta^{1,2}\, f_i''.$$

Pour passer aux jets d'ordre 4, l'ensemble des crochets que l'on peut former s'identifie à la collection des déterminants 2×2 de la matrice matrice 2×5 :

$$\left\| \begin{matrix} f_1' & f_2' & 3\,\Lambda^3 & 5\,\Lambda_1^5 & 5\,\Lambda_2^5 \\ \mathsf{D} f_1' & \mathsf{D} f_2' & \mathsf{D}\Lambda^3 & \mathsf{D}\Lambda_1^5 & \mathsf{D}\Lambda_2^5 \end{matrix} \right\|,$$

ce qui fait au total de $C_5^2 = 10$ crochets, mais en tenant compte du fait que nous connaissons déjà les trois mineurs — calculés à l'étage $\kappa = 3$ — de la sous-matrice :

$$\left\| \begin{matrix} f_1' & f_2' & 3\,\Lambda^3 \\ \mathsf{D} f_1' & \mathsf{D} f_2' & \mathsf{D}\Lambda^3 \end{matrix} \right\|,$$

ce sont exactement sept nouveaux crochets qui apparaissent :

$$\big[\Lambda_i^5,\, f_j'\big], \qquad \big[\Lambda_i^5,\, \Lambda^3\big], \qquad \big[\Lambda_1^5,\, \Lambda_2^5\big].$$

Relations plückeriennes. Cependant, le calcul complet des crochets doit tenir compte des relations de Plücker qui existent au niveau des variables initiales des espaces de jets. En effet, l'idéal des relations plückeriennes entre les $f_i^{(\lambda)}$, $1 \leqslant \lambda \leqslant 5$ et les $\Delta^{\alpha,\beta}$, $1 \leqslant \alpha < \beta \leqslant 5$ est engendré par deux familles quadratiques de relations identiquement satisfaites : dans la première famille :

$$0 \equiv \Delta^{\beta,\gamma} \cdot f_i^{(\alpha)} - \Delta^{\alpha,\gamma} \cdot f_i^{(\beta)} + \Delta^{\alpha,\beta} \cdot f_i^{(\gamma)},$$

i est égal à 1 ou à 2, et les indices supérieurs satisfont $1 \leqslant \alpha < \beta < \gamma \leqslant 5$, ce qui donne $10 \times 2 = 20$ relations ; et dans la seconde famille :

$$0 \equiv \Delta^{\alpha,\delta} \cdot \Delta^{\beta,\gamma} - \Delta^{\alpha,\gamma} \cdot \Delta^{\beta,\delta} + \Delta^{\alpha,\beta} \cdot \Delta^{\gamma,\delta},$$

les indices supérieurs satisfont $1 \leqslant \alpha < \beta < \gamma < \delta \leqslant 5$, ce qui donne 4 relations.

En vérité, seules les deux paires de relations suivantes, extraites de la première familles, seront utiles à l'étage des jets d'ordre $\kappa = 5$:

$$\boxed{\begin{matrix} 0 \equiv \Delta^{2,3}\, f_i' - \Delta^{1,3}\, f_i'' + \underline{\Delta^{1,2}\, f_i'''} \\ 0 \equiv \Delta^{2,4}\, f_i' - \Delta^{1,4}\, f_i'' + \underline{\Delta^{1,2}\, f_i''''} \end{matrix}},$$

et à l'étage $\kappa = 4$, seule la première paire peut être utilisée, tandis qu'aucune relation plückerienne n'intervient aux étages $\kappa \leqslant 3$. Il faut en outre attendre $\kappa = 6$ pour que la première relation de la seconde famille, à savoir : $0 \equiv \Delta^{1,4}\,\Delta^{2,3} - \Delta^{1,3}\,\Delta^{2,4} + \Delta^{1,2}\,\Delta^{3,4}$ commence à interfèrer, mais nous n'entreprendrons pas l'étude de $\mathscr{D}\mathscr{S}_2^6$ dans cet article.

Normalisations préalables des différentielles totales. Ainsi, nous sommes conduits à normaliser $\mathsf{D}\Lambda_i^5$ avant de calculer $\Lambda_{i,j}^7$, en développant tout d'abord :

$$\begin{aligned}\mathsf{D}\Lambda_i^5 &= \Delta^{1,4}\, f_i' + \Delta^{2,3}\, f_i' + \Delta^{1,3}\, f_i'' - 3\,\Delta^{1,3}\, f_i'' - 3, \Delta^{1,2}\, f_i''' \\ &= \Delta^{1,4}\, f_i' + \Delta^{2,3}\, f_i' - 2\,\Delta^{1,3}\, f_i'' - 3\,\Delta^{1,2}\, f_i''',\end{aligned}$$

expression dans laquelle nous pouvons remplacer $\underline{\Delta^{1,2}\, f_i'''}$ par $-\Delta^{2,3}\, f_i' + \Delta^{1,3}\, f_i''$ afin d'éliminer toute présence de f_i''', ce qui nous donne une expression normalisée et compacte ne contenant que trois termes :

$$\mathsf{D}\Lambda_i^5 = \Delta^{1,4}\, f_i' + 4\,\Delta^{2,3}\, f_i' - 5\,\Delta^{1,3}\, f_i''.$$

Achevons donc le calcul de la première famille de crochets $\big[\Lambda_i^5,\, f_j'\big]$ en fournissant tous les détails intermédiaires :

$$\begin{aligned}\big[\Lambda_i^5,\, f_j'\big] &= \mathsf{D}\Lambda_i^5 \cdot f_j' - 5\,\Lambda_i^5 \cdot f_j'' \\ &= \Big(\Delta^{1,4}\, f_i' + 4\,\Delta^{2,3}\, f_i' f_j' - 5\,\Delta^{1,3} f_i''\Big) \cdot f_j' - 5\,\Big(\Delta^{1,3}\, f_i' - 3\,\Delta^{1,2}\, f_i''\Big) \cdot f_j'' \\ &= \Delta^{1,4}\, f_i' f_j' + 4\,\Delta^{2,3}\, f_i' f_j' - 5\,\Delta^{1,3}\big(f_i'' f_j' + f_i' f_j''\big) + 15\,\Delta^{1,2}\, f_i'' f_j'' \\ &=: \Lambda_{i,j}^7.\end{aligned}$$

Ici, la symétrie indicielle $\Lambda_{1,2}^7 = \Lambda_{2,1}^7$ imposée *a priori* par l'identité de Jacobi montre que l'on devrait se dispenser de $\Lambda_{2,1}^7$ (ou de $\Lambda_{1,2}^7$) dans une liste minimale de polynômes invariants fondamentaux.

Invariant de poids 8. Ensuite, grâce à notre normalisation préalable de $\mathsf{D}\Lambda_i^5$, nous pouvons calculer proprement chacun des deux crochets ($i = 1, 2$) :

$$\begin{aligned}\big[\Lambda_i^5,\, \Lambda^3\big] &= 3\,\mathsf{D}\Lambda_i^5 \cdot \Lambda^3 - 5\,\Lambda_i^5 \cdot \mathsf{D}\Lambda^3 \\ &= \Big(3\,\Delta^{1,4}\, f_i' + 12\,\Delta^{2,3}\, f_i' - 15\,\Delta^{1,3}\, f_i''\Big) \cdot \Delta^{1,2} - \\ &\qquad - \Big(5\,\Delta^{1,3}\, f_i' - 15\,\Delta^{1,2}\, f_i''\Big) \cdot \Delta^{1,3} \\ &= 3\,\Delta^{1,4}\,\Delta^{1,2}\, f_i' + 12\,\Delta^{2,3}\,\Delta^{1,2}\, f_i' - 5\,\Delta^{1,3}\,\Delta^{1,3}\, f_i' \\ &= f_i'\big(3\,\Delta^{1,4}\,\Delta^{1,2} + 12\,\Delta^{2,3}\,\Delta^{1,2} - 5\,\Delta^{1,3}\,\Delta^{1,3}\big) \\ &\equiv f_i'\, M^8,\end{aligned}$$

où le nouvel invariant

$$\boxed{\begin{aligned}M^8 &:= \frac{1}{f_i'}\,\big[\Lambda_i^5,\, \Lambda^3\big] \\ &= 3\,\Delta^{1,4}\,\Delta^{1,2} + 12\,\Delta^{2,3}\,\Delta^{1,2} - 5\,\Delta^{1,3}\,\Delta^{1,3}\end{aligned}}$$

doit être introduit, parce que le résultat est divisible par f_i'.

Question. Pourquoi et comment doit-on être conduit à diviser parfois les crochets pour accéder véritablement à de nouveaux invariants fondamentaux?

Fin du passage à l'étage $\kappa = 4$ Enfin, calculons et examinons le dernier crochet possible, à nouveau en fournissant scrupuleusement tous les détails intermédiaires :

$$\begin{aligned}
\big[\Lambda_1^5,\, \Lambda_2^5\big] &= 5\,\mathsf{D}\Lambda_1^5 \cdot \Lambda_2^5 - 5\,\Lambda_1^5 \cdot \mathsf{D}\Lambda_2^5 \\
&= 5\Big(\Delta^{1,4}\, f_1' + 4\,\Delta^{2,3}\, f_1' - 5\,\Delta^{1,3}\, f_1''\Big) \cdot \Big(\Delta^{1,3}\, f_2' - 3\,\Delta^{1,2}\, f_2''\Big) - \\
&\qquad - 5\Big(\Delta^{1,3}\, f_1' - 3\,\Delta^{1,2}\, f_1''\Big) \cdot \Big(\Delta^{1,4}\, f_2' + 4\,\Delta^{2,3}\, f_2' - 5\,\Delta^{1,3}\, f_2''\Big) \\
&= -15\,\Delta^{1,4}\,\Delta^{1,2}\, f_1' f_2'' - 60\,\Delta^{2,3}\,\Delta^{1,2}\, f_1' f_2'' - 25\,\Delta^{1,3}\,\Delta^{1,3}\, f_1'' f_2' + \\
&\quad + 15\,\Delta^{1,4}\,\Delta^{1,2}\, f_2' f_1'' + 60\,\Delta^{2,3}\,\Delta^{1,2}\, f_2' f_1'' + 25\,\Delta^{1,3}\,\Delta^{1,3}\, f_2'' f_1' \\
&= -15\,\Delta^{1,4}\,\Delta^{1,2}\,\Delta^{1,2} - 15\,\Delta^{2,3}\,\Delta^{1,2}\,\Delta^{1,2} + 25\,\Delta^{1,3}\,\Delta^{1,3}\,\Delta^{1,2} \\
&= -5\,\Delta^{1,2}\Big(3\,\Delta^{1,4}\,\Delta^{1,2} + 12\,\Delta^{2,3}\,\Delta^{1,2} - 5\,\Delta^{1,3}\,\Delta^{1,3}\Big) \\
&\equiv -5\,\Lambda^3\, M^8.
\end{aligned}$$

Le résultat étant multiple des deux invariants déjà connus Λ^3 et M^8, il n'apporte rien de nouveau. Toutefois, conservons trace de la relation :

$$\big[\Lambda_1^5,\, \Lambda_2^5\big] = -5\,\Lambda^3\, M^8.$$

Proposition. ([3]) *En dimension $\nu = 2$, les neuf polynômes* :

$$\begin{aligned}
&f_1', \qquad f_2', \qquad \Lambda^3 := \Delta^{1,2}, \qquad \Lambda_1^5 := \Delta^{1,3}\, f_1' - 3\,\Delta^{1,2}\, f_1'', \\
&\Lambda_2^5 := \Delta^{1,3}\, f_2' - 3\,\Delta^{1,2}\, f_2'', \\
&\Lambda_{1,1}^7 := \big(\Delta^{1,4} + 4\,\Delta^{2,3}\big)\, f_1' f_1' - 10\,\Delta^{1,3}\, f_1' f_1'' + 15\,\Delta^{1,2}\, f_1'' f_1'', \\
&\Lambda_{1,2}^7 := \big(\Delta^{1,4} + 4\,\Delta^{2,3}\big)\, f_1' f_2' - 5\,\Delta^{1,3}\big(f_1'' f_2' + f_2'' f_1'\big) + 15\,\Delta^{1,2}\, f_1'' f_2'', \\
&\Lambda_{2,2}^7 := \big(\Delta^{1,4} + 4\,\Delta^{2,3}\big)\, f_2' f_2' - 10\,\Delta^{1,3}\, f_2' f_2'' + 15\,\Delta^{1,2}\, f_2'' f_2'', \\
&\qquad M^8 := 3\,\Delta^{1,4}\,\Delta^{1,2} + 12\,\Delta^{2,3}\,\Delta^{1,2} - 5\,\Delta^{1,3}\,\Delta^{1,3}
\end{aligned}$$

forment un système générateur de polynômes invariants par reparamétrisation pour les jets d'ordre $\kappa = 4$.

Cette proposition sera englobée dans un énoncé plus précis dont la preuve apparaîtra dans la Section 5.

§4. Invariants fondamentaux pour les jets d'ordre 5

Dénombrement des crochets. Pour s'élever de l'étage $\kappa = 4$ à l'étage $\kappa = 5$, l'ensemble des crochets que l'on peut former s'identifie à la collection des déterminants 2×2 de la matrice matrice 2×9 :

$$\left\| \begin{matrix} f_1' & f_2' & 3\,\Lambda^3 & 5\,\Lambda_1^5 & 5\,\Lambda_2^5 & 7\,\Lambda_{1,1}^7 & 7\,\Lambda_{1,2}^7 & 7\,\Lambda_{2,2}^7 & 8\,M^8 \\ \mathsf{D}f_1' & \mathsf{D}f_2' & \mathsf{D}\Lambda^3 & \mathsf{D}\Lambda_1^5 & \mathsf{D}\Lambda_2^5 & \mathsf{D}\Lambda_{1,1}^7 & \mathsf{D}\Lambda_{1,2}^7 & \mathsf{D}\Lambda_{2,2}^7 & \mathsf{D}M^8 \end{matrix} \right\|,$$

ce qui fait au total de $C_9^2 = 36$ crochets, mais il n'y en a en fait que $36 - 10 = 26$ à calculer, en tenant compte du fait que nous connaissons déjà les $C_5^2 = 10$ mineurs — calculés dans la section précédente — de la sous-matrice :

$$\left\| \begin{matrix} f_1' & f_2' & 3\,\Lambda^3 & 5\,\Lambda_1^5 & 5\,\Lambda_2^5 \\ \mathsf{D}f_1' & \mathsf{D}f_2' & \mathsf{D}\Lambda^3 & \mathsf{D}\Lambda_1^5 & \mathsf{D}\Lambda_2^5 \end{matrix} \right\|.$$

Heuristique. Nous sommes par conséquent amenés à penser[63] que tout polynôme invariant $\mathsf{P}\big(j^5 f\big)$ en les jets d'ordre 5 est un polynôme en les neuf précédents polynômes fondamentaux :

$$f_1', \quad f_2', \quad \Lambda^3, \quad \Lambda_1^5, \quad \Lambda_{1,1}^7, \quad \Lambda_{1,2}^7, \quad \Lambda_{2,2}^7, \quad M^8,$$

auxquels on ajoute tous ceux qui sont obtenus par crochets à l'étage supérieur, après simplification, normalisation plückerienne, division éventuelle, et suppression des invariants redondants. On voit immédiatement que les nouveaux crochets à étudier se distribuent en huit familles :

$$\begin{matrix} \big[\Lambda_{i,j}^7,\, f_k'\big], & \big[M^8,\, f_i'\big], \\ \big[\Lambda_{i,j}^7,\, \Lambda^3\big], & \big[M^8,\, \Lambda^3\big], \\ \big[\Lambda_{i,j}^7,\, \Lambda_k^5\big], & \big[M^8,\, \Lambda_i^5\big], \\ \big[\Lambda_{i,j}^7,\, \Lambda_{k,l}^7\big], & \big[M^8,\, \Lambda_{i,j}^7\big]. \end{matrix}$$

Avant de calculer et d'examiner tous ces crochets — tâche substantielle s'il en est —, reprenons en main la liste de tous les crochets précédents (*i.e.* invariants à l'étage $\kappa = 4$), en les écrivant avec des indices :

$$\boxed{\begin{aligned} & f_i' \\ & \Lambda^3 := \Delta^{1,2} \\ & \Lambda_i^5 := \Delta^{1,3}\, f_i' - 3\,\Delta^{1,2}\, f_i'' \\ & \Lambda_{i,j}^7 := \Delta^{1,4}\, f_i' f_j' + 4\,\Delta^{2,3}\, f_i' f_j' - 5\,\Delta^{1,3}\big(f_i'' f_j' + f_i' f_j''\big) + 15\,\Delta^{1,2}\, f_i'' f_j'' \\ & M^8 := 3\,\Delta^{1,4}\,\Delta^{1,2} + 12\,\Delta^{2,3}\,\Delta^{1,2} - 5\,\Delta^{1,3}\,\Delta^{1,3} \end{aligned}}\,.$$

Remarque sur le choix des notations. Nous utiliserons systématiquement les grandes lettres, telles que "Λ" (particulièrement facile à écrire à la main), "M", "H", *etc.*, parce que leur taille les rend disponibles pour recevoir non seulement le nombre total de "$'$" en indice supérieur (poids de l'invariant), mais aussi, en indices inférieurs, la suite *ordonnée* de 1 ou de 2 dont dépend chaque monôme de l'invariant en question.

Normalisation préalable des différentiations totales. Nous avons donc huit familles de crochets à calculer, et pour cela, nous travaillerons avec les représentations indiciées de nos invariants connus à l'étage $\kappa = 4$. Auparavant, nous devons calculer à l'avance les deux expressions dérivées $\mathsf{D}\Lambda_{i,j}^7$ et $\mathsf{D}M^8$, et les normaliser en tenant compte des identités plückeriennes, comme

[63] Cette idée sera discutée plus avant dans la Section 7.

nous l'avons expliqué ci-dessus. Calculons donc, en éliminant $\Delta^{1,2}\, f_i'''$ et $\Delta^{1,2}\, f_j'''$ à la quatrième ligne :

$$\begin{aligned}
\mathsf{D}\Lambda^7_{i,j} &= \Delta^{1,5}\, f_i' f_j' + \Delta^{2,4}\, f_i' f_j' + \Delta^{1,4}\big(f_i'' f_j' + f_i' f_j''\big) + \\
&\quad + 4\,\Delta^{2,4}\, f_i' f_j' + 4\,\Delta^{2,3}\big(f_i'' f_j' + f_i' f_j''\big) - 5\,\Delta^{1,4}\big(f_i'' f_j' + f_i' f_j''\big) - \\
&\quad - 5\,\Delta^{2,3}\big(f_i'' f_j' + f_i' f_j''\big) - 5\,\Delta^{1,3}\big(f_i''' f_j' + 2\, f_i'' f_j'' + f_i' f_j'''\big) + 15\,\Delta^{1,3}\, f_i'' f_j'' + \\
&\quad + 15\,\underline{\Delta^{1,2}\, f_i'''} f_j'' + 15\,\underline{\Delta^{1,2}}\, f_i'' \underline{f_j'''} \\
&= \Delta^{1,5}\, f_i' f_j' + 5\,\Delta^{2,4}\, f_i' f_j' - 4\,\Delta^{1,4}\big(f_i'' f_j' + f_i' f_j''\big) - \Delta^{2,3}\big(f_i'' f_j' + f_i' f_j''\big) - \\
&\quad - 5\,\Delta^{1,3}\big(f_i''' f_j' + f_i' f_j'''\big) + 5\,\Delta^{1,3}\, f_i'' f_j'' - \\
&\quad - 15\,\Delta^{2,3}\, f_i' f_j'' + 15\,\Delta^{1,3}\, f_i'' f_j'' - 15\,\Delta^{2,3}\, f_i'' f_j'' + 15\,\Delta^{1,3}\, f_i'' f_j'' \\
&= \Delta^{1,5}\, f_i' f_j' + 5\,\Delta^{2,4}\, f_i' f_j' - 4\,\Delta^{1,4}\big(f_i'' f_j' + f_i' f_j''\big) - \\
&\quad - 16\,\Delta^{2,3}\big(f_i'' f_j' + f_i' f_j''\big) - 5\,\Delta^{1,3}\big(f_i''' f_j' + f_i' f_j'''\big) + 35\,\Delta^{1,3}\, f_i'' f_j''.
\end{aligned}$$

Ensuite, le calcul de $\mathsf{D}M^8$ est immédiat car il n'implique aucune relation plückerienne :

$$\begin{aligned}
\mathsf{D}M^8 &= 3\,\Delta^{1,5}\,\Delta^{1,2} + 3\,\Delta^{2,4}\,\Delta^{1,2} + 3\,\Delta^{1,4}\,\Delta^{1,3} + \\
&\quad + 12\,\Delta^{2,4}\,\Delta^{1,2} + 12\,\Delta^{2,3}\,\Delta^{1,3} - 10\,\Delta^{1,4}\,\Delta^{1,3} - 10\,\Delta^{2,3}\,\Delta^{1,3} \\
&= 3\,\Delta^{1,5}\,\Delta^{1,2} + 15\,\Delta^{2,4}\,\Delta^{1,2} - 7\,\Delta^{1,4}\,\Delta^{1,3} + 2\,\Delta^{2,3}\,\Delta^{1,3}.
\end{aligned}$$

Tableau des différentiations totales. En résumé, nous obtenons les expressions normalisées suivantes pour nos invariants différentiés :

$$\boxed{\begin{aligned}
\mathsf{D} f_i' &= f_i'' \\
\mathsf{D}\Lambda^3 &= \Delta^{1,3} \\
\mathsf{D}\Lambda^5_i &= \Delta^{1,4}\, f_i' + 4\,\Delta^{2,3}\, f_i' - 5\,\Delta^{1,3}\, f_i'' \\
\mathsf{D}\Lambda^7_{i,j} &= \Delta^{1,5}\, f_i' f_j' + 5\,\Delta^{2,4}\, f_i' f_j' - 4\,\Delta^{1,4}\big(f_i'' f_j' + f_i' f_j''\big) - \\
&\quad - 16\,\Delta^{2,3}\big(f_i'' f_j' + f_i' f_j''\big) - 5\,\Delta^{1,3}\big(f_i''' f_j' + f_i' f_j'''\big) + 35\,\Delta^{1,3}\big(f_i'' f_j''\big) \\
\mathsf{D}M^8 &= 3\,\Delta^{1,5}\,\Delta^{1,2} + 15\,\Delta^{2,4}\,\Delta^{1,2} - 7\,\Delta^{1,4}\,\Delta^{1,3} + 2\,\Delta^{2,3}\,\Delta^{1,3}
\end{aligned}}\,,$$

et nous pouvons maintenant commencer à engendrer la table de multiplication — pondérée par le poids de nos invariants — entre ces deux listes encadrées, afin de découvrir de nouveaux polynômes invariants fondamentaux à l'étage $\kappa = 5$.

Première famille de crochets $\big[\Lambda^7_{i,j},\, f_k'\big]$. Après un calcul direct que nous ne détaillerons pas, mais dans lequel les normalisations plückeriennes

n'interviennent pas, nous obtenons :

$$\begin{aligned}
\big[\Lambda^7_{i,j},\, f'_k\big] &= \mathsf{D}\Lambda^7_{i,j}\cdot f'_k - 7\,\Lambda^7_{i,j}\cdot \mathsf{D}f'_k \\
&= \Delta^{1,5}\, f'_i f'_j f'_k + 5\,\Delta^{2,4}\, f'_i f'_j f'_k - \\
&\quad - 4\,\Delta^{1,4}\big(f''_i f'_j + f'_i f''_j\big)\, f'_k - 7\,\Delta^{1,4}\, f'_i f'_j f''_k - \\
&\quad - 16\,\Delta^{2,3}\big(f''_i f'_j + f'_i f''_j\big) f'_k - 28\,\Delta^{2,3}\, f'_i f'_j f''_k - \\
&\quad - 5\,\Delta^{1,3}\big(f'''_i f'_j + f'_i f'''_j\big)\, f'_k + 35\,\Delta^{1,3}\big(f''_i f''_j f'_k + f''_i f'_j f''_k + f'_i f''_j f''_k\big) - \\
&\quad - 105\,\Delta^{1,2}\, f''_i f''_j f''_k \\
&=: \Lambda^9_{i,j,k}.
\end{aligned}$$

Nous trouvons donc huit nouveaux invariants $\Lambda^9_{1,1,1}$, $\Lambda^9_{1,1,2}$, $\Lambda^9_{1,2,1}$, $\Lambda^9_{1,2,2}$, $\Lambda^9_{2,1,1}$, $\Lambda^9_{2,1,2}$, $\Lambda^9_{2,2,1}$ et $\Lambda^9_{2,2,2}$, qui ne s'expriment clairement pas en fonction de ceux connus à l'étage $\kappa = 4$, à cause par exemple de la présence du déterminant $\Delta^{1,5}$ où apparaissent $f_1^{\prime\prime\prime\prime\prime}$ et $f_2^{\prime\prime\prime\prime\prime}$.

Observation. Cependant, ces huit invariants ne sont pas indépendants entre eux, ne serait-ce que par héritage de la symétrie $\Lambda^7_{1,2} = \Lambda^7_{2,1}$, qui implique les deux relations $\Lambda^9_{1,2,k} = \Lambda^9_{2,1,k}$, $k = 1, 2$. En fait, il y a quatre relations indépendantes, que l'on peut proposer au lecteur de vérifier par un développement direct :

$$\begin{aligned}
\Lambda^9_{1,1,2} &= \Lambda^9_{1,2,1} - f'_1\, M^8 \\
\Lambda^9_{1,2,1} &= \Lambda^9_{2,1,1} \\
\Lambda^9_{1,2,2} &= \Lambda^9_{2,1,2} \\
\Lambda^9_{2,2,1} &= \Lambda^9_{2,1,2} + f'_2\, M^8.
\end{aligned}$$

Toutefois, il est incontestablement préférable d'obtenir ces relations comme suit à partir de l'identité de type Jacobi, en posant tout simplement $\mathsf{P} := f'_i$, $\mathsf{Q} := f'_j$ et $\mathsf{R} := \Lambda^5_k$:

$$0 \equiv \big[\big[f'_i,\, f'_j\big],\, \Lambda^5_k\big] + \big[\big[\Lambda^5_k,\, f'_i\big],\, f'_j\big] + \big[\big[f'_j,\, \Lambda^5_k\big],\, f'_i\big].$$

Si l'on tient compte du fait que $\big[f'_i,\, f'_j\big]$ vaut 0 ou $\pm\Lambda^3$ et si on utilise les relations $\big[\Lambda^5_i,\, \Lambda^3\big] = f'_i\, M^8$, nos quatre relations en découlent. Ainsi, seuls quatre des huit crochets $\big[\Lambda^7_{i,j},\, f'_k\big]$ peuvent être fondamentaux, et on vérifie sans peine que $\Lambda^9_{1,1,1}$, $\Lambda^9_{1,2,1}$, $\Lambda^9_{2,1,2}$ et $\Lambda^9_{2,2,2}$ constituent bien de nouveaux invariants qui sont indépendants entre eux, parce que les trinômes $f'_1 f'_1 f'_1$, $f'_1 f'_2 f'_1$, $f'_2 f'_1 f'_2$ et $f'_2 f'_2 f'_2$ en facteur derrière $\Delta^{1,5} + 5\,\Delta^{2,4}$ le sont.

Deuxième famille de crochets $\big[M^8,\, f'_i\big]$. Le calcul, immédiat, libre d'ambiguïté plückerienne et ne nécessitant aucune réorganisation, fournit

le résultat suivant :

$$\begin{aligned}\big[M^8,\, f_i'\big] &= \mathsf{D}M^8 \cdot f_i' - 8\, M^8 \cdot f_i'' \\ &= \big[3\,\Delta^{1,5}\,\Delta^{1,2} + 15\,\Delta^{2,4}\,\Delta^{1,2} - 7\,\Delta^{1,4}\,\Delta^{1,3} + 2\,\Delta^{2,3}\,\Delta^{1,3}\big]\, f_i' - \\ &\quad - \big[24\,\Delta^{1,4}\,\Delta^{1,2} + 96\,\Delta^{2,3}\,\Delta^{1,2} - 40\,\Delta^{1,3}\,\Delta^{1,3}\big] f_i'' \\ &=: M_i^{10}.\end{aligned}$$

Nous trouvons donc deux nouveaux invariants M_1^{10} et M_2^{10} de poids 10, qui ne s'expriment pas en fonction de ceux déjà connus, à cause de la présence du produit de déterminants $\Delta^{1,5}\,\Delta^{1,2}$ où apparaît $f_1''''' f_2''$.

Troisième famille de crochets $\big[\Lambda_{i,j}^7,\, \Lambda^3\big]$. Bien que le résultat final n'apporte pas de nouvel invariant (*cf. infra*), nous détaillerons ce calcul :

$$\begin{aligned}\big[\Lambda_{i,j}^7,\, \Lambda^3\big] &= 3\,\mathsf{D}\Lambda_{i,j}^7 \cdot \Lambda^3 - 7\,\Lambda_{i,j}^7 \cdot \mathsf{D}\Lambda^3 \\ &= 3\,\Delta^{1,5}\,\Delta^{1,2}\, f_i' f_j' + 15\,\Delta^{2,4}\,\Delta^{1,2}\, f_i' f_j' - 12\,\Delta^{1,4}\,\Delta^{1,2}\big(f_i'' f_j' + f_i' f_j''\big) - \\ &\quad - 48\,\Delta^{2,3}\,\Delta^{1,2}\big(f_i'' f_j' + f_i' f_j''\big) - 15\,\Delta^{1,3}\,\underline{\Delta^{1,2}\, f_i'''} f_j' - \\ &\quad - 15\,\Delta^{1,3}\,\underline{\Delta^{1,2}}\, f_i' \underline{f_j'''} + 105\,\Delta^{1,3}\,\Delta^{1,2}\, f_i'' f_j'' - 7\,\Delta^{1,4}\,\Delta^{1,3}\, f_i' f_j' - \\ &\quad - 28\,\Delta^{2,3}\,\Delta^{1,3}\, f_i' f_j' + 35\,\Delta^{1,3}\,\Delta^{1,3}\big(f_i'' f_j' + f_i' f_j''\big) - 105\,\Delta^{1,3}\,\Delta^{1,2}\, f_i'' f_j''.\end{aligned}$$

Nous utilisons la relation plückerienne pour transformer les deux termes soulignés, qui deviennent :

$$15\,\Delta^{2,3}\,\Delta^{1,3}\, f_i' f_j' - 15\,\Delta^{1,3}\,\Delta^{1,3}\, f_i'' f_j' + 15\,\Delta^{2,3}\,\Delta^{1,3}\, f_i' f_j' - 15\,\Delta^{1,3}\,\Delta^{1,3}\, f_i' f_j'',$$

et ensuite, nous additionnons les monômes égaux et nous regroupons le tout dans un ordre naturel :

$$\begin{aligned}\big[\Lambda_{i,j}^7,\, \Lambda^3\big] &= \Big(3\,\Delta^{1,5}\,\Delta^{1,2} + 15\,\Delta^{2,4}\,\Delta^{1,2} - 7\,\Delta^{1,4}\,\Delta^{1,3} + 2\,\Delta^{2,3}\,\Delta^{1,3}\Big) f_i' f_j' + \\ &\quad + \Big(-12\,\Delta^{1,4}\,\Delta^{1,2} - 48\,\Delta^{2,3}\,\Delta^{1,2} + 20\,\Delta^{1,3}\,\Delta^{1,3}\Big)\big(f_i'' f_j' + f_i' f_j''\big).\end{aligned}$$

Or, *cette troisième famille de crochets n'apporte aucun nouvel invariant*. En effet, considérons la famille d'identités de Jacobi :

$$\begin{aligned}0 &\equiv \big[\big[\Lambda_i^5,\, f_j'\big],\, \Lambda^3\big] + \big[\big[\Lambda^3,\, \Lambda_i^5\big]\, f_j'\big] + \big[\big[f_j',\, \Lambda^3\big],\, \Lambda_i^5\big] \\ &\equiv \big[\Lambda_{i,j}^7,\, \Lambda^3\big] - \big[f_i'\, M^8,\, f_j'\big] - \big[\Lambda_j^5,\, \Lambda_i^5\big].\end{aligned}$$

Si nous faisons tout d'abord $i = j$ ($= 1$ ou $= 2$), en utilisant $\big[f_i'\, M^8,\, f_i'\big] = f_i'\, \big[M^8,\, f_i'\big] = f_i'\, M_i^{10}$, nous obtenons les deux relations :

$$0 \equiv \big[\Lambda_{i,i}^7,\, \Lambda^3\big] - f_i'\, M_i^{10}$$

qui montrent que les deux crochets $\big[\Lambda^7_{i,i},\,\Lambda^3\big]$ pour $i=1,2$ sont superflus. Si nous faisons ensuite $i=1$ et $j=2$, nous obtenons :

$$\begin{aligned}0 &\equiv \big[\Lambda^7_{1,2},\,\Lambda^3\big] - M^8\,\big[f'_1,\,f'_j\big] - f'_1\,\big[M^8,\,f'_2\big] + \big[\Lambda^5_1,\,\Lambda^5_2\big]\\ &\equiv \big[\Lambda^7_{1,2},\,\Lambda^3\big] + M^8\,\Lambda^3 - f'_1\,M^{10}_2 - 5\,\Lambda^3\,M^8\\ &\equiv \big[\Lambda^7_{1,2},\,\Lambda^3\big] - 4\,\Lambda^3\,M^8 - f'_1\,M^{10}_2,\end{aligned}$$

ce qui montre que $\big[\Lambda^7_{1,2},\,\Lambda^3\big]$ est superflu. De la symétrie indicielle $\Lambda^7_{1,2}=\Lambda^7_{2,1}$ on peut déduire sans plus de calcul que le crochet $\big[\Lambda^7_{2,1},\,\Lambda^3\big]$ est lui aussi superflu, mais il est instructif de faire quand même $i=2$ et $j=1$ dans l'identité générale :

$$\begin{aligned}0 &\equiv \big[\Lambda^7_{2,1}\,\Lambda^3\big] - M^8\,\big[f'_2,\,f'_1\big] - f'_2\,\big[M^8,\,f'_1\big] - \big[\Lambda^5_1,\,\Lambda^5_2\big]\\ &\equiv \big[\Lambda^7_{2,1},\,\Lambda^3\big] - M^8\,\Lambda^3 - f'_2\,M^{10}_1 + 5\,\Lambda^3\,M^8\\ &\equiv \big[\Lambda^7_{2,1},\,\Lambda^3\big] + 4\,\Lambda^3\,M^8 - f'_2\,M^{10}_1.\end{aligned}$$

Bien que $\Lambda^7_{1,2}=\Lambda^7_{2,1}$, nous obtenons une relation indépendante de la précédente, et par soustraction, nous obtenons une nouvelle relation, que nous énonçons en passant :

$$0 \equiv f'_2\,M^{10}_1 - f'_1\,M^{10}_2 - 8\,\Lambda^3\,M^8,$$

ce qui anticipe un fait qui va se révéler crucial par la suite : les invariants fondamentaux formés par crochets jouissent d'un très grand nombre de relations algébriques, parfois appelées *syzygies*, qu'il est difficile d'englober dans une combinatoire unifiée. Poursuivons toutefois pour l'instant notre préparation de tous les invariants que l'on peut former par crochets.

Quatrième famille de crochets $\big[M^8,\,\Lambda^3\big]$. Le calcul, "*most elementary*", donne :

$$\begin{aligned}\big[M^8,\,\Lambda^3\big] &= 3\,\mathsf{D}M^8\cdot\Lambda^3 - 8\,M^8\cdot\mathsf{D}\Lambda^3\\ &= 9\,\Delta^{1,5}\,\Delta^{1,2}\,\Delta^{1,2} + 45\,\Delta^{2,4}\,\Delta^{1,2}\,\Delta^{1,2} - 45\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,2} -\\ &\quad - 90\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2} + 40\,\Delta^{1,3}\,\Delta^{1,3}\,\Delta^{1,3}\\ &=: N^{12}.\end{aligned}$$

C'est un nouvel invariant N^{12} de poids 12 qui a la propriété remarquable de s'exprimer seulement en fonction des déterminants $\Delta^{\alpha,\beta}$. En compagnie de Λ^3 et de M^8, il jouera un rôle central dans l'élaboration d'une base de Gröbner pour l'idéal des syzygies entre les invariants fondamentaux.

Cinquième famille de crochets $\big[\Lambda^7_{i,j},\,\Lambda^5_k\big]$. Cette fois-ci, nous ne détaillerons pas les calculs intermédiaires, puisque nous avons déjà évoqué

à présent tous les actes qui permettent de les accomplir. Nous obtenons :

$$\begin{aligned}
\big[\Lambda^7_{i,j},\,\Lambda^5_k\big] &= 5\,\mathsf{D}\Lambda^7_{i,j}\cdot\Lambda^5_k - 7\,\Lambda^7_{i,j}\cdot\mathsf{D}\Lambda^5_k\\
&= 5\,\Delta^{1,5}\,\Delta^{1,3}\,f'_if'_jf'_k + 25\,\Delta^{2,4}\,\Delta^{1,3}\,f'_if'_jf'_k - 7\,\Delta^{1,4}\,\Delta^{1,4}\,f'_if'_jf'_k -\\
&\quad - 56\,\Delta^{2,3}\,\Delta^{1,4}\,f'_if'_jf'_k - 112\,\Delta^{2,3}\,\Delta^{2,3}\,f'_if'_jf'_k - 15\,\Delta^{1,5}\,\Delta^{1,2}\,f'_if'_jf''_k -\\
&\quad - 75\,\Delta^{2,4}\,\Delta^{1,2}\,f'_if'_jf''_k + 15\,\Delta^{1,4}\,\Delta^{1,3}\big(f''_if'_j + f'_if''_j\big)f'_k +\\
&\quad + 35\,\Delta^{1,4}\,\Delta^{1,3}\,f'_if'_jf''_k + 60\,\Delta^{2,3}\,\Delta^{1,3}\big(f''_if'_j + f'_if''_j\big)f'_k -\\
&\quad - 10\,\Delta^{2,3}\,\Delta^{1,3}\,f'_if'_jf''_k - 25\,\Delta^{1,3}\,\Delta^{1,3}\big(f'''_if'_j + f'_if'''_j\big)f'_k +\\
&\quad + 175\,\Delta^{1,3}\,\Delta^{1,3}\,f''_if''_jf'_k - 100\,\Delta^{1,3}\,\Delta^{1,3}\big(f''_if'_j + f'_if''_j\big)f''_k +\\
&\quad + 60\,\Delta^{1,4}\,\Delta^{1,2}\big(f''_if'_j + f'_if''_j\big)f''_k - 105\,\Delta^{1,4}\,\Delta^{1,2}\,f''_if''_jf'_k +\\
&\quad + 240\,\Delta^{2,3}\,\Delta^{1,2}\big(f''_if'_j + f'_if''_j\big)f''_k - 420\,\Delta^{2,3}\,\Delta^{1,2}\,f''_if''_jf'_k.
\end{aligned}$$

Ces six invariants sont nouveaux, à ceci près qu'ils ne sont pas indépendants entre eux. En effet, $(\mathscr{J}ac)$ donne :

$$0 \equiv \big[\big[f'_i,\,\Lambda^5_j\big],\,\Lambda^5_k\big] + \big[\big[\Lambda^5_k,\,f'_i\big],\,\Lambda^5_j\big] + \big[\big[\Lambda^5_j,\,\Lambda^5_k\big],\,f'_i\big],$$

relations qui se réduisent à $0 \equiv 0$ lorsque $j = k$, mais qui fournissent deux relations non triviales lorsque $j \neq k$, à savoir :

$$\begin{aligned}
\big[\Lambda^7_{1,2},\,\Lambda^5_1\big] &= \big[\Lambda^7_{1,1},\,\Lambda^5_2\big] + 5\,M^8\,\Lambda^5_1 + 5\,\Lambda^3\,M^{10}_1,\\
\big[\Lambda^7_{1,2},\,\Lambda^5_2\big] &= \big[\Lambda^7_{2,2},\,\Lambda^5_1\big] - 5\,M^8\,\Lambda^5_2 - 5\,\Lambda^3\,M^{10}_2.
\end{aligned}$$

Celles-ci nous permettent de n'avoir à considérer que les quatre (au lieu de six) nouveaux invariants :

$$\begin{aligned}
K^{13}_{1,1,1} &:= \big[\Lambda^7_{1,1},\,\Lambda^5_1\big], & K^{13}_{1,1,2} &:= \big[\Lambda^7_{1,1},\,\Lambda^5_2\big],\\
K^{13}_{2,2,1} &:= \big[\Lambda^7_{2,2},\,\Lambda^5_1\big], & K^{13}_{2,2,2} &:= \big[\Lambda^7_{2,2},\,\Lambda^5_2\big].
\end{aligned}$$

Cependant, le travail n'est pas terminé. Puisque nous constatons que $K^{13}_{1,1,1}$ est divisible par f'_1, nous devons introduire l'invariant de poids 12 :

$$\begin{aligned}
K^{12}_{1,1} &:= \frac{1}{f'_1}\,\big[\Lambda^7_{1,1},\,\Lambda^5_1\big]\\
&= f'_1f'_1\Big(5\,\Delta^{1,5}\,\Delta^{1,3} + 25\,\Delta^{2,4}\,\Delta^{1,3} - 7\,\Delta^{1,4}\,\Delta^{1,4} - 56\,\Delta^{2,3}\,\Delta^{1,4} -\\
&\quad - 112\,\Delta^{2,3}\,\Delta^{2,3}\Big) + f'_1f''_1\Big(-15\,\Delta^{1,5}\,\Delta^{1,2} - 75\,\Delta^{2,4}\,\Delta^{1,2} +\\
&\quad + 65\,\Delta^{1,4}\,\Delta^{1,3} + 110\,\Delta^{2,3}\,\Delta^{1,3}\Big) + f_1f'''_1\Big(-50\,\Delta^{1,3}\,\Delta^{1,3}\Big) +\\
&\quad + f''_1f''_1\Big(-25\,\Delta^{1,3}\,\Delta^{1,3} + 15\,\Delta^{1,4}\,\Delta^{1,2} + 60\,\Delta^{2,3}\,\Delta^{1,2}\Big).
\end{aligned}$$

Pareillement, $K^{13}_{2,2,2}$ étant divisible par f'_2, nous devons introduire cet invariant défini par $K^{12}_{2,2} := \frac{1}{f'_1}\,\big[\Lambda^7_{2,2},\,\Lambda^5_2\big]$. Mais les deux invariants restants, à savoir $K^{13}_{1,1,2}$ et $K^{13}_{2,2,1}$, ne sont divisibles ni par f'_1 ni par f'_2. Or nous verrons dans la suite qu'il est naturel d'introduire des "polarisations" de certains

invariant spéciaux — tels que $K^{12}_{1,1}$ — qui ne comportent que des "1" en indices inférieurs et que nous appelerons *bi-invariants*, les "polarisations" de ces invariants spéciaux consistant tout simplement à mettre des "1" et des "2" de toutes les manières possibles en indices inférieurs. Par exemple, les polarisations de $\Lambda^7_{1,1}$ sont : $\Lambda^7_{1,2}$, $\Lambda^7_{2,1}$ et $\Lambda^7_{2,2}$. Mais alors, comment donc les deux invariants $K^{13}_{1,1,2}$ et $K^{13}_{2,2,1}$ pourraient-ils être obtenus par polarisation, sachant qu'ils ont trois indices inférieurs? Faut-il revenir à $K^{13}_{1,1,1}$ et le polariser?

Lemme. Si l'on introduit, pour tous i, j appartenant à $\{1, 2\}$, les quatre invariants de poids 12 :

$$\begin{aligned}
K^{12}_{i,j} := {} & f'_i f'_j \Big(5\,\Delta^{1,5}\,\Delta^{1,3} + 25\,\Delta^{2,4}\,\Delta^{1,3} - 7\,\Delta^{1,4}\,\Delta^{1,4} - 56\,\Delta^{2,3}\,\Delta^{1,4} - \\
& - 112\,\Delta^{2,3}\,\Delta^{2,3}\Big) + \frac{(f'_i f''_j + f''_i f'_j)}{2}\Big(-15\,\Delta^{1,5}\,\Delta^{1,2} - 75\,\Delta^{2,4}\,\Delta^{1,2} + \\
& + 65\,\Delta^{1,4}\,\Delta^{1,3} + 110\,\Delta^{2,3}\,\Delta^{1,3}\Big) + \frac{(f'_i f'''_j + f'''_i f'_j)}{2}\Big(-50\,\Delta^{1,3}\,\Delta^{1,3}\Big) + \\
& + f''_i f''_j \Big(-25\,\Delta^{1,3}\,\Delta^{1,3} + 15\,\Delta^{1,4}\,\Delta^{1,2} + 60\,\Delta^{2,3}\,\Delta^{1,2}\Big),
\end{aligned}$$

alors les quatre invariants $K^{13}_{1,1,1}$, $K^{13}_{1,1,2}$, $K^{13}_{2,2,1}$ et $K^{13}_{2,2,2}$ précédents sont réobtenus au moyen des quatre relations :

$$\begin{aligned}
K^{13}_{1,1,1} &= f'_1\,K^{12}_{1,1}, \\
K^{13}_{1,1,2} &= f'_1\,K^{12}_{1,2} - \frac{5}{2}\,\Lambda^3\,M^{10}_1 - 5\,\Lambda^5_1\,M^8, \\
K^{13}_{2,2,1} &= f'_2\,K^{12}_{2,1} + \frac{5}{2}\,\Lambda^3\,M^{10}_2 + 5\,\Lambda^5_2\,M^8, \\
K^{13}_{2,2,2} &= f'_2\,K^{12}_{2,2}.
\end{aligned}$$

Grâce à ce lemme bienvenu, nous pouvons donc introduire l'invariant réduit $K^{12}_{1,1}$ accompagné de ses trois polarisations $K^{12}_{1,2}$, $K^{12}_{2,1}$ et $K^{12}_{2,2}$ (en fait $K^{12}_{1,2} = K^{12}_{2,1}$), et oublier purement et simplement les quatre invariants de poids 13 qui nous étaient fournis par crochets bruts.

Preuve. En considérant la permutation $1 \leftrightarrow 2$ des indices (noter que les $\Delta^{\alpha,\beta}$ changent de signe), il suffit d'établir la deuxième identité. Nous réécrivons tout d'abord, en repartant de l'expression obtenue pour $\big[\Lambda^7_{1,1},\,\Lambda^5_2\big]$:

$$\begin{aligned}
K^{13}_{1,1,2} = {} & f'_1 f'_1 f'_2 \Big(5\,\Delta^{1,5}\,\Delta^{1,3} + 25\,\Delta^{2,4}\,\Delta^{1,3} - 7\,\Delta^{1,4}\,\Delta^{1,4} - 56\,\Delta^{2,3}\,\Delta^{1,4} - \\
& - 112\,\Delta^{2,3}\,\Delta^{2,3}\Big) + f'_1 f'_1 f''_2 \Big(-15\,\Delta^{1,5}\,\Delta^{1,2} - 75\,\Delta^{2,4}\,\Delta^{1,2} + \\
& + 35\,\Delta^{1,4}\,\Delta^{1,3} - 10\,\Delta^{2,3}\,\Delta^{1,3}\Big) + f'_1 f''_1 f'_2 \Big(30\,\Delta^{1,4}\,\Delta^{1,3} + 120\,\Delta^{2,3}\,\Delta^{1,3}\Big) + \\
& + f'_1 f''_1 f'_2 \Big(-50\,\Delta^{1,3}\,\Delta^{1,3}\Big) + f''_1 f''_1 f'_2 \Big(175\,\Delta^{1,3}\,\Delta^{1,3} - 105\,\Delta^{1,4}\,\Delta^{1,2} - \\
& - 420\,\Delta^{2,3}\,\Delta^{1,2}\Big) + f'_1 f''_1 f''_2 \Big(-200\,\Delta^{1,3}\,\Delta^{1,3} + 120\,\Delta^{1,4}\,\Delta^{1,2} + 480\,\Delta^{2,3}\,\Delta^{1,2}\Big).
\end{aligned}$$

Ensuite, nous effectuons la soustraction :

$$\begin{aligned}
&K^{13}_{1,1,1} - f'_1\, K^{12}_{1,2} = \\
&\quad = f'_1 f'_1 f''_2 \Big(-\frac{15}{2}\,\Delta^{1,5}\,\Delta^{1,2} - \frac{75}{2}\,\Delta^{2,4}\,\Delta^{1,2} + \frac{5}{2}\,\Delta^{1,4}\,\Delta^{1,3} - 65\,\Delta^{2,3}\,\Delta^{1,3}\Big) + \\
&\qquad + f'_1 f''_1 f'_2 \Big(\frac{15}{2}\,\Delta^{1,5}\,\Delta^{1,2} + \frac{75}{2}\,\Delta^{2,4}\,\Delta^{1,2} - \frac{5}{2}\,\Delta^{1,4}\,\Delta^{1,3} + 65\,\Delta^{2,3}\,\Delta^{1,3}\Big) + \\
&\qquad + f'_1 f'''_1 f'_2 \Big(-\frac{50}{2}\,\Delta^{1,3}\,\Delta^{1,3}\Big) + \\
&\qquad + f'_1 f'_1 f'''_2 \Big(\frac{50}{2}\,\Delta^{1,3}\,\Delta^{1,3}\Big) + \\
&\qquad + f''_1 f''_1 f'_2 \Big(175\,\Delta^{1,3}\,\Delta^{1,3} - 105\,\Delta^{1,4}\,\Delta^{1,2} - 420\,\Delta^{2,3}\,\Delta^{1,2}\Big) + \\
&\qquad + f''_1 f'_1 f''_2 \Big(-175\,\Delta^{1,3}\,\Delta^{1,3} + 105\,\Delta^{1,4}\,\Delta^{1,2} + 420\,\Delta^{2,3}\,\Delta^{1,2}\Big).
\end{aligned}$$

Remarquablement, les coefficients polynomiaux complexes étant opposés par paires de lignes qui se suivent, nous voyons des déterminants 2×2 se reformer :

$$\begin{aligned}
&K^{13}_{1,1,1} - f'_1\, K^{12}_{1,2} = \\
&\quad = f'_1 \Big(-\frac{15}{2}\,\Delta^{1,5}\,\Delta^{1,2}\,\Delta^{1,2} - \frac{75}{2}\,\Delta^{2,4}\,\Delta^{1,2}\,\Delta^{1,2} + \frac{5}{2}\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,2} - \\
&\qquad\quad - 65\,\Delta^{2,3}\,\Delta^{1,2}\,\Delta^{1,2}\Big) + f'_1 \Big(\frac{50}{2}\,\Delta^{1,3}\,\Delta^{1,3}\,\Delta^{1,2}\Big) + \\
&\qquad + f''_1 \Big(105\,\Delta^{1,4}\,\Delta^{1,2}\,\Delta^{1,2} + 420\,\Delta^{2,3}\,\Delta^{1,2}\,\Delta^{1,2} - 175\,\Delta^{1,3}\,\Delta^{1,3}\,\Delta^{1,3}\Big).
\end{aligned}$$

Les deux premiers termes de la première lignes ressemblant à ceux de $\Lambda^3\, M^{10}_1$, nous pouvons écrire :

$$\begin{aligned}
&\quad = -\frac{5}{2}\,\Lambda^3\, M^{10}_1 + \\
&\quad + f'_1 \Big(-\frac{30}{2}\,\Delta^{1,4}\,\Delta^{1,3}\Delta^{1,2} - 60\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2} + \frac{50}{2}\,\Delta^{1,3}\,\Delta^{1,2}\,\Delta^{1,3}\Big) + \\
&\quad + f''_1 \Big(45\,\Delta^{1,4}\,\Delta^{1,2}\,\Delta^{1,2} + 180\,\Delta^{2,3}\,\Delta^{1,2}\,\Delta^{1,2} - 75\,\Delta^{1,3}\,\Delta^{1,3}\,\Delta^{1,2}\Big).
\end{aligned}$$

Et enfin, nous reconnaissons dans les termes restants l'expression développée de $-5\,\Lambda^5_1\, M^8$, ce qui nous donne bien la relation annoncée, que nous réécrivons : $K^{13}_{1,1,2} - f'_1\, K^{12}_{1,2} = -\frac{5}{2}\,\Lambda^3\, M^{10}_1 - 5\,\Lambda^5_1\, M^8$. □

Sixième famille de crochets $\big[M^8,\, \Lambda^5_i\big]$. Par un calcul facile, court et sans mystère qui nous permer de reprendre haleine avant d'envisager la septième et la plus complexe famille de crochets, nous obtenons :

$$\begin{aligned}
\big[M^8,\, \Lambda^5_i\big] &= 5\,\mathsf{D}M^8 \cdot \Lambda^5_i - 8\, M^8 \cdot \mathsf{D}\Lambda^5_i \\
&= \Big(15\,\Delta^{1,5}\,\Delta^{1,3}\,\Delta^{1,2} + 75\,\Delta^{2,4}\,\Delta^{1,3}\,\Delta^{1,2} + 5\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,3} + \\
&\quad + 170\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,3} - 24\,\Delta^{1,4}\,\Delta^{1,4}\,\Delta^{1,2} - 192\,\Delta^{1,4}\,\Delta^{2,3}\,\Delta^{1,2} - \\
&\quad - 384\,\Delta^{2,3}\,\Delta^{2,3}\,\Delta^{1,2}\Big)\, f'_i + \Big(-45\,\Delta^{1,5}\,\Delta^{1,2}\,\Delta^{1,2} - 225\,\Delta^{2,4}\,\Delta^{1,2}\,\Delta^{1,2} + \\
&\quad + 225\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,2} + 450\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2} - 200\,\Delta^{1,3}\,\Delta^{1,3}\,\Delta^{1,3}\Big)\, f''_i \\
&=: H^{14}_i.
\end{aligned}$$

Le résultat n'est divisible ni par Λ_i^5 (sinon $\mathsf{D}\Lambda_i^5$ le serait), ni par Λ^3, ni par f_i'. Nous trouvons donc deux nouveaux invariants H_1^{14} et H_2^{14} de poids 14.

Septième famille de crochets $\big[\Lambda_{i,j}^7,\, \Lambda_{k,l}^7\big]$. Le calcul complet, que nous détaillons dans la Section 9 parce qu'il est délicat, donne :

$$\begin{aligned}
\frac{\big[\Lambda_{i,j}^7,\, \Lambda_{k,l}^7\big]}{7} &= \mathsf{D}\Lambda_{i,j}^7 \cdot \Lambda_{k,l}^7 - \Lambda_{i,j}^7 \cdot \mathsf{D}\Lambda_{k,l}^7 \\
&= \Big(-5\,\Delta^{1,5}\,\Delta^{1,3} - 25\,\Delta^{2,4}\,\Delta^{1,3} + 4\,\Delta^{1,4}\,\Delta^{1,4} + 32\,\Delta^{1,4}\,\Delta^{2,3} + \\
&\qquad + 64\,\Delta^{2,3}\,\Delta^{2,3}\Big)\big(f_j' f_l'\,\Delta_{i,k}^{1,2} + f_i' f_k'\,\Delta_{j,l}^{1,2}\big) + \\
&+ \Big(15\,\Delta^{1,5}\,\Delta^{1,2} + 75\,\Delta^{2,4}\,\Delta^{1,2} - 35\,\Delta^{1,4}\,\Delta^{1,3} + \\
&\qquad + 10\,\Delta^{2,3}\,\Delta^{1,3}\Big)\big(f_i' f_l''\,\Delta_{j,k}^{1,2} + f_k' f_j''\,\Delta_{i,l}^{1,2}\big) + \\
&+ \Big(-5\,\Delta^{1,4}\,\Delta^{1,3} - 20\,\Delta^{2,3}\,\Delta^{1,3}\Big)\big(f_j' f_l'\,\Delta_{k,i}^{1,3} + f_i' f_k'\,\Delta_{l,j}^{1,3}\big) + \\
&+ \Big(25\,\Delta^{1,3}\,\Delta^{1,3}\Big)\big(f_j' f_l'\,\Delta_{k,i}^{2,3} + f_j' f_k'\,\Delta_{l,i}^{2,3} + f_i' f_l'\,\Delta_{k,j}^{2,3} + f_i' f_k'\,\Delta_{l,j}^{2,3}\big) + \\
&+ \Big(-60\,\Delta^{1,4}\,\Delta^{1,2} - 240\,\Delta^{2,3}\,\Delta^{1,2} + \\
&\qquad + 100\,\Delta^{1,3}\,\Delta^{1,3}\Big)\big(f_i'' f_k''\,\Delta_{j,l}^{1,2} + f_j'' f_l''\,\Delta_{i,k}^{1,2}\big).
\end{aligned}$$

Nous trouvons ainsi trois invariants de poids 15 :

$$\big[\Lambda_{1,1}^7,\, \Lambda_{1,2}^7\big], \qquad \big[\Lambda_{1,1}^7,\, \Lambda_{2,2}^7\big], \qquad \big[\Lambda_{1,2}^7,\, \Lambda_{2,2}^7\big],$$

mais cependant, ces invariants s'expriment en fonction de ceux que nous connaissons déjà. En effet, spécialisons tout d'abord les indices dans la formule générale et nettoyons les expressions obtenues :

$$\begin{aligned}
\frac{\big[\Lambda_{1,1}^7,\, \Lambda_{1,2}^7\big]}{7} &= f_1' f_1'\Big(-5\,\Delta^{1,5}\,\Delta^{1,3}\,\Delta^{1,2} - 25\,\Delta^{2,4}\,\Delta^{1,3}\,\Delta^{1,2} + 4\,\Delta^{1,4}\,\Delta^{1,4}\,\Delta^{1,2} + \\
&\qquad + 32\,\Delta^{1,4}\,\Delta^{2,3}\,\Delta^{1,2} + 64\,\Delta^{2,3}\,\Delta^{2,3}\,\Delta^{1,2} + \\
&\qquad + 5\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,3} - 30\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,3}\Big) + \\
&+ f_1' f_1''\Big(15\,\Delta^{1,5}\,\Delta^{1,2}\,\Delta^{1,2} + 75\,\Delta^{2,4}\,\Delta^{1,2}\,\Delta^{1,2} - \\
&\qquad - 35\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,2} + 10\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2}\Big) + \\
&+ f_1'' f_1''\Big(-60\,\Delta^{1,4}\,\Delta^{1,2}\,\Delta^{1,2} - 240\,\Delta^{2,3}\,\Delta^{1,2}\,\Delta^{1,2} + \\
&\qquad + 100\,\Delta^{1,3}\,\Delta^{1,3}\,\Delta^{1,2}\Big),
\end{aligned}$$

$$\frac{\big[\Lambda_{1,2}^7,\, \Lambda_{2,2}^7\big]}{7} = \text{Idem}\big(\text{indice } \mathbf{1} \longleftrightarrow \text{indice } \mathbf{2}\big),$$

$$\begin{aligned}\frac{\big[\Lambda^7_{1,1},\,\Lambda^7_{2,2}\big]}{7} = f_1'f_2'\Big(&-10\,\Delta^{1,5}\,\Delta^{1,3}\,\Delta^{1,2} - 50\,\Delta^{2,4}\,\Delta^{1,3}\,\Delta^{1,2} + 8\,\Delta^{1,4}\,\Delta^{1,4}\,\Delta^{1,2} + \\ &+ 64\,\Delta^{1,4}\,\Delta^{2,3}\,\Delta^{1,2} + 128\,\Delta^{2,3}\,\Delta^{2,3}\,\Delta^{1,2} + \\ &+ 10\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,3} - 60\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,3}\Big) + \\ + \Big(\frac{f_1'f_2''+f_1''f_2'}{2}\Big)\Big(&30\,\Delta^{1,5}\,\Delta^{1,2}\,\Delta^{1,2} + 150\,\Delta^{2,4}\,\Delta^{1,2}\,\Delta^{1,2} - \\ &- 70\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,2} + 20\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2}\Big) + \\ + f_1''f_2''\Big(&-120\,\Delta^{1,4}\,\Delta^{1,2}\,\Delta^{1,2} - 480\,\Delta^{2,3}\,\Delta^{1,2}\,\Delta^{1,2} + \\ &+ 200\,\Delta^{1,3}\,\Delta^{1,3}\,\Delta^{1,2}\Big).\end{aligned}$$

Si nous examinons le polynôme cubique en Δ qui est multiple de $f_1''f_1''$ dans les deux dernières lignes de l'expression de $\frac{1}{7}\big[\Lambda^7_{1,1},\,\Lambda^7_{1,2}\big]$, nous reconnaissons par exemple $-20\,M^8\,\Delta^{1,2}$, ce qui constitue une coïncidence que nous devrions manifestement exploiter, et ensuite, sans plus tenter de décrire l'ascèse visuelle qui nous permet de deviner des relations algébriques entre de telles expressions, nous trouvons les trois relations suivantes immédiatement vérifiables par développement :

$$\begin{aligned}0 &\equiv 6\,\big[\Lambda^7_{1,1},\,\Lambda^7_{1,2}\big] + 35\,\Lambda^5_1\,M^{10}_1 + f_1'\,H^{14}_1,\\ 0 &\equiv 6\,\big[\Lambda^7_{1,1},\,\Lambda^7_{1,2}\big] + 35\,\big(\Lambda^5_1\,M^{10}_2 + \Lambda^5_2\,M^{10}_1\big) + f_1'\,H^{14}_2 + f_2'\,H^{14}_1,\\ 0 &\equiv 6\,\big[\Lambda^7_{1,2},\,\Lambda^7_{2,2}\big] + 35\,\Lambda^5_2\,M^{10}_2 + f_2'\,H^{14}_2,\end{aligned}$$

qui montrent que les trois invariants $\big[\Lambda^7_{1,1},\,\Lambda^7_{1,2}\big]$, $\big[\Lambda^7_{1,1},\,\Lambda^7_{2,2}\big]$ et $\big[\Lambda^7_{1,2},\,\Lambda^7_{2,2}\big]$ sont en fait superflus. Bien qu'elle nous ait coûté de réels efforts de calculs, cette circonstance n'est pas sans nous déplaire, puisque nous pouvons ainsi réduire de trois unités le nombre d'invariants fondamentaux que nous aurons à considérer ultérieurement.

Huitième famille de crochets $\big[M^8,\,\Lambda^7_{i,j}\big]$. Le calcul, qui implique seulement quelques normalisations plückeriennes et bien sûr aussi de l'arithmétique formelle élémentaire, fournit l'expression massive suivante,

qui est en fait complètement simplifiée :

$$
\begin{aligned}
&[M^8,\,\Lambda^7_{i,j}] = 7\,\mathsf{D}M^8\cdot\Lambda^7_{i,j} - 8\,M^8\cdot\mathsf{D}\Lambda^7_{i,j}\\
&= \Big(-3\,\Delta^{1,5}\,\Delta^{1,4}\,\Delta^{1,2} - 15\,\Delta^{2,4}\,\Delta^{1,4}\,\Delta^{1,2} - 12\,\Delta^{1,5}\,\Delta^{2,3}\,\Delta^{1,2} +\\
&\quad + 40\,\Delta^{1,5}\,\Delta^{1,3}\,\Delta^{1,3} - 60\,\Delta^{2,4}\,\Delta^{2,3}\,\Delta^{1,2} + 200\,\Delta^{2,4}\,\Delta^{1,3}\,\Delta^{1,3} -\\
&\quad - 49\,\Delta^{1,4}\,\Delta^{1,4}\,\Delta^{1,3} - 422\,\Delta^{1,4}\,\Delta^{2,3}\,\Delta^{1,3} - 904\,\Delta^{2,3}\,\Delta^{2,3}\,\Delta^{1,3}\Big) f'_i f'_j +\\
&\quad + \Big(-105\,\Delta^{1,5}\,\Delta^{1,3}\,\Delta^{1,2} - 525\,\Delta^{2,4}\,\Delta^{1,3}\,\Delta^{1,2} + 205\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,3} -\\
&\quad - 230\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,3} + 96\,\Delta^{1,4}\,\Delta^{1,4}\,\Delta^{1,2} + 768\,\Delta^{1,4}\,\Delta^{2,3}\,\Delta^{1,2} +\\
&\quad + 1536\,\Delta^{2,3}\,\Delta^{2,3}\,\Delta^{1,2}\Big)(f''_i f'_j + f'_i f''_j) +\\
&\quad + \Big(-200\,\Delta^{1,3}\,\Delta^{1,3}\,\Delta^{1,3}\Big)(f'''_i f'_j + f'_i f'''_j) +\\
&\quad + \Big(315\,\Delta^{1,5}\,\Delta^{1,2}\,\Delta^{1,2} + 1575\,\Delta^{2,4}\,\Delta^{1,2}\,\Delta^{1,2} - 1575\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,2} -\\
&\quad - 3150\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2} + 1400\,\Delta^{1,3}\,\Delta^{1,3}\,\Delta^{1,3}\Big) f''_i f''_j.
\end{aligned}
$$

Nous obtenons ainsi trois nouveaux invariants de poids 16 :

$$
F^{16}_{1,1} := \big[M^8,\,\Lambda^7_{1,1}\big], \qquad F^{16}_{1,2} := \big[M^8,\,\Lambda^7_{1,2}\big] = F^{16}_{2,1},
$$
$$
F^{16}_{2,2} := \big[M^8,\,\Lambda^7_{2,2}\big].
$$

Système générateur pour les jets d'ordre 5. Il nous faut donc considérer les vingt-cinq polynômes invariants fondamentaux :

$$
\boxed{\begin{array}{r}
f'_1,\ f'_2,\ \Lambda^3,\ \Lambda^5_1,\ \Lambda^5_2,\ \Lambda^7_{1,1},\ \Lambda^7_{1,2},\ \Lambda^7_{2,2},\ M^8,\\
\Lambda^9_{1,1,1},\ \Lambda^9_{1,2,1},\ \Lambda^9_{2,1,2},\ \Lambda^9_{2,2,2},\ M^{10}_1,\ M^{10}_2,\\
N^{12},\ K^{12}_{1,1},\ K^{12}_{1,2},\ K^{12}_{2,1},\ K^{12}_{2,2},\\
H^{14}_1,\ H^{14}_2,\ F^{16}_{1,1},\ F^{16}_{1,2},\ F^{16}_{2,2}.
\end{array}}
$$

Problème. Décrire explicitement l'idéal des relations entre ces vingt-cinq polynômes et établir que tout polynôme invariant par reparamétrisation $\mathsf{P}\big(j^5 f\big)$ se représente comme polynôme $\mathscr{P}\big(f'_1,\dots,M^8,\dots,H^{14}_1,\dots,F^{16}_{2,2}\big)$ en ces vingt-cinq invariants fondamentaux.

§5. Jets d'ordre 4 en dimension 2

Neuf relations fondamentales Cette section et celle qui suit sont consacrées à l'étude plus accessible de $\mathscr{DS}^4_2$. Pour les jets d'ordre 4, on considère les neuf polynômes invariants fondamentaux :

$$
\Big(\ f'_1,\quad f'_2,\quad \Lambda^3,\quad \Lambda^5_1,\quad \Lambda^5_2,\quad \Lambda^7_{1,1},\quad \Lambda^7_{1,2},\quad \Lambda^7_{2,2},\quad M^8\ \Big),
$$

et avant de passer aux jets d'ordre cinq, nous allons démontrer que tout polynôme invariant par reparamétrisation $\mathsf{P}\big(j^4 f\big)$ se représente comme polynôme de la forme $\mathscr{P}\big(f'_1,\dots,\Lambda^5_2,\dots,M^8\big)$.

Trois calculs distincts sur Maple[64] conduisent aux neuf relations fondamentales suivantes entre ces neuf invariants :

$$\Big[0 \overset{1}{\equiv} f_2' \, \Lambda_1^5 - f_1' \, \Lambda_2^5 - 3 \, \Lambda^3 \, \Lambda^3,$$

$$\left[\begin{array}{l} 0 \overset{2}{\equiv} f_2' \, \Lambda_{1,1}^7 - f_1' \, \Lambda_{1,2}^7 - 5 \, \Lambda^3 \, \Lambda_1^5, \\ 0 \overset{3}{\equiv} f_2' \, \Lambda_{1,2}^7 - f_1' \, \Lambda_{2,2}^7 - 5 \, \Lambda^3 \, \Lambda_2^5, \end{array} \right.$$

$$\left[\begin{array}{l} 0 \overset{4}{\equiv} f_1' \, f_1' \, M^8 - 3 \, \Lambda^3 \, \Lambda_{1,1}^7 + 5 \, \Lambda_1^5 \, \Lambda_1^5, \\ 0 \overset{5}{\equiv} f_1' \, f_2' \, M^8 - 3 \, \Lambda^3 \, \Lambda_{1,2}^7 + 5 \, \Lambda_1^5 \, \Lambda_2^5, \\ 0 \overset{6}{\equiv} f_2' \, f_2' \, M^8 - 3 \, \Lambda^3 \, \Lambda_{2,2}^7 + 5 \, \Lambda_2^5 \, \Lambda_2^5, \end{array} \right.$$

$$\left[\begin{array}{l} 0 \overset{7}{\equiv} f_1' \, \Lambda^3 \, M^8 - \Lambda_1^5 \, \Lambda_{1,2}^7 + \Lambda_2^5 \, \Lambda_{1,1}^7, \\ 0 \overset{8}{\equiv} f_2' \, \Lambda^3 \, M^8 - \Lambda_1^5 \, \Lambda_{2,2}^7 + \Lambda_2^5 \, \Lambda_{1,2}^7, \end{array} \right.$$

$$\Big[0 \overset{9}{\equiv} 5 \, \Lambda^3 \, \Lambda^3 \, M^8 - \Lambda_{2,2}^7 \, \Lambda_{1,1}^7 + \Lambda_{1,2}^7 \, \Lambda_{1,2}^7.$$

Dans le cas des jets d'ordre $\kappa = 3$, seule la première relation est présente ; l'idéal des relations est principal et il constitue *per se* la base de Gröbner adéquate.

Question. Comment retrouver et prévoir l'existence de toutes ces relations ?

Indication de réponse. Au niveau $\kappa = 4$, une seule identité de Jacobi peut être formée et elle donne la relation déjà connue $\Lambda_{1,2}^7 = \Lambda_{2,1}^7$, qui est cependant triviale par rapport aux neufs identités listées ci-dessus. Mais en déchiffrant ces neuf équations, ou bien en observant que les deux familles d'identités qui sont satisfaites dans les algèbres plückeriennes et que nous avons déjà utilisées systématiquement pour normaliser l'expression définitive de nos crochets, doivent nécessairement et naturellement "emboîter notre pas" lorsque nous formons tous les déterminants 2×2 de la matrice

$$\left\| \begin{array}{ccccc} f_1' & f_2' & 3 \, \Lambda^3 & 5 \, \Lambda_1^5 & 5 \, \Lambda_2^5 \\ \mathsf{D} f_1' & \mathsf{D} f_2' & \mathsf{D} \Lambda^3 & \mathsf{D} \Lambda_1^5 & \mathsf{D} \Lambda_2^5 \end{array} \right\|,$$

nous devinons[65], ou nous constatons, que le lemme général suivant est vrai.

---

[64] Les résultats que l'on reçoit (après un quart d'heure de calcul environ) dépendent de l'ordre monomial choisi ; ils comprennent d'autres relations superflues, *i.e.* déduites des neuf fondamentales que nous listons, parce que le logiciel doit effectuer algorithmiquement l'opération dite "S-polynôme" sur tous les couples d'identités, suivie d'une division par les éléments présents, afin de compléter le calcul d'une base de Gröbner réduite. L'auteur remercie Erwan Rousseau de lui avoir communiqué cette liste, ainsi que l'invariant M^8.

[65] L'auteur a d'abord deviné et reconstitué les relations générales qui sous-tendent les neufs identités précédentes avant d'arranger synoptiquement la formation de nouveaux invariants par simple calcul de mineurs 2×2.

Lemme. Pour tout quadruplet $\big(\mathsf{P},\,\mathsf{Q},\,\mathsf{R},\,\mathsf{S}\big)$ d'invariants de poids m, n, o, p, les deux identités suivantes de type plückérien sont satisfaites :

$$(\mathscr{P}lck_1) \qquad \boxed{0 \equiv m\,\mathsf{P}\,\big[\mathsf{Q},\,\mathsf{R}\big] + o\,\mathsf{R}\,\big[\mathsf{P},\,\mathsf{Q}\big] + n\,\mathsf{Q}\,\big[\mathsf{R},\,\mathsf{P}\big]},$$

$$(\mathscr{P}lck_2) \qquad \boxed{0 \equiv \big[\mathsf{P},\,\mathsf{Q}\big]\cdot\big[\mathsf{R},\,\mathsf{S}\big] + \big[\mathsf{S},\,\mathsf{P}\big]\cdot\big[\mathsf{R},\,\mathsf{Q}\big] + \big[\mathsf{Q},\,\mathsf{S}\big]\cdot\big[\mathsf{R},\,\mathsf{P}\big]}.$$

Preuve. Si en effet nous développons les deux derniers crochets :

$$o\,\mathsf{R}\big(n\,\mathsf{DP}\cdot\mathsf{Q} - m\,\mathsf{P}\cdot\mathsf{DQ}\big) + n\,\mathsf{Q}\big(m\,\mathsf{DR}\cdot\mathsf{P} - o\,\mathsf{R}\cdot\mathsf{DP}\big),$$

les deux termes extrêmes s'annihilent, tandis que les deux termes centraux :

$$-om\,\mathsf{R}\cdot\mathsf{P}\cdot\mathsf{DQ} + nm\,\mathsf{Q}\cdot\mathsf{DR}\cdot\mathsf{P} \equiv -m\,\mathsf{P}\,\big[\mathsf{Q},\,\mathsf{R}\big]$$

reconstituent l'opposé du premier terme de la première identité $(\mathscr{P}lck_1)$. La seconde $(\mathscr{P}lck_2)$ n'est qu'une reformulation de la relation plückerienne fondamentale qui est satisfaite par les six mineurs 2×2 d'une matrice de taille 2×4. On la vérifie en développant les trois produits de déterminants 2×2, ce qui produit 12 termes constitués de 6 couples s'annihilant. $\square$

Reconstitution des 9 syzygies. Il est très remarquable que les identités $(\mathscr{P}lck_1)$ permettent de reconstituer les huit premières (parmi neuf) des identités listées (voir ci-dessous), et que la neuvième identité "$\overset{9}{\equiv}$" puisse être obtenue comme l'une des identités $(\mathscr{P}lck_2)$.

En effet, au niveau précédent $\kappa = 3$, nous avions cinq polynômes invariants fondamentaux :

$$\Big(\, f_1' \quad f_2' \quad \Lambda^3 \quad \Lambda_1^5 \quad \Lambda_2^5 \,\Big).$$

Par conséquent, le nombre d'identités fondamentales ($\mathscr{P}lck_1$) possibles est égal à $C_5^3 = 10$, et nous les écrivons dans l'ordre suivant :

$$
\begin{aligned}
0 &\overset{a}{\equiv} f_1' \left[f_2',\, \Lambda^3\right] + 3\,\Lambda^3 \left[f_1',\, f_2'\right] + f_2' \left[\Lambda^3,\, f_1'\right],\\
0 &\overset{b}{\equiv} f_1' \left[f_2',\, \Lambda_1^5\right] + 5\,\Lambda_1^5 \left[f_1',\, f_2'\right] + f_2' \left[\Lambda_1^5,\, f_1'\right],\\
0 &\overset{c}{\equiv} f_1' \left[f_2',\, \Lambda_2^5\right] + 5\,\Lambda_2^5 \left[f_1',\, f_2'\right] + f_2' \left[\Lambda_2^5,\, f_1'\right],\\
0 &\overset{d}{\equiv} f_1' \left[\Lambda^3,\, \Lambda_1^5\right] + 5\,\Lambda_1^5 \left[f_1',\, \Lambda^3\right] + 3\,\Lambda^3 \left[\Lambda_1^5,\, f_1'\right],\\
0 &\overset{e}{\equiv} f_1' \left[\Lambda^3,\, \Lambda_2^5\right] + 5\,\Lambda_2^5 \left[f_1',\, \Lambda^3\right] + 3\,\Lambda^3 \left[\Lambda_2^5,\, f_1'\right],\\
0 &\overset{f}{\equiv} f_1' \left[\Lambda_1^5,\, \Lambda_2^5\right] + 5\,\Lambda_2^5 \left[f_1',\, \Lambda_1^5\right] + 5\,\Lambda_1^5 \left[\Lambda_2^5,\, f_1'\right],\\
0 &\overset{g}{\equiv} f_2' \left[\Lambda^3,\, \Lambda_1^5\right] + 5\,\Lambda_1^5 \left[f_2',\, \Lambda^3\right] + 3\,\Lambda^3 \left[\Lambda_1^5,\, f_2'\right],\\
0 &\overset{h}{\equiv} f_2' \left[\Lambda^3,\, \Lambda_2^5\right] + 5\,\Lambda_2^5 \left[f_2',\, \Lambda^3\right] + 3\,\Lambda^3 \left[\Lambda_2^5,\, f_2'\right],\\
0 &\overset{i}{\equiv} f_2' \left[\Lambda_1^5,\, \Lambda_2^5\right] + 5\,\Lambda_2^5 \left[f_2',\, \Lambda_1^5\right] + 5\,\Lambda_1^5 \left[\Lambda_2^5,\, f_2'\right],\\
0 &\overset{j}{\equiv} 3\,\Lambda^3 \left[\Lambda_1^5,\, \Lambda_2^5\right] + 5\,\Lambda_2^5 \left[\Lambda^3,\, \Lambda_1^5\right] + 5\,\Lambda_1^5 \left[\Lambda_2^5,\, \Lambda^3\right].
\end{aligned}
$$

De même, le nombre d'identités plückeriennes ($\mathscr{P}lck_2$) possibles est égal au nombre $C_5^4 = 5$:

$$
\begin{aligned}
0 &\overset{k}{\equiv} \left[f_1',\, f_2'\right] \cdot \left[\Lambda^3,\, \Lambda_1^5\right] + \left[\Lambda_1^5,\, f_1'\right] \cdot \left[\Lambda^3,\, f_2'\right] + \left[f_2',\, \Lambda_1^5\right] \cdot \left[\Lambda^3,\, f_1'\right],\\
0 &\overset{l}{\equiv} \left[f_1',\, f_2'\right] \cdot \left[\Lambda^3,\, \Lambda_1^5\right] + \left[\Lambda_2^5,\, f_1'\right] \cdot \left[\Lambda^3,\, f_2'\right] + \left[f_2',\, \Lambda_2^5\right] \cdot \left[\Lambda^3,\, f_1'\right],\\
0 &\overset{m}{\equiv} \left[f_1',\, f_2'\right] \cdot \left[\Lambda_1^5,\, \Lambda_2^5\right] + \left[\Lambda_2^5,\, f_1'\right] \cdot \left[\Lambda_1^5,\, f_2'\right] + \left[f_2',\, \Lambda_2^5\right] \cdot \left[\Lambda_1^5,\, f_1'\right],\\
0 &\overset{n}{\equiv} \left[f_1',\, \Lambda^3\right] \cdot \left[\Lambda_1^5,\, \Lambda_2^5\right] + \left[\Lambda_2^5,\, f_1'\right] \cdot \left[\Lambda_1^5,\, \Lambda^3\right] + \left[\Lambda^3,\, \Lambda_2^5\right] \cdot \left[\Lambda_1^5,\, f_1'\right],\\
0 &\overset{o}{\equiv} \left[f_2',\, \Lambda^3\right] \cdot \left[\Lambda_1^5,\, \Lambda_2^5\right] + \left[\Lambda_2^5,\, f_1'\right] \cdot \left[\Lambda_1^5,\, \Lambda^3\right] + \left[\Lambda^3,\, \Lambda_2^5\right] \cdot \left[\Lambda_1^5,\, f_2'\right].
\end{aligned}
$$

À présent, nous pouvons réécrire tous ces crochets bruts en utilisant les notations que nous avons introduites pour désigner nos neuf invariants, tout

d'abord dans les dix identités $(\mathscr{P}lck_1)$:

$$
\begin{aligned}
0 &\overset{a}{\equiv} -f_1'\,\Lambda_2^5 - 3\,\Lambda^3\,\Lambda^3 + f_2'\,\Lambda_1^5,\\
0 &\overset{b}{\equiv} -f_1'\,\Lambda_{1,2}^7 - 5\,\Lambda^3\,\Lambda_1^5 + f_2'\,\Lambda_{1,1}^7,\\
0 &\overset{c}{\equiv} -f_1'\,\Lambda_{2,2}^7 - 5\,\Lambda^3\,\Lambda_2^5 + f_2'\,\Lambda_{2,1}^7,\\
0 &\overset{d}{\equiv} -f_1'f_1'\,M^8 - 5\,\Lambda_1^5\,\Lambda_1^5 + 3\,\Lambda^3\,\Lambda_{1,1}^7,\\
0 &\overset{e}{\equiv} -f_1'f_2'\,M^8 - 5\,\Lambda_1^5\,\Lambda_2^5 + 3\,\Lambda^3\,\Lambda_{2,1}^7,\\
0 &\overset{f}{\equiv} -5\,f_1'\,\Lambda^3\,M^8 - 5\,\Lambda_2^5\,\Lambda_{1,1}^7 + 5\,\Lambda_1^5\,\Lambda_{2,1}^7,\\
0 &\overset{g}{\equiv} -f_2'f_1'\,M^8 - 5\,\Lambda_2^5\,\Lambda_1^5 + 3\,\Lambda^3\,\Lambda_{1,2}^7,\\
0 &\overset{h}{\equiv} -f_2'f_2'\,M^8 - 5\,\Lambda_2^5\,\Lambda_2^5 + 3\,\Lambda^3\,\Lambda_{2,2}^7,\\
0 &\overset{i}{\equiv} -5\,f_2'\,\Lambda^3\,M^8 - 5\,\Lambda_2^5\,\Lambda_{1,2}^7 + 5\,\Lambda_1^5\,\Lambda_{2,2}^7,\\
0 &\overset{j}{\equiv} -3\,\Lambda^3\,\Lambda^3\,M^8 - \Lambda_2^5\,f_1'\,M^8 + \Lambda_1^5\,f_2'\,M^8\,;
\end{aligned}
$$

et ensuite dans les cinq identités $(\mathscr{P}lck_2)$:

$$
\begin{aligned}
0 &\overset{k}{\equiv} \Lambda^3\,f_1'\,M^8 + \Lambda_{1,1}^7\,\Lambda_2^5 - \Lambda_{1,2}^7\,\Lambda_1^5,\\
0 &\overset{l}{\equiv} \Lambda^3\,f_2'\,M^8 + \Lambda_{2,1}^7\,\Lambda_2^5 - \Lambda_{2,2}^7\,\Lambda_1^5,\\
0 &\overset{m}{\equiv} 5\,\Lambda^3\,\Lambda^3\,M^8 + \Lambda_{2,1}^7\,\Lambda_{1,2}^7 - \Lambda_{2,2}^7\,\Lambda_{1,1}^7,\\
0 &\overset{n}{\equiv} 5\,\Lambda_1^5\,\Lambda^3\,M^8 + \Lambda_{2,1}^7\,f_1'\,M^8 - \Lambda_{1,1}^7\,f_2'\,M^8,\\
0 &\overset{o}{\equiv} 5\,\Lambda_2^5\,\Lambda^3\,M^8 + \Lambda_{2,2}^7\,f_1'\,M^8 - f_2'\,\Lambda_{1,2}^7\,M^8.
\end{aligned}
$$

Ici, en admettant bien sûr que $\Lambda_{1,2} = \Lambda_{2,1}$, on constate que :

- "$\overset{a}{\equiv}$" fournit "$\overset{1}{\equiv}$" ;
- "$\overset{b}{\equiv}$" fournit "$\overset{2}{\equiv}$" ;
- "$\overset{c}{\equiv}$" fournit "$\overset{3}{\equiv}$" ;
- "$\overset{d}{\equiv}$" fournit "$\overset{4}{\equiv}$" ;
- "$\overset{e}{\equiv}$" fournit "$\overset{5}{\equiv}$" ;
- "$\overset{f}{\equiv}$" fournit "$\overset{7}{\equiv}$" ;
- "$\overset{g}{\equiv}$" est redondant avec "$\overset{e}{\equiv}$" ;
- "$\overset{h}{\equiv}$" fournit "$\overset{6}{\equiv}$" ;
- "$\overset{i}{\equiv}$" fournit "$\overset{8}{\equiv}$" ;

- “$\overset{j}{\equiv}$” redouble “$\overset{1}{\equiv}$” en la multipliant par M^8 ;
- “$\overset{k}{\equiv}$” redouble “$\overset{f}{\equiv}$” ;
- “$\overset{l}{\equiv}$” redouble “$\overset{i}{\equiv}$” ;
- “$\overset{m}{\equiv}$” fournit la dernière identité manquante “$\overset{9}{\equiv}$” ;
- “$\overset{n}{\equiv}$” redouble “$\overset{b}{\equiv}$” en la multipliant par M^8 ;
- “$\overset{o}{\equiv}$” redouble “$\overset{c}{\equiv}$” en la multipliant par M^8.

Conclusion. Toutes les identités algébriques entre les invariants fondamentaux que nous avons trouvées à l'aide de Maple pour les niveaux $\kappa = 3$ et $\kappa = 4$ peuvent en fait être obtenues mécaniquement grâce aux trois familles fondamentales de syzygies :

$$\boxed{\begin{array}{ll} (\mathscr{J}ac) : & 0 \equiv \big[\big[\mathsf{P},\,\mathsf{Q}\big],\,\mathsf{R}\big] + \big[\big[\mathsf{R},\,\mathsf{P}\big],\,\mathsf{Q}\big] + \big[\big[\mathsf{Q},\,\mathsf{R}\big],\,\mathsf{P}\big], \\ (\mathscr{R}\mathscr{F}) : & 0 \equiv m\,\mathsf{P}\,\big[\mathsf{Q},\,\mathsf{R}\big] + o\,\mathsf{R}\,\big[\mathsf{P},\,\mathsf{Q}\big] + n\,\mathsf{Q}\,\big[\mathsf{R},\,\mathsf{P}\big], \\ (\mathscr{P}lck) : & 0 \equiv \big[\mathsf{P},\,\mathsf{Q}\big]\cdot\big[\mathsf{R},\,\mathsf{S}\big] + \big[\mathsf{S},\,\mathsf{P}\big]\cdot\big[\mathsf{R},\,\mathsf{Q}\big] + \big[\mathsf{Q},\,\mathsf{S}\big]\cdot\big[\mathsf{R},\,\mathsf{P}\big] \end{array}}\,.$$

Genèse des syzygies. Crucialement, il semblerait que l'on puisse engendrer toutes les relations entre tous les polynômes invariants que l'on construit récursivement par crochets, juste en développant par récurrence toutes les identités ($\mathscr{J}ac$), ($\mathscr{P}lck_1$) et ($\mathscr{P}lck_2$) possibles lorsqu'on passe d'un étage de jets λ à l'étage supérieur $\lambda+1$. Cette idée conjecturale est renforcée par le fait que dans la théorie classique des invariants, il existe aussi trois procédés automatiques qui engendrent l'idéal des relations entre les invariants, et on démontre rigoureusement ([4]) que tel est bien le cas, sans toutefois poursuivre l'étude plus avant, afin de trouver des bases de Gröbner signifiantes d'un point combinatoire, ou afin de dévoiler des harmonies formelles encore inconnues qui montreraient explicitement en quoi l'algèbre des invariants est de Cohen-Macaulay, ce qui est toujours le cas pour un groupe réductif ([4]).

§6. Décomposition en représentations de Schur

Motivation. La cohomologie des fibrés de Schur sur une variété projective lisse étant connue (*cf. e.g.* [5] et *voir* la Section 8 ci-dessous), nous cherchons maintenant à décomposer en représentations irréductibles de Schur les gradués de poids m de nos deux algèbres d'invariants $\mathscr{D}\mathscr{S}^4_2$ et $\mathscr{D}\mathscr{S}^5_2$.

Action linéaire diagonale sur les jets. À cette fin, sur l'espace des jets d'ordre κ en dimension deux muni des coordonnées $\big(f'_1, f'_2, \ldots, f^{(\lambda)}_1, f^{(\lambda)}_2, \ldots, f^{(\kappa)}_1, f^{(\kappa)}_2\big)$, considérons (*cf.* [5]) l'action du groupe linéaire à deux dimensions $\mathsf{GL}_2(\mathbb{C})$ — constitué des matrices 2×2

de la forme

$$\mathrm{w} := \begin{pmatrix} t & v \\ u & w \end{pmatrix},$$

où $t, u, v, w \in \mathbb{C}$ satisfont $tw - uv \neq 0$ — qui est définie diagonalement par la même transformation évidente sur chaque étage de jets :

$$\begin{aligned} \mathsf{w} \cdot f_1^{(\lambda)} &:= t\, f_1^{(\lambda)} + v\, f_2^{(\lambda)}, \\ \mathsf{w} \cdot f_2^{(\lambda)} &:= u\, f_1^{(\lambda)} + w\, f_2^{(\lambda)}, \end{aligned}$$

pour tout λ tel que $1 \leqslant \lambda \leqslant \kappa$.

Décompositions de Schur. La théorie classique des représentations du groupe linéaire permet alors de décomposer toute représentation de $\mathsf{GL}_2(\mathbb{C})$ comme somme directe de représentations d'un certain type, dites *de Schur*, que l'on repère facilement en recherchant tous les vecteurs qui sont invariants par un certain sous-groupe de $\mathsf{GL}_2(\mathbb{C})$. Énonçons ce que la théorie générale donne dans le cas qui nous intéresse.

Définition. Un polynôme invariant par reparamétrisation $\mathsf{P}\big(j^\kappa f\big)$ est appelé *bi-invariant* s'il est un *vecteur de plus haut poids* pour cette représentation, c'est-à-dire s'il est invariant par l'action du sous-groupe unipotent $\mathsf{U}_2(\mathbb{C})$ constitué des matrices de la forme :

$$\mathrm{u} := \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.$$

Autrement dit, un invariant simple P satisfait $\mathsf{P}\big(j^\kappa(f \circ \phi)\big) = (\phi')^m\, \mathsf{P}\big((j^\kappa f) \circ \phi\big)$ pour un certain $m \geqslant 1$ et c'est un *bi-invariant* si l'on a de plus :

$$\boxed{\mathsf{P}^{2\times\mathrm{inv}}\big(\mathrm{u} \cdot j^\kappa f\big) = \mathsf{P}^{2\times\mathrm{inv}}\big(j^\kappa f\big)},$$

pour toute matrice unipotente $\mathrm{u} \in \mathsf{U}_2(\mathbb{C})$.

Exemples. Puisque l'on a trivialement $\mathrm{u} \cdot f_1' = f_1'$ et $\mathrm{u} \cdot f_1'' = f_1''$, et aussi :

$$\mathrm{u} \cdot \Delta^{\alpha,\beta} = \begin{vmatrix} f_1^{(\alpha)} & f_2^{(\alpha)} + u\, f_1^{(\alpha)} \\ f_1^{(\beta)} & f_2^{(\beta)} + u\, f_1^{(\beta)} \end{vmatrix} = \Delta^{\alpha,\beta},$$

nous voyons immédiatement que f_1', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$ et M^8 sont des bi-invariants (au nombre de cinq), tandis que les quatre invariants restants, à savoir f_2', Λ_2^5, $\Lambda_{1,2}^7$ et $\Lambda_{2,2}^7$ ne sont pas bi-invariants.

Repérage des représentations de Schur. D'après la théorie générale, à tout vecteur $\mathsf{P}^{2\times\mathrm{inv}}$ de plus haut poids correspond alors une et une seule représentation de Schur $\Gamma^{(l_1,l_2)}$, où les deux entiers l_1 et l_2 satisfaisant $l_1 \geqslant l_2$ sont

aisément repérés comme étant les exposants des deux éléments diagonaux qui apparaissent dans la valeur propre

$$\mathsf{t}\cdot\mathsf{P}^{2\times\mathrm{inv}}=\begin{pmatrix} t & 0\\ 0 & w\end{pmatrix}\cdot\mathsf{P}^{2\times\mathrm{inv}}=\mathsf{t}^{l_1}\,\mathsf{w}^{l_2}\,\mathsf{P}^{2\times\mathrm{inv}}$$

dont jouit le bi-invariant — qui est nécessairement vecteur propre — par rapport au sous-groupe des matrices 2×2 diagonales. Très concrètement, l'entier l_1 compte le nombre total d'indices inférieurs "$(\cdot)_1$" qui interviennent dans chaque monôme du bi-invariant en question, et de même, l'entier l_2 compte le nombre d'indices "$(\cdot)_2$", et puisque chaque $\Delta^{\alpha,\beta}$ contribue pour exactement un indice "$(\cdot)_1$" et un indice "$(\cdot)_2$", il est immédiatement clair que nous avons la correspondance suivante entre bi-invariants et représentations de Schur :

$$f_1'\longleftrightarrow\Gamma^{(1,0)},\qquad \Lambda^3\longleftrightarrow\Gamma^{(1,1)},$$
$$\Lambda_1^5\longleftrightarrow\Gamma^{(2,1)},\qquad \Lambda_{1,1}^7\longleftrightarrow\Gamma^{(3,1)},\qquad M^8\longleftrightarrow\Gamma^{(2,2)}.$$

Fait d'expérience. La détermination directe des bi-invariants en dimension $\nu=2$ pour les jets d'ordre $\kappa=4$ ou $\kappa=5$ est *beaucoup moins coûteuse en calcul* que la détermination de la totalité des invariants par reparamétrisation. Voici en effet le premier de nos deux résultats principaux.

Théorème. Pour les jets d'ordre 4 en dimension 2, tout bi-invariant de poids m, $\mathsf{P}^{2\times\mathrm{inv}}\big(j^4f_1,\,j^4f_2\big)$, s'écrit sous forme unique :

$$\mathsf{P}^{2\times\mathrm{inv}}\big(j^4f\big)=\mathscr{Q}^{2\times\mathrm{inv}}\big(f_1',\Lambda^3,\Lambda_{1,1}^7,M^8\big)+\Lambda_1^5\,\mathscr{R}^{2\times\mathrm{inv}}\big(f_1',\Lambda^3,\Lambda_{1,1}^7,M^8\big),$$

où $\mathscr{Q}^{2\times\mathrm{inv}}$ et $\mathscr{R}^{2\times\mathrm{inv}}$ sont deux polynômes absolument arbitraires en leurs arguments qui sont de poids m et de poids $m-5$, respectivement. De plus, l'idéal des relations entre les cinq bi-invariants fondamentaux :

$$\Big(\,f_1'\quad\Lambda^3\quad\Lambda_1^5\quad\Lambda_{1,1}^7\quad M^8\,\Big),$$

est principal, et pour préciser, il est engendré par l'unique[66] relation :

$$0\equiv f_1'f_1'M^8-3\,\Lambda^3\Lambda_{1,1}^7+5\,\Lambda_1^5\Lambda_1^5.$$

Par conséquent, une base de l'espace vectoriel des polynômes de poids m invariant par reparamétrisation et par rapport à l'action de $\mathsf{U}_2(\mathbb{C})$ est constituée de l'ensemble des monômes :

$$(f_1')^\alpha\,\big(\Lambda^3\big)^\beta\,\big(\Lambda_{1,1}^7\big)^\gamma\,\big(M^8\big)^\delta,\qquad \text{avec }\ \alpha+3\beta+7\gamma+8\delta=m,\ \text{ et :}$$
$$\Lambda_1^5\,(f_1')^\alpha\,\big(\Lambda^3\big)^\beta\,\big(\Lambda_{1,1}^7\big)^\gamma\,\big(M^8\big)^\delta,\qquad \text{avec }\ \alpha+3\beta+7\gamma+8\delta=m-5,$$

[66] Ce fait a été confirmé par un calcul de l'idéal des relations sur Maple.

et chacun de ces deux monômes correspond respectivement aux deux représentations de Schur :

$$\Gamma^{\alpha+\beta+3\gamma+2\delta,\ \beta+\gamma+2\delta} \qquad \textit{et} \qquad \Gamma^{2+\alpha+\beta+3\gamma+2\delta,\ 1+\beta+\gamma+2\delta}.$$

Conséquences. Avant d'entreprendre la démonstration de ce premier théorème, notons que l'algèbre complète $\mathscr{DS}_2^4$ des invariants par reparamétrisation s'obtient maintenant facilement en regardant l'orbite, par l'action du groupe complet $\mathsf{GL}_2(\mathbb{C})$, de chacun de nos cinq bi-invariants ; on constate d'ailleurs qu'il suffit de considérer l'action des matrices de la forme

$$\mathrm{v} := \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix},$$

qui nous fournissent immédiatement :

$$\mathrm{v} \cdot f_1' = f_1' + v\, f_2', \qquad \mathrm{v} \cdot \Lambda_1^5 = \Lambda_1^5 + v\, \Lambda_2^5,$$
$$\mathrm{v} \cdot \Lambda_{1,1}^7 = \Lambda_{1,1}^7 + 2v\, \Lambda_{1,2}^7 + v^2\, \Lambda_{2,2}^7,$$

et de cette manière, non seulement nous engendrons facilement les quatre invariants fondamentaux non bi-invariants que nous connaissions déjà, mais encore — et c'est là qu'apparaît *une stratégie crucialement simplifiée que nous ré-exploiterons ultérieurement pour l'étude de* $\mathscr{DS}_2^5$ —, nous déduisons que l'orbite des polynômes arbitraires en $\big(f_1',\, \Lambda^3,\, \Lambda_1^5,\, \Lambda_{1,1}^7,\, M^8\big)$ est juste constituée des polynômes en les neuf invariants par reparamétrisation que nous avions engendrés en calculant méthodiquement des crochets.

Corollaire . Pour les jets d'ordre quatre en dimension deux, l'algèbre $\mathscr{DS}_2^4$ des polynômes invariants par reparamétrisation est polynomialement engendrée par les neuf invariants fondamentaux $\big(f_1',\, f_2',\, \Lambda^3,\, \Lambda_1^5,\, \Lambda_2^5,\, \Lambda_{1,1}^7,\, \Lambda_{1,2}^7,\, \Lambda_{2,2}^7,\, M^8\big)$.

Restrictions. Toutefois, cette manière économique de procéder — étude exclusive et exhaustive des bi-invariants suivie de la déduction raccourcie d'une description partielle de l'algèbre complète des invariants — ne fournit pas de description précise de $\mathscr{DS}_2^4$, c'est-à-dire notamment qu'elle ne fournit pas une écriture unique, en tenant compte des 9 syzygies fondamentales, de tout polynôme général de la forme :

$$\mathscr{P}\big(f_1',\, f_2',\, \Lambda^3,\, \Lambda_1^5,\, \Lambda_2^5,\, \Lambda_{1,1}^7,\, \Lambda_{1,2}^7,\, \Lambda_{2,2}^7,\, M^8\big),$$

et qui plus est, il serait impossible d'obtenir un tel résultat complet, et ce pour une raison profonde, à savoir que l'orbite par $\mathsf{GL}_2(\mathbb{C})$ de l'unique syzygie existant entre les bi-invariants ne couvre pas l'ensemble des neuf syzygies fondamentales qui existent entre les invariants complets.

Heureusement, puisque seule la décomposition en représentations irréductibles de Schur présente un véritable sens algébrique, et aussi, puisque

nous aurons seulement besoin de cette décomposition pour conduire nos calculs de caractéristique d'Euler dans la Section 8, il est en vérité essentiellement inutile de poursuivre plus avant l'étude de $\mathscr{D}\mathscr{S}^4_2$. Nous confierons quand même au lecteur désireux de s'exercer à maîtriser les bases de Gröbner le soin d'établir l'énoncé suivant, ou d'autres énoncés analogues qu'il pourrait formuler en choisissant à sa guise des ordres monomiaux différents.

Proposition. *Tout polynôme en les neuf invariants fondamentaux s'écrit de manière unique comme suit* :

$$\begin{aligned}&\mathscr{P}\big(f_1',\, f_2',\, \Lambda^3,\, \Lambda^7_{1,1},\, \Lambda^7_{2,2}\big) + \Lambda^3\,\mathscr{Q}\big(f_1',\, f_2',\, \Lambda^3,\, \Lambda^7_{1,1},\, \Lambda^7_{2,2}\big) + \\ &+ \Lambda^5_1\,\mathscr{R}\big(f_1',\, f_2',\, \Lambda^3,\, \Lambda^7_{1,1},\, \Lambda^7_{2,2}\big) + \Lambda^5_2\,\mathscr{S}\big(f_1',\, f_2',\, \Lambda^3,\, \Lambda^7_{1,1},\, \Lambda^7_{2,2}\big) + \\ &+ \Lambda^7_{1,2}\,\mathscr{T}\big(f_1',\, f_2',\, \Lambda^3,\, \Lambda^7_{1,1},\, \Lambda^7_{2,2}\big) + \Lambda^3\Lambda^7_{1,2}\,\mathscr{U}\big(f_1',\, f_2',\, \Lambda^3,\, \Lambda^7_{1,1},\, \Lambda^7_{2,2}\big),\end{aligned}$$

où $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{R}$, $\mathscr{S}$, $\mathscr{T}$ et $\mathscr{U}$ sont des polynômes arbitraires en leurs arguments.

Démonstration du premier théorème. Par définition de l'invariance par reparamétrisation d'un polynôme $\mathsf{P} = \mathsf{P}\big(j^4 f\big)$ de poids m, on a :

$$\mathsf{P}\big(j^4(f\circ\phi)\big) = \phi'^m\,\mathsf{P}\big((j^4 f)\circ\phi\big),$$

pour tout biholomorphisme local ϕ de $\mathbb{C}$. En suivant une astuce de [5], nous allons appliquer cette formule à $\phi := f_1^{-1}$ en supposant l'inversibilité, d'où $\phi' = \frac{1}{f_1'}\circ f_1^{-1}$. On a tout d'abord trivialement $\big(f_1\circ f_1^{-1}\big)' = \mathrm{Id}$, d'où $\big(f_1\circ f_1^{-1}\big)^{(\lambda)} = 0$ pour tout $\lambda \geqslant 2$ puis, par des calculs directs dont la teneur est déjà élucidée par notre connaissance préalable des invariants Λ^3, Λ^5_1 et $\Lambda^7_{1,1}$:

$$\begin{aligned}\big(f_2\circ f_1^{-1}\big)' &= \frac{f_2'}{f_1'}\circ f_1^{-1},\\ \big(f_2\circ f_1^{-1}\big)'' &= \frac{\Lambda^3}{(f_1')^3}\circ f_1^{-1},\\ \big(f_2\circ f_1^{-1}\big)''' &= \frac{\Lambda^5_1}{(f_1')^5}\circ f_1^{-1},\\ \big(f_2\circ f_1^{-1}\big)'''' &= \frac{\Lambda^7_{1,1}}{(f_1')^7}\circ f_1^{-1}.\end{aligned}$$

Par conséquent, tout polynôme $\mathsf{P}(j^4 f)$ invariant par reparamétrisation satisfait :

$$\mathsf{P}\bigg(1, \frac{f_2'}{f_1'}, 0, \frac{\Lambda^3}{(f_1')^3}, 0, \frac{\Lambda^5_1}{(f_1')^5}, 0, \frac{\Lambda^7_{1,1}}{(f_1')^7}\bigg)\circ f_1^{-1} = \bigg(\frac{1}{f_1'}\circ f_1^{-1}\bigg)^m\,\mathsf{P}\big((j^4 f\circ f_1^{-1})\big).$$

Recomposons immédiatement avec f_1 pour faire disparaître f_1^{-1}. Si ensuite nous écrivons le polynôme de départ $\mathsf{P} = \mathsf{P}\big(j^4 f\big)$ sous la forme générale

suivante :

$$\sum_{a_1+a_2+2b_1+2b_2+\cdots+4d_2=m} \mathsf{p}_{a_1a_2\cdots d_2} \cdot (f_1')^{a_1}(f_2')^{a_2}(f_1'')^{b_1}(f_2'')^{b_2}(f_1''')^{c_1}(f_2''')^{c_2}(f_1'''')^{d_1}(f_2'''')^{d_2},$$

avec des coefficients $\mathsf{p}_{a_1a_2\cdots d_2} \in \mathbb{C}$, l'identité obtenue à l'instant nous permet alors d'obtenir une représentation générale de P :

$$\begin{aligned}\mathsf{P}\big(j^4f\big) &= (f_1')^m\,\mathsf{P}\bigg(1, \frac{f_2'}{f_1'}, 0, \frac{\Lambda^3}{(f_1')^3}, 0, \frac{\Lambda_1^5}{(f_1')^5}, 0, \frac{\Lambda_{1,1}^7}{(f_1')^7}\bigg)\\ &= (f_1')^m \sum_{a_1+a_2+2b_2+3c_2+4d_2=m} \mathsf{p}_{a_1a_20b_20c_20d_2}\,\frac{1^{a_1}(f_2')^{a_2}\big(\Lambda^3\big)^{b_2}\big(\Lambda_1^5\big)^{c_2}\big(\Lambda_{1,1}^7\big)^{d_2}}{(f_1')^{a_2+3b_2+5c_2+7d_2}}\\ &\qquad\qquad \in \mathbb{C}\big[f_1', f_2', \Lambda^3, \Lambda_1^5, \Lambda_{1,1}^7\big]\Big[\frac{1}{f_1'}\Big],\end{aligned}$$

qui est presque polynomiale, à ceci près qu'on s'autorise à diviser par f_1'. Calculons alors l'ordre maximal en $\frac{1}{f_1'}$ de cette expression rationnelle :

$$\begin{aligned}\max_{a_1+a_2+2b_2+3c_2+4d_2=m}\Big(a_2+3b_2+5c_2+7d_2-m\Big) &= \max_{a_2+2b_2+3c_2+4d_2=m}\Big(b_2+2c_2+3d_2\Big)\\ &= \frac{1}{2}\cdot\max_{2b_2+3c_2+4d_2=m}\Big(2b_2+4c_2+6d_2\Big)\\ &= \frac{m}{2}+\frac{1}{2}\cdot\max_{3c_2+4d_2=m}\Big(c_2+2d_2\Big)\\ &= \frac{3}{4}\,m.\end{aligned}$$

Ainsi, tout polynôme $\mathsf{P}\big(j^4f\big) \in \mathscr{DS}^4_{2,m}$ est de la forme :

$$\sum_{-\frac{3}{4}m\leqslant a\leqslant m} (f_1')^a\,\mathscr{P}_a\big(f_2', \Lambda^3, \Lambda_1^5, \Lambda_{1,1}^7\big).$$

Cependant, toutes les expressions rationnelles de cette forme ne conviennent pas : $\frac{\Lambda^3\,\Lambda^3}{f_1'f_1'}$ avec $m=4$ ne se simplifie pas pour produire un vrai polynôme appartenant $\mathscr{DS}^4_{2,m}$, bien que cette expression rationnelle soit invariante par reparamétrisation. Toutefois, l'énoncé suivant, que nous transférons directement aux jets d'ordre quelconque, est clair.

Lemme. Tout polynôme $\mathsf{P}\big(j^\kappa f\big)$ en le jet strict

$$j^\kappa f := \Big(f_1',\ f_2',\ f_1'',\ f_2'', \ldots\ldots,\ f_1^{(\kappa)},\ f_2^{(\kappa)}\Big)$$

d'ordre $\kappa \geqslant 1$ d'une application holomorphe locale $f=(f_1,f_2):\mathbb{C}\to\mathbb{C}^2$ qui est invariant par reparamétrisation, i.e. qui appartient à $\mathscr{DS}^\kappa_{2,m}$, peut être représenté sous la forme :

$$\mathsf{P}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa}m\leqslant a\leqslant m} (f_1')^a\,\mathscr{P}_a\big(f_2', \Lambda^3, \Lambda_1^5, \Lambda_{1,1}^7, \Lambda_{1,1,1}^9, \ldots, \Lambda_{1,\ldots,1}^{2\kappa-1}\big),$$

avec certains polynômes $\mathscr{P}_a$ de poids $m-a$. Réciproquement, toute expression rationnelle de cette forme qui s'avère être polynomiale en $j^\kappa f$ quand on simplifie numérateur et dénominateur appartient à $\mathscr{DS}^\kappa_{2,m}$. $\square$

Ici, on considère comme précédemment $\Lambda^9_{1,1,1} := \big[\Lambda^7_{1,1},\, f_1'\big]$, et on introduit généralement par récurrence $\Lambda^{2\lambda-1}_{1,\dots,1,1} := \big[\Lambda^{2\lambda-3}_{1,\dots,1},\, f_1'\big]$ pour $3 \leqslant \lambda \leqslant \kappa$.

Décrivons maintenant les polynômes $\mathsf{P} = \mathsf{P}\big(j^\kappa f\big)$ de $\mathscr{DS}^\kappa_{2,m}$ écrits sous une telle forme rationnelle qui sont invariants par l'action de $\mathsf{U}_2(\mathbb{C})$. Par définition, $\mathrm{U} \cdot \mathsf{P} = \mathsf{P}$, *i.e.* explicitement :

$$\mathsf{P}\Big(f_1',\, f_2' + u\, f_1', f_1'',\, f_2'' + u\, f_1'', \dots, f_1^{(\kappa)},\, f_2^{(\kappa)} + u\, f_1^{(\kappa)}\Big) = \mathsf{P}\big(j^\kappa f\big),$$

pour tout $u \in \mathbb{C}$. De manière équivalente,

$$\frac{d}{du}\, \mathsf{P}\Big(f_1',\, f_2' + u\, f_1', f_1'',\, f_2'' + u\, f_1'', \dots, f_1^{(\kappa)},\, f_2^{(\kappa)} + u\, f_1^{(\kappa)}\Big) \equiv 0,$$

ce qui revient à dire que P est annulé identiquement par le champ de vecteurs

$$\underline{\mathscr{U}} := f_1' \frac{\partial}{\partial f_2'} + f_1'' \frac{\partial}{\partial f_2''} + \cdots + f_1^{(\kappa)} \frac{\partial}{\partial f_2^{(\kappa)}},$$

i.e. que l'on a : $0 \equiv \underline{\mathscr{U}}\, \mathsf{P}$. On constate ensuite immédiatement que :

$$\mathrm{U} \cdot \Delta^{\alpha,\beta} = f_1^{(\alpha)}\big(f_2^{(\beta)} + u\, f_1^{(\beta)}\big) - f_1^{(\beta)}\big(f_2^{(\alpha)} + u\, f_1^{(\alpha)}\big) = \Delta^{\alpha,\beta},$$

i.e. : $0 \equiv \underline{\mathscr{U}}\, \Delta^{\alpha,\beta}$, d'où nous déduisons :

$$\mathrm{U} \cdot \Lambda^3 = \Lambda^3, \qquad \mathrm{U} \cdot \Lambda^5_1 = \Lambda^5_1, \qquad \mathrm{U} \cdot \Lambda^7_{1,1} = \Lambda^7_{1,1}, \qquad \mathrm{U} \cdot \Lambda^9_{1,1,1} = \Lambda^7_{1,1}, \qquad \textit{etc.}$$

En appliquant donc cette dérivation $\underline{\mathscr{U}}$ à la représentation rationnelle d'un polynôme quelconque $\mathsf{P}\big(j^\kappa f\big) \in \mathscr{DS}^\kappa_{2,m}$ obtenue à l'instant, nous voyons que l'équation $0 \equiv \underline{\mathscr{U}}\, \mathsf{P}$ est satisfaite si et seulement si chaque $\mathscr{P}_a$ est indépendant de f_2'. Nous pouvons donc résumer comme suit le résultat obtenu.

Lemme . Tout polynôme $\mathsf{P}^{2\times\mathrm{inv}}\big(j^\kappa f\big)$ qui est invariant par reparamétrisation et qui est invariant par rapport à l'action unipotente de $\mathsf{U}_2(\mathbb{C})$ peut être représenté sous la forme :

$$\mathsf{P}^{2\times\mathrm{inv}}\big(j^\kappa f\big) = \sum_{-\frac{\kappa-1}{\kappa} m \leqslant a \leqslant m} (f_1')^a\, \mathscr{P}^{2\times\mathrm{inv}}_a\big(\Lambda^3, \Lambda^5_1, \Lambda^7_{1,1}, \Lambda^9_{1,1,1}, \dots, \Lambda^{2\kappa-1}_{1,\dots,1}\big),$$

avec certains polynômes $\mathscr{P}^{2\times\mathrm{inv}}_a$ de poids $m-a$. Réciproquement, toute expression rationnelle de cette forme, si elle s'avère être polynomiale en $j^\kappa f$ quand on simplifie numérateur et dénominateur, appartient nécessairement à $\mathscr{DS}^\kappa_{2,m}$ et constitue un bi-invariant véritable. $\square$

En revenant à présent aux jets d'ordre 4, utilisons la relation

$$0 \equiv f_1' f_1' \, M^8 - 3 \, \Lambda^3 \, \Lambda_{1,1}^7 + 5 \, \Lambda_1^5 \, \Lambda_1^5$$

pour éliminer toutes les puissances de Λ_1^5 qui sont supérieures ou égales à 2 dans chaque polynôme $\mathscr{P}_a^{2\times\mathrm{inv}}$ et réorganisons le tout en puissances de f_1'. Nous obtenons ainsi une nouvelle représentation :

$$\mathsf{P}^{2\times\mathrm{inv}}\big(j^4 f\big) = \sum_{-\frac{3}{4}m \leqslant a \leqslant m} (f_1')^a \Big[\mathscr{Q}_a^{2\times\mathrm{inv}}\big(\Lambda^3, \Lambda_{1,1}^7, M^8\big) + \Lambda_1^5 \, \mathscr{R}_a^{2\times\mathrm{inv}}\big(\Lambda^3, \Lambda_{1,1}^7, M^8\big) \Big],$$

avec certains polynômes $\mathscr{Q}_a^{2\times\mathrm{inv}}$ de poids $m-a$ et $\mathscr{R}_a^{2\times\mathrm{inv}}$ de poids $m-5-a$.

Maintenant, c'est un fait remarquable qu'une telle représentation (dans laquelle on a tenu compte de l'idéal des relations entre f_1', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$ et M^8) doit nécessairement ne faire apparaître que des puissances positives de f_1', et donc être automatiquement polynomiale : nous l'affirmons.

En effet, si tel n'était pas le cas, en prenant pour a l'exposant le plus négatif tel que $\mathscr{P}_a^{2\times\mathrm{inv}} + \Lambda_1^5 \mathscr{R}_a^{2\times\mathrm{inv}} \not\equiv 0$ et en chassant le dénominateur $\frac{1}{(f_1')^{-a}}$, nous obtiendrions une équation de la forme

$$\mathscr{Q}_a^{2\times\mathrm{inv}}\big(\Lambda^3, \Lambda_{1,1}^7, M^8\big) + \Lambda_1^5 \, \mathscr{R}_a^{2\times\mathrm{inv}}\big(\Lambda^3, \Lambda_{1,1}^7, M^8\big) = \mathrm{O}(f_1'),$$

qui s'annulerait lorsque f_1' est égalé à zéro, circonstance qui est exclue par le lemme suivant.

Lemme. Étant donné deux polynômes quelconques $\mathscr{Q}$ et $\mathscr{R}$ de trois variables complexes, l'identité :

$$0 \equiv \mathscr{Q}\big(\Lambda^3, \Lambda_{1,1}^7, M^8\big) + \Lambda_1^5 \, \mathscr{R}\big(\Lambda^3, \Lambda_{1,1}^7, M^8\big)\Big|_{f_1'=0}$$

est identiquement satisfaite dans $\mathbb{C}\big[f_2', f_1'', f_2'', f_1''', f_1''', f_1'''', f_2''''\big]$, lorsque f_1' est égalé à zéro, si et seulement si $\mathscr{Q}$ et $\mathscr{R}$ sont identiquement nuls.

Preuve. Développons en effet tout d'abord cette identité suivant les puissances de M^8 :

$$0 \equiv \sum_{k \geqslant 0} \big(M^8\big)^k \Big[\mathscr{Q}_k\big(\Lambda^3, \Lambda_{1,1}^7\big) + \Lambda_1^5 \, \mathscr{R}_k\big(\Lambda^3, \Lambda_{1,1}^7\big) \Big]\Big|_{f_1'=0},$$

la somme étant bien entendu finie. Lorsque $f_1' = 0$, les expressions :

$$\begin{aligned}
\Lambda^3\big|_{f_1'=0} &= -f_1'' f_2', \\
\Lambda_1^5\big|_{f_1'=0} &= 3(f_1'' f_2') f_1'', \\
\Lambda_{1,1}^7\big|_{f_1'=0} &= -15\,(f_1'' f_2') f_1'' f_1'', \\
M^8\big|_{f_1'=0} &= 3(f_1'''' f_2')(f_1'' f_2') - 12(f_1''' f_2'' - f_1'' f_2''')(f_1'' f_2') - 5(f_1''' f_2')(f_1''' f_2').
\end{aligned}$$

montrent que $M^8\big|_{f_1'=0}$ est algébriquement indépendant des trois polynômes $\Lambda^3\big|_{f_1'=0}$, $\Lambda_1^5\big|_{f_1'=0}$, $\Lambda_{1,1}^7\big|_{f_1'=0}$. Nous en déduisons que

$$0 \equiv \mathscr{Q}_k\big(\Lambda^3,\, \Lambda_{1,1}^7\big) + \Lambda_1^5\, \mathscr{R}_k\big(\Lambda^3,\, \Lambda_{1,1}^7\big)\Big|_{f_1'=0}$$

pour tout k. L'énoncé suivant permet alors de conclure. $\square$

Lemme. L'identité polynomiale :

$$0 \equiv \mathscr{S}\big(\Lambda^3,\, \Lambda_{1,1}^7\big) + \Lambda_1^5\, \mathscr{T}\big(\Lambda^3,\, \Lambda_{1,1}^7\big)\Big|_{f_1'=0}$$

est satisfaite si et seulement si $\mathscr{S} = \mathscr{T} = 0$.

Preuve. Pour simplifier, introduisons les deux variables algébriquement indépendantes $x := -f_1'' f_2'$ et $y := f_1''$, de telle sorte que

$$\Lambda^3\big|_{f_1'=0} = x, \qquad \Lambda_1^5\big|_{f_1'=0} = 3\, yx, \qquad \Lambda_{1,1}^7\big|_{f_1'=0} = 15\, y^2 x.$$

En développant $\mathscr{S}$ et $\mathscr{T}$ en série de monômes et en regroupant les termes selon les puissances de y, on obtient une identité :

$$0 \equiv \textstyle\sum_l y^{2l} \left(\sum_k 15^l\, \mathsf{s}_{kl}\, x^{k+l}\right) + 3 \sum_l y^{2l+1} \left(\sum_k 15^l\, \mathsf{t}_{kl}\, x^{k+l+1}\right)$$

qui se déploie nécessairement en deux collections d'identités :

$$0 \equiv \textstyle\sum_k 15^l\, \mathsf{s}_{kl}\, x^{k+l} \qquad \text{et} \qquad 0 \equiv \sum_k 15^l\, \mathsf{t}_{kl}\, x^{k+l+1}$$

indexées par l, lesquelles impliquent enfin manifestement l'annulation de tous les coefficients s_{kl} et t_{kl}. $\square$

Ainsi le lemme implique que $\mathscr{Q}_a^{2\times\mathrm{inv}} + \Lambda_1^5\, \mathscr{R}_a^{2\times\mathrm{inv}} \equiv 0$, contradiction. Donc en conclusion, l'expression obtenue :

$$\mathsf{P}^{2\times\mathrm{inv}}\big(j^4 f\big) = \sum_{0\leqslant a\leqslant m} (f_1')^m \left[\mathscr{Q}_a^{2\times\mathrm{inv}}\big(\Lambda^3, \Lambda_{1,1}^7, M^8\big) + \Lambda_1^5\, \mathscr{R}_a^{2\times\mathrm{inv}}\big(\Lambda^3, \Lambda_{1,1}^7, M^8\big)\right]$$

ne fait intervenir que des puissances positives de f_1' : c'est donc un vrai polynôme, et tout polynôme de cette sorte est manifestement invariant par reparamétrisation et par rapport à l'action de $\mathsf{U}_2(\mathbb{C})$. Le théorème est démontré. $\square$

Remarque sur le degré de transcendance. Observons au passage que les quatre polynômes fondamentaux f_1', Λ^3, $\Lambda_{1,1}^7$ et M^8, dont les puissances apparaissent de manière quelconque dans $\mathsf{P}^{2\times\mathrm{inv}}\big(j^4 f\big)$, sont en fait algébriquement indépendants (heureusement!). En effet, en partant des expressions

complètes :

$$\begin{aligned}
f_1' &= f_1' \\
\Lambda^3 &= \Delta^{1,2} \\
\Lambda^7_{1,1} &= \Delta^{1,4}\, f_1' f_1' + 4\,\Delta^{2,3}\, f_1' f_1' - 10\,\Delta^{1,3}\, f_1' f_1'' + 15\,\Delta^{1,2}\, f_1'' f_1'' \\
M^8 &= 3\,\Delta^{1,4}\,\Delta^{1,2} + 12\,\Delta^{2,3}\,\Delta^{1,2} - 5\,\Delta^{1,3}\,\Delta^{1,3},
\end{aligned}$$

si nous introduisons la combinaison algébrique :

$$\begin{aligned}
\widetilde{M}^8 &:= M^8 - 3\,\frac{\Lambda^7_{1,1}\,\Lambda^3}{f_1' f_1'} \\
&= -\,5\,\Delta^{1,3}\,\Delta^{1,3} + 30\,\Delta^{1,3}\,\Delta^{1,2}\,\frac{f_1''}{f_1'} - 45\,\Delta^{1,2}\,\Delta^{1,2}\,\frac{f_1''}{f_1'}\,\frac{f_1''}{f_1'},
\end{aligned}$$

nous voyons immédiatement que f_1', Λ^3, $\Lambda^7_{1,1}$ et $\widetilde{M}^8$ sont algébriquement indépendants, puisque leur expression, de type triangulaire dans les variables de jets, fait successivement apparaître f_1', f_1'', f_1''' et f_1''''.

Lemme. Au dessus de $\mathbb{C}\big[f_1', f_2', f_1'', f_2'', f_1''', f_2''', f_1'''', f_2''''\big]$, le degré de transcendance du corps engendré par les cinq bi-invariants f_1', Λ^3, Λ^5_1, $\Lambda^7_{1,1}$ et M^8 est égal à 4, tandis que celui du corps engendré par les neuf invariants f_1', f_2', Λ^3, Λ^5_1, Λ^5_2, $\Lambda^7_{1,1}$, $\Lambda^7_{1,2}$, $\Lambda^7_{2,2}$ et M^8 est égal à 5.

§7. Jets d'ordre 5 en dimension 2

Idéal des relations. Nous pouvons donc maintenant poursuivre notre étude des polynômes invariants par reparamétrisation au niveau des jets d'ordre $\kappa = 5$. À cet étage, parmi les ving-cinq invariants que nous avons calculés et normalisés dans la Section 4, onze d'entre eux sont bi-invariants de manière évidente, à savoir ceux qui ne comportent que des "$(\cdot)_1$" en indice inférieur :

$$f_1', \quad \Lambda^3, \quad \Lambda^5_1, \quad \Lambda^7_{1,1}, \quad M^8, \quad \Lambda^9_{1,1,1}, \quad M^{10}_1, \quad N^{12}, \quad K^{12}_{1,1}, \quad H^{14}_1, \quad F^{16}_{1,1}.$$

Sachant que nous avons déjà systématiquement tenu compte de l'identité de Jacobi toutes les fois qu'elle nous permettait de réduire le nombre d'invariants indépendants qui doivent être envisagés, l'idéal des relations qui existe entre nos bi-invariants est alors maintenant construit en écrivant méthodiquement les $C^3_5 = 10$ relations ($\mathscr{P}lck_1$) que l'on peut former en sélectionnant trois colonnes abitraires de la matrice :

$$\left\| \begin{array}{ccccc} f_1' & 3\,\Lambda^3 & 5\,\Lambda^5_1 & 7\,\Lambda^5_{1,1} & 8\,M^8 \\ \mathsf{D} f_1' & \mathsf{D}\Lambda^3 & \mathsf{D}\Lambda^5_1 & \mathsf{D}\Lambda^7_{1,1} & \mathsf{D} M^8 \end{array} \right\|,$$

et aussi les $C_5^4 = 5$ relations ($\mathscr{P}lck_2$) du deuxième type associées à chaque choix de quatre colonnes de cette même matrice, ce qui nous donne :

$$
\begin{aligned}
0 &\overset{8}{\equiv} f_1' \left[\Lambda^3,\, \Lambda_1^5\right] + 5\,\Lambda_1^5 \left[f_1',\, \Lambda^3\right] + 3\,\Lambda^3 \left[\Lambda_1^5,\, f_1'\right],\\
0 &\overset{10}{\equiv} f_1' \left[\Lambda^3,\, \Lambda_{1,1}^7\right] + 7\,\Lambda_{1,1}^7 \left[f_1',\, \Lambda^3\right] + 3\,\Lambda^3 \left[\Lambda_{1,1}^7,\, f_1'\right],\\
0 &\overset{13}{\equiv} f_1' \left[\Lambda^3,\, M^8\right] + 8\,M^8 \left[f_1',\, \Lambda^3\right] + 3\,\Lambda^3 \left[M^8,\, f_1'\right],\\
0 &\overset{15}{\equiv} f_1' \left[\Lambda_1^5,\, \Lambda_{1,1}^7\right] + 7\,\Lambda_{1,1}^7 \left[f_1',\, \Lambda_1^5\right] + 5\,\Lambda_1^5 \left[\Lambda_{1,1}^7,\, f_1'\right],\\
0 &\overset{18}{\equiv} f_1' \left[\Lambda_1^5,\, M^8\right] + 8\,M^8 \left[f_1',\, \Lambda_1^5\right] + 5\,\Lambda_1^5 \left[M^8,\, f_1'\right],\\
0 &\overset{25}{\equiv} f_1' \left[\Lambda_{1,1}^7,\, M^8\right] + 8\,M^8 \left[f_1',\, \Lambda_{1,1}^7\right] + 7\,\Lambda_{1,1}^7 \left[M^8,\, f_1'\right],
\end{aligned}
$$

$$
\begin{aligned}
0 &\overset{51}{\equiv} 3\,\Lambda^3 \left[\Lambda_1^5,\, \Lambda_{1,1}^7\right] + 7\,\Lambda_{1,1}^7 \left[\Lambda^3,\, \Lambda_1^5\right] + 5\,\Lambda_1^5 \left[\Lambda_{1,1}^7,\, \Lambda^3\right],\\
0 &\overset{54}{\equiv} 3\,\Lambda^3 \left[\Lambda_1^5,\, M^8\right] + 8\,M^8 \left[\Lambda^3,\, \Lambda_1^5\right] + 5\,\Lambda_1^5 \left[M^8,\, \Lambda^3\right],\\
0 &\overset{61}{\equiv} 3\,\Lambda^3 \left[\Lambda_{1,1}^7,\, M^8\right] + 8\,M^8 \left[\Lambda^3,\, \Lambda_{1,1}^7\right] + 7\,\Lambda_{1,1}^7 \left[M^8,\, \Lambda^3\right],\\
0 &\overset{71}{\equiv} 5\,\Lambda_1^5 \left[\Lambda_{1,1}^7,\, M^8\right] + 8\,M^8 \left[\Lambda_1^5,\, \Lambda_{1,1}^7\right] + 7\,\Lambda_{1,1}^7 \left[M^8,\, \Lambda_1^5\right],
\end{aligned}
$$

$$
\begin{aligned}
0 &\overset{23'}{\equiv} \left[f_1',\, \Lambda^3\right] \cdot \left[\Lambda_1^5,\, \Lambda_{1,1}^7\right] + \left[\Lambda_{1,1}^7,\, f_1'\right] \cdot \left[\Lambda_1^5,\, \Lambda^3\right] + \left[\Lambda^3,\, \Lambda_{1,1}^7\right] \cdot \left[\Lambda_1^5,\, f_1'\right],\\
0 &\overset{26'}{\equiv} \left[f_1',\, \Lambda^3\right] \cdot \left[\Lambda_1^5,\, M^8\right] + \left[M^8,\, f_1'\right] \cdot \left[\Lambda_1^5,\, \Lambda^3\right] + \left[\Lambda^3,\, M^8\right] \cdot \left[\Lambda_1^5,\, f_1'\right],\\
0 &\overset{33'}{\equiv} \left[f_1',\, \Lambda^3\right] \cdot \left[\Lambda_{1,1}^7,\, M^8\right] + \left[M^8,\, f_1'\right] \cdot \left[\Lambda_{1,1}^7,\, \Lambda^3\right] + \left[\Lambda^3,\, M^8\right] \cdot \left[\Lambda_{1,1}^7,\, f_1'\right],\\
0 &\overset{43'}{\equiv} \left[f_1',\, \Lambda_1^5\right] \cdot \left[\Lambda_{1,1}^7,\, M^8\right] + \left[M^8,\, f_1'\right] \cdot \left[\Lambda_{1,1}^7,\, \Lambda_1^5\right] + \left[\Lambda_1^5,\, M^8\right] \cdot \left[\Lambda_{1,1}^7,\, f_1'\right],\\
0 &\overset{98'}{\equiv} \left[\Lambda^3,\, \Lambda_1^5\right] \cdot \left[\Lambda_{1,1}^7,\, M^8\right] + \left[M^8,\, \Lambda^3\right] \cdot \left[\Lambda_{1,1}^7,\, \Lambda_1^5\right] + \left[\Lambda_1^5,\, M^8\right] \cdot \left[\Lambda_{1,1}^7,\, \Lambda^3\right].
\end{aligned}
$$

Dénombrement des syzygies complètes. L'idéal complet des relations ($\mathscr{P}lck_1$) et ($\mathscr{P}lck_2$) existant entre les ving-cinq invariants comporte :

$$C_9^3 + C_9^4 = \mathbf{84} + \mathbf{126} = \mathbf{210},$$

relations que nous avons patiemment développées sur treize pages manuscrites, mais que nous renonçons à recopier dans ce fichier LaTeX, pour la simple raison qu'il suffit, comme nous l'avons argumenté, d'étudier seulement les bi-invariants. Sur les 15 signes "$\equiv$" donnant les syzygies qui existent entre les bi-invariants, nous conservons, pour mémoire, la numérotation de nos $84 + 126$ équations manuscrites.

Énoncé. Voici maintenant l'énoncé que nous devrions attendre comme constituant notre deuxième résultat principal.

Théorème . Pour les jets d'ordre 5 en dimension 2, tout bi-invariant de poids m, $\mathsf{P}^{2\times\mathrm{inv}}\big(j^5 f_1,\, j^5 f_2\big)$, s'exprime polynomialement en fonction de onze polynômes fondamentaux :

$$\boxed{\begin{array}{ccccc} f_1' & \Lambda^3 & \Lambda_1^5 & \Lambda_{1,1}^7 & M^8 \\ & \Lambda_{1,1,1}^9 & M_1^{10} & N^{12} & K_{1,1}^{12} \\ & & & H_1^{14} & F_{1,1}^{16} \end{array}}$$

qui sont donnés explicitement en fonction de $j^5 f$ par les formules calculées à la Section 4, et dont l'idéal des relations est constitué des quinze équations de degré $\leqslant 3$ suivantes :

$$\begin{aligned}
0 &\overset{8}{\equiv} -f_1' f_1'\, M^8 - 5\,\Lambda_1^5\,\Lambda_1^5 + 3\,\Lambda^3\,\Lambda_{1,1}^7,\\
0 &\overset{10}{\equiv} -f_1' f_1'\, M_1^{10} - 7\,\Lambda_1^5\,\Lambda_{1,1}^7 + 3\,\Lambda^3\,\Lambda_{1,1,1}^9,\\
0 &\overset{13}{\equiv} -f_1'\, N^{12} - 8\,\Lambda_1^5\, M^8 + 3\,\Lambda^3\, M_1^{10},\\
0 &\overset{15}{\equiv} -f_1' f_1'\, K_{1,1}^{12} - 7\,\Lambda_{1,1}^7\,\Lambda_{1,1}^7 + 5\,\Lambda_1^5\,\Lambda_{1,1,1}^9,\\
0 &\overset{18}{\equiv} -f_1'\, H_1^{14} - 8\,\Lambda_{1,1}^7\, M^8 + 5\,\Lambda_1^5\, M_1^{10},\\
0 &\overset{25}{\equiv} -f_1'\, F_{1,1}^{16} - 8\, M^8\,\Lambda_{1,1,1}^9 + 7\,\Lambda_{1,1}^7\, M_1^{10},
\end{aligned}$$

$$\begin{aligned}
0 &\overset{51}{\equiv} -3\,\Lambda^3\, K_{1,1}^{12} - 7\,\Lambda_{1,1}^7\, M^8 + 5\,\Lambda_1^5\, M_1^{10},\\
0 &\overset{54}{\equiv} -3\,\Lambda^3\, H_1^{14} - 8\, f_1'\, M^8\, M^8 + 5\,\Lambda_1^5\, N^{12},\\
0 &\overset{61}{\equiv} -3\,\Lambda^3\, F_{1,1}^{16} - 8\, f_1'\, M^8\, M_1^{10} + 7\,\Lambda_{1,1}^7\, N^{12},\\
0 &\overset{71}{\equiv} -5\,\Lambda_1^5\, F_{1,1}^{16} - 8\, f_1'\, M^8\, K_{1,1}^{12} + 7\,\Lambda_{1,1}^7\, H_1^{14},
\end{aligned}$$

$$\begin{aligned}
0 &\overset{23'}{\equiv} \Lambda_1^5\, K_{1,1}^{12} + M^8\,\Lambda_{1,1,1}^9 - \Lambda_{1,1}^7\, M_1^{10},\\
0 &\overset{26'}{\equiv} \Lambda_1^5\, H_1^{14} + f_1'\, M^8\, M_1^{10} - \Lambda_{1,1}^7\, N^{12},\\
0 &\overset{33'}{\equiv} \Lambda_1^5\, F_{1,1}^{16} + f_1'\, M_1^{10}\, M_1^{10} - \Lambda_{1,1,1}^9\, N^{12},\\
0 &\overset{43'}{\equiv} \Lambda_{1,1}^7\, F_{1,1}^{16} + f_1'\, M_1^{10}\, K_{1,1}^{12} - \Lambda_{1,1,1}^9\, H_1^{14},\\
0 &\overset{98'}{\equiv} M^8\, F_{1,1}^{16} + N^{12}\, K_{1,1}^{12} - M_1^{10}\, H_1^{14}.
\end{aligned}$$

Par souci de ne pas alourdir exagérément l'énoncé de ce théorème nous repoussons à la Section 8 l'énoncé précis qui donne les sommes directes de représentations irréductibles de Schur permettant d'entreprendre un calcul de caractéristique d'Euler.

L'action des matrices de la forme :

$$\mathrm{v} := \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$$

faisant "renaître par polarisation" les 14 invariants qui ne sont pas bi-invariants, nous pouvons en déduire une description partielle, mais suffisante pour notre objectif, de $\mathscr{DS}^5_2$.

Corollaire . Tout polynôme $\mathsf{P}\big(j^5 f_1, j^5 f_2\big)$ invariant par reparamétrisation s'exprime polynomialement en fonction de vingt-cinq invariants fondamentaux :

$$\boxed{\begin{array}{ccccccccc} f_1' & f_2' & \Lambda^3 & \Lambda^5_1 & \Lambda^5_2 & \Lambda^7_{1,1} & \Lambda^7_{1,2} & \Lambda^7_{2,2} & M^8 \\ & \Lambda^9_{1,1,1} & & \Lambda^9_{1,2,1} & \Lambda^9_{2,1,2} & \Lambda^9_{2,2,2} & M^{10}_1 & M^{10}_2 & \\ & & & N^{12} & K^{12}_{1,1} & K^{12}_{1,2} & K^{12}_{2,1} & K^{12}_{2,2} & \\ & & & H^{14}_1 & H^{14}_2 & F^{16}_{1,1} & F^{16}_{1,2} & F^{16}_{2,2} & \end{array}},$$

qui sont donnés par les formules explicites normalisées :

$$\begin{aligned} & f_i' \\ & \Lambda^3 := \Delta^{1,2} \\ & \Lambda^5_i := \Delta^{1,3}\, f_i' - 3\,\Delta^{1,2}\, f_i'' \\ & \Lambda^7_{i,j} := \Delta^{1,4}\, f_i' f_j' + 4\,\Delta^{2,3}\, f_i' f_j' - 5\,\Delta^{1,3}\big(f_i'' f_j' + f_i' f_j''\big) + 15\,\Delta^{1,2}\, f_i'' f_j'' \\ & M^8 := 3\,\Delta^{1,4}\,\Delta^{1,2} + 12\,\Delta^{2,3}\,\Delta^{1,2} - 5\,\Delta^{1,3}\,\Delta^{1,3} \end{aligned}$$

$$\begin{aligned} \Lambda^9_{i,j,k} := {} & \Delta^{1,5}\, f_i' f_j' f_k' + 5\,\Delta^{2,4}\, f_i' f_j' f_k' - \\ & - 4\,\Delta^{1,4}\big(f_i'' f_j' + f_i' f_j''\big)\, f_k' - 7\,\Delta^{1,4}\, f_i' f_j' f_k'' - \\ & - 16\,\Delta^{2,3}\big(f_i'' f_j' + f_i' f_j''\big) f_k' - 28\,\Delta^{2,3}\, f_i' f_j' f_k'' - \\ & - 5\,\Delta^{1,3}\big(f_i''' f_j' + f_i' f_j'''\big)\, f_k' + 35\,\Delta^{1,3}\big(f_i'' f_j'' f_k' + f_i'' f_j' f_k'' + f_i' f_j'' f_k''\big) - \\ & - 105\,\Delta^{1,2}\, f_i'' f_j'' f_k'', \end{aligned}$$

$$\begin{aligned} M^{10}_i := {} & \big[3\,\Delta^{1,5}\,\Delta^{1,2} + 15\,\Delta^{2,4}\,\Delta^{1,2} - 7\,\Delta^{1,4}\,\Delta^{1,3} + 2\,\Delta^{2,3}\,\Delta^{1,3}\big]\, f_i' - \\ & - \big[24\,\Delta^{1,4}\,\Delta^{1,2} + 96\,\Delta^{2,3}\,\Delta^{1,2} - 40\,\Delta^{1,3}\,\Delta^{1,3}\big] f_i'', \end{aligned}$$

$$\begin{aligned} N^{12} := {} & 9\,\Delta^{1,5}\,\Delta^{1,2}\,\Delta^{1,2} + 45\,\Delta^{2,4}\,\Delta^{1,2}\,\Delta^{1,2} - 45\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,2} - \\ & - 90\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2} + 40\,\Delta^{1,3}\,\Delta^{1,3}\,\Delta^{1,3}, \end{aligned}$$

$$\begin{aligned} K^{12}_{i,j} := {} & f_i' f_j' \Big(5\,\Delta^{1,5}\,\Delta^{1,3} + 25\,\Delta^{2,4}\,\Delta^{1,3} - 7\,\Delta^{1,4}\,\Delta^{1,4} - 56\,\Delta^{2,3}\,\Delta^{1,4} - 112\,\Delta^{2,3}\,\Delta^{2,3}\Big) + \\ & + \frac{(f_i' f_j'' + f_i'' f_j')}{2}\Big(-15\,\Delta^{1,5}\,\Delta^{1,2} - 75\,\Delta^{2,4}\,\Delta^{1,2} + 65\,\Delta^{1,4}\,\Delta^{1,3} + 110\,\Delta^{2,3}\,\Delta^{1,3}\Big) + \\ & + \frac{(f_i' f_j''' + f_i''' f_j')}{2}\Big(-50\,\Delta^{1,3}\,\Delta^{1,3}\Big) + \\ & + f_i'' f_j''\Big(-25\,\Delta^{1,3}\,\Delta^{1,3} + 15\,\Delta^{1,4}\,\Delta^{1,2} + 60\,\Delta^{2,3}\,\Delta^{1,2}\Big), \end{aligned}$$

$$
\begin{aligned}
H_i^{14} := \Big(&15\,\Delta^{1,5}\,\Delta^{1,3}\,\Delta^{1,2} + 75\,\Delta^{2,4}\,\Delta^{1,3}\,\Delta^{1,2} + 5\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,3} + \\
&+ 170\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,3} - 24\,\Delta^{1,4}\,\Delta^{1,4}\,\Delta^{1,2} - 192\,\Delta^{1,4}\,\Delta^{2,3}\,\Delta^{1,2} - \\
&- 384\,\Delta^{2,3}\,\Delta^{2,3}\,\Delta^{1,2}\Big)\,f_i' + \Big(-45\,\Delta^{1,5}\,\Delta^{1,2}\,\Delta^{1,2} - 225\,\Delta^{2,4}\,\Delta^{1,2}\,\Delta^{1,2} + \\
&+ 225\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,2} + 450\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2} - 200\,\Delta^{1,3}\,\Delta^{1,3}\,\Delta^{1,3}\Big)\,f_i'', \\
F_{i,j}^{16} := \Big(&-3\,\Delta^{1,5}\,\Delta^{1,4}\,\Delta^{1,2} - 15\,\Delta^{2,4}\,\Delta^{1,4}\,\Delta^{1,2} - 12\,\Delta^{1,5}\,\Delta^{2,3}\,\Delta^{1,2} + \\
&+ 40\,\Delta^{1,5}\,\Delta^{1,3}\,\Delta^{1,3} - 60\,\Delta^{2,4}\,\Delta^{2,3}\,\Delta^{1,2} + 200\,\Delta^{2,4}\,\Delta^{1,3}\,\Delta^{1,3} - \\
&- 49\,\Delta^{1,4}\,\Delta^{1,4}\,\Delta^{1,3} - 422\,\Delta^{1,4}\,\Delta^{2,3}\,\Delta^{1,3} - 904\,\Delta^{2,3}\,\Delta^{2,3}\,\Delta^{1,3}\Big)\,f_i'f_j' + \\
&+ \Big(-105\,\Delta^{1,5}\,\Delta^{1,3}\,\Delta^{1,2} - 525\,\Delta^{2,4}\,\Delta^{1,3}\,\Delta^{1,2} + 205\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,3} - \\
&- 230\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,3} + 96\,\Delta^{1,4}\,\Delta^{1,4}\,\Delta^{1,2} + 768\,\Delta^{1,4}\,\Delta^{2,3}\,\Delta^{1,2} + \\
&+ 1536\,\Delta^{2,3}\,\Delta^{2,3}\,\Delta^{1,2}\Big)\,(f_i''f_j' + f_i'f_j'') + \\
&+ \Big(-200\,\Delta^{1,3}\,\Delta^{1,3}\,\Delta^{1,3}\Big)\,(f_i'''f_j' + f_i'f_j''') + \\
&+ \Big(315\,\Delta^{1,5}\,\Delta^{1,2}\,\Delta^{1,2} + 1575\,\Delta^{2,4}\,\Delta^{1,2}\,\Delta^{1,2} - 1575\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,2} - \\
&- 3150\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2} + 1400\,\Delta^{1,3}\,\Delta^{1,3}\,\Delta^{1,3}\Big)\,f_i''f_j'',
\end{aligned}
$$

où les indices i, j et k appartiennent à $\{1, 2\}$.

Remarque. Ce deuxième théorème ainsi que son corollaire doivent être restreints à la sous-algèbre engendrée par les crochets, laquelle s'organise de manière cohérente ([3]) pour former un sous-fibré du fibré des jets de Demailly-Semple au-dessus d'une surface projective algébrique complexe $X^2 \subset P_3(\mathbb{C})$. Nous montrerons en effet à la fin de cette Section 7 que dès les jets d'ordre $\kappa = 5$, il existe des invariants fondamentaux supplémentaires qui ne sont pas obtenus par crochets, et qui s'ajoutent aux nombreux invariants qui apparaissent déjà dans les deux énoncés précédents ; de plus, il pourrait exister une infinité d'invariants par reparamétrisation ainsi que de bi-invariants qui sont fondamentaux, ce qui contredirait *a fortiori* la présomption informelle d'après laquelle les invariants formés par crochets engendrent $\mathscr{DS}_2^5$. Ce phénomène est d'autant plus troublant qu'au niveau $\kappa = 4$, comme nous l'avons démontré, tous les invariants sont engendrés par crochets. Peut-être existe-il des liens mathématiques profonds entre ce phénomène inattendu et le fait que $d = 5$ soit aussi le seuil critique optimal attendu pour la Kobayashi-hyperbolicité des surfaces génériques $X^2 \subset P_3(\mathbb{C})$. L'avenir le dira.

Stratégie. Insistons sur le fait que la stratégie de démonstration que nous allons entreprendre afin de tenter d'établir, comme au niveau $\kappa = 4$, que seuls les invariants formés par crochets existent, *aurait nécessairement dû aboutir si l'algèbre des (bi)invariants engendrés par crochet avait* a priori *coïncidé avec l'algèbre complète des invariants de Demailly-Semple* : ce fait sera argumenté après la fin de nos raisonnements. Ce n'est donc pas cette

stratégie qui est en cause, mais la réalité mathématique, et même si cette dernière contredit parfois nos attentes, il nous faut bien admettre et reconnaître que c'est elle, et seulement elle qui agit en maître, partout et toujours. Et puisque cette réalité précède d'une certaine manière dans ses grandes lignes l'exploration et la recherche, notamment lorsqu'il s'agit de structures algébriques, nous n'aurions jamais pu aboutir à une telle conclusion négative sans entreprendre de considérables efforts de calcul. C'est pourquoi nous convions maintenant notre lecteur à découvrir comment nous comptons généraliser au niveau $\kappa = 5$ notre démonstration qui était valable pour les jets d'ordre 4, avant de dévoiler les interstices dans lesquelles s'insinuent de nombreux invariants fondamentaux supplémentaires (peut-être une infinité) qui ne sont pas engendrés par crochets.

Démonstration du second théorème. Partons de l'expression rationnelle que nous avons obtenue pour tout polynôme bi-invariant

$$\sum_{-\frac{4}{5}m \leqslant a \leqslant m} (f_1')^a \, \mathscr{P}_a\big(\Lambda^3, \Lambda_1^5, \Lambda_{1,1}^7, \Lambda_{1,1,1}^9\big),$$

expression dans laquelle entrent des puissances négatives de f_1'. Le raisonnement que nous avons élaboré pour les jets d'ordre 4 va se généraliser ici, au prix d'une complication supplémentaire mais inévitable, parce que le recours aux bases de Gröbner est en général incontournable pour les idéaux de polynômes à plusieurs variables qui ne sont pas principaux.

Ghost rationality. Observons que les six premières syzygies "$\overset{8}{\equiv}$", "$\overset{10}{\equiv}$", "$\overset{13}{\equiv}$", "$\overset{15}{\equiv}$", "$\overset{18}{\equiv}$" et "$\overset{25}{\equiv}$" entre nos onze bi-invariants fondamentaux font apparaître en première place les six bi-invariants M^8, $\Lambda_{1,1,1}^9$, M_1^{10}, N^{12}, $K_{1,1}^{12}$, H_1^{14} et $F_{1,1}^{16}$ que nous connaissons déjà, mais qui sont invisibles dans le développement en puissances positives et négatives de f_1' que nous venons de rappeler à l'instant, ce dernier ne constituant que la toute première étape de la démonstration. Interprétons donc ces six bi-invariants en les qualifiant intuitivement de "termes fantômes" "cachés" derrière f_1' ou derrière $f_1' f_1'$. Heuristiquement parlant, ce pourrait tout à fait être parce que[67], dans toute expression purement polynomiale en nos onze bi-invariants vers laquelle se dirige notre démonstration :

$$\mathscr{P}\big(f_1', \Lambda^3, \Lambda_1^5, \Lambda_{1,1}^7, M^8, \Lambda_{1,1,1}^9, M_1^{10}, N^{12}, K_{1,1}^{12}, H_1^{14}, F_{1,1}^{16}\big),$$

[67] (et même, "ce *devrait très vraisemblablement être parce que*…", si nous avions déjà achevé la démonstration de notre deuxième théorème!)

l'on peut remplacer ces six bi-invariants spéciaux par leur expression rationnelle en fonction seulement de Λ^3, de Λ_1^5, de $\Lambda_{1,1}^7$ et de $\Lambda_{1,1,1}^9$:

$$\begin{aligned}
M^8 &= \frac{3\,\Lambda^3\,\Lambda_{1,1}^7 - 5\,\Lambda_1^5\,\Lambda_1^5}{f_1'f_1'},\\
M_1^{10} &= \frac{3\,\Lambda^3\,\Lambda_{1,1,1}^9 - 7\,\Lambda_1^5\Lambda_{1,1}^7}{f_1'f_1'},\\
N^{12} &= \frac{-45\,\Lambda^3\,\Lambda_1^5\,\Lambda_{1,1}^7 + 40\,\Lambda_1^5\,\Lambda_1^5\,\Lambda_1^5}{f_1'f_1'f_1'},\\
K_{1,1}^{12} &= \frac{5\,\Lambda_1^5\,\Lambda_{1,1,1}^9 - 7\,\Lambda_{1,1}^7\,\Lambda_{1,1}^7}{f_1'f_1'},\\
H_1^{14} &= \frac{-24\,\Lambda^3\,\Lambda_{1,1}^7\,\Lambda_{1,1}^7 + 5\,\Lambda_1^5\,\Lambda_1^5\,\Lambda_{1,1}^7 + 15\,\Lambda^3\,\Lambda_1^5\,\Lambda_{1,1,1}^9}{f_1'f_1'f_1'},\\
F_{1,1}^{16} &= \frac{-3\,\Lambda^3\,\Lambda_{1,1}^7\,\Lambda_{1,1,1}^9 + 40\,\Lambda_1^5\,\Lambda_1^5\,\Lambda_{1,1,1}^9 - 49\,\Lambda_1^5\,\Lambda_{1,1}^7\,\Lambda_{1,1}^7}{f_1'f_1'f_1'},
\end{aligned}$$

ce qui impose manifestement des divisions par $f_1'f_1'$ ou par $f_1'f_1'f_1'$, ce pourrait donc bien être, disions-nous, pour cette seule et simple raison que notre représentation initiale, (trop) facile à obtenir, d'un bi-invariant sous la forme

$$\sum_{-\frac{4}{5}\,m\leqslant a\leqslant m} (f_1')^a\,\mathscr{P}_a\big(\Lambda^3,\Lambda_1^5,\Lambda_{1,1}^7,\Lambda_{1,1,1}^9\big)$$

faisait inévitablement apparaître des puissances négatives de f_1'. Et pour éliminer ces dénominateurs, rien d'autre ne s'offrirait à nous que d'*injecter les six bi-invariants fantômes dans l'expression rationnelle initiale*. Voilà : nous avons dévoilé une nouvelle idée essentielle qui s'avèrera pertinente et efficiente pour l'étude des invariants de Demailly-Semple à un ordre quelconque.

Élimination des puissances négatives de f_1'. En effet, rappelons-nous tout d'abord que dans le cas des jets d'ordre 4, après avoir injecté le seul bi-invariant "fantôme" existant, à savoir M^8, nous sommes parvenus à éliminer les puissances négatives de f_1' grâce à une normalisation préalable de tout polynôme $\mathscr{P} = \mathscr{P}\big(\Lambda^3,\Lambda_1^5,\Lambda_{1,1}^7,M^8\big)$ sous la forme $\mathscr{Q}\big(\Lambda^3,\Lambda_{1,1}^7,M^8\big) + \Lambda_1^5\,\mathscr{R}\big(\Lambda^3,\Lambda_{1,1}^7,M^8\big)$, ce qui était fort élémentaire, sachant que l'idéal des relations est principal (la théorie des bases de Gröbner est vide dans ce cas), la fin de l'argument reposant seulement sur le fait que lorsqu'on pose $f_1'=0$, aucune relation polynomiale non triviale du type

$$0 \equiv \mathscr{Q}\big(\Lambda^3,\Lambda_1^5,M^8\big) + \Lambda_1^5\,\mathscr{R}\big(\Lambda^3,\Lambda_1^5,M^8\big)\Big|_{f_1'=0}$$

ne peut être satisfaite.

Observation générale cruciale et poursuite de la démonstration. *Aussi est-ce seulement l'idéal des relations entre les bi-invariants restreints à l'hypersurface* $\{f_1' = 0\}$ *qui semble compter*. L'enjeu, ici, après avoir injecté les six bi-invariants fantômes M^8, M_1^{10}, N^{12}, $K_{1,1}^{12}$, H_1^{14} et $F_{1,1}^{16}$ qui étaient cachés derrière des puissances positives de f_1', ce qui nous donne aisément une expression générale du type :

$$\sum_{-\frac{4}{5}m \leqslant a \leqslant m} (f_1')^a \, \mathscr{P}_a\big(\Lambda^3, \Lambda_1^5, \Lambda_{1,1}^7, M^8, \Lambda_{1,1,1}^9, M_1^{10}, N^{12}, K_{1,1}^{12}, H_1^{14}, F_{1,1}^{16}\big),$$

dans laquelle nous supposerons, puisque nous perdons tout contrôle après injection des six bi-invariants supplémentaires, que les nouveaux polynômes $\mathscr{P}_a$ sont arbitraires de poids $m - a$, l'enjeu alors semble être de *parvenir à produire une* écriture normalisée en fonction des syzygies *pour représenter de manière unique tout polynôme* $\mathscr{P}$ *de cette espèce, de façon à ce que toute identité du type* :

$$0 \equiv \text{Écriture unique} \left\{ \mathscr{P}\big(\Lambda^3, \Lambda_1^5, \ldots, H_1^{14}, F_{1,1}^{16}\big) \right\}\Big|_{f_1'=0},$$

implique que le polynôme $\mathscr{P}$ est en fait identiquement nul. Alors l'argument d'élimination de la puissance maximalement négative de f_1', le tout suivi de la restriction à $\{f_1' = 0\}$, cet argument que nous avions utilisé avec succès pour les jets d'ordre 4 fonctionnera à nouveau ici sans modification, et une récurrence immédiate montrera, comme pour les jets d'ordre 4, que les puissances négatives de f_1' n'existent pas, ce que nous désirions obtenir pour achever la démonstration du théorème.

Deux remarques pour mettre un terme à ces considérations heuristiques destinées seulement à dévoiler nettement nos idées en usant du langage spéculatif qui nous a servi de guide pour les élaborer. Premièrement, il est clair que ce plan de démonstration doit fonctionner en toute généralité pour des jets d'ordre quelconque $\kappa \geqslant 4$, et nous montrerons en temps voulu qu'il fonctionne aussi en dimension $\nu \geqslant 3$. Ensuite, notons — puisqu'il est de l'essence des mathématiques d'être "truffées d'obstacles" — que le saut en difficulté, lorsqu'on passe des jets d'ordre 4 aux jets d'ordre 5, est presque trop considérable pour une intuition de généralité habituée aux récurrences régulières et aux combinatoires qui dévoilent progressivement leurs structures : on passe en effet brutalement de une syzygie à quinze, et même de neuf à deux cents dix, pour ce qui concerne les invariants complets ; comment alors ne pas éprouver le sentiment que la complexité algébrique de ce problème explose *trop* rapidement?

Restriction des syzygies. Poser $f_1' = 0$, comme nous devons maintenant le faire, nous donne les 15 équations réduites :

$$
\begin{aligned}
0 &\equiv -5\,\Lambda_1^5\,\Lambda_1^5 + 3\,\Lambda^3\,\Lambda_{1,1}^7\Big|_{f_1'=0},\\
0 &\equiv -7\,\Lambda_1^5\,\Lambda_{1,1}^7 + 3\,\Lambda^3\,\Lambda_{1,1,1}^9\Big|_{f_1'=0},\\
0 &\equiv -8\,\Lambda_1^5\,M^8 + 3\,\Lambda^3\,M_1^{10}\Big|_{f_1'=0},\\
0 &\equiv -7\,\Lambda_{1,1}^7\,\Lambda_{1,1}^7 + 5\,\Lambda_1^5\,\Lambda_{1,1,1}^9\Big|_{f_1'=0},\\
0 &\equiv -8\,\Lambda_{1,1}^7\,M^8 + 5\,\Lambda_1^5\,M_1^{10}\Big|_{f_1'=0},\\
0 &\equiv -8\,M^8\,\Lambda_{1,1,1}^9 + 7\,\Lambda_{1,1}^7\,M_1^{10}\Big|_{f_1'=0},
\end{aligned}
$$

$$
\begin{aligned}
0 &\equiv -3\,\Lambda^3\,K_{1,1}^{12} - 7\,\Lambda_{1,1}^7\,M^8 + 5\,\Lambda_1^5\,M_1^{10}\Big|_{f_1'=0},\\
0 &\equiv -3\,\Lambda^3\,H_1^{14} + 5\,\Lambda_1^5\,N^{12}\Big|_{f_1'=0},\\
0 &\equiv -3\,\Lambda^3\,F_{1,1}^{16} + 7\,\Lambda_{1,1}^7\,N^{12}\Big|_{f_1'=0},\\
0 &\equiv -5\,\Lambda_1^5\,F_{1,1}^{16} + 7\,\Lambda_{1,1}^7\,H_1^{14}\Big|_{f_1'=0},
\end{aligned}
$$

$$
\begin{aligned}
0 &\equiv \Lambda_1^5\,K_{1,1}^{12} + M^8\,\Lambda_{1,1,1}^9 - \Lambda_{1,1}^7\,M_1^{10}\Big|_{f_1'=0},\\
0 &\equiv \Lambda_1^5\,H_1^{14} - \Lambda_{1,1}^7\,N^{12}\Big|_{f_1'=0},\\
0 &\equiv \Lambda_1^5\,F_{1,1}^{16} - \Lambda_{1,1,1}^9\,N^{12}\Big|_{f_1'=0},\\
0 &\equiv \Lambda_{1,1}^7\,F_{1,1}^{16} - \Lambda_{1,1,1}^9\,H_1^{14}\Big|_{f_1'=0},\\
0 &\equiv M^8\,F_{1,1}^{16} + N^{12}\,K^{12} - M_1^{10}\,H_1^{14}\Big|_{f_1'=0}.
\end{aligned}
$$

Base de Gröbner. En choisissant l'ordre purement lexicographique ([1]) sur les monômes de $\mathbb{C}\big[f_1', \Lambda^3, \ldots, H_1^{14}, F_{1,1}^{16}\big]$ qui est déduit de l'ordre suivant sur les monômes élémentaires restreints :

$$\Lambda^3 > \Lambda_1^5 > \Lambda_{1,1}^7 > M^8 > \Lambda_{1,1,1}^9 > M_1^{10} > N^{12} > K_{1,1}^{12} > H_1^{14} > F_{1,1}^{16},$$

(nous sous-entendons ici la mention "$(\cdot)|_{f_1'=0}$"), Maple nous donne la base de Gröbner réduite suivante pour l'idéal complet des syzygies entre nos dix

invariants restreints à $\{f_1' = 0\}$, laquelle est constituée de 21 équations :

$$0 \overset{1}{\equiv} -7\,H_1^{14}\,H_1^{14} + 5\,\underline{N^{12}\,F_{1,1}^{16}}\Big|_{f_1'=0},$$

$$0 \overset{2}{\equiv} -56\,K_{1,1}^{12}\,H_1^{14} + 5\,\underline{M_1^{10}\,F_{1,1}^{16}}\Big|_{f_1'=0},$$

$$0 \overset{3}{\equiv} -8\,N^{12}\,K_{1,1}^{12} + \underline{M_1^{10}\,H_1^{14}}\Big|_{f_1'=0},$$

$$0 \overset{4}{\equiv} -7\,N^{12}\,K_{1,1}^{12} + \underline{M^8\,F_{1,1}^{16}}\Big|_{f_1'=0},$$

$$0 \overset{5}{\equiv} -5\,M_1^{10}\,N^{12} + 8\,\underline{M^8\,H_1^{14}}\Big|_{f_1'=0},$$

$$0 \overset{6}{\equiv} -5\,M_1^{10}\,M_1^{10} + 64\,\underline{M^8\,K_{1,1}^{12}}\Big|_{f_1'=0},$$

$$0 \overset{7}{\equiv} -\Lambda_{1,1,1}^9\,H_1^{14} + \underline{\Lambda_{1,1}^7\,F_{1,1}^{16}}\Big|_{f_1'=0},$$

$$0 \overset{8}{\equiv} -5\,\Lambda_{1,1,1}^9\,N^{12} + 7\,\underline{\Lambda_{1,1}^7\,H_1^{14}}\Big|_{f_1'=0},$$

$$0 \overset{9}{\equiv} -5\,\Lambda_{1,1,1}^9\,M_1^{10} + 56\,\underline{\Lambda_{1,1}^7\,K_{1,1}^{12}}\Big|_{f_1'=0},$$

$$0 \overset{10}{\equiv} -8\,M^8\,\Lambda_{1,1,1}^9 + 7\,\underline{\Lambda_{1,1}^7\,M_1^{10}}\Big|_{f_1'=0},$$

$$0 \overset{11}{\equiv} -\Lambda_{1,1,1}^9\,N^{12} + \underline{\Lambda_1^5\,F_{1,1}^{16}}\Big|_{f_1'=0},$$

$$0 \overset{12}{\equiv} -\Lambda_{1,1}^7\,N^{12} + \underline{\Lambda_1^5\,H_1^{14}}\Big|_{f_1'=0},$$

$$0 \overset{13}{\equiv} -M^8\,\Lambda_{1,1,1}^9 + 7\,\underline{\Lambda_1^5\,K_{1,1}^{12}}\Big|_{f_1'=0},$$

$$0 \overset{14}{\equiv} -8\,\Lambda_{1,1}^7\,M^8 + 5\,\underline{\Lambda_1^5\,M_1^{10}}\Big|_{f_1'=0},$$

$$0 \overset{15}{\equiv} -7\,\Lambda_{1,1}^7\,\Lambda_{1,1}^7 + 5\,\underline{\Lambda_1^5\,\Lambda_{1,1,1}^9}\Big|_{f_1'=0},$$

$$0 \overset{16}{\equiv} -7\,\Lambda_{1,1}^7\,N^{12} + 3\,\underline{\Lambda^3\,F_{1,1}^{16}}\Big|_{f_1'=0},$$

$$0 \overset{17}{\equiv} -5\,\Lambda_1^5\,N^{12} + 3\,\underline{\Lambda^3\,H_1^{14}}\Big|_{f_1'=0},$$

$$0 \overset{18}{\equiv} -\Lambda_{1,1}^7\,M^8 + 3\,\underline{\Lambda^3\,K_{1,1}^{12}}\Big|_{f_1'=0},$$

$$0 \overset{19}{\equiv} -8\,\Lambda_1^5\,M^8 + 3\,\underline{\Lambda^3\,M_1^{10}}\Big|_{f_1'=0},$$

$$0 \overset{20}{\equiv} -7\,\Lambda_1^5\,\Lambda_{1,1}^7 + 3\,\underline{\Lambda^3\,\Lambda_{1,1,1}^9}\Big|_{f_1'=0},$$

$$0 \overset{21}{\equiv} -5\,\Lambda_1^5\,\Lambda_1^5 + 3\,\underline{\Lambda^3\,\Lambda_{1,1}^7}\Big|_{f_1'=0},$$

toutes déduites de nos 15 syzygies restreintes à $\{f_1' = 0\}$, et dont l'ensemble recelle une combinatoire d'une simplicité inattendue qui va se dévoiler à nous dans un instant. Nous avons souligné les monômes de tête pour en extraire l'idéal monomial associé (*voir infra*).

Bien que cette base de Gröbner nous ait été procurée par Maple, il n'est pas nécessaire que nous nous en remettions au calcul formel électronique pour assurer la rigueur du résultat, puisqu'il est ici très aisé de vérifier :

- □ que ces 21 équations sont effectivement conséquence de nos quinze syzygies réduites ;
- □ que ces 21 équations forment effectivement une base de Gröbner.

Le premier point se vérifie sans difficulté ; le paragraphe ci-dessous où nous donnons l'expression des bi-invariants restreints à $\{f_1' = 0\}$ permet d'ailleurs de procéder très rapidement. Pour ce qui est du deuxième point, il nous suffit d'appliquer l'un des nombreux critères caractérisant les bases de Gröbner ([1]), d'après lequel chaque S-polynôme entre deux équations quelconques doit appartenir à l'idéal engendré par les 21 polynômes, et à cette fin, la tâche de calcul manuel est miraculeusement facilitée par le fait que chacune de ces 21 équations ne comporte que deux termes, avec à chaque fois un signe "$-$" et un signe "$+$", ces deux termes étant chacun monomiaux et qui plus est, de degré deux, ce qui fait que *chaque S-polynôme entre*

deux équations ne possède encore que deux termes monomiaux. Par exemple, si on élimine les monômes de tête entre "$\overset{20}{\equiv}$" et "$\overset{21}{\equiv}$" en multipliant par des monômes appropriés et en soustrayant :

$$\begin{aligned}
0 &\equiv \big(-7\,\Lambda_1^5\,\Lambda_{1,1}^7 + 3\,\underline{\Lambda^3\,\Lambda_{1,1,1}^9}\big)\cdot\underline{\Lambda_{1,1}^7} - \big(-5\,\Lambda_1^5\,\Lambda_1^5 + 3\,\underline{\Lambda^3\,\Lambda_{1,1}^7}\big)\cdot\underline{\Lambda_{1,1,1}^9}\Big|_{f_1'=0},\\
&\equiv -7\,\Lambda_1^5\,\Lambda_{1,1}^7\,\Lambda_{1,1}^7 + 5\,\Lambda_1^5\,\Lambda_1^5\,\Lambda_{1,1,1}^9,
\end{aligned}$$

on constate que le S-polynôme obtenu appartient bien à notre idéal, puisqu'il coïncide avec l'équation "$\overset{15}{\equiv}$" multipliée par Λ_1^5. Les 209 autres S-polynômes restants se traitent de la même manière, à la main, en moins de deux heures, après épuration préalable des notations.

Expression des bi-invariants restreints à $\{f_1' = 0\}$. Mais avant de poursuivre, il est instructif d'écrire, d'examiner et de commenter la liste de nos onze bi-invariants restreints :

$$\begin{aligned}
f_1'\big|_0 &= 0,\\
\Lambda^3\big|_0 &= -f_1''f_2' =: \Delta_0^{1,2},\\
\Lambda_1^5\big|_0 &= -3\,\Delta_0^{1,2}\,f_1'',\\
\Lambda_{1,1}^7\big|_0 &= 15\,\Delta_0^{1,2}\,f_1''f_1'',\\
M^8\big|_0 &= 3\,\Delta_0^{1,4}\,\Delta_0^{1,2} + 12\,\Delta_0^{2,3}\,\Delta_0^{1,2} - 5\,\Delta_0^{1,3}\,\Delta^{1,3},\\
\Lambda_{1,1,1}^9\big|_0 &= -105\,\Delta_0^{1,2}\,f_1''f_1''f_1'',\\
M_1^{10}\big|_0 &= -\Big[24\,\Delta_0^{1,4}\,\Delta_0^{1,2} + 96\,\Delta_0^{2,3}\,\Delta_0^{1,2} - 40\,\Delta_0^{1,3}\,\Delta_0^{1,3}\Big]\,f_1'',\\
N^{12}\big|_0 &= 9\,\Delta_0^{1,5}\,\Delta_0^{1,2}\,\Delta_0^{1,2} + 45\,\Delta_0^{2,4}\,\Delta_0^{1,2}\,\Delta_0^{1,2} - 45\,\Delta_0^{1,4}\,\Delta_0^{1,3}\,\Delta_0^{1,2} -\\
&\qquad - 90\,\Delta_0^{2,3}\,\Delta_0^{1,3}\,\Delta_0^{1,2} + 40\,\Delta_0^{1,3}\,\Delta_0^{1,3}\,\Delta_0^{1,3},\\
K_{1,1}^{12}\big|_0 &= \Big[15\,\Delta_0^{1,4}\,\Delta_0^{1,2} + 60\,\Delta_0^{2,3}\,\Delta_0^{1,2} - 25\,\Delta_0^{1,3}\,\Delta_0^{1,3}\Big]\,f_1''f_1'',\\
H_1^{14}\big|_0 &= \Big[-45\,\Delta_0^{1,5}\,\Delta_0^{1,2}\,\Delta_0^{1,2} - 225\,\Delta_0^{2,4}\,\Delta_0^{1,2}\,\Delta_0^{1,2} + 225\,\Delta_0^{1,4}\,\Delta_0^{1,3}\,\Delta_0^{1,2} +\\
&\qquad + 450\,\Delta_0^{2,3}\,\Delta_0^{1,3}\,\Delta_0^{1,2} - 200\,\Delta_0^{1,3}\,\Delta_0^{1,3}\,\Delta_0^{1,3}\Big]\,f_1'',\\
F_{1,1}^{16}\big|_0 &= \Big[315\,\Delta_0^{1,5}\,\Delta_0^{1,2}\,\Delta_0^{1,2} + 1575\,\Delta_0^{2,4}\,\Delta_0^{1,2}\,\Delta_0^{1,2} - 1575\,\Delta_0^{1,4}\,\Delta_0^{1,3}\,\Delta_0^{1,2} -\\
&\qquad - 3150\,\Delta_0^{2,3}\,\Delta_0^{1,3}\,\Delta_0^{1,2} + 1400\,\Delta_0^{1,3}\,\Delta_0^{1,3}\,\Delta_0^{1,3}\Big]\,f_1''f_1''.
\end{aligned}$$

Divisions wronskiennes. En comparant la deuxième et la troisième équation, nous obtenons par exemple $f_1'' = -\frac{1}{3}\,\frac{\Lambda_1^5|_0}{\Lambda^3|_0}$, et aussi $f_1''f_1'' = \frac{1}{15}\,\frac{\Lambda_{1,1}^7|_0}{\Lambda^3|_0}$ si l'on compare la deuxième et la quatrième ligne, et en poursuivant ces

observations, nous pouvons écrire :

$$
\begin{aligned}
&\underline{\Lambda^3}\big|_0,\\
&\underline{\Lambda^5_1}\big|_0,\\
&\Lambda^7_{1,1}\big|_0 = \tfrac{5}{3}\,\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0},\\
&\underline{M^8}\big|_0,\\
&\Lambda^9_{1,1,1}\big|_0 = \tfrac{7}{3}\,\tfrac{\Lambda^5_1|_0\,\Lambda^7_{1,1}|_0}{\Lambda^3|_0} = \tfrac{35}{9}\,\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0\,\Lambda^3|_0},\\
&M^{10}_1\big|_0 = \tfrac{8}{3}\,\tfrac{M^8|_0\,\Lambda^5_1|_0}{\Lambda^3|_0},\\
&\underline{N^{12}}\big|_0,\\
&K^{12}_{1,1}\big|_0 = \tfrac{1}{3}\,\tfrac{M^8|_0\,\Lambda^7_{1,1}|_0}{\Lambda^3|_0} = \tfrac{5}{9}\,\tfrac{M^8|_0\,\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0\,\Lambda^3|_0},\\
&H^{14}_1\big|_0 = \tfrac{5}{3}\,\tfrac{N^{12}|_0\,\Lambda^5_1|_0}{\Lambda^3|_0},\\
&F^{16}_{1,1}\big|_0 = \tfrac{7}{3}\,\tfrac{\Lambda^7_{1,1}|_0\,N^{12}|_0}{\Lambda^3|_0} = \tfrac{35}{9}\,\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,N^{12}|_0}{\Lambda^3|_0\,\Lambda^3|_0},
\end{aligned}
$$

où nous soulignons quatre bi-invariants restreints qui apparaissent fondamentaux, à savoir $\underline{\Lambda^3}\big|_0$, $\underline{\Lambda^5_1}\big|_0$, $\underline{M^8}\big|_0$ et $\underline{N^{12}}\big|_0$, puisque les six autres s'expriment en fonction d'eux, après restriction à $\{f'_1 = 0\}$, lorsqu'on autorise à diviser par le wronskien. Un examen immédiat de l'expression complète de ces quatre bi-invariants restreints fondamentaux $\underline{\Lambda^3}\big|_0$, $\underline{\Lambda^5_1}\big|_0$, $\underline{M^8}\big|_0$ et $\underline{N^{12}}\big|_0$ montre qu'ils sont algébriquement indépendants, puisqu'ils incorporent successivement f''_1, f'''_1, f''''_1 et f'''''_1. Cette indépendance mutuelle resservira ultérieurement.

Triangle harmonieux des monômes de tête. Comme nous le constatons en examinant notre base de Gröbner, les 21 binômes de tête s'organisent, lorsqu'on les range par ordre (lexicographique) croissant, en un triangle remarquable :

$$
\begin{array}{lllll}
\Lambda^3\,\Lambda^7_{1,1} & > \Lambda^3\,\Lambda^9_{1,1,1} & > \Lambda^3\,M^{10}_1 & > \Lambda^3\,K^{12}_{1,1} & > \Lambda^3\,H^{14}_1 & > \Lambda^3\,F^{16}_{1,1}\\
 & > \Lambda^5_1\,\Lambda^9_{1,1,1} & > \Lambda^5_1\,M^{10}_1 & > \Lambda^5_1\,K^{12}_{1,1} & > \Lambda^5_1\,H^{14}_1 & > \Lambda^5_1\,F^{16}_{1,1}\\
 & & > \Lambda^7_{1,1}\,M^{10}_1 & > \Lambda^7_{1,1}\,K^{12}_{1,1} & > \Lambda^7_{1,1}\,H^{14}_1 & > \Lambda^7_{1,1}\,F^{16}_{1,1}\\
 & & & > M^8\,K^{12}_{1,1} & > M^8\,H^{14}_1 & > M^8\,F^{16}_{1,1}\\
 & & & & > M^{10}_1\,H^{14}_1 & > M^{10}_1\,F^{16}_{1,1}\\
 & & & & & > N^{12}\,F^{16}_{1,1}
\end{array}
$$

(on sous-entend la mention "$|_0$" dans ce diagramme) dans lequel nous reconnaissons, à la place des colonnes, les six bi-invariants restreints $\Lambda^7_{1,1}\big|_0$, $\Lambda^9_{1,1,1}\big|_0$, $M^{10}_1\big|_0$, $K^{12}_{1,1}\big|_0$, $H^{14}_1\big|_0$ et $F^{16}_{1,1}\big|_0$, qui s'expriment rationnellement en fonction des quatre bi-invariants restreints fondamentaux $\underline{\Lambda^3}\big|_0$, $\underline{\Lambda^5_1}\big|_0$, $\underline{M^8}\big|_0$ et $\underline{N^{12}}\big|_0$.

Normalisation modulo les syzygies restreintes. Nous pouvons maintenant énoncer et démontrer le lemme sur lequel repose la fin de la démonstration de notre second théorème.

Lemme. Tout polynôme arbitraire en les dix bi-invariants restreints :

$$\mathscr{P}\big(\Lambda^3\big|_0,\ \Lambda^5_1\big|_0,\ \Lambda^7_{1,1}\big|_0,\ M^8\big|_0,\ \Lambda^9_{1,1,1}\big|_0,\ M^{10}_1\big|_0,\ N^{12}\big|_0,\ K^{12}_{1,1}\big|_0,\ H^{14}_1\big|_0,\ F^{16}_{1,1}\big|_0\big)$$

s'écrit de manière unique, en tenant compte des 21 syzygies gröbnérisées ci-dessus, sous la forme unique :

$$\begin{aligned}
&\mathscr{P}_0\big(\Lambda^3\big|_0,\ \Lambda^5_1\big|_0,\ M^8\big|_0,\ N^{12}\big|_0\big) + \Lambda^7_{1,1}\big|_0\,\mathscr{Q}_0\big(\Lambda^5_1\big|_0,\ \Lambda^7_{1,1}\big|_0,\ M^8\big|_0,\ N^{12}\big|_0\big) + \\
&+ \Lambda^9_{1,1,1}\big|_0\,\mathscr{R}_0\big(\Lambda^7_{1,1}\big|_0,\ M^8\big|_0,\ \Lambda^9_{1,1,1}\big|_0,\ N^{12}\big|_0\big) + M^{10}_1\big|_0\,\mathscr{S}_0\big(M^8\big|_0,\ \Lambda^9_{1,1,1}\big|_0,\ M^{10}_1\big|_0,\ N^{12}\big|_0\big) + \\
&+ K^{12}_{1,1}\big|_0\,\mathscr{T}_0\big(\Lambda^9_{1,1,1}\big|_0,\ M^{10}_1\big|_0,\ N^{12}\big|_0,\ K^{12}_{1,1}\big|_0\big) + H^{14}_1\big|_0\,\mathscr{U}_0\big(\Lambda^9_{1,1,1}\big|_0,\ N^{12}\big|_0,\ K^{12}_{1,1}\big|_0,\ H^{14}_1\big|_0\big) + \\
&+ F^{16}_{1,1}\big|_0\,\mathscr{V}_0\big(\Lambda^9_{1,1,1}\big|_0,\ K^{12}_{1,1}\big|_0,\ H^{14}_1\big|_0,\ F^{16}_{1,1}\big|_0\big),
\end{aligned}$$

où $\mathscr{P}_0$, $\mathscr{Q}_0$, $\mathscr{R}_0$, $\mathscr{S}_0$, $\mathscr{T}_0$, $\mathscr{U}_0$ et $\mathscr{V}_0$ sont des polynômes absolument arbitraires en leurs quatre arguments, et pour préciser, toute relation du type :

$$0 \equiv \mathscr{P}_0 + \Lambda^7_{1,1}\big|_0\,\mathscr{Q}_0 + \Lambda^9_{1,1,1}\big|_0\,\mathscr{R}_0 + M^{10}_1\big|_0\,\mathscr{S}_0 + K^{12}_{1,1}\big|_0\,\mathscr{T}_0 + H^{14}_1\big|_0\,\mathscr{U}_0 + F^{16}_{1,1}\big|_0\,\mathscr{V}_0$$

qui est identiquement satisfaite dans $\mathbb{C}\big[f'_2, f''_1, f''_2, f'''_1, f'''_2, f''''_1, f''''_2, f'''''_1, f'''''_2\big]$ lorsqu'on remplace les dix bi-invariants restreints par leur expression en fonction de $j^5 f\big|_0$, implique nécessairement que les sept polynômes $\mathscr{P}_0$, $\mathscr{Q}_0$, $\mathscr{R}_0$, $\mathscr{S}_0$, $\mathscr{T}_0$, $\mathscr{U}_0$ et $\mathscr{V}_0$ s'annulent tous identiquement.

Démonstration du lemme principal. D'après la théorie élémentaire des bases de Gröbner, une base de l'espace vectoriel quotient

$$\mathbb{C}\Big[\Lambda^3\big|_0,\ \Lambda^5_1\big|_0,\ \ldots,\ H^{14}_1\big|_0,\ F^{16}_{1,1}\big|_0\Big]\Big/\big(21 \text{ syzygies restreintes}\big)$$

est constitutée de tous les monômes

$$\begin{aligned}
\big(\Lambda^3\big|_0\big)^a \big(\Lambda^5_1\big|_0\big)^b \big(\Lambda^7_{1,1}\big|_0\big)^c \big(M^8\big|_0\big)^d \big(\Lambda^9_{1,1,1}\big|_0\big)^e \big(M^{10}_1\big|_0\big)^f \big(N^{12}\big|_0\big)^g \\
\big(K^{12}_{1,1}\big|_0\big)^h \big(H^{14}_1\big|_0\big)^i \big(F^{16}_{1,1}\big|_0\big)^j
\end{aligned}$$

qui *n'appartiennent pas* à l'idéal monomial engendré par les 21 monômes de tête que nous avons disposés en triangle, où les exposants $a, b, c, d, e, f, g, h, i, j \in \mathbb{N}$ sont des entiers positifs ou nuls. Or un tel monôme appartient à cet idéal monomial si et seulement si il est divisible par l'un des 21 monômes de tête, ce qui revient à dire que le déca-indice

$$\big(a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i,\ j\big) \in \mathbb{N}^{10}$$

appartient à la réunion des 21 sous-ensembles suivants de $\mathbb{N}^{10}$:

$$\begin{aligned}
\{a \geqslant 1\} \cap \{c \geqslant 1\} \bigcup \{a \geqslant 1\} \cap \{e \geqslant 1\} \bigcup \{a \geqslant 1\} \cap \{f \geqslant 1\} \bigcup \{a \geqslant 1\} \cap \{h \geqslant 1\} \bigcup \{a \geqslant 1\} \cap \{i \geqslant 1\} \bigcup \{a \geqslant 1\} \cap \{j \geqslant 1\} \\
\bigcup \{b \geqslant 1\} \cap \{e \geqslant 1\} \bigcup \{b \geqslant 1\} \cap \{f \geqslant 1\} \bigcup \{b \geqslant 1\} \cap \{h \geqslant 1\} \bigcup \{b \geqslant 1\} \cap \{i \geqslant 1\} \bigcup \{b \geqslant 1\} \cap \{j \geqslant 1\} \\
\bigcup \{c \geqslant 1\} \cap \{f \geqslant 1\} \bigcup \{c \geqslant 1\} \cap \{h \geqslant 1\} \bigcup \{c \geqslant 1\} \cap \{i \geqslant 1\} \bigcup \{c \geqslant 1\} \cap \{j \geqslant 1\} \\
\bigcup \{d \geqslant 1\} \cap \{h \geqslant 1\} \bigcup \{d \geqslant 1\} \cap \{i \geqslant 1\} \bigcup \{d \geqslant 1\} \cap \{j \geqslant 1\} \\
\bigcup \{f \geqslant 1\} \cap \{i \geqslant 1\} \bigcup \{f \geqslant 1\} \cap \{j \geqslant 1\} \\
\bigcup \{g \geqslant 1\} \cap \{j \geqslant 1\}
\end{aligned}$$

Le calcul du complémentaire de cet ensemble est aisé, et il donne :

$$\begin{aligned}
\Big[\{a = 0\} \cup \{c = e = f = h = i = j = 0\}\Big] \bigcap \\
\Big[\{b = 0\} \cup \{e = f = h = i = j = 0\}\Big] \bigcap \\
\Big[\{c = 0\} \cup \{f = h = i = j = 0\}\Big] \bigcap \\
\Big[\{d = 0\} \cup \{h = i = j = 0\}\Big] \bigcap \\
\Big[\{f = 0\} \cup \{i = j = 0\}\Big] \bigcap \\
\Big[\{g = 0\} \cup \{j = 0\}\Big],
\end{aligned}$$

ce qui se simplifie pour donner 7 composantes définies chacune par six équations :

$$\begin{aligned}
\{0 = a = b = c = d = f = g\} \bigcup \\
\{0 = a = b = c = d = f = j\} \bigcup \\
\{0 = a = b = c = d = i = j\} \bigcup \\
\{0 = a = b = c = h = i = j\} \bigcup \\
\{0 = a = b = f = h = i = j\} \bigcup \\
\{0 = a = e = f = h = i = j\} \bigcup \\
\{0 = c = e = f = h = i = j\}.
\end{aligned}$$

(Incidemment, nous avons établi que l'idéal des syzygies restreintes est une intersection complète.) Par conséquent, l'ensemble de tous les monômes qu'il nous reste dans l'espace quotient est constitué des sept listes suivantes :

- $\Big(\Lambda^9_{1,1,1}\big|_0\Big)^e \Big(K^{12}_{1,1}\big|_0\Big)^h \Big(H^{14}_1\big|_0\Big)^i \Big(F^{16}_{1,1}\big|_0\Big)^j$,
- $\Big(\Lambda^9_{1,1,1}\big|_0\Big)^e \Big(N^{12}\big|_0\Big)^g \Big(K^{12}_{1,1}\big|_0\Big)^h \Big(H^{14}_1\big|_0\Big)^i$,
- $\Big(\Lambda^9_{1,1,1}\big|_0\Big)^e \Big(M^{10}_1\big|_0\Big)^f \Big(N^{12}\big|_0\Big)^g \Big(K^{12}_{1,1}\big|_0\Big)^h$,
- $\Big(M^8\big|_0\Big)^d \Big(\Lambda^9_{1,1,1}\big|_0\Big)^e \Big(M^{10}_1\big|_0\Big)^f \Big(N^{12}\big|_0\Big)^g$,
- $\Big(\Lambda^7_{1,1}\big|_0\Big)^c \Big(M^8\big|_0\Big)^d \Big(\Lambda^9_{1,1,1}\big|_0\Big)^e \Big(N^{12}\big|_0\Big)^g$,
- $\Big(\Lambda^5_1\big|_0\Big)^b \Big(\Lambda^7_{1,1}\big|_0\Big)^c \Big(M^8\big|_0\Big)^d \Big(N^{12}\big|_0\Big)^g$,
- $\Big(\Lambda^3\big|_0\Big)^a \Big(\Lambda^5_1\big|_0\Big)^b \Big(M^8\big|_0\Big)^d \Big(N^{12}\big|_0\Big)^g$.

Cependant, puisque ces sept listes se recouvrent partiellement — par exemple : l'intersection de la première et de la deuxième ligne est constitutée des monômes de la forme $\big(\Lambda^9_{1,1,1}\big|_0\big)^e\big(K^{12}_{1,1}\big|_0\big)^h\big(H^{14}_1\big|_0\big)^i$ —, nous devons encore les réorganiser de telle sorte qu'il n'y ait plus aucune intersection entre elles, et si nous désignons ces listes par les septs lettres $\mathtt{A}$, $\mathtt{B}$, $\mathtt{C}$, $\mathtt{D}$, $\mathtt{F}$ et $\mathtt{G}$, il nous suffit en fait tout simplement d'écrire :

$$\begin{aligned}\mathtt{A}\cup\mathtt{B}\cup\mathtt{C}\cup\mathtt{D}\cup\mathtt{E}\cup\mathtt{F}\cup\mathtt{G} = \mathtt{A}\bigcup\mathtt{B}\backslash\mathtt{A}\bigcup\mathtt{C}\backslash(\mathtt{A}\cup\mathtt{B})\bigcup\mathtt{D}\backslash(\mathtt{A}\cup\mathtt{B}\cup\mathtt{C})\bigcup\mathtt{E}\backslash(\mathtt{A}\cup\mathtt{B}\cup\mathtt{C}\cup\mathtt{D})\\ \bigcup\mathtt{F}\backslash(\mathtt{A}\cup\mathtt{B}\cup\mathtt{C}\cup\mathtt{D}\cup\mathtt{E})\bigcup\mathtt{G}\backslash(\mathtt{A}\cup\mathtt{B}\cup\mathtt{C}\cup\mathtt{D}\cup\mathtt{E}\cup\mathtt{G}),\end{aligned}$$

ce qui nous donne immédiatement la représentation énoncée dans notre lemme, au moyen des sept polynômes arbitraires $\mathscr{P}_0$, $\mathscr{R}_0$, $\mathscr{S}_0$, $\mathscr{T}_0$, $\mathscr{U}_0$ et $\mathscr{V}_0$ dont les quatre arguments se différencient successivement d'une unité lorsqu'on saute une ligne (de la ligne 1 à la ligne 7), fait combinatoire aussi remarquable qu'imprévu et que nous aimerions voir se confirmer, se généraliser et se stabiliser lorsque nous étudierons les bi-invariants pour les jets d'ordre 6 — projet ambitieux s'il en est.

Afin d'établir la deuxième assertion du lemme, supposons maintenant qu'une relation du type :

$$0\equiv\mathscr{P}_0+\Lambda^7_{1,1}\big|_0\,\mathscr{Q}_0+\Lambda^9_{1,1,1}\big|_0\,\mathscr{R}_0+M^{10}_1\big|_0\,\mathscr{S}_0+K^{12}_{1,1}\big|_0\,\mathscr{T}_0+H^{14}_1\big|_0\,\mathscr{U}_0+F^{16}_{1,1}\big|_0\,\mathscr{V}_0$$

est identiquement satisfaite, et remplaçons-y alors les six bi-invariants restreints non fondamentaux par leur expression en fonction de ceux qui sont fondamentaux, ce qui nous donne :

$$\begin{aligned}0\equiv{}&\mathscr{P}_0\big(\Lambda^3\big|_0,\,\Lambda^5_1\big|_0,\,M^8\big|_0,\,N^{12}\big|_0\big)+\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0}\,\mathscr{Q}_0\Big(\Lambda^5_1\big|_0,\,\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0},\,M^8\big|_0,\,N^{12}\big|_0\Big)+\\ &+\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0\,\Lambda^3|_0}\,\mathscr{R}_0\Big(\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0},\,M^8\big|_0,\,\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0\,\Lambda^3|_0},\,N^{12}\big|_0\Big)+\\ &+\tfrac{\Lambda^5_1|_0\,M^8|_0}{\Lambda^3|_0}\,\mathscr{S}_0\Big(M^8\big|_0,\,\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0\,\Lambda^3|_0},\,\tfrac{\Lambda^5_1|_0\,M^8|_0}{\Lambda^3|_0},\,N^{12}\big|_0\Big)+\\ &+\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0}\,\mathscr{T}_0\Big(\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0\,\Lambda^3|_0},\,\tfrac{\Lambda^5_1|_0\,M^8|_0}{\Lambda^3|_0},\,N^{12}\big|_0,\,\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0}\Big)+\\ &+\tfrac{\Lambda^5_1|_0\,N^{12}|_0}{\Lambda^3|_0}\,\mathscr{U}_0\Big(\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0\,\Lambda^3|_0},\,N^{12}\big|_0,\,\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0},\,\tfrac{\Lambda^5_1|_0\,N^{12}|_0}{\Lambda^3|_0}\Big)+\\ &\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,N^{12}|_0}{\Lambda^3|_0\,\Lambda^3|_0}\,\mathscr{V}_0\Big(\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0\,\Lambda^3|_0},\,\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0},\,\tfrac{\Lambda^5_1|_0\,N^{12}|_0}{\Lambda^3|_0},\,\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,N^{12}|_0}{\Lambda^3|_0\,\Lambda^3|_0}\Big).\end{aligned}$$

Pour en déduire l'annulation de ces 7 polynômes $\mathscr{P}_0$, $\mathscr{Q}_0$, $\mathscr{R}_0$, $\mathscr{S}_0$, $\mathscr{T}_0$, $\mathscr{U}_0$ et $\mathscr{V}_0$, si nous commençons par multiplier cette identité par le dénominateur $\big(\Lambda^3|_0\big)^\mu$ de son expression réduite, nous obtenons une identité de la forme :

$$0\equiv\big(\Lambda^3\big|\big)^\mu\,\mathscr{P}_0\big(\Lambda^3\big|_0,\,\Lambda^5_1\big|_0,\,M^8\big|_0,\,N^{12}\big|_0\big)+\big(\Lambda^3\big|_0\big)^{\mu-1}\cdot\text{reste polynomial},$$

laquelle montre tout d'abord immédiatement que le premier polynôme $\mathscr{P}_0$ s'annule identiquement, puisque $\Lambda^3\big|_0$, $\Lambda^5_1\big|_0$, $M^8\big|_0$ et $N^{12}\big|_0$ sont algébriquement indépendants. Supprimons donc $\mathscr{P}_0$. Ensuite, en multipliant par une puissance suffisamment élevée $\big(\Lambda^3|_0\big)^\mu$ pour éliminer les puissances négatives de $\Lambda^3|_0$ qui apparaissent dans $\mathscr{Q}_0$, $\mathscr{R}_0$, $\mathscr{S}_0$, $\mathscr{T}_0$, $\mathscr{U}_0$ et $\mathscr{V}_0$, et en divisant

le tout par le facteur $\Lambda_1^5|_0$ présent devant chacun des six polynômes restants, nous obtenons une identité de la forme :

$$0 \equiv \big(\Lambda^3\big|_0\big)^{\mu}\Big[\Lambda_1^5\big|_0\,\mathscr{Q}_0 + M^8\big|_0\,\mathscr{S}_0 + N^{12}\big|_0\,\mathscr{U}_0\Big] + \big(\Lambda^3\big|_0\big)^{\mu-1}\cdot \text{reste polynomial},$$

d'où nous déduisons l'annulation identique du polynôme :

$$\begin{aligned}0 \equiv \big(\Lambda^3\big|_0\big)^{\mu}\Big[&\Lambda_1^5\big|_0\,\mathscr{Q}_0\Big(\Lambda_1^5\big|_0,\ \tfrac{\Lambda_1^5|_0\,\Lambda_1^5|_0}{\Lambda^3|_0},\ M^8\big|_0,\ N^{12}\big|_0\Big)+\\ &+M^8\big|_0\,\mathscr{S}_0\Big(M^8\big|_0,\ \tfrac{\Lambda_1^5|_0\,\Lambda_1^5|_0\,\Lambda_1^5|_0}{\Lambda^3|_0\,\Lambda^3|_0},\ \tfrac{\Lambda_1^5|_0\,M^8|_0}{\Lambda^3|_0},\ N^{12}\big|_0\Big)+\\ &+N^{12}\big|_0\,\mathscr{U}_0\Big(\tfrac{\Lambda_1^5|_0\,\Lambda_1^5|_0\,\Lambda_1^5|_0}{\Lambda^3|_0\,\Lambda^3|_0},\ N^{12}\big|_0,\ \tfrac{\Lambda_1^5|_0\,\Lambda_1^5|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0},\ \tfrac{\Lambda_1^5|_0\,N^{12}|_0}{\Lambda^3|_0}\Big)\Big].\end{aligned}$$

Ensuite, si nous introduisons les développement finis de ces trois polynômes :

$$\mathscr{Q}_0(t_1,t_2,t_3,t_4) = \sum \mathsf{coeff}\cdot t_1^{\alpha}\,t_2^{\beta}\,t_3^{\gamma}\,t_4^{\delta}, \qquad \mathscr{S}_0(t_1,t_2,t_3,t_4) = \sum \mathsf{coeff}\cdot t_1^{\alpha'}\,t_2^{\beta'}\,t_3^{\gamma'}\,t_4^{\delta'},$$
$$\mathscr{U}_0(t_1,t_2,t_3,t_4) = \sum \mathsf{coeff}\cdot t_1^{\alpha''}\,t_2^{\beta''}\,t_3^{\gamma''}\,t_4^{\delta''},$$

nous en déduirons l'annulation de $\mathscr{Q}_0$, de $\mathscr{S}_0$ et de $\mathscr{U}_0$ grâce à l'observation suivante.

Assertion. Les trois familles de monômes :

(i) $\big(\Lambda^3\big|_0\big)^{\mu-\beta}\big(\Lambda_1^5\big|_0\big)^{1+\alpha+2\beta}\big(M^8\big|_0\big)^{\gamma}\big(N^{12}\big|_0\big)^{\delta}$,

(ii) $\big(\Lambda^3\big|_0\big)^{\mu-2\beta'-\gamma'}\big(\Lambda_1^5\big|_0\big)^{3\beta'+\gamma'}\big(M^8\big|_0\big)^{1+\alpha'+\gamma'}\big(N^{12}\big|_0\big)^{\delta'}$,

(iii) $\big(\Lambda^3\big|_0\big)^{\mu-2\alpha''-2\gamma''-\delta''}\big(\Lambda_1^5\big|_0\big)^{3\alpha''+2\gamma''+\delta''}\big(M^8\big|_0\big)^{\gamma''}\big(N^{12}\big|_0\big)^{1+\beta''+\gamma''}$,

ne contiennent aucune redondance, i.e. chaque monôme correspondand à un choix de $(\alpha,\beta,\gamma,\delta)$, ou de $(\alpha',\beta',\gamma',\delta')$, ou encore de $(\alpha'',\beta'',\gamma'',\delta'')$ apparaît une et une seule fois.

En effet, nous vérifions tout d'abord pour la première famille, que l'auto-intersection :

$$\mu-\beta=\mu-\underline{\beta}, \qquad 1+\alpha+2\beta=1+\underline{\alpha}+2\underline{\beta} \qquad \gamma=\underline{\gamma} \qquad \delta=\underline{\delta},$$

est vide, c'est-à-dire implique $\alpha=\underline{\alpha}$, $\beta=\underline{\beta}$, $\gamma=\underline{\gamma}$, $\delta=\underline{\delta}$, puis de même pour la deuxième famille :

$$\mu-2\beta'-\gamma'=\mu-2\underline{\beta}'-\underline{\gamma}', \qquad 3\beta'+\gamma'=3\underline{\beta}'+\underline{\gamma}', \qquad \alpha'+\gamma'=\underline{\alpha}'+\underline{\gamma}', \qquad \delta'=\underline{\delta}',$$

et aussi pour la troisième famille :

$$\begin{aligned}&\mu-2\alpha''-2\gamma''-\delta''=\mu-2\underline{\alpha}''-2\underline{\gamma}''-\underline{\delta}'', \qquad 3\alpha''+2\gamma''+\delta''=3\underline{\alpha}''+2\underline{\gamma}''+\underline{\delta}'',\\ &\gamma''=\underline{\gamma}'', \qquad \beta''+\gamma''=\underline{\beta}''+\underline{\gamma}''.\end{aligned}$$

Ensuite, des formules pour l'intersection entre la première et la deuxième famille :

$$-\beta=-2\beta'-\gamma', \qquad 1+\alpha+2\beta=3\beta'+\gamma', \qquad \gamma=1+\alpha'+\gamma', \qquad \delta=\delta',$$

découle l'équation $\alpha = -\gamma' - \beta' - 1$, impossible parce que l'exposant entier α doit impérativement être $\geqslant 0$. De même, l'intersection entre la première et la troisième famille :

$$-\beta = -2\alpha''-2\gamma''-\delta'', \qquad 1+\alpha+2\beta = 3\alpha''+2\gamma''+\delta'', \qquad \gamma = \gamma'', \qquad \delta = 1+\beta''+\gamma'',$$

implique l'équation impossible $\alpha = -1 - \alpha'' - 2\gamma'' - \delta''$, et enfin aussi, l'intersection entre la deuxième et la troisième famille :

$$\begin{aligned} -2\beta' - \gamma' = -2\alpha'' - 2\gamma'' - \delta'', \qquad & 3\beta' + \gamma' = 3\alpha'' + 2\gamma'' + \delta'', \\ & 1 + \alpha' + \gamma' = \gamma'', \qquad \delta' = 1 + \beta'' + \gamma'', \end{aligned}$$

d'où découle $\gamma' = \delta'' + 2\gamma''$, implique l'équation impossible $\alpha' = -1 - \gamma'' - \delta''$, ce qui démontre l'assertion. $\square$

Ainsi, nous pouvons supprimer $\mathscr{Q}_0$, $\mathscr{S}_0$ et $\mathscr{U}_0$, et nous sommes ramenés à étudier l'identité restante, qui est du type :

$$\begin{aligned} 0 \equiv \big(\Lambda^3\big|_0\big)^{\mu} \Big[& \Lambda^5_1\big|_0 \mathscr{R}_0\Big(\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0}, M^8\big|_0, \tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0\,\Lambda^3|_0}, N^{12}\big|_0\Big) + \\ & + M^8\big|_0 \mathscr{T}_0\Big(\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0\,\Lambda^3|_0}, \tfrac{\Lambda^5_1|_0\,M^8|_0}{\Lambda^3|_0}, N^{12}\big|_0, \tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0}\Big) \\ & + N^{12}\big|_0 \mathscr{V}_0\Big(\tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,\Lambda^5_1|_0}{\Lambda^3|_0\,\Lambda^3|_0}, \tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,M^8|_0}{\Lambda^3|_0\,\Lambda^3|_0}, \tfrac{\Lambda^5_1|_0\,N^{12}|_0}{\Lambda^3|_0}, \tfrac{\Lambda^5_1|_0\,\Lambda^5_1|_0\,N^{12}|_0}{\Lambda^3|_0\,\Lambda^3|_0}\Big)\Big]. \end{aligned}$$

Assertion. Les trois familles de monômes :

(iv) $\big(\Lambda^3\big|_0\big)^{\mu-\alpha-2\gamma}\big(\Lambda^5_1\big|_0\big)^{1+2\alpha+3\gamma}\big(M^8\big|_0\big)^{\beta}\big(N^{12}\big|_0\big)^{\delta}$,

(v) $\big(\Lambda^3\big|_0\big)^{\mu-2\alpha'-\beta'-2\delta'}\big(\Lambda^5_1\big|_0\big)^{3\alpha'+\beta'+2\delta'}\big(M^8\big|_0\big)^{1+\beta'+\delta'}\big(N^{12}\big|_0\big)^{\gamma'}$,

(vi) $\big(\Lambda^3\big|_0\big)^{\mu-2\alpha''-2\beta''-\gamma''-2\delta''}\big(\Lambda^5_1\big|_0\big)^{3\alpha''+2\beta''+\gamma''+2\delta''}\big(M^8\big|_0\big)^{\beta''}\big(N^{12}\big|_0\big)^{1+\gamma''+\delta''}$,

ne contiennent aucune redondance, i.e. chaque monôme correspondand à un choix de $(\alpha, \beta, \gamma, \delta)$, ou de $(\alpha', \beta', \gamma', \delta')$, ou encore de $(\alpha'', \beta'', \gamma'', \delta'')$ apparaît une et une seule fois.

En effet, il est tout d'abord facile de vérifier que chacune des trois auto-intersections est triviale. Ensuite, des formules pour l'intersection entre la première et la deuxième famille :

$$\alpha+2\gamma = 2\alpha'+\beta'+2\delta', \qquad 1+2\alpha+3\gamma = 3\alpha'+\beta'+2\delta', \qquad \beta = 1+\beta'+\delta', \qquad \delta = \gamma',$$

découle l'équation $1+\alpha+\gamma = \alpha'$ dont nous nous servons pour remplacer α' dans la seconde équation, ce qui conduit à l'impossibilité $0 = 2+\alpha+\beta'+2\delta'$. De même, l'intersection entre la première et la troisième famille :

$$\begin{aligned} \alpha + 2\gamma = 2\alpha'' + 2\beta'' + \gamma'' + 2\delta'', \qquad & 1 + 2\alpha + 3\gamma = 3\alpha'' + 2\beta'' + \gamma'' + 2\delta'', \\ & \beta = \beta'', \qquad \delta = 1 + \gamma'' + \delta'', \end{aligned}$$

conduit à l'impossibilité $0 = 2 + \alpha + 2\beta'' + \gamma'' + 2\delta''$ en remplaçant $\alpha'' = 1 + \alpha + \gamma$ dans la seconde équation. Enfin, l'intersection entre la deuxième et la troisième famille :

$$2\alpha' + \beta' + 2\delta' = 2\alpha'' + 2\beta'' + \gamma'' + 2\delta'', \qquad 3\alpha' + \beta' + 2\delta' = 3\alpha'' + 2\beta'' + \gamma'' + 2\delta'',$$
$$1 + \beta' + \delta' = \beta'', \qquad \gamma' = 1 + \gamma'' + \delta'',$$

conduit à $\alpha' = \alpha''$, d'où $\beta' + 2\delta' = 2\beta'' + \gamma'' + 2\delta''$, puis en réécrivant la troisième équation et en y remplaçant $-\beta' - 2\delta'$ par $-2\beta'' - \gamma'' - 2\delta''$, puis $-\beta''$ par $-1 - \beta' - \delta'$:

$$\begin{aligned} 1 &= \beta'' - \beta' - 2\delta' + \delta' \\ &= \beta'' - 2\beta'' - \gamma'' - 2\delta'' + \delta' \\ &= -\beta'' - \gamma'' - 2\delta'' + \delta' \\ &= -1 - \beta' - \gamma'' - 2\delta'', \end{aligned}$$

équation tout aussi impossible que les deux précédentes. Ceci achève la démonstration de notre lemme principal. □

Syzygies complètes et substitutions algébriques. Nous parvenons enfin à la dernière étape de la démonstration de notre second théorème. Soit $\mathsf{P}^{2\times\mathrm{inv}}\big(j^5 f\big)$ un bi-invariant quelconque, et reprenons son développement en puissances positives et négatives de f_1' :

$$\mathsf{P}^{2\times\mathrm{inv}} = \sum_{-\frac{4}{5}m \leqslant a \leqslant m} (f_1')^a \, \mathscr{P}_a\big(\Lambda^3, \Lambda_1^5, \Lambda_{1,1}^7, M^8, \Lambda_{1,1,1}^9, M_1^{10}, N^{12}, K_{1,1}^{12}, H_1^{14}, F_{1,1}^{16}\big),$$

que nous avions obtenu pour le représenter en injectant (artificiellement) les six bi-invariants fantômes dans une expression initiale qui ne montrait que Λ^3, Λ_1^5, $\Lambda_{1,1}^7$ et $\Lambda_{1,1,1}^9$. Choisissons l'exposant a maximalement négatif et examinons le polynôme :

$$\mathscr{P}_a\big(\Lambda^3, \Lambda_1^5, \Lambda_{1,1}^7, M^8, \Lambda_{1,1,1}^9, M_1^{10}, N^{12}, K_{1,1}^{12}, H_1^{14}, F_{1,1}^{16}\big).$$

Notre base de Gröbner pour les bi-invariants restreints est obtenue à partir de nos 15 syzygies fondamentales restreintes, et ce au moyen d'un certain nombre d'opérations algébriques élémentaires : multiplication et addition d'équations, calcul de S-polynômes et divisions euclidiennes subséquentes, opérations autorisées dans toute structure d'idéal algébrique. Il en découle que les mêmes opérations qui produisent les 21 équations normalisées satisfaites sur $\{f_1' = 0\}$ peuvent aussi être conduites *sans poser* $f_1' = 0$, et alors elles produisent, à partir des 15 syzygies complètes, la même liste de 21 équations dans laquelle chaque identité "$0 \equiv$" doit être remplacé par "$\mathrm{O}(f_1') \equiv$", le terme $\mathrm{O}(f_1')$ désignant un reste, variable selon le contexte, qui dépend *a priori* de tous les onze bi-invariants[68] et qui s'annule

[68] Ici, le raisonnement fonctionne seulement si l'algèbre complète des bi-invariant doit coïncider *a priori* avec l'algèbre engendrée par crochets : sous cette condition, le reste derrière f_1' doit alors nécessairement être fonction des onze bi-invariants. Toutefois, nous allons voir dans un instant que

lorsqu'on fait $f_1' = 0$, c'est-à-dire qui est multiple de f_1'. Par conséquent, si nous appliquons à $\mathscr{P}_a$ la même normalisation modulo les syzygies que dans le lemme principal (ce qui revient à substituer toutes les occurences des monômes de tête), mais sans prendre la restriction à $\{f_1' = 0\}$, nous obtenons une expression du même type, et ce, avec un reste :

$$\begin{aligned}\mathscr{P}_a = {} & \mathscr{P}_a\big(\Lambda^3,\, \Lambda_1^5,\, M^8,\, N^{12}\big) + \Lambda_{1,1}^7\, \mathscr{Q}_a\big(\Lambda_1^5,\, \Lambda_{1,1}^7,\, M^8,\, N^{12}\big) + \\ & + \Lambda_{1,1,1}^9\, \mathscr{R}_a\big(\Lambda_{1,1}^7,\, M^8,\, \Lambda_{1,1,1}^9,\, N^{12}\big) + M_1^{10}\, \mathscr{S}_a\big(M^8,\, \Lambda_{1,1,1}^9,\, M_1^{10},\, N^{12}\big) + \\ & + K_{1,1}^{12}\, \mathscr{T}_a\big(\Lambda_{1,1,1}^9,\, M_1^{10},\, N^{12},\, K_{1,1}^{12}\big) + H_1^{14}\, \mathscr{U}_a\big(\Lambda_{1,1,1}^9,\, N^{12},\, K_{1,1}^{12},\, H_1^{14}\big) + \\ & + F_{1,1}^{16}\, \mathscr{V}\big(\Lambda_{1,1,1}^9,\, K_{1,1}^{12},\, H_1^{14},\, F_{1,1}^{16}\big) + \\ & + f_1'\, \mathsf{reste}\big(f_1', \Lambda^3, \Lambda_1^5, \Lambda_{1,1}^7, M^8, \Lambda_{1,1,1}^9, M_1^{10}, N^{12}, K_{1,1}^{12}, H_1^{14}, F_{1,1}^{16}\big),\end{aligned}$$

a priori non contrôlé, mais qui est heureusement repoussé dans les puissances supérieures $(f_1')^{a+1}$, $(f_1')^{a+2}, \ldots$. Ainsi, nous normalisons l'expression du premier polynôme $\mathscr{P}_a$, à savoir celui qui apparaît dans la puissance maximalement négative de f_1'. Et enuite, nous soumettons le nouveau coefficient de $(f_1')^{a+1}$, qui vient de subir l'interférence du reste, et que nous noterons encore $\mathscr{P}_a$, nous soumettons ce nouveau coefficient au même processus de normalisation modulo les 21 syzygies non restreintes, et ainsi de suite, jusqu'à l'exposant maximalement positif (toujours borné par m) de f_1', ce qui nous donne une expression finale de la forme :

$$\begin{aligned}\mathsf{P}^{2\times\mathrm{inv}}(j^5 f) = \sum_{-\frac{4}{5}\, m \leqslant a \leqslant m} & (f_1')^a \Big[\mathscr{P}_a\big(\Lambda^3,\, \Lambda_1^5,\, M^8,\, N^{12}\big) + \Lambda_{1,1}^7\, \mathscr{Q}_a\big(\Lambda_1^5,\, \Lambda_{1,1}^7,\, M^8,\, N^{12}\big) + \\ & + \Lambda_{1,1,1}^9\, \mathscr{R}_a\big(\Lambda_{1,1}^7,\, M^8,\, \Lambda_{1,1,1}^9,\, N^{12}\big) + M_1^{10}\, \mathscr{S}_a\big(M^8,\, \Lambda_{1,1,1}^9,\, M_1^{10},\, N^{12}\big) + \\ & + K_{1,1}^{12}\, \mathscr{T}_a\big(\Lambda_{1,1,1}^9,\, M_1^{10},\, N^{12},\, K_{1,1}^{12}\big) + H_1^{14}\, \mathscr{U}_a\big(\Lambda_{1,1,1}^9,\, N^{12},\, K_{1,1}^{12},\, H_1^{14}\big) + \\ & + F_{1,1}^{16}\, \mathscr{V}_a\big(\Lambda_{1,1,1}^9,\, K_{1,1}^{12},\, H_1^{14},\, F_{1,1}^{16}\big)\Big].\end{aligned}$$

Et maintenant enfin, nous pouvons achever la démonstration : s'il existait des puissances négatives de f_1' dans une telle somme, on multiplierait alors $\mathsf{P}^{2\times\mathrm{inv}}\big(j^5 f\big)$ par la puissance positive minimale $(f_1')^{-a}$ de f_1' qui élimine les dénominateurs, on poserait $f_1' = 0$ et le lemme principal — *which was specially designed on that purpose* — tuerait alors les sept polynômes $\mathscr{P}_a$, $\mathscr{Q}_a$, $\mathscr{R}_a$, $\mathscr{S}_a$, $\mathscr{T}_a$, $\mathscr{U}_a$ et $\mathscr{V}_a$, ce qui contredirait le choix de a. Il n'existe donc que des puissances positives de f_1', et comme tout polynôme de la forme

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de nouveaux bi-invariants fondamentaux "fantômes" se cachent derrière f_1', exactement comme nous avions interprété $\Lambda_{1,1,1}^9$, M_1^{10}, N^{12}, $K_{1,1}^{12}$, H_1^{14} et $F_{1,1}^{16}$, ces nouveaux bi-invariants n'étant pas construits par crochet.

générale

$$\begin{aligned}
&\mathscr{P}\big(f_1', \Lambda^3, \Lambda_1^5, M^8, N^{12}\big) + \Lambda_{1,1}^7\, \mathscr{Q}\big(f_1', \Lambda_1^5, \Lambda_{1,1}^7, M^8, N^{12}\big) + \\
&\quad + \Lambda_{1,1,1}^9\, \mathscr{R}\big(f_1', \Lambda_{1,1}^7, M^8, \Lambda_{1,1,1}^9, N^{12}\big) + M_1^{10}\, \mathscr{S}\big(f_1', M^8, \Lambda_{1,1,1}^9, M_1^{10}, N^{12}\big) + \\
&\quad + K_{1,1}^{12}\, \mathscr{T}\big(f_1', \Lambda_{1,1,1}^9, M_1^{10}, N^{12}, K_{1,1}^{12}\big) + H_1^{14}\, \mathscr{U}\big(f_1', \Lambda_{1,1,1}^9, N^{12}, K_{1,1}^{12}, H_1^{14}\big) + \\
&\quad + F_{1,1}^{16}\, \mathscr{V}\big(f_1', \Lambda_{1,1,1}^9, K_{1,1}^{12}, H_1^{14}, F_{1,1}^{16}\big),
\end{aligned}$$

constitue trivialement un bi-invariant, écrit qui plus est sous forme unique grâce au lemme fondamental, notre second théorème est à présent complètement démontré. □

Observation. Dans cette dernière étape du raisonnement, les termes de reste ci-dessus ont beau être divisibles par f_1', il ne sont pas nécessairement polynomiaux en nos onze bi-invariants fondamentaux ; si cela avait été le cas, la stratégie aurait fonctionné comme pour les jets d'ordre 4. Plus précisément, lorsqu'on compare la liste originale des 15 syzygies restreintes à la liste complète des 21 syzygies restreintes, les 6 syzygies ajoutées sont déduites des 15 initiales en autorisant à diviser une syzygie par tout bi-invariant non identiquement nul qui est en facteur, notamment par le wronskien Λ^3, et c'est pour cette raison que les termes de reste ci-dessus ne sont pas nécessairement polynomiaux en les onze bi-invariants fondamentaux. Après un examen détaillé, on découvre donc l'existence de exactement 7 bi-invariants "fantômes" supplémentaires qui ne sont pas obtenus par crochets et qui se cachent derrière f_1', à savoir :

$$\begin{aligned}
X^{18} &:= \frac{-5\,\Lambda_{1,1,1}^9\, M_1^{10} + 56\,\Lambda_{1,1}^7\, K_{1,1}^{12}}{f_1'} \\
&= f_1' f_1' f_1' \Big(-18816\,\Delta^{1,4}\,[\Delta^{2,3}]^2 - 25088\,[\Delta^{2,3}]^3 - 15\,[\Delta^{1,5}]^2\,\Delta^{1,2} - 150\,\Delta^{1,5}\,\Delta^{2,4}\,\Delta^{1,2} \\
&\quad + 315\,\Delta^{1,5}\,\Delta^{1,4}\,\Delta^{1,3} + 960\,\Delta^{1,5}\,\Delta^{2,3}\,\Delta^{1,3} - 375\,[\Delta^{2,4}]^2\,\Delta^{1,2} + 1575\,\Delta^{2,4}\,\Delta^{1,4}\,\Delta^{1,3} \\
&\quad + 4800\,\Delta^{2,4}\,\Delta^{2,3}\,\Delta^{1,3} - 392\,[\Delta^{1,4}]^3 - 4704\,[\Delta^{1,4}]^2\,\Delta^{2,3}\Big) - f_1' f_1' f_1'' \Big(-2475\,\Delta^{2,4}\,\Delta^{1,4}\,\Delta^{1,2} \\
&\quad - 9900\,\Delta^{2,4}\,\Delta^{2,3}\,\Delta^{1,2} - 2850\,\Delta^{1,5}\,[\Delta^{1,3}]^2 + 51330\,\Delta^{1,4}\,\Delta^{2,3}\,\Delta^{1,3} \\
&\quad + 92760\,[\Delta^{2,3}]^2\,\Delta^{1,3} - 14250\,\Delta^{2,4}\,[\Delta^{1,3}]^2 + 7035\,[\Delta^{1,4}]^2\,\Delta^{1,3} - 495\,\Delta^{1,5}\,\Delta^{1,4}\,\Delta^{1,2} \\
&\quad - 1980\,\Delta^{1,5}\,\Delta^{2,3}\,\Delta^{1,2}\Big) - f_1' f_1' f_1''' \Big(-11100\,\Delta^{2,3}\,[\Delta^{1,3}]^2 - 3150\,\Delta^{1,4}\,[\Delta^{1,3}]^2\Big) \\
&\quad + f_1' f_1'' f_1'' \Big(-109440\,[\Delta^{2,3}]^2\,\Delta^{1,2} - 19050\,\Delta^{2,3}\,[\Delta^{1,3}]^2 - 32325\,\Delta^{1,4}\,[\Delta^{1,3}]^2 \\
&\quad + 11025\,\Delta^{1,5}\,\Delta^{1,3}\,\Delta^{1,2} + 55125\,\Delta^{2,4}\,\Delta^{1,3}\,\Delta^{1,2} - 6840\,[\Delta^{1,4}]^2\,\Delta^{1,2} \\
&\quad - 54720\,\Delta^{1,4}\,\Delta^{2,3}\,\Delta^{1,2}\Big) - f_1' f_1'' f_1''' \Big(+30000\,[\Delta^{1,3}]^3\Big) - f_1'' f_1'' f_1'' \Big(11025\,\Delta^{1,5}\,[\Delta^{1,2}]^2 \\
&\quad - 55125\,\Delta^{2,4}\,[\Delta^{1,2}]^2 + 55125\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,2} + 110250\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2} \\
&\quad - 49000\,[\Delta^{1,3}]^3\Big).
\end{aligned}$$

$$
\begin{aligned}
X^{19} &:= \frac{-5\,M_1^{10}\,M_1^{10} + 64\,M^8\,K_{1,1}^{12}}{f_1'} \\
&= f_1' \Big(1170\,\Delta^{1,5}\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,2} - 45\,\big[\Delta^{1,5}\big]^2\,\big[\Delta^{1,2}\big]^2 - 450\,\Delta^{1,5}\,\Delta^{2,4}\,\big[\Delta^{1,2}\big]^2 \\
&\quad + 74220\,\big[\Delta^{2,3}\big]^2\,\big[\Delta^{1,3}\big]^2 + 3780\,\Delta^{1,5}\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2} - 1600\,\Delta^{1,5}\,\big[\Delta^{1,3}\big]^3 \\
&\quad - 1125\,\big[\Delta^{2,4}\big]^2\,\big[\Delta^{1,2}\big]^2 + 5850\,\Delta^{2,4}\,\Delta^{1,4}\,\Delta^{1,3}\,\Delta^{1,2} + 18900\,\Delta^{2,4}\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2} \\
&\quad - 8000\,\Delta^{2,4}\,\big[\Delta^{1,3}\big]^3 - 1344\,\big[\Delta^{1,4}\big]^3\,\Delta^{1,2} - 16128\,\big[\Delta^{1,4}\big]^2\,\Delta^{2,3}\,\Delta^{1,2} + 1995\,\big[\Delta^{1,4}\big]^2\,\big[\Delta^{1,3}\big]^2 \\
&\quad - 64512\,\Delta^{1,4}\,\big[\Delta^{2,3}\big]^2\,\Delta^{1,2} + 27660\,\Delta^{1,4}\,\Delta^{2,3}\,\big[\Delta^{1,3}\big]^2 - 86016\,\big[\Delta^{2,3}\big]^3\,\Delta^{1,2}\Big) \\
&\quad + f_1''\Big(-74400\,\Delta^{2,3}\,\big[\Delta^{1,3}\big]^3 - 10800\,\Delta^{2,4}\,\Delta^{1,4}\,\big[\Delta^{1,2}\big]^2 - 2160\,\Delta^{1,5}\,\Delta^{1,4}\,\big[\Delta^{1,2}\big]^2 \\
&\quad - 8640\,\Delta^{1,5}\,\Delta^{2,3}\,\big[\Delta^{1,2}\big]^2 + 3600\,\Delta^{1,5}\,\big[\Delta^{1,3}\big]^2\,\Delta^{1,2} + 64800\,\Delta^{1,4}\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2} \\
&\quad - 43200\,\Delta^{2,4}\,\Delta^{2,3}\,\big[\Delta^{1,2}\big]^2 + 18000\,\Delta^{2,4}\,\big[\Delta^{1,3}\big]^2\,\Delta^{1,2} + 10800\,\big[\Delta^{1,4}\big]^2\,\Delta^{1,3}\,\Delta^{1,2} \\
&\quad - 27600\,\Delta^{1,4}\,\big[\Delta^{1,3}\big]^3 + 86400\,\big[\Delta^{2,3}\big]^2\,\Delta^{1,3}\,\Delta^{1,2}\Big) + f_1'''\Big(16000\,\big[\Delta^{1,3}\big]^4\Big).
\end{aligned}
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\begin{aligned}
X^{21} &:= \frac{-5\,M_1^{10}\,N^{12} + 8\,M^8\,H_1^{14}}{f_1'} \\
&= -135\,\big[\Delta^{1,5}\big]^2\,\big[\Delta^{1,2}\big]^3 - 1350\,\Delta^{1,5}\,\Delta^{2,4}\,\big[\Delta^{1,2}\big]^3 + 1350\,\Delta^{1,5}\,\Delta^{1,4}\,\Delta^{1,3}\,\big[\Delta^{1,2}\big]^2 \\
&\quad + 2700\,\Delta^{1,5}\,\Delta^{2,3}\,\Delta^{1,3}\,\big[\Delta^{1,2}\big]^2 - 1200\,\Delta^{1,5}\,\big[\Delta^{1,3}\big]^3\,\Delta^{1,2} - 3375\,\big[\Delta^{2,4}\big]^2\,\big[\Delta^{1,2}\big]^3 \\
&\quad + 6750\,\Delta^{2,4}\,\Delta^{1,4}\,\Delta^{1,3}\,\big[\Delta^{1,2}\big]^2 + 13500\,\Delta^{2,4}\,\Delta^{2,3}\,\Delta^{1,3}\,\big[\Delta^{1,2}\big]^2 - 6000\,\Delta^{2,4}\,\big[\Delta^{1,3}\big]^3\,\Delta^{1,2} \\
&\quad - 576\,\big[\Delta^{1,4}\big]^3\,\big[\Delta^{1,2}\big]^2 - 6912\,\big[\Delta^{1,4}\big]^2\,\Delta^{2,3}\,\big[\Delta^{1,2}\big]^2 - 495\,\big[\Delta^{1,4}\big]^2\,\big[\Delta^{1,3}\big]^2\,\Delta^{1,2} \\
&\quad - 27648\,\Delta^{1,4}\,\big[\Delta^{2,3}\big]^2\,\big[\Delta^{1,2}\big]^2 + 9540\,\Delta^{1,4}\,\Delta^{2,3}\,\big[\Delta^{1,3}\big]^2\,\Delta^{1,2} + 1200\,\Delta^{1,4}\,\big[\Delta^{1,3}\big]^4 \\
&\quad - 36864\,\big[\Delta^{2,3}\big]^3\,\big[\Delta^{1,2}\big]^2 + 32580\,\big[\Delta^{2,3}\big]^2\,\big[\Delta^{1,3}\big]^2\,\Delta^{1,2} - 7200\,\Delta^{2,3}\,\big[\Delta^{1,3}\big]^4.
\end{aligned}
$$

$$
\begin{aligned}
X^{23} &:= \frac{-7\,N^{12}\,K_{1,1}^{12} + M^8\,F_{1,1}^{16}}{f_1'} \\
&= f_1' \Big(432\,\Delta^{1,5}\,\big[\Delta^{1,4}\big]^2\,\big[\Delta^{1,2}\big]^2 + 3456\,\Delta^{1,5}\,\Delta^{1,4}\,\Delta^{2,3}\,\big[\Delta^{1,2}\big]^2 + 1710\,\Delta^{1,5}\,\Delta^{1,4}\,\big[\Delta^{1,3}\big]^2\,\Delta^{1,2} \\
&\quad - 3150\,\Delta^{1,5}\,\Delta^{2,4}\,\Delta^{1,3}\,\big[\Delta^{1,2}\big]^2 + 540\,\Delta^{1,5}\,\Delta^{2,3}\,\big[\Delta^{1,3}\big]^2\,\Delta^{1,2} - 1600\,\Delta^{1,5}\,\big[\Delta^{1,3}\big]^4 \\
&\quad - 7875\,\big[\Delta^{2,4}\big]^2\,\Delta^{1,3}\,\big[\Delta^{1,2}\big]^2 + 6912\,\Delta^{1,5}\,\big[\Delta^{2,3}\big]^2\,\big[\Delta^{1,2}\big]^2 - 8000\,\Delta^{2,4}\,\big[\Delta^{1,3}\big]^4 \\
&\quad - 2352\,\big[\Delta^{1,4}\big]^3\,\Delta^{1,3}\,\Delta^{1,2} - 23904\,\big[\Delta^{1,4}\big]^2\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2} + 2205\,\big[\Delta^{1,4}\big]^2\,\big[\Delta^{1,3}\big]^3 \\
&\quad - 78336\,\Delta^{1,4}\,\big[\Delta^{2,3}\big]^2\,\Delta^{1,3}\,\Delta^{1,2} + 34740\,\Delta^{1,4}\,\Delta^{2,3}\,\big[\Delta^{1,3}\big]^3 - 81408\,\big[\Delta^{2,3}\big]^3\,\Delta^{1,3}\,\Delta^{1,2} \\
&\quad + 72180\,\big[\Delta^{2,3}\big]^2\,\big[\Delta^{1,3}\big]^3 + 2160\,\Delta^{2,4}\,\big[\Delta^{1,4}\big]^2\,\big[\Delta^{1,2}\big]^2 + 17280\,\Delta^{2,4}\,\Delta^{1,4}\,\Delta^{2,3}\,\big[\Delta^{1,2}\big]^2 \\
&\quad + 8550\,\Delta^{2,4}\,\Delta^{1,4}\,\big[\Delta^{1,3}\big]^2\,\Delta^{1,2} + 34560\,\Delta^{2,4}\,\big[\Delta^{2,3}\big]^2\,\big[\Delta^{1,2}\big]^2 + 2700\,\Delta^{2,4}\,\Delta^{2,3}\,\big[\Delta^{1,3}\big]^2\,\Delta^{1,2} \\
&\quad - 315\,\big[\Delta^{1,5}\big]^2\,\Delta^{1,3}\,\big[\Delta^{1,2}\big]^2\Big) + f_1''\Big(23625\,\big[\Delta^{2,4}\big]^2\,\big[\Delta^{1,2}\big]^3 - 47250\,\Delta^{2,4}\,\Delta^{1,4}\,\Delta^{1,3}\,\big[\Delta^{1,2}\big]^2 \\
&\quad - 94500\,\Delta^{2,4}\,\Delta^{2,3}\,\Delta^{1,3}\,\big[\Delta^{1,2}\big]^2 + 42000\,\Delta^{2,4}\,\big[\Delta^{1,3}\big]^3\,\Delta^{1,2} + 576\,\big[\Delta^{1,4}\big]^3\,\big[\Delta^{1,2}\big]^2 \\
&\quad + 6912\,\big[\Delta^{1,4}\big]^2\,\Delta^{2,3}\,\big[\Delta^{1,2}\big]^2 + 20745\,\big[\Delta^{1,4}\big]^2\,\big[\Delta^{1,3}\big]^2\,\Delta^{1,2} + 27648\,\Delta^{1,4}\,\big[\Delta^{2,3}\big]^2\,\big[\Delta^{1,2}\big]^2 \\
&\quad + 945\,\big[\Delta^{1,5}\big]^2\,\big[\Delta^{1,2}\big]^3 + 9450\,\Delta^{1,5}\,\Delta^{2,4}\,\big[\Delta^{1,2}\big]^3 - 9450\,\Delta^{1,5}\,\Delta^{1,4}\,\Delta^{1,3}\,\big[\Delta^{1,2}\big]^2 \\
&\quad - 18900\,\Delta^{1,5}\,\Delta^{2,3}\,\Delta^{1,3}\,\big[\Delta^{1,2}\big]^2 + 8400\,\Delta^{1,5}\,\big[\Delta^{1,3}\big]^3\,\Delta^{1,2} + 71460\,\Delta^{1,4}\,\Delta^{2,3}\,\big[\Delta^{1,3}\big]^2\,\Delta^{1,2} \\
&\quad - 37200\,\Delta^{1,4}\,\big[\Delta^{1,3}\big]^4 + 36864\,\big[\Delta^{2,3}\big]^3\,\big[\Delta^{1,2}\big]^2 + 48420\,\big[\Delta^{2,3}\big]^2\,\big[\Delta^{1,3}\big]^2\,\Delta^{1,2} \\
&\quad - 64800\,\Delta^{2,3}\,\big[\Delta^{1,3}\big]^4\Big) + f_1'''\Big(16000\,\big[\Delta^{1,3}\big]^5\Big).
\end{aligned}
$$

$$
\begin{aligned}
Y^{23} &:= \frac{-8\,N^{12}\,K_{1,1}^{12} + M_1^{10}\,H_1^{14}}{f_1'} \\
&= X^{23}.
\end{aligned}
$$

$$
\begin{aligned}
X^{25} &:= \frac{-56\,K_{1,1}^{12}\,H_1^{14} + 5\,M_1^{10}\,F_{1,1}^{16}}{f_1'} \\
&= f_1'\,f_1'\,\Big(-45\,[\Delta^{1,5}]^2\,\Delta^{1,4}\,[\Delta^{1,2}]^2 - 180\,[\Delta^{1,5}]^2\,\Delta^{2,3}\,[\Delta^{1,2}]^2 - 3600\,[\Delta^{1,5}]^2\,[\Delta^{1,3}]^2\,\Delta^{1,2} \\
&\quad - 2800\,\Delta^{1,5}\,\Delta^{1,4}\,[\Delta^{1,3}]^3 - 83200\,\Delta^{1,5}\,\Delta^{2,3}\,[\Delta^{1,3}]^3 - 1125\,[\Delta^{2,4}]^2\,\Delta^{1,4}\,[\Delta^{1,2}]^2 \\
&\quad - 4500\,[\Delta^{2,4}]^2\,\Delta^{2,3}\,[\Delta^{1,2}]^2 - 90000\,[\Delta^{2,4}]^2\,[\Delta^{1,3}]^2\,\Delta^{1,2} - 14000\,\Delta^{2,4}\,\Delta^{1,4}\,[\Delta^{1,3}]^3 \\
&\quad - 416000\,\Delta^{2,4}\,\Delta^{2,3}\,[\Delta^{1,3}]^3 - 150528\,[\Delta^{1,4}]^3\,\Delta^{2,3}\,\Delta^{1,2} - 903168\,[\Delta^{1,4}]^2\,[\Delta^{2,3}]^2\,\Delta^{1,2} \\
&\quad + 163800\,[\Delta^{1,4}]^2\,\Delta^{2,3}\,[\Delta^{1,3}]^2 - 2408448\,\Delta^{1,4}\,[\Delta^{2,3}]^3\,\Delta^{1,2} + 1129500\,\Delta^{1,4}\,[\Delta^{2,3}]^2\,[\Delta^{1,3}]^2 \\
&\quad - 9408\,[\Delta^{1,4}]^4\,\Delta^{1,2} + 3675\,[\Delta^{1,4}]^3\,[\Delta^{1,3}]^2 - 2408448\,[\Delta^{2,3}]^4\,\Delta^{1,2} + 2132400\,[\Delta^{2,3}]^3\,[\Delta^{1,3}]^2 \\
&\quad - 450\,\Delta^{1,5}\,\Delta^{2,4}\,\Delta^{1,4}\,[\Delta^{1,2}]^2 - 1800\,\Delta^{1,5}\,\Delta^{2,4}\,\Delta^{2,3}\,[\Delta^{1,2}]^2 \\
&\quad - 36000\,\Delta^{1,5}\,\Delta^{2,4}\,[\Delta^{1,3}]^2\,\Delta^{1,2} + 11970\,\Delta^{1,5}\,[\Delta^{1,4}]^2\,\Delta^{1,3}\,\Delta^{1,2} \\
&\quad + 187920\,\Delta^{1,5}\,[\Delta^{2,3}]^2\,\Delta^{1,3}\,\Delta^{1,2} + 59850\,\Delta^{2,4}\,[\Delta^{1,4}]^2\,\Delta^{1,3}\,\Delta^{1,2} \\
&\quad + 939600\,\Delta^{2,4}\,[\Delta^{2,3}]^2\,\Delta^{1,3}\,\Delta^{1,2} + 474300\,\Delta^{2,4}\,\Delta^{1,4}\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2} \\
&\quad + 94860\,\Delta^{1,5}\,\Delta^{1,4}\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2}\Big) + f_1'\,f_1''\,\Big(-2556600\,\Delta^{1,4}\,\Delta^{2,3}\,[\Delta^{1,3}]^3 \\
&\quad - 5014200\,[\Delta^{2,3}]^2\,[\Delta^{1,3}]^3 - 187950\,[\Delta^{1,4}]^2\,[\Delta^{1,3}]^3 + 5621760\,\Delta^{1,4}\,[\Delta^{2,3}]^2\,\Delta^{1,3}\,\Delta^{1,2} \\
&\quad + 5652480\,[\Delta^{2,3}]^3\,\Delta^{1,3}\,\Delta^{1,2} - 2764800\,\Delta^{2,4}\,[\Delta^{2,3}]^2\,[\Delta^{1,2}]^2 \\
&\quad + 99000\,\Delta^{2,4}\,\Delta^{2,3}\,[\Delta^{1,3}]^2\,\Delta^{1,2} + 500000\,\Delta^{2,4}\,[\Delta^{1,3}]^4 + 174720\,[\Delta^{1,4}]^3\,\Delta^{1,3}\,\Delta^{1,2} \\
&\quad + 1751040\,[\Delta^{1,4}]^2\,\Delta^{2,3}\,\Delta^{1,3}\,\Delta^{1,2} - 276480\,\Delta^{1,5}\,\Delta^{1,4}\,\Delta^{2,3}\,[\Delta^{1,2}]^2 \\
&\quad - 105300\,\Delta^{1,5}\,\Delta^{1,4}\,[\Delta^{1,3}]^2\,\Delta^{1,2} - 552960\,\Delta^{1,5}\,[\Delta^{2,3}]^2\,[\Delta^{1,2}]^2 \\
&\quad + 19800\,\Delta^{1,5}\,\Delta^{2,3}\,[\Delta^{1,3}]^2\,\Delta^{1,2} + 100000\,\Delta^{1,5}\,[\Delta^{1,3}]^4 + 551250\,[\Delta^{2,4}]^2\,\Delta^{1,3}\,[\Delta^{1,2}]^2 \\
&\quad - 172800\,\Delta^{2,4}\,[\Delta^{1,4}]^2\,[\Delta^{1,2}]^2 - 1382400\,\Delta^{2,4}\,\Delta^{1,4}\,\Delta^{2,3}\,[\Delta^{1,2}]^2 \\
&\quad - 526500\,\Delta^{2,4}\,\Delta^{1,4}\,[\Delta^{1,3}]^2\,\Delta^{1,2} - 34560\,\Delta^{1,5}\,[\Delta^{1,4}]^2\,[\Delta^{1,2}]^2 + 22050\,[\Delta^{1,5}]^2\,\Delta^{1,3}\,[\Delta^{1,2}]^2 \\
&\quad + 220500\,\Delta^{1,5}\,\Delta^{2,4}\,\Delta^{1,3}\,[\Delta^{1,2}]^2\Big) \\
&\quad + f_1'\,f_1'''\,\Big(28000\,\Delta^{1,4}\,[\Delta^{1,3}]^4 + 472000\,\Delta^{2,3}\,[\Delta^{1,3}]^4\Big) \\
&\quad + f_1''\,f_1''\,\Big(330750\,\Delta^{1,5}\,\Delta^{1,4}\,\Delta^{1,3}\,[\Delta^{1,2}]^2 + 661500\,\Delta^{1,5}\,\Delta^{2,3}\,\Delta^{1,3}\,[\Delta^{1,2}]^2 \\
&\quad - 294000\,\Delta^{1,5}\,[\Delta^{1,3}]^3\,\Delta^{1,2} - 330750\,\Delta^{1,5}\,\Delta^{2,4}\,[\Delta^{1,2}]^3 \\
&\quad + 1653750\,\Delta^{2,4}\,\Delta^{1,4}\,\Delta^{1,3}\,[\Delta^{1,2}]^2 + 3307500\,\Delta^{2,4}\,\Delta^{2,3}\,\Delta^{1,3}\,[\Delta^{1,2}]^2 \\
&\quad - 1470000\,\Delta^{2,4}\,[\Delta^{1,3}]^3\,\Delta^{1,2} - 2880\,[\Delta^{1,4}]^3\,[\Delta^{1,2}]^2 - 34560\,[\Delta^{1,4}]^2\,\Delta^{2,3}\,[\Delta^{1,2}]^2 \\
&\quad - 812475\,[\Delta^{1,4}]^2\,[\Delta^{1,3}]^2\,\Delta^{1,2} - 138240\,\Delta^{1,4}\,[\Delta^{2,3}]^2\,[\Delta^{1,2}]^2 - 33075\,[\Delta^{1,5}]^2\,[\Delta^{1,2}]^3 \\
&\quad + 1446000\,\Delta^{1,4}\,[\Delta^{1,3}]^4 - 184320\,[\Delta^{2,3}]^3\,\Delta^{1,2}]^2 - 3077100\,[\Delta^{2,3}]^2\,[\Delta^{1,3}]^2\,\Delta^{1,2} \\
&\quad + 2844000\,\Delta^{2,3}\,[\Delta^{1,3}]^4 - 826875\,[\Delta^{2,4}]^2\,[\Delta^{1,2}]^3 - 3192300\,\Delta^{1,4}\,\Delta^{2,3}\,[\Delta^{1,3}]^2\,\Delta^{1,2}\Big) \\
&\quad + f_1''\,f_1'''\,\Big(-640000\,[\Delta^{1,3}]^5\Big).
\end{aligned}
$$

$$
\begin{aligned}
X^{27} &:= \frac{-7\,H_1^{14}\,H_1^{14} + 5\,N^{12}\,F_{1,1}^{16}}{f_1'} \\
&= f_1' \Big(-1032192\,\Delta^{1,4}\,[\Delta^{2,3}]^3\,[\Delta^{1,2}]^2 - 186300\,\Delta^{1,4}\,\Delta^{2,3}\,[\Delta^{1,3}]^4 - 3375\,[\Delta^{2,4}]^2\,\Delta^{1,4}\,[\Delta^{1,2}]^3 \\
&\quad + 5625\,[\Delta^{2,4}]^2\,[\Delta^{1,3}]^2\,[\Delta^{1,2}]^2 - 540\,[\Delta^{1,5}]^2\,\Delta^{2,3}\,[\Delta^{1,2}]^3 + 12705\,[\Delta^{1,4}]^3\,[\Delta^{1,3}]^2\,\Delta^{1,2} \\
&\quad + 1320720\,[\Delta^{2,3}]^3\,[\Delta^{1,3}]^2\,\Delta^{1,2} - 64512\,[\Delta^{1,4}]^3\,\Delta^{2,3}\,[\Delta^{1,2}]^2 + 225\,[\Delta^{1,5}]^2\,[\Delta^{1,3}]^2\,[\Delta^{1,2}]^2 \\
&\quad - 135\,[\Delta^{1,5}]^2\,\Delta^{1,4}\,[\Delta^{1,2}]^3 - 13500\,[\Delta^{2,4}]^2\,\Delta^{2,3}\,[\Delta^{1,2}]^3 - 387072\,[\Delta^{1,4}]^2\,[\Delta^{2,3}]^2\,[\Delta^{1,2}]^2 \\
&\quad - 1350\,\Delta^{1,5}\,\Delta^{2,4}\,\Delta^{1,4}\,[\Delta^{1,2}]^3 - 5400\,\Delta^{1,5}\,\Delta^{2,4}\,\Delta^{2,3}\,[\Delta^{1,2}]^3 \\
&\quad + 2250\,\Delta^{1,5}\,\Delta^{2,4}\,[\Delta^{1,3}]^2\,[\Delta^{1,2}]^2 + 3510\,\Delta^{1,5}\,[\Delta^{1,4}]^2\,\Delta^{1,3}\,[\Delta^{1,2}]^2 \\
&\quad - 10650\,\Delta^{1,5}\,\Delta^{1,4}\,[\Delta^{1,3}]^3\,\Delta^{1,2} + 45360\,\Delta^{1,5}\,[\Delta^{2,3}]^2\,\Delta^{1,3}\,[\Delta^{1,2}]^2 \\
&\quad - 38100\,\Delta^{1,5}\,\Delta^{2,3}\,[\Delta^{1,3}]^3\,\Delta^{1,2} + 17550\,\Delta^{2,4}\,[\Delta^{1,4}]^2\,\Delta^{1,3}\,[\Delta^{1,2}]^2 \\
&\quad - 53250\,\Delta^{2,4}\,\Delta^{1,4}\,[\Delta^{1,3}]^3\,\Delta^{1,2} + 226800\,\Delta^{2,4}\,[\Delta^{2,3}]^2\,\Delta^{1,3}\,[\Delta^{1,2}]^2 \\
&\quad - 190500\,\Delta^{2,4}\,\Delta^{2,3}\,[\Delta^{1,3}]^3\,\Delta^{1,2} + 187560\,[\Delta^{1,4}]^2\,\Delta^{2,3}\,[\Delta^{1,3}]^2\,\Delta^{1,2} \\
&\quad + 877140\,\Delta^{1,4}\,[\Delta^{2,3}]^2\,[\Delta^{1,3}]^2\,\Delta^{1,2} + 8000\,\Delta^{1,5}\,[\Delta^{1,3}]^5 + 40000\,\Delta^{2,4}\,[\Delta^{1,3}]^5 \\
&\quad - 4032\,[\Delta^{1,4}]^4\,[\Delta^{1,2}]^2 - 9975\,[\Delta^{1,4}]^2\,[\Delta^{1,3}]^4 - 1032192\,[\Delta^{2,3}]^4\,[\Delta^{1,2}]^2 \\
&\quad - 563100\,[\Delta^{2,3}]^2\,[\Delta^{1,3}]^4 + 126900\,\Delta^{2,4}\,\Delta^{1,4}\,\Delta^{2,3}\,\Delta^{1,3}\,[\Delta^{1,2}]^2 \\
&\quad + 25380\,\Delta^{1,5}\,\Delta^{1,4}\,\Delta^{2,3}\,\Delta^{1,3}\,[\Delta^{1,2}]^2\Big) + f_1''\Big(-259200\,\Delta^{2,4}\,\Delta^{1,4}\,\Delta^{2,3}\,[\Delta^{1,2}]^3 \\
&\quad - 518400\,\Delta^{2,4}\,[\Delta^{2,3}]^2\,[\Delta^{1,2}]^3 + 432000\,\Delta^{2,4}\,\Delta^{2,3}\,[\Delta^{1,3}]^2\,\Delta^{1,2}]^2 \\
&\quad - 90000\,\Delta^{2,4}\,[\Delta^{1,3}]^4\,\Delta^{1,2} + 32400\,[\Delta^{1,4}]^3\,\Delta^{1,3}\,[\Delta^{1,2}]^2 + 324000\,[\Delta^{1,4}]^2\,\Delta^{2,3}\,\Delta^{1,3}\,[\Delta^{1,2}]^2 \\
&\quad - 136800\,[\Delta^{1,4}]^2\,[\Delta^{1,3}]^3\,\Delta^{1,2} + 1036800\,\Delta^{1,4}\,[\Delta^{2,3}]^2\,\Delta^{1,3}\,[\Delta^{1,2}]^2 \\
&\quad - 878400\,\Delta^{1,4}\,\Delta^{2,3}\,[\Delta^{1,3}]^3\,\Delta^{1,2} + 186000\,\Delta^{1,4}\,[\Delta^{1,3}]^5 + 1036800\,[\Delta^{2,3}]^3\,\Delta^{1,3}\,[\Delta^{1,2}]^2 \\
&\quad - 1324800\,[\Delta^{2,3}]^2\,[\Delta^{1,3}]^3\,\Delta^{1,2} + 564000\,\Delta^{2,3}\,[\Delta^{1,3}]^5 + 108000\,\Delta^{2,4}\,\Delta^{1,4}\,[\Delta^{1,3}]^2\,[\Delta^{1,2}]^2 \\
&\quad - 103680\,\Delta^{1,5}\,[\Delta^{2,3}]^2\,[\Delta^{1,2}]^3 - 6480\,\Delta^{1,5}\,[\Delta^{1,4}]^2\,[\Delta^{1,2}]^3 - 51840\,\Delta^{1,5}\,\Delta^{1,4}\,\Delta^{2,3}\,[\Delta^{1,2}]^3 \\
&\quad + 21600\,\Delta^{1,5}\,\Delta^{1,4}\,[\Delta^{1,3}]^2\,[\Delta^{1,2}]^2 + 86400\,\Delta^{1,5}\,\Delta^{2,3}\,[\Delta^{1,3}]^2\,[\Delta^{1,2}]^2 \\
&\quad - 18000\,\Delta^{1,5}\,[\Delta^{1,3}]^4\,\Delta^{1,2} - 32400\,\Delta^{2,4}\,[\Delta^{1,4}]^2\,[\Delta^{1,2}]^3\Big) + f_1'''\Big(-80000\,\big[\Delta^{1,3}\big]^6\Big).
\end{aligned}
$$

Toutes les tentatives de calcul algébrique pour égaler l'un de ces 6 bi-invariants supplémentaires à un certain polynôme entre les 11 bi-invariants que nous connaissons déjà conduisent à un échec. Toutefois, nous vérifierons dans un instant que

$$X^{27} = M^8\,X^{19},$$

et nous établirons que les $11 + 5 = 16$ bi-invariants :

$$
\begin{array}{ccccccccccc}
f_1', & \Lambda^3, & \Lambda_1^5, & \Lambda_{1,1}^7, & M^8, & \Lambda_{1,1,1}^9, & M_1^{10}, & N^{12}, & K_{1,1}^{12}, & H_1^{14}, & F_{1,1}^{16} \\
 & & & & & & X^{18}, & X^{19}, & X^{21}, & X^{23}, & X^{25}
\end{array}
$$

sont mutuellement indépendants. De surcroît, nous allons constater que de nouveaux bi-invariants "fantômes" doivent encore nécessairement apparaître.

Déduction de relations impliquant le wronskien et $X^{18}, \ldots, X^{27}$. Sans poser $f_1' = 0$, multiplions l'équation "$\overset{10}{\equiv}$" par M^{10} :

$$0 \equiv -7\,\underline{\Lambda_1^5}\,\Lambda_{1,1}^7\,\underline{M_1^{10}} + 3\,\Lambda^3\,\Lambda_{1,1,1}^9\,M_1^{10} - f_1' f_1'\,M_1^{10}\,M_1^{10}.$$

Éliminons le binôme souligné $\underline{\Lambda_1^5\,M_1^{10}}$ grâce à "$\overset{18}{\equiv}$" multipliée par $\Lambda_{1,1}^7$:

$$0 \equiv -\frac{56}{5}\,\Lambda_{1,1}^7\,\Lambda_{1,1}^7\,M^8 - \frac{7}{5}\,f_1'\,\Lambda_{1,1}^7\,H_1^{14} + 3\,\Lambda^3\,\Lambda_{1,1,1}^9\,M_1^{10} - f_1' f_1'\,M_1^{10}\,M_1^{10}.$$

Enfin, si nous éliminons le binôme souligné $\underline{\Lambda_{1,1}^7\,M^8}$ grâce à l'équation (immédiatement déduite de "$\overset{51}{\equiv}$" et de "$\overset{18}{\equiv}$") suivante :

$$0 \overset{\widetilde{51}}{\equiv} -\Lambda_{1,1}^7\,M^8 + 3\,\Lambda^3\,K_{1,1}^{12} - f_1'\,H_1^{14}$$

multipliée par $\Lambda_{1,1}^7$, nous obtenons une identité :

$$0 \equiv \Lambda^3\big(-5\Lambda_{1,1,1}^9\,M_1^{10} + 56\,\Lambda_{1,1}^7\,K_{1,1}^{12}\big) + f_1'\big(\tfrac{5}{3}\,f_1'\,M_1^{10}\,M_1^{10} - \tfrac{49}{3}\,\Lambda_{1,1}^7\,H_1^{14}\big)$$

dans laquelle apparaît $f_1' X^{18}$.

En procédant de manière analogue pour X^{19}, X^{21}, X^{23}, X^{25} et X^{27}, et en éliminant à la fin le facteur non identiquement nul f_1', nous obtenons de nouvelles syzygies impliquant nos six nouveaux bi-invariants :

$$\begin{aligned}
0 &\overset{a}{\equiv} -49\,\Lambda_{1,1}^7\,H_1^{14} + 3\,\Lambda^3\,X^{18} + 5\,f_1'\,M_1^{10}\,M_1^{10},\\
0 &\overset{b}{\equiv} 5\,M_1^{10}\,N^{12} - 56\,M^8\,H_1^{14} + 3\,\Lambda^3\,X^{19},\\
0 &\overset{c}{\equiv} 5\,N^{12}\,N^{12} + 64\,M^8\,M^8\,M^8 + 3\,\Lambda^3\,X^{21},\\
0 &\overset{d}{\equiv} 7\,N^{12}\,H_1^{14} + 8\,M^8\,M^8\,M_1^{10} + 3\,\Lambda^3\,X^{23},\\
0 &\overset{e}{\equiv} 35\,N^{12}\,F_{1,1}^{16} - 448\,M^8\,M^8\,K_{1,1}^{12} + 40\,M^8\,M_1^{10}\,M_1^{10} + 3\,\Lambda^3\,X^{25},\\
0 &\overset{f}{\equiv} 5\,M^8\,M_1^{10}\,N^{12} - 56\,M^8\,M^8\,H_1^{14} + 3\,\Lambda^3\,X^{27}.
\end{aligned}$$

Ce ne sont pas les seules syzygies supplémentaires : par exemple :

$$\begin{aligned}
0 &\equiv \Lambda_{1,1,1}^9\,H_1^{14} - 8\,\Lambda_{1,1}^7\,F_{1,1}^{16} + \Lambda_1^5\,X^{18},\\
0 &\equiv M_1^{10}\,H_1^{14} - 8\,M^8\,F_{1,1}^{16} + \Lambda_1^5\,X^{19},\\
0 &\equiv N^{12}\,H_1^{14} + 8\,M^8\,M^8\,M_1^{10} + \Lambda_1^5\,X^{21},\\
0 &\equiv N^{12}\,F_{1,1}^{16} + M^8\,M_1^{10}\,M_1^{10} + \Lambda_1^5\,X^{23}.
\end{aligned}$$

et d'autres encore peuvent être formées, que nous ne rechercherons pas ici.

Restriction à $\{f_1' = 0\}$. L'expression en fonction de $j^5 f$ de nos six nouveaux bi-invariants se simplifie lorsqu'on pose $f_1' = 0$:

$$X^{18}\big|_0 = f_1'' f_1'' f_1'' \Big(11025\,\Delta_0^{1,5}\,\big[\Delta_0^{1,2}\big]^2 + 55125\,\Delta_0^{2,4}\,\big[\Delta_0^{1,2}\big]^2 - 55125\,\Delta_0^{1,4}\,\Delta_0^{1,3}\,\Delta_0^{1,2} \\ - 110250\,\Delta_0^{2,3}\,\Delta_0^{1,3}\,\Delta_0^{1,2} + 49000\,\big[\Delta_0^{1,3}\big]^3\Big),$$

et l'on a des expressions similaires pour $X^{19}\big|_0$, $X^{21}\big|_0$, $X^{23}\big|_0$, $X^{25}\big|_0$ et $X^{27}\big|_0$. En fait, grâce à nos six nouvelles syzygies "$\overset{a}{\equiv}$", "$\overset{b}{\equiv}$", "$\overset{c}{\equiv}$", "$\overset{d}{\equiv}$", "$\overset{e}{\equiv}$" et "$\overset{f}{\equiv}$" faisant intervenir le wronskien, nous pouvons exprimer ces restrictions en fonction seulement des quatre bi-invariants restreints algébriquement indépendants que sont $\Lambda^3\big|_0$, $\Lambda_1^5\big|_0$, $M^8\big|_0$ et $N^{12}\big|_0$ (tout en rappelant les expressions les expressions que nous connaissons déjà) :

$$\underline{\Lambda^3}\big|_0, \qquad \underline{\Lambda_1^5}, \qquad \Lambda_{1,1}^7\big|_0 = \tfrac{5}{3}\,\tfrac{\Lambda_1^5\,\Lambda_1^5}{\Lambda^3}\Big|_0, \qquad \underline{M^8}\big|_0,$$
$$\Lambda_{1,1,1}^9\big|_0 = \tfrac{35}{9}\,\tfrac{\Lambda_1^5\,\Lambda_1^5\,\Lambda_1^5}{\Lambda^3\,\Lambda^3}\Big|_0, \qquad M_1^{10}\big|_0 = \tfrac{8}{3}\,\tfrac{M^8\,\Lambda_1^5}{\Lambda^3}\Big|_0, \qquad \underline{N^{12}}\big|_0,$$
$$K_{1,1}^{12}\big|_0 = \tfrac{5}{9}\,\tfrac{M^8\,\Lambda_1^5\,\Lambda_1^5}{\Lambda^3\,\Lambda^3}\Big|_0, \qquad H_1^{14}\big|_0 = \tfrac{5}{3}\,\tfrac{N^{12}\,\Lambda_1^5}{\Lambda^3}\Big|_0, \qquad F_{1,1}^{16}\big|_0 = \tfrac{35}{9}\,\tfrac{\Lambda_1^5\,\Lambda_1^5\,N^{12}}{\Lambda^3\,\Lambda^3}\Big|_0,$$

$$X^{18}\big|_0 = 1225\,\tfrac{\Lambda_1^5\,\Lambda_1^5\,\Lambda_1^5\,N^{12}}{\Lambda^3\,\Lambda^3\,\Lambda^3}\Big|_0, \qquad X^{19}\big|_0 = \tfrac{80}{3}\,\tfrac{\Lambda_1^5\,M^8\,N^{12}}{\Lambda^3\,\Lambda^3}\Big|_0,$$
$$X^{21}\big|_0 = -\tfrac{5}{3}\,\tfrac{N^{12}\,N^{12}}{\Lambda^3}\Big|_0 - \tfrac{64}{3}\,\tfrac{M^8\,M^8\,M^8}{\Lambda^3}\Big|_0, \qquad X^{23}\big|_0 = -\tfrac{35}{9}\,\tfrac{\Lambda_1^5\,N^{12}\,N^{12}}{\Lambda^3\,\Lambda^3}\Big|_0 - \tfrac{64}{9}\,\tfrac{\Lambda_1^5\,M^8\,M^8\,M^8}{\Lambda^3}\Big|_0,$$
$$X^{25}\big|_0 = -1225\,\tfrac{\Lambda_1^5\,\Lambda_1^5\,N^{12}\,N^{12}}{\Lambda^3\,\Lambda^3\,\Lambda^3}\Big|_0 - \tfrac{320}{27}\,\tfrac{\Lambda_1^5\,\Lambda_1^5\,M^8\,M^8\,M^8}{\Lambda^3\,\Lambda^3\,\Lambda^3}\Big|_0, \qquad X^{27}\big|_0 = \tfrac{80}{3}\,\tfrac{\Lambda_1^5\,M^8\,M^8\,N^{12}}{\Lambda^3\,\Lambda^3}\Big|_0.$$

Assertion . Le bi-invariant X^{18} ne s'exprime pas comme un certain polynôme en les onze bi-invariants construits par crochets f_1', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$, M^8, $\Lambda_{1,1,1}^9$, M_1^{10}, N^{12}, $K_{1,1}^{12}$, H_1^{14} et $F_{1,1}^{16}$.

Preuve. Par l'absurde, supposons que

$$X^{18} = \sum \mathsf{coeff} \cdot \big(f_1'\big)^a \big(\Lambda^3\big)^b \big(\Lambda_1^5\big)^c \big(\Lambda_{1,1}^7\big)^d \big(M^8\big)^e \big(\Lambda_{1,1,1}^9\big)^f \big(M_1^{10}\big)^g \\ \big(N^{12}\big)^h \big(K_{1,1}^{12}\big)^i \big(H_1^{14}\big)^j \big(F_{1,1}^{16}\big)^k,$$

avec des exposants entiers a, b, c, d, e, f, g, h, i, j et k tous $\geqslant 0$, et posons $f_1' = 0$ pour en déduire une relation de la forme :

$$\tfrac{(\Lambda^5)^3\,N^{12}}{(\Lambda^3)^3}\Big|_0 = \sum \mathsf{coeff} \cdot \big(\Lambda^3\big)^b \big(\Lambda_1^5\big)^c \Big(\tfrac{\Lambda_1^5\,\Lambda_1^5}{\Lambda^3}\Big)^d \big(M^8\big)^e \Big(\tfrac{\Lambda_1^5\,\Lambda_1^5\,\Lambda_1^5}{\Lambda^3\,\Lambda^3}\Big)^f \Big(\tfrac{\Lambda_1^5\,M^8}{\Lambda^3}\Big)^g \\ \big(N^{12}\big)^h \Big(\tfrac{\Lambda_1^5\,\Lambda_1^5\,M^8}{\Lambda^3\,\Lambda^3}\Big)^i \Big(\tfrac{\Lambda_1^5\,N^{12}}{\Lambda^3}\Big)^j \Big(\tfrac{\Lambda_1^5\,\Lambda_1^5\,N^{12}}{\Lambda^3\,\Lambda^3}\Big)^k\Big|_0 \\ = \sum \mathsf{coeff} \cdot \big(\Lambda^3\big)^{b-d-2f-g-2i-j-2k} \big(\Lambda_1^5\big)^{c+2d+3f+g+2i+j+2k} \\ \big(M^8\big)^{e+g+i} \big(N^{12}\big)^{h+j+k}.$$

Si nous identifions alors les exposants des quatre quantités algébriquement indépendantes :

$$\begin{cases} 3 = -b + d + 2f + g + 2i + j + 2k, \\ 3 = c + 2d + 3f + g + 2i + j + 2k, \\ 0 = e + g + i, \\ 1 = h + j + k, \end{cases}$$

nous déduisons $e = g = i = 0$ de la troisième ligne, puis $0 = c + b + d + f$ en soustrayant la première de la seconde, d'où $c = b = d = f = 0$, ce qui fait que première et quatrième ligne se simplifient comme :

$$3 = j + 2k \qquad \text{et} \qquad 1 = h + j + k,$$

d'où $h = 0$, puis $k = 2$ et enfin $3 = j + 4$, ce qui est impossible. □

Assertion . Le bi-invariant X^{19} ne s'exprime pas comme un certain polynôme en les douze bi-invariants f_1', Λ^3, Λ_1^5, $\Lambda_{1,1}^7$, M^8, $\Lambda_{1,1,1}^9$, M_1^{10}, N^{12}, $K_{1,1}^{12}$, H_1^{14}, $F_{1,1}^{16}$ X^{18}.

Preuve. Le même raisonnement que pour l'assertion précédente teste l'existence d'une représentation restreinte de la forme :

$$\frac{\Lambda^5\, M^8\, N^{12}}{\Lambda^3\, \Lambda^3}\Big|_0 = \sum \mathsf{coeff} \cdot \big(\Lambda^3\big)^{b-d-2f-g-2i-j-2k-3l} \big(\Lambda_1^5\big)^{c+2d+3f+g+2i+j+2k+3l} \big(M^8\big)^{e+g+i} \big(N^{12}\big)^{h+j+k+l}\Big|_0,$$

laquelle conduit au système suivant de quatre équations entre entiers $\geqslant 0$:

$$\begin{cases} 2 - b + d + 2f + g + 2i + j + 2k + 3l, \\ 1 = c + 2d + 3f + g + 2i + j + 2k + 3l, \\ 1e + g + i, \\ 1h + j + k, \end{cases}$$

et ici, la contradiction se voit immédiatement en soustrayant la première équation de la seconde, ce qui nous donne l'équation impossible $-1 = c + b + d + f$. □

Suite. Par des raisonnements similaires, on établit — comme annoncé — qu'*il est réellement nécessaire d'introduire les 5 bi-invariants supplémentaires* :

$$X^{18}, \quad X^{19}, \quad X^{21}, \quad X^{23}, \quad X^{25},$$

lesquels ne sont pas engendrés par crochets.

Poursuite du processus d'engendrement. Mais ce n'est pas tout : grâce à la liste des valeurs que prennent nos 16 bi-invariants en $f_1' = 0$, nous constatons qu'il nous faut introduire encore d'autres bi-invariants, notamment :

$$\frac{-7\,\Lambda_{1,1}^7\,F_{1,1}^{16}+\Lambda_1^5\,X^{18}}{f_1'},\qquad \frac{-5\,\Lambda_{1,1,1}^9\,F_{1,1}^{16}+\Lambda_{1,1}^7\,X^{18}}{f_1'},\qquad \frac{-49\,K_{1,1}^{12}\,H_1^{14}+M^8\,X^{18}}{f_1'},$$
$$\frac{-56\,K_{1,1}^{12}\,F_{1,1}^{16}+M_1^{10}\,X^{18}}{f_1'},\qquad \frac{-7\,H_1^{14}\,F_{1,1}^{16}+N^{12}\,X^{18}}{f_1'},\qquad \frac{-5\,F_{1,1}^{16}\,F_{1,1}^{16}+H_1^{14}\,X^{18}}{f_1'},$$
$$\frac{-48\,M^8\,N^{12}+\Lambda^3\,X^{19}}{f_1'},\qquad \frac{-6\,M_1^{10}\,H_1^{14}+\Lambda_1^5\,X^{19}}{f_1'},\qquad \frac{-48\,K_{1,1}^{12}\,H_1^{14}+\Lambda_{1,1}^7\,X^{19}}{f_1'},$$
$$\frac{-48\,K_{1,1}^{12}\,F_{1,1}^{16}+\Lambda_{1,1,1}^9\,X^{19}}{f_1'},$$

et ensuite, il nous faut encore soumettre chacune de ces expressions au test de savoir si elle ne s'exprime pas polynomialement en fonction d'une liste croissante de bi-invariants connus à l'étape précédente.

Question. L'algèbre $\mathscr{D}\mathscr{S}_2^5$ possède-t-elle une infinité de bi-invariants fondamentaux?

§8. Calculs de caractéristique d'Euler

Surfaces algébriques complexes projectives dans $P_3(\mathbb{C})$. Soit $X \subset P_3(\mathbb{C})$ une surface algébrique complexe projective lisse de degré $d \geqslant 1$. D'après [2], lorsque x parcourt X, la réunion des fibres $\big(\mathscr{D}\mathscr{S}_{2,m}^{\kappa}\big)_x$ que nous avons étudiées d'un point de vue purement algébrique en un point fixé, s'organise de manière cohérente en un sous-fibré $\mathscr{D}\mathscr{S}_{2,m}^{\kappa}T_X^*$ du fibré $J^{\kappa}(\mathbb{C}, X)$ des jets d'ordre κ d'applications holomorphes de $\mathbb{C}$ à valeurs dans X. Nous renvoyons le lecteur à [2, 5] pour de plus amples informations géométriques. Pour fixer les idées, choisissons maintenant $\kappa = 4$.

À chaque monôme bi-invariant parmi les deux listes fournies par le premier théorème, à savoir :

$$(f_1')^a\,\big(\Lambda^3\big)^b\,\big(\Lambda_{1,1}^7\big)^d\,\big(M^8\big)^e \qquad \text{ou} \qquad \Lambda_1^5\,(f_1')^a\,\big(\Lambda^3\big)^b\,\big(\Lambda_{1,1}^7\big)^d\,\big(M^8\big)^e$$

correspond alors le *fibré de Schur*[69] :

$$\Gamma^{(a+b+3d+2e,\,b+d+2e)}\,T_X^* \qquad \text{ou} \qquad \Gamma^{(2+a+b+3d+2e,\,1+b+d+2e)}\,T_X^*,$$

de telle sorte que $\mathscr{D}\mathscr{S}_{2,m}^4T_X^*$ est isomorphe à la somme directe de ces deux familles de fibrés de Schur, où le quadruplet d'entiers positifs ou nuls (a, b, d, e) prend toutes les valeurs telles que $a + 3b + 7d + 8e = m$ pour la première famille, et où il prend toutes les valeurs satisfaisant $5 + a + 3b + 7d + 8e = m$ pour la deuxième famille.

[69] Pour obtenir les deux entiers l_1 et l_2 de $\Gamma^{(l_1,\,l_2)}T_X^*$, rappellons qu'il suffit de compter le nombre de fois qu'apparaissent les indices "$(\cdot)_1$ et "$(\cdot)_2$" dans chacun de ces monômes, sachant qu'ils apparaissent chacun exactement une fois dans tout déterminant $\Delta^{\alpha,\beta}$.

Retour sur le choix d'une base de Gröbner pour les jets d'ordre 4. Nous n'avons pas encore fait remarquer que notre choix de $\underline{\Lambda_1^5\,\Lambda_1^5}\big|_{f_1'=0}$ comme monôme de tête dans l'unique syzygie restreinte qui existe entre les cinq bi-invariants fondamentaux restreints $\Lambda^3\big|_0$, $\Lambda_1^5\big|_0$, $\Lambda_{1,1}^7\big|_0$ et $M^8\big|_0$ au niveau $\kappa = 4$ n'était pas en harmonie avec le choix d'ordre lexicographique que nous avions fait au niveau $\kappa = 5$ et qui nous avait fournit une base de 21 syzygies faisant apparaître un *triangle remarquable de monômes de tête*. En effet, si nous voulions rétablir la cohérence entre les deux niveaux $\kappa = 4$ et $\kappa = 5$, nous devrions, au niveau $\kappa = 4$, choisir plutôt l'ordre purement lexicographique déduit de l'ordre suivant entre bi-invariants restreints :

$$\Lambda^3 \,>\, \Lambda_1^5 \,>\, \Lambda_{1,1}^7 \,>\, M^8$$

(nous sous-entendons ici la mention "$(\cdot)_{f_1'=0}$"), ce qui conduit à *changer de monôme de tête* dans l'unique syzygie restreinte existante :

$$0 \equiv -5\Lambda_1^5\,\Lambda_1^5 + 3\,\underline{\Lambda^3\,\Lambda_{1,1}^7}\big|_{f_1'=0}.$$

Lemme. Avec ce nouveau choix d'ordre qui anticipe une harmonie avec le niveau suivant $\kappa = 5$, tout polynôme bi-invariant de poids m dans $\mathscr{D}\mathscr{S}_{2,m}^4$ s'écrit de manière unique :

$$\begin{aligned}
\mathsf{P}^{2\times\mathrm{inv}}\big(j^4 f\big) &= \mathscr{P}\big(f_1', \Lambda^3, \Lambda_1^5, M^8\big) + \Lambda_{1,1}^7\,\mathscr{Q}\big(f_1', \Lambda_1^5, \Lambda_{1,1}^7, M^8\big) \\
&= \sum_{a+3b+5c+8e=m} \mathsf{coeff}\cdot (f_1')^a\,\big(\Lambda^3\big)^b\,\big(\Lambda_1^5\big)^c\,\big(M^8\big)^e + \\
&\qquad + \sum_{7+a+5c+7d+8e=m} \mathsf{coeff}\cdot \Lambda_{1,1}^7\,(f_1')^a\,\big(\Lambda_1^5\big)^c\,\big(\Lambda_{1,1}^7\big)^d\,\big(M^8\big)^e,
\end{aligned}$$

et par conséquent, le fibré de Demailly-Semple $\mathscr{D}\mathscr{S}_{2,m}^4 T_X^*$ est isomorphe à la somme directe suivante de fibrés de Schur :

$$\begin{aligned}
\mathscr{D}\mathscr{S}_{2,m}^4 T_X^* = &\bigoplus_{a+3b+5c+8e=m} \Gamma^{(a+b+2c+2e,\,b+c+2e)}\,T_X^* \\
&\bigoplus_{7+a+5c+7d+8e=m} \Gamma^{(3+a+2c+3d+2e,\,1+c+d+2e)}\,T_X^*.
\end{aligned}$$

Caractéristique d'Euler des fibrés de Schur. Si $\mathsf{c}_i = \mathsf{c}_i(T_X)$, $i = 1, 2$, désignent les classes de Chern de d'une surface complexe X, on a la formule suivante ([5]) :

$$\boxed{\chi\big(X,\,\Gamma^{(l_1,l_2)}\,T_X^*\big) = \frac{1}{6}\,\mathsf{c}_1^2\,\big[l_1^3 - l_2^3\big] - \frac{1}{6}\,\mathsf{c}_2\,(l_1 - l_2)^3 + \mathrm{O}\big(|\lambda|^2\big)}$$

pour la caractéristique d'Euler du fibré de Schur $\Gamma^{(l_1,l_2)}T_X^*$. Mentionnons au passage que pour X de dimension trois, la formule devient :

$$\begin{aligned}
&-\chi\big(X,\,\Gamma^{(l_1,l_2,l_3)}\,T_X^*\big) = \\
&= \frac{-\mathsf{c}_3}{1!\;2!\;3!}\begin{vmatrix} l_1 & l_2 & l_3 \\ l_1^2 & l_2^2 & l_3^2 \\ l_1^3 & l_2^3 & l_3^3 \end{vmatrix} + \frac{-\mathsf{c}_1\mathsf{c}_2+\mathsf{c}_3}{0!\;2!\;4!}\begin{vmatrix} 1 & 1 & 1 \\ l_1^2 & l_2^2 & l_3^2 \\ l_1^4 & l_2^4 & l_3^4 \end{vmatrix} + \\
&\qquad + \frac{-\mathsf{c}_1^3+2\,\mathsf{c}_1\mathsf{c}_2-\mathsf{c}_3}{0!\;1!\;5!}\begin{vmatrix} 1 & 1 & 1 \\ l_1 & l_2 & l_3 \\ l_1^5 & l_2^5 & l_3^5 \end{vmatrix} + \mathrm{O}\big(|l|^5\big),
\end{aligned}$$

puis pour X de dimension quatre :

$$\begin{aligned}
&\chi\big(X,\,\Gamma^{(l_1,l_2,l_3,l_4)}\,T_X^*\big) = \frac{-\mathsf{c}_1^4+3\,\mathsf{c}_1^2\mathsf{c}_2-\mathsf{c}_2^2-2\,\mathsf{c}_1\mathsf{c}_3+\mathsf{c}_4}{0!\;1!\;2!\;7!}\begin{vmatrix} 1 & 1 & 1 & 1 \\ l_1 & l_2 & l_3 & l_4 \\ l_1^2 & l_2^2 & l_3^2 & l_4^2 \\ l_1^7 & l_2^7 & l_3^7 & l_4^7 \end{vmatrix} + \\
&+ \frac{-\mathsf{c}_4}{1!\;2!\;3!\;4!}\begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ l_1^2 & l_2^2 & l_3^2 & l_4^2 \\ l_1^3 & l_2^3 & l_3^3 & l_4^3 \\ l_1^4 & l_2^4 & l_3^4 & l_4^4 \end{vmatrix} + \frac{-\mathsf{c}_1\mathsf{c}_3+\mathsf{c}_4}{0!\;2!\;3!\;5!}\begin{vmatrix} 1 & 1 & 1 & 1 \\ l_1^2 & l_2^2 & l_3^2 & l_4^2 \\ l_1^3 & l_2^3 & l_3^3 & l_4^3 \\ l_1^5 & l_2^5 & l_3^5 & l_4^5 \end{vmatrix} + \\
&+ \frac{-\mathsf{c}_1^2\mathsf{c}_2+\mathsf{c}_2^2+\mathsf{c}_1\mathsf{c}_3-\mathsf{c}_4}{0!\;1!\;3!\;6!}\begin{vmatrix} 1 & 1 & 1 & 1 \\ l_1 & l_2 & l_3 & l_4 \\ l_1^3 & l_2^3 & l_3^3 & l_4^3 \\ l_1^6 & l_2^6 & l_3^6 & l_4^6 \end{vmatrix} + \frac{\mathsf{c}_1\mathsf{c}_3-\mathsf{c}_2^2}{0!\;1!\;4!\;5!}\begin{vmatrix} 1 & 1 & 1 & 1 \\ l_1 & l_2 & l_3 & l_4 \\ l_1^4 & l_2^4 & l_3^4 & l_4^4 \\ l_1^5 & l_2^5 & l_3^5 & l_4^5 \end{vmatrix} + \mathrm{O}\big(|l|^9\big),
\end{aligned}$$

et enfin, ajoutons qu'il ne serait pas difficile de fournir la formule générale, valable en dimension quelconque.

Sommations de caractéristiques. En revenant à la dimension deux, nous déduisons tout d'abord trivialement du lemme précédent la formule sommatoire suivante pour la caractéristique d'Euler de la première somme directe de fibrés de Schur :

$$\begin{aligned}
&\chi\Big(X,\bigoplus_{a+3b+5c+8e=m}\Gamma^{(a+b+2c+2e,\,b+c+2e)}\,T_X^*\Big) \\
&\qquad\qquad = \sum_{a+3b+5c+8e=m}\chi\Big(X,\;\Gamma^{(a+b+2c+2e,\,b+c+2e)}\,T_X^*\Big),
\end{aligned}$$

et ensuite, grâce à une table de calculs linéaires destinée à éliminer l'exposant a :

$$\begin{aligned}
m &= a+3b+5c+8e, \\
l_1 &= a+b+2c+2e \\
&= m-2b-3c-6e, \\
l_2 &= \qquad\;\; b+c+2e, \\
l_1-l_2 &= m-3b-4c-8e,
\end{aligned}$$

ce qui nous permet de remplacer $\sum_{a+3b+5c+8e=m}$ par $\sum_{3b+5c+8e\leqslant m}$, nous pouvons calculer les deux coefficients rationnels $\mathtt{A}_1 \in m^6\cdot\mathbb{Q}$ et $\mathtt{A}_2 \in m^6\cdot\mathbb{Q}$ qui apparaissent devant c_1^2 et devant $-\mathsf{c}_2$ lorsqu'on effectue la première

somme de caractéristiques, que nous appellerons “A” (la seconde s’appellera “B”) :

$$\begin{aligned}
\mathsf{A}_1 &= \frac{1}{6}\int_0^{\frac{m}{8}} de \int_0^{\frac{m-8e}{5}} dc \int_0^{\frac{m-5c-8e}{3}} db \Big[(m-2b-3c-6e)^3-(b+c+2e)^3\Big] + \mathrm{O}(m^5)\\
&= \frac{937\ m^6}{28\,800\,000},\\
\mathsf{A}_2 &= \frac{1}{6}\int_0^{\frac{m}{8}} de \int_0^{\frac{m-8e}{5}} dc \int_0^{\frac{m-5c-8e}{3}} db\,(m-3b-4c-8e)^3 + \mathrm{O}(m^5)\\
&= \frac{13\ m^6}{900\,000},
\end{aligned}$$

où nous avons utilisé le fait que les sommes de Riemann sont suffisamment bien approximées par des intégrales si l’on s’intéresse seulement au coefficient de m^6.

Ensuite, si nous procédons de la même manière pour la deuxième somme directe de fibrés de Schur :

$$\begin{aligned}
&\chi\Big(X, \bigoplus_{7+a+5c+7d+8e=m} \Gamma^{(3+a+2c+3d+2e,\,1+c+d+2e)}\,T_X^*\Big)\\
&\qquad = \sum_{7+a+5c+7d+8e=m} \chi\Big(X,\ \Gamma^{(3+a+2c+3d+2e,\,1+c+d+2e)}\,T_X^*\Big)\\
&\qquad = \sum_{a+5c+7d+8e=m} \chi\Big(X,\ \Gamma^{(a+2c+3d+2e,\,c+d+2e)}\,T_X^*\Big) + \mathrm{O}(m^5),
\end{aligned}$$

en observant que décalage de 7 dans le poids m, ainsi que les deux décalages de 3 et de 1 dans l_1 et dans l_2 ne contribuent en fait qu’en $\mathrm{O}(m^5)$ dans le résultat final, ce qui nous permet de les négliger, nous pouvons dresser une table de calculs élémentaires analogue à la précédente :

$$\begin{aligned}
m &= a+5c+7d+8e,\\
l_1 &= a+2c+3d+2e\\
&= m-3c-4d-6e,\\
l_2 &= \qquad c+d+2e,\\
l_1-l_2 &= m-4c-5d-8e,
\end{aligned}$$

remplacer $\sum_{a+5c+7d+8e=m}$ par $\sum_{5c+7d+8e\leqslant m}$ après avoir éliminé a, de telle sorte que nous sommes ramenés à calculer les deux intégrales suivantes :

$$\begin{aligned}
\mathsf{B}_1 &= \int_0^{\frac{m}{8}} de \int_0^{\frac{m-8e}{7}} dd \int_0^{\frac{m-7d-8e}{5}} dc \Big[(m-3c-4d-6e)^3-(c+d+2e)^3\Big] + \mathrm{O}(m^5)\\
&= \frac{559\,819\ m^6}{34\,574\,400\,000} + \mathrm{O}(m^5),\\
\mathsf{B}_2 &= \int_0^{\frac{m}{8}} de \int_0^{\frac{m-8e}{7}} dd \int_0^{\frac{m-7d-8e}{5}} dc\,(m-4c-5d-8e)^3 + \mathrm{O}(m^5)\\
&= \frac{36949\ m^6}{4\,321\,800\,000} + \mathrm{O}(m^5).
\end{aligned}$$

En définitive, nous obtenons les deux coefficients rationnels totaux $\mathscr{C}_1 \in m^6 \cdot \mathbb{Q}$ et $\mathscr{C}_2 \in m^6\mathbb{Q}$ de c_1^2 et de $-\mathsf{c}_2$:

$$\mathscr{C}_1 = \mathsf{A}_1 + \mathsf{B}_1 = \frac{1797\ m^6}{36\,879\,360} + \mathrm{O}(m^5),$$
$$\mathscr{C}_2 = \mathsf{A}_2 + \mathsf{B}_2 = \frac{848\ m^6}{36\,879\,360} + \mathrm{O}(m^5).$$

Application. Les classes de Chern $\mathsf{c}_i = \mathsf{c}_i(T_X)$, $i = 1, 2$, de X sont reliées au degré d de X par $\mathsf{c}_1^2 = (4-d)^2 d$ et $\mathsf{c}_2 = d\,(d^2 - 4d + 6)$.

Proposition. *En dimension deux pour les jets d'ordre quatre, la caractéristique d'Euler de $\mathscr{D}\mathscr{S}^4_{2,m}T^*_X$ vaut :*

$$\chi\big(X, \mathscr{D}\mathscr{S}^4_{2,m}T^*_X\big) = \frac{m^6}{36\,879\,360}\big(1\,797\,\mathsf{c}_1^2 - 848\,\mathsf{c}_2\big) + \mathrm{O}(m^5),$$

donc si l'on pose :

$$\mathscr{C} := \tfrac{1\,797}{848} = 2,119\cdots,$$

alors en réexprimant le tout en fonction du degré, on a l'équivalence :

$$\chi\big(X, \mathscr{D}\mathscr{S}^4_{2,m}T^*_X\big) \sim \frac{1\,797\,m^6}{36\,879\,360}\,d\,\mathsf{q}_{\mathscr{C}}(d),$$

quand $m \to \infty$, avec un polynôme quadratique

$$\boxed{\mathsf{q}_{\mathscr{C}}(d) := d^2(\mathscr{C} - 1) - d(8\,\mathscr{C} - 4) + 16\,\mathscr{C} - 6}$$

qui est positif pour tout degré $d \geqslant \mathbf{9}$.

Remarque. Le quotient $\mathscr{C}$ des coefficients de c_1^2 et de c_2 vaut $\frac{47}{26} = 1,807\cdots$ pour les jets d'ordre 3, d'où $\mathsf{q}_{\mathscr{C}}(d)$ est positif pour tout $d \geqslant \mathbf{11}$ (*cf.* [5]), et il vaut $\frac{13}{9} = 1,444\cdots$ pour les jets d'ordre 2 (*cf.* [2]), d'où $\mathsf{q}_{\mathscr{C}}(d)$ est positif pour tout $d \geqslant \mathbf{15}$. Demailly a conjecturé que ce quotient tend vers l'infini avec κ. Les valeurs numériques $1,44\cdots$, $1,80\cdots$ et $2,12\cdots$ suggèrent une certaine lenteur de la convergence potentielle.

Corollaire. Si A est un fibré en droites ample sur X, pour tout m suffisamment grand, il y a des sections globales de $\mathscr{D}\mathscr{S}^4_{2,m}T^*_X \otimes A^{-1}$ lorsque $d \geqslant \mathbf{9}$, et toute courbe entière $f = \mathbb{C} \to X$ doit satisfaire l'équation différentielle globale correspondante.

Passage aux jets d'ordre 5. Maintenant, la correspondance entre bi-invariants fondamentaux et représentations de Schur :

$$f_1' \longleftrightarrow \Gamma^{(1,0)}, \qquad \Lambda^3 \longleftrightarrow \Gamma^{(1,1)},$$
$$\Lambda_1^5 \longleftrightarrow \Gamma^{(2,1)}, \qquad \Lambda_{1,1}^7 \longleftrightarrow \Gamma^{(3,1)}, \qquad M^8 \longleftrightarrow \Gamma^{(2,2)},$$
$$\Lambda_{1,1,1}^9 \longleftrightarrow \Gamma^{(4,1)}, \qquad M_1^{10} \longleftrightarrow \Gamma^{(3,2)}, \qquad N^{12} \longleftrightarrow \Gamma^{(3,3)},$$
$$K_{1,1}^{12} \longleftrightarrow \Gamma^{(4,2)}, \qquad H_1^{14} \longleftrightarrow \Gamma^{(4,3)}, \qquad F_{1,1}^{16} \longleftrightarrow \Gamma^{(5,3)},$$

Étant donné qu'il existe des invariants fondamentaux supplémentaires, nous pourrions attendre encore avant d'entreprendre un calcul de Riemann-Roch, mais nous avons quand même l'opportunité de nous restreindre à la sous-algèbre engendrée par les crochets.

Base de Gröbner. En choisissant l'ordre purement lexicographique sur les monômes de $\mathbb{C}\big[\Lambda^3, \dots, H_1^{14}, F_{1,1}^{16}, f_1'\big]$ qui est déduit de l'ordre suivant sur les monômes élémentaires restreints (noter que f_1' est placé en dernière position[70]) :

$$\Lambda^3 > \Lambda_1^5 > \Lambda_{1,1}^7 > M^8 > \Lambda_{1,1,1}^9 > M_1^{10} > N^{12} > K_{1,1}^{12} > H_1^{14} > F_{1,1}^{16} > f_1'$$

Maple nous donne la base de Gröbner réduite suivante pour l'idéal complet des syzygies entre nos onze bi-invariants restreints à $\{f_1' = 0\}$, laquelle est constituée de 26 équations :

$$\begin{aligned}
0 &= -5\, f_1' \big(M_1^{10}\big)^2 N^{12}\, K_{1,1}^{12}\, F_{1,1}^{16} + 5\, f_1'\, N^{12} \big(F_{1,1}^{16}\big)^3 - 64\, f_1' \big(M_1^{10}\big)^2 K_{1,1}^{12} \big(H_1^{14}\big)^2 + 5\, \underline{f_1' \big(M_1^{10}\big)^3 H_1^{14}\, F_{1,1}^{16}} + \\
&\quad + 128\, f_1'\, M_1^{10}\, N^{12} \big(K_{1,1}^{12}\big)^2 H_1^{14} - 7\, f_1' \big(H_1^{14}\big)^2 \big(F_{1,1}^{16}\big)^2 - 64\, f_1' \big(N^{12}\big)^2 \big(K_{1,1}^{12}\big)^3, \\
0 &= 15\, f_1'\, M_1^{10}\, K_{1,1}^{12}\, H_1^{14} - 7\, \Lambda_{1,1,1}^9 \big(H_1^{14}\big)^2 - 5\, f_1' \big(M_1^{10}\big)^2 F_{1,1}^{16} + 5\, \underline{\Lambda_{1,1,1}^9\, N^{12}\, F_{1,1}^{16}} - 8\, f_1'\, N^{12} \big(K_{1,1}^{12}\big)^2, \\
0 &= 7\, f_1' \big(M_1^{10}\big)^2 K_{1,1}^{12} + \Lambda_{1,1,1}^9\, M_1^{10}\, H_1^{14} + f_1' \big(F_{1,1}^{16}\big)^2 - 8\, \Lambda_{1,1,1}^9\, N^{12}\, K_{1,1}^{12}, \\
0 &= N^{12}\, K_{1,1}^{12} - M_1^{10}\, H_1^{14} + \underline{M^8\, F_{1,1}^{16}}, \\
0 &= 64\, \underline{f_1'\, M^8\, M_1^{10}\, K_{1,1}^{12}\, H_1^{14}} + 7\, f_1' \big(H_1^{14}\big)^2 F_{1,1}^{16} + 5\, f_1' \big(M_1^{10}\big)^2 N^{12}\, K_{1,1}^{12} - 64\, f_1'\, M^8\, N^{12}\, K_{1,1}^{12} - \\
&\quad - 5\, f_1' \big(M_1^{10}\big)^3 H_1^{14} - 5\, f_1'\, N^{12} \big(F_{1,1}^{16}\big)^2, \\
0 &= f_1'\, H_1^{14}\, F_{1,1}^{16} + 8\, \underline{M^8\, \Lambda_{1,1,1}^9\, H_1^{14}} + 5\, f_1' \big(M_1^{10}\big)^3 - 5\, \Lambda_{1,1,1}^9\, M_1^{10}\, N^{12} - 8\, f_1'\, M^8\, M_1^{10}\, K_{1,1}^{12}, \\
0 &= 64\, \underline{M^8\, \Lambda_{1,1,1}^9\, N^{12}\, K_{1,1}^{12}} - 64\, f_1'\, M^8 \big(M_1^{10}\big)^2 K_{1,1}^{12} - 5\, \Lambda_{1,1,1}^9 \big(M_1^{10}\big)^2 N^{12} + 5\, f_1' \big(M_1^{10}\big)^4 - \\
&\quad - 7\, f_1'\, M_1^{10}\, H_1^{14}\, F_{1,1}^{16} + 8\, f_1'\, N^{12}\, K_{1,1}^{12}\, F_{1,1}^{16},
\end{aligned}$$

$$\begin{aligned}
0 &= 7\, f_1' \big(H_1^{14}\big)^2 - 5\, f_1'\, M^8 \big(M_1^{10}\big)^2 + 64\, \underline{f_1' \big(M^8\big)^2 K_{1,1}^{12}} - 5\, f_1'\, N^{12}\, F_{1,1}^{16}, \\
0 &= 64 \underline{\big(M^8\big)^2 \Lambda_{1,1,1}^9\, K_{1,1}^{12}} - 5\, M^8\, \Lambda_{1,1,1}^9 \big(M_1^{10}\big)^2 - 5\, f_1' \big(M_1^{10}\big)^2 F_{1,1}^{16} + 15\, f_1'\, M_1^{10}\, K_{1,1}^{12}\, H_1^{14} - 8\, f_1'\, N^{12} \big(K_{1,1}^{12}\big)^2, \\
0 &= -\Lambda_{1,1,1}^9\, H_1^{14} + \underline{\Lambda_{1,1}^7\, F_{1,1}^{16}} + f_1'\, M_1^{10}\, K_{1,1}^{12}, \\
0 &= 5\, f_1' \big(M_1^{10}\big)^2 - 5\, \Lambda_{1,1,1}^9\, N^{12} - 8\, f_1'\, M^8\, K_{1,1}^{12} + 7\, \underline{\Lambda_{1,1}^7\, H_1^{14}}, \\
0 &= 56\, \underline{\Lambda_{1,1}^7\, N^{12}\, K_{1,1}^{12}} - 64\, f_1'\, M^8\, M_1^{10}\, K_{1,1}^{12} - 5\, \Lambda_{1,1,1}^9\, M_1^{10}\, N^{12} + 5\, f_1' \big(M_1^{10}\big)^3 - 7\, f_1'\, H_1^{14}\, F_{1,1}^{16}, \\
0 &= -8\, M^8\, \Lambda_{1,1,1}^9 + 7\, \underline{\Lambda_{1,1}^7\, M_1^{10}} - f_1'\, F_{1,1}^{16}, \\
0 &= 56\, \underline{\Lambda_{1,1}^7\, M^8\, K_{1,1}^{12}} - 5\, M^8\, \Lambda_{1,1,1}^9\, M_1^{10} - 5\, f_1'\, M_1^{10}\, F_{1,1}^{16} + 7\, f_1'\, K_{1,1}^{12}\, H_1^{14}, \\
0 &= 49 \underline{\big(\Lambda_{1,1}^7\big)^2 K_{1,1}^{12}} - 5\, M^8 \big(\Lambda_{1,1,1}^9\big)^2 - 5\, f_1'\, \Lambda_{1,1,1}^9\, F_{1,1}^{16} + 7 \big(f_1'\big)^2 \big(K_{1,1}^{12}\big)^2, \\
0 &= -\Lambda_{1,1,1}^9\, N^{12} + \underline{\Lambda_1^5\, F_{1,1}^{16}} + f_1'\, M_1^{10}\, M_1^{10},
\end{aligned}$$

[70] Sinon, les bases fournies contiennent plus d'une soixantaine d'équations.

$$
\begin{aligned}
0 &= -\Lambda_{1,1}^7\, N^{12} + \underline{\Lambda_1^5\, H_1^{14}} + f_1'\, M^8\, M_1^{10},\\
0 &= 7\,\underline{\Lambda_1^5\, K_{1,1}^{12}} - M^8\, \Lambda_{1,1,1}^9 - f_1'\, F_{1,1}^{16},\\
0 &= -8\,\Lambda_{1,1}^7\, M^8 + 5\,\underline{\Lambda_1^5\, M_1^{10}} - f_1'\, H_1^{14},\\
0 &= -7\,\Lambda_{1,1}^7\, \Lambda_{1,1}^7 + 5\,\underline{\Lambda_1^5\, \Lambda_{1,1,1}^9} - f_1' f_1'\, K_{1,1}^{12},\\
0 &= -7\,\Lambda_{1,1}^7\, N^{12} + 3\,\underline{\Lambda^3\, F_{1,1}^{16}} + 8\, f_1'\, M^8\, M_1^{10},\\
0 &= -5\,\Lambda_1^5\, N^{12} + 3\,\underline{\Lambda^3\, H_1^{14}} + 8\, f_1'\, M^8\, M^8,\\
0 &= 3\,\underline{\Lambda^3\, K_{1,1}^{12}} - \Lambda_{1,1}^7\, M^8 - f_1'\, H_1^{14},\\
0 &= -8\,\Lambda_1^5\, M^8 + 3\,\underline{\Lambda^3\, M_1^{10}} - f_1'\, N^{12},\\
0 &= -7\,\Lambda_1^5\, \Lambda_{1,1}^7 + 3\,\underline{\Lambda^3\, \Lambda_{1,1,1}^9} - f_1' f_1'\, M_1^{10},\\
0 &= -5\big(\Lambda_1^5\big)^2 + 3\,\underline{\Lambda^3\, \Lambda_{1,1}^7} - f_1' f_1'\, M^8.
\end{aligned}
$$

D'après la théorie des bases de Gröbner, une base de l'espace vectoriel :

$$\mathbb{C}\big[\Lambda^3,\, \Lambda_1^5,\, \ldots,\, H_1^{14},\, F_{1,1}^{16},\, f_1'\big]\Big/\big(\text{26 équations précédentes}\big)$$

est constituée de tous les monômes

$$\big(\Lambda^3\big)^a\big(\Lambda_1^5\big)^b\big(\Lambda_{1,1}^7\big)^c\big(M^8\big)^d\big(\Lambda_{1,1,1}^9\big)^e\big(M_1^{10}\big)^f\big(N^{12}\big)^g\big(K_{1,1}^{12}\big)^h\big(H_1^{14}\big)^i\big(F_{1,1}^{16}\big)^j\big(f_1'\big)^k$$

qui n'appartiennent pas à l'idéal monomial engendré par les 26 monômes de tête de chacun des 26 générateurs que nous avons soulignés. Or un tel monôme appartient à cet idéal monomial si et seulement si il est divisible par l'un des 26 monômes de tête, ce qui revient à dire que le muti-indice

$$\big(a,\, b,\, c,\, d,\, e,\, f,\, g,\, h,\, i,\, j,\, k\big) \in \mathbb{N}^{11}$$

appartient à la *réunion* des 26 sous-ensembles suivants de $\mathbb{N}^{11}$:

$$
\begin{array}{rrr}
\{f \geqslant 3\} \cap \{i \geqslant 1\} \cap \{j \geqslant 1\} \cap \{k \geqslant 1\}, & \{e \geqslant 1\} \cap \{g \geqslant 1\} \cap \{j \geqslant 1\}, & \\
 & \{e \geqslant 1\} \cap \{f \geqslant 1\} \cap \{i \geqslant 1\}, & \\
\multicolumn{3}{l}{\{d \geqslant 1\} \cap \{j \geqslant 1\},} \\
\multicolumn{3}{l}{\{d \geqslant 1\} \cap \{f \geqslant 1\} \cap \{h \geqslant 1\} \cap \{i \geqslant 1\} \cap \{k \geqslant 1\},} \\
\multicolumn{3}{l}{\{d \geqslant 1\} \cap \{e \geqslant 1\} \cap \{i \geqslant 1\},} \\
\multicolumn{3}{l}{\{d \geqslant 1\} \cap \{e \geqslant 1\} \cap \{g \geqslant 1\} \cap \{h \geqslant 1\},} \\
\multicolumn{3}{l}{\{d \geqslant 2\} \cap \{h \geqslant 1\} \cap \{k \geqslant 1\},} \\
\multicolumn{3}{l}{\{d \geqslant 2\} \cap \{e \geqslant 1\} \cap \{h \geqslant 1\},} \\
\{c \geqslant 1\} \cap \{j \geqslant 1\}, & \{b \geqslant 1\} \cap \{j \geqslant 1\}, & \{a \geqslant 1\} \cap \{j \geqslant 1\}, \\
\{c \geqslant 1\} \cap \{i \geqslant 1\}, & \{b \geqslant 1\} \cap \{i \geqslant 1\}, & \{a \geqslant 1\} \cap \{i \geqslant 1\}, \\
\{c \geqslant 1\} \cap \{g \geqslant 1\} \cap \{h \geqslant 1\}, & \{b \geqslant 1\} \cap \{h \geqslant 1\}, & \{a \geqslant 1\} \cap \{h \geqslant 1\}, \\
\{c \geqslant 1\} \cap \{f \geqslant 1\}, & \{b \geqslant 1\} \cap \{f \geqslant 1\}, & \{a \geqslant 1\} \cap \{f \geqslant 1\}, \\
\{c \geqslant 1\} \cap \{d \geqslant 1\} \cap \{h \geqslant 1\}, & \{b \geqslant 1\} \cap \{e \geqslant 1\}, & \{a \geqslant 1\} \cap \{e \geqslant 1\}, \\
\{c \geqslant 2\} \cap \{h \geqslant 1\}, & & \{a \geqslant 1\} \cap \{c \geqslant 1\}.
\end{array}
$$

Pour calculer le complémentaire de cette réunion, on procède comme dans la Section 7, en regroupant séparément les 6 intersections commençant par $\{a \geqslant 1\}$, puis les 5 commençant par $\{b \geqslant 1\}$, puis les 6 commençant par $\{c \geqslant 1\}$, puis les 6 commençant par $\{d \geqslant 1\}$, puis les 2 commançant par

$\{e \geqslant 1\}$, et puis enfin la dernière, qui commence par $\{f \geqslant 1\}$. Trouver le complémentaire global reviendra donc à calculer l'*intersection* de six sous-ensembles de $\mathbb{N}^{11}$.
Clairement, le premier et le deuxième complémentaires sont donnés par :

$$\begin{aligned}
\mathscr{N}_1 &:= \{a = 0\} \cup \{c = e = f = g = h = i = j = 0\},\\
\mathscr{N}_2 &:= \{b = 0\} \cup \{e = f = h = i = j = 0\}.
\end{aligned}$$

Ensuite, calculons le troisième complémentaire, en simplifiant progressivement les intersections, et ce, en partant du dernier terme :

$$\begin{aligned}
\mathscr{N}_3 := \{c = 1\} &\cup \{c = 0\} \cup \{h = 0\} \bigcap \{c = 0\} \cup \{d = 0\} \cup \{h = 0\} \bigcap \{c = 0\} \cup \{f = 0\}\\
&\bigcap \{c = 0\} \cup \{g = 0\} \cup \{h = 0\} \bigcap \{c = 0\} \cup \{i = 0\} \bigcap \{c = 0\} \cup \{j = 0\}\\
= \{c = 1\} &\cup \{c = 0\} \cup \{h = 0\} \bigcap \{c = 0\} \cup \{d = 0\} \cup \{h = 0\} \bigcap \{c = 0\} \cup \{f = 0\}\\
&\bigcap \{c = 0\} \cup \{c = g = i = j = 0\} \cup \{h = i = j = 0\}\\
= \{c = 0\} &\cup \{f = h = i = j = 0\} \cup \{c = 1,\, d = f = g = i = j = 0\}.
\end{aligned}$$

Les calculs suivants donnent :

$$\begin{aligned}
\mathscr{N}_4 := \{d = 0\} &\cup \{e = h = j = 0\} \cup \{e = j = k = 0\} \cup \{h = i = j = 0\} \cup\\
&\cup \{d = 1,\, e = i = j = 0\} \cup \{d = 1,\, e = f = j = 0\} \cup \{d = 1,\, g = i = j = 0\} \cup\\
&\cup \{d = 1,\, e = h = j = 0\} \cup \{d = 1,\, h = i = j = 0\} \cup \{d = 1,\, e = j = k = 0\},\\
\mathscr{N}_5 := \{e = 0\} &\cup \{f = g = 0\} \cup \{f = j = 0\} \cup \{g = i = 0\} \cup \{i = j = 0\},\\
\mathscr{N}_6 := \{f = 2\} &\cup \{f = 1\} \cup \{f = 0\} \cup \{i = 0\} \cup \{j = 0\} \cup \{k = 0\}.
\end{aligned}$$

En développant l'intersection finale :

$$\mathscr{N}_1 \cap \mathscr{N}_2 \cap \mathscr{N}_3 \cap \mathscr{N}_4 \cap \mathscr{N}_5 \cap \mathscr{N}_6,$$

nous pouvons négliger toutes les composantes qui incorporent un nombre $\geqslant$ 7 d'équations, puisque dans la sommation de fibrés de Schur, la contribution ne sera qu'en $\mathrm{O}(m^7)$, les termes principaux étant multiples rationnels non nuls de m^8. Ainsi, en négligeant de tels termes, nous obtenons exactement 16 composantes de dimension 5 définies par 6 équations :

$$\begin{aligned}
&\{a = b = c = d = e = 0,\, f = 2\} \cup \{a = b = c = d = e = 0,\, f = 1\}\\
&\cup \{a = b = c = d = e = f = 0\} \cup \{a = b = c = d = e = i = 0\}\\
&\cup \{a = b = c = d = e = j = 0\} \cup \{a = b = c = d = e = k = 0\}\\
&\cup \{a = b = c = d = f = g = 0\} \cup \{a = b = c = d = f = i = 0\}\\
&\cup \{a = b = c = d = g = i = 0\} \cup \{a = b = c = d = i = j = 0\}\\
&\cup \{a = b = c = e = h = j = 0\} \cup \{a = b = c = e = j = k = 0\}\\
&\cup \{a = b = c = h = i = j = 0\} \cup \{a = b = f = h = i = j = 0\}\\
&\cup \{a = e = f = h = i = j = 0\} \cup \{c = e = f = h = i = j = 0\}.
\end{aligned}$$

Les 16 familles de monômes correspondant à ces équations peuvent être rangées dans un tableau :

A :	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$(M_1^{10})^2$	$(N^{12})^g$	$(K_{1,1}^{12})^h$	$(H_1^{14})^i$	$(F_{1,1}^{16})^j$	$(f_1')^k$
B :	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	M_1^{10}	$(N^{12})^g$	$(K_{1,1}^{12})^h$	$(H_1^{14})^i$	$(F_{1,1}^{16})^j$	$(f_1')^k$
C :	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$(N^{12})^g$	$(K_{1,1}^{12})^h$	$(H_1^{14})^i$	$(F_{1,1}^{16})^j$	$(f_1')^k$
D :	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$(M_1^{10})^f$	$(N^{12})^g$	$(K_{1,1}^{12})^h$	$\bullet$	$(F_{1,1}^{16})^j$	$(f_1')^k$
E :	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$(M_1^{10})^f$	$(N^{12})^g$	$(K_{1,1}^{12})^h$	$(H_1^{14})^i$	$\bullet$	$(f_1')^k$
F :	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$(M_1^{10})^f$	$(N^{12})^g$	$(K_{1,1}^{12})^h$	$(H_1^{14})^i$	$(F_{1,1}^{16})^j$	$\bullet$
G :	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$(\Lambda_{1,1,1}^9)^e$	$\bullet$	$\bullet$	$(K_{1,1}^{12})^h$	$(H_1^{14})^i$	$(F_{1,1}^{16})^j$	$(f_1')^k$
H :	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$(\Lambda_{1,1,1}^9)^e$	$\bullet$	$(N^{12})^g$	$(K_{1,1}^{12})^h$	$\bullet$	$(F_{1,1}^{16})^j$	$(f_1')^k$
I :	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$(\Lambda_{1,1,1}^9)^e$	$(M_1^{10})^f$	$\bullet$	$(K_{1,1}^{12})^h$	$\bullet$	$(F_{1,1}^{16})^j$	$(f_1')^k$
J :	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$(\Lambda_{1,1,1}^9)^e$	$(M_1^{10})^f$	$(N^{12})^g$	$(K_{1,1}^{12})^h$	$\bullet$	$\bullet$	$(f_1')^k$
K :	$\bullet$	$\bullet$	$\bullet$	$(M^8)^d$	$\bullet$	$(M_1^{10})^f$	$(N^{12})^g$	$\bullet$	$(H_1^{14})^i$	$\bullet$	$(f_1')^k$
L :	$\bullet$	$\bullet$	$\bullet$	$(M^8)^d$	$\bullet$	$(M_1^{10})^f$	$(N^{12})^g$	$(K_{1,1}^{12})^h$	$(H_1^{14})^i$	$\bullet$	$\bullet$
M :	$\bullet$	$\bullet$	$\bullet$	$(M^8)^d$	$(\Lambda_{1,1,1}^9)^e$	$(M_1^{10})^f$	$(N^{12})^g$	$\bullet$	$\bullet$	$\bullet$	$(f_1')^k$
N :	$\bullet$	$\bullet$	$(\Lambda_{1,1}^7)^c$	$(M^8)^d$	$(\Lambda_{1,1,1}^9)^e$	$\bullet$	$(N^{12})^g$	$\bullet$	$\bullet$	$\bullet$	$(f_1')^k$
O :	$\bullet$	$(\Lambda_1^5)^b$	$(\Lambda_{1,1}^7)^c$	$(M^8)^d$	$\bullet$	$\bullet$	$(N^{12})^g$	$\bullet$	$\bullet$	$\bullet$	$(f_1')^k$
P :	$(\Lambda^3)^a$	$(\Lambda_1^5)^b$	$\bullet$	$(M^8)^d$	$\bullet$	$\bullet$	$(N^{12})^g$	$\bullet$	$\bullet$	$\bullet$	$(f_1')^k$

Ensuite, lorsqu'on effectue la somme des caractéristiques d'Euler des fibrés de Schur correspondants, il n'est pas nécessaire de réorganiser ces familles de telle sorte qu'elles soient d'intersection vide, puisque de toute façon, chaque intersection entre deux familles ne contribuera au final qu'en $\mathrm{O}(m^7)$. Nous pouvons donc additionner les seize couples d'intégrales correspondantes. Voici les deux premières, que l'on confie aisément à Maple :

$$\begin{aligned}
\mathsf{A}_1 &= \frac{1}{6}\int_0^m dk \int_0^{\frac{m-k}{16}} dj \int_0^{\frac{m-16j-k}{14}} di \int_0^{\frac{m-14i-16j-k}{12}} dh \int_0^{\frac{m-12h-14i-16j-k}{12}} dg \\
&\qquad \big[(m-9g-8h-10i-11j)^3-(3g+2h+3i+3j)^3\big]+\mathrm{O}(m^7) \\
&= \frac{36562817\ m^8}{4933428814282752}+\mathrm{O}(m^7)\,; \\
\mathsf{A}_2 &= \frac{1}{6}\int_0^m dk \int_0^{\frac{m-k}{16}} dj \int_0^{\frac{m-16j-k}{14}} di \int_0^{\frac{m-14i-16j-k}{12}} dh \int_0^{\frac{m-12h-14i-16j-k}{12}} dg \\
&\qquad (m-12g-10h-13i-14j)^3+\mathrm{O}(m^7) \\
&= \frac{5015441\ m^8}{1233357203570688}+\mathrm{O}(m^7)\,;
\end{aligned}$$

les quinze autres fournissent des expressions similaires. Nous déduisons donc dans chacun des deux cas par sommation de seize nombres rationnels :

$$\mathscr{C}_1 = \frac{159897336810563}{356792619604377600000}+\mathrm{O}(m^7), \qquad \mathscr{C}_2 = \frac{784698232169}{3303635366707200000}+\mathrm{O}(m^7),$$

et finalement, le quotient significatif :

$$\mathscr{C} = \frac{\mathscr{C}_1}{\mathscr{C}_2} = 1,\,887\cdots$$

est inférieur à celui de $\mathscr{D}\mathscr{S}_2^4 T_X^*$: confirmation supplémentaire de l'inadéquation et de l'insuffisance du sous-fibré engendrée par les crochets.

Problème ouvert. Changer d'optique quant au calcul de Riemann-Roch, en tenant compte d'une étude préalable, reprise à partir de zéro, de la structure spécifique de $\mathscr{D}\mathscr{S}^5_2 T^*_X$. Le procédé de division par f'_1 des générateurs de l'idéal des relations entre invariants restreints à $\{f'_1 = 0\}$ constitue un procédé adéquat et complet d'engendrement qui doit être poussé au-delà de X^{25}.

§9. Appendice 1 : jets d'ordre 3 en dimension 3

Expression initiale. Comme annoncé dans la Section 4, nous détaillons ici le calcul (délicat) des trois crochets entre les trois invariants $\Lambda^7_{1,1}$, $\Lambda^7_{1,2}$, $\Lambda^7_{2,2}$ de poids 7:

$$\frac{\big[\Lambda^7_{i,j},\, \Lambda^7_{k,l}\big]}{7} = \mathsf{D}\Lambda^7_{i,j} \cdot \Lambda^7_{k,l} - \Lambda^7_{i,j} \cdot \mathsf{D}\Lambda^7_{k,l}.$$

Développement économe. N'écrivons que le premier produit, en résumant le second par le symbole $(i,j) \longleftrightarrow (k,l)$, parce qu'il s'en déduit par ce simple changement d'indices:

$$\begin{aligned}
&= \Big(\Delta^{1,5}\, f_i f'_j + 5\,\Delta^{2,4}\, f'_i f'_j - 4\,\Delta^{1,4}\big(f''_i f'_j + f'_i f''_j\big) - 16\,\Delta^{2,3}\big(f''_i f'_j + f'_i f''_j\big) - \\
&\quad - 5\,\Delta^{1,3}\big(f'''_i f'_j + f'_i f'''_j\big) + 35\,\Delta^{1,3}\, f''_i f''_j\Big) \cdot \Big(\Delta^{1,4}\, f'_k f'_l + 4\,\Delta^{2,3}\, f'_k f'_l - \\
&\quad - 5\,\Delta^{1,3}\big(f''_k f'_l + f'_k f''_l\big) + 15\,\Delta^{1,2}\, f''_k f''_l\Big) - \\
&\quad - (i,j) \longleftrightarrow (k,l)
\end{aligned}$$

$$\begin{aligned}
= {}& \underline{\Delta^{1,5}\,\Delta^{1,4}\, f'_i f'_j f'_k f'_l}_{\circ} + \underline{5\,\Delta^{2,4}\,\Delta^{1,4}\, f'_i f'_j f'_k f'_l}_{\circ} - 4\,\Delta^{1,4}\,\Delta^{1,4}\big(f''_i f'_j + f'_i f''_j\big)\, f'_k f'_l - \\
& - 16\,\Delta^{2,3}\,\Delta^{1,4}\big(f''_i f'_j + f'_i f''_j\big)\, f'_k f'_l - 5\,\Delta^{1,4}\,\Delta^{1,3}\big(f'''_i f'_j + f'_i f'''_j\big)\, f'_k f'_l + \\
& + 35\,\Delta^{1,4}\,\Delta^{1,3}\, f''_i f''_j f'_k f'_l + \underline{4\,\Delta^{1,5}\,\Delta^{2,3}\, f'_i f'_j f'_k f'_l}_{\circ} + \underline{20\,\Delta^{2,4}\,\Delta^{2,3}\, f'_i f'_j f'_k f'_l}_{\circ} - \\
& - 16\,\Delta^{1,4}\,\Delta^{2,3}\big(f''_i f'_j + f'_i f''_j\big)\, f'_k f'_l - 64\,\Delta^{2,3}\,\Delta^{2,3}\big(f''_i f'_j + f'_i f''_j\big)\, f'_k f'_l - \\
& - 20\,\Delta^{2,3}\,\Delta^{1,3}\big(f'''_i f'_j + f'_i f'''_j\big)\, f'_k f'_l + 140\,\Delta^{2,3}\,\Delta^{1,3}\, f''_i f''_j f'_k f'_l - \\
& - 5\,\Delta^{1,5}\,\Delta^{1,3}\, f'_i f'_j\big(f''_k f'_l + f'_k f''_l\big) - 25\,\Delta^{2,4}\,\Delta^{1,3}\, f'_i f'_j\big(f''_k f'_l + f'_k f''_l\big) + \\
& + \underline{20\,\Delta^{1,4}\,\Delta^{1,3}\big(f''_i f'_j + f'_i f''_j\big)\big(f''_k f'_l + f'_k f''_l\big)}_{\circ} + \underline{80\,\Delta^{2,3}\,\Delta^{1,3}\big(f''_i f'_j + f'_i f''_j\big)\big(f''_k f'_l + f'_k f''_l\big)}_{\circ} + \\
& + 25\,\Delta^{1,3}\,\Delta^{1,3}\big(f'''_i f'_j + f'_i f'''_j\big)\big(f''_k f'_l + f'_k f''_l\big) - 175\,\Delta^{1,3}\,\Delta^{1,3}\, f''_i f''_j\big(f''_k f'_l + f'_k f''_l\big) + \\
& + 15\,\Delta^{1,5}\,\Delta^{1,2}\, f'_i f'_j f''_k f''_l + 75\,\Delta^{2,4}\,\Delta^{1,2}\, f'_i f'_j f''_k f''_l - 60\,\Delta^{1,4}\,\Delta^{1,2}\big(f''_i f'_j + f'_i f''_j\big)\, f''_k f''_l - \\
& - 240\,\Delta^{2,3}\,\Delta^{1,2}\big(f''_i f'_j + f'_i f''_j\big)\, f''_k f''_l - 75\,\Delta^{1,3}\,\Delta^{1,2}\big(f'''_i f'_j + f'_i f'''_j\big)\, f''_k f''_l + \\
& + \underline{525\,\Delta^{1,3}\,\Delta^{1,2}\, f''_i f''_j f''_k f''_l}_{\circ} - \\
& - (i,j) \longleftrightarrow (k,l).
\end{aligned}$$

Nous soulignons (en ajoutant un petit cercle) les termes qui s'annihilent avec ceux qui leur correspondent dans la permutation $(i,j) \longleftrightarrow (k,l)$. Nous utilisons les relations plückeriennes pour remplacer le dernier terme restant, à savoir:

$$-75\,\Delta^{1,3}\,\Delta^{1,2}\big(f'''_i f'_j + f'_i f'''_j\big)\, f''_k f''_l + 75\,\Delta^{1,3}\,\Delta^{1,2}\big(f'''_k f'_l + f'_k f'''_l\big)\, f''_i f''_j$$

par:

$$- 75\,\Delta^{1,3}\,\Delta^{1,3}\,f_i''f_j'f_k''f_l'' + 75\,\Delta^{1,3}\,\Delta^{2,3}\,f_i'f_j'f_k''f_l''-$$
$$- 75\,\Delta^{1,3}\,\Delta^{1,3}\,f_i'f_j''f_k''f_l'' + 75\,\Delta^{2,3}\,\Delta^{1,3}\,f_i'f_j'f_k''f_l'',$$

et nous additionnons tous ces termes en effectuant des regroupements qui n'impliquent que des sommations de nombres entiers:

$$\begin{aligned}
&= -5\,\Delta^{1,5}\,\Delta^{1,3}\,f_i'f_j'(f_k''f_l' + f_k'f_l'') - 25\,\Delta^{2,4}\,\Delta^{1,3}\,f_i'f_j'(f_k''f_l' + f_k'f_l'') + 15\,\Delta^{1,5}\,\Delta^{1,2}\,f_i'f_j'f_k''f_l''+\\
&\quad + 75\,\Delta^{2,4}\,\Delta^{1,2}\,f_i'f_j'f_k''f_l'' - 4\,\Delta^{1,4}\,\Delta^{1,4}(f_i''f_j' + f_i'f_j'')\,f_k'f_l' - 32\,\Delta^{1,4}\,\Delta^{2,3}(f_i''f_j' + f_i'f_j'')\,f_k'f_l'-\\
&\quad - 64\,\Delta^{2,3}\,\Delta^{2,3}(f_i''f_j' + f_i'f_j'')\,f_k'f_l' - 5\,\Delta^{1,4}\,\Delta^{1,3}(f_i'''f_j' + f_i'f_j''')\,f_k'f_l' + 35\,\Delta^{1,4}\,\Delta^{1,3}\,f_i''f_j''f_k'f_l'-\\
&\quad - 20\,\Delta^{2,3}\,\Delta^{1,3}(f_i'''f_j' + f_i'f_j''')\,f_k'f_l' + 140\,\Delta^{2,3}\,\Delta^{1,3}\,f_i''f_j''f_k'f_l' + 150\,\Delta^{2,3}\,\Delta^{1,3}\,f_i'f_j'f_k''f_l''-\\
&\quad - 60\,\Delta^{1,4}\,\Delta^{1,2}(f_i''f_j' + f_i'f_j'')\,f_k''f_l'' - 240\,\Delta^{2,3}\,\Delta^{1,2}(f_i''f_j' + f_i'f_j'')\,f_k''f_l''+\\
&\quad + 25\,\Delta^{1,3}\,\Delta^{1,3}(f_i'''f_j' + f_i'f_j''')(f_k''f_l' + f_k'f_l'') - 175\,\Delta^{1,3}\,\Delta^{1,3}\,f_i''f_j''(f_k''f_l' + f_k'f_l'')-\\
&\quad - 75\,\Delta^{1,3}\,\Delta^{1,3}(f_i''f_j' + f_i'f_j'')\,f_k''f_l''-\\
&\quad - (i,j) \longleftrightarrow (k,l).
\end{aligned}$$

Synthèse de déterminants. Maintenant, la soustraction suivie de la permutation fait apparaître des déterminants 2×2: on a en effet cinq relations immédiatement vérifiables par développement:

$$\begin{aligned}
f_i'f_j'(f_k''f_l' + f_k'f_l'') - f_k'f_l'(f_i''f_j' + f_i'f_j'') &= f_j'f_l'\,\Delta^{1,2}_{i,k} + f_i'f_k'\,\Delta^{1,2}_{j,l},\\
f_i'f_j'f_k''f_l'' - f_i''f_j''f_k'f_l' &= f_i'f_l''\,\Delta^{1,2}_{j,k} + f_k'f_j''\,\Delta^{1,2}_{i,l},\\
(f_i'''f_j' + f_i'f_j''')\,f_k'f_l' - (f_k'''f_l' + f_k'f_l''')\,f_i'f_j' &= f_j'f_l'\,\Delta^{1,3}_{k,i} + f_i'f_k'\,\Delta^{1,3}_{l,j},\\
(f_i''f_j' + f_i'f_j'')\,f_k''f_l'' - (f_k''f_l' + f_k'f_l'')f_i''f_j'' &= f_i''f_k''\,\Delta^{1,2}_{j,l} + f_j''f_l''\,\Delta^{1,2}_{i,k},
\end{aligned}$$

$$\begin{aligned}
&(f_i'''f_j' + f_i'f_j''')(f_k''f_l' + f_k'f_l'') - (f_k'''f_l' + f_k'f_l''')(f_i''f_j' + f_i'f_j'')\\
&\qquad = f_j'f_l'\,\Delta^{2,3}_{k,i} + f_j'f_k'\,\Delta^{2,3}_{l,i} + f_i'f_l'\,\Delta^{2,3}_{k,j} + f_i'f_k'\,\Delta^{2,3}_{l,j}.
\end{aligned}$$

En regroupant les termes, on obtient l'expression finale de $\frac{1}{7}\left[\Lambda^7_{i,j},\,\Lambda^7_{k,l}\right]$. $\square$

§10. Appendice 2 : jets d'ordre 3 en dimension 3

Jets d'ordre $\kappa = 3$ en dimension $\nu = 3$. Pour terminer, donnons une description "à la main" des générateurs de $\mathscr{D}\mathscr{S}^3_3$ qui ne fasse pas appel à des arguments raffinés de théorie des invariants (*cf.* [5]).

Recherchons directement les bi-invariants, *i.e.* les polynômes invariants par reparamétrisation qui sont aussi invariants par l'action du sous-groupe unipotent $\mathsf{U}_3(\mathbb{C}) \subset \mathsf{GL}_3(\mathbb{C})$ constitué des matrices de la forme:

$$\mathrm{U} := \begin{pmatrix} 1 & 0 & 0\\ u_a & 1 & 0\\ u_c & u_b & 1 \end{pmatrix},$$

qui est définie par $\mathrm{U}\cdot f_1^{(\lambda)} := f_1^{(\lambda)}$, puis $\mathrm{U}\cdot f_2^{(\lambda)} := f_2^{(\lambda)} + u_a f_1^{(\lambda)}$ et enfin $\mathrm{U}\cdot f_3^{(\lambda)} := f_3^{(\lambda)} + u_b\,f_2^{(\lambda)} + u_c\,f_1^{(\lambda)}$, pour $\lambda = 1, 2, 3$. Un premier raisonnement

initial entièrement similaire à celui que nous avons tiré de [5] pour $\mathscr{DS}^4_{2,m}$ fournit une représentation de tout $\mathsf{P} \in \mathscr{DS}^3_{3,m}$ sous la forme rationnelle

$$\mathsf{P}\big(j^3 f\big) = \sum_{-\frac{2}{3}m \leqslant a \leqslant m} (f_1')^a \, \mathscr{P}_a\big(f_2', f_3', \Lambda^3_{1,2}, \Lambda^3_{1,3}, \Lambda^5_{1,2;1}, \Lambda^5_{1,3;1}\big),$$

où les invariants $\Lambda^3_{i,j}$ et $\Lambda^5_{i,j;k}$ sont simplement définis par:

$$\Lambda^3_{i,j} := \Delta^{1,2}_{i,j} \qquad \text{et} \qquad \Lambda^5_{i,j;k} := \Delta^{1,3}_{i,j} \, f_k' - 3 \, \Delta^{1,2}_{i,j} \, f_k''.$$

Ensuite, si nous considérons le sous-groupe de $\mathsf{U}_3(\mathbb{C})$ constitué des matrices de la forme:

$$\overline{\mathsf{U}} := \begin{pmatrix} 1 & 0 & 0 \\ u_a & 1 & 0 \\ u_c & 0 & 1 \end{pmatrix},$$

lesquelles stabilisent tous les invariants qui apparaissent dans notre première expression rationnelle:

$$\begin{aligned} \overline{\mathsf{U}} \cdot \Lambda^3_{1,2} &= \Lambda^3_{1,2}, & \overline{\mathsf{U}} \cdot \Lambda^5_{1,2;1} &= \Lambda^5_{1,2;1}, \\ \overline{\mathsf{U}} \cdot \Lambda^3_{1,3} &= \Lambda^3_{1,3}, & \overline{\mathsf{U}} \cdot \Lambda^5_{1,3;1} &= \Lambda^5_{1,3;1}, \end{aligned}$$

mais agissent en perturbant f_2' et f_3' par $\overline{\mathsf{U}} \cdot f_2' = f_2' + u_a \, f_1'$ et $\overline{\mathsf{U}} \cdot f_3' = f_3' + u_c \, f_1'$, nous voyons que si P est aussi invariant par l'action unipotente, alors chaque polynôme $\mathscr{P}_a$ ci-dessus doit en fait être indépendant de f_2' et de f_3', d'où:

$$\mathsf{P}^{2\times\mathrm{inv}}\big(j^3 f\big) = \sum_{-\frac{2}{3}m \leqslant a \leqslant m} (f_1')^a \, \mathscr{P}_a\big(\Lambda^3_{1,2}, \, \Lambda^3_{1,3}, \, \Lambda^5_{1,2;1}, \, \Lambda^5_{1,3;1}\big).$$

Mais ce n'est pas terminé, car il faut encore s'assurer de l'invariance par l'action du sous-groupe constitué des matrices de la forme:

$$\mathsf{U}_b := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u_b & 1 \end{pmatrix},$$

lesquelles agissent comme suit sur les quatre invariants algébriquement indépendants qui apparaissent:

$$\begin{aligned} \mathsf{U}_b \cdot \Lambda^3_{1,2} &= \Lambda^3_{1,2}, & \mathsf{U}_b \cdot \Lambda^5_{1,2;1} &= \Lambda^5_{1,2;1}, \\ \mathsf{U}_b \cdot \Lambda^3_{1,3} &= \Lambda^3_{1,3} + u_b \, \Lambda^3_{1,2}, & \mathsf{U}_b \cdot \Lambda^5_{1,3;1} &= \Lambda^5_{1,3;1} + u_b \, \Lambda^5_{1,2;1}. \end{aligned}$$

D'après un calcul direct, le troisième et dernier invariant fondamental pour cette action, à savoir:

$$\Lambda^3_{1,2} \, \Lambda^5_{1,3;1} - \Lambda^3_{1,3} \, \Lambda^5_{1,2;1} \equiv f_1' f_1' \, D^6_{1,2,3}$$

fait naître, en tant que nouveau bi-invariant "fantôme" caché derrière $(f_1')^2$, le déterminant wronskien en dimension trois:

$$D^6_{1,2,3} := \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix},$$

et en injectant ce nouveau bi-invariant, nous obtenons une nouvelle expression pour le bi-invariant quelconque dont nous étions partis:

$$\mathsf{P}^{2\times\mathrm{inv}}\big(j^3 f\big) = \sum_{-\frac{2}{3}m \leqslant a \leqslant m} (f_1')^a \, \mathscr{P}_a\big(\Lambda^3_{1,2},\, \Lambda^5_{1,2;1},\, D^6_{1,2,3}\big),$$

laquelle incorpore toujours des puissances négatives de f_1'. Mais pour terminer, nous affirmons qu'il n'existe en fait aucune puissance négative de f_1', car sinon, après multiplication par la puissance maximalement négative de f_1' (*cf.* le raisonnement conduit dans la Section 6) et après restriction à $\{f_1' = 0\}$, nous obtiendrions une identité du type:

$$\begin{aligned} 0 &\equiv \mathscr{P}_a\big(\Lambda^3_{1,2},\, \Lambda^5_{1,2;1},\, D^6_{1,2,3}\big)\Big|_{f_1'=0} \\ &= \mathscr{P}_a\Big(- f_1'' f_2',\, 3\, f_1'' f_2' f_1'',\, \begin{vmatrix} 0 & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix} \Big), \end{aligned}$$

qui impliquerait immédiatement $\mathscr{P}_a \equiv 0$, par indépendance algébrique de ses trois arguments.

En conclusion, nous avons redémontré (*cf.* [5]) qu'en dimension $\nu = 3$ pour les jets d'ordre $\kappa = 3$, les polynômes bi-invariants s'écrivent:

$$\mathscr{P}^{2\times\mathrm{inv}}\big(j^3 f\big) = \mathscr{P}\big(f_1',\, \Lambda^3_{1,2},\, \Lambda^5_{1,2;1},\, D^6_{1,2,3}\big),$$

où $\mathscr{P}$ est un polynôme arbitraire, aucune syzygie n'existant entre ces quatre bi-invariants fondamentaux, et par conséquent, en polarisant les indices — ce qui revient à faire agir le groupe complet $\mathsf{GL}_3(\mathbb{C})$ —, nous déduisons que les polynômes généraux invariants par reparamétrisation s'écrivent comme polynômes quelconques en fonction de 16 invariants fondamentaux:

$$\begin{aligned} \mathsf{P}\big(j^3 f\big) = \mathscr{P}\big(&f_1',\, f_2',\, f_3', \Lambda^3_{1,2},\, \Lambda^3_{1,3},\, \Lambda^3_{2,3},\, \Lambda^5_{1,2;1},\, \Lambda^5_{1,2;2},\, \Lambda^5_{1,2;3}, \\ &\Lambda^5_{1,3;1},\, \Lambda^5_{1,3;2},\, \Lambda^5_{1,3;3},\, \Lambda^5_{2,3;1},\, \Lambda^5_{2,3;2},\, \Lambda^5_{2,3;3},\, D^6_{1,2,3}\big). \end{aligned}$$

On vérifie aussi que parmi les 62 syzygies existant entre ces 16 invariants qui ont été obtenues par un calcul sur Maple (*cf.* les références dans [5]), 30 d'entre elles sont fondamentales et qu'elles proviennent toutes des trois procédures que nous avons décrites. □

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